



Lecture 3

Local Feature Descriptors and Matching

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Content

- Scale Invariant Feature Transform
- Case Study: Homography Estimation
 - Matrix Differentiation
 - Lagrange Multiplier
 - Least-squares for Linear Systems
 - Problem of Homography Estimation
 - RANSAC-based Homography Estimation



SIFT

- Scale Invariant Feature Transform
 - Proposed in [1]
 - It uses extrema of DoG to detect key points and the associated characteristic scales
 - It uses SIFT to describe a key point

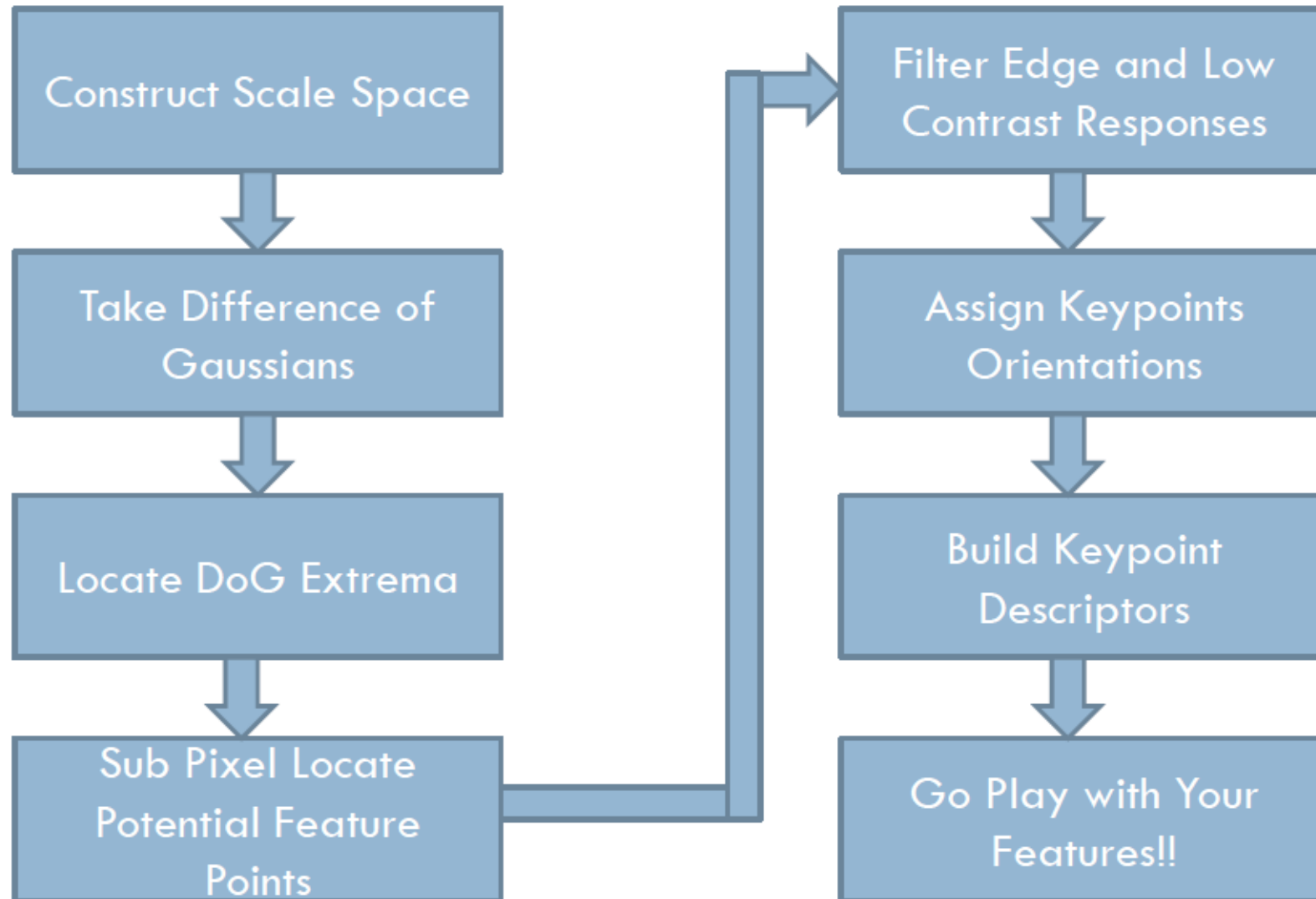
[1] D.G. Lowe, Distinctive image features from scale-invariant keypoints, *IJCV* 60 (2), pp. 91-110, 2004



Prof. David Lowe
University of British Columbia

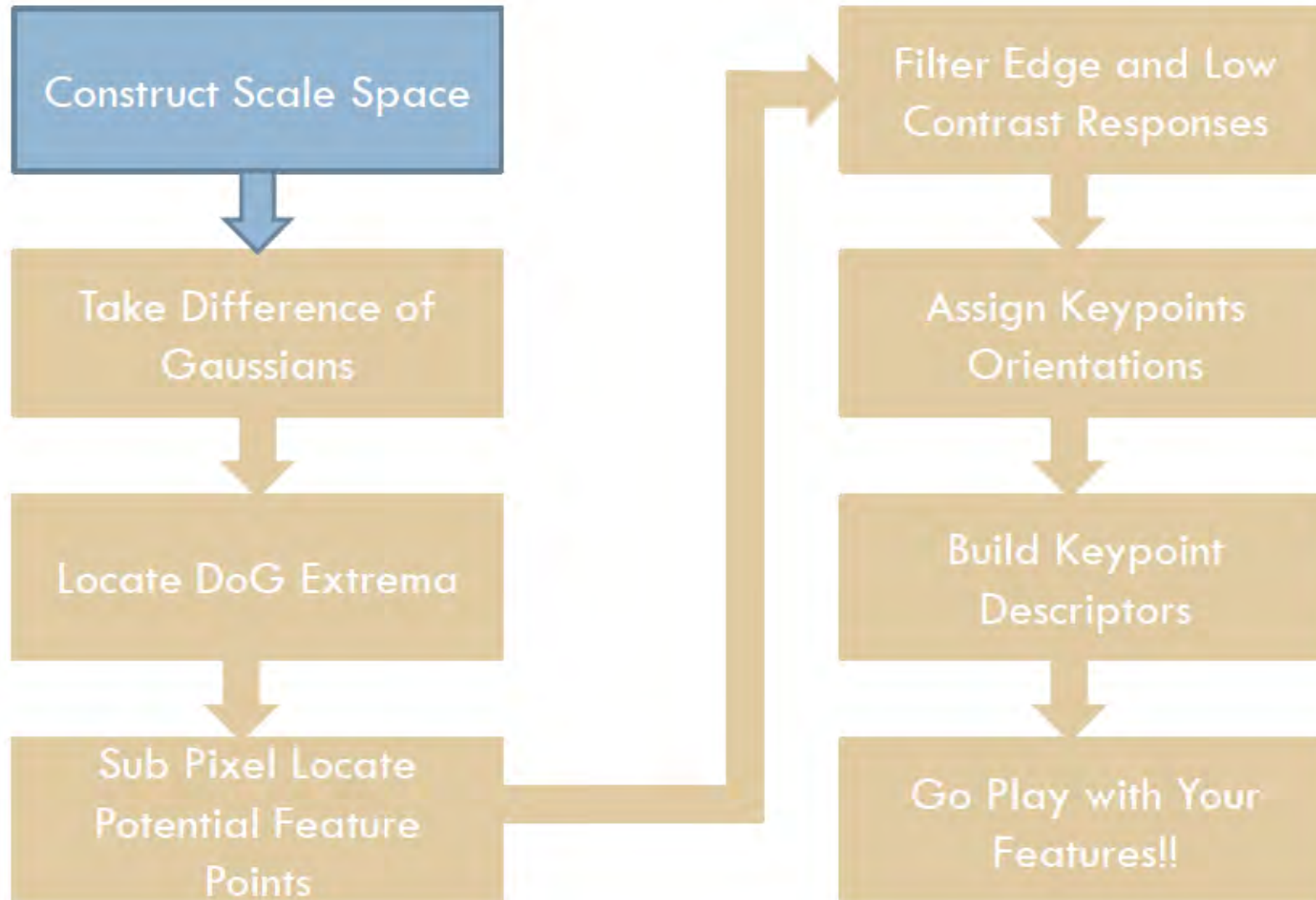


SIFT



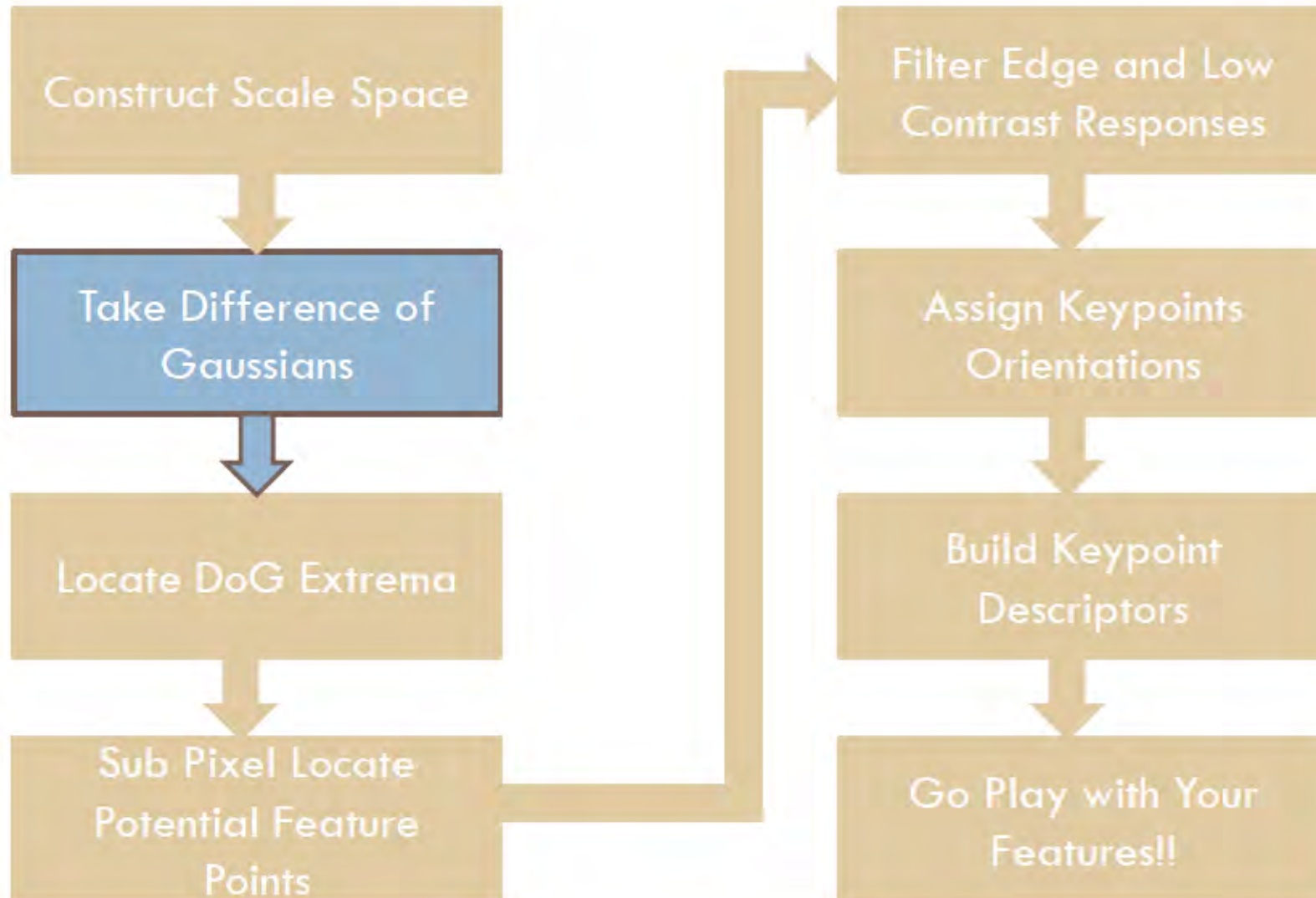


SIFT



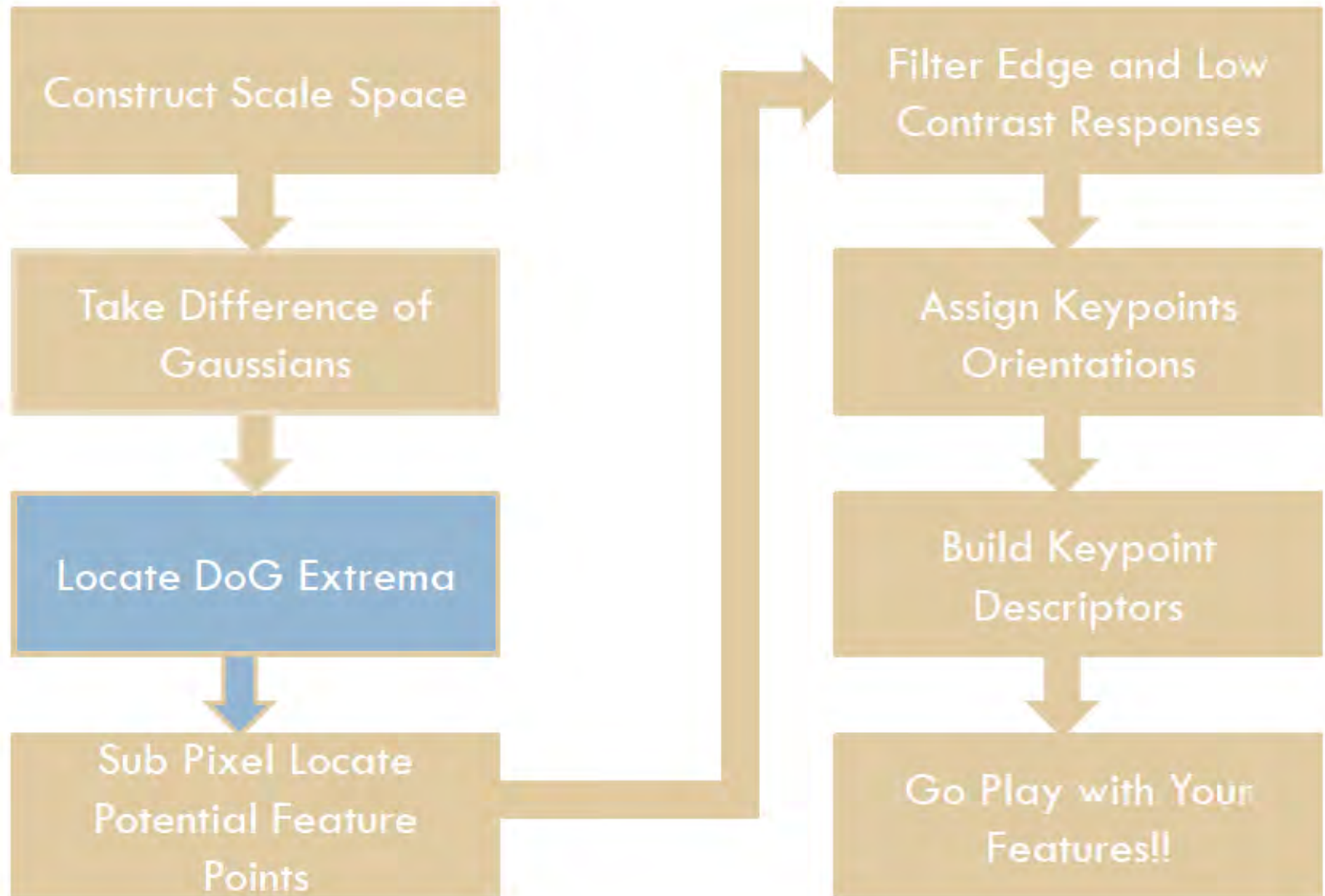


SIFT





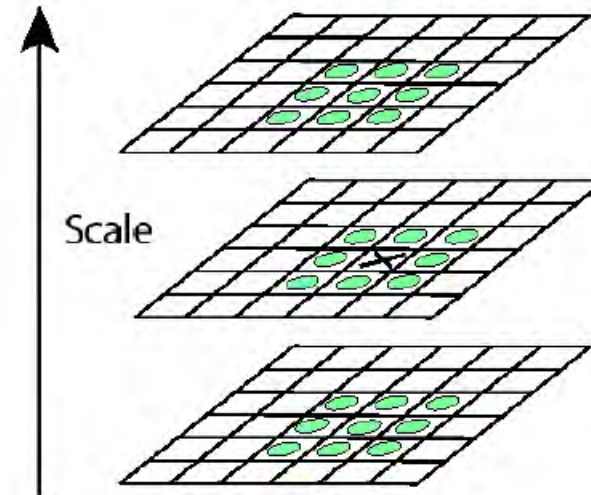
SIFT





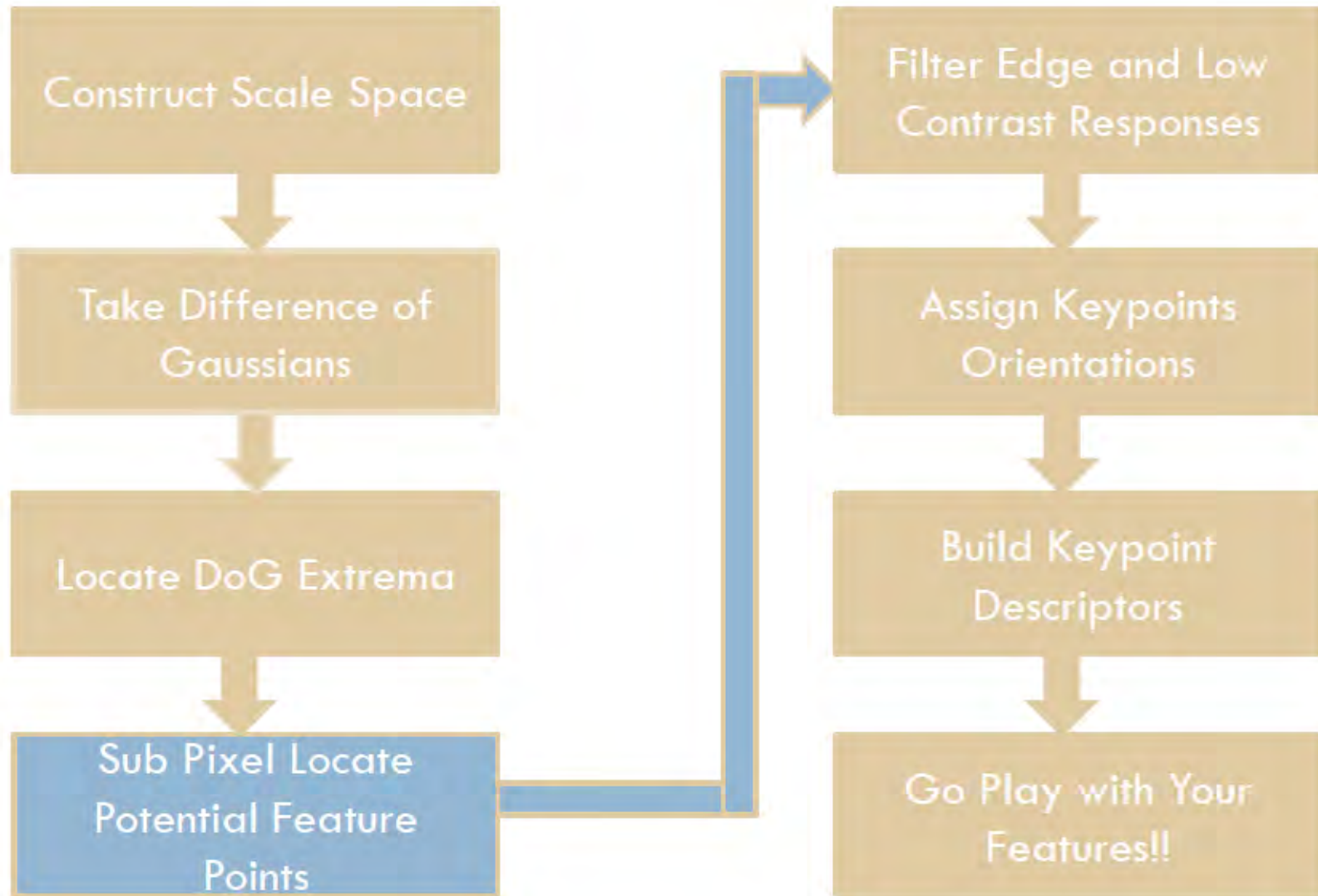
SIFT

- Scan each DOG image
 - ▣ Look at all neighboring points (including scale)
 - ▣ Identify Min and Max
 - 26 Comparisons



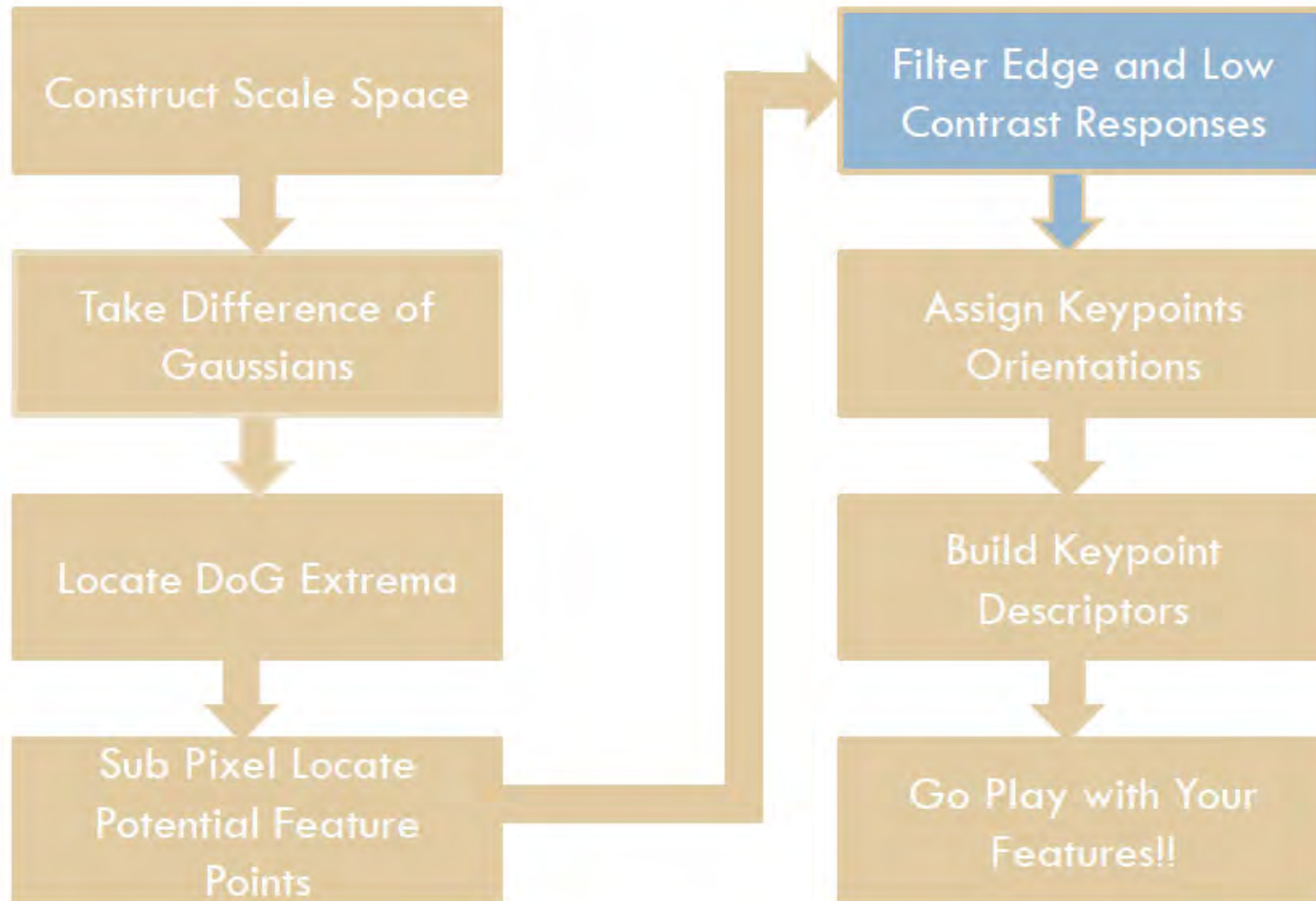


SIFT



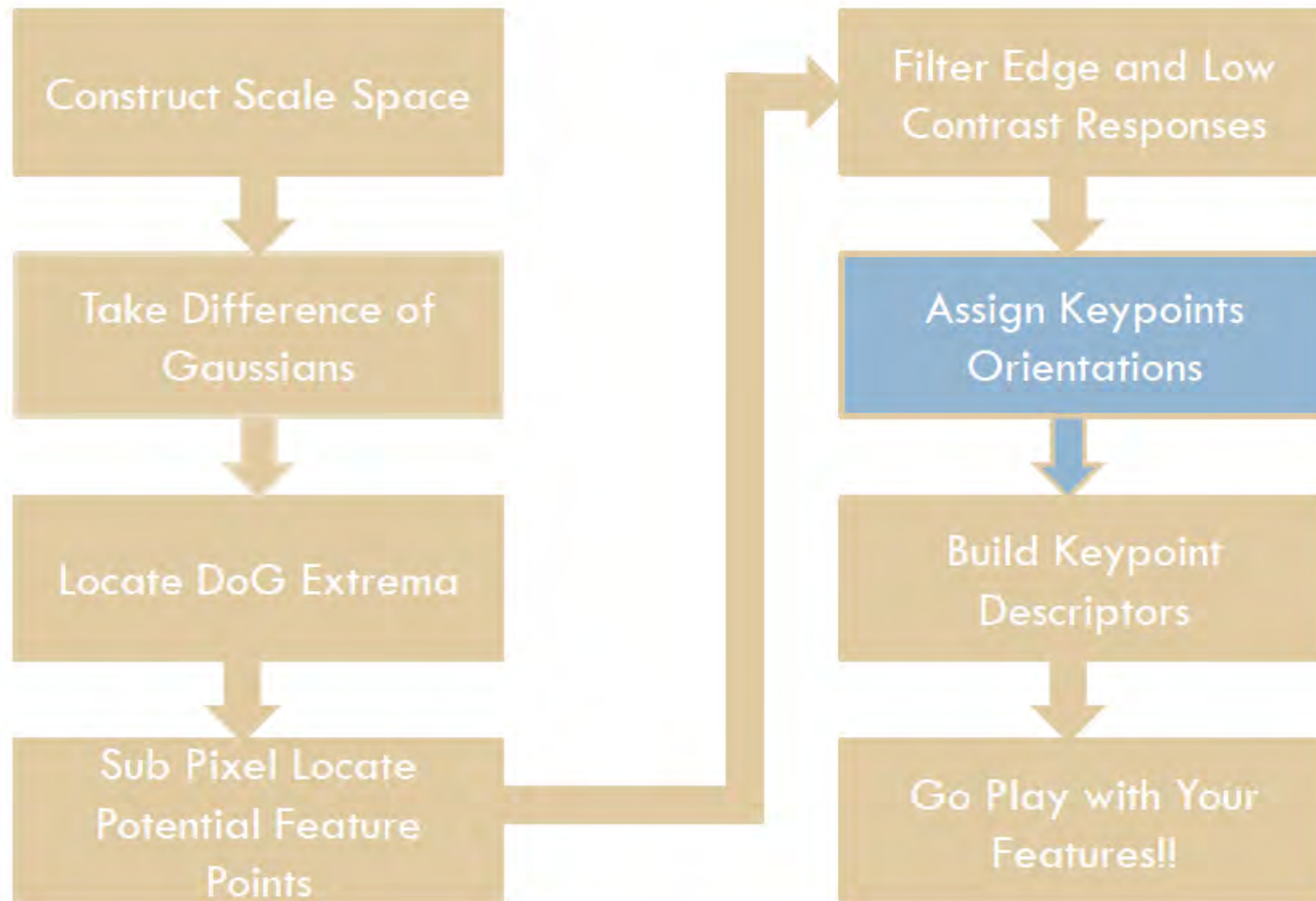


SIFT





SIFT



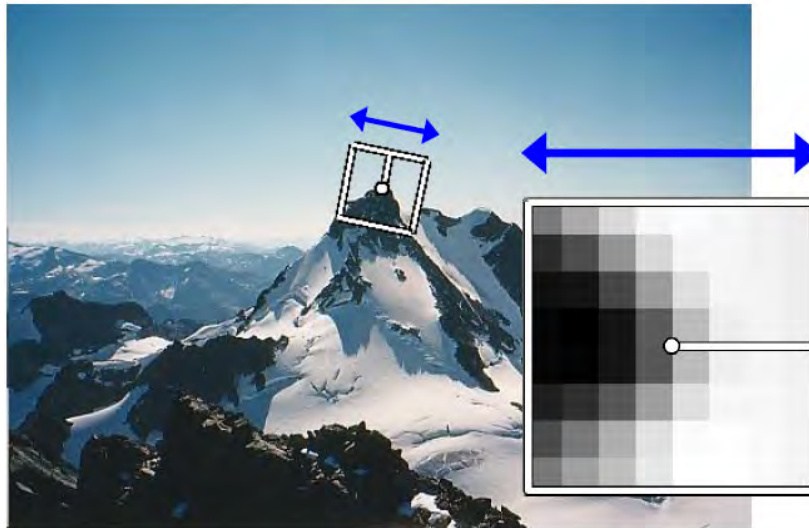


Assign Keypoints Orientations

- Assign orientation to the keypoint
 - Find local orientation: dominant orientation of gradient for the image patch (its size is determined by the characteristic scale)



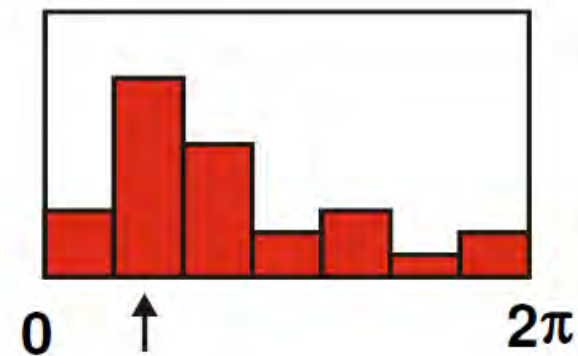
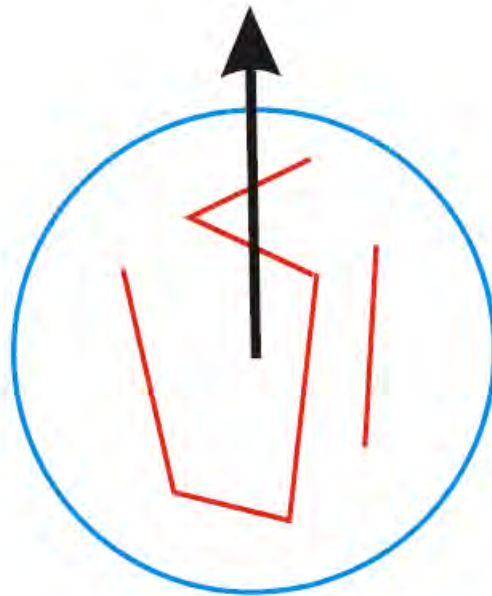
- Rotate the patch according to this angle; this can achieve rotation invariance description





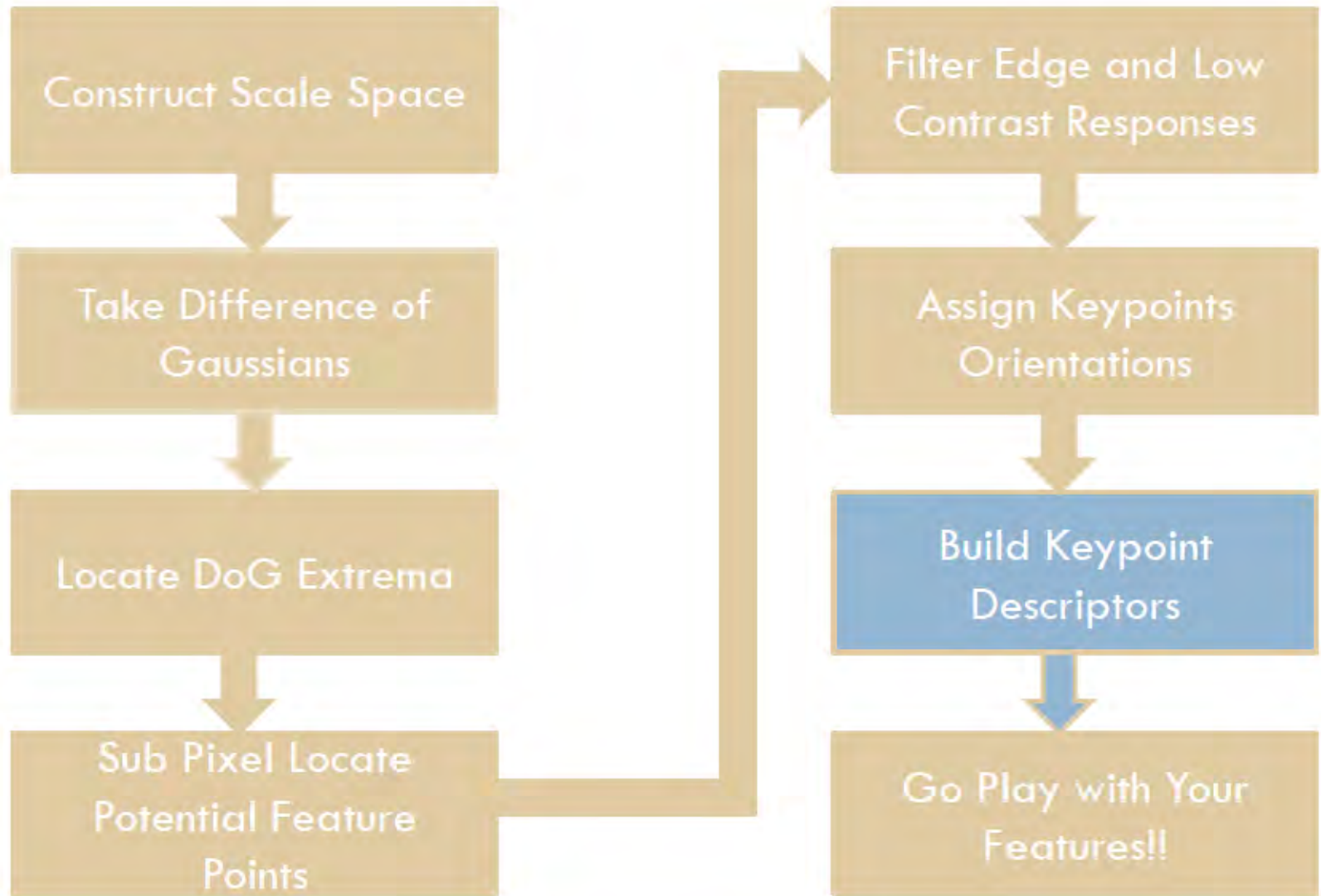
Assign Keypoints Orientations

- Orientation normalization
 - Compute orientation histogram
 - Select dominant orientation
 - Normalization: rotate the patch to the selected orientation





SIFT





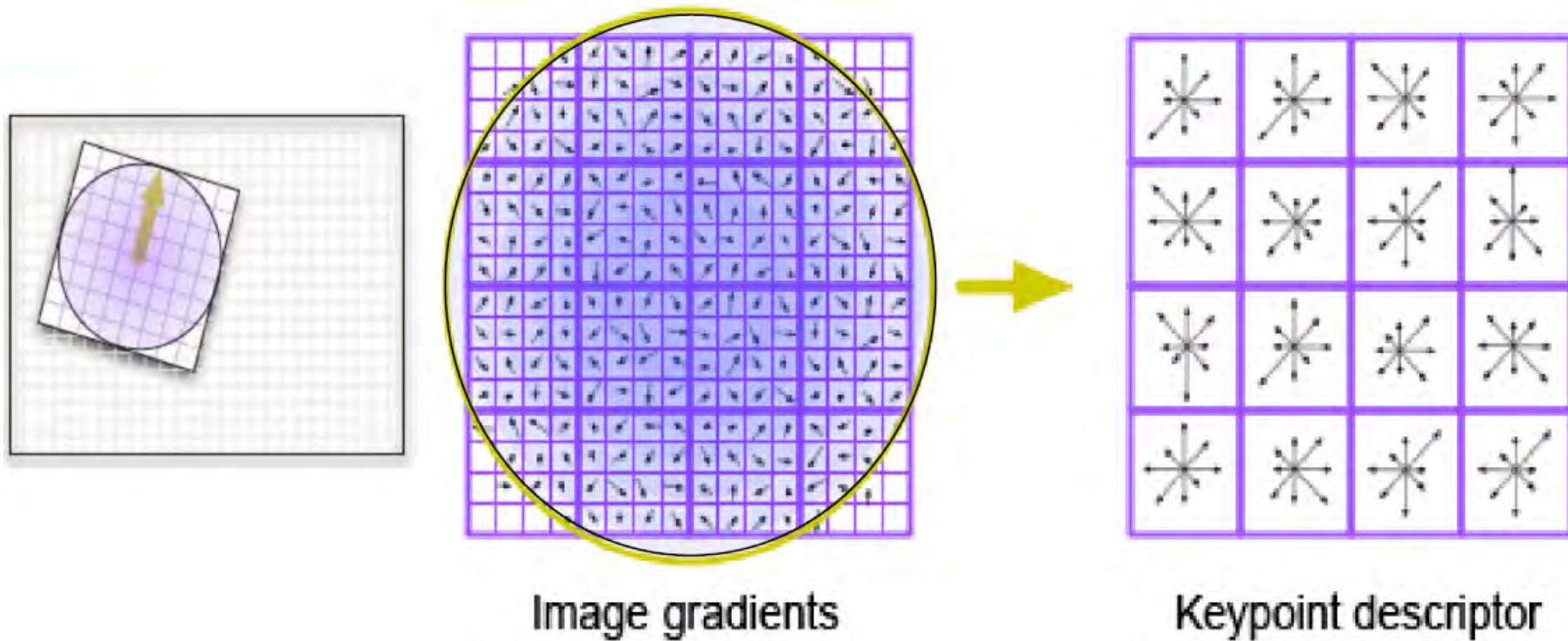
SIFT

- Building the descriptor
 - Sample the points around the keypoint
 - Rotate the gradients and coordinates by the previously computed orientation
 - Separate the region in to 4×4 sub-regions
 - Create gradient-orientation histogram for each sub-region with 8 bins (In real implementation, each sample point is weighted by a Gaussian)



SIFT

- Building the descriptor



- Actual implementation uses 4×4 sub regions which lead to a $4 \times 4 \times 8 = 128$ element vector



SIFT

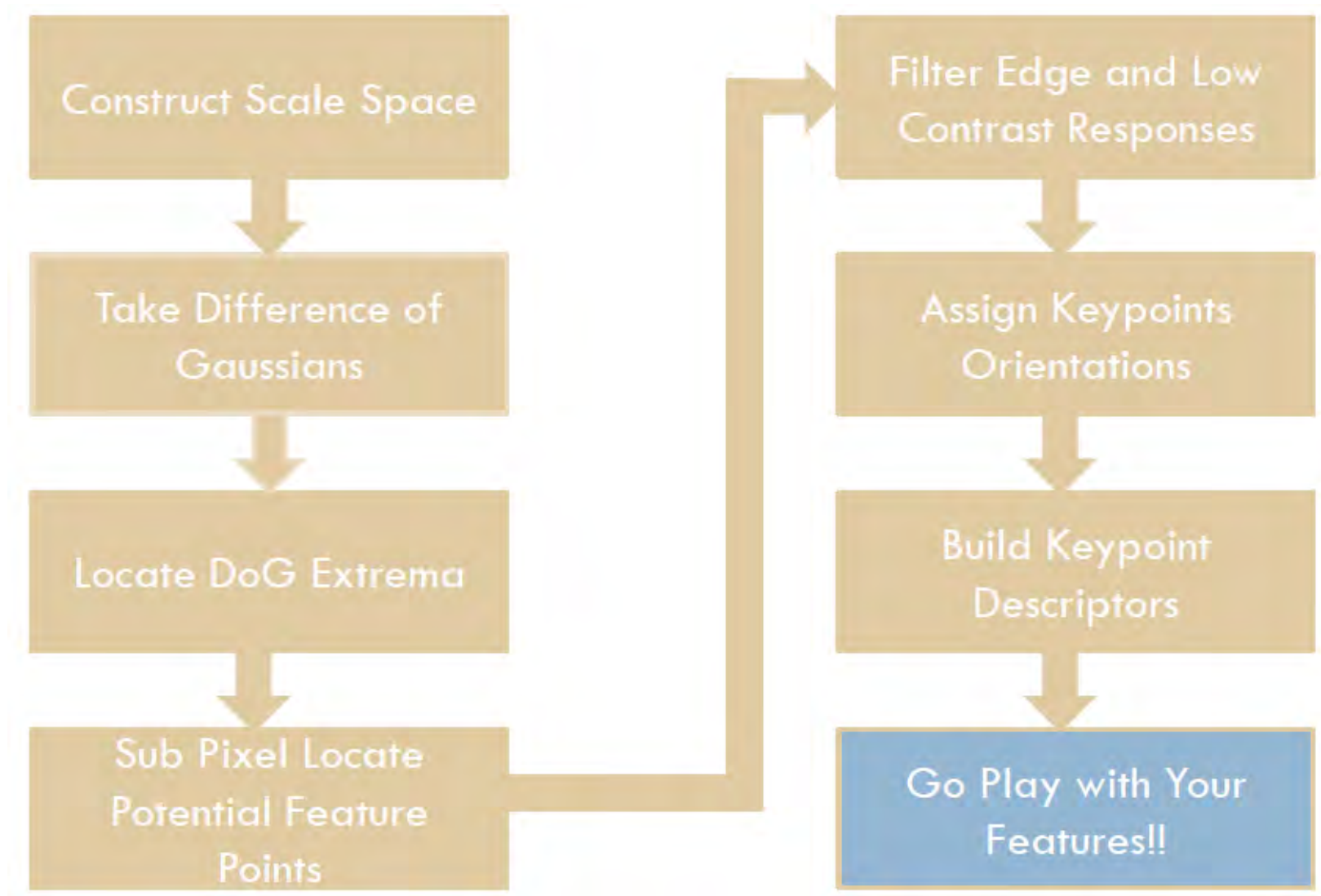
- One image yields:

- n 128-dimensional descriptors: each one is a histogram of the gradient orientations within a patch
 - $[n \times 128 \text{ matrix}]$
- n scale parameters specifying the size of each patch
 - $[n \times 1 \text{ vector}]$
- n orientation parameters specifying the angle of the patch
 - $[n \times 1 \text{ vector}]$
- n 2D points giving positions of the patches
 - $[n \times 2 \text{ matrix}]$





SIFT





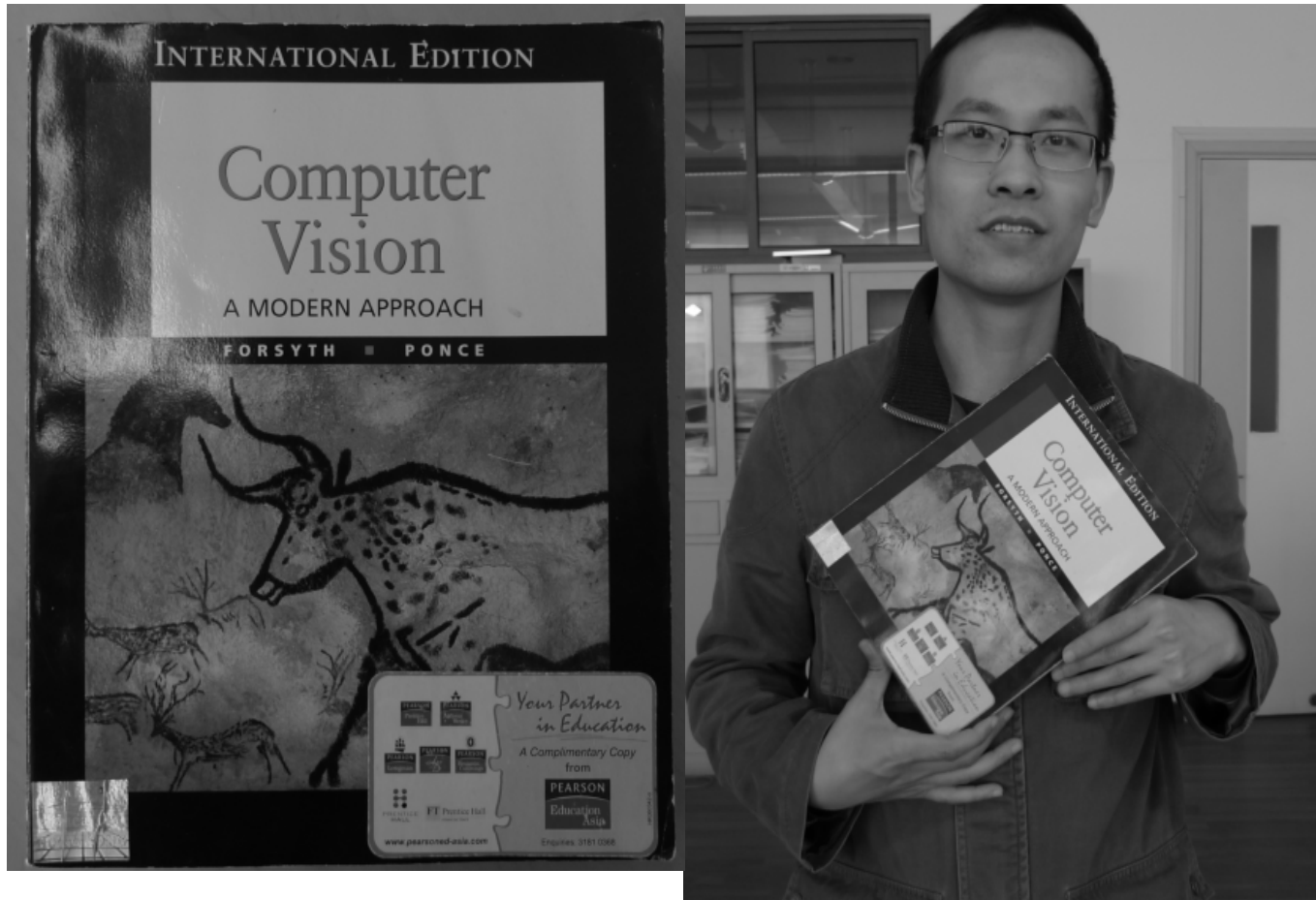
Applications of SIFT

- Object recognition
- Robot localization and mapping
- Panorama stitching
- 3D scene modeling, recognition and tracking
- Analyzing the human brain in 3D magnetic resonance images



Applications of SIFT

- Object recognition





Applications of SIFT

- Object recognition





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Matrix differentiation

- Function is a vector and the variable is a scalar

$$\mathbf{f}(t) = [f_1(t), f_2(t), \dots, f_n(t)]^T$$

Definition

$$\frac{d\mathbf{f}}{dt} = \left[\frac{df_1(t)}{dt}, \frac{df_2(t)}{dt}, \dots, \frac{df_n(t)}{dt} \right]^T$$



Matrix differentiation

- Function is a matrix and the variable is a scalar

$$\mathbf{F}(t) = \begin{bmatrix} f_{11}(t) & f_{12}(t), \dots, f_{1m}(t) \\ f_{21}(t) & f_{22}(t), \dots, f_{2m}(t) \\ \vdots & \\ f_{n1}(t) & f_{n2}(t), \dots, f_{nm}(t) \end{bmatrix} = \left[f_{ij}(t) \right]_{n \times m}$$

Definition

$$\frac{d\mathbf{F}}{dt} = \begin{bmatrix} \frac{df_{11}(t)}{dt} & \frac{df_{12}(t)}{dt}, \dots, \frac{df_{1m}(t)}{dt} \\ \frac{df_{21}(t)}{dt} & \frac{df_{22}(t)}{dt}, \dots, \frac{df_{2m}(t)}{dt} \\ \vdots & \\ \frac{df_{n1}(t)}{dt} & \frac{df_{n2}(t)}{dt}, \dots, \frac{df_{nm}(t)}{dt} \end{bmatrix} = \left[\frac{df_{ij}(t)}{dt} \right]_{n \times m}$$



Matrix differentiation

- Function is a scalar and the variable is a vector

$$f(\mathbf{x}), \mathbf{x} = (x_1, x_2, \dots, x_n)^T$$

Definition

$$\frac{df}{d\mathbf{x}} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]^T$$

In a similar way,

$$f(\mathbf{x}), \mathbf{x} = (x_1, x_2, \dots, x_n)$$

$$\frac{df}{d\mathbf{x}} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]$$



Matrix differentiation

- Function is a vector and the variable is a vector

$$\mathbf{x} = [x_1, x_2, \dots, x_n]^T, \mathbf{y} = [y_1(\mathbf{x}), y_2(\mathbf{x}), \dots, y_m(\mathbf{x})]^T$$

Definition

$$\frac{d\mathbf{y}}{d\mathbf{x}^T} = \begin{bmatrix} \frac{\partial y_1(\mathbf{x})}{\partial x_1}, \frac{\partial y_1(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial y_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial y_2(\mathbf{x})}{\partial x_1}, \frac{\partial y_2(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial y_2(\mathbf{x})}{\partial x_n} \\ \vdots \\ \frac{\partial y_m(\mathbf{x})}{\partial x_1}, \frac{\partial y_m(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial y_m(\mathbf{x})}{\partial x_n} \end{bmatrix}_{m \times n}$$



Matrix differentiation

- Function is a vector and the variable is a vector

$$\mathbf{x} = [x_1, x_2, \dots, x_n]^T, \mathbf{y} = [y_1(\mathbf{x}), y_2(\mathbf{x}), \dots, y_m(\mathbf{x})]^T$$

In a similar way,

$$\frac{d\mathbf{y}^T}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial y_1(\mathbf{x})}{\partial x_1}, \frac{\partial y_2(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial y_m(\mathbf{x})}{\partial x_1} \\ \frac{\partial y_1(\mathbf{x})}{\partial x_2}, \frac{\partial y_2(\mathbf{x})}{\partial x_2}, \dots, \frac{\partial y_m(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial y_1(\mathbf{x})}{\partial x_n}, \frac{\partial y_2(\mathbf{x})}{\partial x_n}, \dots, \frac{\partial y_m(\mathbf{x})}{\partial x_n} \end{bmatrix}_{n \times m}$$



Matrix differentiation

- Function is a vector and the variable is a vector

Example:

$$\mathbf{y} = \begin{bmatrix} y_1(\mathbf{x}) \\ y_2(\mathbf{x}) \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y_1(\mathbf{x}) = x_1^2 - x_2, y_2(\mathbf{x}) = x_3^2 + 3x_2$$

$$\frac{d\mathbf{y}^T}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial y_1(\mathbf{x})}{\partial x_1} & \frac{\partial y_1(\mathbf{x})}{\partial x_2} & \frac{\partial y_1(\mathbf{x})}{\partial x_3} \\ \frac{\partial y_2(\mathbf{x})}{\partial x_1} & \frac{\partial y_2(\mathbf{x})}{\partial x_2} & \frac{\partial y_2(\mathbf{x})}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 & -1 & 0 \\ 0 & 3 & 2x_3 \end{bmatrix}$$



Matrix differentiation

- Function is a scalar and the variable is a matrix

$$f(\mathbf{X}), \mathbf{X} \in \mathbb{R}^{m \times n}$$

Definition

$$\frac{df(\mathbf{X})}{d\mathbf{X}} = \begin{bmatrix} \frac{\partial f}{\partial x_{11}} & \frac{\partial f}{\partial x_{12}} & \dots & \frac{\partial f}{\partial x_{1n}} \\ \dots & & & \\ \frac{\partial f}{\partial x_{m1}} & \frac{\partial f}{\partial x_{m2}} & \dots & \frac{\partial f}{\partial x_{mn}} \end{bmatrix}$$



Matrix differentiation

- Useful results

(1) $\mathbf{x}, \mathbf{a} \in \mathbb{R}^{n \times 1}$

Then,

$$\frac{d\mathbf{a}^T \mathbf{x}}{d\mathbf{x}} = \mathbf{a}, \frac{d\mathbf{x}^T \mathbf{a}}{d\mathbf{x}} = \mathbf{a}$$



How to prove?



Matrix differentiation

- Useful results

$$(2) \quad \mathbf{x} \in \mathbb{R}^{n \times 1} \quad \text{Then,} \quad \frac{d\mathbf{x}^T \mathbf{x}}{d\mathbf{x}} = 2\mathbf{x}$$

$$(3) \quad \mathbf{y}(\mathbf{x}) \in \mathbb{R}^{m \times 1}, \quad \mathbf{x} \in \mathbb{R}^{n \times 1}, \quad \frac{d\mathbf{y}^T(\mathbf{x})}{d\mathbf{x}} = \left(\frac{d\mathbf{y}(\mathbf{x})}{d\mathbf{x}^T} \right)^T$$

$$(4) \quad A \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^{n \times 1} \quad \text{Then,} \quad \frac{dA\mathbf{x}}{d\mathbf{x}^T} = A$$

$$(5) \quad A \in \mathbb{R}^{m \times n}, \mathbf{x} \in \mathbb{R}^{n \times 1} \quad \text{Then,} \quad \frac{d\mathbf{x}^T A^T}{d\mathbf{x}} = A^T$$

$$(6) \quad A \in \mathbb{R}^{n \times n}, \mathbf{x} \in \mathbb{R}^{n \times 1} \quad \text{Then,} \quad \frac{d\mathbf{x}^T A \mathbf{x}}{d\mathbf{x}} = (A + A^T) \mathbf{x}$$

$$(7) \quad \mathbf{X} \in \mathbb{R}^{m \times n}, \mathbf{a} \in \mathbb{R}^{m \times 1}, \mathbf{b} \in \mathbb{R}^{n \times 1} \quad \text{Then,} \quad \frac{d\mathbf{a}^T \mathbf{X} \mathbf{b}}{d\mathbf{X}} = \mathbf{a} \mathbf{b}^T$$



Matrix differentiation

- Useful results

$$(8) \quad \mathbf{X} \in \mathbb{R}^{n \times m}, \mathbf{a} \in \mathbb{R}^{m \times 1}, \mathbf{b} \in \mathbb{R}^{n \times 1} \quad \text{Then, } \frac{d\mathbf{a}^T \mathbf{X}^T \mathbf{b}}{d\mathbf{X}} = \mathbf{b} \mathbf{a}^T$$

$$(9) \quad \mathbf{X} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{n \times m} \quad \text{Then, } \frac{d(\text{tr} \mathbf{X} \mathbf{B})}{d\mathbf{X}} = \mathbf{B}^T$$

$$(10) \quad \mathbf{X} \in \mathbb{R}^{n \times n}, \mathbf{X} \text{ is invertible, } \frac{d|\mathbf{X}|}{d\mathbf{X}} = |\mathbf{X}| (\mathbf{X}^{-1})^T$$



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Lagrange multiplier

- Single-variable function

$f(x)$ is differentiable in (a, b) . At $x_0 \in (a, b)$, $f(x)$ achieves an extremum

$$\longrightarrow \frac{df}{dx} \Big|_{x_0} = 0$$

- Two-variables function

$f(x, y)$ is differentiable in its domain. At (x_0, y_0) , $f(x, y)$ achieves an extremum

$$\longrightarrow \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = 0, \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} = 0$$



Lagrange multiplier

- In general case

If $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^{n \times 1}$ achieves a local extremum at \mathbf{x}_0 and it is derivable at \mathbf{x}_0 , then \mathbf{x}_0 is a stationary point of $f(\mathbf{x})$, i.g.,

$$\frac{\partial f}{\partial x_1} \Big|_{\mathbf{x}_0} = 0, \frac{\partial f}{\partial x_2} \Big|_{\mathbf{x}_0} = 0, \dots, \frac{\partial f}{\partial x_n} \Big|_{\mathbf{x}_0} = 0$$

Or in other words,

$$\nabla f(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x}_0} = \mathbf{0}$$



Lagrange multiplier

- Lagrange multiplier is a strategy for finding **all the possible** extremum points of a function subject to equality constraints

Problem: find **all the possible** extremum points for $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^{n \times 1}$

under m constraints $g_k(\mathbf{x}) = 0, k = 1, 2, \dots, m$

Solution: $F(\mathbf{x}; \lambda_1, \dots, \lambda_m) = f(\mathbf{x}) + \sum_{k=1}^m \lambda_k g_k(\mathbf{x})$

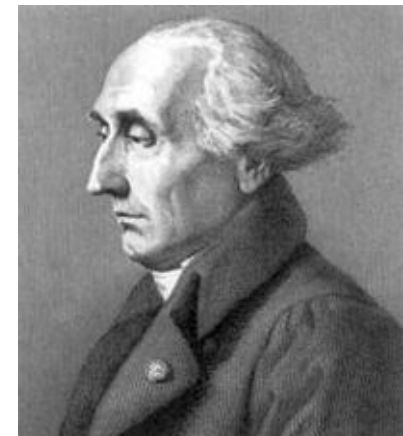
If \mathbf{x}_0 is an extremum point of $f(\mathbf{x})$ under constraints



$\exists \lambda_{10}, \lambda_{20}, \dots, \lambda_{m0}$, making $(\mathbf{x}_0, \lambda_{10}, \lambda_{20}, \dots, \lambda_{m0})$

a stationary point of F

Thus, by identifying the stationary points of F , we can get all the possible extremum points of $f(\mathbf{x})$ under equality constraints



Joseph-Louis Lagrange
Jan. 25, 1736~Apr.10, 1813



Lagrange multiplier

- Lagrange multiplier is a strategy for finding **all the possible** extremum points of a function subject to equality constraints

Problem: find **all the possible** extremum points for $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^{n \times 1}$

under m constraints $g_k(\mathbf{x}) = 0, k = 1, 2, \dots, m$

Solution: $F(\mathbf{x}; \lambda_1, \dots, \lambda_m) = f(\mathbf{x}) + \sum_{k=1}^m \lambda_k g_k(\mathbf{x})$

$(\mathbf{x}_0, \lambda_{10}, \dots, \lambda_{m0})$ is a stationary point of F 

$$\underbrace{\frac{\partial F}{\partial x_1} = 0, \frac{\partial F}{\partial x_2} = 0, \dots, \frac{\partial F}{\partial x_n} = 0, \frac{\partial F}{\partial \lambda_1} = 0, \frac{\partial F}{\partial \lambda_2} = 0, \dots, \frac{\partial F}{\partial \lambda_m} = 0}_{\text{at that point}}$$

at that point

$n + m$ equations!



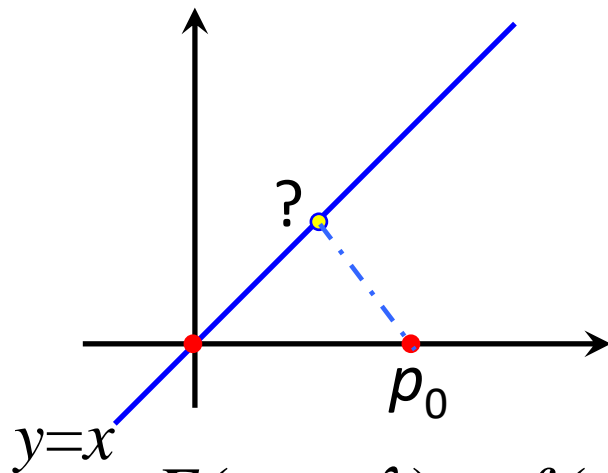
\mathbf{x}_0 is a possible extremum point of $f(\mathbf{x})$ under equality constraints



Lagrange multiplier

- Example

Problem: for a given point $p_0 = (1, 0)$, among all the points lying on the line $y=x$, identify the one having the least distance to p_0 .



The distance is

$$f(x, y) = (x-1)^2 + (y-0)^2$$

Now we want to find the global minimizer of $f(x, y)$ under the constraint

$$g(x, y) = y - x = 0$$

According to Lagrange multiplier method, construct the Lagrange function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = (x-1)^2 + y^2 + \lambda(y-x)$$

Find the stationary point of $F(x, y, \lambda)$





Lagrange multiplier

• Example
$$\begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ \frac{\partial F}{\partial \lambda} = 0 \end{cases} \rightarrow \begin{cases} 2(x-1) + \lambda = 0 \\ 2y - \lambda = 0 \\ x - y = 0 \end{cases} \rightarrow \begin{cases} x = 0.5 \\ y = 0.5 \\ \lambda = 1 \end{cases}$$

Thus, $(0.5, 0.5, 1)$ is the only stationary point of $F(x, y, \lambda)$

$(0.5, 0.5)$ is the only possible extremum point of $f(x, y)$ under constraints

The global minimizer of $f(x, y)$ under constraints exists

$(0.5, 0.5)$ is the global minimizer of $f(x, y)$ under constraints



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LS for Inhomogeneous Linear System

Consider the following linear equations system

$$\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + x_2 = 4 \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\mathbf{b}}$$

Matrix form: $\mathbf{Ax} = \mathbf{b}$

It can be easily solved $\begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$



LS for Inhomogeneous Linear System

How about the following one?

$$\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + x_2 = 4 \\ x_1 + 2x_2 = 6 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

It does not have a solution!



What is the condition for a linear equation system $A\mathbf{x} = \mathbf{b}$ can be solved?

Can we solve it in an approximate way?

A: we can use least squares technique!



Carl Friedrich Gauss



LS for Inhomogeneous Linear System

Let's consider a system of m linear equations with n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = \mathbf{b}_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = \mathbf{b}_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = \mathbf{b}_m \end{cases} \Leftrightarrow A_{m \times n} \mathbf{x}_{n \times 1} = \mathbf{b}_{m \times 1}$$

↑
unknowns

We consider the case: $\text{rank}(A)=n$, and $\text{rank}([A; \mathbf{b}])=n+1$

In general case, there is no solution!

Instead, we want to find a vector \mathbf{x} that minimizes the error:

$$E(\mathbf{x}) \equiv \sum_{i=1}^m (a_{i1}x_1 + \dots + a_{in}x_n - \mathbf{b}_i)^2 = \|A\mathbf{x} - \mathbf{b}\|_2^2$$




LS for Inhomogeneous Linear System

$$\mathbf{x}^* = \arg \min_{\mathbf{x}} E(\mathbf{x}) = \arg \min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2^2$$

The stationary point of $E(\mathbf{x})$ is $\mathbf{x}_s = (A^T A)^{-1} A^T \mathbf{b}$

Since $E(\mathbf{x})$ is a **convex** function, its stationary point is the global minimizer^[1]


$$\mathbf{x}^* = \mathbf{x}_s = \underbrace{(A^T A)^{-1} A^T}_{\text{Pseudoinverse of } A} \mathbf{b}$$

[1] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004, pp. 69



LS for Homogeneous Linear System

Let's consider a system of m linear equations with n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases} \Leftrightarrow A_{m \times n} \mathbf{x}_{n \times 1} = \mathbf{0}$$

↑
unknowns

We consider the case: $m \geq n$, and $\text{rank}(A) = n$

Theoretically, there is only a trivial solution: $\mathbf{x} = \mathbf{0}$

We can add a constraint $\|\mathbf{x}\|_2 = 1$ to avoid the trivial solution





LS for Homogeneous Linear System

We want to minimize $E(\mathbf{x}) = \|A\mathbf{x}\|_2^2$, subject to $\|\mathbf{x}\|_2 = 1$

$$\mathbf{x}^* = \arg \min_{\mathbf{x}} E(\mathbf{x}), \text{ s.t. }, \|\mathbf{x}\|_2 = 1 \quad (1)$$

Construct the Lagrange function,

$$L(\mathbf{x}, \lambda) = \|A\mathbf{x}\|_2^2 + \lambda(1 - \|\mathbf{x}\|_2^2) \quad (2)$$

Solving the stationary point $(\mathbf{x}_0, \lambda_0)$ of $L(\mathbf{x}, \lambda)$,

$$\begin{cases} \frac{\partial [\|A\mathbf{x}\|_2^2 + \lambda(1 - \|\mathbf{x}\|_2^2)]}{\partial \mathbf{x}} = \mathbf{0} \\ \frac{\partial [\|A\mathbf{x}\|_2^2 + \lambda(1 - \|\mathbf{x}\|_2^2)]}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} A^T A \mathbf{x}_0 = \lambda_0 \mathbf{x}_0 \\ \mathbf{x}_0^T \mathbf{x}_0 = 1 \end{cases}$$

Note: the stationary point of $L(\mathbf{x}, \lambda)$ is not unique



LS for Homogeneous Linear System

Suppose that $(\mathbf{x}_i, \lambda_i)$ is a stationary point of L , then \mathbf{x}_i is a possible extremum point of $E(\mathbf{x})$ under the equality constraint and we have

$$E(\mathbf{x}_i) = \|A\mathbf{x}_i\|_2^2 = \mathbf{x}_i^T A^T A \mathbf{x}_i = \mathbf{x}_i^T \lambda_i \mathbf{x}_i = \lambda_i$$



The global minimum of $E(\mathbf{x})$ is $\min \{\lambda_i\}$ and the global minimizer of $E(\mathbf{x})$ is the unit eigen-vector of $A^T A$ associated with its least eigen value



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Problem of Homography Estimation

Problem definition:

On two projective planes π_1 and π_2 , there is a set of corresponding points $\{\mathbf{x}_i, \mathbf{x}'_i\}_{i=1}^n$, and we suppose that there is a homography matrix linking the two planes,

$$c\mathbf{x}'_i = H\mathbf{x}_i, i = 1, 2, \dots, n$$

Coordinates of $\{\mathbf{x}_i\}_i^n$ and $\{\mathbf{x}'_i\}_{i=1}^n$ are known, we need to find H

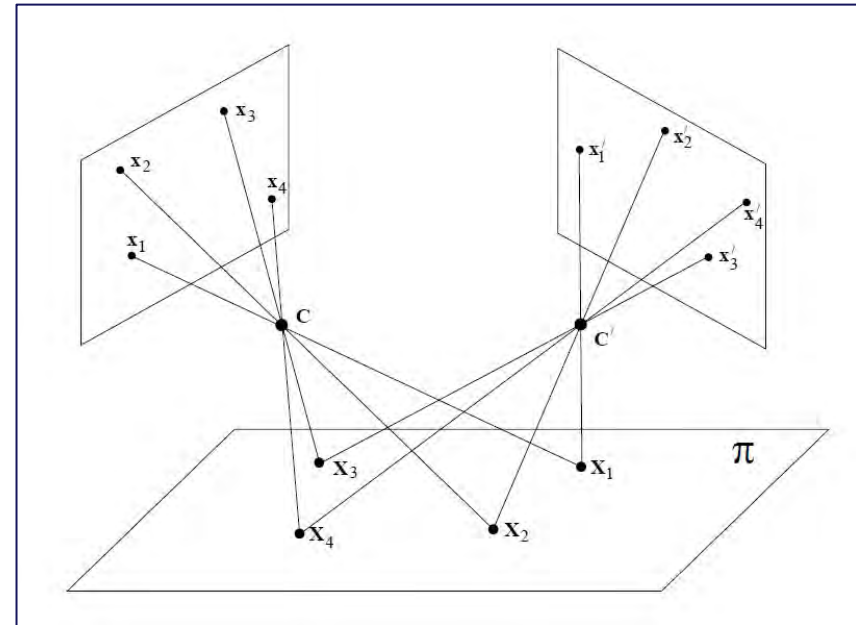
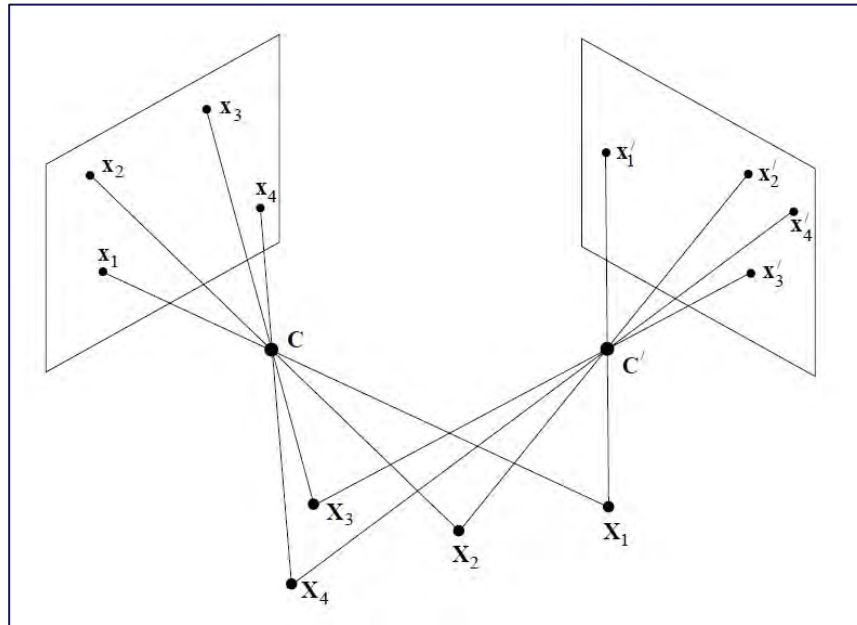
$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Note: H is defined up to a scale factor. In other words, it has 8 DOFs



Problem of Homography Estimation

Note: Theoretically speaking, homography can only be estimated between two planes, i.e., when you use such a technique to stitch two images, image contents should be roughly on the same plane





Problem of Homography Estimation

4 point-correspondence pairs can uniquely determine a homography matrix since each correspondence pair solves two degrees of freedom

$$c \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \quad \rightarrow \quad \begin{cases} h_{11}u + h_{12}v + h_{13} = cx \\ h_{21}u + h_{22}v + h_{23} = cy \\ h_{31}u + h_{32}v + h_{33} = c \end{cases}$$

$$\rightarrow \begin{cases} \frac{h_{11}u + h_{12}v + h_{13}}{h_{31}u + h_{32}v + h_{33}} = x \\ \frac{h_{21}u + h_{22}v + h_{23}}{h_{31}u + h_{32}v + h_{33}} = y \end{cases}$$

Note: here we assume that the matching points are all finite points (no points at infinity)



Problem of Homography Estimation

4 point-correspondence pairs can uniquely determine a homography matrix since each correspondence pair solves two degrees of freedom

$$\begin{pmatrix} u & v & 1 & 0 & 0 & 0 & -ux & -vx & -x \\ 0 & 0 & 0 & u & v & 1 & -uy & -vy & -y \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{pmatrix} = 0$$

Thus, four correspondence pairs
generate 8 equations



Problem of Homography Estimation

4 point-correspondence pairs can uniquely determine a homography matrix since each correspondence pair solves two degrees of freedom

$$\begin{array}{ccc} & A\mathbf{x} = 0 & (1) \\ & \swarrow \quad \searrow & \\ 8 \times 9 & & 9 \times 1 \end{array}$$

Normally, $\text{Rank}(A) = 8$; thus (1) has 1 (9-8) solution vector (linear independent) in its solution space

In our case, since we have $n > 4$ point pairs, we get

$$\mathbf{A}_{2n \times 9} \mathbf{h}_{9 \times 1} = \mathbf{0}$$

It is an overdetermined homogeneous linear equation system



Problem of Homography Estimation

Since only the ratios among the elements of H take effect, in another way we can fix $h_{33}=1$ (suppose that $h_{33} \neq 0$),

$$c \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \rightarrow \begin{cases} h_{11}u + h_{12}v + h_{13} = cx \\ h_{21}u + h_{22}v + h_{23} = cy \\ h_{31}u + h_{32}v + 1 = c \end{cases} \rightarrow \begin{cases} \frac{h_{11}u + h_{12}v + h_{13}}{h_{31}u + h_{32}v + 1} = x \\ \frac{h_{21}u + h_{22}v + h_{23}}{h_{31}u + h_{32}v + 1} = y \end{cases}$$

$$\rightarrow \begin{pmatrix} u & v & 1 & 0 & 0 & 0 & -ux & -vx \\ 0 & 0 & 0 & u & v & 1 & -uy & -vy \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \text{Since we have } n > 4 \text{ point pairs, we get}$$

$$\mathbf{A}_{2n \times 8} \mathbf{h}_{8 \times 1} = \mathbf{b}_{2n \times 1}$$

It is an overdetermined inhomogeneous linear equation system



Content

- Scale Invariant Feature Transform
- Case Study: Homography Estimation
 - Matrix Differentiation
 - Lagrange Multiplier
 - Least-squares for Linear Systems
 - Problem of Homography Estimation
 - RANSAC-based Homography Estimation



RANSAC-based Homography Estimation

- When there are more than 4 correspondence pairs, is it a proper way to use the LS method to solve the model directly?
 - NO! Because usually, outliers exist among the correspondence pairs

RANdom SAmple Consensus (RANSAC) is an iterative framework to estimate a parametric model from observations with noisy outliers



RANSAC-based Homography Estimation

Objective

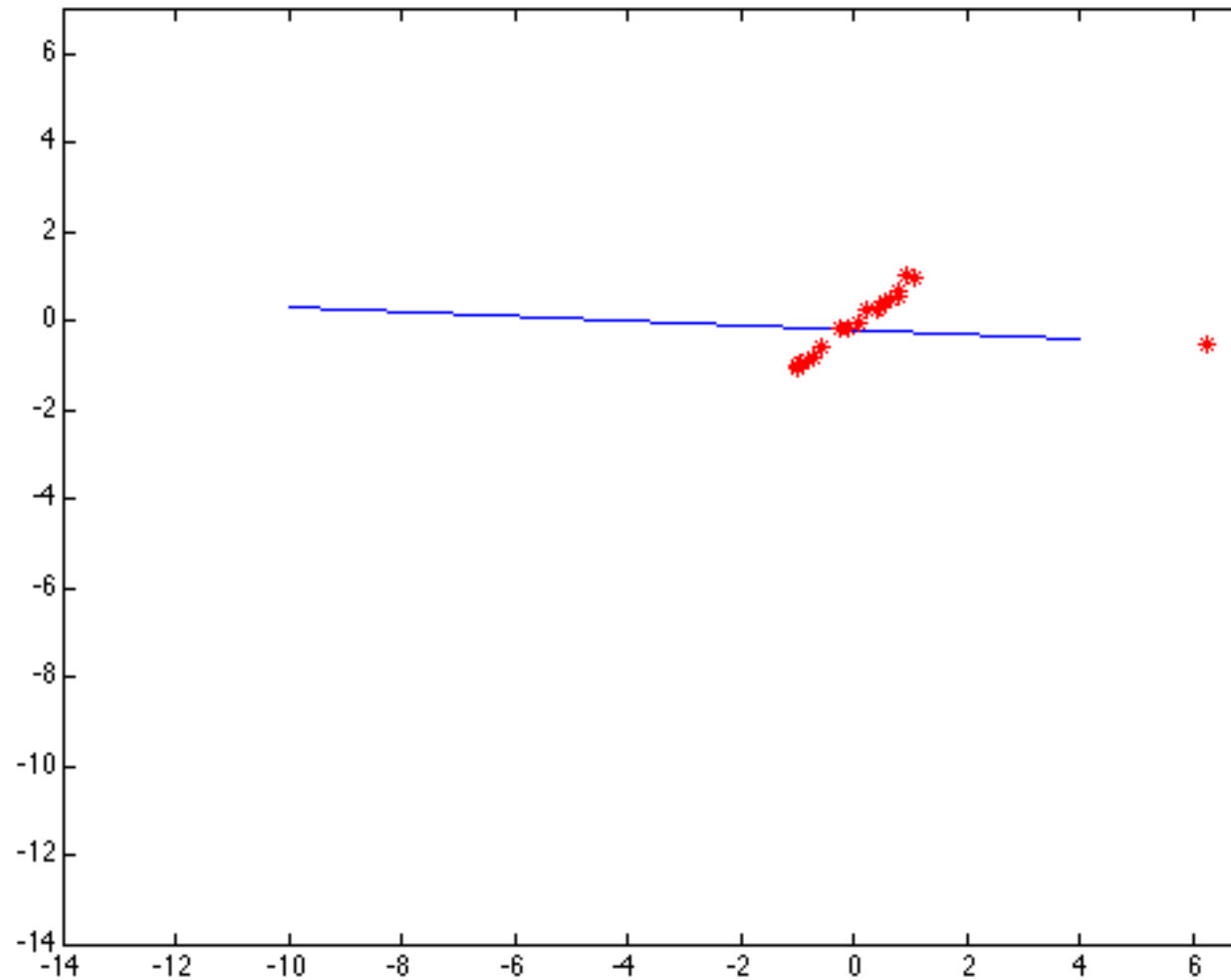
Robust fit a model to a data set S which contains outliers

Algorithm

- (1) Randomly select a sample of s data points from S and instantiate the model from this subset
- (2) Determine the set of data points S_i which are within a distance threshold t of the model. The set S_i is the consensus set of the sample and defines the inliers of S
- (3) If the size of S_i (the number of inliers) is greater than some threshold T , re-estimate the model using all points in S_i and terminate
- (4) If the size of S_i is less than T , select a new subset and repeat the above
- (5) After N trials the largest consensus set S_i is selected, and the model is re-estimated using all points in the subset S_i

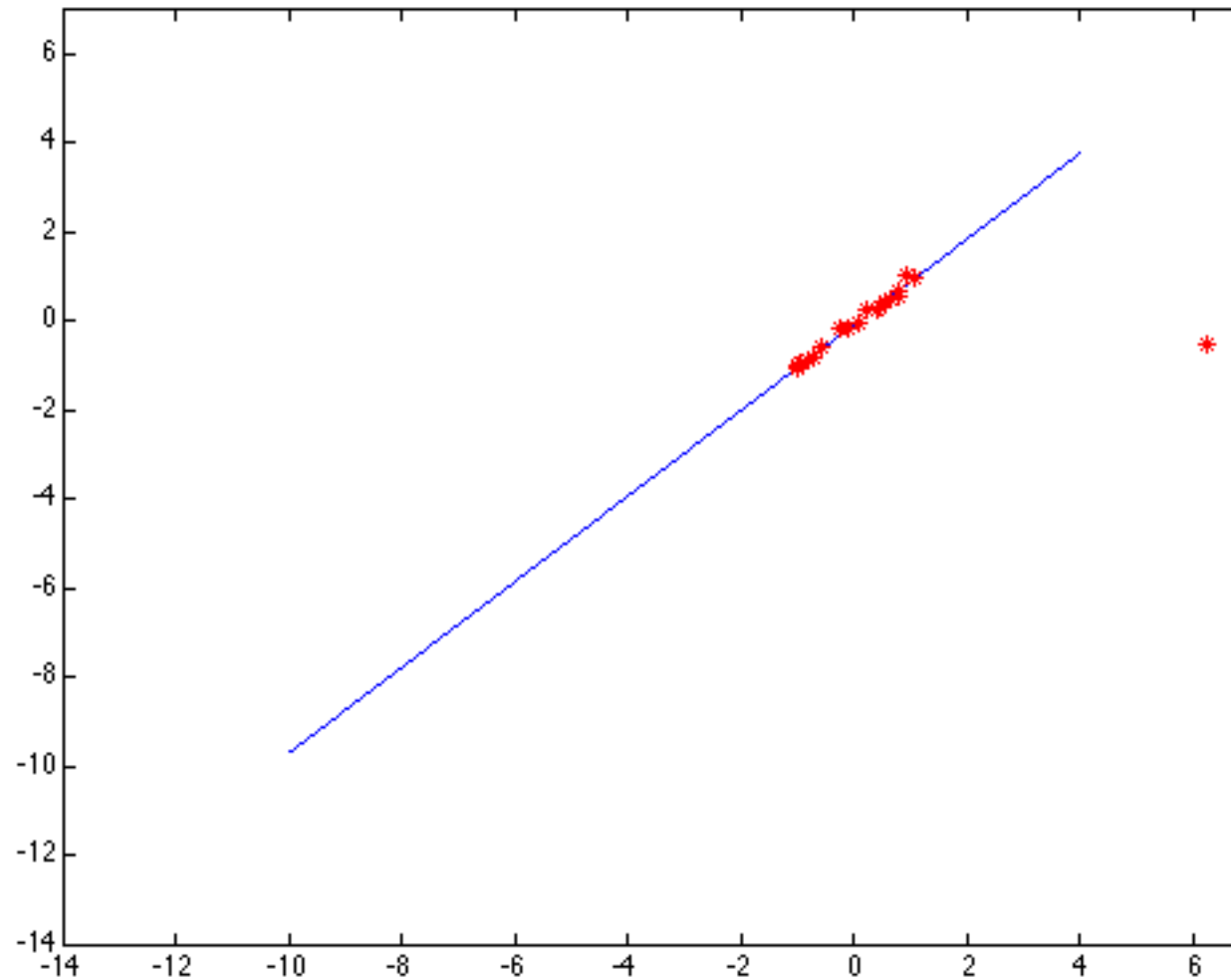


RANSAC-based Homography Estimation





RANSAC-based Homography Estimation





RANSAC-based Homography Estimation

Line fitting by RANSAC

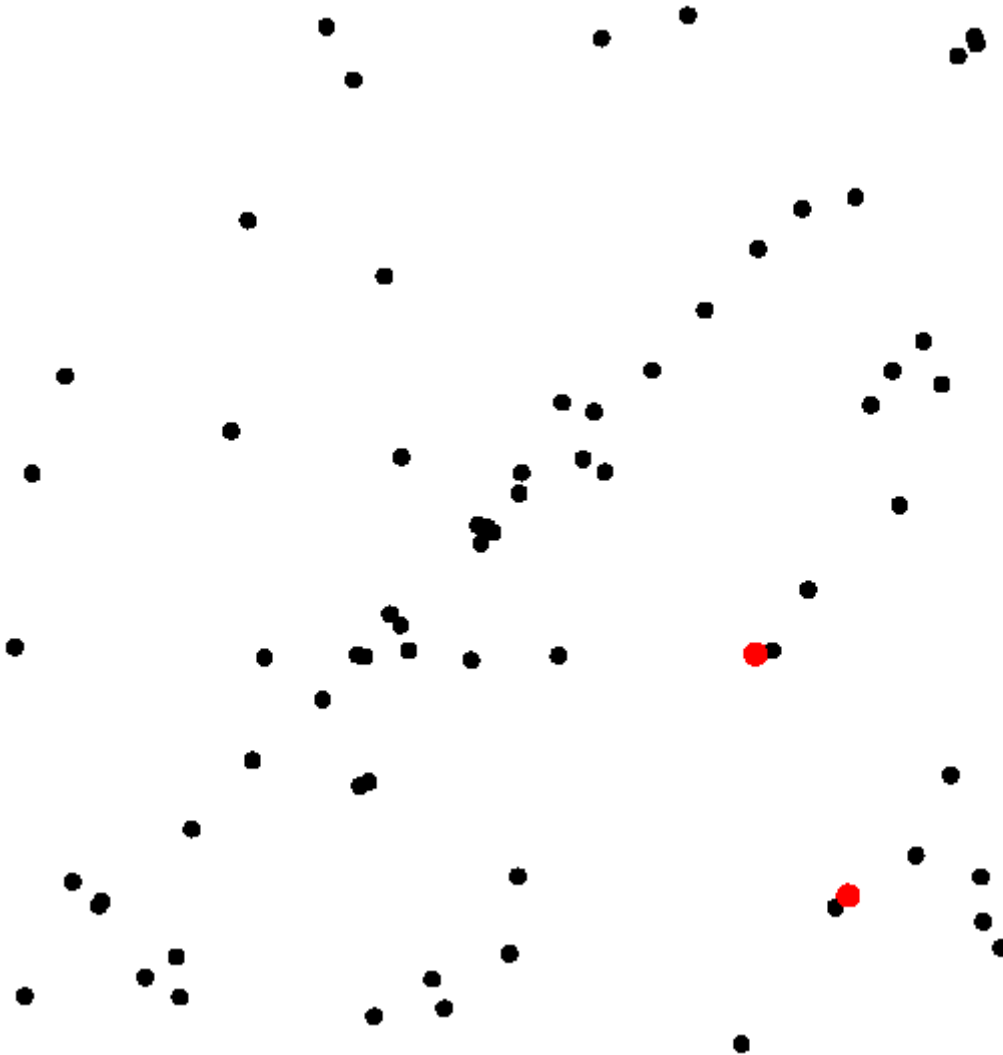




RANSAC-based Homography Estimation

Line fitting by RANSAC

- Randomly select two points

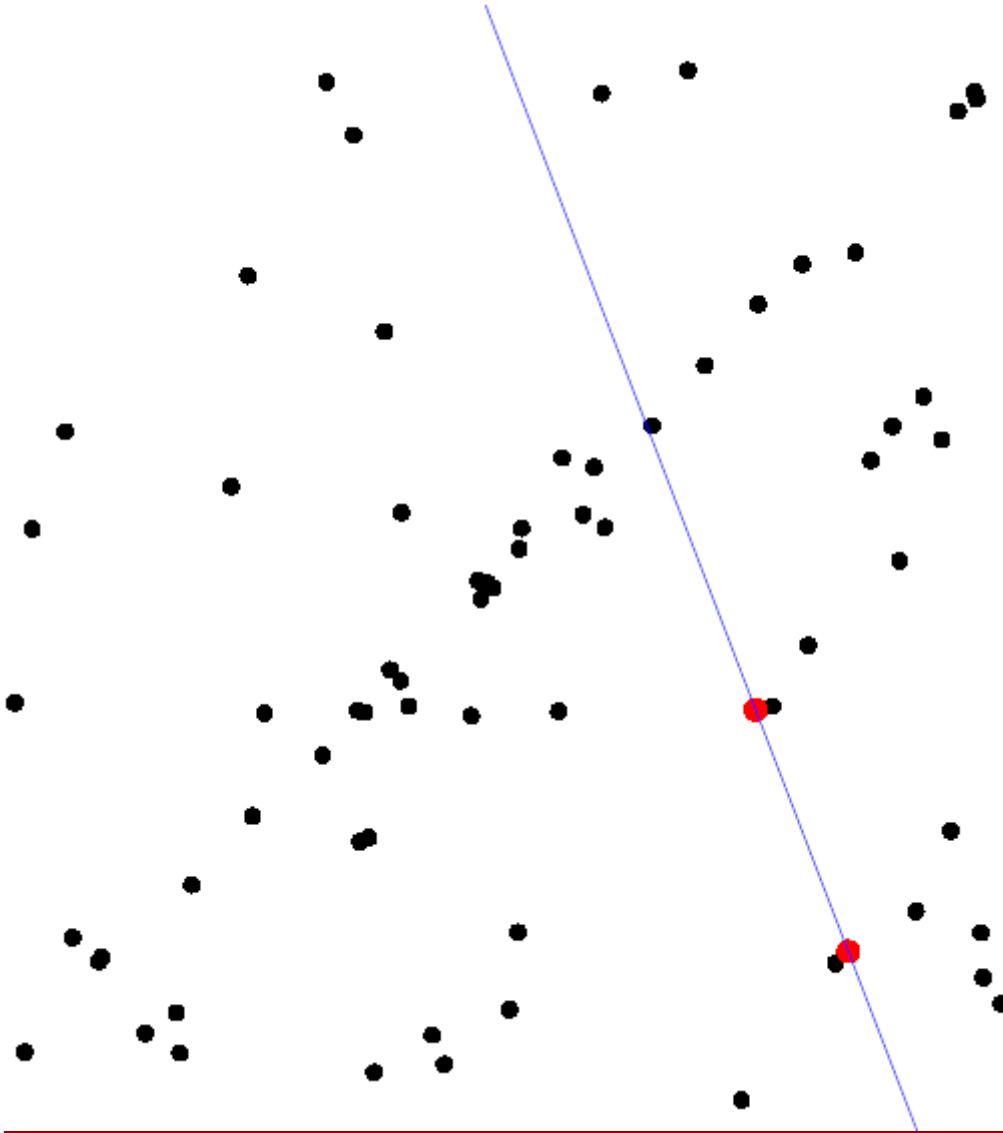




RANSAC-based Homography Estimation

Line fitting by RANSAC

- Randomly select two points
- The hypothesized model is the line passing through the two points

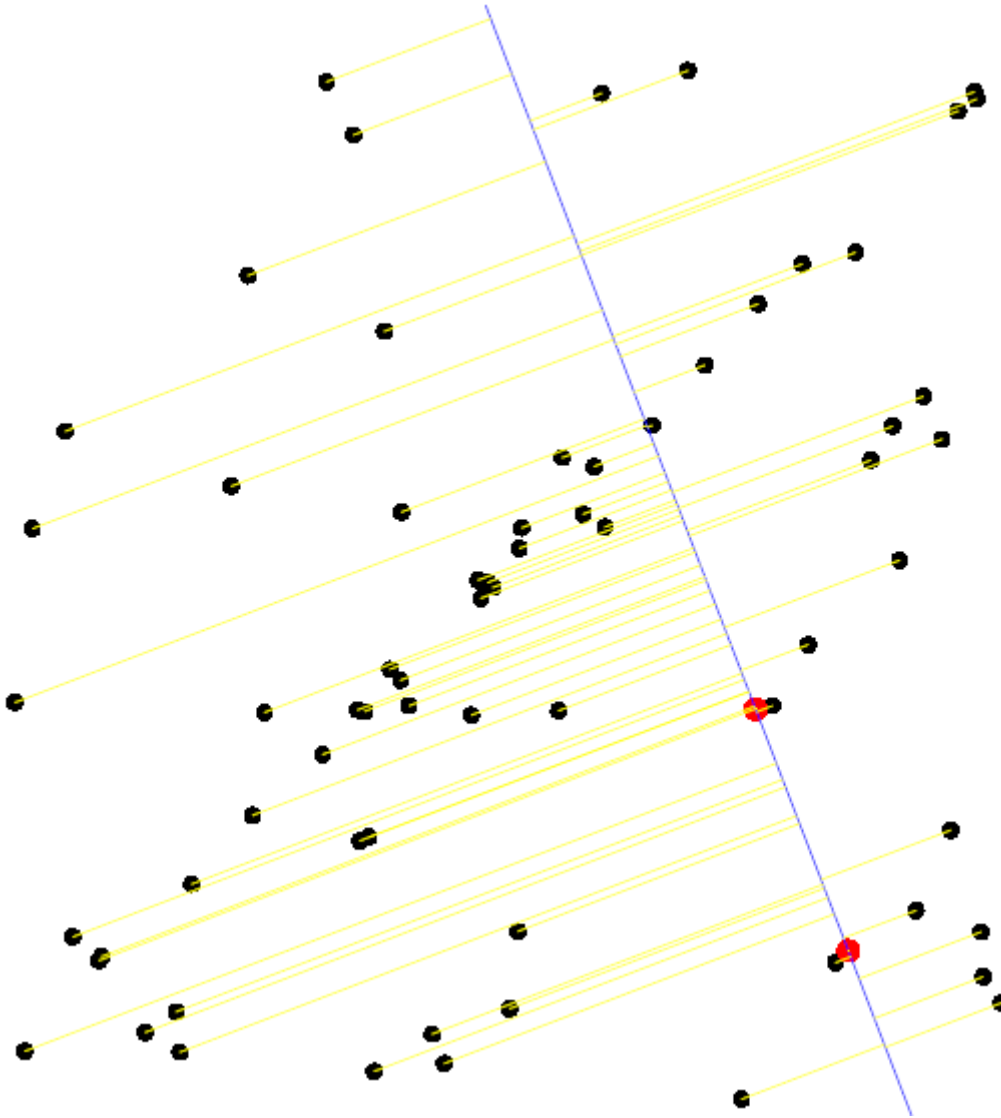




RANSAC-based Homography Estimation

Line fitting by RANSAC

- Randomly select two points
- The hypothesized model is the line passing through the two points

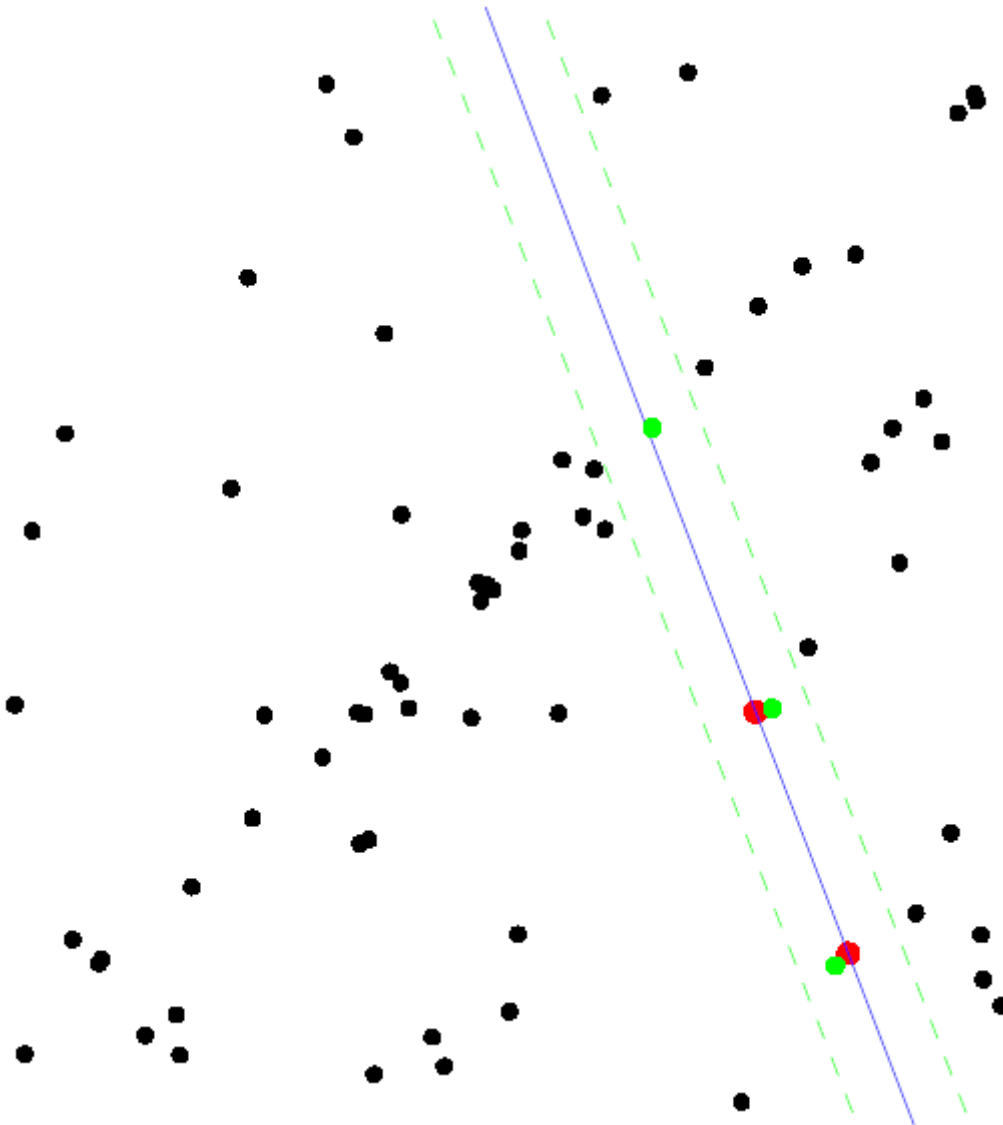




RANSAC-based Homography Estimation

Line fitting by RANSAC

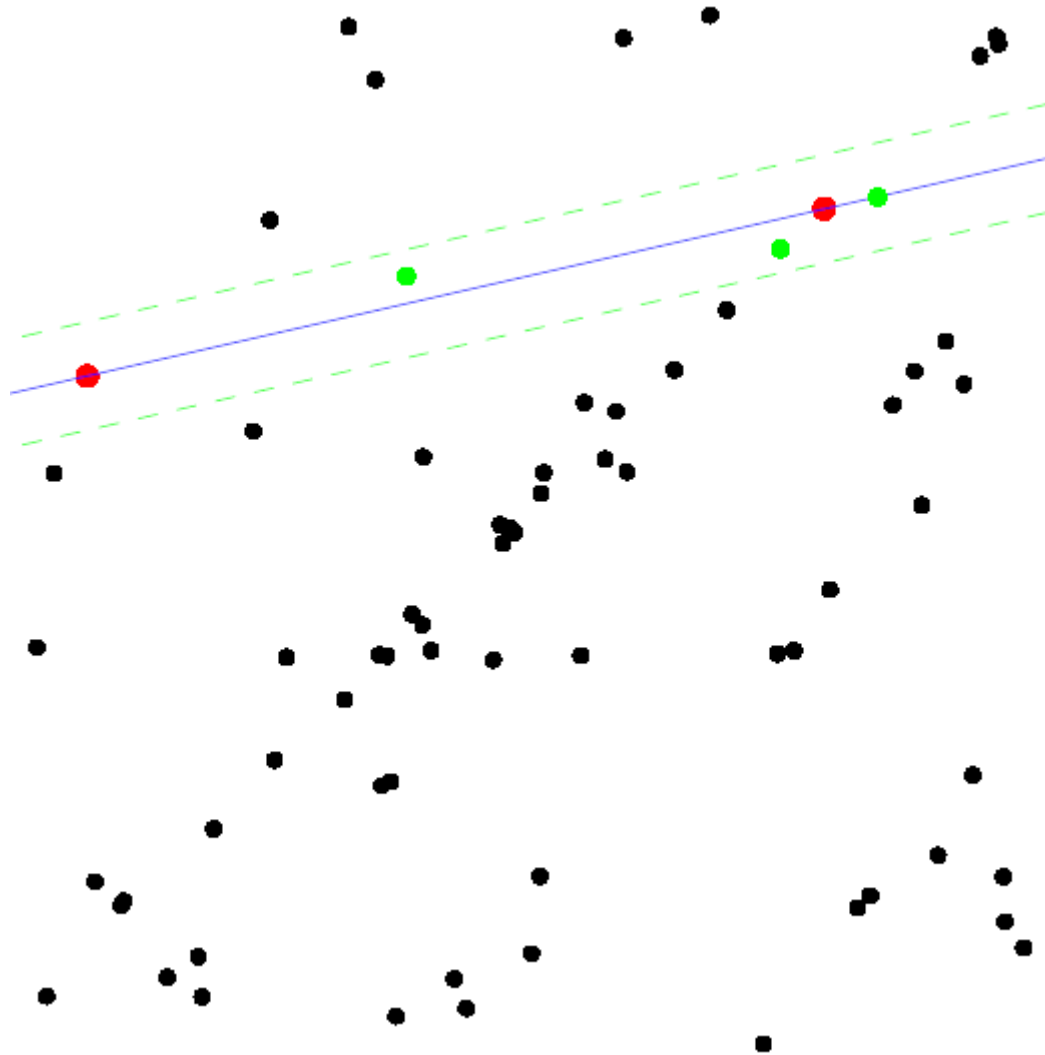
- Randomly select two points
- The hypothesized model is the line passing through the two points





RANSAC-based Homography Estimation

Line fitting by RANSAC



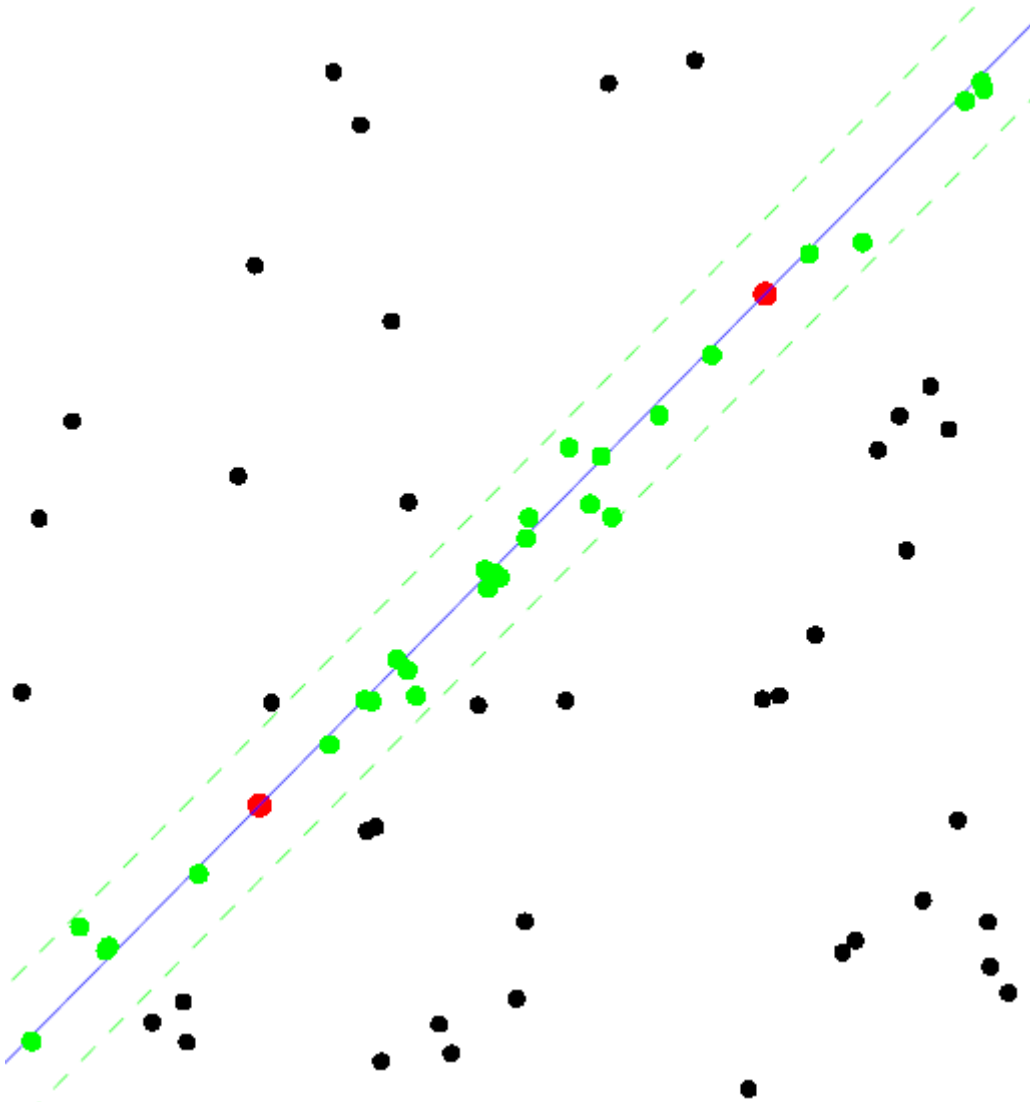
- Test another two points



RANSAC-based Homography Estimation

Line fitting by RANSAC

- The final fitting result

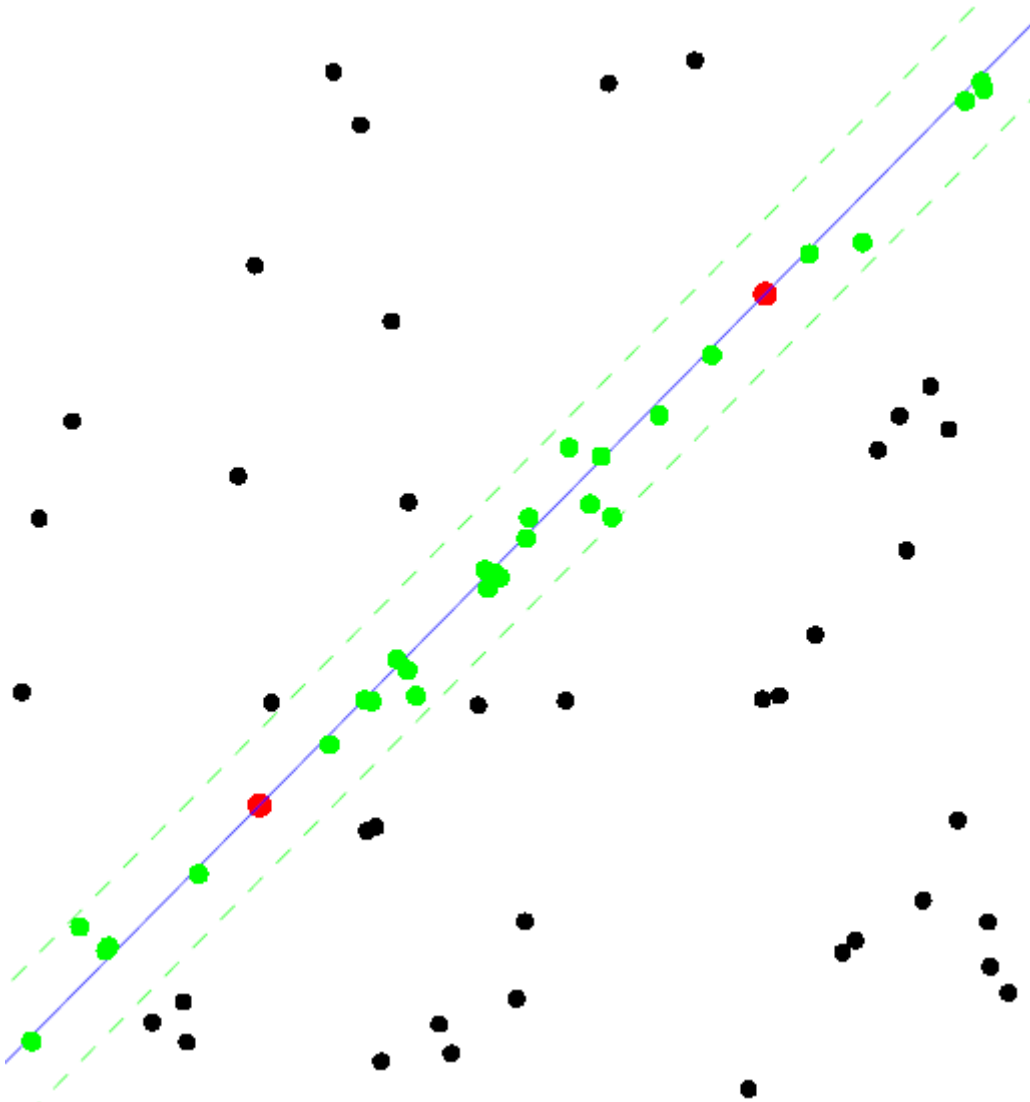




RANSAC-based Homography Estimation

Line fitting by RANSAC

- The final fitting result





RANSAC-based Homography Estimation



*Can you describe the steps of
homography estimation when
using RANSAC?*



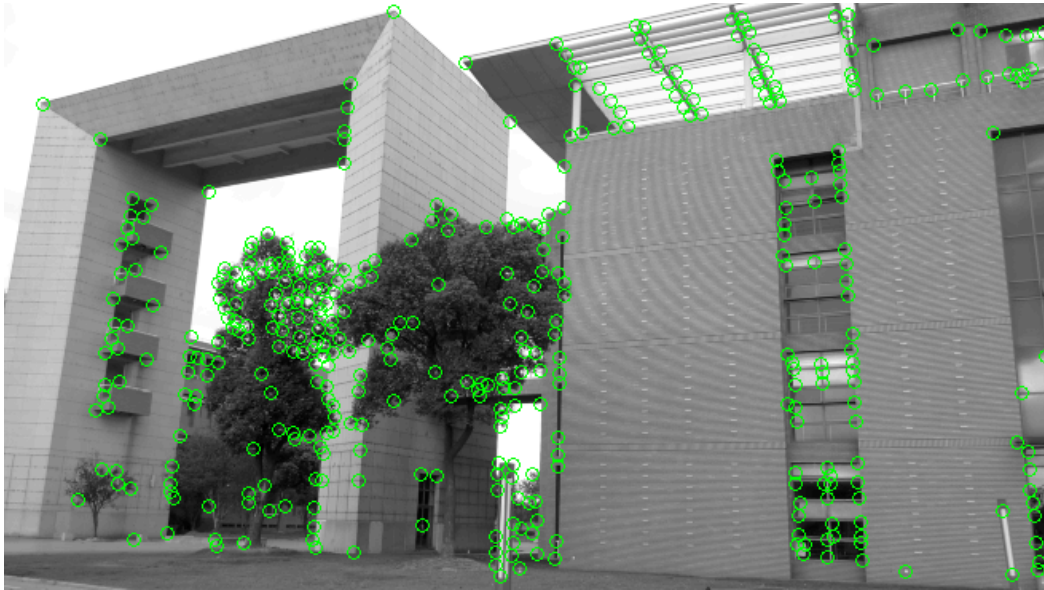
Homography Estimation: Example 1



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Homography Estimation: Example 1



Interest points detection

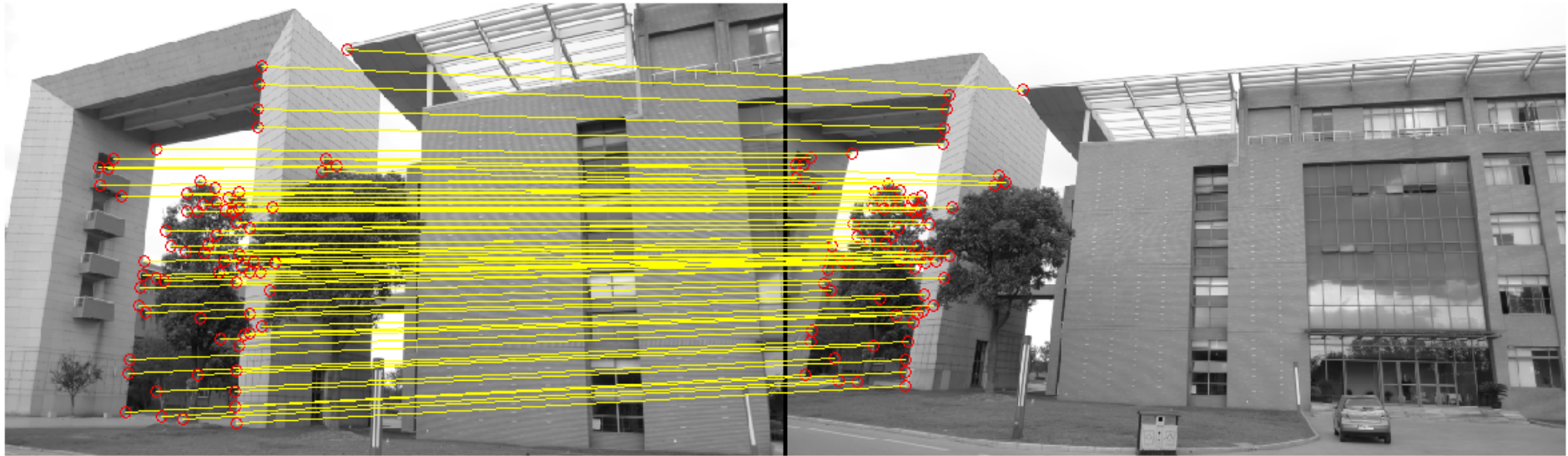




Homography Estimation: Example 1

Correspondence estimation

Then, the homography matrix can be estimated by using the correspondence pairs with RANSAC





Homography Estimation: Example 1

Transform image one using the estimated homography matrix





Homography Estimation: Example 1

Finally, stitch the transformed image one with image two





Homography Estimation: Example 2





Homography Estimation: Example 2

Interest points detection



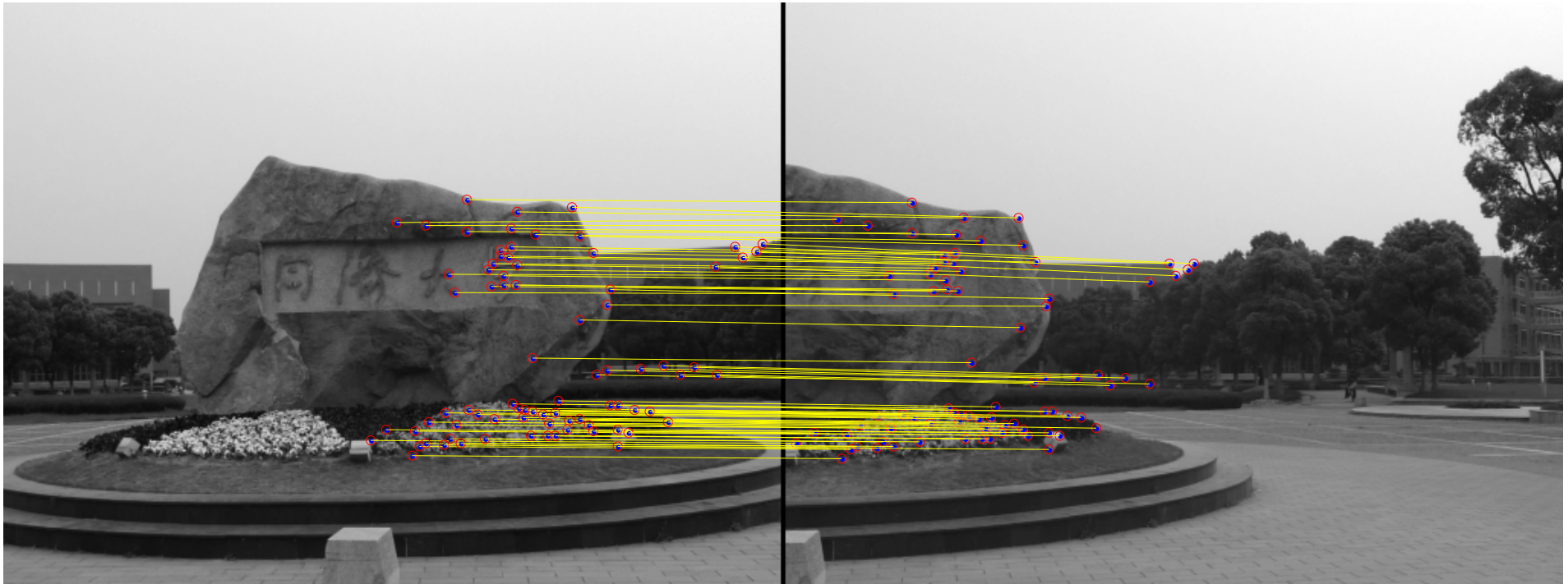
Lin ZHANG, SSE, Tongji Univ.



Homography Estimation: Example 2

Correspondence estimation

Then, the homography matrix can be estimated by using the correspondence pairs with RANSAC





Homography Estimation: Example 2

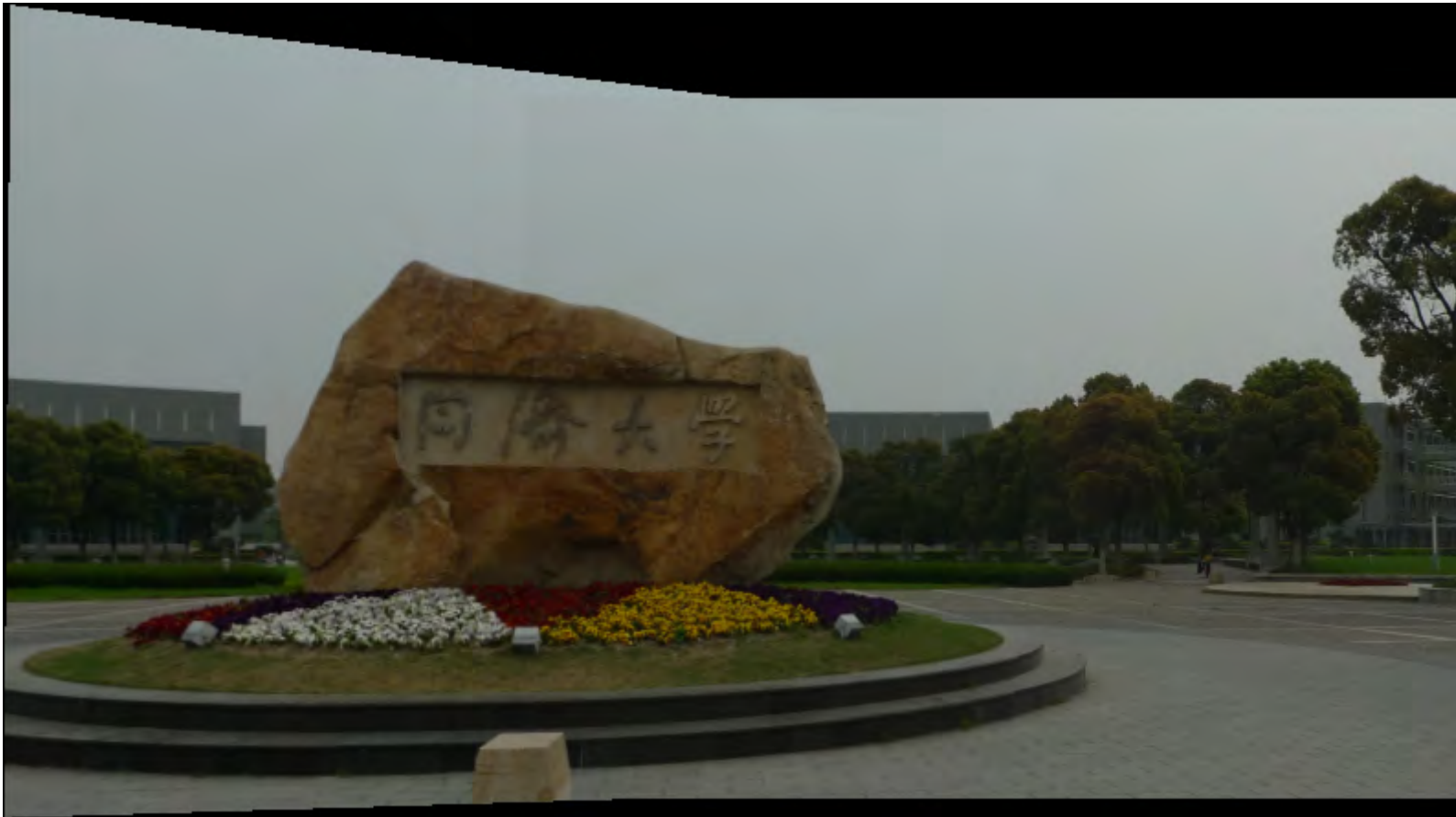
Transform image one using the estimated homography matrix





Homography Estimation: Example 2

Finally, stitch the transformed image one with image two





Homography Estimation: Example 3

Project products of students from 2009 Media&Arts



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