Assignment2

Course:	Computer Vision
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Q1. In the augmented Euclidean plane, there is a line $x - 3y + 4 = 0$, what is the homogeneous coordinate of the infinity point of this line?

Ans.

In the augmented Euclidean plane，the point at infinity on a line exists both on the line itself and on the line at infinity. Therefore, the point at infinity of a line is the intersection point of the line and the line at infinity.

Now the homogeneous coordinates of two lines are: $\mathbf{l} = (1, -3, 4)$ and $\mathbf{l}' = (0, 0, 1)$.

Therefore, the homogeneous coordinates of the intersection point x is $\mathbf{l} \times \mathbf{l}' = (-3, -1, 0)$.

From the definition of a point's homogeneous coordinates, if $(-3, -1, 0)$ is a homogeneous coordinate of a point, then $(k(-3, -1, 0))$ is also a homogeneous coordinate of the same point, where $k \neq 0$.

So the homogeneous coordinates of the infinity point of the line is $k(-3, -1, 0)(k \neq 0)$.

Q2. On the normalized retinal plane, suppose that p_n is an ideal point of projection without considering distortion. If distortion is considered, $p_n = (x, y)^T$ is mapped to $p_d = (x_d, y_d)^T$ which is also on the normalized retinal plane. Their relationship is

$x_d = x(1 + k_1r^2 + k_2r^4) + 2\rho_1 xy + \rho_2(r^2 + 2x^2) + xk_3r^6$

$y_d = y(1 + k_1r^2 + k_2r^4) + 2\rho_2 xy + \rho_1(r^2 + 2y^2) + yk_3r^6$

where $r^2 = x^2 + y^2$

For performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian matrix of p_d w.r.t p_n, i.e., $\frac{dp_d}{dp_n^T}$

It should be noted that in this question p_d is the function of p_n and all the other parameters can be regarded as constants.

Ans.

$$\frac{dp_d}{dp_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix}$$

$$r^2 = x^2 + y^2 \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$
\begin{aligned}
\frac{\partial x_d}{\partial x} &= (1 + k_1 r^2 + k_2 r^4) + x(k_1 2r\frac{\partial r}{\partial x} + k_2 4r^3\frac{\partial r}{\partial x}) + 2\rho_1 y + \rho_2(2r\frac{\partial r}{\partial x} + 4x) + k_3(r^6 + x6r^5\frac{\partial r}{\partial x}) \\
&= (1 + k_1 r^2 + k_2 r^4) + (2k_1 x^2 + 4k_2 r^2 x^2) + 2\rho_1 y + 6\rho_2 x + (k_3 r^6 + 6k_3 r^4 x^2) \\
\frac{\partial x_d}{\partial y} &= x(k_1 2r\frac{\partial r}{\partial y} + k_2 4r^3\frac{\partial r}{\partial y}) + 2\rho_1 x + \rho_1(2r\frac{\partial r}{\partial y}) + xk_3(6r^5\frac{\partial r}{\partial y}) \\
&= (2k_1 xy + 4k_2 r^2 xy) + 2\rho_1 x + 2\rho_1 y + 6k_3 r^4 xy \\
\frac{\partial y_d}{\partial x} &= y(k_1 2r\frac{\partial r}{\partial x} + k_2 4r^3\frac{\partial r}{\partial x}) + 2\rho_2 y + \rho_1(2r\frac{\partial r}{\partial x}) + yk_3 6r^5\frac{\partial r}{\partial x} \\
&= (2k_1 xy + 4k_2 r^2 xy) + 2\rho_2 y + 2\rho_1 x + 6k_3 r^4 xy \\
\frac{\partial y_d}{\partial y} &= (1 + k_1 r^2 + k_2 r^4) + y(k_1 2r\frac{\partial r}{\partial y} + k_2 4r^3\frac{\partial r}{\partial y}) + 2\rho_2 x + \rho_1(2r\frac{\partial r}{\partial y} + 4y) + k_3(r^6 + y6r^5\frac{\partial r}{\partial y}) \\
&= (1 + k_1 r^2 + k_2 r^4) + (2k_1 y^2 + 4k_2 r^2 y^2) + 2\rho_2 x + 6\rho_1 y + (k_3 r^6 + 6k_3 r^4 y^2)
\end{aligned}
$$

So the answer is following:

$$
\frac{dp_d}{dp_n^T} = \begin{bmatrix} (1 + k_1 r^2 + k_2 r^4) + (2k_1 x^2 + 4k_2 r^2 x^2) + 2\rho_1 y + 6\rho_2 x + (k_3 r^6 + 6k_3 r^4 x^2) & (2k_1 xy + 4k_2 r^2 xy) + 2\rho_1 x + 2\rho_1 y + 6k_3 r^4 xy \\ (2k_1 xy + 4k_2 r^2 xy) + 2\rho_2 y + 2\rho_1 x + 6k_3 r^4 xy & (1 + k_1 r^2 + k_2 r^4) + (2k_1 y^2 + 4k_2 r^2 y^2) + 2\rho_2 x + 6\rho_1 y + (k_3 r^6 + 6k_3 r^4 y^2) \end{bmatrix}
$$

Q3. In our lecture, we mentioned that for performing nonlinear optimization in the pipeline of camera calibration, we need to compute the Jacobian of the rotation matrix (represented in a vector) w.r.t its axis-angle representation. In this question, your task is to derive the concrete formula of this Jacobian matrix. Suppose that

$$
\mathbf{d} = \theta \mathbf{n} \in \mathbb{R}^{3\times 1}
$$

where $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is a 3D unit vector and θ is a real number denoting the rotation angle.

With Rodrigues formula, $\mathbf{d}$ can be converted to its rotation matrix form:

$$
\mathbf{R} = \cos\theta \mathbf{I} + (1 - \cos\theta)\mathbf{n}\mathbf{n}^T + \sin\theta \mathbf{n}^\wedge
$$

and obviously $\mathbf{R} \triangleq \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ is a 3×3 matrix.

Denote $\mathbf{r}$ by the vectorized form of $\mathbf{R}$, i.e. $\mathbf{r} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{21} & r_{22} & r_{23} & r_{31} & r_{32} & r_{33} \end{bmatrix}^T$. Please give the concrete form of Jacobian matrix of $\mathbf{r}$ w.r.t $\mathbf{d}$, i.e., $\frac{d\mathbf{r}}{d\mathbf{d}^T} \in \mathbb{R}^{9\times 3}$.

In order to make it easy to check your result, please follow the following notation requirements, $\alpha \triangleq \sin\theta$, $\beta \triangleq \cos\theta$, $\gamma = 1 - \cos\theta$.

In other words, the ingredients appearing in your formula are restricted to α, β, γ, θ, n_1, n_2, and n_3.

Ans.

$$R = \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} n_1^2 & n_1n_2 & n_1n_3 \\ n_1n_2 & n_2^2 & n_2n_2 \\ n_1n_3 & n_2n_3 & n_3^2 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$$\begin{aligned} \gamma &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{21} & r_{22} & r_{23} & r_{31} & r_{32} & r_{33} \end{bmatrix}^T \\ &= \begin{bmatrix} \beta + \gamma n_1^2 \\ \gamma n_1 n_2 - \alpha n_3 \\ \gamma n_1 n_3 + \alpha n_2 \\ \gamma n_1 n_2 + \alpha n_3 \\ \beta + \gamma n_2^2 \\ \gamma n_2 n_3 - \alpha n_1 \\ \gamma n_1 n_3 - \alpha n_2 \\ \gamma n_2 n_3 + \alpha n_1 \\ \beta + \gamma n_3^2 \end{bmatrix} \end{aligned}$$

$$\mathbf{d} = \theta\mathbf{n} \Rightarrow \begin{cases} d_1 &= \theta n_1 \\ d_2 &= \theta n_2 \\ d_3 &= \theta n_3 \end{cases}$$

Since $\mathbf{n}$ is a 3D unit vector, we can know:

$$(\frac{d_1}{\theta})^2 + (\frac{d_2}{\theta})^2 + (\frac{d_3}{\theta})^2 + 1 \Rightarrow \theta = \sqrt{d_1^2 + d_2^2 + d_3^2}$$

$$n_i = \frac{d_1}{\sqrt{d_1^2 + d_2^2 + d_3^2}}, \quad \text{where } i \in \{1, 2, 3\}$$

In this problem, we consider d_1, d_2, and d_3 as independent variables.

$$\frac{\partial\theta}{\partial d_i} = \frac{d_i}{\sqrt{d_1^2 + d_2^2 + d_3^2}}, \quad \text{where } i \in \{1, 2, 3\}$$

$$\begin{cases} \frac{\partial\alpha}{\partial d_i} = \frac{\partial\alpha}{\partial\theta}\frac{\partial\theta}{\partial d_i} = \cos\theta\frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}} = \beta n_i, \quad \text{where } i \in \{1, 2, 3\} \\ \frac{\partial\beta}{\partial d_i} = \frac{\partial\beta}{\partial\theta}\frac{\partial\theta}{\partial d_i} = -\sin\theta\frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}} = -\alpha n_i, \quad \text{where } i \in \{1, 2, 3\} \\ \frac{\partial\gamma}{\partial d_i} = \frac{\partial\gamma}{\partial\theta}\frac{\partial\theta}{\partial d_i} = \sin\theta\frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}} = \alpha n_i, \quad \text{where } i \in \{1, 2, 3\} \end{cases}$$

$$
\begin{aligned}
\frac{\partial n_j}{\partial d_i} &= \begin{cases} \frac{\sqrt{d_1^2+d_2^2+d_3^2}-d_i\frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}}}{d_1^2+d_2^2+d_3^2} & \text{if } i=j \\ -\frac{d_j\frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}}}{d_1^2+d_2^2+d_3^2} & \text{if } i\neq j \end{cases} \\
&= \begin{cases} \frac{1-n_i^2}{\theta} & \text{if } i=j \\ -\frac{n_i n_j}{\theta} & \text{if } i\neq j \end{cases}
\end{aligned}
$$

The answer is following, and the detail is in the next page:

$$
\begin{aligned}
\frac{d\mathbf{r}}{d\mathbf{d}^T} &= \begin{bmatrix}
\frac{\partial r_{11}}{\partial d_1} & \frac{\partial r_{11}}{\partial d_2} & \frac{\partial r_{11}}{\partial d_3} \\
\frac{\partial r_{12}}{\partial d_1} & \frac{\partial r_{12}}{\partial d_2} & \frac{\partial r_{12}}{\partial d_3} \\
\frac{\partial r_{13}}{\partial d_1} & \frac{\partial r_{13}}{\partial d_2} & \frac{\partial r_{13}}{\partial d_3} \\
\frac{\partial r_{21}}{\partial d_1} & \frac{\partial r_{21}}{\partial d_2} & \frac{\partial r_{21}}{\partial d_3} \\
\frac{\partial r_{22}}{\partial d_1} & \frac{\partial r_{22}}{\partial d_2} & \frac{\partial r_{22}}{\partial d_3} \\
\frac{\partial r_{23}}{\partial d_1} & \frac{\partial r_{23}}{\partial d_2} & \frac{\partial r_{23}}{\partial d_3} \\
\frac{\partial r_{31}}{\partial d_1} & \frac{\partial r_{31}}{\partial d_2} & \frac{\partial r_{31}}{\partial d_3} \\
\frac{\partial r_{32}}{\partial d_1} & \frac{\partial r_{32}}{\partial d_2} & \frac{\partial r_{32}}{\partial d_3} \\
\frac{\partial r_{33}}{\partial d_1} & \frac{\partial r_{33}}{\partial d_2} & \frac{\partial r_{33}}{\partial d_3}
\end{bmatrix} \\
&= \begin{bmatrix}
\alpha n_1(n_1^2-1)+\frac{2\gamma n_1(1-n_1^2)}{\theta} & \alpha n_2(n_1^2-1)-\frac{2\gamma n_1^2 n_2}{\theta} & \alpha n_3(n_1^2-1)-\frac{2\gamma n_1^2 n_3}{\theta} \\
n_1(\alpha n_1 n_2-\beta n_3)+\frac{\gamma n_2(1-2n_1^2)+\alpha n_1 n_3}{\theta} & n_2(\alpha n_1 n_2-\beta n_3)+\frac{\gamma n_1(1-2n_2^2)+\alpha n_2 n_3}{\theta} & n_3(\alpha n_1 n_2-\beta n_3)-\frac{2\gamma n_1 n_2 n_3+\alpha(1-n_3^2)}{\theta} \\
n_1(\alpha n_1 n_3+\beta n_2)+\frac{\gamma n_3(1-2n_1^2)-\alpha n_1 n_2}{\theta} & n_2(\alpha n_1 n_3+\beta n_2)+\frac{\alpha(1-n_2^2)-2\gamma n_1 n_2 n_3}{\theta} & n_3(\alpha n_1 n_3+\beta n_2)+\frac{\gamma n_1(1-2n_3^2)-\alpha n_2 n_3}{\theta} \\
n_1(\alpha n_1 n_2+\beta n_3)+\frac{\gamma n_2(1-2n_1^2)-\alpha n_1 n_3}{\theta} & n_2(\alpha n_1 n_2+\beta n_3)+\frac{\gamma n_1(1-2n_2^2)-\alpha n_2 n_3}{\theta} & n_3(\alpha n_1 n_2+\beta n_3^2)+\frac{\alpha(1-n_3^2)-2\gamma n_1 n_2 n_3}{\theta} \\
\alpha n_1(n_2^2-1)-\frac{2\gamma n_1 n_2^2}{\theta} & \alpha n_2(n_2^2-1)+\frac{2\gamma n_2(1-n_2^2)}{\theta} & \alpha n_3(n_2^2-1)-\frac{2\gamma n_2^2 n_3}{\theta} \\
n_1(\alpha n_2 n_3-\beta n_1)-\frac{\alpha(1-n_1^2)+2\gamma n_1 n_2 n_3}{\theta} & n_2(\alpha n_2 n_3-\beta n_1)+\frac{\gamma n_3(1-2n_2^2)+\alpha n_1 n_2}{\theta} & n_3(\alpha n_2 n_3-\beta n_1)+\frac{\gamma n_2(1-2n_3^2)+\alpha n_1 n_3}{\theta} \\
n_1(\alpha n_1 n_3-\beta n_2)+\frac{\gamma n_3(1-2n_1^2)+\alpha n_1 n_2}{\theta} & n_2(\alpha n_1 n_3-\beta n_2)-\frac{\alpha(1-n_2^2)+2\gamma n_1 n_2 n_3}{\theta} & n_3(\alpha n_1 n_3-\beta n_2)+\frac{\gamma n_1(1-2n_3^2)+\alpha n_2 n_3}{\theta} \\
n_1(\alpha n_2 n_3+\beta n_1)+\frac{\alpha(1-n_1^2)-2\gamma n_1 n_2 n_3}{\theta} & n_2(\alpha n_2 n_3+\beta n_1)+\frac{\gamma n_3(1-2n_2^2)-\alpha n_1 n_2}{\theta} & n_3(\alpha n_2 n_3+\beta n_1)+\frac{\gamma n_2(1-2n_3^2)-\alpha n_1 n_3}{\theta} \\
\alpha n_1(n_3^2-1)-\frac{2\gamma n_1 n_3^2}{\theta} & \alpha n_2(n_3^2-1)-\frac{2\gamma n_2 n_3^2}{\theta} & \alpha n_3(n_3^2-1)+\frac{2\gamma n_3(1-n_3^2)}{\theta}
\end{bmatrix}
\end{aligned}
$$

$$
\frac{d\mathbf{r}}{d\mathbf{d}^T} = \begin{bmatrix}
\frac{\partial\beta}{\partial d_1} + \frac{\partial\gamma}{\partial d_1}n_1^2 + \gamma 2n_1\frac{\partial n_1}{\partial d_1} & \frac{\partial\beta}{\partial d_2} + \frac{\partial\gamma}{\partial d_2}n_1^2 + \gamma 2n_1\frac{\partial n_1}{\partial d_2} & \frac{\partial\beta}{\partial d_3} + \frac{\partial\gamma}{\partial d_3}n_1^2 + \gamma 2n_1\frac{\partial n_1}{\partial d_3} \\
\frac{\partial\gamma}{\partial d_1}n_1n_2 + \gamma\frac{\partial n_1}{\partial d_1}n_2 + \gamma n_1\frac{\partial n_2}{\partial d_1} - \frac{\partial\alpha}{\partial d_1}n_3 - \alpha\frac{\partial n_3}{\partial d_1} & \frac{\partial\gamma}{\partial d_2}n_1n_2 + \gamma\frac{\partial n_1}{\partial d_2}n_2 + \gamma n_1\frac{\partial n_2}{\partial d_2} - \frac{\partial\alpha}{\partial d_2}n_3 - \alpha\frac{\partial n_3}{\partial d_2} & \frac{\partial\gamma}{\partial d_3}n_1n_2 + \gamma\frac{\partial n_1}{\partial d_3}n_2 + \gamma n_1\frac{\partial n_2}{\partial d_3} - \frac{\partial\alpha}{\partial d_3}n_3 - \alpha\frac{\partial n_3}{\partial d_3} \\
\frac{\partial\gamma}{\partial d_1}n_1n_3 + \gamma\frac{\partial n_1}{\partial d_1}n_3 + \gamma n_1\frac{\partial n_3}{\partial d_1} + \frac{\partial\alpha}{\partial d_1}n_2 + \alpha\frac{\partial n_2}{\partial d_1} & \frac{\partial\gamma}{\partial d_2}n_1n_3 + \gamma\frac{\partial n_1}{\partial d_2}n_3 + \gamma n_1\frac{\partial n_3}{\partial d_2} + \frac{\partial\alpha}{\partial d_2}n_2 + \alpha\frac{\partial n_2}{\partial d_2} & \frac{\partial\gamma}{\partial d_3}n_1n_3 + \gamma\frac{\partial n_1}{\partial d_3}n_3 + \gamma n_1\frac{\partial n_3}{\partial d_3} + \frac{\partial\alpha}{\partial d_3}n_2 + \alpha\frac{\partial n_2}{\partial d_3} \\
\frac{\partial\gamma}{\partial d_1}n_1n_2 + \gamma\frac{\partial n_1}{\partial d_1}n_2 + \gamma n_1\frac{\partial n_2}{\partial d_1} + \frac{\partial\alpha}{\partial d_1}n_3 + \alpha\frac{\partial n_3}{\partial d_1} & \frac{\partial\gamma}{\partial d_2}n_1n_2 + \gamma\frac{\partial n_1}{\partial d_2}n_2 + \gamma n_1\frac{\partial n_2}{\partial d_2} + \frac{\partial\alpha}{\partial d_2}n_3 + \alpha\frac{\partial n_3}{\partial d_2} & \frac{\partial\gamma}{\partial d_3}n_1n_2 + \gamma\frac{\partial n_1}{\partial d_3}n_2 + \gamma n_1\frac{\partial n_2}{\partial d_3} + \frac{\partial\alpha}{\partial d_3}n_3 + \alpha\frac{\partial n_3}{\partial d_3} \\
\frac{\partial\beta}{\partial d_1} + \frac{\partial\gamma}{\partial d_1}n_2^2 + \gamma 2n_2\frac{\partial n_2}{\partial d_1} & \frac{\partial\beta}{\partial d_2} + \frac{\partial\gamma}{\partial d_2}n_2^2 + \gamma 2n_2\frac{\partial n_2}{\partial d_2} & \frac{\partial\beta}{\partial d_3} + \frac{\partial\gamma}{\partial d_3}n_2^2 + \gamma 2n_2\frac{\partial n_2}{\partial d_3} \\
\frac{\partial\gamma}{\partial d_1}n_2n_3 + \gamma\frac{\partial n_2}{\partial d_1}n_3 + \gamma n_2\frac{\partial n_3}{\partial d_1} - \frac{\partial\alpha}{\partial d_1}n_1 - \alpha\frac{\partial n_1}{\partial d_1} & \frac{\partial\gamma}{\partial d_2}n_2n_3 + \gamma\frac{\partial n_2}{\partial d_2}n_3 + \gamma n_2\frac{\partial n_3}{\partial d_2} - \frac{\partial\alpha}{\partial d_2}n_1 - \alpha\frac{\partial n_1}{\partial d_2} & \frac{\partial\gamma}{\partial d_3}n_2n_3 + \gamma\frac{\partial n_2}{\partial d_3}n_3 + \gamma n_2\frac{\partial n_3}{\partial d_3} - \frac{\partial\alpha}{\partial d_3}n_1 - \alpha\frac{\partial n_1}{\partial d_3} \\
\frac{\partial\gamma}{\partial d_1}n_1n_3 + \gamma\frac{\partial n_1}{\partial d_1}n_3 + \gamma n_1\frac{\partial n_3}{\partial d_1} - \frac{\partial\alpha}{\partial d_1}n_2 - \alpha\frac{\partial n_2}{\partial d_1} & \frac{\partial\gamma}{\partial d_2}n_1n_3 + \gamma\frac{\partial n_1}{\partial d_2}n_3 + \gamma n_1\frac{\partial n_3}{\partial d_2} - \frac{\partial\alpha}{\partial d_2}n_2 - \alpha\frac{\partial n_2}{\partial d_2} & \frac{\partial\gamma}{\partial d_3}n_1n_3 + \gamma\frac{\partial n_1}{\partial d_3}n_3 + \gamma n_1\frac{\partial n_3}{\partial d_3} - \frac{\partial\alpha}{\partial d_3}n_2 - \alpha\frac{\partial n_2}{\partial d_3} \\
\frac{\partial\gamma}{\partial d_1}n_2n_3 + \gamma\frac{\partial n_2}{\partial d_1}n_3 + \gamma n_2\frac{\partial n_3}{\partial d_1} + \frac{\partial\alpha}{\partial d_1}n_1 + \alpha\frac{\partial n_1}{\partial d_1} & \frac{\partial\gamma}{\partial d_2}n_2n_3 + \gamma\frac{\partial n_2}{\partial d_2}n_3 + \gamma n_2\frac{\partial n_3}{\partial d_2} + \frac{\partial\alpha}{\partial d_2}n_1 + \alpha\frac{\partial n_1}{\partial d_2} & \frac{\partial\gamma}{\partial d_3}n_2n_3 + \gamma\frac{\partial n_2}{\partial d_3}n_3 + \gamma n_2\frac{\partial n_3}{\partial d_3} + \frac{\partial\alpha}{\partial d_3}n_1 + \alpha\frac{\partial n_1}{\partial d_3} \\
\frac{\partial\beta}{\partial d_1} + \frac{\partial\gamma}{\partial d_1}n_3^2 + \gamma 2n_3\frac{\partial n_3}{\partial d_1} & \frac{\partial\beta}{\partial d_2} + \frac{\partial\gamma}{\partial d_2}n_3^2 + \gamma 2n_3\frac{\partial n_3}{\partial d_2} & \frac{\partial\beta}{\partial d_3} + \frac{\partial\gamma}{\partial d_3}n_3^2 + \gamma 2n_3\frac{\partial n_3}{\partial d_3}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-\alpha n_1 + \alpha n_1^3 + \frac{2\gamma n_1(1-n_1^2)}{\theta} & -\alpha n_2 + \alpha n_1^2 n_2 - \frac{2\gamma n_1^2 n_2}{\theta} & -\alpha n_3 + \alpha n_1^2 n_3 - \frac{2\gamma n_1^2 n_3}{\theta} \\
\alpha n_1^2 n_2 + \frac{\gamma n_2(1-n_1^2)}{\theta} - \frac{\gamma n_1^2 n_2}{\theta} - \beta n_1 n_3 + \frac{\alpha n_1 n_3}{\theta} & \alpha n_1 n_2^2 - \frac{\gamma n_1 n_2^2}{\theta} + \frac{\gamma n_1(1-n_2^2)}{\theta} - \beta n_2 n_3 + \frac{\alpha n_2 n_3}{\theta} & \alpha n_1 n_2 n_3 - \frac{\gamma n_1 n_2 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} - \beta n_3^2 - \frac{\alpha(1-n_3^2)}{\theta} \\
\alpha n_1^2 n_3 + \frac{\gamma n_3(1-n_1^2)}{\theta} - \frac{\gamma n_1^2 n_3}{\theta} + \beta n_1 n_2 - \frac{\alpha n_1 n_2}{\theta} & \alpha n_1 n_2 n_3 - \frac{\gamma n_1 n_2 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} + \beta n_2^2 + \frac{\alpha(1-n_2^2)}{\theta} & \alpha n_1 n_3^2 - \frac{\gamma n_1 n_3^2}{\theta} + \frac{\gamma n_1(1-n_3^2)}{\theta} + \beta n_2 n_3 - \frac{\alpha n_2 n_3}{\theta} \\
\alpha n_1^2 n_2 + \frac{\gamma n_2(1-n_1^2)}{\theta} - \frac{\gamma n_1^2 n_2}{\theta} + \beta n_1 n_3 - \frac{\alpha n_1 n_3}{\theta} & \alpha n_1 n_2^2 - \frac{\gamma n_1 n_2^2}{\theta} + \frac{\gamma n_1(1-n_2^2)}{\theta} + \beta n_2 n_3 - \frac{\alpha n_2 n_3}{\theta} & \alpha n_1 n_2 n_3 - \frac{\gamma n_1 n_2 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} + \beta n_3^2 + \frac{\alpha(1-n_3^2)}{\theta} \\
-\alpha n_1 + \alpha n_1 n_2^2 - \frac{2\gamma n_1 n_2^2}{\theta} & -\alpha n_2 + \alpha n_2^3 + \frac{2\gamma n_2(1-n_2^2)}{\theta} & -\alpha n_3 + \alpha n_2^2 n_3 - \frac{2\gamma n_2^2 n_3}{\theta} \\
\alpha n_1 n_2 n_3 - \frac{\gamma n_1 n_2 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} - \beta n_1^2 - \frac{\alpha(1-n_1^2)}{\theta} & \alpha n_2^2 n_3 + \frac{\gamma n_3(1-n_2^2)}{\theta} - \frac{\gamma n_2^2 n_3}{\theta} - \beta n_1 n_2 + \frac{\alpha n_1 n_2}{\theta} & \alpha n_2 n_3^2 - \frac{\gamma n_2 n_3^2}{\theta} + \frac{\gamma n_2(1-n_3^2)}{\theta} - \beta n_1 n_3 + \frac{\alpha n_1 n_3}{\theta} \\
\alpha n_1^2 n_3 + \frac{\gamma n_3(1-n_1^2)}{\theta} - \frac{\gamma n_1^2 n_3}{\theta} - \beta n_1 n_2 + \frac{\alpha n_1 n_2}{\theta} & \alpha n_1 n_2 n_3 - \frac{\gamma n_1 n_2 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} - \beta n_2^2 - \frac{\alpha(1-n_2^2)}{\theta} & \alpha n_1 n_3^2 - \frac{\gamma n_1 n_3^2}{\theta} + \frac{\gamma n_1(1-n_3^2)}{\theta} - \beta n_2 n_3 + \frac{\alpha n_2 n_3}{\theta} \\
\alpha n_1 n_2 n_3 - \frac{\gamma n_1 n_2 n_3}{\theta} - \frac{\gamma n_1 n_2 n_3}{\theta} + \beta n_1^2 + \frac{\alpha(1-n_1^2)}{\theta} & \alpha n_2^2 n_3 + \frac{\gamma n_3(1-n_2^2)}{\theta} - \frac{\gamma n_2^2 n_3}{\theta} + \beta n_1 n_2 - \frac{\alpha n_1 n_2}{\theta} & \alpha n_2 n_3^2 - \frac{\gamma n_2 n_3^2}{\theta} + \frac{\gamma n_2(1-n_3^2)}{\theta} + \beta n_1 n_3 - \frac{\alpha n_1 n_3}{\theta} \\
-\alpha n_1 + \alpha n_1 n_3^2 - \frac{2\gamma n_1 n_3^2}{\theta} & -\alpha n_2 + \alpha n_2 n_3^2 - \frac{2\gamma n_2 n_3^2}{\theta} & -\alpha n_3 + \alpha n_3^3 + \frac{2\gamma n_3(1-n_3^2)}{\theta}
\end{bmatrix}
$$