Unsupervised GRU-Augmented EKF via Innovation Likelihood for Multi-Anchor Range-Only Target Tracking under Motion-Model Mismatch

Lei Li, Coauthor Name
University of Bologna, Italy
Email: lei.li@studio.unibo.it

***Abstract*—We study multi-anchor range-only target tracking in regimes where the motion model is imperfect and process-noise statistics are uncertain. We propose an unsupervised estimator that preserves the analytic EKF measurement update and augments the prediction step with a lightweight GRU. The GRU outputs a bounded residual correction to the prior mean and a constrained diagonal scaling of the prior covariance. Training uses unlabeled measurement sequences only and maximizes the one-step predictive likelihood, implemented as an innovation negative log-likelihood with Cholesky-stable linear algebra. We evaluate EKF, UKF, particle filtering, and the proposed GRU-augmented EKF on the Exercise 5.5 range-only scenario under matched and maneuver-mismatched motion.**

***Index Terms*—Target tracking, range-only measurements, extended Kalman filter, innovation likelihood, unsupervised learning, GRU.**

I. Introduction

Multi-anchor ranging is widely used for localization and tracking in indoor positioning, infrastructure-assisted navigation, and time-of-flight sensing. A standard solution combines a kinematic prior with a Kalman-type recursion. In practice, performance degrades when the motion prior is mismatched, or when process-noise statistics are unknown or time-varying, often yielding long-horizon drift and miscalibrated uncertainty.

Model-based nonlinear filters such as the extended Kalman filter (EKF) and the unscented Kalman filter (UKF) can accommodate nonlinear measurements, but they still rely on a reasonably accurate transition model and well-tuned covariances. Particle filters (PFs) can handle stronger nonlinearities and non-Gaussian posteriors, but they increase computational cost and can suffer from weight degeneracy. Learning-based estimators can capture unmodeled dynamics, yet many require supervised state labels or replace the Bayesian update, reducing interpretability and transparency of uncertainty.

We target a hybrid design that retains the analytic EKF measurement update and learns only prediction-side corrections from unlabeled measurements. Concretely, we augment the EKF prediction with a GRU that outputs a residual on the prior mean and a diagonal scaling on the prior covariance. The model is trained end-to-end by minimizing an innovation negative log-likelihood derived from the one-step predictive distribution.

Contributions.

- A GRU-augmented EKF that preserves the closed-form EKF update while producing explicit covariance estimates.
- An unsupervised training objective based solely on innovation predictive likelihood, computed stably via Cholesky factorization.
- An experimental study on Exercise 5.5 range-only tracking, comparing EKF/UKF/PF and the proposed NLL-trained GRU-augmented EKF under motion-model mismatch.

II. System Model

We consider a discrete-time nonlinear state-space model (SSM) with latent state $\boldsymbol{x}_k \in \mathbb{R}^m$ and measurement $\boldsymbol{y}_k \in \mathbb{R}^n$:

$$\boldsymbol{x}_k = f(\boldsymbol{x}_{k-1}) + \boldsymbol{w}_{k-1}, \qquad \boldsymbol{w}_{k-1} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}), \tag{1a}$$

$$\boldsymbol{y}_k = h(\boldsymbol{x}_k) + \boldsymbol{v}_k, \qquad \boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}). \tag{1b}$$

Here $f(\cdot)$ and $h(\cdot)$ are (possibly nonlinear) transition and measurement functions. We use $\boldsymbol{\Sigma}_{k|k}$ and $\boldsymbol{\Sigma}_{k|k-1}$ to denote the posterior and prior error covariance, respectively.

A. Extended Kalman Filter (EKF)

Given $(\hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{\Sigma}_{k-1|k-1})$, EKF linearizes f and h using Jacobians

$$\boldsymbol{J}^f_{k-1} \triangleq \left.\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_{k-1|k-1}}, \qquad \boldsymbol{J}^h_k \triangleq \left.\frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_{k|k-1}}. \tag{2}$$

Prediction.

$$\hat{\boldsymbol{x}}_{k|k-1} = f(\hat{\boldsymbol{x}}_{k-1|k-1}), \tag{3a}$$

$$\boldsymbol{\Sigma}_{k|k-1} = \boldsymbol{J}^f_{k-1}\boldsymbol{\Sigma}_{k-1|k-1}(\boldsymbol{J}^f_{k-1})^{\mathsf{T}} + \boldsymbol{Q}. \tag{3b}$$

Update. Define $\hat{\boldsymbol{y}}_{k|k-1} = h(\hat{\boldsymbol{x}}_{k|k-1})$ and $\Delta \boldsymbol{y}_k = \boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k|k-1}$. Then

$$\boldsymbol{S}_k = \boldsymbol{J}_k^h \boldsymbol{\Sigma}_{k|k-1} (\boldsymbol{J}_k^h)^\top + \boldsymbol{R}, \tag{4a}$$

$$\boldsymbol{K}_k = \boldsymbol{\Sigma}_{k|k-1} (\boldsymbol{J}_k^h)^\top \boldsymbol{S}_k^{-1}, \tag{4b}$$

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}_k \Delta \boldsymbol{y}_k. \tag{4c}$$

For numerical stability, we use the Joseph-form covariance update:

$$\boldsymbol{\Sigma}_{k|k} = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{J}_k^h) \boldsymbol{\Sigma}_{k|k-1} (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{J}_k^h)^\top + \boldsymbol{K}_k \boldsymbol{R} \boldsymbol{K}_k^\top. \tag{5}$$

III. Proposed Approach

We assume the measurement model is reliable, while the transition model used by the filter is only *nominal* and may be mismatched. We preserve the analytic EKF update (4) and learn prediction-side corrections from unlabeled measurements.

A. Nominal EKF Prior

Let f_{nom} denote the nominal transition used by the filter. Given $(\hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{\Sigma}_{k-1|k-1})$, define

$$\boldsymbol{J}_{k-1}^{f,\mathrm{nom}} \triangleq \left. \frac{\partial f_{\mathrm{nom}}(\boldsymbol{x})}{\partial \boldsymbol{x}} \right|_{\boldsymbol{x} = \hat{\boldsymbol{x}}_{k-1|k-1}}. \tag{6}$$

The nominal EKF prior is

$$\hat{\boldsymbol{x}}_{k|k-1}^{\mathrm{nom}} = f_{\mathrm{nom}}(\hat{\boldsymbol{x}}_{k-1|k-1}), \tag{7a}$$

$$\boldsymbol{\Sigma}_{k|k-1}^{\mathrm{nom}} = \boldsymbol{J}_{k-1}^{f,\mathrm{nom}} \boldsymbol{\Sigma}_{k-1|k-1} (\boldsymbol{J}_{k-1}^{f,\mathrm{nom}})^\top + \boldsymbol{Q}_0, \tag{7b}$$

where $\boldsymbol{Q}_0$ is a fixed nominal process covariance (estimated by warm-start; see Sec. III-C).

B. GRU-Augmented Prediction

We augment the nominal prior mean and covariance using a GRU. Let $\boldsymbol{z}_k$ be the GRU hidden state. We form an input feature vector from the previous posterior and the previous innovation:

$$\boldsymbol{u}_k \triangleq \begin{bmatrix} \hat{\boldsymbol{x}}_{k-1|k-1} \\ \Delta \boldsymbol{y}_{k-1} \end{bmatrix}, \qquad \boldsymbol{z}_k = \mathrm{GRU}(\boldsymbol{z}_{k-1}, \boldsymbol{u}_k). \tag{8}$$

Two lightweight heads map $\boldsymbol{z}_k$ to (i) a bounded mean correction $\boldsymbol{\delta}_k$ and (ii) a constrained diagonal scaling vector $\boldsymbol{\alpha}_k$:

$$\boldsymbol{\delta}_k = c \tanh(\boldsymbol{W}_\delta \boldsymbol{z}_k + \boldsymbol{b}_\delta), \tag{9a}$$

$$\boldsymbol{\alpha}_k = \alpha_{\min} \mathbf{1} + (\alpha_{\max} - \alpha_{\min})\, \sigma(\boldsymbol{W}_\alpha \boldsymbol{z}_k + \boldsymbol{b}_\alpha), \tag{9b}$$

where $c > 0$ controls the maximum correction magnitude and $\alpha_{\min} \le \alpha_{k,i} \le \alpha_{\max}$ enforces bounded scaling.

Define $\boldsymbol{A}_k \triangleq \mathrm{diag}(\boldsymbol{\alpha}_k)$. The GRU-augmented prior is

$$\hat{\boldsymbol{x}}_{k|k-1} = \hat{\boldsymbol{x}}_{k|k-1}^{\mathrm{nom}} + \boldsymbol{\delta}_k, \tag{10a}$$

$$\boldsymbol{\Sigma}_{k|k-1} = \boldsymbol{A}_k\, \boldsymbol{\Sigma}_{k|k-1}^{\mathrm{nom}}\, \boldsymbol{A}_k. \tag{10b}$$

We then apply the standard EKF update (4) with this augmented prior, thereby retaining a closed-form Bayesian measurement update and explicit covariance recursion.

C. Warm-Start of the Process Covariance

Training is unsupervised and uses only measurement sequences. We estimate a nominal process covariance on a held-out warm-start subset $\mathbb{D}_{\mathrm{warm}}$ and keep it fixed during subsequent GRU training.

We adopt an isotropic parameterization $\boldsymbol{Q}_0 = q_0^2 \boldsymbol{I}_{d_x}$. We select q_0 by minimizing the innovation negative log-likelihood on $\mathbb{D}_{\mathrm{warm}}$ using a short one-dimensional search.

For a candidate q_0, we run a nominal EKF (without GRU augmentation) on $\mathbb{D}_{\mathrm{warm}}$ and compute the one-step predictive distribution $\boldsymbol{y}_k \mid \boldsymbol{Y}_{1:k-1} \approx \mathcal{N}(\hat{\boldsymbol{y}}_{k|k-1}, \boldsymbol{S}_k)$, where $\Delta \boldsymbol{y}_k = \boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k|k-1}$ and

$$\boldsymbol{S}_k = \boldsymbol{H}_k \boldsymbol{\Sigma}_{k|k-1} \boldsymbol{H}_k^\top + \boldsymbol{R}. \tag{11}$$

Discarding constants, the per-step innovation NLL is

$$\ell_k(q_0) = \Delta \boldsymbol{y}_k^\top \boldsymbol{S}_k^{-1} \Delta \boldsymbol{y}_k + \log\det(\boldsymbol{S}_k). \tag{12}$$

We aggregate over the warm-start set

$$\mathcal{L}_{\mathrm{warm}}(q_0) \triangleq \frac{1}{N_{\mathrm{warm}}} \sum_{k \in \mathbb{D}_{\mathrm{warm}}} \ell_k(q_0), \tag{13}$$

and select

$$q_0^\star \triangleq \arg\min_{q_0 \in \mathcal{G}} \mathcal{L}_{\mathrm{warm}}(q_0), \qquad \boldsymbol{Q}_0 \leftarrow (q_0^\star)^2 \boldsymbol{I}_{d_x}, \tag{14}$$

where $\mathcal{G}$ is a short log-spaced grid. All terms are computed via Cholesky factorization of $\boldsymbol{S}_k$ to avoid explicit matrix inversion and to obtain $\log\det(\boldsymbol{S}_k)$ stably.

The resulting $\boldsymbol{Q}_0$ is then fixed for subsequent GRU-augmented training.

D. Unsupervised Training via Innovation Negative Log-Likelihood

Given a measurement sequence $\boldsymbol{Y}_{1:T} = \{\boldsymbol{y}_k\}_{k=1}^T$, the EKF defines a one-step predictive distribution

$$p(\boldsymbol{y}_k \mid \boldsymbol{Y}_{1:k-1}) \approx \mathcal{N}\big(\hat{\boldsymbol{y}}_{k|k-1},\, \boldsymbol{S}_k\big), \tag{15}$$

where $\hat{\boldsymbol{y}}_{k|k-1} = h(\hat{\boldsymbol{x}}_{k|k-1})$ and $\boldsymbol{S}_k$ is given by (4a) computed using the augmented prior (10). Applying the chain rule,

$$p(\boldsymbol{Y}_{1:T}) = \prod_{k=1}^{T} p(\boldsymbol{y}_k \mid \boldsymbol{Y}_{1:k-1}), \tag{16}$$

and taking negative log-likelihood yields (up to an additive constant independent of parameters):

$$\mathcal{L}_{\mathrm{NLL}}(\theta) = \frac{1}{T} \sum_{k=1}^{T} \left(\Delta \boldsymbol{y}_k^\top \boldsymbol{S}_k^{-1} \Delta \boldsymbol{y}_k + \log\det(\boldsymbol{S}_k) \right), \tag{17}$$

where θ denotes all trainable GRU and head parameters, and $\Delta \boldsymbol{y}_k = \boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k|k-1}$. In practice, $\boldsymbol{S}_k^{-1} \Delta \boldsymbol{y}_k$ and $\log\det(\boldsymbol{S}_k)$ are computed via Cholesky factorization to avoid explicit matrix inversion.

IV. Experiments

This section specifies the Exercise 5.5 scenario, the professor-provided range observation convention, and the evaluation protocol. All concrete model matrices, anchor geometry, and noise settings are defined here.

A. Scenario: Exercise 5.5 Constant-Velocity Model

We use the 2D constant-velocity state

$$\boldsymbol{x}_k \triangleq [x_k,\ y_k,\ \dot{x}_k,\ \dot{y}_k]^\top \in \mathbb{R}^4. \tag{18}$$

With sampling period Δt, the nominal linear transition is

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{x}_{k-1} + \boldsymbol{q}_{k-1}, \qquad \boldsymbol{q}_{k-1} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}), \tag{19}$$

with

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{20}$$

The process noise covariance follows the discretized Wiener velocity model:

$$\boldsymbol{Q} = \begin{bmatrix} q_1^c \frac{\Delta t^3}{3} & 0 & q_1^c \frac{\Delta t^2}{2} & 0 \\ 0 & q_2^c \frac{\Delta t^3}{3} & 0 & q_2^c \frac{\Delta t^2}{2} \\ q_1^c \frac{\Delta t^2}{2} & 0 & q_1^c \Delta t & 0 \\ 0 & q_2^c \frac{\Delta t^2}{2} & 0 & q_2^c \Delta t \end{bmatrix}. \tag{21}$$

B. Multi-Anchor Range-Only Measurements (Professor Convention)

Let M anchors have known fixed 2D positions $\boldsymbol{s}^{(i)} = [s_x^{(i)},\ s_y^{(i)}]^\top$, $i \in \{1, \ldots, M\}$. At time k, anchor i produces one scalar range measurement using the professor's "plus" convention:

$$\theta_k^{(i)} = \sqrt{\left(s_x^{(i)} + x_k\right)^2 + \left(s_y^{(i)} + y_k\right)^2} + r_k^{(i)}, \qquad r_k^{(i)} \sim \mathcal{N}(0, \sigma_{r,i}^2). \tag{22}$$

Stacking $\boldsymbol{y}_k = [\theta_k^{(1)}, \ldots, \theta_k^{(M)}]^\top$ yields

$$\boldsymbol{y}_k = \boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{r}_k, \qquad \boldsymbol{r}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}), \quad \boldsymbol{R} = \mathrm{diag}(\sigma_{r,1}^2, \ldots, \sigma_{r,M}^2). \tag{23}$$

Here $\boldsymbol{h}(\boldsymbol{x}_k) = [h^{(1)}(\boldsymbol{x}_k), \ldots, h^{(M)}(\boldsymbol{x}_k)]^\top$ and

$$h^{(i)}(\boldsymbol{x}_k) = \sqrt{\left(s_x^{(i)} + x_k\right)^2 + \left(s_y^{(i)} + y_k\right)^2}. \tag{24}$$

C. EKF Measurement Jacobian for the Professor Range Form

For anchor i, define

$$\Delta x_k^{(i)} \triangleq s_x^{(i)} + \hat{x}_{k|k-1}, \qquad \Delta y_k^{(i)} \triangleq s_y^{(i)} + \hat{y}_{k|k-1}, \qquad d_k^{(i)} \triangleq \sqrt{(\Delta x_k^{(i)})^2 + (\Delta y_k^{(i)})^2}.$$

Clamping $d_k^{(i)} \leftarrow \max(d_k^{(i)}, \varepsilon_d)$ avoids division by zero. Then the i-th Jacobian row (w.r.t. $\boldsymbol{x}_k = [x_k, y_k, \dot{x}_k, \dot{y}_k]^\top$) is

$$\boldsymbol{H}_k^{(i)} = \begin{bmatrix} \frac{\Delta x_k^{(i)}}{d_k^{(i)}} & \frac{\Delta y_k^{(i)}}{d_k^{(i)}} & 0 & 0 \end{bmatrix}. \tag{25}$$

Stacking rows gives $\boldsymbol{H}_k \in \mathbb{R}^{M\times 4}$ used in EKF/UKF linearization steps.

D. Data Generation and Motion-Model Mismatch (Exercise 5.5-Style Maneuvers)

We generate trajectories from (19)–(21) outside maneuver segments. To induce controlled mismatch while keeping the filter nominal constant-velocity, we inject turn segments by rotating the *true* velocity vector:

$$\begin{bmatrix} \dot{x}_{k+1} \\ \dot{y}_{k+1} \end{bmatrix} = \begin{bmatrix} \cos(\omega\Delta t) & -\sin(\omega\Delta t) \\ \sin(\omega\Delta t) & \cos(\omega\Delta t) \end{bmatrix} \begin{bmatrix} \dot{x}_k \\ \dot{y}_k \end{bmatrix}, \tag{26}$$

followed by position integration. Outside these segments, the true dynamics follow (19). We evaluate:

- **Matched:** $\omega = 0$ for all k.
- **Mild mismatch:** $\omega = \omega_1$ on designated segments, 0 otherwise.
- **Strong mismatch:** $\omega = \omega_2$ on the same segments, 0 otherwise.

E. Baselines

We compare four methods:

- **EKF (nominal):** EKF with nominal $(\boldsymbol{F}, \boldsymbol{Q}_0)$ and range model (22).
- **UKF (nominal):** UKF with the same nominal $(\boldsymbol{F}, \boldsymbol{Q}_0)$ and $\boldsymbol{R}$.
- **PF (bootstrap):** particle filter with transition (19) and Gaussian process noise, and importance weights from the Gaussian likelihood implied by (23).
- **GRU-augmented EKF (ours, NLL-only):** Sec. III with warm-started $\boldsymbol{Q}_0$ and training loss (17).

F. Training Protocol

Training uses only measurement sequences $\{\boldsymbol{y}_k\}$. We optimize θ by minimizing (17) with truncated backpropagation through time (TBPTT) over short windows. Warm-start estimation of $\boldsymbol{Q}_0$ is performed once on a held-out warm-start subset, and then $\boldsymbol{Q}_0$ is fixed.

G. Metrics

We report position RMSE over N test trajectories of length T:

$$\mathrm{RMSE}_{\mathrm{pos}} = \sqrt{\frac{1}{NT}\sum_{n=1}^{N}\sum_{k=1}^{T}\left\|\boldsymbol{p}_k^{(n)} - \hat{\boldsymbol{p}}_{k|k}^{(n)}\right\|_2^2}, \qquad \boldsymbol{p}_k = [x_k, y_k]^\top. \tag{27}$$

We additionally report the mean innovation NLL on the test set, computed from (17). Optionally, we monitor the mean NIS, $\mathrm{NIS}_k = \Delta\boldsymbol{y}_k^\top \boldsymbol{S}_k^{-1} \Delta\boldsymbol{y}_k$, as a diagnostic of innovation-level scale calibration.

V. Conclusion

We presented an unsupervised GRU-augmented EKF for multi-anchor range-only tracking that preserves the analytic EKF measurement update and learns prediction-side corrections using innovation likelihood only. By combining residual mean correction and diagonal covariance scaling within the EKF prior, the proposed method improves robustness under controlled maneuver-induced mismatch, while retaining explicit covariance outputs. Experiments on the Exercise 5.5 range-only scenario compare EKF, UKF, PF, and the proposed NLL-trained GRU-augmented EKF.

REFERENCES