

# Unsupervised GRU-Augmented EKF via Innovation Likelihood for Multi-Anchor Range-Only Target Tracking under Motion-Model Mismatch

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**Abstract**—We study multi-anchor range-only target tracking in regimes where the motion model is imperfect and process-noise statistics are uncertain. We propose an unsupervised estimator that preserves the analytic EKF measurement update and augments the prediction step with a lightweight GRU. The GRU outputs a bounded residual correction to the prior mean and a constrained diagonal scaling of the prior covariance. Training uses unlabeled measurement sequences only and maximizes the one-step predictive likelihood, implemented as an innovation negative log-likelihood with Cholesky-stable linear algebra. We evaluate EKF, UKF, particle filtering, and the proposed GRU-augmented EKF on the Exercise 5.5 range-only scenario under matched and maneuver-mismatched motion.

**Index Terms**—Target tracking, range-only measurements, extended Kalman filter, innovation likelihood, unsupervised learning, GRU.

## I. INTRODUCTION

Multi-anchor ranging is widely used for localization and tracking in indoor positioning, infrastructure-assisted navigation, and time-of-flight sensing. A standard solution combines a kinematic prior with a Kalman-type recursion. In practice, performance degrades when the motion prior is mismatched, or when process-noise statistics are unknown or time-varying, often yielding long-horizon drift and miscalibrated uncertainty.

Model-based nonlinear filters such as the extended Kalman filter (EKF) and the unscented Kalman filter (UKF) can accommodate nonlinear measurements, but they still rely on a reasonably accurate transition model and well-tuned covariances. Particle filters (PFs) can handle stronger nonlinearities and non-Gaussian posteriors, but they increase computational cost and can suffer from weight degeneracy. Learning-based estimators can capture unmodeled dynamics, yet many require supervised state labels or replace the Bayesian update, reducing interpretability and transparency of uncertainty.

We target a hybrid design that retains the analytic EKF measurement update and learns only prediction-side corrections from unlabeled measurements. Concretely, we augment the EKF prediction with a GRU that outputs a residual on the prior mean and a diagonal scaling on the prior covariance. The model is trained end-to-end by minimizing an innovation

negative log-likelihood derived from the one-step predictive distribution.

## Contributions.

- A GRU-augmented EKF that preserves the closed-form EKF update while producing explicit covariance estimates.
- An unsupervised training objective based solely on innovation predictive likelihood, computed stably via Cholesky factorization.
- An experimental study on Exercise 5.5 range-only tracking, comparing EKF/UKF/PF and the proposed NLL-trained GRU-augmented EKF under motion-model mismatch.

## II. SYSTEM MODEL

We consider a discrete-time nonlinear state-space model (SSM) with latent state  $\mathbf{x}_k \in \mathbb{R}^m$  and measurement  $\mathbf{y}_k \in \mathbb{R}^n$ :

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}) + \mathbf{w}_{k-1}, \quad \mathbf{w}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}), \quad (1a)$$

$$\mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{v}_k, \quad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}). \quad (1b)$$

Here  $f(\cdot)$  and  $h(\cdot)$  are (possibly nonlinear) transition and measurement functions. We use  $\Sigma_{k|k}$  and  $\Sigma_{k|k-1}$  to denote the posterior and prior error covariance, respectively.

### A. Extended Kalman Filter (EKF)

Given  $(\hat{\mathbf{x}}_{k-1|k-1}, \Sigma_{k-1|k-1})$ , EKF linearizes  $f$  and  $h$  using Jacobians

$$\mathbf{J}_{k-1}^f \triangleq \left. \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1|k-1}}, \quad \mathbf{J}_k^h \triangleq \left. \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}. \quad (2)$$

### Prediction.

$$\hat{\mathbf{x}}_{k|k-1} = f(\hat{\mathbf{x}}_{k-1|k-1}), \quad (3a)$$

$$\Sigma_{k|k-1} = \mathbf{J}_{k-1}^f \Sigma_{k-1|k-1} (\mathbf{J}_{k-1}^f)^T + \mathbf{Q}. \quad (3b)$$

**Update.** Define  $\hat{y}_{k|k-1} = h(\hat{x}_{k|k-1})$  and  $\Delta\mathbf{y}_k = \mathbf{y}_k - \hat{y}_{k|k-1}$ . Then

$$\mathbf{S}_k = \mathbf{J}_k^h \Sigma_{k|k-1} (\mathbf{J}_k^h)^\top + \mathbf{R}, \quad (4a)$$

$$\mathbf{K}_k = \Sigma_{k|k-1} (\mathbf{J}_k^h)^\top \mathbf{S}_k^{-1}, \quad (4b)$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \Delta\mathbf{y}_k. \quad (4c)$$

For numerical stability, we use the Joseph-form covariance update:

$$\Sigma_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{J}_k^h) \Sigma_{k|k-1} (\mathbf{I} - \mathbf{K}_k \mathbf{J}_k^h)^\top + \mathbf{K}_k \mathbf{R} \mathbf{K}_k^\top. \quad (5)$$

### III. PROPOSED APPROACH

We assume the measurement model is reliable, while the transition model used by the filter is only *nominal* and may be mismatched. We preserve the analytic EKF update (??) and learn prediction-side corrections from unlabeled measurements.

#### A. Nominal EKF Prior

Let  $f_{\text{nom}}$  denote the nominal transition used by the filter. Given  $(\hat{\mathbf{x}}_{k-1|k-1}, \Sigma_{k-1|k-1})$ , define

$$\mathbf{J}_{k-1}^{f,\text{nom}} \triangleq \frac{\partial f_{\text{nom}}(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1|k-1}}. \quad (6)$$

The nominal EKF prior is

$$\hat{\mathbf{x}}_{k|k-1}^{\text{nom}} = f_{\text{nom}}(\hat{\mathbf{x}}_{k-1|k-1}), \quad (7a)$$

$$\Sigma_{k|k-1}^{\text{nom}} = \mathbf{J}_{k-1}^{f,\text{nom}} \Sigma_{k-1|k-1} (\mathbf{J}_{k-1}^{f,\text{nom}})^\top + \mathbf{Q}_0, \quad (7b)$$

where  $\mathbf{Q}_0$  is a fixed nominal process covariance (estimated by warm-start; see Sec. ??).

#### B. GRU-Augmented Prediction

We augment the nominal prior mean and covariance using a GRU. Let  $\mathbf{z}_k$  be the GRU hidden state. We form an input feature vector from the previous posterior and the previous innovation:

$$\mathbf{u}_k \triangleq \begin{bmatrix} \hat{\mathbf{x}}_{k-1|k-1} \\ \Delta\mathbf{y}_{k-1} \end{bmatrix}, \quad \mathbf{z}_k = \text{GRU}(\mathbf{z}_{k-1}, \mathbf{u}_k). \quad (8)$$

Two lightweight heads map  $\mathbf{z}_k$  to (i) a bounded mean correction  $\delta_k$  and (ii) a constrained diagonal scaling vector  $\alpha_k$ :

$$\delta_k = c \tanh(\mathbf{W}_\delta \mathbf{z}_k + \mathbf{b}_\delta), \quad (9a)$$

$$\alpha_k = \alpha_{\min} \mathbf{1} + (\alpha_{\max} - \alpha_{\min}) \sigma(\mathbf{W}_\alpha \mathbf{z}_k + \mathbf{b}_\alpha), \quad (9b)$$

where  $c > 0$  controls the maximum correction magnitude and  $\alpha_{\min} \leq \alpha_{k,i} \leq \alpha_{\max}$  enforces bounded scaling.

Define  $\mathbf{A}_k \triangleq \text{diag}(\alpha_k)$ . The GRU-augmented prior is

$$\hat{\mathbf{x}}_{k|k-1} = \hat{\mathbf{x}}_{k|k-1}^{\text{nom}} + \delta_k, \quad (10a)$$

$$\Sigma_{k|k-1} = \mathbf{A}_k \Sigma_{k|k-1}^{\text{nom}} \mathbf{A}_k. \quad (10b)$$

We then apply the standard EKF update (??) with this augmented prior, thereby retaining a closed-form Bayesian measurement update and explicit covariance recursion.

#### C. Warm-Start of the Process Covariance

Training is unsupervised and uses only measurement sequences. We estimate a nominal process covariance on a held-out warm-start subset  $\mathbb{D}_{\text{warm}}$  and keep it fixed during subsequent GRU training.

We adopt an isotropic parameterization  $\mathbf{Q}_0 = q_0^2 \mathbf{I}_{d_x}$ . We select  $q_0$  by minimizing the innovation negative log-likelihood on  $\mathbb{D}_{\text{warm}}$  using a short one-dimensional search.

For a candidate  $q_0$ , we run a nominal EKF (without GRU augmentation) on  $\mathbb{D}_{\text{warm}}$  and compute the one-step predictive distribution  $\mathbf{y}_k | \mathbf{Y}_{1:k-1} \approx \mathcal{N}(\hat{\mathbf{y}}_{k|k-1}, \mathbf{S}_k)$ , where  $\Delta\mathbf{y}_k = \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}$  and

$$\mathbf{S}_k = \mathbf{H}_k \Sigma_{k|k-1} \mathbf{H}_k^\top + \mathbf{R}. \quad (11)$$

Discarding constants, the per-step innovation NLL is

$$\ell_k(q_0) = \Delta\mathbf{y}_k^\top \mathbf{S}_k^{-1} \Delta\mathbf{y}_k + \log \det(\mathbf{S}_k). \quad (12)$$

We aggregate over the warm-start set

$$\mathcal{L}_{\text{warm}}(q_0) \triangleq \frac{1}{N_{\text{warm}}} \sum_{k \in \mathbb{D}_{\text{warm}}} \ell_k(q_0), \quad (13)$$

and select

$$q_0^* \triangleq \arg \min_{q_0 \in \mathcal{G}} \mathcal{L}_{\text{warm}}(q_0), \quad \mathbf{Q}_0 \leftarrow (q_0^*)^2 \mathbf{I}_{d_x}, \quad (14)$$

where  $\mathcal{G}$  is a short log-spaced grid. All terms are computed via Cholesky factorization of  $\mathbf{S}_k$  to avoid explicit matrix inversion and to obtain  $\log \det(\mathbf{S}_k)$  stably.

The resulting  $\mathbf{Q}_0$  is then fixed for subsequent GRU-augmented training.

#### D. Unsupervised Training via Innovation Negative Log-Likelihood

Given a measurement sequence  $\mathbf{Y}_{1:T} = \{\mathbf{y}_k\}_{k=1}^T$ , the EKF defines a one-step predictive distribution

$$p(\mathbf{y}_k | \mathbf{Y}_{1:k-1}) \approx \mathcal{N}(\hat{\mathbf{y}}_{k|k-1}, \mathbf{S}_k), \quad (15)$$

where  $\hat{\mathbf{y}}_{k|k-1} = h(\hat{\mathbf{x}}_{k|k-1})$  and  $\mathbf{S}_k$  is given by (??) computed using the augmented prior (??). Applying the chain rule,

$$p(\mathbf{Y}_{1:T}) = \prod_{k=1}^T p(\mathbf{y}_k | \mathbf{Y}_{1:k-1}), \quad (16)$$

and taking negative log-likelihood yields (up to an additive constant independent of parameters):

$$\mathcal{L}_{\text{NLL}}(\theta) = \frac{1}{T} \sum_{k=1}^T (\Delta\mathbf{y}_k^\top \mathbf{S}_k^{-1} \Delta\mathbf{y}_k + \log \det(\mathbf{S}_k)), \quad (17)$$

where  $\theta$  denotes all trainable GRU and head parameters, and  $\Delta\mathbf{y}_k = \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}$ . In practice,  $\mathbf{S}_k^{-1} \Delta\mathbf{y}_k$  and  $\log \det(\mathbf{S}_k)$  are computed via Cholesky factorization to avoid explicit matrix inversion.

## IV. EXPERIMENTS

This section specifies the Exercise 5.5 scenario, the standard range observation convention, and the evaluation protocol. All concrete model matrices, anchor geometry, and noise settings are defined here.

#### A. Scenario: Exercise 5.5 Constant-Velocity Model

We use the 2D constant-velocity state

$$\mathbf{x}_k \triangleq [x_k, y_k, \dot{x}_k, \dot{y}_k]^\top \in \mathbb{R}^4. \quad (18)$$

With sampling period  $\Delta t$ , the nominal linear transition is

$$\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \quad \mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}), \quad (19)$$

with

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (20)$$

The process noise covariance follows the discretized Wiener velocity model:

$$\mathbf{Q} = \begin{bmatrix} q_1^c \frac{\Delta t^3}{3} & 0 & q_1^c \frac{\Delta t^2}{2} & 0 \\ 0 & q_2^c \frac{\Delta t^3}{3} & 0 & q_2^c \frac{\Delta t^2}{2} \\ q_1^c \frac{\Delta t^2}{2} & 0 & q_1^c \Delta t & 0 \\ 0 & q_2^c \frac{\Delta t^2}{2} & 0 & q_2^c \Delta t \end{bmatrix}. \quad (21)$$

#### B. Multi-Anchor Range-Only Measurements (Standard Form)

Let  $M$  anchors have known fixed 2D positions  $\mathbf{s}^{(i)} = [s_x^{(i)}, s_y^{(i)}]^\top$ ,  $i \in \{1, \dots, M\}$ . At time  $k$ , anchor  $i$  produces one scalar range measurement in the standard “target minus anchor” form:

$$\theta_k^{(i)} = \sqrt{(x_k - s_x^{(i)})^2 + (y_k - s_y^{(i)})^2} + r_k^{(i)}, \quad r_k^{(i)} \sim \mathcal{N}(0, \sigma_{r,i}^2) \quad (22)$$

Stacking  $\mathbf{y}_k = [\theta_k^{(1)}, \dots, \theta_k^{(M)}]^\top$  yields

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k, \quad \mathbf{r}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}), \quad \mathbf{R} = \text{diag}(\sigma_{r,1}^2, \dots, \sigma_{r,M}^2). \quad (23)$$

Here  $\mathbf{h}(\mathbf{x}_k) = [h^{(1)}(\mathbf{x}_k), \dots, h^{(M)}(\mathbf{x}_k)]^\top$  and

$$h^{(i)}(\mathbf{x}_k) = \sqrt{(x_k - s_x^{(i)})^2 + (y_k - s_y^{(i)})^2}. \quad (24)$$

#### C. EKF Measurement Jacobian for the Standard Range Form

For anchor  $i$ , define

$$\begin{aligned} \Delta x_k^{(i)} &\triangleq \hat{x}_{k|k-1} - s_x^{(i)}, & \Delta y_k^{(i)} &\triangleq \hat{y}_{k|k-1} - s_y^{(i)}, \\ d_k^{(i)} &\triangleq \sqrt{(\Delta x_k^{(i)})^2 + (\Delta y_k^{(i)})^2}. \end{aligned}$$

Clamping  $d_k^{(i)} \leftarrow \max(d_k^{(i)}, \varepsilon_d)$  avoids division by zero. Then the  $i$ -th Jacobian row (w.r.t.  $\mathbf{x}_k = [x_k, y_k, \dot{x}_k, \dot{y}_k]^\top$ ) is

$$\mathbf{H}_k^{(i)} = \begin{bmatrix} \frac{\Delta x_k^{(i)}}{d_k^{(i)}} & \frac{\Delta y_k^{(i)}}{d_k^{(i)}} & 0 & 0 \end{bmatrix}. \quad (25)$$

Stacking rows gives  $\mathbf{H}_k \in \mathbb{R}^{M \times 4}$  used in EKF/UKF linearization steps.

#### D. Data Generation and Motion-Model Mismatch (Exercise 5.5-Style Maneuvers)

Let  $\mathbf{p}(t) = [x(t), y(t)]^\top$  and  $\mathbf{v}(t) = [\dot{x}(t), \dot{y}(t)]^\top$ . We use a fixed sampling period  $\Delta t = 0.01$  s throughout both data generation and filtering.

Outside maneuver segments, ground-truth trajectories are generated from the continuous-time white-noise-acceleration (WNA) model

$$\dot{\mathbf{p}}(t) = \mathbf{v}(t), \quad \dot{\mathbf{v}}(t) = \mathbf{w}(t), \quad (26)$$

where  $\mathbf{w}(t)$  is zero-mean white Gaussian noise with spectral density  $\mathbf{Q}_c = \text{diag}(q_1^c, q_2^c)$ . Sampling (??) with period  $\Delta t$  yields the discrete-time constant-velocity model (??) with  $\mathbf{F}$  in (??) and  $\mathbf{Q}$  in (??).

To induce a controlled model mismatch while keeping the filter nominal constant-velocity, we inject turn segments in the *true* dynamics via a constant turn-rate velocity model

$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}(\omega)\mathbf{v}(t), \quad \boldsymbol{\Omega}(\omega) = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad (27)$$

with  $\dot{\mathbf{p}}(t) = \mathbf{v}(t)$  for position integration. The exact discretization of (??) over  $\Delta t$  gives

$$\mathbf{v}_{k+1} = \exp(\boldsymbol{\Omega}(\omega)\Delta t)\mathbf{v}_k = \begin{bmatrix} \cos(\omega\Delta t) & -\sin(\omega\Delta t) \\ \sin(\omega\Delta t) & \cos(\omega\Delta t) \end{bmatrix} \mathbf{v}_k. \quad (28)$$

We use three maneuver windows (each of length  $L$  discrete steps) and choose a single constant turn rate  $\omega$  such that the velocity direction rotates by  $90^\circ$  over each window:

$$\omega = \frac{\pi/2}{L \Delta t}. \quad (29)$$

In our implementation, each window contains  $L = 50$  update steps (left-closed, right-open), and  $\Delta t = 0.01$  s, hence

$$\omega = \frac{\pi/2}{50 \cdot 0.01} = \pi \text{ rad/s}. \quad (30)$$

Across three such windows, the ground-truth velocity accumulates a total rotation of  $270^\circ$ . During these windows, the filter still assumes the nominal CV transition (??) (i.e.,  $\omega = 0$ ), which creates the intended motion-model mismatch.

We evaluate:

- **Matched:** no turn segments (equivalently  $\omega = 0$  for all  $k$ ).
- **Mismatch:** three  $90^\circ$  turn segments with  $\omega = \pi$  rad/s in the ground truth, while the filter assumes  $\omega = 0$ .

#### E. Baselines

We compare four methods:

- **EKF (nominal):** EKF with nominal ( $\mathbf{F}, \mathbf{Q}_0$ ) and range model (??).
- **UKF (nominal):** UKF with the same nominal ( $\mathbf{F}, \mathbf{Q}_0$ ) and  $\mathbf{R}$ .
- **PF (bootstrap):** particle filter with transition (??) and Gaussian process noise, and importance weights from the Gaussian likelihood implied by (??).
- **GRU-augmented EKF (ours, NLL-only):** Sec. III with warm-started  $\mathbf{Q}_0$  and training loss (??).

#### F. Training Protocol

Training uses only measurement sequences  $\{\mathbf{y}_k\}$ . We optimize  $\theta$  by minimizing (??) with truncated backpropagation through time (TBPTT) over short windows. Warm-start estimation of  $\mathbf{Q}_0$  is performed once on a held-out warm-start subset, and then  $\mathbf{Q}_0$  is fixed.

#### G. Metrics

We report position RMSE over  $N$  test trajectories of length  $T$ :

$$\text{RMSE}_{\text{pos}} = \sqrt{\frac{1}{NT} \sum_{n=1}^N \sum_{k=1}^T \left\| \mathbf{p}_k^{(n)} - \hat{\mathbf{p}}_{k|k}^{(n)} \right\|_2^2}, \quad \mathbf{p}_k = [x_k, y_k]^\top. \quad (31)$$

We additionally report the mean innovation NLL on the test set, computed from (??). Optionally, we monitor the mean NIS,  $\text{NIS}_k = \Delta \mathbf{y}_k^\top \mathbf{S}_k^{-1} \Delta \mathbf{y}_k$ , as a diagnostic of innovation-level scale calibration.

## V. CONCLUSION

We presented an unsupervised GRU-augmented EKF for multi-anchor range-only tracking that preserves the analytic EKF measurement update and learns prediction-side corrections using innovation likelihood only. By combining residual mean correction and diagonal covariance scaling within the EKF prior, the proposed method improves robustness under controlled maneuver-induced mismatch, while retaining explicit covariance outputs. Experiments on the Exercise 5.5 range-only scenario compare EKF, UKF, PF, and the proposed NLL-trained GRU-augmented EKF.

## REFERENCES