Stability of Flows on Networks

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Summary. The problems of traffic flow forecasting on complex traffic networks are still almost not explored. However these problems are very actual for scientists as well as for traffic engineers. In this paper we consider problems of stability of particle (car) flows on networks. The definitions of critical, stable and unstable flow states on networks are obtained as properties of solutions of nonlinear differential equations on graphs. For networks with different geometry the necessary and sufficient conditions of flow stability on networks are found. The perspective problems of exploration of qualitative properties of flows on networks are formulated.

1 Introduction

The rapid growth of the motorization level in the world provoked interest of scientists and engineers in problems of stability of traffic flows [1-5]. However, more difficult problems, such as forecasting of stable states of flows on complex traffic networks, are still almost not explored [6-8].
In this paper we consider problems of stability of particle (car) flows on networks. A network is an oriented graph with edges corresponding to road sections and vertices corresponding to road junctions. A state of flow on networks is defined by the vector-function of densities $\rho(t) = \{\rho_i(t)\}$. The time-dependence of each coordinate $\rho(t)$ is simulated by a system of differential equations and expresses the following physical principle: *the rate of density change on an edge is proportional to the difference of the intensities (flux) of cumulative input and output flows from the edge.*
The flow intensity $q_i(t)$ on edge i is a function of the density $\rho_i(t)$, which is defined by the fundamental diagram $q_i(t) = \lambda_i \rho_i(t)(\rho_i^* - \rho_i(t))$, where ρ_i^* is the maximal density on edge i. Then $q_i^* = \lambda_i \left(\rho_i^*\right)^2 /4$ is the maximal intensity (*the highway capability of the edge)*. At last if l_i is the length of edge i, then $C_i^* = l_i \rho_i^*$ is the *edge capacity* and $C^* = \sum_i C_i^*$ the *network capacity.*
The flow regime $\bar{\rho} = \bar{\rho}(t)$ *in the time* t_0 *is called critical,* if an i-edge exists with $\rho_i(t_0) = \rho_i^*$. According to the physical principle of the model the flow regime will be critical also for $t \geq t_0$, because $q_i(t) \equiv 0$, $t \geq t_0$.

The flow regime is called T-critical, if for $t \geq T$ the flow regime is critical, otherwise *the flow regime is called T-uncritical.* So *the flow state $\bar{\rho}(t)$ is called T-critical point or T-uncritical point accordingly.* Let $T_*(\bar{\rho}(0))$ denote $\min\{\tau \geq 0/$ the flow regime is critical for $t \geq \tau\}$.
The point $\bar{\rho}(0)$ is called stable T-uncritical point, if all points belonging to some neighborhood of $\bar{\rho}(0)$ and considered as initial states will be T-uncritical. Otherwise this *T-uncritical point is called unstable T-uncritical point.*
Let $\bar{\rho}(0)$ *be a stationary uncritical point of flow*, i.e. the uncritical flow state is not changing during time. Then $\bar{\rho}(0)$ is an ∞-uncritical point as $T_*(\bar{\rho}(0)) = \infty$. An *uncritical flow state $\bar{\rho}(0)$ is locally stable*, if for small changes of the flow state the flow returns to the state $\bar{\rho}(0)$ when $t \to \infty$. Clearly $\bar{\rho}(0)$ is a stationary point.
We consider *closed and open networks.* Let the *flow mass C be the total quantity of particles on the network.* If the network is closed then C is constant. If the network is open then particles can arrive to and depart from the network, thus $C = C(t)$. It is clear that the open network can be considered as part of a more complex closed network.

2 Closed Networks

2.1 Flow on Ring Consisting of Two Identical Sections

Let us consider a unidirectional flow on a ring consisting of two identical sections (see Fig. 1). Let $l = l_1 = l_2$ be the lengths of sections, $\rho^* = \rho_1^* = \rho_2^*$ be the maximal densities. Then $C_1^* = C_2^* = \rho^* l$ are the capacities of sections and $C^* = 2\rho^* l$ is the network capacity. In the interval of admissible values $\rho \in [0, \rho^*]$ the dependence of the intensity on density is $\lambda\rho(\rho^* - \rho)$. In this case the exact expression for the flow density dependence can be found.
The flow densities on the sections satisfy a system of two differential equations and a normalization condition expressing the constancy of the flow mass.
Whether the flow regime is critical or not, $T_*(\bar{\rho}(0)) = \infty$, will depend on the ratio between the ring capacity C^* and the flow mass C. It means that:
If the flow mass is less than half of the ring capacity, $C < C^*/2$, then the flow

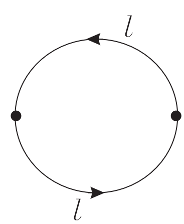

Fig. 1. Ring consisting of two identical sections

regime is stable ∞-uncritical. In this case flow state converges to stationary regime $\rho_1 = \rho_2 = C/2$ when t grows and $\left|\rho_i(t) - \frac{C}{2}\right| \searrow 0, \quad i = 1, 2.$
If $C > C^*/2$, $\rho_1(0) \neq \rho_2(0)$, then the flow regime is critical for any initial admissible conditions and $T_*(\rho_0) \leq \frac{l}{2\lambda\rho^*\left(\frac{2C}{C^*}-1\right)} \ln \frac{(\rho^* - \frac{C\rho^*}{C^*})}{(\rho_0 - \frac{C\rho^*}{C^*})}$, where $\rho_0 = \rho_1(0)$.
Indeed, since in the case of identical sections $\rho_1(0) + \rho_2(0) = C/l$, then either $\rho_1(0) = \rho_2(0) = C/2l$, or on one of the sections the initial density is greater then $C/2l$. In the first case the flow regime is unstable uncritical regime (the stationary point). In the second case the flow state is critical. Let us find the time required to reach the critical regime. Assume that $\rho_0 = \rho_1(0) > C/2l > \rho_2(0)$. Until the critical regime is not reached the time-dependence of the flow density on the first section is given by $\rho(t) = \frac{C}{2l} + (\rho_0 - \frac{C}{2l})e^{\frac{2\lambda}{l}\left(\frac{C}{l} - \rho^*\right)t}$. For the time $T_*(\rho_0)$ to reach the critical regime we have $\frac{C}{2l} + (\rho_0 - \frac{C}{2l})e^{\frac{2\lambda}{l}\left(\frac{C}{l} - \rho^*\right)T_*} = \rho^*$, i.e. $e^{\frac{2\lambda}{l}\left(\frac{C}{l} - \rho^*\right)T_*} = \frac{\rho^* - \frac{C}{2l}}{\rho_0 - \frac{C}{2l}}$ and $\frac{2\lambda}{l}\left(\frac{C}{l} - \rho^*\right)T_* = \ln \frac{\rho^* - \frac{C}{2l}}{\rho_0 - \frac{C}{2l}}$. Thus we get $T_*(\rho_0) = \frac{l}{2\lambda\rho^*\left(\frac{2C}{C^*}-1\right)} \ln \frac{(\rho^* - \frac{C\rho^*}{C^*})}{(\rho_0 - \frac{C\rho^*}{C^*})}$.
If $C = C^*/2$, then the flow state is stationary and unstable at any initial condition $l\left(\rho_1(0) + \rho_2(0)\right) = C$.

2.2 Ring Consisting of Two Non-Identical Sections

Let us consider the flow on a ring consisting of two non-identical sections ($i = 1, 2$) with lengths l_i, maximal densities ρ_i^* and intensities $q_i = \lambda_i\rho_i(\rho_i^* - \rho_i)$ (see Fig. 2).
The system of differential equations for the flow is then

$$\begin{aligned} l_1 \frac{d\rho_1}{dt} &= -\lambda_1\rho_1(\rho_1^* - \rho_1) + \lambda_2\rho_2(\rho_2^* - \rho_2) \\ l_2 \frac{d\rho_2}{dt} &= \lambda_1\rho_1(\rho_1^* - \rho_1) - \lambda_2\rho_2(\rho_2^* - \rho_2). \end{aligned} \tag{1}$$

We have the condition of flow mass constancy

$$l_1\rho_1 + l_2\rho_2 = C, \tag{2}$$

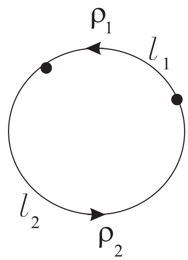

Fig. 2. A ring consisting of two non-identical sections.

and the set of admissible densities $\mathcal{D}$:

$$0 \le \rho_1 \le \rho_1^*, 0 \le \rho_2 \le \rho_2^*. \tag{3}$$

The stationary points are the solutions of the equation

$$-\lambda_1\rho_1(\rho_1^* - \rho_1) + \lambda_2\rho_2(\rho_2^* - \rho_2) = 0. \tag{4}$$

We put $x = \rho_1 - \rho_1^*/2$, $y = \rho_2 - \rho_2^*/2$. Hence in (4) we obtain $\lambda_1(x^2 - (\rho_1^*)^2/4) - \lambda_2(y^2 - (\rho_2^*)^2/4) = 0$ or

$$-\lambda_1 x^2 + \lambda_2 y^2 = -\lambda_1(\rho_1^*)^2/4 + \lambda_2(\rho_2^*)^2/4 = -q_1^* + q_2^*. \tag{5}$$

Assume that $q_2^* > q_1^*$, i.e. the highway capacity of the second section is greater than the capacity of the first section. Then (5) is a hyperbola with focuses on the $y-$axis, i.e. in initial coordinates on $\mathcal{D}$ (Fig. 3).
The direction field of velocities $(\dot{\rho_1}, \dot{\rho_2})$ is parallel to the line $l_1\rho_1 + l_2\rho_2 = C$, and its direction is defined by the sign $\dot{\rho_1}$ in Fig. 3.
In the point $(\rho_1^*, 0)$ the normal to the line (4) is $(\lambda_1\rho_1^*, \lambda_2\rho_2^*)$ and in the point $(0, \rho_2^*)$ is symmetrical and is equal to $(-\lambda_1\rho_1^*, -\lambda_2\rho_2^*)$. *Assume that*

$$\frac{\lambda_2\rho_2^*}{\lambda_1\rho_1^*} > \frac{l_2}{l_1}, \tag{6}$$

i.e. the tangent of the angle of slope of the normal to the hyperbola in the point $(\rho_1^*, 0)$ is greater than the tangent of the angle of slope of the normal to (2). Then when C changes the line (2) can not meet one of hyperbola branches (4) twice. Thus, we get the qualitative sketch shown in Fig. 4.
Condition (6) is equivalent to

$$v_2^* = \frac{\lambda_2\rho_2^*}{l_2} > \frac{\lambda_1\rho_1^*}{l_1} = v_1^*. \tag{7}$$

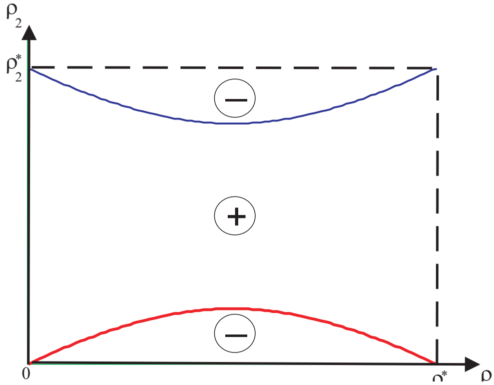

Fig. 3. Signs of $\dot{\rho}$

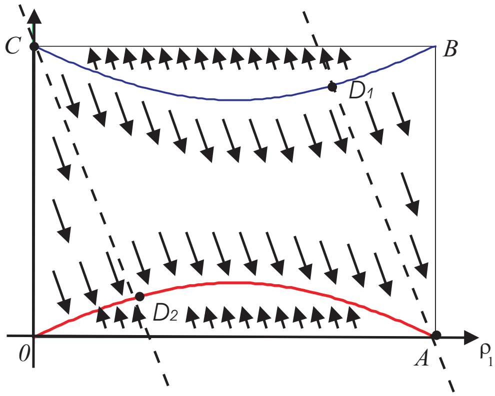

Fig. 4. Case $v_2^* > v_1^*$

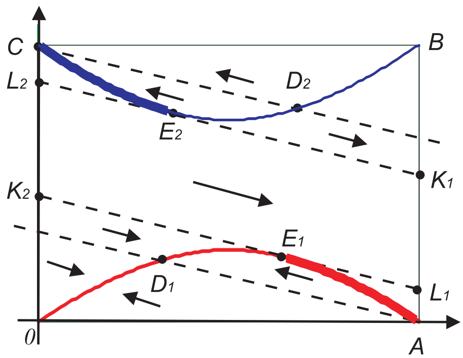

Fig. 5. Case $v_2^* < v_1^*$

Condition (7) means that *the rate of density change on the second section in the neighborhood of the boundaries* $[0, \rho_2^*]$ *is greater than the corresponding rate on the first section*. The opposite case, $(v_2^* < v_1^*)$, is shown in Fig. 5.
The distinctive feature of the second case is *the existence of stable and unstable fragments of stationary points on each hyperbola branch. Otherwise, in the first case, when* $v_2^* > v_1^*$ *(Fig. 4), the upper branch is unstable and the lower branch is stable.*
Let us assume that $q_1^* = q_2^*$, i.e. $\lambda_1(\rho_1^*)^2 = \lambda_2(\rho_2^*)^2$. The hyperbola (4) degenerates to the pair of lines $\lambda_1 \left(\rho_1 - \rho_1^*/2\right)^2 = \lambda_2 \left(\rho_2 - \rho_2^*/2\right)^2$. It is equal to

$$\sqrt{\lambda_1}\left|\rho_1 - \frac{\rho_1^*}{2}\right| = \sqrt{\lambda_2}\left|\rho_2 - \frac{\rho_2^*}{2}\right|. \tag{8}$$

These lines meet the opposite vertices of the rectangle $\mathcal{D}$ (Fig. 6).
Depending on the velocity ratio v_1^* *and* v_2^**, (7), we have either the upper branch* CDB*, or the right stationary branch* ADB *unstable, and the corresponding adjunct is stable.*

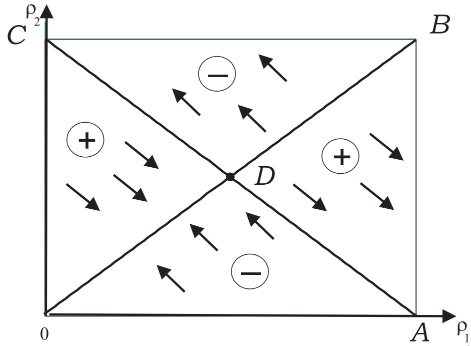

Fig. 6. Degeneration, $q_1^* = q_2^*$

3 Open Unidirectional Edge

Let us consider an edge with length l, which receives a flow of constant intensity q (Fig. 7). Due to the fact that as a rule an edge in the network has an

Fig. 7. Elementary open section

exit, the intensity of the output flow is fixed. Thus,

$$l\frac{d\rho}{dt} = \Theta\left(\rho_{\max} - \rho\right) - \min(q_{\max}, f(\rho)), \tag{9}$$

where $\Theta\left(\rho\right) = \{1; \rho > 0; 0; \rho \leq 0\}$, $f(\rho) = \lambda\rho(\rho^* - \rho)$, $0 \leq \rho \leq \rho^*$.
It is clear, that if $q > \min(q_{\max}, q^*)$, *than the flow is critical and*

$$T_*(\rho(0)) \leq \frac{l(\rho^* - \rho)}{q - \min(q_{\max}, q^*)}, \tag{10}$$

where $q^* = \frac{\lambda(\rho^*)^2}{4}$ *is highway capability of the edge.*
Then, let us assume that $q \leq \min(q_{\max}, q^*)$. At the beginning $q^* \leq q_{\max}$. Then $\min(q_{\max}, f(\rho)) = f(\rho)$, and equation (9) reads $l\frac{d\rho}{dt} = q - f(\rho)$.
It is obvious that the point $(\rho_1(q), q)$ (Fig. 8) is a stable stationary point, the point $(\rho_2(q), q)$ is an unstable stationary point and $\rho(0) \in (\rho_2(q), \rho^*)$ is the set of critical points. So

$$l\int\limits_{\rho(0)}^{\rho^*} \frac{d\rho}{q - f(\rho)} = \int\limits_{0}^{T_*(\rho(0))} dt = T_*\left(\rho(0)\right). \tag{11}$$

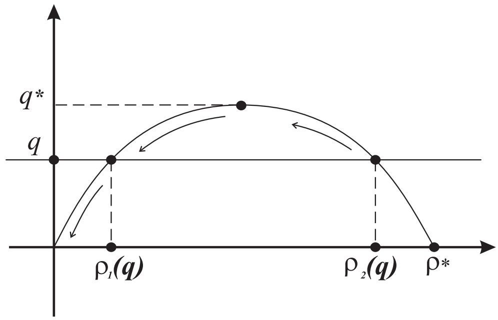

Fig. 8. $q \leq q^* \leq q_{\max}$

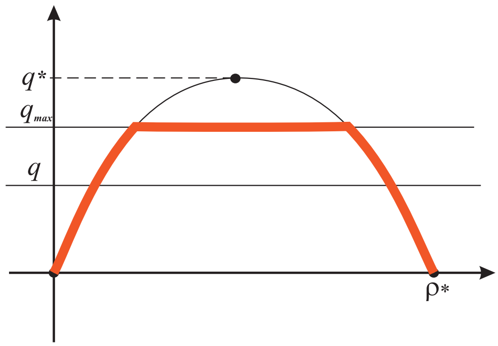

Fig. 9. $q \leq q_{\max} \leq q^*$

Let us consider the remaining case $q_{\max} < q^*$, i.e. $q < q_{\max} < q^*$. The main qualitative characteristics of the flow's behaviour are similar to that of the previous case, with the only difference that instead of $f(\rho)$ (Fig. 8), $\min\left(f(\rho), q_{\max}\right)$ is used (Fig. 9).

4 Problems

4.1 Ring of n Sections

Let us consider unidirectional movement on a ring consisting of n sections $(n > 2,)$ and l is the section length (Fig. 10).
The following statements are true:

- If $C > nC^*/2$, C^* is *capacity of a section, then the flow regime becomes critical in finite time for any initial conditions,* except for the case of stationary points.
- If $C < C^*$, *i.e. flow mass is less than a section capacity, then the movement is uncritical and converges to the stationary point.*
- However, if $C^* < C < nC^*/2$, *then qualitative characteristics of the flow depend on its initial conditions* $\bar{\rho}(0) = \left(\rho_1(0), \cdots, \rho_n(0)\right), l \sum_{i=1}^{n} \rho_i(0) = C$.

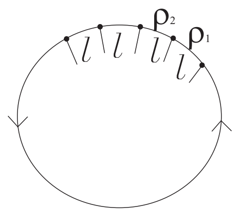

Fig. 10. Ring of n sections

For example, when $n = 2m$ and $\rho_{2i-1}(0) = \rho_1$, $\rho_{2i}(0) = \rho_2$, $i = 1, \cdots, m$ and the initial conditions are periodic, then the vector $\bar{\rho}(t)$ will be periodic due to its uniqueness at any point of time t.

Thus, *the flow on a ring with $2m$ sections will be equivalent to the flow on m rings of 2 sections.* Therefore at $l\,(\rho_1 + \rho_2) < C^*$ the movement is ∞-uncritical, $T_* = \infty$, but at $l\,(\rho_1 + \rho_2) > C^*$, except for stationary conditions, the movement is critical.

The following problems are to be explored: How to describe a flow in the common case n and at any initial conditions? What are sufficient conditions for $\bar{\rho}(0)$ at which the flow converts to the critical regime in a finite period of time?

4.2 Ring of 2 Sections with Control

Another generalization of a ring model of 2 sections is unidirectional movement on a ring of 2 sections with “traffic lights” (Fig. 11). In this model an alternation of 2 phases is considered. During the first phase, the movement of particles from one section to another is allowed, and during the second phase it is prohibited. The flow can be controlled by choosing the length of phases to prevent the transition to the critical regime.

What other stationary states can be generated and in what ways by using above described controls?

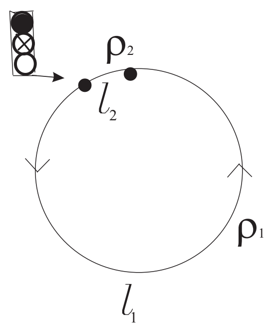

Fig. 11. Circular movement with control

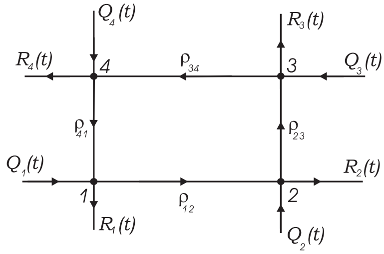

Fig. 12. Crossroads without mixing

4.3 Crossroads

Let us consider the model in Fig. 12. Assume that the edges and main diagrams of the graph are identical. The input flows have intensities $Q_i(t)$, $i = 1, 2, 3, 4$. In nodes 1-4 the Boolean control functions $U_i(t)$ are introduced $(i = 1, 2, 3, 4)$ with $U_i(t) = 0$, if horizontal movement is allowed, and $U_i(t) = 1$ if vertical movement is allowed (Fig. 12).
The problem is in the description of control methods, ensuring uncritical regimes and fixed intensities of output flows $R_i(t)$, $i = 1, \ldots, 4$.

References

1. A.P. Buslaev, A.V. Novikov, V.M. Prikhodko, A.G. Tatashev, M.V. Yashina: Stochastical and Imitation Approch to Traffic Movement. (Mir, Moscow 2003)
2. V.N. Lukanin, A.P. Buslaev, A.V. Novikov, M.V. Yashina: Traffic Flows Modelling and the Evaluation of Energy-Ecological Parameters. Part I. Int. J. of Vehicle Design (2001)
3. V.N. Lukanin , A.P. Buslaev, A.V. Novikov, M.V. Yashina: Traffic Flows Modelling and the Evaluation of Energy-Ecological Parameters. Part II. Int. J. of Vehicle Design (2001)
4. I. Lubashevski, R. Mahnke, P. Wagner, S. Kalenkov: Phys. Rev. E **66**, 016117 (2002).
5. I. Lubashevsky, P. Wagner, R. Mahnke: Eur. Phys. J. B **32**, 243–247 (2003)
6. Yu.V. Pokorny, E.N. Povorotova, O.M. Penkin: On spectre of some vector boundary problems. Problems of the qualitative theory differential equations. ed. by V.M. Matrosov (Nauka, Novosibirsk 1988)
7. S. Nicaise: Some results on spectral theory over networks, applied to nerve impulse transmission. Lecture Notes in Math. **1171**, pp. 532–541 (Springer 1985)
8. A.P. Buslaev, A.G. Tatashev, M.V. Yashina: On properties of a class of systems of non-linear differential equations on graphs. Vladikavkaz Math. J. **4** (2004)
9. A.P. Buslaev, V.M. Prikhodko, A.G. Tatashev, M.V. Yashina: Deterministic-stochastic flow model. (2005) ArXiv.org/0504139
10. A.P. Buslaev, A.G. Tatashev, M.V. Yashina: Traffic flow stochastic model 2*2 with discrete set of states and continuous time. (2004) ArXiv.org/0405471