APPROXIMATION BY NEURAL NETWORK OPERATORS OF CONVOLUTION TYPE ACTIVATED BY DEFORMED AND PARAMETRIZED HALF HYPERBOLIC TANGENT FUNCTION

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ABSTRACT. Here, we introduce three kinds of neural network operators of convolution type which are activated by q-deformed and β -parametrized half hyperbolic tangent function. We obtain quantitative convergence results to the identity operator with the use of modulus of continuity. Global smoothness preservation of our operators are also presented and the iterated versions of them are taken into the consideration.

1. Introduction and Motivation

The main idea behind neural networks and artificial neural networks is to act like human brain by understanding the biological nature and function of it. Neural and artificial neural networks lay in the centre of machine learning and the studies on artificial intelligence that can learn and solve problems go back to 1950's [29]. Recently, the power of computers, accessing the big data, studies on neural network, machine learning and deep learning gain speed and take noteworthy attention since they give us opportunity to handle complex, nonlinear relations in big data, by making them suitable for the duties such as image recognition, natural language processing, optimization, process control system, forecasting, approximation of functions and solutions of curve fitting [1], [2], [3], [16], [17], [18], [20], [21], [22], [23], [28].

Quantitative approximation of positive linear operators to the unit operator has been studied by G. A. Anastassiou since 1985 [4], [5], [6], [14]. By originating from the quantitative weak convergence of finite positive measures to the unit Dirac measure, having as a method the geometric moment theory [5], he has obtained best upper bounds and these studies have been considered from all possible perspectives, univariate and multivariate cases by many authors. In the present study, we introduce three kinds of convolution operators with the kernel depending on symmetrized, q-deformed and β -parametrized half hyperbolic tangent function. It is noteworthy to mention that this activation function is frequently used in neural networks and they can be interpreted as positive linear operators, thus we use the methods of positive linear operator theory in our proofs. We present quantitative

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convergence results to the identity operator by using modulus of continuity and global smoothness preservation of our operators are also obtained. Furthermore, iterated versions of these operators are taken into consideration.

The outline of the paper is as follows: In section 1, we explain the history of neural network theory and our motivation. Section 2 is devoted to preliminaries. In section 3, we present convergence results to the unit by our operators. In section 4, the iterated versions of these operators are considered. We find it is valuable to mention that general motivation comes from seminal works [15], [19], [24], [25], [26], [27], [30].

2. Preliminaries

Let us recall the following deformed and parametrized half hyperbolic tangent function and its basic properties as below [8]:

$$\nu_q(x) = \frac{1 - qe^{-\beta x}}{1 + qe^{-\beta x}}, \ x \in \mathbb{R}, \ q, \ \beta > 0,$$
 (2.1)

- $\begin{array}{l} \bullet \ \nu_q(+\infty) = 1, \ \nu_{q,\beta}(-\infty) = -1, \\ \bullet \ \nu_q(-x) = -\nu_{\frac{1}{q}}(x), \ for \ every \ x \in (-\infty, \infty), \end{array}$
- $\nu_q(0) = \frac{1-q}{1+q}$.

Then with the use of ν , the following activation function has been considered and its properties have been given in [8]:

$$G_{q,\beta}(x) = \frac{1}{4} \left(\nu_q(x+1) - \nu_q(x-1) \right), \ x \in (-\infty, \infty).$$
 (2.2)

One can easily notice that

$$G_{q,\beta}(-x) = \frac{1}{4} \left(\nu_{\frac{1}{q},\beta}(x+1) - \nu_{\frac{1}{q},\beta}(x-1) \right) = G_{\frac{1}{q},\beta}(x),$$

$$G_{q,\beta}(-x) = G_{\frac{1}{q},\beta}(x), \qquad (2.3)$$

which means that we have a deformed symmetry and the global maximum of $G_{q,\beta}(x)$ is

$$G_{q,\beta}\left(\frac{\ln q}{\beta}\right) = \frac{1 - e^{-\beta}}{2(1 + e^{-\beta})}.$$

It is known from [8] that $G_{q,\beta}$ is a density function on $(-\infty,\infty)$ since

$$\int_{-\infty}^{\infty} G_{q,\beta}(x)dx = 1.$$

By observing the following symmetry

$$\left(G_{q,\beta}+G_{\frac{1}{q},\beta}\right)(-x)=\left(G_{q,\beta}+G_{\frac{1}{q},\beta}\right)(x),$$

we introduce

$$\Psi = \frac{G_{q,\beta} + G_{\frac{1}{q},\beta}}{2}.$$

Since our aim is to get a symmetrized density function, this definition of Ψ serves to our aim, i.e.,

$$\Psi(x) = \Psi(-x), \ x \in (-\infty, \infty)$$

and

$$\int_{-\infty}^{\infty} \Psi(x) dx = 1,$$

implying

$$\int_{-\infty}^{\infty} \Psi(nx - v) dv = 1, \text{ for every } n \in \mathbb{N}, \ x \in (-\infty, \infty).$$
 (2.4)

3. Main Results

Here, we introduce the following convolution type operators activated by symmetrized, q-deformed and parametrized half hyperbolic tangent function. Our aim in this section is to examine their approximation properties and improve these results.

Now by considering the above new density function Ψ , we introduce the following neural network operators of convolution type for $f \in C_B(\mathbb{R})$ consisting of the functions which are continuous and bounded on \mathbb{R} and $n \in \mathbb{N}$:

$$B_n(f)(x) := \int_{-\infty}^{\infty} f\left(\frac{v}{n}\right) \Psi(nx - v) dv, \tag{3.1}$$

$$B_n^*(f)(x) := n \int_{-\infty}^{\infty} \left(\int_{\frac{v}{n}}^{\frac{v+1}{n}} f(h) dh \right) \Psi(nx - v) dv$$

$$= n \int_{-\infty}^{\infty} \left(\int_{0}^{\frac{1}{n}} f\left(h + \frac{v}{n}\right) dh \right) \Psi(nx - v) dv,$$
(3.2)

$$\overline{\mathbb{B}_n}(f)(x) := \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s f\left(\frac{v}{n} + \frac{s}{nr}\right) \right) \Psi(nx - v) dv, \tag{3.3}$$

where $w_s \ge 0$, $\sum_{s=1}^r w_s = 1$ and (3.2) is called activated Kantorovich type of (3.1),

(3.3) is called activated Quadrature type of (3.1).

Here is our first result.

Theorem 1.

$$\int_{\{v \in \mathbb{R}: |nx-v| > n^{1-\alpha}\}} \Psi(nx-v) dv < \frac{\left(q + \frac{1}{q}\right)}{e^{\beta(n^{1-\alpha}-1)}}$$
(3.4)

holds for $0 < \alpha < 1$, $n \in \mathbb{N}$ such that $n^{1-\alpha} > 2$.

Proof. Since

$$G_{q,\beta}(x) = \frac{1}{4} \left(\nu_{q,\beta}(x+1) - \nu_{q,\beta}(x-1) \right), \ x \in (-\infty, \infty), \tag{3.5}$$

first letting $x \in [1, \infty)$, and using Mean Value Theorem, we get

$$G_{q,\beta}(x) = \frac{1}{4}\nu'_{q,\beta}(\xi)2 = \frac{1}{2}\nu'_{q,\beta}(\xi) = \frac{q\beta e^{\beta\xi}}{(e^{\beta\xi} + q)^2}, \ 0 \le x - 1 < \xi < x + 1.$$
 (3.6)

Also, we obtain that

$$G_{q,\beta}(x) < \frac{q\beta}{e^{\beta\xi}} < \frac{q\beta}{e^{\beta(x-1)}}$$

and similarly,

$$G_{\frac{1}{q},\beta}(x) < \frac{\frac{1}{q}\beta}{e^{\beta\xi}} < \frac{\frac{1}{q}\beta}{e^{\beta(x-1)}}$$

which imply

$$\Psi < \frac{1}{2} \left(\frac{q\beta}{e^{\beta(x-1)}} + \frac{\frac{1}{q}\beta}{e^{\beta(x-1)}} \right)$$

$$\Psi < \frac{1}{2} \left(q + \frac{1}{q} \right) \beta e^{-\beta(x-1)}.$$

Let

$$F := \{ v \in \mathbb{R} : |nx - v| \ge n^{1 - \alpha} \}$$

and then we can write that

$$\begin{split} \int_F \Psi(nx-v) dv &= \int_F \Psi(|nx-v|) dv < \frac{1}{2} \left(q + \frac{1}{q}\right) \beta \int_F e^{-\beta(|nx-v|-1)} dv \\ &= 2 \cdot \frac{1}{2} \left(q + \frac{1}{q}\right) \beta \int_{n^{1-\alpha}}^\infty e^{-\beta(x-1)} dx \\ &= \left(q + \frac{1}{q}\right) \beta \int_{n^{1-\alpha}-1}^\infty e^{-\beta y} dy \\ &= \left(q + \frac{1}{q}\right) e^{-\beta y} \Big|_\infty^{n^{\alpha-1}-1} = \left(q + \frac{1}{q}\right) e^{-\beta(n^{1-\alpha}-1)} \\ &= \frac{\left(q + \frac{1}{q}\right)}{e^{\beta(n^{1-\alpha}-1)}}. \end{split}$$

Hence, we complete the proof.

Recall the modulus of continuity of a function f for $\theta > 0$ on $\mathbb R$ as follows:

$$\omega(f,\theta) := \sup_{\substack{x,y \in \mathbb{R} \\ |x-y| \le \theta}} |f(x) - f(y)|.$$

Now we are ready to present quantitative convergence results for our operators. Note that for Theorems 2, 3, 4, we let $0 < \alpha < 1$, $n \in \mathbb{N}$ such that $n^{1-\alpha} > 2$.

Theorem 2.

$$|\mathsf{B}_n(f)(x) - f(x)| \le \omega \left(f, \frac{1}{n^\alpha}\right) + \frac{2\left(q + \frac{1}{q}\right) \|f\|_\infty}{e^{\beta(n^{1-\alpha}-1)}} =: \mathsf{T}$$

and

$$\|\mathbf{B}_n(f) - f\|_{\infty} \leq \mathbf{T}$$

hold for $f \in C_B(\mathbb{R})$. Furthermore, we have that $\lim_{n\to\infty} B_n(f) = f$, pointwise and uniformly for $f \in C_{UB}(\mathbb{R})$, consisting of the functions which are uniformly continuous and bounded on \mathbb{R} .

Proof. Let $F_1 := \left\{ v \in \mathbb{R} : \left| \frac{v}{n} - x \right| < \frac{1}{n^{\alpha}} \right\}$, $F_2 := \left\{ v \in \mathbb{R} : \left| \frac{v}{n} - x \right| \ge \frac{1}{n^{\alpha}} \right\}$. Then $F_1 \cup F_2 = \mathbb{R}$ and we can write by using Theorem 1 that

$$\begin{aligned} |\mathbb{B}_{n}(f)(x) - f(x)| &= \left| \int_{-\infty}^{\infty} f\left(\frac{v}{n}\right) \Psi(nx - v) dv - f(x) \int_{-\infty}^{\infty} \Psi(nx - v) dv \right| \\ &\leq \int_{-\infty}^{\infty} \left| f\left(\frac{v}{n}\right) - f(x) \right| \Psi(nx - v) dv \\ &= \int_{F_{1}} \left| f\left(\frac{v}{n}\right) - f(x) \right| \Psi(nx - v) dv + \int_{F_{2}} \left| f\left(\frac{v}{n}\right) - f(x) \right| \Psi(nx - v) dv \\ &\leq \int_{F_{1}} \omega \left(f, \left| \frac{v}{n} - x \right| \right) \Psi(nx - v) dv + 2 ||f||_{\infty} \int_{F_{2}} \Psi(nx - v) dv \\ &\leq \omega \left(f, \frac{1}{n^{\alpha}} \right) + \frac{2 ||f||_{\infty} \left(q + \frac{1}{q} \right)}{e^{\beta(n^{1-\alpha} - 1)}} \end{aligned}$$

which gives the desired results.

Theorem 3.

$$|\mathrm{B}_n^*(f)(x) - f(x)| \leq \omega \left(f, \frac{1}{n} + \frac{1}{n^\alpha}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_\infty}{e^{\beta(n^{1-\alpha}-1)}} =: \mathrm{E}$$

and

$$\|\mathbf{B}_n^*(f) - f\|_{\infty} \leq \mathbf{E}$$

hold for $f \in C_B(\mathbb{R})$. Furthermore, we have that $\lim_{n \to \infty} B_n^*(f) = f$, pointwise and uniformly for $f \in C_{UB}(\mathbb{R})$.

Proof. Let F_1 and F_2 be defined as above. Then we obtain by using Theorem 1 that

$$\begin{split} |\mathbf{B}_{n}^{*}(f)(x) - f(x)| &= \left| n \int_{-\infty}^{\infty} \left(\int_{\frac{v}{n}}^{\frac{v+1}{n}} f\left(t\right) dt \right) \Psi(nx - v) dv - \int_{-\infty}^{\infty} f(x) \Psi(nx - v) dv \right| \\ &= \left| n \int_{-\infty}^{\infty} \left(\int_{\frac{v}{n}}^{\frac{v+1}{n}} f\left(t\right) dt \right) \Psi(nx - v) dv - n \int_{-\infty}^{\infty} \left(\int_{\frac{v}{n}}^{\frac{v+1}{n}} f(x) dt \right) \Psi(nx - v) dv \right| \\ &\leq n \int_{-\infty}^{\infty} \left(\int_{\frac{v}{n}}^{\frac{v+1}{n}} |f(t) - f(x)| dt \right) \Psi(nx - v) dv \\ &= n \int_{-\infty}^{\infty} \left(\int_{0}^{\frac{1}{n}} \left| f\left(t + \frac{v}{n}\right) - f(x) \right| dt \right) \Psi(nx - v) dv \\ &\leq \int_{F_{1}} \left(n \int_{0}^{\frac{1}{n}} \left| f\left(t + \frac{v}{n}\right) - f(x) \right| dt \right) \Psi(nx - v) dv \\ &+ \int_{F_{2}} \left(n \int_{0}^{\frac{1}{n}} \left| f\left(t + \frac{v}{n}\right) - f(x) \right| dt \right) \Psi(nx - v) dv \\ &\leq \int_{F_{1}} \left(n \int_{0}^{\frac{1}{n}} \omega \left(f, |t| + \frac{1}{n^{\alpha}} \right) dt \right) \Psi(nx - v) dv + 2 ||f||_{\infty} \int_{F_{2}} \Psi(nx - v) dv \\ &\leq \omega \left(f, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) ||f||_{\infty}}{e^{\beta(n^{1-\alpha} - 1)}} \end{split}$$

which gives the desired results.

Theorem 4.

$$\left|\overline{\mathrm{B}_n}(f)(x) - f(x)\right| \leq \omega \left(f, \frac{1}{n} + \frac{1}{n^\alpha}\right) + \frac{2\left(q + \frac{1}{q}\right) \|f\|_\infty}{e^{\beta(n^{1-\alpha}-1)}} = \mathrm{E}$$

and

$$\|\overline{\mathtt{B}_n}(f) - f\|_{\infty} \le \mathtt{E}$$

hold for $f \in C_B(\mathbb{R})$. Furthermore we have $\lim_{n \to \infty} \overline{B_n}(f) = f$, pointwise and uniformly for $f \in C_{UB}(\mathbb{R})$.

Proof. Again, we can write by using Theorem 1 that

$$\begin{aligned} |\overline{\mathbf{B}_n}(f)(x) - f(x)| &= \left| \int_{-\infty}^{\infty} \sum_{s=1}^r w_s f\left(\frac{v}{n} + \frac{s}{nr}\right) \Psi(nx - v) dv - \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s f(x)\right) \Psi(nx - v) dv \right| \\ &\leq \int_{-\infty}^{\infty} \sum_{s=1}^r w_s \left| f\left(\frac{v}{n} + \frac{s}{nr}\right) - f(x) \right| \Psi(nx - v) dv \\ &\leq \int_{F_1} \sum_{s=1}^r w_s \left| f\left(\frac{v}{n} + \frac{s}{nr}\right) - f(x) \right| \Psi(nx - v) dv \\ &+ \int_{F_2} \sum_{s=1}^r w_s \left| f\left(\frac{v}{n} + \frac{s}{nr}\right) - f(x) \right| \Psi(nx - v) dv \\ &\leq \omega \left(f, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) ||f||_{\infty}}{e^{\beta(n^{1-\alpha}-1)}} \end{aligned}$$

which gives the desired results.

Proposition 1.

$$\int_{-\infty}^{\infty} |h|^k \Psi(h) dh \le \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q}\right) \frac{e^{\beta}}{\beta^k} \Gamma(k+1) < \infty$$

holds for $k \in \mathbb{N}$.

Proof. We can easily compute that

$$\begin{split} \int_{-\infty}^{\infty} |h|^k \Psi(h) dh &= 2 \int_{0}^{\infty} h^k \Psi(h) dh \\ &= 2 \left(\int_{0}^{1} h^k \Psi(h) dh + \int_{1}^{\infty} h^k \Psi(h) dh \right) \\ &\leq 2 \left(\frac{1 - e^{-\beta}}{2(1 + e^{-\beta})} \int_{0}^{1} h^k dh + \int_{1}^{\infty} h^k \frac{1}{2} \left(q + \frac{1}{q} \right) \beta e^{-\beta(h-1)} dh \right) \\ &\leq 2 \left(\frac{1 - e^{-\beta}}{2(1 + e^{-\beta})} \frac{1}{k+1} + \frac{1}{2} \left(q + \frac{1}{q} \right) \beta \int_{1}^{\infty} h^k e^{-\beta(h-1)} dh \right) \\ &\leq 2 \left(\frac{1 - e^{-\beta}}{2(1 + e^{-\beta})} \frac{1}{k+1} + \frac{1}{2} \left(q + \frac{1}{q} \right) \frac{\beta e^{\beta}}{\beta^k} \int_{0}^{\infty} h^k \beta^k e^{-\beta h} \beta dh \right) \\ &\leq \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta}}{\beta^k} \Gamma(k+1) \\ &\leq \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^k} < \infty. \end{split}$$

Theorem 5.

$$\omega(\mathsf{B}_n(f),\theta) \le \omega(f,\theta), \ \theta > 0 \tag{3.7}$$

holds for $f \in C_B(\mathbb{R}) \cup C_U(\mathbb{R})$. Furthermore, we have $B_n(f) \in C_U(\mathbb{R})$ for $f \in C_U(\mathbb{R})$ where $C_U(\mathbb{R})$ is the set of all uniformly continuous functions on \mathbb{R} .

Proof. Since

$$\mathrm{B}_n(f)(x) = \int_{-\infty}^{\infty} f\left(x - \frac{h}{n}\right) \Psi(h) dh,$$

let $x, y \in \mathbb{R}$ and then we can write that

$$B_n(f)(x) - B_n(f)(y) = \int_{-\infty}^{\infty} \left(f\left(x - \frac{h}{n}\right) - f\left(y - \frac{h}{n}\right) \right) \Psi(h) dh.$$

Hence

$$\begin{aligned} |\mathbf{B}_n(f)(x) - \mathbf{B}_n(f)(y)| &\leq \int_{-\infty}^{\infty} \left| f\left(x - \frac{h}{n}\right) - f\left(y - \frac{h}{n}\right) \right| \Psi(h) dh \\ &\leq \omega(f, |x - y|) \int_{-\infty}^{\infty} \Psi(h) dh = \omega(f, |x - y|) \end{aligned}$$

holds and by letting $|x-y| \le \theta$, $\theta > 0$ then we obtain the desired results.

Remark 1. It is clear that the equality in (3.7) holds for f = identity map =: id, and we have that

$$|B_n(id)(x) - B_n(id)(y)| = |id(x) - id(y)| = |x - y|,$$

$$\omega(\mathsf{B}_n(id),\theta) = \omega(id,\theta) = \theta > 0.$$

Also, we have that

$$\begin{split} \mathbf{B}_n(id)(x) &= \int_{-\infty}^{\infty} \left(x - \frac{h}{n} \right) \Psi(h) dh \\ &= x \int_{-\infty}^{\infty} \Psi(h) dh - \frac{1}{n} \int_{-\infty}^{\infty} h \Psi(h) dh = x - \frac{1}{n} \int_{-\infty}^{\infty} h \Psi(h) dh \end{split}$$

and for fixed $x \in \mathbb{R}$,

$$|\mathbf{B}_{n}(id)(x)| \leq |x| + \frac{1}{n} \int_{-\infty}^{\infty} |h| \Psi(h) dh$$

$$\leq |x| + \frac{1}{n} \left[\frac{1 - e^{-\beta}}{2(1 + e^{-\beta})} + \frac{\left(q + \frac{1}{q}\right) e^{\beta}}{\beta} \right] < \infty.$$

It is obvious that $id \in C_U(\mathbb{R})$.

Theorem 6.

$$\omega(\mathsf{B}_n^*(f),\theta) \le \omega(f,\theta), \ \theta > 0 \tag{3.8}$$

holds for $f \in C_B(\mathbb{R}) \cup C_U(\mathbb{R})$. Furthermore, we have $B_n^*(f) \in C_U(\mathbb{R})$ for $f \in C_U(\mathbb{R})$.

Proof. Notice that

$$\mathrm{B}_n^*(f)(x) = n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} f\left(t + \left(x - \frac{h}{n}\right)\right) dt \right) \Psi(h) dh$$

and

$$\mathrm{B}_n^*(f)(y) = n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} f\left(t + \left(y - \frac{h}{n}\right)\right) dt \right) \Psi(h) dh$$

for $x, y \in \mathbb{R}$.

Hence $|B_n^*(f)(x) - B_n^*(f)(y)|$

$$\begin{split} & \leq n \int_{-\infty}^{\infty} \left(\int_{0}^{\frac{1}{n}} \left| f \left(t + \left(x - \frac{h}{n} \right) \right) - f \left(t + \left(y - \frac{h}{n} \right) \right) \right| dt \right) \Psi(h) dh \\ & \leq & \omega(f, |x - y|) \int_{-\infty}^{\infty} \Psi(h) dh = \omega(f, |x - y|) \end{split}$$

and again implies the desired results. Furthermore,

$$|B_n^*(id)(x) - B_n^*(id)(y)| = |x - y|$$

and

$$\begin{split} |\mathsf{B}_n^*(id)(x)| &\leq n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} \left(|t| + |x| + \frac{|h|}{n} \right) dt \right) \Psi(h) dh \\ &\leq \int_{-\infty}^{\infty} \left(\frac{1}{n} + |x| + \frac{|h|}{n} \right) \Psi(h) dh \\ &= \frac{1}{n} + |x| + \frac{1}{n} \int_{-\infty}^{\infty} |h| \Psi(h) dh < \infty. \end{split}$$

This proves the attainability of (3.8).

Theorem 7.

$$\omega(\overline{B_n}(f), \theta) \le \omega(f, \theta), \ \theta > 0$$
 (3.9)

holds for $f \in C_B(\mathbb{R}) \cup C_U(\mathbb{R})$. Furthermore, we have $\overline{B_n}(f) \in C_U(\mathbb{R})$ for $f \in C_U(\mathbb{R})$.

Proof. Notice that

$$\overline{B_n}(f)(x) = \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s f\left(\left(x - \frac{h}{n} \right) + \frac{s}{nr} \right) \right) \Psi(h) dh$$

and

$$\overline{B_n}(f)(y) = \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s f\left(\left(y - \frac{h}{n} \right) + \frac{s}{nr} \right) \right) \Psi(h) dh$$

for $x, y \in \mathbb{R}$.

Hence $|\overline{B_n}(f)(x) - \overline{B_n}(f)(y)|$

$$= \left| \int_{-\infty}^{\infty} \sum_{s=1}^{r} w_{s} \left[f\left(\left(x - \frac{h}{n}\right) + \frac{s}{nr}\right) - f\left(\left(y - \frac{h}{n}\right) + \frac{s}{nr}\right) \right] \Psi(h) dh \right|$$

$$\leq \int_{-\infty}^{\infty} \sum_{s=1}^{r} w_{s} \left| f\left(\left(x - \frac{h}{n}\right) + \frac{s}{nr}\right) - f\left(\left(y - \frac{h}{n}\right) + \frac{s}{nr}\right) \right| \Psi(h) dh$$

$$\leq \omega(f, |x - y|) \int_{-\infty}^{\infty} \Psi(h) dh = \omega(f, |x - y|)$$

and this implies the desired results. Furthermore,

$$|\overline{B_n}(id)(x) - \overline{B_n}(id)(y)| = |x - y|$$

and

$$|\overline{B}_{n}(id)(x)| \leq \int_{-\infty}^{\infty} \left(\sum_{s=1}^{r} w_{s} \left(|x| + \frac{|h|}{n} + \frac{1}{n} \right) \right) \Psi(h) dh$$

$$\leq \int_{-\infty}^{\infty} \left(|x| + \frac{|h|}{n} + \frac{1}{n} \right) \Psi(h) dh = \frac{1}{n} + |x| + \frac{1}{n} \int_{-\infty}^{\infty} |h| dh < \infty$$

proving the attainability of (3.9).

Remark 2. Fix $s \in \mathbb{N}$ and let $f \in C^{(s)}(\mathbb{R})$ such that $f^{(k)} \in C_B(\mathbb{R})$ for k = 1, 2, ..., s. Then we can write that

$$B_n(f)(x) = \int_{-\infty}^{\infty} f\left(x - \frac{h}{n}\right) \Psi(h) dh$$

and with the use of Leibniz's rule

$$\begin{split} \frac{\partial^{s} \mathbf{B}_{n}(x)}{\partial x^{s}} &= \int_{-\infty}^{\infty} f^{(s)} \left(x - \frac{h}{n} \right) \Psi(h) dh \\ &= \int_{-\infty}^{\infty} f^{(s)} \left(\frac{v}{n} \right) \Psi(nx - v) dv = \mathbf{B}_{n}(f^{(s)})(x), \ for \ all \ x \in \mathbb{R}. \end{split}$$

Clearly the followings also hold:

$$\frac{\partial^s \mathsf{B}_n^*(f)(x)}{\partial x^s} = \mathsf{B}_n^*(f^{(s)})(x); \ \frac{\partial^s \overline{\mathsf{B}_n}(f)(x)}{\partial x^s} = \overline{\mathsf{B}_n}(f^{(s)})(x).$$

Remark 3. Fix $s \in \mathbb{N}$ and let $f \in C^{(s)}(\mathbb{R})$ such that $f^{(k)} \in C_B(\mathbb{R})$ for k = 1, 2, ..., s. We derive that

$$(B_n(f))^{(k)}(x) = B_n(f^{(k)})(x),$$

$$(B_n^*(f))^{(k)}(x) = B_n^*(f^{(k)})(x),$$

$$(\overline{B_n}(f))^{(k)}(x) = \overline{B_n}(f^{(k)})(x), \text{ for all } x \in \mathbb{R} \text{ and } k = 1, 2, \dots, s.$$

Then we have the followings under the assumptions of $0 < \alpha < 1$ and $n \in \mathbb{N}$ such that $n^{1-\alpha} > 2$.

Theorem 8. Let $f^{(k)} \in C_B(\mathbb{R})$ for $k = 1, 2, \dots, s \in \mathbb{N}$. Then we have the followings:

(i)

$$\left| (\mathsf{B}_n(f))^{(k)}(x) - f^{(k)}(x) \right| \le \omega \left(f^{(k)}, \frac{1}{n^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f^{(k)}\|_{\infty}}{e^{\beta(n^{1-\alpha}-1)}} =: \mathsf{T}_k$$
and

$$\|(\mathbf{B}_n(f))^{(k)} - f^{(k)}\|_{\infty} \le \mathbf{T}_k,$$

(ii)

$$\left| (\mathsf{B}_{n}^{*}(f))^{(k)}(x) - f^{(k)}(x) \right| \leq \omega \left(f^{(k)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f^{(k)}\|_{\infty}}{e^{\beta(n^{1-\alpha}-1)}} =: \mathsf{E}_{k}$$
and
$$\| (\mathsf{B}_{n}^{*}(f))^{(k)} - f^{(k)} \|_{\infty} \leq \mathsf{E}_{k},$$

(iii)

$$\left| (\overline{\mathbf{B}_{n}}(f))^{(k)}(x) - f^{(k)}(x) \right| \leq \omega \left(f^{(k)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f^{(k)}\|_{\infty}}{e^{\beta(n^{1-\alpha}-1)}} = \mathbf{E}_{k}$$
and
$$\| (\overline{\mathbf{B}_{n}}(f))^{(k)} - f^{(k)}\|_{\infty} \leq \mathbf{E}_{k}.$$

Proof. Taking into consideration of Theorems 2, 3, 4, we immediately obtain the proof. \Box

Now, we present a similar result on the preservation of global smoothness.

Theorem 9. Let $f^{(k)} \in C_B(\mathbb{R}) \cup C_U(\mathbb{R})$, for $k = 0, 1, \dots, s \in \mathbb{N}$. Then we have the followings:

(i)
$$\omega\left(\left(\mathsf{B}_{n}(f)\right)^{(k)},\theta\right) \leq \omega\left(f^{(k)},\theta\right),\ \theta>0,$$
 also $\left(\mathsf{B}_{n}(f)\right)^{(k)} \in C_{U}(\mathbb{R})$ when $f^{(k)} \in C_{U}(\mathbb{R})$, (ii)
$$\omega\left(\left(\mathsf{B}_{n}^{*}(f)\right)^{(k)},\theta\right) \leq \omega\left(f^{(k)},\theta\right),\ \theta>0,$$
 also $\left(\mathsf{B}_{n}^{*}(f)\right)^{(k)} \in C_{U}(\mathbb{R})$ when $f^{(k)} \in C_{U}(\mathbb{R})$, (iii)
$$\omega\left(\left(\overline{\mathsf{B}_{n}}(f)\right)^{(k)},\theta\right) \leq \omega\left(f^{(k)},\theta\right),\ \theta>0,$$
 also $\left(\overline{\mathsf{B}_{n}}(f)\right)^{(k)} \in C_{U}(\mathbb{R})$ when $f^{(k)} \in C_{U}(\mathbb{R})$.

Proof. Taking into consideration of Theorems 5, 6, 7, we immediately obtain the proof. \Box

Now, we present results dealing with the improvement of the rate of convergence of our operators by assuming the differentiability of functions.

Theorem 10. If $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)} \in C_B(\mathbb{R})$, then the followings hold:

(i)

$$\left| (\mathsf{B}_{n}(f))(x) - f(x) - \sum_{k=1}^{N} \frac{f^{(k)}(x)}{k!} \left(\mathsf{A}_{n} \left((\cdot - x)^{k} \right) \right) (x) \right|$$

$$\leq \frac{\omega \left(f^{(N)}, \frac{1}{n^{\alpha}} \right)}{n^{\alpha N} N!} + \frac{2^{N+2} \|f^{(N)}\|_{\infty} e^{\beta} \left(q + \frac{1}{q} \right)}{n^{N} \beta^{N}} e^{\frac{-\beta n^{1-\alpha}}{2}} \to 0, \ as \ n \to \infty,$$

(ii) if
$$f^{(k)}(x) = 0$$
, $k = 1, 2, \dots, N$ then

$$|\mathsf{B}_n(f)(x) - f(x)| \le \frac{\omega\left(f^{(N)}, \frac{1}{n^{\alpha}}\right)}{n^{\alpha N} N!} + \frac{2^{N+2} ||f^{(N)}||_{\infty} e^{\beta}\left(q + \frac{1}{q}\right)}{n^N \beta^N} e^{\frac{-\beta n^{1-\alpha}}{2}},$$

(iii)

$$\begin{split} |\mathbf{B}_n(f)(x) - f(x)| &\leq \sum_{k=1}^N \frac{\left| f^{(k)}(x) \right|}{k!} \frac{1}{n^k} \left[\frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^k} \right] \\ &+ \frac{\omega \left(f^{(N)}, \frac{1}{n^{\alpha}} \right)}{n^{\alpha N} N!} + \frac{2^{N+2} \|f^{(N)}\|_{\infty} e^{\beta} \left(q + \frac{1}{q} \right)}{n^N \beta^N} e^{\frac{-\beta n^{1-\alpha}}{2}}, \end{split}$$

(iv)

$$\|\mathbf{B}_{n}(f) - f\|_{\infty} \leq \sum_{k=1}^{N} \frac{\|f^{(k)}\|_{\infty}}{k!} \frac{1}{n^{k}} \left[\frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q}\right) \frac{e^{\beta} k!}{\beta^{k}} \right] + \frac{\omega \left(f^{(N)}, \frac{1}{n^{\alpha}}\right)}{n^{\alpha N} N!} + \frac{2^{N+2} \|f^{(N)}\|_{\infty} B^{\beta} \left(q + \frac{1}{q}\right)}{n^{N} \beta^{N} (\ln B)^{N}} B^{\frac{-\beta n^{1-\alpha}}{2}}.$$

Proof. Since it is already known that

$$f\left(\frac{v}{n}\right) = \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!} \left(\frac{v}{n} - x\right)^{k} + \int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt$$

and

$$f\left(\frac{v}{n}\right)\Psi(nx-v) = \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!} \Psi(nx-v) \left(\frac{v}{n} - x\right)^{k} + \Psi(nx-v) \int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt,$$

we can write that

$$\begin{split} \mathbf{B}_n(f)(x) &= \int_{-\infty}^{\infty} f\left(\frac{v}{n}\right) \Psi(nx - v) dv \\ &= \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!} \int_{-\infty}^{\infty} \Psi(nx - v) \left(\frac{v}{n} - x\right)^k dv \\ &+ \int_{-\infty}^{\infty} \Psi(nx - v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt\right) dv \end{split}$$

and

$$B_n(f)(x) - f(x) = \sum_{k=1}^{N} \frac{f^{(k)}(x)}{k!} \int_{-\infty}^{\infty} \Psi(nx - v) \left(\frac{v}{n} - x\right)^k dv + \int_{-\infty}^{\infty} \Psi(nx - v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt\right) dv.$$

Let

$$R_n(x) := \int_{-\infty}^{\infty} \Psi(nx - v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt \right) dv$$

and

$$\gamma(v) := \int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt.$$

Now, we have two cases:

1.
$$\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}$$

$$2. \left| \frac{v}{n} - x \right| \ge \frac{1}{n^{\alpha}}$$

Let us start with Case 1.

Case 1 (i): When $\frac{v}{n} \ge x$ we have

$$\begin{split} |\gamma(v)| &\leq \omega \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \left(\int_{x}^{\frac{v}{n}} \frac{\left(\frac{v}{n} - t \right)^{N-1}}{(N-1)!} dt \right) \\ &= \omega \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{v}{n} - x \right)^{N}}{N!} \leq \omega \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!}. \end{split}$$

Case 1 (ii): When $\frac{v}{n} < x$, we have

$$|\gamma(v)| \leq \omega\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{\left(x - \frac{v}{n}\right)^{N}}{N!} \leq \omega\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!}.$$

Therefore $|\gamma(v)| \leq \omega\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!}$ holds whenever $\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}$. Then we have that

$$\begin{split} \left| \int_{\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}} \Psi(nx - v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt \right) dv \right| \\ & \leq \int_{\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}} \Psi(nx - v) |\gamma(v)| dv \leq \omega \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!}. \end{split}$$

Case 2: In this case, we can write that

$$\begin{split} &\left| \int_{\left|\frac{v}{n}-x\right| \geq \frac{1}{n^{\alpha}}} \Psi(nx-v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n}-t\right)^{N-1}}{(N-1)!} dt \right) dv \right| \\ &\leq \int_{\left|\frac{v}{n}-x\right| \geq \frac{1}{n^{\alpha}}} \Psi(nx-v) \left| \int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n}-t\right)^{N-1}}{(N-1)!} dt \right| dv \\ &= \int_{\left|\frac{v}{n}-x\right| \geq \frac{1}{n^{\alpha}}} \Psi(nx-v) |\gamma(v)| := \mathtt{P}. \end{split}$$

If $\frac{v}{n} \ge x$ then $|\gamma(v)| \le 2 \|f^{(N)}\|_{\infty} \frac{\left(\frac{v}{n} - x\right)^N}{N!}$ and if $\frac{v}{n} < x$ then $|\gamma(v)| \le 2 \|f^{(N)}\|_{\infty} \frac{\left(x - \frac{v}{n}\right)^N}{N!}$. So by combining them, we can also write $|\gamma(v)| \le 2 \|f^{(N)}\|_{\infty} \frac{\left|\frac{v}{n} - x\right|^N}{N!}$ and by using Theorem 1 that

$$\begin{split} \mathbf{P} & \leq \frac{2 \|f^{(N)}\|_{\infty}}{N!} \int_{|\frac{v}{n} - x| \geq \frac{1}{n^{\alpha}}} \Psi(nx - v) \left| x - \frac{v}{n} \right|^{N} dv \\ & = \frac{2 \|f^{(N)}\|_{\infty}}{n^{N} N!} \int_{|nx - v| \geq n^{1 - \alpha}} \Psi(|nx - v|) \left| nx - v \right|^{N} dv \\ & \leq \frac{2 \|f^{(N)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) \beta \int_{n^{1 - \alpha}}^{\infty} e^{-\beta(x - 1)} x^{N} dx \\ & = \frac{2 \|f^{(N)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) \frac{e^{\beta}}{\beta^{N}} \int_{n^{1 - \alpha}}^{\infty} e^{-\beta x} (\beta x)^{N} (\beta dx) \\ & = \frac{2 \|f^{(N)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) \frac{e^{\beta}}{\beta^{N}} \int_{\beta n^{1 - \alpha}}^{\infty} e^{-t} t^{N} dt \\ & \leq \frac{2 \|f^{(N)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) \frac{e^{\beta} 2^{N} N!}{\beta^{N}} \int_{\beta n^{1 - \alpha}}^{\infty} e^{-t} e^{\frac{t}{2}} dt \\ & = \frac{2^{N+1} \|f^{(N)}\|_{\infty} \left(q + \frac{1}{q} \right)}{n^{N}} \frac{e^{\beta}}{\beta^{N}} \int_{\beta n^{1 - \alpha}}^{\infty} e^{-\frac{t}{2}} dt \\ & = \frac{2^{N+1} \|f^{(N)}\|_{\infty} \left(q + \frac{1}{q} \right)}{n^{N}} \frac{e^{\beta}}{\beta^{N}} (-2) e^{-\frac{t}{2}} \Big|_{\beta n^{1 - \alpha}}^{\infty} \\ & = \frac{2^{N+2} \|f^{(N)}\|_{\infty} e^{\beta} \left(q + \frac{1}{q} \right)}{n^{N} \beta^{N}} e^{-\frac{\beta n^{1 - \alpha}}{2}}. \end{split}$$

Then we get that

$$\left| \int_{\left|\frac{v}{n} - x\right| \ge \frac{1}{n^{\alpha}}} \Psi(nx - v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt \right) dv \right|$$

$$\le \frac{2^{N+2} \|f^{(N)}\|_{\infty} e^{\beta} \left(q + \frac{1}{q} \right)}{n^{N} \beta^{N}} e^{\frac{-\beta n^{1-\alpha}}{2}} \to 0, \ n \to \infty.$$

Finally we conclude that,

$$|R_{n}(x)| \leq \left| \int_{\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}} \Psi(nx - v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt \right) dv \right|$$

$$+ \left| \int_{\left|\frac{v}{n} - x\right| \ge \frac{1}{n^{\alpha}}} \Psi(nx - v) \left(\int_{x}^{\frac{v}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} - t\right)^{N-1}}{(N-1)!} dt \right) dv \right|$$

$$\leq \omega \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{2^{N+2} ||f^{(N)}||_{\infty} e^{\beta} \left(q + \frac{1}{q} \right)}{n^{N} \beta^{N}} e^{\frac{-\beta n^{1-\alpha}}{2}} \to 0, \ n \to \infty.$$

Now letting $k = 1, 2, \dots, N$ and h = nx - v then we obtain that

$$\begin{aligned} \left| \mathbb{B}_{n} \left((\cdot - x)^{k} \right) (x) \right| &= \left| \int_{-\infty}^{\infty} \Psi(nx - v) \left(\frac{v}{n} - x \right)^{k} dv \right| \\ &\leq \int_{-\infty}^{\infty} \left| \frac{v}{n} - x \right|^{k} \Psi(nx - v) dv \\ &= \frac{1}{n^{k}} \int_{-\infty}^{\infty} \left| nx - v \right|^{k} \Psi(nx - v) dv = \frac{1}{n^{k}} \int_{-\infty}^{\infty} \left| h \right|^{k} \Psi(h) dh \\ &\leq \frac{1}{n^{k}} \left[\frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^{k}} \right] \to 0, \ as \ n \to \infty. \end{aligned}$$

Hence we complete the proof.

Theorem 11. If $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$ such that $f^{(N)} \in C_B(\mathbb{R})$, then the followings hold:

$$\begin{split} \left| \mathsf{B}_{n}^{*}(f)(x) - f(x) - \sum_{k=1}^{N} \frac{f^{(k)}(x)}{k!} \left(\mathsf{A}_{n}^{*} \left((\cdot - x)^{k} \right) \right) (x) \right| \\ & \leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!} \\ & + \frac{2^{N} \left\| f^{(N)} \right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right] \to 0, \ n \to \infty, \end{split}$$

$$(ii) \ if \ f^{(k)}(x) = 0, \ k = 1, 2, \cdots, N \ then \\ \left| \mathsf{B}_{n}^{*}(f)(x) - f(x) \right| \leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!} \\ & + \frac{2^{N} \left\| f^{(N)} \right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right], \end{split}$$

$$\begin{split} |\mathbf{B}_{n}^{*}(f)(x) - f(x)| &\leq \sum_{k=1}^{N} \frac{\left|f^{(k)}(x)\right|}{k!} \frac{2^{k-1}}{n^{k}} \left[1 + \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q}\right) \frac{e^{\beta}k!}{\beta^{k}} \right] \\ &+ \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}}\right)^{N}}{N!} \\ &+ \frac{2^{N} \left\| f^{(N)} \right\|_{\infty}}{n^{N}N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1}N!}{\beta^{N}} \right], \end{split}$$

(iv)

$$\begin{split} \|\mathbf{B}_{n}^{*}(f) - f\|_{\infty} &\leq \sum_{k=1}^{N} \frac{\|f^{(k)}\|_{\infty}}{k!} \frac{2^{k-1}}{n^{k}} \left[1 + \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^{k}} \right] \\ &+ \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!} \\ &+ \frac{2^{N} \|f^{(N)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right]. \end{split}$$

Proof. Since it is already known that

$$f\left(t + \frac{v}{n}\right) = \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!} \left(t + \frac{v}{n} - x\right)^{k} + \int_{x}^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds$$

and

$$\int_0^{\frac{1}{n}} f\left(t + \frac{v}{n}\right) dt = \sum_{k=0}^N \frac{f^{(k)}(x)}{k!} \int_0^{\frac{1}{n}} \left(t + \frac{v}{n} - x\right)^k dt + \int_0^{\frac{1}{n}} \left(\int_x^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds\right) dt,$$

we can write that

$$\begin{split} n & \int_{-\infty}^{\infty} \left(\int_{0}^{\frac{1}{n}} f\left(t + \frac{v}{n}\right) dt \right) \Psi(nx - v) dv \\ & = \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!} n \int_{-\infty}^{\infty} \left(\int_{0}^{\frac{1}{n}} f\left(t + \frac{v}{n} - x\right)^{k} dt \right) \Psi(nx - v) dv \\ & + n \int_{-\infty}^{\infty} \left(\int_{0}^{\frac{1}{n}} \left(\int_{x}^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \right) \Psi(nx - v) dv \end{split}$$

and

$$\mathsf{B}_n^*(f)(x) - f(x) = \sum_{k=1}^N \frac{f^{(k)}(x)}{k!} \mathsf{A}_n^* \left((\cdot - x)^k \right)(x) + R_n(x).$$

Here, let

$$R_n(x) := n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} \left(\int_x^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \right) \Psi(nx - v) dv$$

and

$$\gamma(v) := n \int_0^{\frac{1}{n}} \left(\int_x^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt.$$

Now we have two cases:

1.
$$\left| \frac{v}{n} - x \right| < \frac{1}{n^{\alpha}},$$

2. $\left| \frac{v}{n} - x \right| \ge \frac{1}{n^{\alpha}}.$

Let us start with Case 1.

Case 1 (i): When $t + \frac{v}{n} \ge x$, we have

$$\begin{split} |\gamma(v)| &\leq n \int_0^{\frac{1}{n}} \left(\int_x^{t+\frac{v}{n}} \left| f^{(N)}(s) - f^{(N)}(x) \right| \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \\ &\leq n \int_0^{\frac{1}{n}} \omega \left(f^{(N)}, |t| + \left| \frac{v}{n} - x \right| \right) \left(\int_x^{t + \frac{v}{n}} \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \\ &\leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) n \int_0^{\frac{1}{n}} \frac{\left(|t| + \left| \frac{v}{n} - x \right| \right)^N}{N!} dt \\ &\leq \frac{\omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right)}{N!} \left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^N. \end{split}$$

Case 1 (ii): When $t + \frac{v}{n} < x$, we have

$$\begin{aligned} |\gamma(v)| &= n \left| \int_0^{\frac{1}{n}} \left(\int_x^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(\left(t + \frac{v}{n} \right) - s \right)^{N-1}}{(N-1)!} ds \right) dt \right| \\ &\leq n \int_0^{\frac{1}{n}} \left(\int_{t + \frac{v}{n}}^x \left| f^{(N)}(s) - f^{(N)}(x) \right| \frac{\left(s - \left(t + \frac{v}{n} \right) \right)^{N-1}}{(N-1)!} ds \right) dt \\ &\leq n \int_0^{\frac{1}{n}} \omega \left(f^{(N)}, |t| + \left| \frac{v}{n} - x \right| \right) \left(\int_{t + \frac{v}{n}}^x \frac{\left(s - \left(t + \frac{v}{n} \right) \right)^{N-1}}{(N-1)!} ds \right) dt \\ &\leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) n \int_0^{\frac{1}{n}} \frac{\left(x - \left(t + \frac{v}{n} \right) \right)^N}{N!} dt \\ &\leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) n \int_0^{\frac{1}{n}} \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^N}{N!} dt \\ &= \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^N}{N!}. \end{aligned}$$

By considering the above inequalities, we find that

$$|\gamma(v)| \le \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}}\right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}}\right)^N}{N!}.$$

Then we conclude that

$$\left| n \int_{\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}}^{\infty} \left(\int_{0}^{\frac{1}{n}} \left(\int_{x}^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \right) \Psi(nx - v) dv \right|$$

$$\leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}}\right)^{N}}{N!}.$$

Case 2: In this case, we can write that

$$\left| \int_{\left| \frac{v}{n} - x \right| \ge \frac{1}{n^{\alpha}}} n \left(\int_{0}^{\frac{1}{n}} \left(\int_{x}^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \right) \Psi(nx - v) dv \right|$$

$$\leq \int_{\left|\frac{v}{n} - x\right| \geq \frac{1}{n^{\alpha}}} \left| n \left(\int_{0}^{\frac{1}{n}} \left(\int_{x}^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \right) \right| \Psi(nx - v) dv =: \Gamma(x) + \frac{1}{n^{\alpha}} \left(\int_{0}^{\frac{1}{n}} \left(\int_{x}^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt \right) dt$$

and

$$|\gamma(v)| \le n \int_0^{\frac{1}{n}} \left| \int_x^{t+\frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N-1}}{(N-1)!} ds \right| dt.$$

Now, if $t + \frac{v}{n} \ge x$, then

$$|\gamma(v)| \le 2 \left\| f^{(N)} \right\|_{\infty} n \int_0^{\frac{1}{n}} \frac{\left(t + \frac{v}{n} - x\right)^N}{N!} dt$$

$$\le 2 \left\| f^{(N)} \right\|_{\infty} n \int_0^{\frac{1}{n}} \frac{\left(|t| + \left|\frac{v}{n} - x\right|\right)^N}{N!} dt$$

$$\le 2 \frac{\left\| f^{(N)} \right\|_{\infty}}{N!} \left(\frac{1}{n} + \left|\frac{v}{n} - x\right|\right)^N,$$

and if $t + \frac{v}{n} < x$ then $|\gamma(v)| \le 2 \|f^{(N)}\|_{\infty} \frac{\left(\frac{1}{n} + \left|\frac{v}{n} - x\right|\right)^N}{N!}$. By considering the above inequalities we conclude that

$$|\gamma(v)| \le 2 \frac{\left\|f^{(N)}\right\|_{\infty}}{N!} \left(\frac{1}{n} + \left|\frac{v}{n} - x\right|\right)^{N}$$

and by using Theorem 1

$$\begin{split} &\Gamma \leq \left(\int_{\left|\frac{v}{n} - x\right| \geq \frac{1}{n^{\alpha}}} \left(\frac{1}{n} + \left|\frac{v}{n} - x\right| \right)^{N} \Psi(nx - v) dv \right) \frac{2 \left\|f^{(N)}\right\|_{\infty}}{N!} \\ &\leq \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{N!} \int_{\left|nx - v\right| \geq n^{1-\alpha}} \left(\frac{1}{n^{N}} + \frac{\left|nx - v\right|^{N}}{n^{N}} \right) \Psi(nx - v) dv \\ &\leq \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \int_{\left|nx - v\right| \geq n^{1-\alpha}} \left(1 + \left|nx - v\right|^{N} \right) \Psi(\left|nx - v\right|) dv \\ &\leq \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \int_{F} \frac{1}{2} \left(q + \frac{1}{q} \right) \beta e^{-\beta(\left|nx - v\right| - 1)} (1 + \left|nx - v\right|)^{N} dv, \ F = \{v \in \mathbb{R} : \left|nx - v\right| \geq n^{1-\alpha} \} \\ &\leq \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \frac{1}{2} \left(q + \frac{1}{q} \right) \beta \int_{F} e^{-\beta(\left|nx - v\right| - 1)} (1 + \left|nx - v\right|)^{N} dv \\ &= \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \frac{1}{2} \left(q + \frac{1}{q} \right) \beta \int_{n^{1-\alpha}}^{\infty} e^{-\beta(x - 1)} \left(1 + x^{N} \right) dx \\ &= \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} \beta \left[\int_{n^{1-\alpha}}^{\infty} e^{-\beta x} dx + \int_{n^{1-\alpha}}^{\infty} e^{-\beta x} x^{N} dx \right] \\ &= \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} \beta \left[\frac{e^{-\beta(n^{1-\alpha})}}{\beta} \right]_{n^{1-\alpha}}^{\infty} + \int_{n^{1-\alpha}}^{\infty} e^{-\beta x} x^{N} dx \\ &= \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} \beta \left[\frac{e^{-\beta(n^{1-\alpha})}}{\beta} + \int_{n^{1-\alpha}}^{\infty} e^{-\beta x} x^{N} dx \right] \\ &= \frac{2^{N} \left\|f^{(N)}\right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} \beta \left[\frac{e^{-\beta(n^{1-\alpha})}}{\beta} + \int_{n^{1-\alpha}}^{\infty} e^{-\beta x} x^{N} dx \right] = \mathbb{M} \end{split}$$

and

$$\begin{split} \int_{n^{1-\alpha}}^{\infty} e^{-\beta x} x^N dx &= \frac{1}{\beta^{N+1}} \int_{n^{1-\alpha}}^{\infty} e^{-\beta x} (\beta x)^N (\beta dx) \\ &= \frac{1}{\beta^{N+1}} \int_{\beta n^{1-\alpha}}^{\infty} e^{-t} t^N dt \\ &\leq \frac{1}{\beta^{N+1}} \int_{\beta n^{1-\alpha}}^{\infty} e^{-t} e^{\frac{t}{2}} 2^N N! dt \\ &= \frac{2^N N!}{\beta^{N+1}} \int_{\beta n^{1-\alpha}}^{\infty} e^{-\frac{t}{2}} dt \\ &= \frac{-2^{N+1} N!}{\beta^{N+1}} e^{-\frac{t}{2}} \Big|_{\beta n^{1-\alpha}}^{\infty} \\ &= \frac{2^{N+1} N!}{\beta^{N+1}} e^{-\frac{\beta n^{1-\alpha}}{2}}. \end{split}$$

Then we obtain

$$\mathtt{M} \leq \frac{2^N \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) e^{\beta} \beta \left\lceil \frac{e^{-\beta (n^{1-\alpha})}}{\beta} + \frac{2^{N+1} N!}{\beta^{N+1}} e^{-\frac{\beta n^{1-\alpha}}{2}} \right\rceil,$$

and since

$$\beta n^{1-\alpha} > \frac{\beta n^{1-\alpha}}{2} \Rightarrow -\beta n^{1-\alpha} < \frac{-\beta n^{1-\alpha}}{2} \Rightarrow e^{-\beta(n^{1-\alpha})} < e^{-\frac{\beta n^{1-\alpha}}{2}}$$

we can easily write that

$$\begin{split} \mathbf{M} & \leq \frac{2^N \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) e^{\beta} \beta \left[\frac{e^{-\beta (n^{1-\alpha})}}{\beta} + \frac{2^{N+1} N!}{\beta^{N+1}} e^{-\frac{\beta n^{1-\alpha}}{2}} \right] \\ & = \frac{2^N \left\| f^{(N)} \right\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^N} \right]. \end{split}$$

Hence, we find

$$\left| \int_{\left|\frac{v}{n} - x\right| \ge \frac{1}{n^{\alpha}}} n \left(\int_{0}^{\frac{1}{n}} \left(\int_{x}^{t + \frac{v}{n}} \left(f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{v}{n} - s\right)^{N - 1}}{(N - 1)!} ds \right) dt \right) \Psi(nx - v) dv \right|$$

$$\leq \frac{2^{N} \left\| f^{(N)} \right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1 - \alpha}}{2}} \left[1 + \frac{2^{N + 1} N!}{\beta^{N}} \right] \to 0, \ n \to \infty.$$

Also

$$|R_n(x)| \le \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^N}{N!} + \frac{2^N \|f^{(N)}\|_{\infty}}{n^N N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^N} \right] \to 0, \ n \to \infty.$$

So we notice for $k = 1, 2, \dots, N$ that

$$\begin{split} \left| \mathbf{A}_n^* \left((\cdot - x)^k \right) \right| &= \left| n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} \left(t + \frac{v}{n} - x \right)^k dt \right) \Psi(nx - v) dv \right| \\ &\leq n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} \left| t + \frac{v}{n} - x \right|^k dt \right) \Psi(nx - v) dv \\ &\leq n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} \left(\left| t \right| + \left| \frac{v}{n} - x \right| \right)^k dt \right) \Psi(nx - v) dv \\ &\leq \int_{-\infty}^{\infty} \left(\frac{1}{n} + \left| \frac{v}{n} - x \right| \right)^k \Psi(nx - v) dv \\ &= \frac{1}{n^k} \left[\int_{-\infty}^{\infty} \left(1 + \left| nx - v \right| \right)^k \Psi(nx - v) dv \right] \\ &\leq \frac{2^{k-1}}{n^k} \left[1 + \int_{-\infty}^{\infty} \left| nx - v \right|^k \Psi(nx - v) dv \right] \\ &\leq \frac{2^{k-1}}{n^k} \left[1 + \int_{-\infty}^{\infty} \left| h \right|^k \Psi(h) dh \right] \\ &\leq \frac{2^{k-1}}{n^k} \left[1 + \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^k} \right] \to 0, \ as \ n \to \infty \end{split}$$

and we complete the proof.

Our next result deals with activated Quadrature operators.

Theorem 12. If $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$, $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$ with $f^{(N)} \in C_B(\mathbb{R})$. Then the followings hold:

(i)

$$\left| \overline{\mathbf{B}_n}(f)(x) - f(x) - \sum_{k=1}^N \frac{f^{(k)}(x)}{k!} \left(\overline{\mathbf{A}_n} \left((\cdot - x)^k \right) \right) (x) \right|$$

$$\leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!} + \frac{2^{N} \| f^{(N)} \|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right] \to 0, \ as \ n \to \infty,$$

(ii) if
$$f^{(k)}(x) = 0$$
, $k = 1, 2, \dots, N$ then

$$|\overline{B}_{n}(f)(x) - f(x)| \leq \omega \left(f, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!} + \frac{2^{N} \|f^{(N)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right],$$

(iii)

$$|\overline{B}_{n}(f)(x) - f(x)| \leq \sum_{k=1}^{N} \frac{|f^{(k)}(x)|}{k!} \frac{2^{k-1}}{n^{k}} \left[1 + \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^{k}} \right]$$

$$+ \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!}$$

$$+ \frac{2^{N} ||f^{(N)}||_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right],$$

(iv)

$$\begin{split} \|\overline{\mathbf{B}_{n}}(f) - f\|_{\infty} &\leq \sum_{k=1}^{N} \frac{\|f^{(k)}\|_{\infty}}{k!} \frac{2^{k-1}}{n^{k}} \left[1 + \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^{k}} \right] \\ &+ \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!} \\ &+ \frac{2^{N} \|f^{(N)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right]. \end{split}$$

Proof. Since it is already known that

$$f\left(\frac{v}{n} + \frac{s}{nr}\right) = \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!} \left(\frac{v}{n} + \frac{s}{nr} - x\right)^{k} + \int_{x}^{\frac{v}{n} + \frac{s}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - t\right)^{N-1}}{(N-1)!} dt$$

and

$$\sum_{s=1}^{r} w_s f\left(\frac{v}{n} + \frac{s}{nr}\right) = \sum_{k=0}^{N} \frac{f^{(k)}(x)}{k!} \sum_{s=1}^{r} w_s \left(\frac{v}{n} + \frac{s}{nr} - x\right)^k + \sum_{s=1}^{r} w_s \int_{x}^{\frac{v}{n} + \frac{s}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - t\right)^{N-1}}{(N-1)!} dt,$$

we can write that

$$\begin{split} \overline{\mathbf{B}_n}(f)(x) &= \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s f\left(\frac{v}{n} + \frac{s}{nr}\right) \right) \Psi(nx - v) dv \\ &= \sum_{k=0}^N \frac{f^{(k)}(x)}{k!} \left(\int_{-\infty}^{\infty} \sum_{s=1}^r w_s \left(\frac{v}{n} + \frac{s}{nr} - x\right)^k \right) \Psi(nx - v) dv \\ &+ \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s \int_x^{\frac{v}{n} + \frac{s}{nr}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - t\right)^{N-1}}{(N-1)!} dt \right) \Psi(nx - v) dv, \end{split}$$

and

$$\overline{\mathbf{B}_n}(f)(x) - f(x) = \sum_{k=1}^N \frac{f^{(k)}(x)}{k!} \left(\overline{\mathbf{A}_n} \left((\cdot - x)^k \right) (x) \right) + R_n(x),$$

where

$$R_n(x) := \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s \int_x^{\frac{v}{n} + \frac{s}{nr}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - t\right)^{N-1}}{(N-1)!} dt \right) \Psi(nx - v) dv$$

and calling

$$\gamma(v) := \sum_{s=1}^{r} w_s \int_{x}^{\frac{v}{n} + \frac{s}{nr}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - t\right)^{N-1}}{(N-1)!} dt.$$

Now we have two cases:

1.
$$\left| \frac{v}{n} - x \right| < \frac{1}{n^{\alpha}},$$

2. $\left| \frac{v}{n} - x \right| \ge \frac{1}{n^{\alpha}}.$

Let us start with Case 1.

Case 1 (i): When $\frac{v}{n} + \frac{s}{nr} \ge x$, we have

$$|\gamma(v)| \le \sum_{s=1}^{r} w_s \omega \left(f^{(N)}, \left| \frac{v}{n} + \frac{s}{nr} - x \right| \right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - x \right)^N}{N!}$$
$$\le \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^N}{N!}.$$

Case 1 (ii): When $\frac{v}{n} + \frac{s}{nr} < x$, we have

$$|\gamma(v)| = \left| \sum_{s=1}^{r} w_{s} \int_{\frac{v}{n} + \frac{s}{nr}}^{x} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \left(\frac{v}{n} + \frac{s}{nr} \right) \right)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \sum_{s=1}^{r} w_{s} \int_{\frac{v}{n} + \frac{s}{nr}}^{x} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left(t - \left(\frac{v}{n} + \frac{s}{nr} \right) \right)^{N-1}}{(N-1)!} dt$$

$$\leq \sum_{s=1}^{r} w_{s} \omega \left(f^{(N)}, \left| x - \left(\frac{v}{n} + \frac{s}{nr} \right) \right| \right) \frac{\left(x - \left(\frac{v}{n} + \frac{s}{nr} \right) \right)^{N}}{N!}$$

$$\leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!}.$$

By considering the above inequalities, we find that

$$|\gamma(v)| \le \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}}\right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}}\right)^N}{N!}$$

and

$$\left| \int_{\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}} \left(\sum_{s=1}^{r} w_{s} \int_{x}^{\frac{v}{n} + \frac{s}{nr}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - t\right)^{N-1}}{(N-1)!} dt \right) \Psi(nx - v) dv \right|$$

$$\leq \int_{\left|\frac{v}{n} - x\right| < \frac{1}{n^{\alpha}}} \Psi(nx - v) |\gamma(v)| dv \leq \omega \left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}}\right)^{N}}{N!}.$$

Case 2: In this case, we can write that

$$\left| \int_{\left|\frac{v}{n} - x\right| \ge \frac{1}{n^{\alpha}}} \Psi(nx - v) \left(\sum_{s=1}^{r} w_s \int_{x}^{\frac{v}{n} + \frac{s}{nr}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{v}{n} + \frac{s}{nr} - t\right)^{N-1}}{(N-1)!} dt \right) dv \right|$$

$$\leq \int_{\left|\frac{v}{n} - x\right| \ge \frac{1}{n^{\alpha}}} \Psi(nx - v) |\gamma(v)| dv =: \xi.$$

First let us consider the Case $\frac{v}{n} + \frac{s}{nr} \ge x$. Then we have

$$|\gamma(v)| \le \frac{2\|f^{(N)}\|_{\infty}}{N!} \sum_{s=1}^{r} w_s \left(\frac{v}{n} + \frac{s}{nr} - x\right)^N.$$

Now let us assume that $\frac{v}{n} + \frac{s}{nr} < x$. Then we have

$$|\gamma(v)| = \left| \sum_{s=1}^{r} w_{s} \int_{\frac{v}{n} + \frac{s}{nr}}^{x} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \left(\frac{v}{n} + \frac{s}{nr} \right) \right)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \sum_{s=1}^{r} w_{s} \int_{\frac{v}{n} + \frac{s}{nr}}^{x} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left(t - \left(\frac{v}{n} + \frac{s}{nr} \right) \right)^{N-1}}{(N-1)!} dt$$

$$\leq \frac{2 \|f^{(N)}\|_{\infty}}{N!} \sum_{s=1}^{r} w_{s} \left(x - \left(\frac{v}{n} + \frac{s}{nr} \right) \right)^{N}$$

and consequently we can write for all cases

$$|\gamma(v)| \leq \frac{2 \left\|f^{(N)}\right\|_{\infty}}{N!} \left(\left|x - \frac{v}{n}\right| + \frac{1}{n}\right)^{N}.$$

Similarly, as in the earlier theorem, we have that

$$\left| \int_{\left| \frac{v}{n} - x \right| \ge \frac{1}{n^{\alpha}}} \Psi(nx - v) \gamma(v) dv \right| \le \xi$$

and

$$\xi \le \frac{2^N \|f^{(N)}\|_{\infty}}{n^N N!} \left(q + \frac{1}{q}\right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^N}\right] \to 0, \ as \ n \to \infty.$$

We also see for $k = 1, 2, \dots, N$ that

$$\begin{aligned} \left| \overline{\mathbb{B}_n} \left((\cdot - x)^k \right) (x) \right| &= \left| \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s \left(\frac{v}{n} + \frac{s}{nr} - x \right)^k \right) \Psi(nx - v) dv \right| \\ &\leq \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s \left| \frac{v}{n} + \frac{s}{nr} - x \right|^k \right) \Psi(nx - v) dv \\ &\leq \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s \left(\left| \frac{v}{n} - x \right| + \frac{s}{nr} \right)^k \right) \Psi(nx - v) dv \\ &\leq \int_{-\infty}^{\infty} \left(\frac{1}{n} + \left| \frac{v}{n} - x \right| \right)^k \Psi(nx - v) dv \\ &= \frac{1}{n^k} \int_{-\infty}^{\infty} \left(1 + |nx - v| \right)^k \Psi(nx - v) dv \\ &\leq \frac{2^{k-1}}{n^k} \left[1 + \frac{1 - e^{-\beta}}{(1 + e^{-\beta})} \frac{1}{k+1} + \left(q + \frac{1}{q} \right) \frac{e^{\beta} k!}{\beta^k} \right] \to 0, \ as \ n \to \infty. \end{aligned}$$

Hence we complete the proof.

Theorem 13. If $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$, $f^{(k)} \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, $k = 0, 1, \dots, s \in \mathbb{N}$ with $f^{(N+k)} \in C_B(\mathbb{R})$. Then

(i)
$$\left| (\mathsf{B}_{n}(f))^{(k)}(x) - f^{(k)}(x) - \sum_{m=1}^{N} \frac{f^{(k+m)}(x)}{m!} \left(\mathsf{A}_{n} \left((\cdot - x)^{m} \right) \right) (x) \right|$$

$$\leq \frac{\omega \left(f^{(N+k)}, \frac{1}{n^{\alpha}} \right)}{n^{\alpha N} N!} + \frac{2^{N+2} \|f^{(N+k)}\|_{\infty} e^{\beta} \left(q + \frac{1}{q} \right)}{n^{N} \beta^{N}} e^{\frac{-\beta n^{1-\alpha}}{2}},$$
(ii)

$$\left| (\mathbf{B}_{n}^{*}(f))^{(k)}(x) - f^{(k)}(x) - \sum_{m=1}^{N} \frac{f^{(k+m)}(x)}{m!} \left(\mathbf{A}_{n}^{*} \left((\cdot - x)^{m} \right) \right) (x) \right|$$

$$\leq \omega \left(f^{(N+k)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!}$$

$$+ \frac{2^{N} \left\| f^{(N+k)} \right\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right],$$

(iii)
$$\left| (\overline{\mathbf{B}_{n}}(f))^{(k)}(x) - f^{(k)}(x) - \sum_{m=1}^{N} \frac{f^{(k+m)}(x)}{m!} \left(\overline{\mathbf{A}_{n}} \left((\cdot - x)^{m} \right) \right) (x) \right|$$

$$\leq \omega \left(f^{(N+k)}, \frac{1}{n} + \frac{1}{n^{\alpha}} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\alpha}} \right)^{N}}{N!}$$

$$+ \frac{2^{N} \|f^{(N+k)}\|_{\infty}}{n^{N} N!} \left(q + \frac{1}{q} \right) e^{\beta} e^{-\frac{\beta n^{1-\alpha}}{2}} \left[1 + \frac{2^{N+1} N!}{\beta^{N}} \right].$$

Proof. By using Theorems 10, 11, 12, we immediately obtain the proof.

4. Iterated Version of
$$B_n$$
, B_n^* $\overline{B_n}$

In this section, we consider the iterated versions of our operators and examine approximation properties of them under the light of [7], [9], [10], [11], [12], [13], [14].

Remark 4. (About Iterated Convolution)

Notice that

$$B_n(f)(x) = \int_{-\infty}^{\infty} f\left(x - \frac{h}{n}\right) \Psi(h) dh, \text{ for } f \in C_B(\mathbb{R})$$

and let $x_k \to x$, as $k \to \infty$, and

$$\mathtt{B}_n(f)(x_k) - \mathtt{B}_n(f)(x) = \int_{-\infty}^{\infty} \left[f\left(x_k - \frac{h}{n}\right) - f\left(x - \frac{h}{n}\right) \right] \Psi(h) dh.$$

We have that

$$f\left(x_k - \frac{h}{n}\right)\Psi(h) \to f\left(x - \frac{h}{n}\right)\Psi(h), \ for \ every \ h \in \mathbb{R}, \ k \to \infty.$$

Furthermore

$$|\mathbf{B}_n(f)(x_k) - \mathbf{B}_n(f)(x)| \le \int_{-\infty}^{\infty} \left| f\left(x_k - \frac{h}{n}\right) - f\left(x - \frac{h}{n}\right) \right| \Psi(h)dh \to 0, \ k \to \infty,$$

by Dominated Convergence Theorem, since

$$\left| f\left(x_k - \frac{h}{n}\right) \right| \Psi(h)dh \le ||f||_{\infty} \Psi(h)$$

and $||f||_{\infty}\Psi(h)$ is integrable over $(-\infty,\infty)$, for every $h\in(-\infty,\infty)$. Thus $\mathtt{B}_n(f)\in C_B(\mathbb{R})$.

Also we have that

$$|\mathsf{B}_n(f)(x)| \le ||f||_{\infty} \int_{-\infty}^{\infty} \Psi(h)dh = ||f||_{\infty}$$

i.e.,

$$\|\mathbf{B}_n(f)\|_{\infty} \le \|f\|_{\infty}$$

which means that B_n is bounded and linear for $n \in \mathbb{N}$.

Remark 5. Let $r \in \mathbb{N}$. Since

$$B_n^r f - f = (B_n^r f - B_n^{r-1} f) + (B_n^{r-1} f + B_n^{r-2} f) + (B_n^{r-2} f + B_n^{r-3} f) + \dots + (B_n^r f - B_n f) + (B_n^r f - f).$$

We have that $\|\mathbf{B}_n^r f - f\|_{\infty} \le r \|\mathbf{B}_n f - f\|_{\infty}$ and

$$\begin{split} \mathbf{B}_{k_r}(\mathbf{B}_{k_{r-1}}(\dots \mathbf{B}_{k_2}(\mathbf{B}_{k_1}f))) - f &= \dots = \mathbf{B}_{k_r}(\mathbf{B}_{k_{r-1}}(\dots \mathbf{B}_{k_2}))(\mathbf{B}_{k_1} - f) \\ &+ \mathbf{B}_{k_r}(\mathbf{B}_{k_{r-1}}(\dots \mathbf{B}_{k_3}))(\mathbf{B}_{k_2} - f) \\ &+ \mathbf{B}_{k_r}(\mathbf{B}_{k_{r-1}}(\dots \mathbf{B}_{k_4}))(\mathbf{B}_{k_3} - f)\mathbf{B}_{k_r} \\ &+ \dots + (\mathbf{B}_{k_{r-1}}f - f) + \mathbf{B}_{k_r}f - f \end{split}$$

where $k_1, k_2, \ldots, k_r \in \mathbb{N} : k_1 \leq k_2 \leq \ldots \leq k_r$ and

$$\|\mathbf{B}_{k_r}(\mathbf{B}_{k_{r-1}}(\dots\mathbf{B}_{k_2}(\mathbf{B}_{k_1}f))) - f\|_{\infty} \le \sum_{m=1}^r \|\mathbf{B}_{k_m}f - f\|_{\infty}$$

as in [10]. The similar results can be obtained for \mathtt{B}_n^* and $\overline{\mathtt{B}_n}$ using the same technique.

Remark 6. Notice that

$$\mathsf{B}_n^*(f)(x) = n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} f\left(t + \left(x - \frac{h}{n}\right)\right) dt \right) \Psi(h) dh, \ f \in C_B(\mathbb{R}),$$

and let $x_k \to x$, as $k \to \infty$,

and

$$\begin{split} & |\mathrm{B}_n^*(f)(x_k) - \mathrm{B}_n^*(f)(x)| \\ \leq & n \int_{-\infty}^{\infty} \left(\int_0^{\frac{1}{n}} \left| f\left(t + \left(x_k - \frac{h}{n}\right)\right) - f\left(t + \left(x - \frac{h}{n}\right)\right) \right| dt \right) \Psi(h) dh, \end{split}$$

by Bounded Convergence Theorem, we get that:

$$x_k \to x \leadsto t + \left(x_k - \frac{h}{n}\right) \to t + \left(x - \frac{h}{n}\right)$$

and

$$f\left(t + \left(x_k - \frac{h}{n}\right)\right) \to f\left(t + \left(x - \frac{h}{n}\right)\right)$$

$$\left|f\left(t + \left(x_k - \frac{h}{n}\right)\right)\right| \le ||f||_{\infty}$$

and $\left[0,\frac{1}{n}\right]$ is finite. Hence

$$n\int_0^{\frac{1}{n}} \left| f\left(t + \left(x_k - \frac{h}{n}\right)\right) - f\left(t + \left(x - \frac{h}{n}\right)\right) \right| dt \to 0, \ as \ k \to \infty.$$

Therefore it holds that

$$n\int_0^{\frac{1}{n}} f\left(t + \left(x_k - \frac{h}{n}\right)\right) dt \to n\int_0^{\frac{1}{n}} f\left(t + \left(x - \frac{h}{n}\right)\right) dt \to 0, \ as \ k \to \infty$$

and

$$\left(n\int_0^{\frac{1}{n}} f\left(t + \left(x_k - \frac{h}{n}\right)\right) dt\right) \Psi(h) \to \left(n\int_0^{\frac{1}{n}} f\left(t + \left(x - \frac{h}{n}\right)\right) dt\right) \Psi(h),$$

as $k \to \infty$, for every $h \in (-\infty, \infty)$. Also, we get

$$\left| \left(n \int_0^{\frac{1}{n}} f\left(t + \left(x_k - \frac{h}{n}\right)\right) dt \right) \Psi(h) \right| \le ||f||_{\infty} \Psi(h).$$

Again by Dominated Convergence Theorem,

$$B_n^*(f)(x_k) \to B_n^*(f)(x), \text{ as } k \to \infty.$$

Thus $B_n^*(f)(x)$ is bounded and continuous in $x \in (-\infty, \infty)$ and the iterated facts hold for B_n^* as in the $B_n(f)$ case, all the same.

Remark 7. Next we observe that: $f \in C_B(\mathbb{R})$, and

$$\overline{B_n}(f)(x) := \int_{-\infty}^{\infty} \left(\sum_{s=1}^r w_s f\left(\left(x - \frac{h}{n}\right) + \frac{s}{nr}\right) \right) \Psi(h) dh. \tag{4.1}$$

Let $x_k \to x$, as $k \to \infty$.

Then

$$\left| \overline{\mathbf{B}_{n}}(f)(x_{k}) - \overline{\mathbf{B}_{n}}(f)(x) \right| = \left| \int_{-\infty}^{\infty} \left(\sum_{s=1}^{r} w_{s} \left(f\left(\left(x_{k} - \frac{h}{n} \right) + \frac{s}{nr} \right) - f\left(\left(x - \frac{h}{n} \right) + \frac{s}{nr} \right) \right) \right) \Psi(h) dh \right|$$

$$\leq \int_{-\infty}^{\infty} \left| \sum_{s=1}^{r} w_{s} \left(f\left(\left(x_{k} - \frac{h}{n} \right) + \frac{s}{nr} \right) - f\left(\left(x - \frac{h}{n} \right) + \frac{s}{nr} \right) \right) \right| \Psi(h) dh \to 0, \text{ as } k \to \infty$$

The last is from Dominated Convergence Theorem:

$$\left(x_k - \frac{h}{n}\right) + \frac{s}{nr} \to \left(x - \frac{h}{n}\right) + \frac{s}{nr}$$

and

$$\sum_{s=1}^{r} w_s f\left(\left(x_k - \frac{h}{n}\right) + \frac{s}{nr}\right) \to \sum_{s=1}^{r} w_s f\left(\left(x - \frac{h}{n}\right) + \frac{s}{nr}\right)$$

and

$$\left(\sum_{s=1}^{r} w_s f\left(\left(x_k - \frac{h}{n}\right) + \frac{s}{nr}\right)\right) \Psi(h) \to \left(\sum_{s=1}^{r} w_s \left(\left(x - \frac{h}{n}\right) + \frac{s}{nr}\right)\right) \Psi(h),$$

as $k \to \infty, \forall h \in (-\infty, \infty)$. Furthermore, we have

$$\left| \sum_{s=1}^{r} w_s f\left(\left(x_k - \frac{h}{n} \right) + \frac{s}{nr} \right) \right| \Psi(h) \le ||f||_{\infty} \Psi(h)$$

which the last one function is integrable over $(-\infty, \infty)$.

Therefore

$$\overline{B_n}(f)(x_k) \to \overline{B_n}(f)(x), \ as \ k \to \infty.$$

Thus $\overline{B_n}(f)(x)$ is bounded and continuous in $x \in (-\infty, \infty)$ and the iterated facts holds all the same.

Theorem 14. If $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $r \in \mathbb{N}$, $f \in C_B(\mathbb{R})$, then we have the followings:

(i)

$$\|\mathbf{B}_n^r(f) - f\|_{\infty} \le r \|\mathbf{B}_n f - f\|_{\infty} \le r \left[\omega\left(f, \frac{1}{n^{\alpha}}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_{\infty}}{e^{\beta(n^{1-\alpha}-1)}}\right],$$

(ii)

$$\|\mathbf{B}_n^{*r}(f) - f\|_{\infty} \leq r \, \|\mathbf{B}_n^* f - f\|_{\infty} \leq r \, \left[\omega \left(f, \frac{1}{n} + \frac{1}{n^{\alpha}}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_{\infty}}{e^{\beta(n^{1-\alpha}-1)}}\right],$$

(iii)

$$\left\|\overline{\mathbf{B}_n}^r(f) - f\right\|_{\infty} \le r \left\|\overline{\mathbf{B}_n}f - f\right\|_{\infty} \le r \left[\omega \left(f, \frac{1}{n} + \frac{1}{n^{\alpha}}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_{\infty}}{e^{\beta(n^{1-\alpha}-1)}}\right].$$

The rate of convergence of B_n^r , B_n^{*r} , $\overline{B_n}^r$ to identity is not worse than the rate of convergence of B_n , B_n^* , $\overline{B_n}$.

Proof. The proof follows from Theorems 2, 3, 4.

Theorem 15. If $0 < \alpha < 1$, $n \in \mathbb{N}; k_1, k_2, \dots, k_r \in \mathbb{N} : k_1 \le k_2 \le \dots \le k_r$, $k_p^{1-\alpha} > 2$, $p = 1, 2, \dots, r$; $f \in C_B(\mathbb{R})$, then we have the followings:

(i)

$$\begin{split} \|\mathbf{B}_{k_r}(\mathbf{B}_{k_{r-1}}(\dots \mathbf{B}_{k_2}(\mathbf{B}_{k_1}f))) - f\|_{\infty} &\leq \sum_{p=1}^r \|\mathbf{B}_{k_p}f - f\|_{\infty} \\ &\leq \sum_{p=1}^r \left[\omega\left(f, \frac{1}{k_p^{\alpha}}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_{\infty}}{e^{\beta(k_p^{1-\alpha} - 1)}} \right] \\ &\leq r \left[\omega\left(f, \frac{1}{k_1^{\alpha}}\right) + \frac{2\left(q + \frac{1}{q}\right)\|f\|_{\infty}}{e^{\beta(k_1^{1-\alpha} - 1)}} \right], \end{split}$$

(ii)

$$\begin{split} \|\mathbf{B}_{k_r}^*(\mathbf{B}_{k_{r-1}}^*(\dots \mathbf{B}_{k_2}^*(\mathbf{B}_{k_1}^*f))) - f\|_{\infty} &\leq \sum_{p=1}^r \|\mathbf{B}_{k_p}^*f - f\|_{\infty} \\ &\leq \sum_{p=1}^r \left[\omega \left(f, \frac{1}{k_p} + \frac{1}{k_p^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f\|_{\infty}}{e^{\beta(k_1^{1-\alpha}-1)}} \right] \\ &\leq r \left[\omega \left(f, \frac{1}{k_1} + \frac{1}{k_1^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f\|_{\infty}}{e^{\beta(k_1^{1-\alpha}-1)}} \right], \end{split}$$

(iii)

$$\begin{split} \|\overline{\mathbf{B}}_{k_{r}}(\mathbf{B}_{k_{r-1}}^{*}(\dots\overline{\mathbf{B}}_{k_{2}}(\mathbf{B}_{k_{1}}^{*}f))) - f\|_{\infty} &\leq \sum_{p=1}^{r} \|\overline{\mathbf{B}}_{k_{p}}f - f\|_{\infty} \\ &\leq \sum_{p=1}^{r} \left[\omega \left(f, \frac{1}{k_{p}} + \frac{1}{k_{p}^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f\|_{\infty}}{e^{\beta(k_{p}^{1-\alpha} - 1)}} \right] \\ &\leq r \left[\omega \left(f, \frac{1}{k_{1}} + \frac{1}{k_{1}^{\alpha}} \right) + \frac{2\left(q + \frac{1}{q} \right) \|f\|_{\infty}}{e^{\beta(k_{1}^{1-\alpha} - 1)}} \right]. \end{split}$$

Proof. The proof follows from Theorems 2, 3, 4.

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