TMUA Homework 3 Solutions

10 Questions

40 Minutes

Answers: FEDDDACCCD

Question 1

We first note that the object fits within a 3×3 cube. We begin with the plan view of the object: all of the cubes must be within three layers fitting within this L-shape.

Let us show the three layers:

		?				?				?
		?				?				?
?	?	?		?	?	?		?	?	?
bottom layer				middle layer				top layer		

We have used '?' to indicate that we do not yet know where there are cubes. We do, however, know that there must be a cube in at least one of the three layers for each of the five positions.

Next, let us look at the front elevation. This tells us that in the left hand column, we can see two cubes. Since there is only one possible location for the cubes in this column, the first two layers must have a cube and the third not. Likewise, for the middle column, there is only a cube in the bottom layer. In the right hand column, there must be at least one cube in each layer, but it is not yet clear where these would be, as there are three possible positions in each layer. We now know the following:

		?				?				?
		?				?				?
Y	Y	?		Y	x	?		x	x	?
bottom layer				middle layer				top layer		

Next, let us consider the side elevation. The left hand column, which corresponds to the bottom row of the plan view layers, has height 2, which means that the bottom right corner of the top layer cannot have a cube. The middle column, which corresponds to the middle row of the plan view layers, has height three, so all three of these cubes must be present (as there is only one possible position in the middle row). Finally, the right column, corresponding to the top row of the plan view layers, has height one, so only the bottom layer has a cube in this position.

We thus reach this position:

		Y				x				x
		Y				Y				Y
Y	Y	?		Y	x	?		x	x	x
bottom layer				middle layer				top layer		

One of the '?' cubes must be present, to give the required plan view, and the other one may or may not be – it makes no difference to any of the views. So there are either 8 or 9 cubes, and the answer is option F. (Note that we would need the corner of the bottom layer to be 'Y' if it is a single object.)

Question 2

We can find the interior angle of a regular polygon with n sides as follows. The exterior angle is $\frac{360°}{n}$, so the interior angle is $180° - \frac{360°}{n}$. Therefore the given condition can be written as

$$180° - \frac{360°}{n} = \frac{3}{4}\left(180° - \frac{360°}{m}\right).$$

We start by dividing by 180° for simplicity, giving

$$1 - \frac{2}{n} = \frac{3}{4}\left(1 - \frac{2}{m}\right).$$

Now multiplying both sides by 4 and expanding the brackets gives

$$4 - \frac{8}{n} = 3 - \frac{6}{m},$$

which rearranges to

$$\frac{8}{n} - \frac{6}{m} = 1. \tag{$*$}$$

We will look at two methods of solving this equation, starting with a case-based approach.

We know that n and m must each be at least 3. If $n \geqslant 8$, then $\frac{8}{n} \leqslant 1$, and so this cannot give a solution to ($*$). So we need only consider $n = 3$, 4, 5, 6 and 7. For each of these, we can solve the equation to find m; if m is an integer greater than 2, we have a solution.

n	m
3	$\frac{18}{5}$
4	6
5	10
6	18
7	42

There are four pairs of n and m, and the answer is E.

A second approach to solving ($*$) is to multiply both sides by mn to eliminate the fractions; this gives

$$8m - 6n = mn.$$

We can rearrange this to give $mn + 6n - 8m = 0$, and we can factorise the left hand side by adding or subtracting a constant to get

$$(m + 6)(n - 8) = -48.$$

So we need to find two integers which multiply to -48. Since m and n are both positive, we must have $n - 8 < 0$ and $m + 6 \geqslant 9$ (as $m \geqslant 3$), so the possibilities are:

$m+6$	$n-8$	m	n
12	-4	6	4
16	-3	10	5
24	-2	18	6
48	-1	42	7

as before.

Question 3

I Every valid code begins with a ‘U’ and ends with a ‘D’, so the reverse code begins with a ‘D’, meaning it immediately dips below sea level. So it is never a valid code, and this statement is false.

II As with I, the first letter of this substituted code will always be ‘D’, which will never give a valid code, so this statement is false.

III The new ‘U’ at the beginning of the code and ‘D’ at the end mean that the whole middle part of the profile is lifted one unit higher than the original profile. So this will never dip below sea level, and will always be a valid code. So this statement is true.

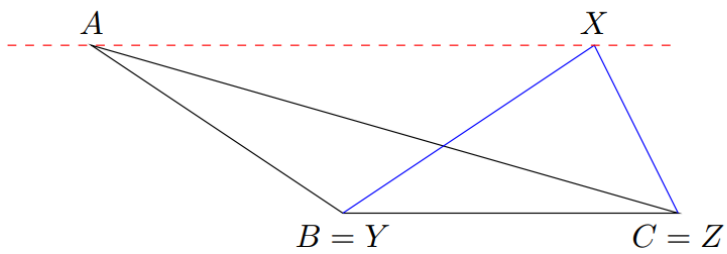
Therefore the correct answer is D.

Question 4

We work through the conditions in the order given. We recall three relevant conditions for these two triangles to be congruent:

- side-side-side (SSS): all three corresponding pairs of sides are equal in length
- side-angle-side (SAS): two corresponding pairs of sides are equal in length, and the corresponding included angles are equal
- angle-side-angle (ASA): two corresponding pairs of angles are equal, and the corresponding sides between these angles are equal in length

(1) If we knew that either $AC = XZ$ or the angle B equals the angle Y, then they would certainly be congruent by SSS or SAS. We know that the areas are equal, so we have $\frac{1}{2}AB.BC\sin B = \frac{1}{2}XY.YZ\sin Y$, and so $\sin B = \sin Y$. This does not imply that $B = Y$ though: we could have $Y = 180° - B$, for example:



(2) We now have an angle at the end of the given side, so we can use the equal-area property to work out a second side: the angle $\angle ABC$ is included between AB and BC, and likewise for XYZ. Therefore, as the areas are $\frac{1}{2}AB.BC\sin(\angle ABC)$ and $\frac{1}{2}XY.YZ\sin(\angle XYZ)$, it follows that $BC = YZ$. The triangles are therefore congruent by SAS.

(3) We now have two pairs of corresponding equal angles, and so the third pair of corresponding angles is also equal and the triangles are similar. Since the areas are equal, they must be congruent.

Hence conditions (2) and (3) each imply that the triangles are congruent, and the answer is option D.

Question 5

We start by finding the roots of $\mathrm{f}(x) = 0$. They are given by the quadratic formula, and are

$$r_1, r_2 = \frac{2p \pm \sqrt{(-2p)^2 - 4q}}{2} = p \pm \sqrt{p^2 - q}.$$

The roots are real if and only if the discriminant is non-negative, so if and only if $p^2 - q \geqslant 0$.

The difference between the roots is $r_2 - r_1 = 2\sqrt{p^2 - q}$, so the condition $(*)$ is true if and only if

$$p^2 - q \geqslant 0 \text{ and } 2 < 2\sqrt{p^2 - q} < 4$$

so it is true if and only if

$$p^2 - q \geqslant 0 \text{ and } 1 < \sqrt{p^2 - q} < 2$$

which is true if and only if

$$p^2 - q \geqslant 0 \text{ and } 1 < p^2 - q < 4$$

The first condition is implied by the second condition, so both conditions are true if and only if $1 < p^2 - q < 4$, which we can rearrange to $q + 1 < p^2 < q + 4$. Subtracting 1 gives $q < p^2 - 1 < q + 3$, which is condition D.

We can also show that the other four statements are not true if and only if $(*)$ is true, as follows.

Statement A fails, because the left inequality $q < p^2$ is not the same as $q < p^2 - 1$. Statement A is true if $(*)$ is true, but $(*)$ could be true without statement A being true; in other words "statement A is true only if $(*)$ is true" is false.

Statement B fails, because this requires $p > 0$, whereas we could have $p < 0$; thus, statement B is true only if $(*)$ is true, but "statement B is true if $(*)$ is true" is false.

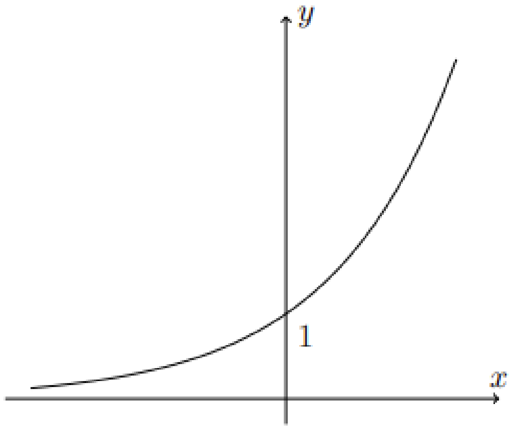
Statement C fails, because this allows for the difference between the roots to be exactly 2 or 4. So statement C is true if $(*)$ is true, but "statement C is true only if $(*)$ is true" is false.

Finally, statement E fails, because the second inequality is equivalent to $p^2 - 1 < q + 4$, so similarly to statement A, statement E is true if $(*)$ is true, but "statement E is true only if $(*)$ is true" is false.

So the correct option is D.

Question 6

We sketch the graph of $y = 2^x$. We know that $y = mx + c$ is a straight line with gradient m and a y-intercept of c, and we will imagine this line for different values of m and c. Solutions to the equation $2^x = mx + c$ correspond to the x-coordinates where these two graphs meet.



I Recall that 'P **only if** Q' is the same as saying '**if** P **then** Q', so in this case, the statement becomes '**if** the equation $2^x = mx + c$ has a negative real solution, **then** $c > 1$'. But we see that the line $y = -x$, with $c = 0$, has a solution with negative real x, so this statement is false.

II We can rewrite this statement as '**if** $c > 1$ **then** the equation has two distinct real solutions'. So we imagine straight lines passing through some point above $(0, 1)$, and we see that if this straight line has a negative gradient, then it can only intercept the curve once. So this statement is false.

III If $c \leqslant 1$, then a line with a negative gradient will still only intercept the curve once, so this is also false.

Therefore none of the statements is true, and the correct option is A.

Question 7

We work through the given statements in turn.

A This is a statement about numbers of the form N^2. If we write N as a product of prime factors, then N^2 will have the same prime factors, but each to double the power. For example, if $N = 3 \times 17^3 \times 41^4$ then $N^2 = 3^2 \times 17^6 \times 41^8$. So each prime factor occurs to an even power in its prime factorisation, so it is a squaresum.

Alternatively, we can ignore the theorem completely, and note that $N^2 = N^2 + 0^2$, so N^2 is a squaresum.

So this statement is true.

B If N is a squaresum, then each of its awkward prime factors occurs to an even power in its prime factorisation, by the theorem. Likewise, each of the awkward prime factors of M occurs to an even power in its prime factorisation. The prime factorisation of NM is the product of the prime factorisations of M and N, so each awkward prime occurs to an even power in the prime factorisation of NM, and therefore NM is a squaresum.

Thus this statement is also true.

C If NM is a squaresum, then each of its awkward prime factors occurs to an even power in its prime factorisation. But it is not clear why that would mean that each of the awkward prime factors of M and N each occur to an even power in the factorisations of M and N. And we can find a counterexample to this statement: 23 is awkward, so if $N = M = 23$ then $NM = 23^2$ is a squaresum, but neither N nor M is.

So this statement is false.

D As N is not a squaresum, some awkward prime factor occurs to an odd power in its prime factorisation. If we multiply N by this awkward prime factor, then the result will have this awkward prime factor to an even power. But N might have more than one awkward prime factor occurring to an odd power, so we multiply by all such awkward prime factors. The result will be a number with every awkward prime factor occurring to an even power. Thus letting k be the product of all the awkward prime factors of N which occur to an odd power, kN is a squaresum.

Hence this statement is true.

The only false statement is C.

Question 8

We could start by trying a few examples to get a feeling for the statement.

- $n = 0$ gives $0 + 1 + 2 + 3 = 6$, so $(*)$ is true in this case.
- $n = 1$ gives $1 + 2 + 3 + 4 = 10$, so $(*)$ is false in this case.
- $n = 2$ gives $2 + 3 + 4 + 5 = 14$, so $(*)$ is false in this case.
- $n = 3$ gives $3 + 4 + 5 + 6 = 18$, so $(*)$ is true in this case.

Of the given options, this exploration shows: A is false; B is false (as $(*)$ is false for $n = 1$); C may be true; D is false (as $(*)$ is true for $n = 3$), and E is false.

So by elimination, the correct answer must be C.

We can also prove this. The sum of the four consecutive integers is

$$n + (n + 1) + (n + 2) + (n + 3) = 4n + 6.$$

This is a multiple of 6 whenever $4n$ is a multiple of 6. An integer is a multiple of 6 if and only if it is a multiple of both 2 and 3. $4n$ is always a multiple of 2, and is a multiple of 3 when n is a multiple of 3 and for no other value of n. Hence option C is correct.

Question 9

We work through the possibilities systematically.

- Urn P's statement is true, so the urns each contain one or four balls. As Urn Q has a false statement, there cannot be two or four balls in the urns, and so there is one ball in each urn. But then Urn S has a true statement, which is impossible.
- Urn Q's statement is true, so the urns each contain two or four balls. As Urn P has a false statement, there cannot be one or four balls in each urn, so there are two balls in each urn. But then Urn S has a true statement, which is impossible.
- Urn R's statement is true, so there are three or four balls in each urn. As Urn P's statement is false, there cannot be four balls in the urn, so there are three balls in each urn. This means that Urn Q, Urn S and Urn T each have a false statement, and therefore this is possible.
- Urn S's statement is true, so the urns each contain one or two balls. As Urn P's statement is false, there cannot be one ball in each urn, so there are two balls in each urn. But then Urn Q has a true statement, which is impossible.
- Urn T's statement is true, so the urns each contain one or two balls. But then Urn S's statement is true, which is impossible.

Therefore the only urn which has a true statement is Urn R, so option C is correct.

Question 10

For ease of explanation, we say 'Mr X is true/false' as a shorthand for 'Mr X's statement is true/false'.

Of these statements, Ms Q and Mr T give the most explicit information, so we start with them. They cannot both be true, as they contradict each other, though they could both be false. So we look at these three possibilities in turn.

- Ms Q true, Mr T false.

  Since Ms Q is true, Ms S is also true, which means that either Mr P or Mr R is true (as Mr T is false).

  If Mr R is true, then Mr P is also true, contradicting Ms S. So Mr R must be false.

  Since Ms S is true, Mr P must be true; this does not pose a problem with Mr R being false, as it could be that Mr R's first name is not Robert. We must, though, check whether Mr P's statement being true is consistent with the remaining statements.

  We have Mr P, Ms Q and Ms S true and the other two false, giving three true statements. Thus Mr P's statement is, indeed, true, and this is a possible set of truth values for the statements.

- Ms Q false, Mr T true.

  Since Mr T is true, Ms S is false, and so either Mr P, Mr R or both must be true (as Mr T is already true).

  If Mr P is true, there are two true statements from Mr P, Ms Q, Ms S and Mr T, which is an even number, so Mr R must also be true. This is consistent, as Mr R says that Mr P's statement is true, which it is.

  If Mr R is true, then Mr P must also be true, and this is feasible as we have just seen.

  So in this case, we have Mr T, Mr P and Mr R all true, with the other two false, giving three true statements.

- Both Ms Q and Mr T false.

  Then exactly one statement made by a woman must be true, hence Ms S must be true.

  This requires exactly one of Mr P and Mr R to be true (as Mr T is false). If Mr R is true, then Mr P would also be true, which is not possible.

  If, on the other hand, Mr P is true and Mr R is false (say his first name is not Robert, as before), then we would have only Mr P and Ms S being true, and that is an even number of true statements. So this is not possible (as Mr P says there are an odd number of true statements).

Hence we must have one of the first two possibilities, and there are exactly three true statements; the answer is option D.