TMUA Homework 2 Solutions

10 Questions

40 Minutes

Answers: HBCCABDDED

Question 1

We expand $(ax+b)^3$ and compare with the given expansion.

$$(ax+b)^3 = a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3 = 8x^3 - px^2 + 18x - 3\sqrt{3}.$$

So we must have:

$$\begin{aligned} a^3 &= 8 \\ 3a^2b &= -p \\ 3ab^2 &= 18 \\ b^3 &= -3\sqrt{3} \end{aligned}$$

The final equation can be rewritten as $b^3 = -3^{\frac{3}{2}}$, so $b = -3^{\frac{1}{2}} = -\sqrt{3}$.

Then the third equation becomes $3a \times 3 = 18$, so $a = 2$.

Finally, the second equation gives $3 \times 2^2 \times (-\sqrt{3}) = -p$, so $p = 12\sqrt{3}$ and the answer is H.

Question 2

We expand the first expression up to powers of x^3 to find its coefficient. We can think of the first expression as $(1+(2x+3x^2))^6$ and use the binomial theorem to expand it. We note that the first few binomial coefficients are

$$\binom{6}{0}=1; \qquad \binom{6}{1}=6; \qquad \binom{6}{2}=\frac{6\times 5}{2!}=15; \qquad \binom{6}{3}=\frac{6\times 5\times 4}{3!}=20.$$

We thus have

$$\begin{aligned}(1+(2x+3x^2))^6 &= 1+\tbinom{6}{1}(2x+3x^2)+\tbinom{6}{2}(2x+3x^2)^2+\tbinom{6}{3}(2x+3x^2)^3+\cdots \\ &= 1+6(2x+3x^2)+15((2x)^2+2(2x)(3x^2)+\cdots)+20((2x)^3+\cdots)+\cdots\end{aligned}$$

where we have stopped when the powers reach 3. We can read off the coefficient of x^3 from this; it is

$$15\times 2\times 2\times 3+20\times 8=340.$$

We can expand $(1-ax^2)^5$ similarly, and obtain

$$1+\tbinom{5}{1}(-ax^2)+\tbinom{5}{2}(-ax^2)^2+\cdots=1-5ax^2+10a^2x^4+\cdots$$

so the coefficient of x^4 is $10a^2$.

We are told how these two coefficients relate to each other: we have

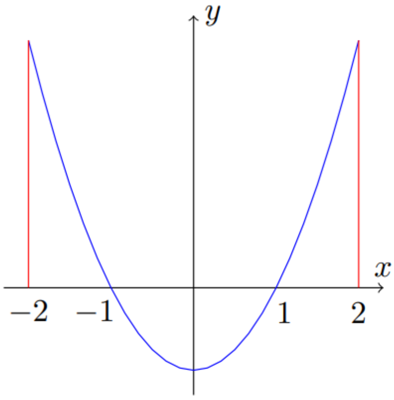
$$340=2\times 10a^2$$

so $a^2=17$ and $a=\pm\sqrt{17}$, giving the answer as option B.

Question 3

To answer this question, it is worth drawing a sketch.

The graph is of $y = x^2 - 1 = (x+1)(x-1)$, so the parabola intersects the x-axis at $(1,0)$ and $(-1,0)$:



To find the area enclosed, we integrate over the three separate regions, from -2 to -1, from -1 to 1, and from 1 to 2:

$$\begin{aligned}
\int_{-2}^{-1} x^2 - 1 \, \mathrm{d}x &= \left[\tfrac{1}{3}x^3 - x\right]_{-2}^{-1} \\
&= \left(-\tfrac{1}{3} + 1\right) - \left(-\tfrac{8}{3} + 2\right) \\
&= \tfrac{4}{3} \\
\int_{-1}^{1} x^2 - 1 \, \mathrm{d}x &= \left[\tfrac{1}{3}x^3 - x\right]_{-1}^{1} \\
&= \left(\tfrac{1}{3} - 1\right) - \left(-\tfrac{1}{3} + 1\right) \\
&= -\tfrac{4}{3} \\
\int_{1}^{2} x^2 - 1 \, \mathrm{d}x &= \left[\tfrac{1}{3}x^3 - x\right]_{1}^{2} \\
&= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) \\
&= \tfrac{4}{3}
\end{aligned}$$

Thus the three areas are each $\frac{4}{3}$, and the total area is 4, so the answer is C.

Question 4

We start by expanding the brackets to get

$$y = (2x + a)(x^2 - 4ax + 4a^2) = 2x^3 - 7ax^2 + 4a^2x + 8a^3$$

so the derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 14ax + 4a^2.$$

Therefore at $x = 1$, the gradient is $6 - 14a + 4a^2$. To find the least possible value of this as a varies, we can either differentiate this expression with respect to a or complete the square. The latter approach gives

$$\begin{aligned} 4a^2 - 14a + 6 &= 4(a^2 - \tfrac{7}{2}a) + 6 \\ &= 4\big((a - \tfrac{7}{4})^2 - (\tfrac{7}{4})^2\big) + 6 \\ &= 4(a - \tfrac{7}{4})^2 - \tfrac{49}{4} + 6 \\ &= 4(a - \tfrac{7}{4})^2 - \tfrac{25}{4} \end{aligned}$$

so the minimum value is $-\frac{25}{4}$ and the answer is C.

Question 5

This requires us to first differentiate the function. We therefore write

$$y = \frac{1-x}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}} - x^{\frac{1}{3}}$$

which we can differentiate to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{2}{3}x^{-\frac{5}{3}} - \tfrac{1}{3}x^{-\frac{2}{3}}$$

This is zero when

$$-\tfrac{2}{3}x^{-\frac{5}{3}} - \tfrac{1}{3}x^{-\frac{2}{3}} = 0,$$

so multiplying by $3x^{\frac{5}{3}}$ to clear fractions in both the coefficients and in the powers gives

$$-2 - x = 0$$

so $x = -2$. (We could also have obtained this by writing $-\frac{2}{3}x^{-\frac{5}{3}} = \frac{1}{3}x^{-\frac{2}{3}}$ and dividing one side by the other.)

We next need to determine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ in each region. We note that the function is not defined at $x = 0$, so we have to deal with $x < 0$ and $x > 0$ separately. It is also not clear how to find the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ directly from its given form, so we first factorise it, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{1}{3}x^{-\frac{5}{3}}(2+x).$$

(Incidentally, this gives yet another way to see that the derivative is zero at $x = -2$.) We can now work out the signs of the two factors, and hence of the derivative, in the various ranges:

	$x < -2$	$x = -2$	$-2 < x < 0$	$x > 0$
$x^{-\frac{5}{3}}$	$-$	$-$	$-$	$+$
$2 + x$	$-$	0	$+$	$+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+$	0	$-$	$+$

It is therefore increasing in the region $x < -2$ and $x > 0$. (It it not increasing at $x = -2$, but rather it is stationary at that point.) The correct answer is therefore A (though the question mistakenly says $x \leqslant -2$).

Question 6

We find the equation of the perpendicular bisector first, and then find the x-coordinate of this line when $y = 0$.

The midpoint of the line segment joining $(2, -6)$ and $(5, 4)$ is $(\frac{7}{2}, -1)$, and the gradient of this line segment is $\frac{4-(-6)}{5-2} = \frac{10}{3}$.

Therefore the perpendicular bisector passes through $(\frac{7}{2}, -1)$ and has gradient $-\frac{3}{10}$. Its equation is therefore

$$y - (-1) = -\tfrac{3}{10}\left(x - \tfrac{7}{2}\right).$$

We could expand and simplify this equation, but that is not necessary for our purposes. Instead, we substitute $y = 0$ to obtain

$$1 = -\tfrac{3}{10}\left(x - \tfrac{7}{2}\right)$$

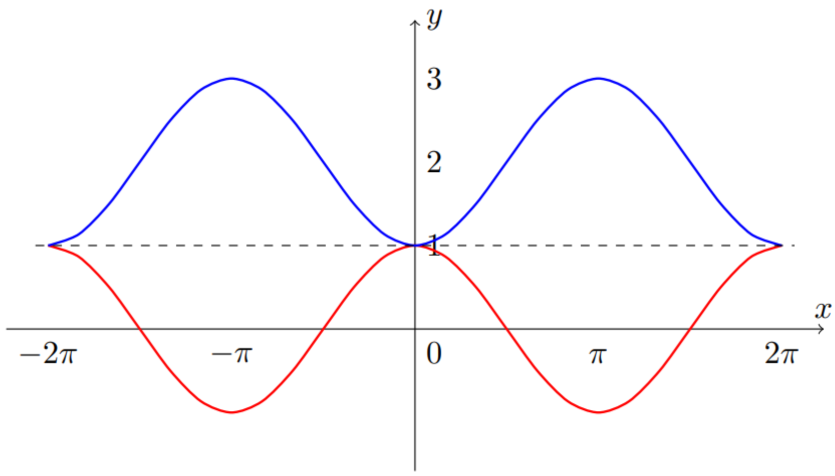
giving

$$x - \tfrac{7}{2} = -\tfrac{10}{3}$$

so $x = \frac{7}{2} - \frac{10}{3} = \frac{21}{6} - \frac{20}{6} = \frac{1}{6}$ showing that the correct answer is option B.

Question 7

When the graph is reflected in the line $y = 1$, we obtain the following:



We can find the equation of the reflected curve in a variety of ways. One is to observe that the reflection is centred about $y = 2$, as the values go from $y = 1$ to $y = 3$. The cosine curve is 'upside-down', so the equation must be $y = -(\cos x) + 2 = 2 - \cos x$.

Another way to see this is as follows. The refletion of $y = \mathrm{f}(x)$ in the x-axis ($y = 0$) is $y = -\mathrm{f}(x)$. When the line of reflection is translated, the whole reflection is translated. Since $y = 0$ is transformed to $y = 2$, the reflection of the function must be translated to give $y = 2 - \mathrm{f}(x)$.

The result is then translated by $\frac{\pi}{4}$ in the positive x-direction, so that x is replaced by $x - \frac{\pi}{4}$. (We can see this because the new $x = \frac{\pi}{4}$ corresponds to the old $x = 0$.)

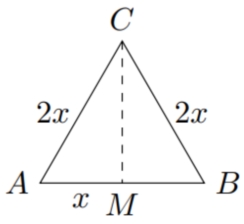
Therefore, the equation of the resulting graph is

$$y = 2 - \cos\left(x - \frac{\pi}{4}\right).$$

Thus the correct answer is option D.

Question 8

The volume of the prism is the length times the cross-sectional area, so we need to work out the area of the equilateral triangle:



We could use trigonometry and the formula for the area of a triangle, area $= \frac{1}{2}ab\sin C$; this gives the area as $\frac{1}{2}(2x)(2x)\sin 60° = x^2\sqrt{3}$.

Alternatively, we could find the length CM using Pythagoras's theorem, giving $CM^2 = (2x)^2 - x^2 = 3x^2$, so $CM = x\sqrt{3}$. Thus the area of the triangle is $\frac{1}{2}AB \times CM = x^2\sqrt{3}$.

Thus the volume of the prism is $T = x^2 d\sqrt{3}$.

The total surface area of the prism is twice the area of the triangle, plus the area of the three rectangular faces, so

$$T = 2x^2\sqrt{3} + 3(2xd) = 2x^2\sqrt{3} + 6xd.$$

These expressions for T are equal, so

$$x^2 d\sqrt{3} = 2x^2\sqrt{3} + 6xd.$$

Collecting the d terms to the left hand side gives

$$x^2 d\sqrt{3} - 6xd = 2x^2\sqrt{3}$$

so

$$d(x^2\sqrt{3} - 6x) = 2x^2\sqrt{3}$$

hence

$$d = \frac{2x^2\sqrt{3}}{x^2\sqrt{3} - 6x} = \frac{2x\sqrt{3}}{x\sqrt{3} - 6}.$$

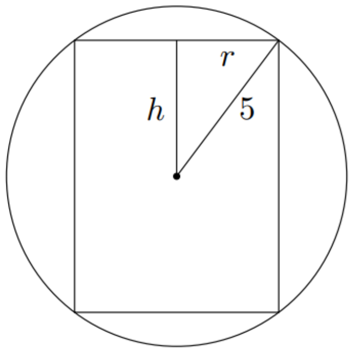
If we now divide the numerator and denominator by $\sqrt{3}$, we obtain

$$d = \frac{2x}{x - 2\sqrt{3}},$$

which is option D.

Question 9

Let the radius of the cylinder be r. Then the diagram is as follows (with all measurements in cm), where we have drawn in two extra lines:



The 5cm is the radius of the sphere and hence of the circle shown (as the cross section is through the centre of the sphere).

Pythagoras's theorem gives $h^2 + r^2 = 5^2$, and the volume of the cylinder is $V = \pi r^2(2h) = \pi(5^2 - h^2)(2h) = 2\pi(25h - h^3)$. We can maximise this by differentiating with respect to h. (We could write everything in terms of r instead, but that would involve square roots.)

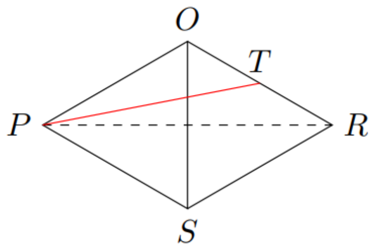
We have $\dfrac{\mathrm{d}V}{\mathrm{d}h} = 2\pi(25 - 3h^2)$, which is zero when $25 = 3h^2$, so $h = \dfrac{5}{\sqrt{3}}$. Substituting this into the formula for V gives the largest possible V as

$$V = 2\pi(5^2 - h^2)h = 2\pi\left(25 - \frac{25}{3}\right)\frac{5}{\sqrt{3}} = \frac{500}{3\sqrt{3}}\pi = \frac{500\sqrt{3}}{9}\pi$$

hence the answer is E.

Question 10

We now look at the other route, which we can do by drawing the triangles OPS and OSR:



Using the height of the triangle as calculated above, P lies $10\sqrt{3}$ to the left of OS and T lies $5\sqrt{3}$ to its right, while T lies 5 above P (being a quarter of OS). Pythagoras's theorem therefore gives

$$\begin{aligned} PT^2 &= (15\sqrt{3})^2 + 5^2 \\ &= 225 \times 3 + 25 \\ &= 700 \end{aligned}$$