TMUA Homework 1 Solutions

10 Questions

40 Minutes

Answers: DDDACDEBFD

Question 1

We can rearrange the first equation to

$$x = 3y - 1.$$

Substituting this into the second equation gives

$$3(3y-1)^2 - 7(3y-1)y = 5$$

which expands to

$$27y^2 - 18y + 3 - 21y^2 + 7y = 5$$

so

$$6y^2 - 11y - 2 = 0.$$

We can either factorise this as $(y-2)(6y+1) = 0$, giving $y = 2$ or $y = -\frac{1}{6}$ or use the quadratic formula to obtain

$$y = \frac{11 + \sqrt{11^2 - 4 \times 6 \times (-2)}}{12} = \frac{11 \pm 13}{12}$$

giving $y = 2$ or $y = -\frac{1}{6}$ again.

These give $x = 5$ or $x = -\frac{3}{2}$, so the sum is 3.5, and the answer is D.

Alternatively, we could have rearranged the first equation to get

$$y = \frac{x+1}{3}$$

and then substitute this into the second equation. This has the advantage that we obtain the values of x directly, but the disadvantage that there are fractions involved throughout.

Question 2

We rewrite $\sin^2\theta$ as $1-\cos^2\theta$ to give

$$1-\cos^2\theta+3\cos\theta=3.$$

This is now a quadratic in $u=\cos\theta$, giving

$$1-u^2+3u=3$$

or $u^2-3u+2=0$. This factorises as $(u-1)(u-2)=0$, so $u=1$ or $u=2$, that is, $\cos\theta=1$ or $\cos\theta=2$.

$\cos\theta=2$ has no real solutions. The solutions of $\cos\theta=1$ in the given range are $\theta=0$, $\theta=2\pi$, $\theta=4\pi$. So there are three solutions, and the answer is option D.

Question 3

Since $x + 2$ is a factor, substituting $x = -2$ into the polynomial must yield zero by the factor theorem:

$$(-2)^3 + 4c(-2)^2 + (-2)(c+1)^2 - 6 = 0.$$

Simplifying gives

$$-8 + 16c - 2(c^2 + 2c + 1) - 6 = 0$$

so

$$-2c^2 + 12c - 16 = 0.$$

Dividing by -2 now gives

$$c^2 - 6c + 8 = 0$$

so $(c-2)(c-4) = 0$ and the roots are $c = 2$ and $c = 4$, with a sum of 6. Hence the answer is D.

We could also find the sum of the roots directly from the quadratic $c^2 - 6c + 8 = 0$ without solving it: if the roots are $c = p$ and $c = q$, then the quadratic can be written as $(c-p)(c-q) = 0$, which expands to $c^2 - (p+q)c + pq = 0$. So the sum of the roots is the negative of the c coefficient, which is 6, and the product of the roots is the constant, which is 7.

Question 4

The roots of the equation, using the quadratic formula, are

$$x = \frac{11 \pm \sqrt{11^2 - 8c}}{4}$$

and these differ by

$$x = \frac{2\sqrt{11^2 - 8c}}{4} = 2$$

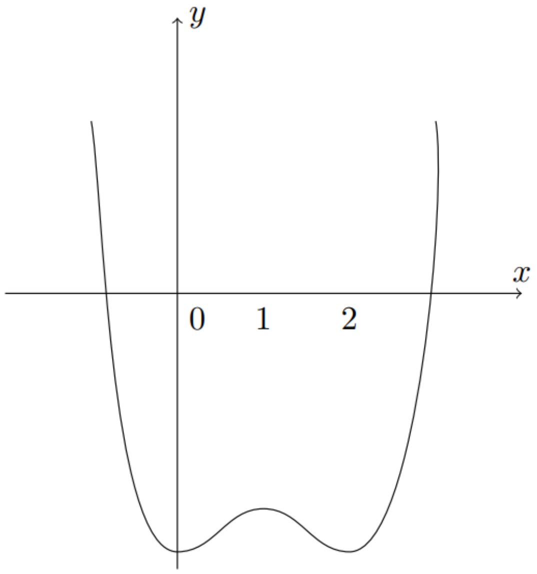
so we require $\sqrt{11^2 - 8c} = 4$, or $121 - 8c = 16$. Thus $8c = 105$, so $c = \frac{105}{8}$ and the answer is A.

Question 5

We can determine the answer to this question by locating the stationary points of the function $f(x) = x^4 - 4x^3 + 4x^2 - 10$. We have

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2).$$

There are therefore stationary points at $x = 0$, $x = 1$ and $x = 2$. Substituting these values into $f(x)$ gives us the y-coordinates of these points on the graph of $y = f(x)$: they are $(0, -10)$, $(1, -9)$ and $(2, -10)$. Since $f(x)$ tends to $+\infty$ as x tends to $+\infty$ or $-\infty$, the graph looks roughly like this:



There are thus only two real solutions to $f(x) = 0$, and the answer is option C.

Question 6

If the graph is a straight line, then we must have

$$(\log y) = m(\log x) + c$$

for some m and c; this is just the usual straight-line equation with x and y replaced by $\log x$ and $\log y$.

This can be written as

$$\log y = \log(x^m) + c.$$

If we now write $c = \log C$ for some C, then this becomes

$$\log y = \log x^m + \log C = \log(Cx^m).$$

Exponentiating both sides gives

$$y = Cx^m$$

which is in the form of option D.

An alternative to writing $c = \log C$ is just to exponentiate the equation $\log y = \log(x^m) + c$. If we suppose that the base of the logarithms is k, then this gives

$$y = x^m k^c$$

(using $k^{u+v} = k^u k^v$ and $k^{\log u} = u$ for any u and v). Since k^c is a constant, this equation can be rewritten as $y = ax^m$ where $a = k^c$, and we again obtain option D.

Question 7

We begin by simplifying this expression; we write each term as a product of prime powers to obtain:

$$\begin{aligned}
\frac{10^{c-2d} \times 20^{2c+d}}{8^c \times 125^{c+d}} &= \frac{(2 \times 5)^{c-2d} \times (2^2 \times 5)^{2c+d}}{(2^3)^c \times (5^3)^{c+d}} \\
&= \frac{2^{c-2d+2(2c+d)} \times 5^{c-2d+(2c+d)}}{2^{3c} \times 5^{3(c+d)}} \\
&= \frac{2^{5c)} \times 5^{3c-d}}{2^{3c} \times 5^{3c+3d}} \\
&= 2^{2c} \times 5^{-4d}
\end{aligned}$$

For this to be an integer, we require $2c$ and $-4d$ to be non-negative integers. Since c and d are non-zero integers, we need $c > 0$ and $d < 0$, which is option E.

(In fact, this is an "if and only if" condition; options C, D and F would make the expression non-integer, as would A and G; while conditions B and H are necessary, they are not sufficient: if $d < 0$, it is still possible that $c < 0$, so it is not true that the given expression is (necessarily) an integer if $d < 0$.)

Question 8

Commentary: *This looks quite scary! It is unlikely that you have ever seen a sequence looking like this, so a sensible thing to do is to work out the first few values and look for any patterns.*

We calculate the first few terms of the sequence:

$$\begin{aligned}
a_1 &= (-1)^1 - (-1)^0 + (-1)^3 = (-1) - 1 + (-1) = -3 \\
a_2 &= (-1)^2 - (-1)^1 + (-1)^4 = 1 - (-1) + 1 = 3 \\
a_3 &= (-1)^3 - (-1)^2 + (-1)^5 = (-1) - 1 + (-1) = -3 \\
a_4 &= (-1)^4 - (-1)^3 + (-1)^6 = 1 - (-1) + 1 = 3
\end{aligned}$$

The pattern is now clear (and we could prove it if we wished to): the sequence goes -3, 3, -3, 3, -3, 3 and so on. So the sum of each pair of terms is zero: $a_1 + a_2 = 0$, $a_3 + a_4 = 0$, ..., $a_{37} + a_{38} = 0$. Thus

$$\begin{aligned}
\sum_{n=1}^{39} a_n &= (a_1 + a_2) + (a_3 + a_4) + \cdots + (a_{37} + a_{38}) + a_{39} \\
&= 0 + 0 + \cdots + 0 + (-3) \\
&= -3
\end{aligned}$$

and the answer is B.

Question 9

We note that the only angle involved is $2x$, and $0° \leqslant x \leqslant 360°$ gives $0° \leqslant 2x \leqslant 720°$.

We start by writing everything in terms of $\sin 2x$, giving:

$$(1 - \sin^2 2x) + \sqrt{3}\sin 2x - \tfrac{7}{4} = 0$$

which rearranges to give

$$\sin^2 2x - \sqrt{3}\sin 2x + \tfrac{3}{4} = 0.$$

We can apply the quadratic formula to this to obtain

$$\sin 2x = \frac{\sqrt{3} \pm \sqrt{3-3}}{2} = \frac{\sqrt{3}}{2}$$

and hence the possible values of $2x$ in the range are $2x = 60°$, $120°$, $420°$ and $480°$. Thus the largest possible value of x in the range $0° \leqslant x \leqslant 360°$ is $240°$, and the answer is F.

Question 10

The formula for y is the product of two factors, namely $1 + 2\cos x$ and $\cos 2x$. The whole expression is negative when one of the two factors is positive and the other is negative. The factors change sign when they cross a point where the factor is zero, and we can find these points:

- $1 + 2\cos x = 0$ when $\cos x = -\frac{1}{2}$, which is when $x = \frac{2\pi}{3}$ (within the range $0 < x < \pi$).
- $\cos 2x = 0$ when $x = \frac{\pi}{4}$ and when $x = \frac{3\pi}{4}$.

We can now make a table showing the signs of the two factors in the different parts of the interval $0 < x < \pi$. (This is a useful technique in general.)

	$0 < x < \frac{\pi}{4}$	$x = \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{2\pi}{3}$	$x = \frac{2\pi}{3}$	$\frac{2\pi}{3} < x < \frac{3\pi}{4}$	$x = \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \pi$
$1 + 2\cos x$	$+$	$+$	$+$	0	$-$	$-$	$-$
$\cos 2x$	$+$	0	$-$	$-$	$-$	0	$+$
y	$+$	0	$-$	0	$+$	0	$-$

Therefore y is negative when $\frac{\pi}{4} < x < \frac{2\pi}{3}$ and when $\frac{3\pi}{4} < x < \pi$, and so the answer is D.