Activity 9

Chaotic dynamics

Goals

Nonlinear dynamical systems presents unstable bounded orbits. To be both unstable and bounded, the system must present particularly erratic behaviours in the neighbourhood of such these orbits. These behaviours are called chaos. We want to characterise properly this notion.

Theory

Definitions of chaos

Definition 45 (Sensitivity to initial conditions (SIC))**.** *A dynamical system (Γ, φ, μ) is said to be sensitive to initial conditions in the neighbourhood of $X \in \Gamma$, if $\exists \delta > 0$ such that $\forall \epsilon > 0$, $\exists X' \in \Gamma$, $\|X - X'\| < \epsilon$ and $\exists t > 0$ with $\|\varphi^t(X) - \varphi^t(X')\| > \delta$.*

If a dynamical system is SIC, whatever precision ϵ with which the initial condition X is estimated by X', at some instant the prediction concerning the orbit of X' be far from the true orbit at least by a distance δ. SIC systems are deterministic but not predictable for long term. Obviously, this definition is close to the Lyapunov instability:

Property 6. *A dynamical system with a positive main asymptotic Lyapunov exponent $\underline{\lambda}(X) > 0$ is SIC in the neighbourhood of X.*

The two notions are nevertheless different, there are systems presenting a zero main Lyapunov exponent but being SIC.

Definition 46 (Chaotic dynamical systems)**.** *A dynamical system (Γ, φ, μ) is said to be chaotic:*

- *in the meaning of Li-Yorke, if there is a compact subset S of Γ (said scrambled set) such that*
  - $\limsup_{n \to +\infty} \|\varphi^n(X) - \varphi^n(X')\| > 0$, $\forall X, X' \in S$, $X \neq X'$;
  - $\liminf_{n \to +\infty} \|\varphi^n(X) - \varphi^n(X')\| = 0$, $\forall X, X' \in S$, $X \neq X'$;
  - $\limsup_{n \to +\infty} \|\varphi^n(X) - \varphi^n(X_*)\| > 0$, $\forall X \in S$ *and* $\forall X_* \in \Gamma$ *p-cyclic.*
- *in the meaning of Block-Coppel, if $\exists m \in \mathbb{N}^*$, and A a compact subset of Γ invariant for φ^m ($\varphi^m(A) \subset A$), such that there is a map $h : A \to \Sigma$ satisfying $h \circ \varphi^m = \sigma \circ h$ on A, where Σ is the set of infinite words written with the symbols R and L, and $\sigma : \Sigma \to \Sigma$ is the shift map: $\sigma(a_0 a_1 a_2 ...) = a_1 a_2 ...$ $a_i \in \{R, L\}$.*
- *in the meaning of Devaney, if there is an invariant compact subset A of Γ ($\varphi(A) \subset A$) such that*

  - $\varphi_{|A}$ *is topologically transitive;*
  - $\overline{P_\varphi(A)} = A$ *where* $P_{\varphi(A)}$ *is the set of periodic points of* A*;*
  - *the system is SIC in* A*.*

- *in the meaning of Wiggins, if there is an invariant compact subset* A *of* Γ *such that*
  - $\varphi_{|A}$ *is mixing;*
  - *the system is SIC in* A*.*

- *in the meaning of Lyapunov, if there is an invariant compact subset* A *of* Γ *such that*
  - $\varphi_{|A}$ *is mixing;*
  - $\underline{\lambda}(X) > 0$, $\forall X \in A$.

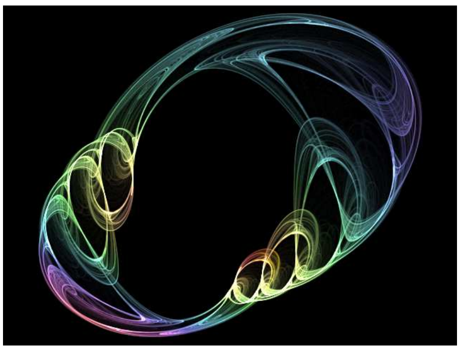
Roughly speaking, a dynamical system is chaotic if it has a compact component on which it is unstable and where it "mixes" the trajectories during the evolution. The different meanings of the chaos correspond to the degree of instability (Lyapunov, SIC or $\limsup \|\varphi^n(X) - \varphi^n(X')\| > 0$ whereas $\underline{\liminf \|\varphi^n(X) - \varphi^n(X')\| = 0}$) and to the degree of "mixing" (true mixing, topological transitivity with $\overline{P_\varphi(A)} = A$, or orbits cannot being asymptotically closed to cyclic orbits).

The attractors of the chaotic dissipative systems are strange. Reciprocally, a dissipative system presenting a strange attractor A is chaotic in A and in its basin of attraction. We can expect that the sum of Lyapunov exponents of a dissipative chaotic system be positive (the unstable behaviour being stronger to the stable behaviour for chaotic systems).

A bifurcation cascade of a logistic flow is a signature of chaos.

Autonomous 2D continuous time systems cannot be chaotic (three dimensions are needed to have chaos with autonomous continuous time systems), but this is possible for 1D or 2D discrete time systems and for driven 2D continuous time systems.

Important remark: in numerical simulations of continuous time systems, the numerical integrator can add its own instability. It is then possible to have a fake chaos induced by a bad numerical method. This is particularly true with the Euler integrator. This one can transform a 2D autonomous system to a (discrete time, since numerical methods discretise the time) chaotic system.



Typical morphology of a fake strange attractor induced by the Euler method with a too large time step.

Criteria of chaos

It is in general not obvious to prove the chaoticity of a system by using the definitions. We have three results, called Li-Yorke, Shilnikov and Chirikov criteria to help.

Theorem 11 (Li-Yorke theorem)**.** *Let (I, f, dx) be a logistic flow. If there exists a point x_* such that $f^3(x_*) \leq x_* < f(x_*) < f^2(x_*)$ (or $f^3(x_*) \geq x_* > f(x_*) > f^2(x_*)$), then*

- *there are k-cyclic points for all period k;*
- *the system is chaotic in the meaning of Li-Yorke.*

The assumption of the theorem is automatically satisfied if there exists a 3-cyclic point x_*.

Theorem 12 (Shilnikov (Ši'lnikov) theorems)**.** *Let $(\mathbb{R}^3, \varphi, \mu)$ be an autonomous three dimensional continuous time system, where the generator of the flow F is a $\mathcal{C}^2$-function. Let X_* be a hyperbolic fixed point and $\lambda_0 = \gamma$, $\lambda_\pm = \rho \pm \imath\omega$ be the local Lyapunov values at X_*. The system is chaotic in the meaning of Lyapunov if*

- $|\gamma| > |\rho|$,
- *there is a homoclinic orbit attached to X_*.*

Let X_{*1} and X_{*2} be two hyperbolic fixed points and $\lambda_{0,i} = \gamma_i$, $\lambda_{\pm,i} = \rho_i \pm \imath\omega_i$ be the local Lyapunov values at X_{*i}. The system is chaotic in the meaning of Lyapunov if

- $|\gamma_i| > |\rho_i|$ *for* $i = 1, 2$,
- $\gamma_1\gamma_2 > 0$ *or* $\rho_1\rho_2 > 0$,
- *there is a heteroclinic orbit linking X_{*1} and X_{*2}.*

Theorem 13 (Chirikov "theorem")**.** *Let (Γ, φ, μ) be a 2D periodically driven dynamical system described by an Hamiltonian $\mathscr{H}(\theta, I, t) = \mathscr{H}_0(\theta, I) + KV(\theta, I, t)$ (where $\mathscr{H}_0$ is the Hamiltonian of an autonomous system and KV is a periodic perturbation, $K \in \mathbb{R}$, $V \sim 1$); the variable θ being supposed to be an angle and the conjugated momentum I to be an angular momentum. The phase space is then a cylinder $\Gamma = \mathbb{S}^1 \times \mathbb{R}$ or a torus $\Gamma = \mathbb{T}^2$. If the phase portrait of the unperturbed system presents islands of cycles (resonances), then the system is chaotic in the meaning of Chirikov if*

$$K \simeq S^2 > 1 \quad \text{with } S = \frac{\Delta\omega_r}{\Omega_d}$$

where $\Delta\omega_r$ is the sum of the half-widths in the direction I of the islands and where Ω_d is the distance in the direction I between the centres of the islands.

Note that the Chirikov criterion does not permit to relate the system to a rigorous definition of chaos. A system satisfying this criterion is then said to be chaotic in the meaning of Chirikov although this is not a rigorous definition.

Fourier analysis

Let $f \in L^\infty(\Gamma, d\mu)$ be a suitable observable of the system, and $f(t) \equiv f(\varphi^t(X))$ be the evolution of this one for an orbit. Let $\nu \mapsto |TF[f](\nu)|^2$ be the power spectrum of f ($TF[f]$ being the Fourier transform of f). If the orbit is on a cycle or on an invariant torus, $f(t)$ is periodic or quasiperiodic. Its power spectrum is then characterised by isolated peaks at the fundamental frequencies and their harmonics. But if the orbit is chaotic, the erratic behaviour presents a continuous infinite number of frequencies, so the power spectrum of f is characterised by large continuous bands. Numerically, the Fourier transform can be computed by using a FFT algorithm (available in a lot of computing libraries), but it can be difficult to make the difference between a continuous band and a large set of closed peaks, and between a continuous band and a continuous background corresponding to white noise.
In a same way, the autocorrelation function of f, $R_f(t) = \int_{-\infty}^{+\infty} \overline{f(t-\tau)} f(\tau) d\tau = TF^{-1}[|TF[f]|^2](\tau)$ presents oscillations with a periodic or quasi-periodic orbit, whereas it falls to zero with small autocorrelation peaks appearing erratically (corresponding to the Poincaré recurrences) with a chaotic orbit. In contrast, the autocorrelation function of a white noise is a single thin peak at 0.

Work to be done

Find parameters for which your system is chaotic and try to characterise and to illustrate the chaoticity of your system.