Activity 6

Asymptotic stability and asymptotic Lyapunov exponents

Goals

We want to study the stability of any orbit of a dynamical system under perturbation and not only of the special orbits (fixed points, cycles). We want then generalise the notion of Lyapunov exponents.

Theory

General discussion

Let (Γ, φ, μ) be a dynamical system. We define the main asymptotic Lyapunov exponent at $X \in \Gamma$ as being

$$\underline{\lambda}(X) = \limsup_{t \to +\infty} \lim_{\delta X \to 0} \frac{1}{t} \ln \frac{\|\varphi^t(X + \delta X) - \varphi^t(X)\|}{\|\delta X\|}$$

The supremum limit being the limit onto the upper bounds if the function (or the sequence) has no limit by an oscillating behaviour at infinity. $\underline{\lambda}(X)$ measures the average logarithmic increase of the error on the orbit of X when X is perturbed by an infinitely small shift δX. We have then for n sufficiently large:

$$\|\varphi^t(X + \delta X) - \varphi^t(X)\| \simeq e^{t\underline{\lambda}(X)} \|\delta X\|$$

Asymptotically, the error exponentially increases or decreases with a speed fixed by $\underline{\lambda}$:

- If $\underline{\lambda}(X) > 0$ then $\mathrm{Orb}(X)$ is asymptotically unstable (a perturbed orbit diverges exponentially at infinity from the non-perturbed orbit).
- If $\underline{\lambda}(X) = 0$ then $\mathrm{Orb}(X)$ is asymptotically marginally stable (a perturbed orbit remains in a neighbourhood of the non-perturbed orbit at infinity).
- If $\underline{\lambda}(X) < 0$ then $\mathrm{Orb}(X)$ is asymptotically exponentially stable (a perturbed orbit exponentially converges to the non-perturbed orbit at infinity).

Definition 29 (Lyapunov horizon of predictability). *Let $\|\delta X\| > 0$ be a fixed non-zero initial error (magnitude of perturbation) and $\epsilon > \|\delta X\|$ be the maximal expected error onto the prediction concerning the orbit of a point X. The Lyapunov horizon of predictability of the dynamical system for the initial condition X is the quantity:*

$$t_H = \frac{1}{\max(\underline{\lambda}(X), 0)} (\ln \epsilon - \ln \|\delta X\|)$$

t_H is a time for continuous time dynamical system; for discrete time dynamical system $\lfloor t_H \rfloor$ (integer part of t_H) is a number of iterations.

If the error concerning the measure of the initial condition is of magnitude $\|\delta X\|$ and if the admissible error concerning the predictions about the dynamical system is ϵ, the Lyapunov horizon is the time or the number of iterations for which the predictions will be satisfactory. After the Lyapunov horizon, we cannot predict the behaviour of the system with sufficient accuracy and it needs to remake a measurement of the position of the system. For stable orbits, the Lyapunov horizon is infinite.

Important remark: if possible do not use the formula $\underline{\lambda}(X) = \limsup_t \lim_{\delta X \to 0} \frac{1}{t} \ln \frac{\|\varphi^t(X+\delta X) - \varphi^t(X)\|}{\|\delta X\|}$ to compute numerically the main asymptotic Lyapunov exponent, in general this one does not converge properly and the numerical computation provides a fake zero value. So if you use this formula and find the result 0, you cannot conclude (maybe the exponent is well 0, but maybe it is positive or negative but the computation is not correctly converged).

Continuous and discrete time dynamical systems

Definition 30 (Main asymptotic Lyapunov exponent)**.** *Let (Γ, φ, μ) be a continuous or discrete time dynamical system. We define the main asymptotic Lyapunov exponent at $X \in \Gamma$ by*

$$\underline{\lambda}(X) = \limsup_{n \to +\infty} \frac{1}{2n\Delta t} \ln \left(\mathrm{tr} \left(U_n^\dagger U_n \right) \right)$$

where $U_n = \partial\varphi_{X_{n-1}} ... \partial\varphi_{X_1} \partial\varphi_{X_0}$ (with $X_n = \varphi^{n\Delta t}(X)$).
$\Delta t = 1$ for the discrete time systems and is a sufficiently small discretisation time step for the continuous time systems. Moreover for continuous time systems $\partial\varphi_{X_i} \equiv e^{\partial F_{X_i} \Delta t}$.

U_n is the evolution matrix of the linearisation of the flow at each point of the orbit of X. So $\mathrm{tr}(U_n^\dagger U_n)$ is the sum of the squared eigenvalues of U_n. Since the eigenvalues are exponentials, they are negligible with respect to the one with the larger value. $\ln \mathrm{tr}(U_n^\dagger U_n)$ is then approximately equal to the larger value. This one changes at each step, but the asymptotic value is its average at infinity.

With this formula we have only the main exponent, we can also have the other ones:

Definition 31 (Asymptotic Lyapunov exponents)**.** *Let (Γ, φ, μ) be a dynamical system. We define the asymptotic Lyapunov exponents at X, $\underline{\lambda}_1(X) \geq \underline{\lambda}_2(X) \geq ... \geq \underline{\lambda}_N(X)$ by*

$$\begin{aligned}
\underline{\lambda}_1(X) &= \sup_{v \in T_X\Gamma} \limsup_{n \to +\infty} \frac{1}{n\Delta t} \ln \|U_n v\| = \sup_{v \in T_X\Gamma} \limsup_{n \to +\infty} \frac{1}{n\Delta t} \ln \|\nabla_v \varphi^{n\Delta t}(X)\| \\
\underline{\lambda}_2(X) &= \sup_{v \in [v_1]^\perp} \limsup_{n \to +\infty} \frac{1}{n\Delta t} \ln \|U_n v\| = \sup_{v \in [v_1]^\perp} \limsup_{n \to +\infty} \frac{1}{n\Delta t} \ln \|\nabla_v \varphi^{n\Delta t}(X)\| \\
\underline{\lambda}_2(X) &= \sup_{v \in [v_1, v_2]^\perp} \limsup_{n \to +\infty} \frac{1}{n\Delta t} \ln \|U_n v\| = \sup_{v \in [v_1, v_2]^\perp} \limsup_{n \to +\infty} \frac{1}{n\Delta t} \ln \|\nabla_v \varphi^{n\Delta t}(X)\| \\
&\vdots
\end{aligned}$$

where $U_n = \partial\varphi_{X_{n-1}} ... \partial\varphi_{X_1} \partial\varphi_{X_0}$ (with $X_n = \varphi^{n\Delta t}(X)$), anew $\Delta t = 1$ for discrete time systems and $\partial\varphi_{X_i} \equiv e^{\partial F_{X_i} \Delta t}$ for continuous time systems. $T_X\Gamma$ is vector space of tangent vectors of Γ at X, $\nabla_v = \sum_i v^i \partial_i$ is the derivative in the direction v, $[v_1, ..., v_j]^\perp$ is the supplementary subspace orthogonal to the vector space generated by $(v_1, ..., v_j)$ (with the canonical inner product of $\mathbb{R}^N$), v_1 is the vector such that $\limsup_{n \to +\infty} \frac{1}{n} \ln \|U_n v\|$ reaches its upper bound in $T_X\Gamma$, v_2 is the vector such that $\limsup_{n \to +\infty} \frac{1}{n} \ln \|U_n v\|$ reaches its upper bound in $[v_1]^\perp$, etc.

$\underline{\lambda}_1(X)$ is the main asymptotic Lyapunov exponent (the two definitions coincides).

- If $\underline{\lambda}_i(X) > 0$, v_i indicates an asymptotic unstable direction at X.
- If $\underline{\lambda}_i(X) = 0$, v_i indicates an asymptotic marginally stable direction at X.
- If $\underline{\lambda}_i(X) < 0$, v_i indicates an asymptotic exponentially stable direction at X.

We have then

$$T_X\Gamma = E_s(X) \oplus E_u(X) \oplus E_0(X)$$

where $E_s(X)$ is the subspace generated by the vectors indicating exponentially stable directions, $E_u(X)$ is the subspace generated by the vectors indicating unstable directions, and $E_0(X)$ is the subspace generated by the vectors indicating marginally stable directions. Note that for the continuous time dynamical system $\dim E_0(X) \geq 1$, we have at least one tangent vector indicating a marginally stable direction, $v = (F^1(X), ..., F^n(X))$ (with F the generator of the flow), which is the direction of propagation of the flow.

Definition 32 (Hyperbolic point). *A point $X \in \Gamma$ is said hyperbolic if it has at least one positive asymptotic Lyapunov exponent, at least one negative asymptotic Lyapunov exponent and no zero asymptotic Lyapunov exponent (except the one associated with the direction of propagation of the flow for the continuous time systems).*

Definition 33 (Stable and unstable manifolds). *Let $X \in \Gamma$ be a hyperbolic point. We define the global unstable manifold of X by*

$$W_u(X) = \{Y \in \Gamma, \liminf_{n\to+\infty} \frac{1}{n\Delta t} \ln ||\varphi^{-n\Delta t}(Y) - \varphi^{-n\Delta t}(X)\| < 0\}$$

The global stable manifold of X is defined by

$$W_s(X) = \{Y \in \Gamma, \liminf_{n\to+\infty} \frac{1}{n\Delta t} \ln ||\varphi^{n\Delta t}(Y) - \varphi^{n\Delta t}(X)\| < 0\}$$

Definition 34 (Anosov system). *A dynamical system is said to be of Anosov type if μ-almost all points of its phase space are hyperbolic.*

Symbolic dynamical systems

Due to the discrete nature of the phase spaces of the symbolic systems, the notion of perturbation is meaningless for these systems.

The L-systems being defined by an inflation rule, they cannot be considered as stable. An equivalent of the main Lyapunov exponent, is then the mean logarithm rate of growth of its word:

$$\underline{\lambda} = \lim_{n\to+\infty} \ln \frac{\ell(w_{n+1})}{\ell(w_n)}$$

where w_n is the word at iteration n, and $\ell(w)$ is number of symbols in the word w. So for n large, we have $\ell(w_{n+1}) \simeq e^{\underline{\lambda}}\ell(w_n)$. We can relate this quantity to the linearisation of the L-system $\partial\varphi$. To simplify we suppose that $\mathscr{A} = \{R, L\}$ (this can generalised to a larger number of symbols). If $\begin{pmatrix} n_R \\ n_L \end{pmatrix}$ are the numbers of symbols R and L at some iteration, $\partial\varphi \begin{pmatrix} n_R \\ n_L \end{pmatrix}$ are the numbers at the next iteration. We diagonalise the matrix

$$\partial\varphi v^{\pm} = e^{\underline{\lambda}_{\pm}} v^{\pm}$$

with $\underline{\lambda}_+ > \underline{\lambda}_-$. If for some normalisation, $v^\pm = \begin{pmatrix} v_R^\pm \\ v_L^\pm \end{pmatrix} \in \mathbb{N}^2$, $(v_R^\pm, v_L^\pm)$ are the two possible compositions of an initial word for which the ratio of the numbers of symbols R and symbols L is constant during the iterations. Even if such a normalisation does not exist, we however can write that for any initial composition (n_R, n_L):

$$\exists \alpha, \beta \in \mathbb{R}, \quad \begin{pmatrix} n_R \\ n_L \end{pmatrix} = \alpha v^+ + \beta v^-$$

So after a large number of iterations, we have

$$(\partial\varphi)^n \begin{pmatrix} n_R \\ n_L \end{pmatrix} = \alpha e^{n\underline{\lambda}_+} v^+ + \beta e^{n\underline{\lambda}_-} v^- \simeq \alpha e^{n\underline{\lambda}_+} v^+$$

and so

$$\lim_{n\to+\infty} \ln \frac{\ell(w_{n+1})}{\ell(w_n)} = \lim_{n\to+\infty} \ln \frac{\alpha e^{(n+1)\underline{\lambda}_+}(v_R^+ + v_L^+)}{\alpha e^{n\underline{\lambda}_+}(v_R^+ + v_L^+)} = \underline{\lambda}_+$$

$\underline{\lambda}_+$ is the main Lyapunov exponent of the L-system ($\underline{\lambda}_-$ is the other Lyapunov exponent).

For a cellular automaton, it needs to introduce a definition compatible with the discrete topology:

Definition 35 (Lyapunov-Shereshevsky exponents)**.** *We consider a one dimensional cellular automaton of alphabet $\mathscr{A}$ and global transition map φ. $\forall c \in \mathscr{A}^\mathbb{Z}$ and $s \in \mathbb{N}$, we set:*

$$\begin{aligned} W_s^+(c) &= \{k \in \mathscr{A}^\mathbb{Z} \textit{ such that } k_n = c_n, \forall n \geq s\} \\ W_s^-(c) &= \{k \in \mathscr{A}^\mathbb{Z} \textit{ such that } k_n = c_n, \forall n \leq -s\} \end{aligned}$$

We remark that $W_s^\pm(c) \subset W_{s+1}^\pm(c)$. $\forall n$ we set:

$$\tilde{\Lambda}_n^\pm(c) = \min\{s \geq 0 \textit{ such that } \varphi^n(W_0^\pm(c)) \subset W_s^\pm(\varphi^n(c))\}$$

Let $\sigma : \mathscr{A}^\mathbb{Z} \to \mathscr{A}^\mathbb{Z}$ be the shift map: $\sigma(c)_n = c_{n-1}$. We set

$$\Lambda_n^\pm(c) = \max_{j\in\mathbb{Z}} \tilde{\Lambda}_n^\pm(\sigma^j(c))$$

The Lyapunov-Shereshevsky exponents of the cellular automaton are the quantities

$$\underline{\lambda}^\pm(c) = \limsup_{n\to+\infty} \frac{\Lambda_n^\pm(c)}{n}$$

$\varphi^n(W_0^+(c))$ is the set of all sequences identical to c from the rank 0 after n iterations, $W_s^+(\varphi^n(c))$ is the set of all sequences identical from the rank s to the n-th iteration of c. $\tilde{\Lambda}_n^+(c)$ is then the rank from which the sequences initially identical to c from the rank 0, are still identical to c after n iterations. This is then a measure of stability. If $\tilde{\Lambda}_n^+(c)$ is close to 0, the initially close sequences (on the right), are still close after n iterations because it is not necessary to see far in rank to find similarities. The point of view is centred on 0 with $\tilde{\Lambda}_n^+(c)$, $\Lambda_n^+(c)$ permits to consider any point of view. $\underline{\lambda}^+(c)$ is then the asymptotic mean speed of movement of the rank from which the sequences similar on the right remain similar. $\underline{\lambda}^-(c)$ is the same thing on the left.

This definition can be applied to two dimensional cellular automata by transforming them in 1D systems with the Cantor pairing map. This definition must be used with an infinite number of cells (the exponents are trivially zero with finite grids).

This definition can be not easy to applied. There exists another approach closer to the classical notion of Lyapunov exponent. Let $c \in \mathscr{A}^{\mathbb{Z}^2}$ be a configuration of a cellular automaton. We call defect a change of colours of some cells of c such that $d(c, c + \delta c)$ be small, where δc is the list of cells with a change of colour ($+\delta c$ denoting the application of the changes), d being the Tychonoff distance. We can define a main Lyapunov exponent as

$$\underline{\lambda}(c) = \limsup_{n \to +\infty} \frac{1}{n} \ln \frac{\mathcal{N}(\varphi^n(c + \delta c), \varphi^n(c))}{\mathcal{N}(c + \delta c, c)}$$

where $\mathcal{N}(c^*, c)$ is the number of cells of c^* which have a colour different from the corresponding one in c (φ being the global transition map). Another possibility consists to use directly the Tychonoff distance:

$$\underline{\lambda}(c) = \limsup_{n \to +\infty} \frac{1}{n} \ln \frac{d(\varphi^n(c + \delta c), \varphi^n(c))}{d(c + \delta c, c)}$$

In that case, we have then the same definition than the general one. But note that nothing ensure that the three definitions (Shereshevsky, with the number of defects and with the Tychonoff distance) provide the same result.

Algorithm

An efficient algorithm to compute the main asymptotic Lyapunov exponent is the Sprott algorithm:

- We initialise the algorithm with $\tilde{X}_0 = X + \delta X$ and $d_0 = \|\delta X\|$ for some perturbation δX.
- At each step, we compute $\hat{X}_n = \varphi^{\Delta t}(\tilde{X}_{n-1})$ and the distance $d_n = \|\hat{X}_n - \varphi^{n\Delta t}(X)\|$.
- We renormalise the distance $\tilde{X}_n = \varphi^{n\Delta t}(X) + \frac{d_0}{d_n}(\hat{X}_n - \varphi^{n\Delta t}(X))$.
- The main asymptotic Lyapunov exponent is then

$$\underline{\lambda}(X) = \limsup_{n \to +\infty} \frac{1}{n\Delta t} \sum_{k=1}^{n} \ln \frac{d_k}{d_0}$$

To find the supremum limit, draw the plot of the sequence $(\frac{1}{n\Delta t} \sum_{k=1}^{n} \ln \frac{d_k}{d_0})_{n \in \mathbb{N}^*}$ within an horizontal line $y = \lambda$. Modify the value of λ until the line be on the upper bounds of the oscillations of the sequence at large values of n. This value is $\underline{\lambda}(X)$.

It is recommended to test several random values of δX (and if possible to consider the average value of $\underline{\lambda}(X)$ with respect to these different random perturbations).

Work to be done

- If your system is continuous or discrete in time:
  1. Code a function to compute the main asymptotic Lyapunov exponent at any point of the phase space.
  2. Code a function to colour the phase space (or a 2D Poincaré section) with a colour gradient indicating the value of the asymptotic Lyapunov exponent (or of the Lyapunov horizon).
  3. Find the borders of the regions for which the orbits are stable, and of the regions for which the orbits are unstable.

4. If you have found hyperbolic fixed points, draw their stable and unstable manifolds.
5. Code a function to compute the complete list of asymptotic Lyapunov exponents and the associated directions. Code a function to show the stable and unstable directions at any point of the phase space.

- If your system is a L-system:
  Compute the Lyapunov exponents governing the rate of growth of the word.
- If your system is a cellular automaton:
  Code functions to compute an evaluation of the mean Lyapunov exponent and compare the results for different definitions.