**Activity 1**

# Equation setting and linearisation

## Goals

Any Hamiltonian system has the behaviour of an harmonic oscillator in the neighbourhood of its stable equilibrium points. The harmonic oscillator model is then universal. But its use is restricted to linear phenomena. There is no a single universal model to describe nonlinear phenomena. But there exists a large set of nonlinear models which can be classified into some families of similar behaviours. None of these models is completely universal, but each of them has applications in a large amount of different applications (which can be in physics, chemistry, economy, biology,...).
The goal of this chapter is the introduction of a first kind of classification of the nonlinear dynamical systems and to study the linear part of these ones.

## Theory

### General definitions

**Definition 1** (Dynamical system)**.** *A dynamical system is the three kinds of data* $(\Gamma, \varphi, \mu)$ *where* $\Gamma$ *is a phase space describing the possible configurations (classical states) of the system,* $\varphi : \Gamma \to \Gamma$ *is a map called the flow of the dynamical system and describes the time evolution, and* $\mu$ *is a integration measure onto a* $\sigma$*-algebra* $\mathscr{T}$ *of* $\Gamma$*.*

A $\sigma$-algebra is a definition of the set of measurable open subsets of $\Gamma$, the measure $\mu$ is needed to define summons, averages and probabilities concerning the observables of the dynamical system:

**Definition 2** (Observables of a dynamical system)**.** *An observable of a dynamical system is a* $\mathbb{C}$*-valued function of* $\Gamma$ $\mu$*-almost everywhere bounded. The space of the observables is then*

$$L^\infty(\Gamma, d\mu) = \{f : \Gamma \to \mathbb{C}, \exists \mathrm{Dom} f \in \mathscr{T}, \mathrm{Dom} f \simeq \Gamma, \sup_{X \in \mathrm{Dom} f} |f(X)| < +\infty\}$$

"Almost everywhere" means that the eventual set $\varsigma$ on which the property is false has a zero measure $\mu(\varsigma) = 0$ (the set $\varsigma$ weights nothing in $\Gamma$ and has no physical meaning). $A \simeq B$ (with $A$ and $B$ two sets) means that the two sets are equal except on a part of zero measure ($\mu(A \backslash (A \cap B)) = \mu(B \backslash (A \cap B)) = 0$).

There is three kinds of dynamical systems:

- The continuous time dynamical systems which are characterised by a phase space being $\mathbb{R}^n$ (or a submanifold of $\mathbb{R}^n$) and by a flow which is solution of a first order differential equation.
- The discrete time dynamical systems which are characterised by a phase space being $\mathbb{R}^n$ (or a submanifold of $\mathbb{R}^n$) and by a flow defining a sequence by a first order recurrence relation.
- The symbolic dynamical systems which are characterised by a discrete phase space and by a flow defining rules of symbolic manipulation.

## Continuous time dynamical systems

**Definition 3** (Continuous time dynamical system)**.** *A continuous time dynamical system is the three kinds of data* $(\Gamma, \varphi^{t,t_0}, \mu)$ *where the phase space is* $\Gamma \subseteq \mathbb{R}^n$*, where* $\varphi^{t,t_0} : \Gamma \to \Gamma$ *is a continuous map such that* $\forall X \in \Gamma$*:*

- $\varphi^{t,t}(X) = X$
- $\varphi^{t,s} \circ \varphi^{s,r}(X) = \varphi^{t,s}(\varphi^{s,r}(X)) = \varphi^{t,r}(X)$

*(the flow is said to be a continuous semi-group of transformations), and where* $\mu$ *is a measure onto the Borel* $\sigma$*-algebra of* $\Gamma$ *(generally* $\mu$ *is the Lebesgue measure).*

(We recall that the Borel $\sigma$-algebra is the $\sigma$-algebra generated by the rectangles of $\mathbb{R}^n$ ($]a_1, b_1[\times ... \times]a_n, b_n[$); and that $d\mu(X) = w(X)dX^1...dX^n$ with $w : \Gamma \to \mathbb{R}^+$ a weight function, in the case of the Lebesgue measure with a cartesian coordinate system $w(X) = 1\ \forall X$).

Alternatively, the flow can be defined by a differential equation:

$$\begin{aligned} &X(t) \equiv \varphi^{t,t_0}(X(t_0)) \qquad X(t_0) \in \Gamma \\ &\dot{X}(t) = F(t, X(t)) \end{aligned}$$

where $F : \mathbb{R} \times \Gamma \to \Gamma$ is a continuous map called the generator of $\varphi$.

If the flow depends only on the duration $t - t_0$ and not on the initial time, $\varphi^{t,t_0} = \varphi^{t-t_0}$ (equivalently if $F$ is independent of the time), the dynamical system is said autonomous (in the contrary case, it is said driven). In fact, all driven dynamical system is equivalent to an autonomous dynamical system by the following transformation: we consider the extended phase space $\Gamma^+ = \mathbb{R} \times \Gamma$, with $Y = (Y^0, X) \in \mathbb{R} \times \Gamma$ ($Y^0$ being an auxiliary variable for the time), $F^+(Y) = (1, F(Y^0, X))$ and $d\mu^+(Y) = dY^0 d\mu(X)$. We have then

$$\dot{Y}(t) = F^+(Y(t)) \iff \dot{X}(t) = F(t, X(t)) \text{ with } Y^0(t) = t$$

For periodically driven system ($F(t + T, X) = F(t, X)$), it is more useful to choose $\Gamma^+ = \mathbb{S}^1 \times \Gamma$ with $\mathbb{S}^1$ the circle of the phase angle $\theta$ with $\theta(t) = \omega t \mod 2\pi$ ($\omega = \frac{2\pi}{T}$), $F^+(Y) = (\omega, F(\theta/\omega, X))$ and $d\mu^+(Y) = \frac{d\theta}{2\pi} d\mu(X)$.
Without loss of generality, we can then consider only the case of autonomous dynamical systems, for which the flow satisfies:

- $\varphi^0 = \mathrm{id}_\Gamma$
- $\varphi^t \circ \varphi^s = \varphi^{s+t}$
- $(\varphi^t)^{-1} = \varphi^{-t}$

A system modelled by a second order differential equation is equivalent to a dynamical system by the following transformation:

$$\begin{aligned} &x \in \mathbb{R}^\ell, \qquad \ddot{x} = K(\dot{x}, x, t) \\ &\iff \\ &X \equiv (x, p) \text{ with } p = \dot{x}, \qquad \dot{X} = F(X, t) \equiv (p, K(p, x, t)) \end{aligned}$$

The phase space is then $\Gamma = \mathbb{R}^{2\ell}$. A mechanical system described by a Hamiltonian $\mathscr{H}(q, p) = \sum_{i=1}^{\ell} \frac{(p^i)^2}{2m} + V(q)$ defines than an autonomous dynamical system $(\mathbb{R}^{2\ell}, F, dq^1...dq^\ell dp^1...dp^\ell)$ with $F(q, p) =$

$\left(\frac{\partial \mathscr{H}}{\partial p}, -\frac{\partial \mathscr{H}}{\partial q}\right)$ and which is governed by the Hamilton equations:

$$\begin{array}{rcl} \dot{q}^i & = & \dfrac{\partial \mathscr{H}}{\partial p^i} \\ \dot{p}^i & = & -\dfrac{\partial \mathscr{H}}{\partial q^i} \end{array}$$

We can note that $\mathscr{H} \in L^\infty(\mathbb{R}^{2\ell}, dq^1...dq^\ell dp^1...dp^\ell)$.

**Definition 4** (Conservative dynamical system)**.** *An autonomous dynamical system $(\Gamma, \varphi^t, \mu)$ is said to be conservative (or Hamiltonian) if the flow preserves the measure, i.e. $\forall O \in \mathscr{T}$, $\forall t$, $\mu(\varphi^t(O)) = \mu(O)$.*

**Theorem 1** (Liouville Theorem)**.** *The dynamical systems defined by Hamiltonians $\mathscr{H} \in L^\infty(\mathbb{R}^{2\ell}, dqdp)$ are conservative.*

***Proof :***

$$\mu(\varphi^t(D)) = \int_{\varphi^t(D)} dqdp = \int_D |\mathcal{J}(q,p|q_{-t},p_{-t})| dq_{-t}dp_{-t}$$

with $(q_{-t}, p_{-t}) = \varphi^{-t}(q,p)$ and the following Jacobian of transformation

$$\mathcal{J}(q,p|q_{-t},p_{-t}) = \begin{vmatrix} \frac{\partial q}{\partial q_{-t}} & \frac{\partial p}{\partial q_{-t}} \\ \frac{\partial q}{\partial p_{-t}} & \frac{\partial p}{\partial p_{-t}} \end{vmatrix}$$

For all invertible matrix we have $\det(A^{-1}) = \frac{1}{\det(A)}$ and by substitution of $t$ by $-t$, we can study the following Jacobian

$$\mathcal{J}(q_t,p_t|q,p) = \begin{vmatrix} \frac{\partial q_t}{\partial q} & \frac{\partial p_t}{\partial q} \\ \frac{\partial q_t}{\partial p} & \frac{\partial p_t}{\partial p} \end{vmatrix}$$

Let $h$ be in a neighbourhood of zero. We have

$$\begin{array}{rcl} \mathcal{J}(q_{t+h},p_{t+h}|q,p) & = & \det\left(\begin{pmatrix} \frac{\partial q_t}{\partial q} & \frac{\partial p_t}{\partial q} \\ \frac{\partial q_t}{\partial p} & \frac{\partial p_t}{\partial p} \end{pmatrix} + h \begin{pmatrix} \frac{\partial \dot{q}_t}{\partial q} & \frac{\partial \dot{p}_t}{\partial q} \\ \frac{\partial \dot{q}_t}{\partial p} & \frac{\partial \dot{p}_t}{\partial p} \end{pmatrix} + \mathcal{O}(h^2)\right) \\ & = & \mathcal{J}(q_t,p_t|q,p) \det\left(id + h \begin{pmatrix} \frac{\partial q}{\partial q_t} & \frac{\partial p}{\partial q_t} \\ \frac{\partial q}{\partial p_t} & \frac{\partial p}{\partial p_t} \end{pmatrix} \begin{pmatrix} \frac{\partial \dot{q}_t}{\partial q} & \frac{\partial \dot{p}_t}{\partial q} \\ \frac{\partial \dot{q}_t}{\partial p} & \frac{\partial \dot{p}_t}{\partial p} \end{pmatrix} + \mathcal{O}(h^2)\right) \\ & = & \mathcal{J}(q_t,p_t|q,p) \det\left(id + h \begin{pmatrix} \frac{\partial \dot{q}_t}{\partial q_t} & \frac{\partial \dot{p}_t}{\partial q_t} \\ \frac{\partial \dot{q}_t}{\partial p_t} & \frac{\partial \dot{p}_t}{\partial p_t} \end{pmatrix} + \mathcal{O}(h^2)\right) \end{array}$$

but $\det(1 + hA) = 1 + h\mathrm{tr}(A) + \mathcal{O}(h^2)$ for all matrix $A$. We have then

$$\mathcal{J}(q_{t+h},p_{t+h}|q,p) = \mathcal{J}(q_t,p_t|q,p)\left(1 + h\sum_i \left(\frac{\partial \dot{q}_t^i}{\partial q_t^i} + \frac{\partial \dot{p}_{it}}{\partial p_{it}}\right) + \mathcal{O}(h^2)\right)$$

By using the Hamilton equations, we have

$$\frac{\partial \dot{q}_{it}}{\partial q_{it}} + \frac{\partial \dot{p}_{it}}{\partial p_{it}} = \frac{\partial^2 \mathcal{H}}{\partial p_{it} \partial q_t^i} - \frac{\partial^2 \mathcal{H}}{\partial p_{it} \partial q_t^i} = 0$$

Finally

$$\frac{d}{dt}\mathcal{J}(q_t,p_t|q,p) = \lim_{h \to 0} \frac{\mathcal{J}(q_{t+h},p_{t+h}) - \mathcal{J}(q_t,p_t|q,p)}{h} = 0$$

Then

$$\mathcal{J}(q_t,p_t|q,p) = \mathcal{J}(q_0,p_0|q,p) = \mathcal{J}(q,p|q,p) = 1$$

We conclude by

$$\mu(\varphi^t(D)) = \int_D dq_{-t}dp_{-t} = \mu(D)$$

since the integration variables are dummy. ■

**Definition 5** (Orbit)**.** *Let $\varphi^t$ be a flow on a phase space $\Gamma$. We call orbit of $X_0 \in \Gamma$ the whole future phase trajectory starting from $X_0$, i.e.* $\mathrm{Orb}(X_0) = \{\varphi^t(X_0)\}_{t \in [0,+\infty[}$.

**Definition 6** (Fixed point)**.** *Let $\varphi^t$ be a flow on a phase space $\Gamma$ (of generator $F$). A fixed point $X_* \in \Gamma$ is a point such that $\forall t$, $\varphi^t(X_*) = X_*$ (the orbit of $X_*$ is reduced to a single point), or equivalently such that $F(X_*) = 0$.*

Let $t \mapsto X(t)$ be a phase trajectory in the neighbourhood of a fixed point $X_*$. We set

$$X(t) = X_* + \delta X(t) + \mathcal{O}(\|\delta X\|^2)$$

Then:

$$\begin{aligned}
\delta \dot{X} &= \dot{X} + \mathcal{O}(\|\delta X\|^2) \\
&= F(X_* + \delta X) + \mathcal{O}(\|\delta X\|^2) \\
&= F(X_*) + \sum_{a=1}^{n} \left. \frac{\partial F}{\partial X^a} \right|_{X=X_*} \delta X^a + \mathcal{O}(\|\delta X\|^2) \\
&= \sum_{a=1}^{n} \left. \frac{\partial F}{\partial X^a} \right|_{X=X_*} \delta X^a + \mathcal{O}(\|\delta X\|^2)
\end{aligned}$$

Let $\partial F_{X_*}$ be the Jacobian matrix of $F$ at $X_*$:

$$(\partial F_{X_*})^a{}_b = \left. \frac{\partial F^a}{\partial X^b} \right|_{X=X_*}$$

$\delta X(t)$ is described by a linear dynamical system of equation

$$\delta \dot{X} = (\partial F_{X_*}) \delta X \iff \delta \dot{X}^a = \sum_{b=1}^{n} (\partial F_{X_*})^a{}_b \delta X^b$$

with a neighbourhood of $0 \in \mathbb{R}^n$ as phase space. This system is called linearisation of $(\Gamma, F, \mu)$ in the neighbourhood of $X_*$.

## Discrete time dynamical systems

**Definition 7** (Discrete time dynamical system)**.** *A discrete time dynamical system is the three kinds of data $(\Gamma, \varphi, \mu)$ where the phase space is $\Gamma \subseteq \mathbb{R}^n$, where $\varphi : \Gamma \to \Gamma$ is a continuous map defining sequences in $\Gamma$ by the recurrence relation:*

$$X_{n+1} = \varphi(X_n)$$

*and where $\mu$ is measure onto the Borel $\sigma$-algebra of $\Gamma$ (generally $\mu$ is the Lebesgue measure).*

(We recall that the Borel $\sigma$-algebra is the $\sigma$-algebra generated by the rectangles of $\mathbb{R}^n$ ($]a_1, b_1[\times \ldots \times ]a_n, b_n[$); and that $d\mu(X) = w(X) dX^1 \ldots dX^n$ with $w : \Gamma \to \mathbb{R}^+$ a weight function, in the case of the Lebesgue measure with a cartesian coordinate system $w(X) = 1 \ \forall X$).

The map $F : \Gamma \to \Gamma$ defined by $F(X) = \varphi(X) - X$ ($F(X_n) = X_{n+1} - X_n$) is called generator of $\varphi$ by finite difference.

There is a special class of discrete time dynamical system:

**Definition 8** (Logistic flow). *A logistic flow is a dynamical system* $([a,b], f, dx)$ *where* $[a,b] \subset \mathbb{R}$ *and* $f : [a,b] \to [a,b]$ *is a continuous unimodal function (which means* $f$ *has one and only one local extremum in* $]a,b[$, $x_c$ *called critical point).*

**Definition 9** (Conservative dynamical system). *A dynamical system* $(\Gamma, \varphi, \mu)$ *is said to be conservative (or Hamiltonian) if the flow preserves the measure, i.e.* $\forall O \in \mathscr{T}$, $\mu(\varphi(O)) = \mu(O)$.

**Definition 10** (Orbit). *Let* $\varphi$ *be a flow on a phase space* $\Gamma$. *We call orbit of* $X_0 \in \Gamma$ *the whole future phase trajectory starting from* $X_0$, *i.e.* $\mathrm{Orb}(X_0) = \{\varphi^n(X_0)\}_{n\in\mathbb{N}}$ *where* $\varphi^n = \varphi \circ ... \circ \varphi$ *and* $\varphi^0 = \mathrm{id}_\Gamma$..

**Definition 11** (Fixed point). *Let* $\varphi$ *be a flow on a phase space* $\Gamma$. *A fixed point* $X_* \in \Gamma$ *is a point such that* $\varphi(X_*) = X_*$ *(the orbit of* $X_*$ *is reduced to a single point), or equivalently such that* $F(X_*) = 0$ *(*$F = \varphi - \mathrm{id}_\Gamma$*).*

Let $(X_n)$ be a phase trajectory in the neighbourhood of a fixed point $X_*$. We set $X_n = X_* + \delta X_n + \mathcal{O}(\|\delta X\|^2)$. By a Taylor expansion we have

$$\varphi(X_* + \delta X_n) = \varphi(X_*) + (\partial\varphi_{X_*})\delta X_n + \mathcal{O}(\|\delta X\|^2)$$

where $\partial\varphi_{X_0}$ is the Jacobian matrix of $\varphi$ at $X_0$:

$$(\partial\varphi_{X_*})^a{}_b = \left.\frac{\partial\varphi^a}{\partial X^b}\right|_{X=X_*}$$

We have then

$$X_* + \delta X_{n+1} = X_* + (\partial\varphi_{X_*})\delta X_n + \mathcal{O}(\|\delta X\|^2)$$

$(\delta X_n)$ is then described by a linear dynamical system of recurrence relation

$$\delta X_{n+1} = (\partial\varphi_{X_*})\delta X_n$$

with a neighbourhood of $0 \in \mathbb{R}^n$ as phase space. This system is called linearisation of $(\Gamma, \varphi, \mu)$ in the neighbourhood of $X_*$.

In some situations, it is useful to introduce the generator of the linearisation $\partial F_{X_*}$ defined by $\partial\varphi_{X_*} = e^{\partial F_{X_*}}$ (or equivalently $\partial F_{X_*} = \mathrm{Log}(\partial\varphi_{X_*})$, with the matrix exponential and the matrix logarithm).

## Symbolic dynamical systems

**Definition 12** (Symbolic dynamical systems). *A symbolic dynamical system is the three kinds of data* $(\Gamma, \varphi, \mu)$ *where* $\Gamma$ *is a set of sequences or strings of elements chosen in an alphabet* $\mathscr{A}$ *of several symbols, where* $\varphi$ *is a generating rule permitting to built a sequence of strings of characters in* $\mathscr{A}$ *(so called words), and where* $\mu$ *is the cardinality onto the power set of* $\Gamma$.

For a set $O$ of elements in $\Gamma$, $\mu(O)$ is just the number of elements in $O$.

Simple examples of symbolic dynamical systems are the L-systems which are defined by an inflation rule: Let $\mathscr{A} = \{R, L\}$ be an alphabet of two symbols, $\Gamma = \bigcup_{n\geq 1} \mathscr{A}^n$, and $\varphi$ a substitution rule:

$$\varphi : R \to \rho \;\&\; L \to \lambda$$

where $\rho, \lambda \in \bigcup_{n\geq 1} \mathscr{A}^n$ are strings of several symbols. By example consider the rule $\varphi : R \to RL$ & $L \to R$ with a string initialised with a single symbol $w_1 = R$. The iterative construction of the word of the dynamical system is

$$\begin{aligned} w_1 &= R \\ w_2 &= RL \\ w_3 &= RLR \\ w_4 &= RLRRL \\ w_5 &= RLRRLRLR \\ w_6 &= RLRRLRLRRLRRL \\ &\vdots \end{aligned}$$

A L-system defined by an inflation rule $\varphi : R \to \rho$ & $L \to \lambda$ is associated with a matrix

$$\partial\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{N})$$

where $a$ is the number of $R$ in the pattern $\rho$, $b$ is the number of $R$ in the pattern $\lambda$, $c$ is the number of $L$ in the pattern $\rho$ and $d$ is the number of $L$ in the pattern $\lambda$ (this can be easily generalised with an alphabet with a larger number of symbols). If at some iteration a word presents $n_R$ symbols $R$ and $n_L$ symbols $L$, at the following iteration the numbers of the symbols are:

$$\begin{pmatrix} n_R \\ n_L \end{pmatrix} \to \partial\varphi \begin{pmatrix} n_R \\ n_L \end{pmatrix} = \begin{pmatrix} an_R + bn_L \\ cn_R + dn_L \end{pmatrix}$$

The dynamical system obtained by the recurrence rule

$$X_{n+1} = \partial\varphi X_n$$

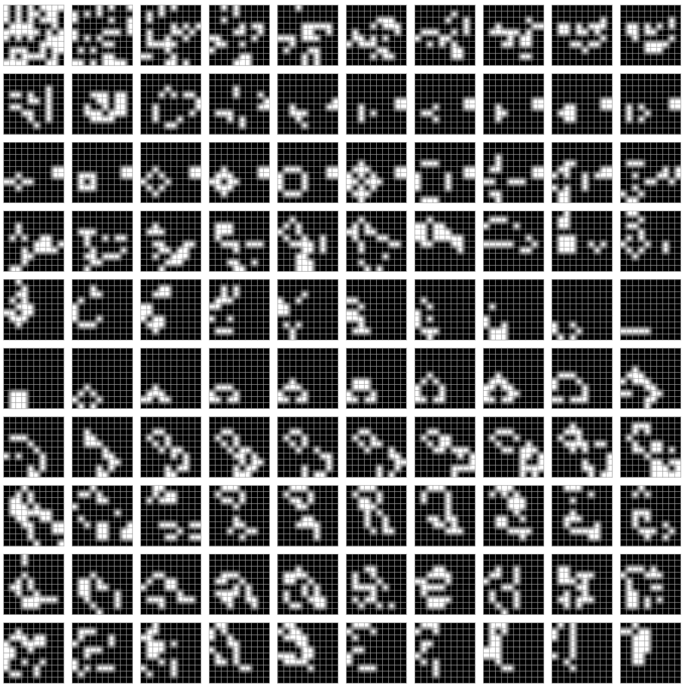
with $X_n$ a two dimensional $\mathbb{N}$-valued vector, is the linearisation of the L-system.
For some L-systems, we distinguish in the alphabet the variable symbols (which are replaced by a pattern by the substitution rule) and the constant symbols (which are unchanged by the substitution rule).

**Definition 13** (Cellular automata). *A cellular automaton is an homogeneous grid $\mathfrak{X}$ of cells which can have $p$ states called colours, and an evolution rule permitting to built the grid at the iteration $n+1$ knowing the colours of the cells at the iteration $n$.*

Often, we consider only two colours (white and black), and the rule defines the colour of a particular cell at the iteration $n+1$ with respect to the colours of the adjacent cells at the iteration $n$.

An example of cellular automaton is the Game of Life. $\mathfrak{X}$ is a 2D grid, the two colours are said alive and dead, and the evolution rule is the following:

- if a cell is alive and is surrounded by two or three alive cells, then it remains alive at the next iteration, otherwise it dies.
- if a cell is dead and is surrounded by exactly three alive cells, then it reborn, otherwise it remains dead.



*Evolution of the Game of Live automaton (from left to right and from top to bottom) with a random initial condition.*

Formally, a 2D cellular automaton is defined by

- an alphabet $\mathscr{A} = \{0, 1, ..., p-1\}$ (with $p \geq 1$) representing the $p$ colours (for example $p = 0$ for black and $p = 1$ for white).
- the phase space is $\Gamma = \mathscr{A}^{\mathbb{Z}^2} = \{(c_{nm})_{(nm)\in\mathbb{Z}^2}, c_{nm} \in \mathscr{A}\}$ ($\mathscr{A}$-valued sequences with double indices).
- a local rule $F : \mathscr{A}^s \to \mathscr{A}$ providing the colour of a cell at the next iteration knowing the colours of the $s$ cells used in the evolution rule.
- the global transition rule $\varphi : \mathscr{A}^{\mathbb{Z}^2} \to \mathscr{A}^{\mathbb{Z}^2}$ defined by

  $$\forall c \in (\mathscr{A})^{\mathbb{Z}^2}, \quad \varphi(c)_{n_1,n_2} = F(c_{n_1+q_1^1,n_2+q_2^1}, ..., c_{n_1+q_1^s,n_2+q_2^s})$$

  where $(q_1^i, q_2^i) \in \mathbb{Z}^2$ are the relative coordinates of the $i$-th cell involved by the evolution rule. $\varphi(c)_{n_1,n_2}$ is the colour of the cell of coordinates $(n_1, n_2)$ after one iteration of the evolution starting from the configuration $c$.

A 2D cellular automaton ($\Gamma = \mathscr{A}^{\mathbb{Z}^2}$) can be converted to a 1D cellular automaton ($\Gamma = \mathscr{A}^{\mathbb{Z}}$) by using the Cantor pairing map $b : \mathbb{N}^2 \to \mathbb{N}$:

$$b(n_1, n_2) = \frac{(n_1 + n_2 + 1)(n_1 + n_2)}{2} + n_1$$

and the map $p : \mathbb{Z} \to \mathbb{N}$:

$$p(n) = \begin{cases} 2n & \text{si } n \geq 0 \\ -2n + 1 & \text{si } n < 0 \end{cases}$$

$p^{-1} \circ b \circ (p \times p) : \mathbb{Z}^2 \to \mathbb{Z}$ is bijective. The Cantor pairing map corresponds to :

| $n_2 \downarrow$ $n_1 \rightarrow$ | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 3 | 6 | |
| 1 | 2 | 4 | 7 | 11 | |
| 2 | 5 | 8 | 12 | 17 | |
| 3 | 9 | 13 | 18 | 24 | |
| $\vdots$ | | | | | |

$p^{-1} \circ b \circ (p \times p)$ induces a map $B : \mathscr{A}^{\mathbb{Z}^2} \to \mathscr{A}^{\mathbb{Z}}$ defined by $B(c)_n = c_{(p^{-1} \times p^{-1}) \circ b^{-1} \circ p(n)}$.
Note that for a finite 2D grid, $\mathbb{Z}^2$ can be replaced by $\{1, 2, ..., N_r\} \times \{1, 2, ..., N_c\}$ (where $N_r$ and $N_d$ are the numbers of rows and of columns of the grid). In that case, the conversion to a 1D grid can be also realised by concatenate the rows one after the other.

Simple examples of 1D cellular automata are the "Wolfram rules" which are defined by an alphabet of two colours (black and white) and a substitution rule $\varphi : c_{i-1}c_ic_{i+1} \to xf(c_{i-1}, c_i, c_{i+1})x$ which defines the colour of a cell at the next iteration with respect to the colours of itself and of the two adjacent cells at the previous iteration.

We can introduce the addition of two sequences of $\mathscr{A}^{\mathbb{Z}}$ by $(c+q)_n = c_n + q_n \mod p$ (with $p$ the number of colours of $\mathscr{A}$). Generally, $\varphi$ is not linear: $\varphi(c+q) \neq \varphi(c) + \varphi(q)$. Since $\mathscr{A}$ is a discrete set, it is not possible to linearise $\varphi$.

## Algorithms

A fixed point $X_* \in \mathbb{R}^n$ is solution of $F(X_*) = 0$. We can solve numerically this equation by the Newton algorithm which consists to consider the following sequence:

$$X_{n+1} = X_n - (\partial F_{X_n})^{-1} F(X_n)$$

where $\partial F_{X_n}$ is the Jacobian matrix of $F$. We see that $X_*$ is a fixed point of the sequence, and then if it converges in the neighbourhood of $X_*$, it is to $X_*$.

**Theorem 2** (Convergence of the Newton method)**.** *If $F$ is $\mathcal{C}^2$ and if $\det \partial F_{X_*} \neq 0$ then $\exists b > 0$ (called radius of convergence) and $\exists \beta > 0$ such that*

- $\|X_0 - X_*\| < b \Rightarrow \|X_n - X_*\| < b$ *($\forall n$).*
- $\lim_{n \to +\infty} X_n = X_*$
- $\|X_{n+1} - X_*\| \leq \beta \|X_n - X_*\|^2$

This theorem ensures that if we choose $X_0$ at a distance lower than $b$ from the searched solution, then the Newton sequence of approximations of $X_*$ remains in this neighbourhood and converges quadratically to $X_*$.
The difficulty is that $b$ is generally unknown and that it needs to choose $X_0$ as a rough approximation of $X_*$.

## Work to be done

1. Choose a dynamical system to study in the list of the proposed systems on the moodle page:
   https://moodle.univ-fcomte.fr/mod/glossary/view.php?id=346060
   and choose an application field (a concrete phenomenon modelled by the dynamical system).

2. Your chosen dynamical system is described by an equation (by an Hamiltonian, or by symbolic rules) concerning some variables and depending on some parameters (appearing in the coefficients of the equation but not as variables). By a bibliographic research, find the concrete physical interpretations of these different variables and of these different parameters in your chosen application field.
   Your dynamical system can be a predictive model for your application, its equation comes from the fundamental laws of the application field by simplifications, approximations and idealisations. If it is the case, search in the literature the initial fundamental laws and the framework of simplifications, approximations and idealisations for which your model is given (it is not necessary to find the rigorous mathematical calculus to pass from the fundamental laws to the equation of the dynamical system).
   But your dynamical system can be a phenomenological model of your application, its equation has been set to reproduce experimental or observational results without reference to fundamental laws. In that case, search in the literature the experimental or observational conditions which are described by your dynamical system.

3. If it is not directly the case, rewrite the definition of your chosen system in the standard form of a dynamical system, i.e. find explicitly the phase space $\Gamma$ (and the alphabet $\mathscr{A}$ for a symbolic system) and find:
   - the generator $F$ of the autonomous system if your system is continuous in time (if your system is driven, find its autonomous representation);
   - the flow $\varphi$ if your system is discrete in time;
   - or the generating rule if your system is symbolic (if this one is a cellular automaton find the local rule $F$ and the global transition rule $\varphi$).

4. - For the non-symbolic systems, try to find analytically the fixed points of your dynamical system. Code a small program to find the fixed points of your system and compare with the analytic results (if these were successful).
   - For symbolic systems, try to find patterns stable by the iteration (maybe by simplifying your system by reduction of the generating rule and/or by choosing a small grid). Code a small program searching stable patterns by testing successively all the possibilities or by testing a lot of random configurations (generally, these needs to consider a simplified version of the system). You can also search in the literature the existence of known stable patterns, in that case verify explicitly the stability by applying the evolution rule.

5. Compute analytically the linearisation of your system (the Jabobian matrix $\partial F$ at each fixed point for the continuous time systems, the Jacobian matrix $\partial\varphi$ at each fixed point for the discrete time systems, or the matrix of the inflation rule $\partial\varphi$ for a L-system).