Activity 4

Typology of bounded orbits

Goals

Dynamical systems exhibits special orbits which are the manifestation of the specific properties of the system. Until now, the only viewed special orbits are the fixed points (orbits reduced to a single point). We are now interested by the other special orbits as cyclic orbits (the orbit returns to its starting point) or quasi-cyclic orbits (the orbit returns to the neighbourhood of its starting point).
This activity does not concern the L-systems.

Theory

General results and definitions

Theorem 4 (Poincaré recurrence theorem)**.** *Let (Γ, φ, μ) be a conservative dynamical system (discrete or continuous in time). We suppose that there exists a region $\Sigma \in \mathscr{T}$, of finite measure $\mu(\Sigma) < +\infty$ and stable by the flow: $\forall t$, $\varphi^t(\Sigma) \subset \Sigma$. Let $X_* \in \Sigma$. Then μ-almost all points of the neighbourhood U of X_* will return in this neighbourhood, i.e. there exists a time $T > 0$ (real for continuous time systems or integer for discrete time systems) such that $\varphi^T(X) \in U$ for μ-almost all $X \in U$.*

Proof : We proceed by *reductio ad absurdum*. Let $W \subset U$ be the set of points which never return in U, that is to say for which $\forall n > 0$, $\varphi^n(W) \cap U = \varnothing$. We moreover suppose that this region is not of zero measure $\mu(W) > 0$. Let $n \in \mathbb{N}$, $n \geq 1$. $\varphi^n(W) \cap \varphi^{n-1}(W) = \varnothing$ because $\varphi^{-(n-1)}\left(\varphi^n(W) \cap \varphi^{n-1}(W)\right) = \varphi(W) \cap W = \varnothing$. We have then

$$\mu\left(\bigcup_{n=1}^{+\infty} \varphi^n(W)\right) = \sum_{n=1}^{+\infty} \mu(\varphi^n(W))$$

(the domains being disjoint, the measure of their region is the addition of their measures). But the system being conservative, we have $\mu(\varphi^n(W)) = \mu(W)$, and it follows that

$$\mu\left(\bigcup_{n=1}^{+\infty} \varphi^n(W)\right) = \sum_{n=1}^{+\infty} \mu(W) = +\infty$$

but

$$\bigcup_{n=1}^{+\infty} \varphi^n(W) \subset \Sigma \Rightarrow \mu\left(\bigcup_{n=1}^{+\infty} \varphi^n(W)\right) \leq \mu(\Sigma) \Rightarrow +\infty \leq \mu(\Sigma)$$

Σ being of finite measure, we have a contradiction. The starting hypothesis is then false, and then $\mu(W) = 0$. This proof can be applied to any time discretisation, its result can be then extended to the continuous time systems. ■

Due to this theorem, for the conservative systems bounded in a finite region of the phase space (with respect to the measure), μ-almost all orbits are recurrent. The observables are then recurrent (including periodic and quasi-periodic cases) without transient regime. For $\epsilon > 0$, the Poincaré recurrence time $T_\epsilon(X)$ at a point $X \in \Sigma$ (not a fixed point) is defined as

$$T_\epsilon(X) = \inf\left\{t > t_\epsilon^{\text{exit}}(X) \text{ s.t. } \|\varphi^t(X) - X\| < \epsilon\right\} \text{ with } t_\epsilon^{\text{exit}} = \inf\{t > 0 \text{ s.t. } \|\varphi^t(X) - X\| > \epsilon\}$$

Note that ϵ must be sufficiently small to t_ϵ^{exit} exists. If $T_\epsilon(X)$ is independent of ϵ, X is periodic (the orbit returns to X after a time $T(X)$). If $T_\epsilon(X)$ depends on ϵ but with $|T_\epsilon(X) - T_\epsilon(X')| < \epsilon$ if $\|X - X'\| < \epsilon$ (for ϵ sufficiently small), X is quasi-periodic (the orbit returns in the neighbourhood of X after $T_\epsilon(X)$ and anew after $2T_\epsilon(X)$ and anew after $3T_\epsilon(X)$, etc). If $T_\epsilon(X)$ depends on ϵ and strongly on X, X is aperiodic (the orbit returns first in the neighbourhood of X after $T_\epsilon(X)$ but the second return is after a time completely different from $T_\epsilon(X)$, the third return is after a different time from the two previous ones, etc). For quasi-periodic and aperiodic points, $\lim_{\epsilon\to 0} T_\epsilon(X) = +\infty$ (for periodic point by definition $\lim_{\epsilon\to 0} T_\epsilon(X) = T(X) < +\infty$).

The interesting cases of non-conservative dynamical systems are the dissipative ones for which $\mu(\varphi^t(D)) \leq \mu(D)$ ($\forall t,\ D \in \mathscr{T}$), since their orbits can be bounded. This kind of systems present a transient regime (during which a dissipation occurs) and a permanent regime characterised by dynamics taking place in small regions of the phase space:

Definition 18 (Attractor). *Let (Γ, φ, μ) be a (discrete or continuous) time dynamical system. An attractor of the system is a region of the phase space $A \subset \Gamma$ in which converges all phase trajectory passing in a particular neighbourhood Ω of A, called basin of attraction, in other words $\lim_{t\to+\infty} \varphi^t(\Omega) = A$ (Ω being the maximal set of $\mathscr{T}$ satisfying this property).*

A dynamics reaches the permanent regime when the orbit reaches the attractor. The attractors are classified into four categories:

- If the attractor is reduced to be a single point, it is called limit fixed point.
- If the attractor is a closed curve, it is called limit cycle.
- If the attractor is topologically a torus, it is called limit torus.
- If the attractor does not correspond to any of the previous morphologies, it is called strange attractor.

Note that a topological torus is a manifold which can be transformed to a regular torus $\mathbb{T}^m = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ (cartesian product of m circles) by continuous deformations:

A regular 2-torus $\mathbb{T}^2$ and two other manifolds topologically equivalent to a 2-torus.

The notion of attractor can be extended to symbolic systems. For example if after some iterations, the word becomes the repetition of a pattern, this one is a limit cycle. We can have also periodic or quasiperiodic appearance of a pattern in a cellular automaton.

Continuous time dynamical systems

Definition 19 (Cycle)**.** *A closed curve C in Γ is a cycle of period $T \in \mathbb{R}^{+*}$, if $\forall X_* \in C$, $\varphi^t(X_*) \in C$ ($\forall t$) and $\varphi^T(X_*) = X_*$.*

The points of a cycle are called cyclic points. By definition if X_* belongs to a cycle C, then $\mathrm{Orb}(X_*) = C$. The observables onto a cycle are T-periodic.

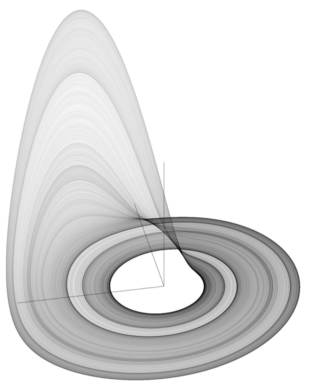
Definition 20 (Invariant torus)**.** *A manifold $A \subset \Gamma$ topologically equivalent to a n-torus $\mathbb{T}^n$ ($n \in \mathbb{N}_{\geq 2}$) is an invariant torus if $\forall X_* \in A$, $\varphi^t(X_*) \in A$ ($\forall t$).*

A dynamics onto an invariant n-torus is characterised by n frequencies. For example with A a 2-torus and $X_* \in A$, we have two periods T_1 and T_2 (times needed to the orbit makes one turn respectively around the minor circle and around the major circle of the torus). If $\frac{T_1}{T_2} = \frac{q}{p} \in \mathbb{Q}$ (with $q, p \in \mathbb{N}$ and $\frac{q}{p}$ the irreducible fraction), then $pT_1 = qT_2$ and the orbit make p turns around the minor circle along with q turns around the major circle. So $\varphi^{pT_1}(X_*) = \varphi^{qT_2}(X_*) = X_*$ and so $\mathrm{Orb}(X_*)$ is a cycle of period $qT_1 = pT_2$ warps around the torus A. But if $\frac{T_1}{T_2} \notin \mathbb{Q}$, $\mathrm{Orb}(X_*) = A$ (the orbit visits all points of A - we said that it is ergodic in A - and is an open infinite curve warped around A). In this case, X_* is quasi-periodic.
Limit cycles and limit tori of dissipative systems are cycles and invariant tori.

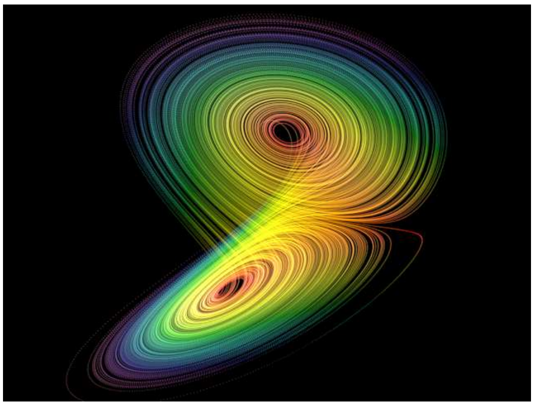
Definition 21 (Homoclinic and heteroclinic orbits)**.** *Let $O = \{\varphi^t(X_0)\}_{t \in \mathbb{R}}$ be a (past and future) orbit. We said that:*

- *O is homoclinic if $\lim_{t \to \pm\infty} \varphi^t(X_0) = X_*$ where X_* is a fixed point of the flow.*
- *O is heteroclinic if $\lim_{t \to +\infty} = X_{*+}$ and $\lim_{t \to -\infty} = X_{*-}$ with X_{*+} and X_{*-} two distinct fixed points.*

Homoclinic orbits are kind of cycles of infinite period (since the movement stop at X_* in the past and in the future). A heteroclinic orbit links two fixed points, it flees the first one and goes to the second one. Generally (always ?) there exists a twin heteroclinic orbit which flees the second fixed point and goes to the first one.
In practice it is highly difficult to find the homoclinic and heteroclinic orbits of a dynamical system (if they exist). But the orbits in the neighbourhood of these special orbits have very specific morphologies. In the neighbourhood of a homoclinic orbit, the other orbits have the shape of a "leaf-seashell":



In the neighbourhood of a heteroclinic orbit, the other orbits have the shape of "butterfly wings":



The "seashell" and the "butterfly" are strange attractors of the dynamical system.

Discrete time dynamical systems

Definition 22 (Cyclic point)**.** *A point $X_* \in \Gamma$ is said to be cyclic of period $p \in \mathbb{N}_{\geq 2}$ if $\varphi^p(X_*) = X_*$.*

By definition, the orbit of a p-cyclic points X_* has only p elements: $\mathrm{Orb}(X_*) = \{X_*, \varphi(X_*), ..., \varphi^{p-1}(X_*)\}$.

Definition 23 (Invariant cycle)**.** *A closed curve C in Γ is an invariant cycle, if $\forall X_* \in C$, $\varphi(X_*) \in C$.*

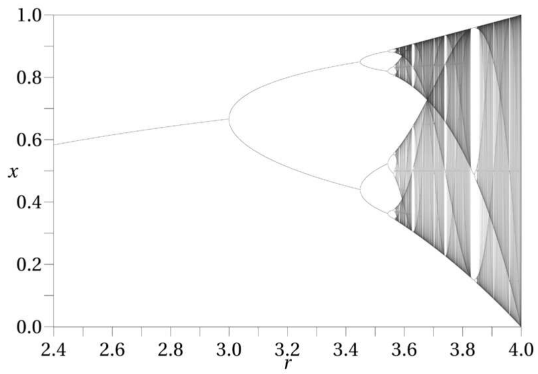
The points on C can be cyclic, but in general they are only quasi-periodic.

Definition 24 (Invariant torus)**.** *A manifold $A \subset \Gamma$ topologically equivalent to a n-torus $\mathbb{T}^n$ ($n \in \mathbb{N}_{\geq 2}$) is an invariant torus if $\forall X_* \in A$, $\varphi(X_*) \in A$.*

Anew, in general the points of an invariant torus are quasi-periodic.
Limit cycles and limit tori are invariant cycles and invariant tori.

Theorem 5 (Fatou theorem)**.** *Let (I, f, dx) be a logistic flow where f is a degree 2 polynomial. If the flow has a stable periodic orbit, this one includes the critical point x_c (single local extremum of f into I).*

Generally, the critical point of a logistic flow is then a cyclic point.
If the sequence $x_{n+1} = f(x_n)$ converges, the limit is a fixed point x_*. If the sequence neither converges nor diverges, after a transient regime it can oscillates with p-cyclic points $\{x_{*1}, ...x_{*p}\}$ with $x_{*n+1} = f(x_{*n})$ (and $f(x_{*p}) = x_{*1}$), which is a limit cyclic orbit. It can also oscillate erratically. To enlighten the behaviour of a logistic flow, we can draw its bifurcation diagram. For a large set of initial conditions x_0, we compute the limit values x_N (for a very large N), and we plot the position of all x_N. We reproduce this computation by varying a parameter r of the logistic flow. The obtained figure (with position of the different x_N for ordinates and values of r for abscissae) is called bifurcation diagram:



In this example, we see that for values of $r < 3$ we have only one limit value, and so the sequence converges to a limit fixed point. But for $3 < r < 3.4$, we have two limit values, corresponding to 2-periodic cyclic points. At $r \approx 3$, the topology of the attractor changes (the limit fixed point becomes a limit 2-periodic orbit). This point is said to be a bifurcation. For large values of r, we observe a lot of bifurcations implying that the sequence has now an erratic behaviour. This phenomenon is called a cascade of bifurcations.

Symbolic dynamical systems

Due to the discrete nature of the phase space of a cellular automaton $\Gamma = \mathscr{A}^{\mathbb{Z}^2}$, it needs to introduce a distance associated with this topology in order to define the attractors.

Definition 25 (Tychonoff distance)**.**

$$\forall c, k \in \mathscr{A}^{\mathbb{Z}^2}, \quad d(c,k) = \begin{cases} 0 & \textit{if } c = k \\ 2^{-\langle c|k \rangle_\infty} & \textit{otherwise} \end{cases}$$

with

$$\langle c|k \rangle_\infty = \min\{\max(|n_1|, |n_2|), \ \textit{for } (n_1, n_2) \ \textit{such that } c_{n_1,n_2} \neq k_{n_1,n_2}\}$$

$\langle c|k \rangle_\infty$ is the distance on $\mathbb{Z}^2$ between the cell $(0,0)$ and the closest cell having a colour different between the configurations c and k. Implicitly, we suppose that the point of view on the automaton is centred on the cell $(0,0)$. So if $\langle c|k \rangle_\infty$ is small, we see quickly differences, and then c and k are highly distant $d(c,k) \simeq 1$; whereas if $\langle c|k \rangle_\infty \gg 1$ the two configuration are almost the same for the point of view and then $d(c,k) \simeq 0$.

A non-empty set of configurations $A \subset \mathscr{A}^{\mathbb{Z}^2}$ is an attractor of the cellular automaton, if it exists a neighbourhood Ω of A (neighbourhood defined for the Tychonoff distance) such that $\varphi(\Omega) \subset \Omega$ (φ being the global transition map) and $A = \bigcap_{n=0}^{+\infty} \varphi^n(\Omega)$.

Because it is difficult to treat the question of periodic or quasi-periodic configurations with cellular automata, it is more useful to consider global behaviours of patterns appearing in the automata. There is a classification of the cellular automata with respect to these behaviours:

Definition 26 (Wolfram classification)**.** *A cellular automata is:*

- *class I, if almost all initial configuration provides an homogenous state, no stable nor periodic pattern arises;*
- *class II, if simple stable or periodic patterns arise;*
- *class III, if patterns arise erratically (aperiodic patterns) with a stabilisation of the arising frequency in the long term;*
- *class IV, if oscillating or moving patterns arise and/or if complicated pattern preserving their self-organisation arise.*

Work to be done

1. By varying the initial condition and parameters of your dynamical system, try to find invariant cycles or tori (periodic or quasi-periodic orbits), orbits close to homoclinic or heteroclinic orbits (for continuous time systems), or stable, periodic, aperiodic or moving patterns (for cellular automata). You can search in the literature particular choices of initial conditions and of parameters providing such special orbits. Note that for discrete time dynamical system, you can find the p-periodic cyclic points by searching the fixed points of φ^p. Search the cyclic points with small periods.

2. - If your system is conservative (Hamiltonian):
     a. Code a function computing for any point X of the phase space, the Poincaré recurrence time $T_\epsilon(X)$ for a precision ϵ. Code a function colouring the phase space with a colour gradient indicating the value of T_ϵ (for example, choose a lot of random points, compute T_ϵ for each point and attribute a colour at a pixel on the considered point).
     b. Let M be the space in which the parameters of your system vary (M is called control space of the dynamical system). Code a function colouring M (reduced to two varying parameters) with a colour gradient indicating the value of $T_\epsilon(X_0)$ with respect to the different values of the parameters, with X_0 a chosen fixed initial point of the orbit.
   - If your system is dissipative:
     a. By computing long time orbits, search attractors of your system (and their basin of attraction). Modify the values of the parameters of your system to find changes of shape or of topology of these attractors. If you observe topological changes (strange attractor to limit torus, limit circle to limit fixed point, or something else), try to draw in the space of variation of the parameters (so-called control space M), reduced to two varying parameters, the curves of topological changes (called bifurcation curves) (the scheme of the bifurcation curves is equivalent of a phase diagram in thermodynamics).
     b. Code a function computing for any point X of a basin of attraction, the time needed to the orbit starting from X reaches the attractor up to a precision ϵ (duration of the transient regime). Code a function colouring the phase space with a gradient indicating the duration of the transient regime (for example, choose a lot of random points, compute duration of the transient regime for each point and attribute a colour at a pixel on the considered point). If your system has more than 2 dimensions, colour the phase space of a 2D Poincaré section of your system.
3. If your system can be reduced to logistic flows, code functions to draw their bifurcation diagrams.
4. If your system is a cellular automaton, code a function computing the Tychonoff distance between any two configurations.