Activity 3

Reduction of dynamical systems

Goals

Nonlinear dynamical systems have complicated behaviours which are difficult to analyse and to enlighten by simple plots. We might want to reduce the complexity of a dynamical system by finding a reduced system having similar properties but being more simple (smaller number of variables, discrete time in place of continuous time, symbolic system in place of discrete time system).

Theory

Continuous time dynamical systems

Definition 16 (Poincaré section)**.** *Let (Γ, φ^t, μ) be a continuous time dynamical system, and S be a submanifold of Γ (generally a plane) of smaller dimension. We call Poincaré section of the flow by S for an initial condition X_0, the intersection of the orbit of X_0 with S: $\mathrm{Orb}(X_0) \cap S$.*

Let t_n be the date of the nth passage of $\varphi^t(X_0)$ by S. The Poincaré section is the set of points $\{\varphi^{t_n}(X_0)\}_{n\in\mathbb{N}}$ (with $t_0 = 0$), which can be viewed as an orbit of a discrete time flow in S:

$$\varphi_S(\varphi^{t_n}(X_0)) = \varphi^{t_{n+1}}(X_0)$$

Definition 17 (First return map)**.** *The discrete flow $\varphi_S : S \to S$ defined by $\varphi_S(X) = \varphi^{\tau_X}(X)$ where $\tau_X = \inf\{t \in]0, +\infty[, s.t.\ \varphi^t(X) \in S\}$ (for $X \in S$), is called Poincaré first return map.*

We have then a discrete time dynamical system $(S, \varphi_S, \mu_{|S})$ built from (Γ, φ^t, μ). Generally, this one does not correspond to a uniform discretisation of time (and this discretisation is generally different from an orbit to another one). If S is appropriately chosen, the properties of $(S, \varphi_S, \mu_{|S})$ and of (Γ, φ^t, μ) are similar. Because $\dim S < \dim \Gamma$, the Poincaré section is a dimensional reduction of the original system. An appropriate choice for S generally corresponds to a physically meaningful constraint.

Let F be the generator of the flow. Let S be the submanifold of Γ such that $F^i(X) = 0$ for some i. By definition, the passages of the flow by S correspond to local extrema of $X^i(t)$ ($\dot{X}^i(t_n) = 0$ for t_n a date of passage of the flow by S). This kind of choices for S provides approximate choices to define a Poincaré section. Moreover we can consider the one dimensional Poincaré section corresponding to the sequence $(X^i_n)_{n\in\mathbb{N}}$ of the successive passages by S. We have then a one dimensional discrete time dynamical system (I, f, dX^i) where $I \subset \mathbb{R}$ is the open interval of variations of X^i, and f is the first return map such that $X^i_{n+1} = f(X^i_n)$. Numerically, we can draw f as a map $y = f(x)$, by interpolating the sequence of points $\{(X^i_n, X^i_{n+1})\}_{n\in\mathbb{N}}$. If f is unimodal (f has one and only one local extremum onto I), (I, f, dX^i) is said to be a logistic flow.

Let $(\Gamma, \varphi^{t,t_0}, \mu)$ be a T-periodic driven continuous time dynamical system: $\varphi^{t+T,t_0+T} = \varphi^{t,t_0}$ or equivalently $F(t+T, X) = F(t, X)$ (with F the generator of the flow). We call stroboscopic representation of the flow, the discrete time flow defined by $\varphi_S(X) = \varphi^{T,0}(X)$ (so that $\varphi_S^n(X) = \varphi^{nT,0}(X)$). By adding to the phase space the auxiliary variable $\theta = \omega t \mod 2\pi$ ($\omega = 2\pi/T$), the stroboscopic dynamics is in fact the Poincaré section of the extended autonomous system by the hyperplane S defined by $\theta = 0$.

Discrete time dynamical systems

Let (Γ, φ, μ) be a discrete time dynamical system and $h \in L^\infty(\Gamma, d\mu)$ be a real valued observable (the choice $h(X) = X^i$ for some i is not excluded). The sequence $(h(\varphi^n(X_0)))_{n\in\mathbb{N}}$ defines a one dimensional discrete time dynamical system (I, f, dh) where I is the interval of variations of h, and f is defined by $h_{n+1} = f(h_n)$ with $h_n \equiv h(\varphi^n(X_0))$. Numerically, we can draw f as a map $y = f(x)$, by interpolating the sequence of points $\{(h_n, h_{n+1})\}_{n\in\mathbb{N}}$. If h is appropriately chosen, the properties of (I, f, dh) and of (Γ, φ, μ) are similar. But the dimension has been reduced to one. An appropriate choice of h generally corresponds to a physically meaningful observable.

If f is unimodular, (I, f, dh) is a logistic flow. If it is not the case, we can split $I = \bigcup_m I_m$ in several intervals I_m on which f is unimodular, the border points being inflection points of f. $(I_m, f_{|I_m}, dh)$ is then a logistic flow ($\forall m$).

Let (I, f, dx) be a logistic flow, with x_c the critical point (the single local extremum of f in I). The sequence $(x_n)_{n\in\mathbb{N}}$ with $x_{n+1} = f(x_n)$ defines a symbolic dynamical system of alphabet $\mathscr{A} = \{R, L, C\}$ with:

- if $x_n > x_c$, the n-th symbol of the word is R,
- if $x_n < x_c$, the n-th symbol of the word is L,
- if $x_n = x_c$, the n-th symbol of the word is C.

The symbolic dynamical system $(\bigcup_{n\geq 1} \mathscr{A}^n, \varphi_{LCR}, \sharp)$ has properties similar to the logistic flow (I, f, dx), the generating rule φ_{LCR} corresponding to the previous rules concerning the position of x_n with respect to x_c.

For a multimodular flow f, we can add the symbols $1, 2, ..., m$ to the alphabet, the symbol i meaning that the flow leaves the current interval to go in the interval I_i.

General scheme of reduction of dynamical systems

Continuous time dynamical systems

$\Big\downarrow$ Poincaré section

Discrete time dynamical systems

$\Big\downarrow$ single observable choice

Logistic flows

$\Big\downarrow$ LCR coding

symbolic dynamical systems

Algorithms

Poincaré section

To built numerically the Poincaré section of a continuous time dynamical system by a submanifold S:

1. Use a numerical integrator to find a sequence (X_n) approaching $X(t_n)$ with Δt sufficiently small. (X_n) is an approximation of the orbit $\{\varphi^t(X_0)\}_{t\geq 0}$.
2. S is supposed splitting Γ in two parts Γ_+ and Γ_- (if S is open and non-compact, as a plane, Γ_+ and Γ_- are the half spaces on either side of S; if S is closed and compact, as a sphere, Γ_+ and Γ_- are the exterior and the interior of S). For each point X_n find if $X_n \in \Gamma_+$, $X_n \in \Gamma_-$ or $X_n \in S$.
3. Let P be the list of points of the Poincaré section:
   - If $X_n \in S$, then X_n must be added to P.
   - If $X_n \in \Gamma_-$ and $X_{n+1} \in \Gamma_+$ or if $X_n \in \Gamma_+$ and $X_{n+1} \in \Gamma_-$, an intermediate point $X_{n+1/2}$ must be added to P. This one can be approximated by $X_{n+1/2} = \frac{X_n + X_{n+1}}{2}$, or can be searched by dichotomy (use the numerical integrator with a smaller value of Δt from X_n to X_{n+1} to locate more precisely the point on S).
4. If $X_n, X_{n+1} \in \Gamma_+$ or $\in \Gamma_-$, then no point is to be added to P.

First return map of a 1D discrete time flow

We search a function $f : I \to \mathbb{R}$ known only by a list of points $\{(h_n, h_{n+1})\}_{n\in\mathbb{N}}$ with $h_{n+1} = f(h_n)$. To compute $f(x)$ if $x \notin \{h_n\}_n$ we can use a linear interpolation:

1. find in $\{h_n\}_n$ the values closest to x: $h_{n_-} < x < h_{n_+}$
2. compute $\alpha = \frac{x - h_{n_-}}{h_{n_+} - h_{n_-}}$ and $\beta = \frac{h_{n_+} - x}{h_{n_+} - h_{n_-}}$ (we have $x = \alpha h_{n_+} + \beta h_{n_-}$).
3. $f(x) \simeq \alpha f(h_{n_+}) + \beta f(h_{n_-}) = \alpha h_{n_+ +1} + \beta h_{n_- +1}$.

This interpolation algorithm is very not accurate. There is a lot of better interpolation algorithms which are implemented in scientific computation libraries.

Work to be done

1. If your dynamical system is continuous in time, find a pertinent submanifold S to realise a Poincaré section of your system, and code a program to compute this Poincaré section. Read the sections of activities 1 and 3 concerning the discrete time dynamical systems.
2. If you have a discrete time dynamical system (chosen system or Poincaré section of your chosen system), find a pertinent observable h, and code a program to compute logistic flows associated with this one. Code a program computing the LCR coding of these logistic flows. Read the sections of activities 1 and 3 concerning symbolic dynamical systems.
3. - If your dynamical system is discrete in time, read the sections of activities 1 and 3 concerning continuous time dynamical systems, and find out if your system is not an idealisation/approximation of the Poincaré section of a continuous time dynamical system.
   - If your dynamical system is symbolic, read the sections of activities 1 and 3 concerning continuous and discrete time dynamical systems, and find out if your system is not an idealisation/approximation of a continuous or discrete time dynamical system.