Activity 5

Stability of fixed points and local Lyapunov exponents

Goals

Considering a fixed point X_* of a dynamical system, after a perturbation moving to a point in a neighbourhood of X_*, we want to know if the orbit returns to X_* (stable fixed point) or if the orbit leaves the neighbourhood of X_* (unstable fixed point).
This activity does not concern the symbolic dynamical systems.

Theory

Lyapunov stability

Definition 27 (Stability within the meaning of Lyapunov). *Let (Γ, F, μ) be a discrete or continuous time dynamical system. Let X_* be a fixed point of the flow ($\varphi^t(X_*) = X_*$). We say that X_* is:*

- *stable within the meaning of Lyapunov, if $\forall \epsilon > 0$, $\exists d_\epsilon$ such that $\forall t > 0$*

$$\|X - X_*\| < d_\epsilon \Rightarrow \|\varphi^t(X) - X_*\| < \epsilon$$

- *asymptotically stable within the meaning of Lyapunov, if it is stable and if $\exists \eta > 0$,*

$$\|X - X_*\| < \eta \Rightarrow \lim_{t \to +\infty} \varphi^t(X) = X_*$$

- *exponentially stable within the meaning of Lyapunov, if it is stable and if $\exists \alpha, \beta, \eta > 0$*

$$\|X - X_*\| < \eta \Rightarrow \|\varphi^t(X) - X_*\| \leq \alpha \|X - X_*\| e^{-\beta t}$$

A fixed point is said to be unstable within the meaning of Lyapunov if it is not stable.

A perturbation of a marginally stable fixed point X_* (stable but not asymptotically stable) induces an orbit remaining in the neighbourhood of X_* but not returning to it. An exponential stability means that the perturbed orbit returns to the fixed point at an exponential speed.

Theorem 6. *The stability or the instability of a fixed point X_* of a nonlinear dynamical system is equivalent to the stability or the instability of 0 for the linearisation of the dynamical system at X_*, except in the case of a marginal stability where nonlinearities can eventually destabilise the system.*

Proof : We consider the perturbative expansion $X(t) = X_* + \delta X(t) + \sum_{n=2}^{+\infty} D_n \delta X(t)...\delta X(t)$ where D_n is a rank $n+1$ tensor, with $X(0) = X_* + \delta X(0)$.
Suppose that the linearised system is asymptotically stable: $\exists \eta > 0$, $\|\delta X(0)\| < \eta \Rightarrow \lim_{t\to+\infty} \delta X(t) = 0$, then $\|X(0) - X_*\| < \eta \Rightarrow \lim_{t\to+\infty} X(t) = X_*$. The dynamical system is then asymptotically stable.
Suppose that the linearised system is only marginally stable: $\forall \epsilon > 0$, $\exists d_\epsilon$, $\|\delta X(0)\| < d_\epsilon \Rightarrow \|\delta X(t)\| < \epsilon$. Then $\|X(t) - X_*\| = \|\delta X + \sum_{n=2}^{+\infty} D_n \delta X...\delta X\| \leq \epsilon + \sum_{n=2}^{+\infty} \|D_n\|\epsilon^n$. Even if ϵ is small, nothing ensures the convergence of the series $\sum_{n=2}^{+\infty} \|D_n\|\epsilon^n$. We cannot then conclude.
Suppose that the linearised system is unstable: $\lim_{t\to+\infty} \delta X(t) = +\infty$ (only unstable behaviour of a linear system), then the perturbative series diverges (this does not necessarily mean that $\lim_{t\to+\infty} X(t) = +\infty$), the convergence radius of the series is then zero and the behaviour of the system cannot be represented only in the neighbourhood of X_*. The system is then unstable. ■

Theorem 7 (Lyapunov theorem)**.** *Let (Γ, φ, μ) be a dynamical system, X_* be fixed point, ∂F_{X_*} be the Jacobian matrix at X_* of the generator F of the flow (for a continuous time system) or $\partial \varphi_{X_*}$ be the Jacobian matrix at X_* of the flow (for a discrete time system). We suppose that ∂F_{X_*} (or $\partial \varphi_{X_*}$) is diagonalisable. Let $\{\Re\mathrm{e}(\lambda_i)\}_i$ be the set of real parts of the eigenvalues of ∂F_{X_*}, or be such that $\{e^{\lambda_i}\}_i$ is the set of the eigenvalues of $\partial \varphi_{X_*}$. These quantities are called local Lyapunov exponents. Let $(n_+, n_0, n_-) \in \mathbb{N}^3$ be the signature of the Lyapunov exponents: n_+ is the number of positive Lyapunov exponents, n_0 is the number of zero Lyapunov exponents and n_- is the number of negative Lyapunov exponents (with the eigenvalues counted as many times as they are degenerated).*

- *If $(n_+, n_0, n_-) = (0, 0, n)$, then X_* is a stable fixed point (exponentially stable for the linearisation).*
- *If $(n_+, n_0, n_-) = (0, 1, n-1)$, then X_* is stable, or is on a limit cycle.*
- *If $(n_+, n_0, n_-) = (0, 2, n-2)$ then X_* is stable and is surrounded by limit cycles, or is on a limit torus.*
- *If $(n_+, n_0, n_-) = (0, 3, n-3)$ then X_* is stable and is on a limit torus.*
- *If $(n_+, n_0, n_-) = (0, 4, n-4)$ then X_* is stable and is surrounded by limit tori, or is on a limit torus.*
- *If $(n_+, n_0, n_-) = (0, n_0, n-n_0)$ with $n_0 > 4$ then X_* is stable and is surrounded by limit tori.*
- *If $n_+ \neq 0$ then X_* is unstable.*

Proof : Let $\partial \varphi^t_{X_*}$ be the t power of the Jacobian matrix of the flow if the system is discrete in time, or be $\partial \varphi^t_{X_*} = e^{\partial F_{X_*} t}$ if the system is continuous in time. Let $\{e^{\lambda_i t}\}_i$ be the eigenvalues of $\partial \varphi^t_{X_*}$ and $(\chi_{i(a)})_{i,a}$ be the associated eigenvectors (a varies from 1 to the degeneracy level of λ_i). We suppose firstly than $\ker \partial F_{X_*} = \{0\}$ (the vector subspace of vectors invariant by $\partial \varphi^t_{X_*}$ is reduced to $\{0\}$). $\partial \varphi^t_{X_*}$ being diagonalisable, the set of the eigenvectors is a basis of $\mathbb{C}^n$ (n being the dimension of the phase space), and then

$$\forall \delta X(0) \in \mathbb{R}^n, \exists \alpha^{i(\alpha)} \in \mathbb{C} \quad \delta X(0) = \sum_i \sum_a \alpha^{i(a)} \chi_{i(a)}$$

$$\delta X(t) = \partial \varphi^t_{X_*} \delta X(0) = \sum_i \sum_a \alpha^{i(a)} e^{\Re\mathrm{e}(\lambda_i)t} e^{\imath \Im\mathrm{m}(\lambda_i)t} \chi_{i(a)}$$

if $\forall i$, $\Re\mathrm{e}(\lambda_i) < 0$ we have obviously

$$\|\delta X(t)\| = \mathcal{O}(e^{\max_i \Re\mathrm{e}(\lambda_i)t})$$

0 is then an exponentially stable fixed point. If at least one value λ_j has a positive real part, then 0 is unstable because:

$$\forall t \in \mathcal{V}(+\infty), \quad \delta X(t) \simeq \sum_{j|\Re\mathrm{e}(\lambda_j)>0} \sum_a \alpha^{j(a)} e^{\Re\mathrm{e}(\lambda_j)t} \chi_{j(a)}$$

Suppose now that $\forall i$, $\Re\mathrm{e}(\lambda_i) \leq 0$. Note that the number of eigenvalues with zero real part is necessarily even. Indeed $\partial F_{X_*} \in \mathfrak{M}_{n\times n}(\mathbb{R})$, and then if λ_i is eigenvalue of ∂F_{X_*} then $\overline{\lambda_i}$ is also eigenvalue ($\partial F \chi_{i(a)} = \lambda_i \chi_{i(a)} \Rightarrow \partial F \overline{\chi_{i(a)}} = \overline{\lambda_i \chi_{i(a)}}$). The eigenvalues are conjugated. We have then $\forall t$ in the neighbourhood of $+\infty$,

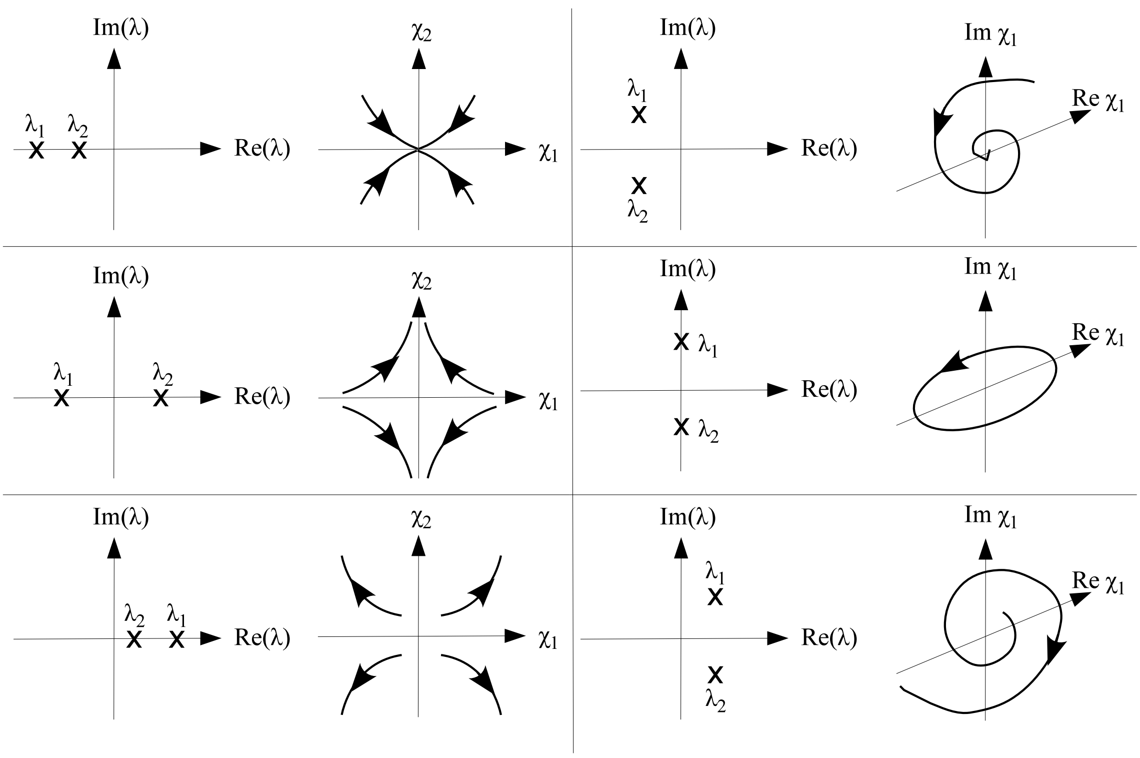
$$\begin{aligned}
\delta X(t) &\simeq \sum_{i|\Re\mathrm{e}(\lambda_i)=0 \text{ and } \Im\mathrm{m}(\lambda_i)>0} \sum_a 2\Re\mathrm{e}\left(\alpha^{i(a)} e^{\imath \Im\mathrm{m}(\lambda_i)t} \chi_{i(a)}\right) \\
&\simeq 2 \sum_{i|\Re\mathrm{e}(\lambda_i)=0 \text{ and } \Im\mathrm{m}(\lambda_i)>0} \sum_a \left(\cos(\Im\mathrm{m}(\lambda_i)t)\Re\mathrm{e}(\alpha^{i(a)}\chi_{i(a)}) - \sin(\Im\mathrm{m}(\lambda_i)t)\Im\mathrm{m}(\alpha^{i(a)}\chi_{i(a)})\right)
\end{aligned}$$

We recall that the parametric equation of an ellipse in $\mathbb{R}^2$ is:

$$A\cos(2\pi t) + B\sin(2\pi t) = \begin{pmatrix} A_x\cos(2\pi t) + B_x \sin(2\pi t) \\ A_y \cos(2\pi t) + B_y \sin(2\pi t) \end{pmatrix}$$

(with the half-axis $\overrightarrow{OA}$ and $\overrightarrow{OB}$). We see that the trajectory of $\delta X(t)$ converges to a limit torus.
If $\dim\ker \partial F_{X_*} > 0$: $\ker \partial F_{X_*}$ is a set of fixed points, and then the system is stable by perturbation inner to $\ker \partial F_{X_*}$. This space is then a part of the attractor. If $\dim \ker \partial F_{X_*} = 1$ we have a stable cycle (the linearisation is valid only in the neighbourhood of X_*, and so a straight line for the linearised is a small part of a curve for the nonlinear system). If $\dim\ker\partial F_{X_*} = 2$ we have a stable torus. The previous analysis can be then remake for ∂F_{X_*} restricted to $(\ker \partial F_{X_*})^\perp$ (orthogonal supplement of $\ker \partial F_{X_*}$). ■

It is more easy to represent the situation for a two dimensional phase space:



Schemes of the phase trajectories in the neighbourhood of a fixed point X_* of a two dimensional dynamical system with respect to the eigenvalues ∂F_{X_*} in the complex plane.

If $\partial\varphi^t_{X_*}$ is not diagonalisable, we can use the following property:

Property 1. *$\forall A \in \mathfrak{M}_{n\times n}(\mathbb{C})$, $\exists D \in \mathfrak{M}_{n\times n}(\mathbb{C})$ a diagonalisable matrix and $\exists N \in \mathfrak{M}_{n\times n}(\mathbb{C})$ a nilpotent matrix (i.e. $\exists p \in \mathbb{N}$ such that $N^p = 0$) such that*

$$A = D + N \quad \textit{and} \quad e^A = e^D e^N = e^D \sum_{i=0}^{p} \frac{N^p}{p!}$$

So if $\partial\varphi^t_{X_*}$ is not diagonalisable, by using the decomposition of ∂F_{X_*} we see that $\partial\varphi^t_{X_*}$ is the product of a diagonalisable matrix by a polynomial of degree $p > 0$ with respect to t. If 0 is an exponentially stable fixed point for the diagonalisable part, it is an asymptotically stable fixed point for the whole linearised dynamics (since the convergence to 0 is as $t^p e^{-\alpha t}$). If 0 is a marginally stable fixed point for the diagonalisable part (or is unstable), then it is unstable for the whole linearised dynamics.

Remark: Since $\partial\varphi^t_{X_*} = e^{t\partial F_{X_*}}$ and $\{\lambda_i\} = \mathrm{Sp}(\partial F_{X_*})$, then for the continuous time systems ($t \in \mathbb{R}^+$), the Lyapunov values and exponents are inverses of times; whereas for discrete time systems ($t \in \mathbb{N}$), the Lyapunov values and exponents are without physical dimension.

Hyperbolic fixed points

Definition 28 (Hyperbolic point). *Let (Γ, φ, μ) be a (discrete or continuous time) dynamical system. A fixed point X_* is said to be a hyperbolic point if the signature of the local Lyapunov exponents at X_* is $(n_+, 0, n_-)$ with $n_\pm \neq 0$.*

In the figure representing the two dimensional case, the case of an hyperbolic point is represented 1st column 2nd row.

Let $\{\chi_i\}_i$ be the eigenvectors associated with the Lyapunov values $\{\lambda_i\}$ of an hyperbolic point X_*. If $\Re\mathrm{e}\lambda_i < 0$, χ_i corresponds to a stable direction of the flow in the neighbourhood of X_*; whereas if $\Re\mathrm{e}\lambda_i > 0$, χ_i corresponds to an unstable direction of the flow in the neighbourhood of X_*. We can then set

$$T_{X_*}\Gamma = E_s \oplus E_u$$

where $T_{X_*}\Gamma$ is the vector space of tangent vectors of Γ at X_*, $E_s = \mathrm{Lin}_{\mathbb{R}}\{\Re\mathrm{e}\chi_i, \Im\mathrm{m}\chi_i | \Re\mathrm{e}\lambda_i < 0\}$ is the vector subspace of stable directions, and $E_u = \mathrm{Lin}_{\mathbb{R}}\{\Re\mathrm{e}\chi_i, \Im\mathrm{m}\chi_i | \Re\mathrm{e}\lambda_i > 0\}$ is the vector subspace of unstable directions.

Theorem 8 (Stable manifold theorem). *Let (Γ, φ, μ) be a dynamical system with X_* an hyperbolic point (and with Γ of class $\mathcal{C}^\infty$ if it is a submanifold of $\mathbb{R}^n$), such that $\varphi : \Gamma \to \Gamma$ be a diffeomorphism (i.e. an infinitely differentiable bijection of Γ to Γ with an infinitely differentiable inverse map). Then there is a submanifold W_s of Γ containing X_* on which φ is a contraction ($\|\varphi(X) - \varphi(Y)\| \leq \|X - Y\|$, $\forall X, Y \in W_s$) and such that $\partial\varphi_{X_*} : T_{X_*}W_s \to T_{X_*}\Gamma$ is an isomorphism of vector spaces between $T_{X_*}W_s$ and E_s. W_s is called stable manifold of X_*. We define W_u the unstable manifold of X_* as the stable manifold associated to the inverse flow φ^{-1} (stable manifold for the flow to the past).*

Because $\dim W_s + \dim W_u = \dim \Gamma$, $W_s \cup W_u$ cover Γ in the neighbourhood of X_*, but generally $W_s \cup W_u \neq \Gamma$ (some points far from X_* may not belong to W_s or W_u).

By definition, the fixed points associated to homoclinic and heteroclinic orbits are hyperbolic. An homoclinic orbit leaves a fixed point along its unstable manifold and returns to it along its stable manifold. An heteroclinic orbit leaves a fixed point along its unstable manifold and goes to another fixed point along its stable manifold.

Stability of invariant cycles and invariant tori

Let C be a cycle of a continuous time dynamical system of period T. By definition $\forall X_* \in \Gamma$, $\varphi^T(X_*) = X_*$. So X_* is a fixed point of the discrete time dynamical system (Γ, Φ, μ) with $\Phi(X) = \varphi^T(X)$ ($\Phi^n(X) = \varphi^{nT}(X)$). The stability of the cycle C for the flow φ^t is then equivalent to the stability of the fixed point X_* for the stroboscopic flow Φ. The choice of X_* on C is free (the stability is the same for all points on C if φ is infinitely differentiable).

Let X_* be a cyclic point of a discrete time dynamical system of period p. By definition $\varphi^p(X_*) = X_*$. So X_* is a fixed point of the discrete time dynamical system (Γ, Φ, μ) with $\Phi(X) = \varphi^p(X)$ ($\Phi^n(X) = \varphi^{np}(X)$). The stability of the cyclic orbit of X_* for the flow φ is then equivalent to the stability of the fixed point X_* for the stroboscopic flow Φ (the stability is the same for all points of the cyclic orbit if φ is infinitely differentiable).

The stability of invariant cycles of a discrete time system which are not composed of cyclic orbits, and of invariant tori is more difficult. It is governed by the KAM (Kolmogorov-Arnold-Moser) theorem. This theorem is not on the program of the present course.

Work to be done

1. Diagonalise the Jacobian matrix ∂F_{X_*} or $\partial \varphi_{X_*}$ at each fixed points of your dynamical system, and find the local Lyapunov values $\{\lambda_i\}$ and the Lyapunov eigendirections $\{\Re\mathrm{e}\chi_i, \Im\mathrm{m}\chi_i\}$ for each fixed points.
2. Determine the nature of each fixed points (exponentially, asymptotically or marginally stable; unstable, hyperbolic).
3. Draw a figure showing the stable and unstable directions in the neighbourhood of the hyperbolic points.
4. If you have found cycles (for continuous time systems) or cyclic orbits (for discrete time systems), remake the three previous points for studying the stability of these orbits.