Activity 8

Ergodicity and mixing

Goals

We want now characterise the topology of a dynamical system, so we want answer to the following questions:

1. Is a region Ω of the phase space necessarily visited by the flow? For what initial conditions?
2. If two regions of the phase space A and B are initially well separated, do their evolutions remain separate or do they become indistinguishable after a long time?

This activity does not concern symbolic systems.

Theory

Definition 42 (Ergodicity)**.** *Let (Γ, φ, μ) be a dynamical system and $\Omega \subset \Gamma$ be a region of the phase space (in an attractor if the system is dissipative). The flow is said to be ergodic in Ω if*

$$\forall A, B \in \mathscr{T}_{|\Omega}, \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(\varphi^{k\Delta t}(B) \cap A) = \mu(A)\mu(B)$$

with $\Delta t = 1$ for a discrete time system or is a discretisation time step for a continuous time system.

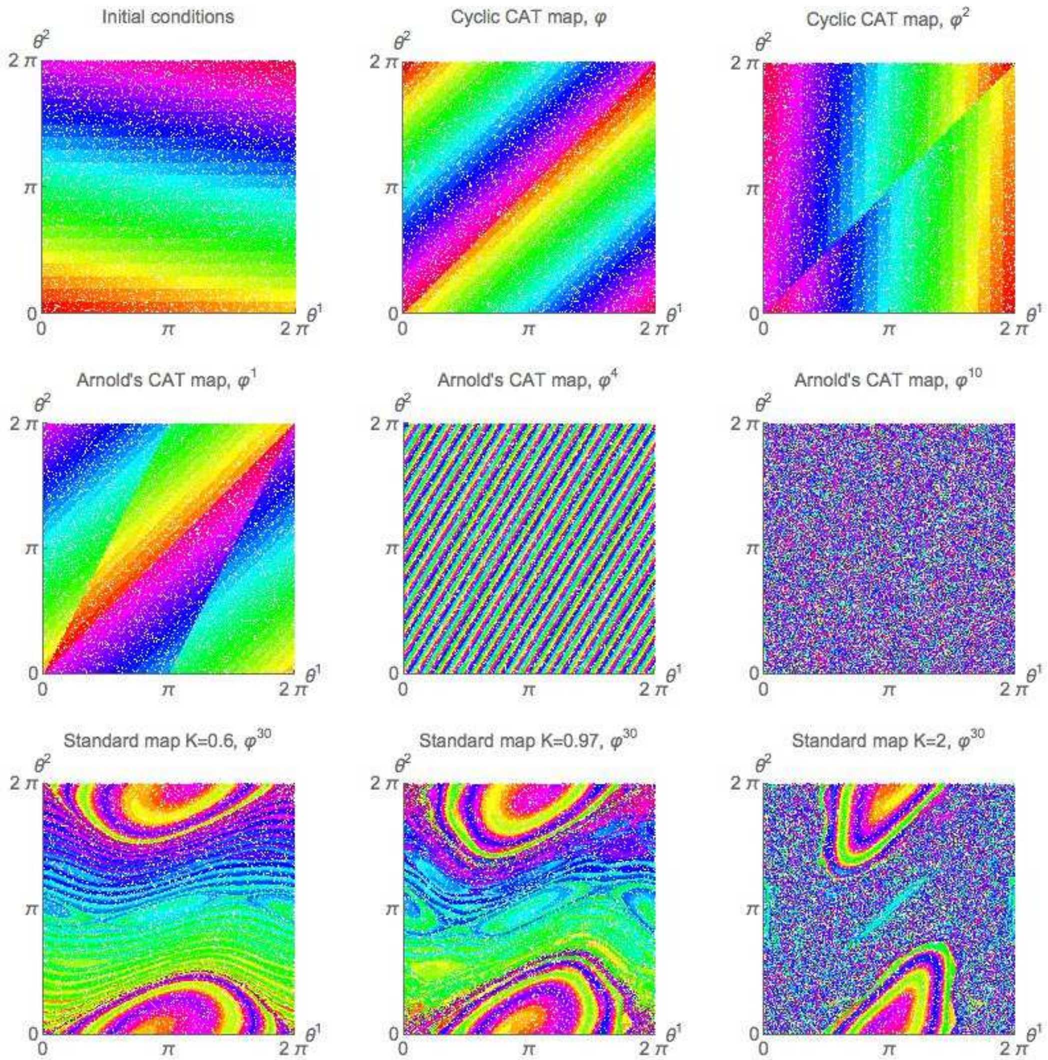
This means that in average at long term, the evolution of any subset B of Ω meets any subset A of Ω, with the mean value of covering proportional to the product of the weighs of A and B (for the measure μ).
If the system is conservative, it is possible to decompose the phase space into ergodic components: $\Gamma = \bigcup_i \Gamma_i$ such that Γ_i is a maximal region on which the flow is ergodic (this one is connected (in one piece) for the continuous time systems).

Definition 43 (Mixing)**.** *Let (Γ, φ, μ) be a dynamical system and $\Omega \subset \Gamma$ be a region of the phase space (in an attractor if the system is dissipative). The flow is said to be mixing in Ω if*

$$\forall A, B \in \mathscr{T}_{|\Omega}, \lim_{n\to+\infty} \mu(\varphi^{n\Delta t}(B) \cap A) = \mu(A)\mu(B)$$

with $\Delta t = 1$ for a discrete time system or is a discretisation time step for a continuous time system.

If the flow is mixing in Ω, the evolution of any subset B of Ω meets at long term any subset A of Ω, with a covering proportional to the product of the weights of A and B. This property is stronger that the ergodicity. For a flow just ergodic, $\varphi^{n\Delta t}$ with n large does not necessarily meet A, this is only in average onto a large number of steps whereas if the flow is mixing, $\varphi^{n\Delta t}$ meets A for n sufficiently large. It is clear that if the flow is mixing in Ω this implies that it is also ergodic (but the reciprocal is false).
To illustrate the difference between the two notions, we can consider the following examples of discrete time systems:



The phase space is the torus $\Gamma = \mathbb{T}^2$ (represented unfolded). At the initial time, we have coloured with different colours some small squares covering Γ. The first raw represents the evolution induced by a 3-cyclic map ($\varphi^3 = \mathrm{id}$). This flow is not ergodic in Γ (we see for example that the green squares do not meet the upper left corner). Almost all points belong to a 3-cyclic orbits. The phase space is then composed to an infinite continuous number of ergodic components (the 3-cyclic orbits) each having only three points. In such a case, the ergodic decomposition is meaningless and it is more pertinent to say that the flow is not ergodic.
The second row represents the evolution induced by a mixing flow in Γ. We see that after a relatively large number of iterations, all colours meet each region.
The last row represents the state after a large number of iterations for a parameter dependent flow. For $K = 0.6$ the phase space is decomposed in a lot of ergodic components appearing as cycles (of uniform colour). For $K = 2$, the phase space is decomposed into three regions. A large elliptic region around $(\pi, 0)$ which can be decomposed into ergodic components appearing as cycles. A double small elliptic region around $(0, \pi)$ and (π, π) which can also be decomposed into ergodic components appearing as "cycles". In fact, the orbits "jump" from a small elliptic part to the other at each iteration. It is then more consistent to see this double region as a flat disconnected torus (and the ergodic components as flat disconnected invariant tori). These two regions are called islands of stability (the cycles and the tori are stable). These two islands are surrounded by another ergodic region where the flow is mixing. This one is called chaotic sea.

Definition 44 (Topological transitivity). *Let (Γ, φ, μ) be a dynamical system and $\Omega \subset \Gamma$ be a region* *of the phase space (in an attractor if the system is dissipative). The flow is said to be topologically transitive in Ω if*

$$\forall A, B \in \mathscr{T}_{|\Omega}, \exists n \in \mathbb{N}^*, \varphi^{n\Delta t}(B) \cap A \neq \varnothing$$

The flow is topologically transitive in Ω if the evolution of any subset B of Ω meets sometime any subset A of Ω. This property is both stronger that the mixing property (since the step of the meet is finite) and weaker that the mixing property (since nothing ensure that the meet continues at long term). Note that "topologically mixing" is synonymous of "topologically transitive" but to avoid any confusion, we prefer to use "transitive".

Property 4. *A conservative dynamical system is ergodic on Ω, with μ a probability measure of Ω ($\mu(\Omega) = 1$) if and only if $\forall D \in \mathscr{T}_{|\Omega}$, $\varphi(D) = D \Rightarrow \mu(D) = 1$ or $\mu(D) = 0$.*

Proof : Suppose that the flow is ergodic. Let $D \in \mathscr{T}$ be a subset invariant by the flow. Then $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu(\varphi^{k\Delta t}(D) \cap D) = \mu(D)^2$. But D being invariant, $\varphi^{k\Delta t}(D) \cap D = D$, and then $\mu(D) = \mu(D)^2 \Rightarrow \mu(D) = 0$ or 1. The reciprocal results from the Birkhoff ergodic theorem. ■

Theorem 10 (Birkhoff ergodic theorem)**.** *Let φ be a conservative flow ergodic in Ω, with μ a probability measure on Ω ($\mu(\Omega) = 1$). $\forall f \in L^1(\Omega, d\mu)$ summable observable ($\int_\Omega |f(X)| d\mu(X) < \infty$) we have*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^{k\Delta t}(X_0)) = \int_\Omega f(X) d\mu(X)$$

for μ-almost all $X_0 \in \Omega$

If the flow is ergodic in Ω, then the time mean value of f is equal to its statistical mean value.

Proof :

Lemma 1. *All map $f \in \mathcal{C}^0(\Omega)$ such that almost everywhere $f(\varphi(X)) = f(X)$, is almost everywhere constant.*

If f is not almost everywhere constant, $\exists x \in \mathbb{R}$ such that $f^{-1}([x, +\infty[) = A$ with $0 < \mu(A) < 1$. $f \circ \varphi = f \Rightarrow \varphi^{-1}(A) \simeq A$ ($\simeq$ means that the set are equal except eventually on a part of zero measure). Let $A' = \bigcap_{p\in\mathbb{N}} \bigcup_{n\geq p} \varphi^{-n}(A)$. By construction $A' \simeq A$ and then $0 < \mu(A') < 1$. But $\varphi^{-1}(A') = A'$, the system being ergodic, we should have $\mu(A') = 0$ or 1.

We return to the Birkhoff theorem. We set $\tilde{f}(X) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^k(X))$. $\tilde{f}(\varphi(X)) = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} f(\varphi^k(X))$. But

$$\frac{n+1}{n}\frac{1}{n+1}\sum_{k=0}^{n} f(\varphi^k(X)) - \frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^{k+1}(X)) = \frac{1}{n} f(X)$$

By considering the limit $n \to +\infty$ in this expression we find $\tilde{f}(X) - \tilde{f}(\varphi(X)) = 0$. By applying the lemma, we deduce that $\tilde{f}$ is almost everywhere constant.

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1}\int_\Omega f(\varphi^k(X)) d\mu(X) &= \frac{1}{n}\sum_{k=0}^{n-1}\int_{\varphi^{-k}(\Omega)} f(X) d\mu(X) \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_\Omega f(X) d\mu(X) \\
&= \int_\Omega f(X) d\mu(X)
\end{aligned}$$

since $\varphi^{-n}(\Omega) \simeq \Omega$ because $\mu(\Omega) = 1$, $\mu(\varphi^{-1}(\Omega)) = 1$ (φ preserve μ) and the system is ergodic. It follows that $\int_\Omega \tilde{f}(X)d\mu(X) = \int_\Omega f(X)d\mu(X)$. Finally $\tilde{f}$ being almost everywhere constant, $\int_\Omega \tilde{f}(X)d\mu(X) = \tilde{f}(X)$. It remains to prove that the series $\frac{1}{n}\sum_{k=0}^{n-1} f(\varphi^k(X))$ converges (we admit this point in this course). ■

An interpretation of the ergodicity is the following. If we apply the ergodic theorem to the characteristic function $\mathbb{I}_D$ ($D \in \mathscr{T}$ in an ergodic region, $\mathbb{I}_D(X) = 1$ if $X \in D$, otherwise $\mathbb{I}_D(X) = 0$ with $X \notin D$), the theorem states that $\nu = \mu(D)$ is the time ratio where the system is in D. Ergodic systems spend in each region D a time proportional to the probability of occupation of the region.

Property 5 (Density of the orbits). *If the probability measure is ergodic for the flow in $\Omega \subset \Gamma$, then for μ-almost all $X \in \Omega$ we have $\mu\left(\overline{\mathrm{Orb}(X)}\right) = 1$ (where $\overline{D}$ denotes D completed by the points being at its border).*

This means that if the flow is ergodic in Ω, almost all phase trajectory of Ω visits the whole of Ω.

Proof : We proceed by *reductio ad absurdum*. We suppose that there is an open set O of Ω, of non zero measure $\mu(O) \neq 0$ and which is not reached by the flow (if $\mu\left(\overline{\{\varphi^n(X), n \in \mathbb{N}\}}\right) < 1$ then $\mu\left(\Omega \setminus \overline{\{\varphi^n(X), n \in \mathbb{N}\}}\right) > 0$). Let $\tilde{\mathbb{I}}_O(X)$ be a smooth characteristic function of O ($\tilde{\mathbb{I}}_O$ is continuous, such that $\tilde{\mathbb{I}}_O(X) = 1$ for $X \in F$ a closed subset in O, $\tilde{\mathbb{I}}_O(X) = 0$ for $X \notin O$, F being sufficiently large to $\mu(F) > 0$). We apply the Birkhoff ergodic theorem to $\tilde{\mathbb{I}}_O$:

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \tilde{\mathbb{I}}_O(\varphi^n(X)) = \int_\Omega \tilde{\mathbb{I}}_O(X')d\mu(X') \geq \mu(F) > 0$$

But by definition $\tilde{\mathbb{I}}_O(\varphi^n(X)) = 0$. We have then a contradiction. ■

Work to be done

1. If your system is conservative, find the decomposition of the phase space into ergodic components. Vary the parameters of the system to find the changes in the ergodic decomposition.
2. Study each ergodic component or each attractor of your system, to determine if this one is only ergodic or if it is mixing.
3. Choose a region in which your system is ergodic and a pertinent observable. Compute numerically the time average of this observable for an orbit, and compare with the statistical average of it (computed by a Monte Carlo method).