# Gravitational astrophysics: resonances among the Saturnian satellites

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## 1 Introduction

This project aims at exploring the dynamics of the satellites of Saturn. For that, you will work from the simplified system, which is composed of

- Saturn
- Mimas
- Tethys
- Titan.

Mimas, Tethys and Titan are natural satellites of Saturn. You will simulate their orbital motions around Saturn.

## 2 Integration of the two-body problem

The first step consists in integrating the motion of these 3 satellites, in cartesian coordinates ($x$,$y$,$z$). This means integrating the following system of equations

$$\frac{dX}{dt} = F(t, X),$$

where $X$ is a state vector, which is composed of 18 variables. These variables are, for each of these satellites: $x_i$, $y_i$, $z_i$, $\dot{x}_i$, $\dot{y}_i$ et $\dot{z}_i$. The four body are, in this section, considered to be point masses.

Express the function $F$.

What are the required parameters?

For planetary bodies, it is advised to use the product $\mathcal{G}M$ instead of the mass. You can find these data on the server Horizons (Jet Propulsion Laboratory).

Specify the numbers you use.

Once the equations are written, you will need a numerical integrator, and initial conditions. Usually you already have a numerical integrator. Please specify the one you use, and its order.

You can find the initial conditions on JPL / Horizons. Beware of the units, and the used reference frame. Once you have the integrator and the initial conditions, you have everything you need to simulate the keplerian trajectories of the 3 satellites around Saturn. Please specify the time step you use. Which test permits you to check the validity of the integration?

## 3 Switching from cartesian to keplerian elements, and conversely

It is advisable to use the keplerian elements to display the results. The elements you will use are

- $a$ : semimajor axis,
- $e$ : eccentricity,
- $I$ : orbital inclination,
- $\Omega$ : longitude of the ascending node,
- $\varpi = \omega + \Omega$ : longitude of the pericentre,
- $\lambda = \mathcal{M} + \varpi$ : mean longitude.

I provide you a routine for that.

## 4 Including the mutual perturbations

Being realistic on the orbital motions of the satellites requires to include the mutual perturbations, i.e. each of these three satellites affects the motion of the other two. Rewrite the equations and implement them.

Check how the orbital elements are affected. Why is Titan only marginally affected with respect to Mimas and Tethys?

## 5 The flattening of Saturn

In fact, Saturn is an oblate body, and you will now consider that. Its gravity potential in spherical coordinates reads

$$\mathcal{U}(r,\lambda,\phi) = \frac{\mathcal{G}M}{r}\left[1 - \left(\frac{R}{r}\right)^2 J_2 P_2(\sin\phi)\right]$$

with

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},$$

and $R$ is Saturn's equatorial radius. Use $J_2 = 1.629 \times 10^{-2}$. Moreover, the cartesian equations associated with this motion are

$$\begin{aligned}
\frac{d^2x}{dt^2} &= -\frac{3}{2}\mathcal{G}(M+M_i)J_2R^2\frac{x}{r^7}\left(x^2+y^2-4z^2\right), \\
\frac{d^2y}{dt^2} &= -\frac{3}{2}\mathcal{G}(M+M_i)J_2R^2\frac{y}{r^7}\left(x^2+y^2-4z^2\right), \\
\frac{d^2z}{dt^2} &= -\frac{3}{2}\mathcal{G}(M+M_i)J_2R^2\frac{z}{r^7}\left(3x^2+3y^2-2z^2\right).
\end{aligned}$$

What do you notice in checking the time evolutions of $\Omega$ and $\varpi$? Estimate their precession periods.

You will check that

$$
\begin{aligned}
\dot{\varpi} &\approx \frac{3}{2} J_2 n \left(\frac{R}{a}\right)^2, \\
-\dot{\Omega} &\approx \frac{3}{2} J_2 n \left(\frac{R}{a}\right)^2.
\end{aligned}
$$

You should get some differences between the numbers given by your codes and the ones coming from the formulae above. Where should they come from?

If you had to obtain these analytical formulae, how would you have proceeded?

# 6 Orbital resonances in the Saturnian system

Plot the argument $2\lambda_1 - 4\lambda_3 + \Omega_1 + \Omega_3$, where the index 1 stands for Mimas, and 3 for Tethys. What do you notice? (if no resonance is obvious, feel free to slightly alter the initial conditions).

Which relation should the orbital elements satisfy at the exact resonance? I recall you the Third Kepler Law

$$
\frac{T^2}{a^3} = \frac{4\pi^2}{\mathcal{G}(M_{♄} + M)},
$$

with $nT = 2\pi$.

# A Deriving the keplerian elements from the cartesian coordinates

You will use the following elements

- the semimajor axis $a$,
- the orbital inclination $I$,
- the longitude of the ascending node $\Omega$,
- the eccentricity $e$,
- the longitude of the pericentre $\varpi$,
- the mean longitude $\lambda$.

## A.1 The semimajor axis $a$

You can get the semimajor axis straightforwardly from the total energy $K$

$$
K = -M \frac{\mathcal{G}(M_{♄} + M)}{2a},
$$

with

$$
\begin{aligned}
K &= T + V, \\
T &= \frac{Mv^2}{2}, \\
V &= -M \frac{\mathcal{G}(M_{♄} + M)}{2a}.
\end{aligned}
$$

## A.2 The orbital inclination $I$ and the longitude of the ascending node ☊

You can get them from the angular momentum of the body $B$ with respect to the body $A$, which is here Saturn:

$$
\begin{aligned}
\overrightarrow{\sigma_A} &= \vec{r} \times (M\vec{v}) \\
&= Mr^2\dot{\theta}^2 \left(\sin I \sin \text{☊} \widehat{e_x} - \sin I \cos \text{☊} \widehat{e_y} + \cos I \widehat{e_z}\right),
\end{aligned}
$$

from which you get $I$ and ☊.

## A.3 The eccentricity $e$ and the longitude of the pericentre $\varpi$

Please accept that

$$\vec{e} = \frac{\vec{v} \times \overrightarrow{\sigma_A}}{M\mathcal{G}(M_{\text{♄}} + M)} - \widehat{e_r}$$

and we have, in the reference frame in which $(\widehat{e_X},\widehat{e_Y})$ is the orbital plane

$$\vec{e} = e\left(\cos\omega\widehat{e_X} + \sin\omega\widehat{e_Y}\right),$$

from which you derive $e$ and $\omega$ straightforwardly. Moreover, $\varpi = \omega + \text{☊}$. You can get this reference frame from 2 rotations from the initial frame, the first one of axis 3 and angle ☊, and the second one of axis 1 and angle $I$.

You can also get the eccentricity $e$ from the energy, i.e.

$$e = \sqrt{1 + 2\frac{KC^2}{M\mathcal{G}(M_{\text{♄}} + M)}},$$

where $C = r^2\dot{\theta} = \sigma_A/M$.

## A.4 The mean longitude

The Kepler equation give the mean anomaly $\mathcal{M}$ from the eccentric one $E$

$$\mathcal{M} = E - e\sin E,$$

and

$$
\begin{aligned}
x_1 &= a\cos E - ae, \\
y_1 &= a\sqrt{1-e^2}\sin E, \\
r &= a(1 - e\cos E),
\end{aligned}
$$

where $x_1$ and $y_1$ are the coordinates of the body $B$ in the reference frame which is bound to the orbit, and in which the semimajor axis is also the axis of abscissa. This gives

$$E = \arctan 2\left(\frac{y_1}{a\sqrt{1-e^2}}, e + \frac{x_1}{a}\right).$$

And you finally have $\lambda = \mathcal{M} + \varpi$.

# B From the osculating to the cartesian elements

## B.1 Kepler's equation

The first step is to solve Kepler's equation

$$\mathcal{M} = E - e \sin E.$$

$\mathcal{M}$ is straightforwardly derived from $\lambda - \varpi$. You can get the eccentric anomaly $E$ in iterating the following scheme:

$$\begin{aligned} E_0 &= \mathcal{M} \\ E_{n+1} &= \mathcal{M} + e \sin E_n \end{aligned}$$

You can optimize this algorithm with

$$\begin{aligned} E_0 &= \mathcal{M} + e \sin \mathcal{M} \\ E_{n+1} &= E_n - \frac{E_n - e \sin E_n - \mathcal{M}}{1 - e \cos E_n}, \end{aligned}$$

which is Newton's algorithm:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}.$$

Of course, the convergence of the scheme must be carefully checked.

We get straightforwardly from the eccentric anomaly

$$\begin{aligned} x_1 &= a \cos E - ae, \\ y_1 &= a\sqrt{1 - e^2} \sin E, \\ r &= a(1 - e \cos E). \end{aligned}$$

where $x_1$ and $y_1$ locate the body $B$ in the reference frame which is bound to the orbit, the semimajor axis being the axis of abscissa. We then get the cartesian coordinates $x$, $y$ et $z$ in the original frame from 3 rotations:

1. axis 3, angle $\omega = \varpi - \Omega$
2. axis 1, angle $I$
3. axis 3, angle $\Omega$.

And you get the velocity from the formula

$$\vec{v} = \frac{\overrightarrow{\sigma_A} \wedge (\widehat{e_r} + \vec{e})}{Ma(1 - e^2)}.$$