Estimating within-cluster and between-cluster spillover effects in randomized saturation designs

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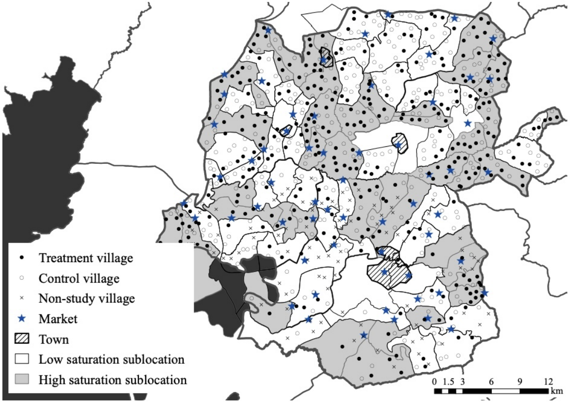
Joint work with Lei Shi and Peng Ding

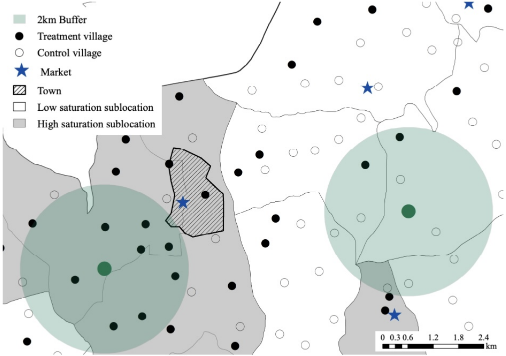
Outline

Motivating example: large-scale cash transfer experiment in rural Kenya (Egger et al., 2022)

- Two-stage *randomized saturation* design:
  - **Stage 1:** Sublocations randomized to high and low saturation.
  - **Stage 2:** Villages randomized to treatment at the assigned saturation rate.
- Geography (markets, labor, mobility) induces interference *within* and *between* sublocations.



Spatial exposure to treatment



- Evidence of meaningful spillovers on non-recipient households and local markets.
- Positive effects on local enterprise activity and consumption reported.

Randomized saturation designs

- Randomize treatment in two stages (Hudgens and Halloran, 2008; Baird et al., 2018; Basse and Feller, 2018):
  1. Stage 1 (cluster level): Randomize clusters (communities, schools, villages, networks) to different saturation levels (high vs. low coverage).
  2. Stage 2 (unit level): Randomize individual units within each cluster to treatment, according to the assigned saturation rate.
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  2. Stage 2 (unit level): Randomize individual units within each cluster to treatment, according to the assigned saturation rate.
- Enables identification of both direct and spillover effects.
- Widely-used in public health and social sciences: deworming on health (Miguel and Kremer, 2004), vaccination on health (Ali et al., 2005), labor market policies on displacement (Crépon et al., 2013), school governance on learning (Pradhan et al., 2014), microcredit on business outcomes (Banerjee et al., 2015), education on fertility (Duflo et al., 2015), cash transfers on school attendance (Barrera-Osorio et al., 2019).

Motivation

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- Common assumption in prior work: spillovers **only** occur within clusters, no interference between clusters.
- Not reasonable in many cases:
  - Real-world clusters are rarely isolated.
  - Networks, mobility, markets, or environmental channels can connect units across cluster boundaries.
  - Ignoring between-cluster interference can bias effect estimates and mislead policy conclusions.

Our contribution and roadmap

- Formalize analysis in randomized saturation designs, allowing for *both* within- and between-cluster spillovers.

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- Estimands: Direct effect, within-cluster indirect effect, and between-cluster indirect effect.
- Estimators: Design-based Horvitz–Thompson and Hájek estimators by inverse probability weighting.
- Inference: Closed-form asymptotic variance, conservative variance estimators, CLTs, and Wald CIs.

Notation and exposure mapping

- i: unit/village index.
- $A_i \in \{0, 1\}$: treatment indicator.
- $Y_i(\boldsymbol{a})$ for $\boldsymbol{a} = (a_1, \ldots, a_n)^{\mathrm{T}}$: 2^n potential outcomes.

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- Potential outcome reduces to $Y_i(a, s, h)$.
- Finite population regime: fixed potential outcomes, randomness from treatment.

Outline

Controlled direct and indirect effects

- The controlled direct effect of treatment A_i on the outcome, while holding the exposure variables (S_i, H_i) fixed at values (s, h):

$$\mathrm{DE}(s, h) = n^{-1} \sum_{i=1}^{n} Y_i(1, s, h) - n^{-1} \sum_{i=1}^{n} Y_i(0, s, h).$$

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- The within-cluster controlled indirect effect of S_i on the outcome, holding H_i fixed at h:

$$\mathrm{WIE}(s, s', h) = n^{-1} \sum_{i=1}^{n} Y_i(0, s, h) - n^{-1} \sum_{i=1}^{n} Y_i(0, s', h).$$

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- The between-cluster controlled indirect effect of H_i on the outcome, holding S_i fixed at s:

$$\mathrm{BIE}(s, h, h') = n^{-1} \sum_{i=1}^{n} Y_i(0, s, h) - n^{-1} \sum_{i=1}^{n} Y_i(0, s, h').$$

Marginal direct and indirect effects

- The marginal direct effect of treatment A_i, marginalizing over the distribution of treatments for all other villages:

$$\mathrm{DE} = n^{-1} \sum_{i=1}^{n} \underbrace{E_{\boldsymbol{A}_{(-i)}} \{Y_i(1, S_i, H_i)\}}_{\text{distribution of } \boldsymbol{A}_{(-i)}} - n^{-1} \sum_{i=1}^{n} E_{\boldsymbol{A}_{(-i)}} \{Y_i(0, S_i, H_i)\}.$$

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$$\mathrm{WIE}(s, s') = n^{-1} \sum_{i=1}^{n} \underbrace{E_{\boldsymbol{A}_{(-i)} | S_i = s} \{ Y_i(0, s, H_i) \}}_{\text{conditional distribution } \boldsymbol{A}_{(-i)} \text{ given } S_i = s} - n^{-1} \sum_{i=1}^{n} E_{\boldsymbol{A}_{(-i)} | S_i = s'} \{ Y_i(0, s', H_i) \} .$$

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- The between-cluster marginal indirect effect:

$$\mathrm{BIE}(h, h') = n^{-1} \sum_{i=1}^{n} E_{\boldsymbol{A}_{(-i)} \mid H_i = h} \{Y_i(0, S_i, h)\} - n^{-1} \sum_{i=1}^{n} E_{\boldsymbol{A}_{(-i)} \mid H_i = h'} \{Y_i(0, S_i, h')\}.$$

Policy-specific direct and indirect effects

- For a given treatment policy ψ, define the policy-specific direct and indirect effects:

$$\mathrm{DE}_{\psi} = n^{-1}\sum_{i=1}^{n} E_{\psi}\{Y_i(1,S_i,H_i)\} - n^{-1}\sum_{i=1}^{n} E_{\psi}\{Y_i(0,S_i,H_i)\},$$
$$\mathrm{WIE}_{\psi} = n^{-1}\sum_{i=1}^{n} E_{\psi}\{Y_i(0,1,H_i)\} - n^{-1}\sum_{i=1}^{n} E_{\psi}\{Y_i(0,0,H_i)\},$$
$$\mathrm{BIE}_{\psi} = n^{-1}\sum_{i=1}^{n} E_{\psi}\{Y_i(0,S_i,1)\} - n^{-1}\sum_{i=1}^{n} E_{\psi}\{Y_i(0,S_i,0)\},$$

  - subscript ψ: expectation under distribution of (S_i, H_i) conditional on $A_i = a$ induced by policy ψ for $a = 0, 1$.

- Comparison between different policies ψ_1 vs ψ_2, e.g., $\mathrm{DE}_{\psi_1} - \mathrm{DE}_{\psi_2}$.

Outline

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  - Horvitz–Thompson estimator:

$$\hat{Y}^{\mathrm{ht}}(a,s,h) = n^{-1}\sum_{i=1}^{n} \frac{\mathbb{I}_i(a,s,h)}{\pi_i(a,s,h)} Y_i.$$

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  - Hájek estimator:

$$\hat{Y}^{\mathrm{haj}}(a, s, h) = n^{-1} \sum_{i=1}^{n} \frac{\mathbb{I}_i(a, s, h)}{\pi_i(a, s, h)} Y_i \Big/ n^{-1} \sum_{i=1}^{n} \frac{\mathbb{I}_i(a, s, h)}{\pi_i(a, s, h)}.$$

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- Controlled effect can be estimated by differences between these average estimators.

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$$\hat{Y}^{\text{ht}}(a, s, h; \Gamma) \quad = \quad n^{-1} \sum_{i=1}^{n} \frac{\mathbb{I}_i(a, s, h)\gamma_i(a, s, h)}{\pi_i(a, s, h)} Y_i.$$

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- Hájek-type estimator

$$\hat{Y}^{\text{haj}}(a, s, h) = n^{-1} \sum_{i=1}^{n} \gamma_i(a, s, h) \cdot n^{-1} \sum_{i=1}^{n} \frac{\mathbb{I}_i(a, s, h)\gamma_i(a, s, h)}{\pi_i(a, s, h)} Y_i \Big/ n^{-1} \sum_{i=1}^{n} \frac{\mathbb{I}_i(a, s, h)\gamma_i(a, s, h)}{\pi_i(a, s, h)}.$$

Choice of weights

- Choose weights to target policy ψ for different effects:

$$\gamma_{i,\psi}^{\mathrm{DE}}(a, s, h) = \pi_{i,\psi}(\cdot, s, h), \quad \gamma_{i,\psi}^{\mathrm{WIE}}(a, s, h) = \pi_{i,\psi}(\cdot, \cdot, h), \quad \gamma_{i,\psi}^{\mathrm{BIE}}(a, s, h) = \pi_{i,\psi}(\cdot, s, \cdot).$$

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- All of the above weights can be simulated based on the design.
- Can be used to construct estimators for the policy-specific effects: for $* \in \{\mathrm{ht}, \mathrm{haj}\}$,

$$\begin{aligned}
\hat{\mathrm{DE}}_{\psi}^{*} &= \sum_{s,h=0,1}\{\hat{Y}^{*}(1,s,h;\Gamma_{\psi}^{\mathrm{DE}})-\hat{Y}^{*}(0,s,h;\Gamma_{\psi}^{\mathrm{DE}})\},\\
\hat{\mathrm{WIE}}_{\psi}^{*} &= \sum_{h=0,1}\{\hat{Y}^{*}(0,1,h;\Gamma_{\psi}^{\mathrm{WIE}})-\hat{Y}^{*}(0,0,h;\Gamma_{\psi}^{\mathrm{WIE}})\},\\
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\end{aligned}$$

- Marginal direct/indirect effects are a special case with the implemented policy.

Outline

Asymptotic variance: Horvitz–Thompson

- For the Horvitz–Thompson estimator:

$$
\begin{aligned}
\mathrm{var}\{\hat{Y}^{\mathrm{ht}}(a,s,h;\Gamma)\} = {} & \underbrace{n^{-2}\sum_{i=1}^{n}\frac{1-\pi_i(a,s,h)}{\pi_i(a,s,h)}\{\gamma_i(a,s,h)Y_i(a,s,h)\}^2}_{\text{contribution from the variation of unit } i} \\
& + n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{\pi_{ij}(a,s,h;a,s,h)-\pi_i(a,s,h)\pi_j(a,s,h)}{\pi_i(a,s,h)\pi_j(a,s,h)} \\
& \underbrace{\qquad\qquad \times\gamma_i(a,s,h)\gamma_j(a,s,h)Y_i(a,s,h)Y_j(a,s,h).}_{\text{contribution from the covariance between units due to interference}}
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$$

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$$+ \underbrace{n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{\pi_{ij}(a,s,h;a,s,h)-\pi_i(a,s,h)\pi_j(a,s,h)}{\pi_i(a,s,h)\pi_j(a,s,h)} \times \gamma_i(a,s,h)\gamma_j(a,s,h)Y_i(a,s,h)Y_j(a,s,h).}_{\text{contribution from the covariance between units due to interference}}$$

- Can be written compactly as:

$$\mathrm{var}\{\hat{Y}^{\mathrm{ht}}(a,s,h;\Gamma)\} = n^{-2}\boldsymbol{Y}(a,s,h)^{\mathrm{T}}\boldsymbol{\Gamma}(a,s,h)\boldsymbol{\Omega}(a,s,h)\boldsymbol{\Gamma}(a,s,h)\boldsymbol{Y}(a,s,h).$$

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  - The centered potential outcomes are:

$$\tilde{Y}_i(a, s, h) = Y_i(a, s, h) - \frac{n^{-1} \sum_{i=1}^{n} \gamma_i(a, s, h) Y_i(a, s, h)}{n^{-1} \sum_{i=1}^{n} \gamma_i(a, s, h)}.$$

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  - Compactly: $n^{-2} \tilde{\boldsymbol{Y}}(a, s, h)^{\mathrm{T}} \boldsymbol{\Gamma}(a, s, h) \boldsymbol{\Omega}(a, s, h) \boldsymbol{\Gamma}(a, s, h) \tilde{\boldsymbol{Y}}(a, s, h)$.

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  - Compactly: $n^{-2}\tilde{\boldsymbol{Y}}(a,s,h)^{\mathrm{T}}\boldsymbol{\Gamma}(a,s,h)\boldsymbol{\Omega}(a,s,h)\boldsymbol{\Gamma}(a,s,h)\tilde{\boldsymbol{Y}}(a,s,h)$.
- Conditions for the consistency of the estimators (Aronow and Samii, 2017).

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  - Same form of the Horvitz–Thompson variance, with $\boldsymbol{Y}(a, s, h)$ replaced by the **centered** potential outcomes $\tilde{\boldsymbol{Y}}(a, s, h)$.
  - The centered potential outcomes are:

$$\tilde{Y}_i(a, s, h) = Y_i(a, s, h) - \frac{n^{-1} \sum_{i=1}^{n} \gamma_i(a, s, h) Y_i(a, s, h)}{n^{-1} \sum_{i=1}^{n} \gamma_i(a, s, h)}.$$

  - Compactly: $n^{-2} \tilde{\boldsymbol{Y}}(a, s, h)^{\mathrm{T}} \boldsymbol{\Gamma}(a, s, h) \boldsymbol{\Omega}(a, s, h) \boldsymbol{\Gamma}(a, s, h) \tilde{\boldsymbol{Y}}(a, s, h)$.
- Conditions for the consistency of the estimators (Aronow and Samii, 2017).
- The centering property explains why empirically the Hájek estimator is usually more stable.

Covariance structure

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\begin{aligned}
&\operatorname{cov}\{\hat{Y}^{\mathrm{ht}}(a,s,h;\Gamma),\, \hat{Y}^{\mathrm{ht}}(a',s',h';\Gamma)\} \\
&= \underbrace{-n^{-2}\sum_{i=1}^{n}\gamma_i(a,s,h)\gamma_i(a',s',h')Y_i(a,s,h)Y_i(a',s',h')}_{\text{Covariance of different potential outcomes for the same unit } i} \\
&\quad \underbrace{+\, n^{-2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{\pi_{ij}(a,s,h;a',s',h') - \pi_i(a,s,h)\pi_j(a',s',h')}{\pi_i(a,s,h)\pi_j(a',s',h')} \times \gamma_i(a,s,h)\gamma_j(a',s',h')Y_i(a,s,h)Y_j(a',s',h')}_{\text{Covariance of different potential outcomes between units}}
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&\quad + n^{-2} \sum_{i=1}^{n} \sum_{j \neq i} \underbrace{\frac{\pi_{ij}(a, s, h; a', s', h') - \pi_i(a, s, h)\pi_j(a', s', h')}{\pi_i(a, s, h)\pi_j(a', s', h')} \times \gamma_i(a, s, h)\gamma_j(a', s', h') Y_i(a, s, h) Y_j(a', s', h')}_{\text{Covariance of different potential outcomes between units}}
\end{aligned}
$$

- Hájek covariance: Same form but with centered potential outcomes $\tilde{Y}_i(a, s, h)$ and $\tilde{Y}_j(a', s', h')$

CLT and confidence intervals

- Ongoing: to develop central limit theorems and confidence intervals for the proposed estimators.
- Central limit theorem: use Stein's method to capture correlation due to network interference and characterize asymptotic distributions.
- Confidence interval: relies on constructing valid variance estimators.

Outline

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- **Partially saturated regression:**

$$Y_i \sim 1 + A_i + \tilde{S}_i + \tilde{H}_i + A_i\tilde{S}_i + A_i\tilde{H}_i,$$

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- See Basse and Feller (2018); Zhao and Ding (2022).

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  - Adapting the framework to the current setting is a future step to enhance the analysis.
- Experimental validation:
  - The Cash Transfer experiment has a publicly available dataset.
  - Working on experimental validation with this study.

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