Calculation of BR and BZ in GTS Loader

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Abstract

A description of the algebra used in calculation of BR and BZ starting from flux coordinates quantities obtained in GTS and ESI functions.

1 Flux coordinates used in GTS

GTS is using straight field line flux coordinates while keep toroidal coordinate ϕ still corresponding to toroidal angle in Tokamak cylindrical coordinates. Radial coordinate is chosen to be the square root of normalized poloidal flux, i.e. $a = \sqrt{\frac{\psi_p}{\psi_{p,e}}}$, where $\psi_{p,e}$ is the poloidal flux at the last closed flux surface. Poloidal angle coordinate θ is chosen so that along the field line $\frac{d\theta}{d\phi} = constant$. $\theta = 0$ at the outer mid-plane and increases counter-clockwise if view along $\hat{\Phi}$ from the right-side cut of the torus. $\{a, \theta, \phi\}$ forms a right-handed coordinate system. Compare to the cylindrical coordinates $\{R, \Phi, Z\}$, $\hat{\phi}$ in flux coordinates is on the opposite direction of $\hat{\Phi}$.

2 Useful quantities provided by ESI subroutines

ESI code provides calculation of the following quantities on given points specified by their flux coordinates (a, θ, ϕ)(with their variable names in ESI code):

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1. Cylindrical coordinates

$$R(a,\theta), Z(a,\theta) \qquad (rj, zj)$$

2. Derivative of the poloidal flux

$$\frac{d\psi_p}{da} \qquad (-gYaj)$$

3. Partial derivatives of components of cylindrical coordinates respect to flux coordinates

$$\frac{\partial R}{\partial \theta} \qquad (rqj)$$
$$\frac{\partial R}{\partial a} \qquad (raj)$$
$$\frac{\partial Z}{\partial \theta} \qquad (zqj)$$
$$\frac{\partial Z}{\partial a} \qquad (zaj)$$

3 Quantities related to the coordinates transformation between cylindrical and flux coordinates

Relevant quantities are:

3.1 Jacobian

The Jacobian is defined as:

$$J_{a\theta\phi} \equiv (\nabla a \times \nabla\theta \cdot \nabla\phi)^{-1} \tag{1}$$

To Calculate $J_{a\theta\phi}$, we notice that:

$$\begin{aligned} J_{RZ\Phi} \equiv & (\nabla R \times \nabla Z \cdot \nabla\Phi)^{-1} \\ = & (\hat{R} \times \hat{Z} \cdot \frac{\hat{\Phi}}{R})^{-1} \\ = & R \end{aligned} \tag{2}$$

On the other hand:

$$
\begin{aligned}
J_{RZ\Phi} \equiv& (\nabla R \times \nabla Z \cdot \nabla \Phi)^{-1} \\
=& [(\frac{\partial R}{\partial a}\nabla a + \frac{\partial R}{\partial \theta}\nabla \theta + \frac{\partial R}{\partial \phi}\nabla \phi) \times (\frac{\partial Z}{\partial a}
\end{aligned}
$$

$$
\begin{aligned}
g^{aa} &= \nabla a \cdot \nabla a \\
g^{a\theta} &= \nabla a \cdot \nabla \theta \\
g^{\theta\theta} &= \nabla \theta \cdot \nabla \theta \\
g^{\phi\phi} &= \nabla \phi \cdot \nabla \phi
\end{aligned}
$$

The last term is straight forward, since $\nabla\phi = \frac{1}{R}\hat{\phi}$:

$$
g^{\phi\phi} = \frac{1}{R^2}\hat{\phi} \cdot \hat{\phi} = \frac{1}{R^2} \tag{7}
$$

Now, we need to express the other three terms in terms of the ESI quantities. We know that

$$
\begin{aligned}
\nabla a &= \frac{\partial a}{\partial R}\nabla R + \frac{\partial a}{\partial Z}\nabla Z \\
\nabla \theta &= \frac{\partial \theta}{\partial R}\nabla R + \frac{\partial \theta}{\partial Z}\nabla Z
\end{aligned} \tag{8}
$$

thus:

$$
\begin{aligned}
g^{aa} &= (\frac{\partial a}{\partial R})^2 + (\frac{\partial a}{\partial Z})^2 \\
g^{\theta\theta} &= (\frac{\partial \theta}{\partial R})^2 + (\frac{\partial \theta}{\partial Z})^2 \\
g^{a\theta} &= \frac{\partial a}{\partial R}\frac{\partial \theta}{\partial R} + \frac{\partial a}{\partial Z}\frac{\partial \theta}{\partial Z}
\end{aligned} \tag{9}
$$

above, we used the fact that $\nabla R \cdot \nabla R = \nabla Z \cdot \nabla Z = 1$ and $\nabla R \cdot \nabla Z = 0$.

In order to calculate the partial derivatives on the right hand side of Eq.9, we use the fact that our coordinate transformation from cynlindrical to flux coordinates is actually $(R, Z) \to (a, \theta)$ and $\Phi \to -\phi$, which is essentially a 2D mapping times a 1D mapping. The Jacobian Matrix of the 2D mapping is:

$$
\overleftrightarrow{J}_{RZ\to a\theta} = \begin{pmatrix} \frac{\partial R}{\partial a} & \frac{\partial R}{\partial \theta} \\ \frac{\partial Z}{\partial a} & \frac{\partial Z}{\partial \theta} \end{pmatrix} \tag{10}
$$

The Jacobian Matrix of the inverse mapping $(a, \theta) \to (R, Z)$ must be the inverse of this matrix, i.e. :

$$
\overleftrightarrow{J}_{a\theta\to RZ} = \begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial Z} \end{pmatrix} = \overleftrightarrow{J}^{-1}_{RZ\to a\theta} \tag{11}
$$

Using the matrix inversion formula, we then have:

$$\begin{aligned}
\frac{\partial a}{\partial R} &= \frac{1}{J_{RZ}}(\frac{\partial Z}{\partial \theta}) \\
\frac{\partial a}{\partial Z} &= \frac{1}{J_{RZ}}(-\frac{\partial R}{\partial \theta}) \\
\frac{\partial \theta}{\partial R} &= \frac{1}{J_{RZ}}(-\frac{\partial Z}{\partial a}) \\
\frac{\partial \theta}{\partial Z} &= \frac{1}{J_{RZ}}(\frac{\partial R}{\partial a})
\end{aligned} \tag{12}$$

where

$$J_{RZ} \equiv det(\overleftrightarrow{J}_{RZ \to a\theta}) = \frac{\partial R}{\partial a} \cdot \frac{\partial Z}{\partial \theta} - \frac{\partial R}{\partial \theta} \cdot \frac{\partial Z}{\partial a}$$

Finally, we substitute Eq.12 into Eq.9, we arrive at the final form of the metric tensor elements:

$$\begin{aligned}
g^{aa} &= \frac{1}{J_{RZ}^2}\left[\left(\frac{\partial Z}{\partial \theta}\right)^2 + \left(\frac{\partial R}{\partial \theta}\right)^2\right] \\
g^{\theta\theta} &= \frac{1}{J_{RZ}^2}\left[\left(\frac{\partial Z}{\partial a}\right)^2 + \left(\frac{\partial R}{\partial a}\right)^2\right] \\
g^{a\theta} &= -\frac{1}{J_{RZ}^2}\left(\frac{\partial Z}{\partial \theta}\frac{\partial Z}{\partial a} + \frac{\partial R}{\partial \theta}\frac{\partial R}{\partial a}\right)
\end{aligned} \tag{13}$$

4 Calculation of B_R and B_Z

We are now prepared to calculate B_R and B_Z in terms of ESI quantities. Here I'll show how B_R is obtained, B_Z will follow the exactly same argument.

$$\begin{aligned}
B_R = \frac{\overrightarrow{B} \cdot \nabla R}{|\nabla R|} &= \frac{\overrightarrow{B} \cdot (\frac{\partial R}{\partial \theta}\nabla\theta + \frac{\partial R}{\partial a}\nabla a)}{|\nabla R|} \\
&= \frac{(\overrightarrow{B} \cdot \nabla\theta)\frac{\partial R}{\partial \theta} + (\overrightarrow{B} \cdot \nabla a)\frac{\partial R}{\partial a}}{|\nabla R|} \\
&= \frac{(\overrightarrow{B} \cdot \nabla\theta)\frac{\partial R}{\partial \theta}}{|\nabla R|}
\end{aligned} \tag{14}$$

Here we used the fact that a is the label of flux surface, so ∇a is perpendicular to $\overrightarrow{B}$.

Note that for Clebsch representation, $\overrightarrow{B}$ can be written as:

$$\overrightarrow{B} = I\nabla\phi + \nabla\psi_p \times \nabla\phi \tag{15}$$

where

$$\nabla\psi_p = \frac{d\psi_p}{da}\nabla a$$

substituting Eq.15 to Eq.14, we have:

$$\begin{aligned} B_R &= \frac{\frac{d\psi_p}{da}(\nabla a \times \nabla\phi \cdot \nabla\theta) \cdot \frac{\partial R}{\partial\theta}}{|\nabla R|} \\ &= \frac{\frac{d\psi_p}{da}(-J_{a\theta\phi}^{-1}) \cdot \frac{\partial R}{\partial\theta}}{|\nabla R|} \end{aligned} \tag{16}$$

where $J_{a\theta\phi} \equiv (\nabla a \times \nabla\theta \cdot \nabla phi)^{-1}$ is the Jacobian of our flux coordinate system defined in Section 3.1.

For the denominator, we have:

$$\begin{aligned} |\nabla R| &= |\frac{\partial R}{\partial a}\nabla a + \frac{\partial R}{\partial\theta}\nabla\theta| \\ &= \sqrt{\left(\frac{\partial R}{\partial a}\right)^2 (\nabla a \cdot \nabla a) + \left(\frac{\partial R}{\partial\theta}\right)^2 (\nabla\theta \cdot \nabla\theta) + 2\frac{\partial R}{\partial a}\frac{\partial R}{\partial\theta}\nabla a \cdot \nabla\theta} \\ &= \sqrt{\left(\frac{\partial R}{\partial a}\right)^2 g^{aa} + \left(\frac{\partial R}{\partial\theta}\right)^2 g^{\theta\theta} + 2\frac{\partial R}{\partial a}\frac{\partial R}{\partial\theta}g^{a\theta}} \end{aligned} \tag{17}$$

where g^{aa},$g^{a\theta}$, and $g^{\theta\theta}$ are metric tensor elements defined in Section3.2.

Now, we have successfully expressed B_R in terms of the quantities provided by ESI routines.

Similar calculation will show that:

$$B_Z = \frac{\frac{d\psi_p}{da}(-J_{a\theta\phi}^{-1}) \cdot \frac{\partial Z}{\partial\theta}}{|\nabla Z|} \tag{18}$$

in which,

$$|\nabla Z| = \sqrt{\left(\frac{\partial Z}{\partial a}\right)^2 g^{aa} + \left(\frac{\partial Z}{\partial\theta}\right)^2 g^{\theta\theta} + 2\frac{\partial Z}{\partial a}\frac{\partial Z}{\partial\theta}g^{a\theta}} \tag{19}$$