**Midterm**
**Math 715 Fall 2018**
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**Name:** Key

You must **show your work clearly and justify everything** to receive credit.

**Problem 1** [10 points] Consider the matrix decomposition:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Label the columns of the leftmost matrix and the rows of the rightmost matrix, so as to write $A$ as a sum of rank-one matrices. Show how to modify this sum so as to find the singular value decomposition of $A$. Also, write down a matrix $A_1$ which is a best rank 1 approximation of $A$, and compute the condition number of $A$.

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \end{bmatrix} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} - & 2b_1^T & - \\ - & (-3)b_2^T & - \end{bmatrix}$$

$$= 2a_1b_1^T - 3a_2b_2^T = 2\|a_1\|\|b_1\| \left(\frac{a_1}{\|a_1\|}\right)\left(\frac{b_1}{\|b_1\|}\right)^T + 3\|a_2\|\|b_2\| \left(\frac{a_2}{\|a_2\|}\right)\left(\frac{-b_2}{\|b_2\|}\right)^T$$

$$\|a_1\| = \sqrt{6} \quad \|a_2\| = \sqrt{3} \quad \|b_1\| = \sqrt{5} = \|b_2\| \Rightarrow \begin{cases} 2\|a_1\|\|b_1\| = 2\sqrt{30} = \sqrt{120} \\ 3\|a_2\|\|b_2\| = 3\sqrt{15} = \sqrt{135} \end{cases}$$

Let $u_1 = \frac{a_2}{\|a_2\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $u_2 = \frac{a_1}{\|a_1\|} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$, $v_1 = \frac{-b_2}{\|b_2\|} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$

$$v_2 = \frac{b_1}{\|b_1\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \Rightarrow A = 3\sqrt{15}\, u_1 v_1^T + 2\sqrt{30}\, u_2 v_2^T$$

$$K(A) = \frac{3\sqrt{15}}{2\sqrt{30}} = \frac{3}{2\sqrt{2}}$$

$$A_1 = 3\sqrt{15}\, u_1 v_1^T = 3\sqrt{15} \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} = 3 \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 3 & -6 \\ 3 & -6 \end{bmatrix}$$

**Problem 2** [10 points] Consider a Householder reflector $H = I - 2uu^T$, with $\|u\|_2 = 1$. Compute $H^T$ and $HH^T$. Also pick an ONB for $\langle u\rangle^\perp$ and produce an ONB of eigenvectors for $H$, and identify the corresponding eigenvalues. What is $\det(H)$?

* $H^T = (I - 2uu^T)^T = I^T - 2u^{T^T}u^T = I - 2uu^T = H$

* $HH^T = (I-2uu^T)(I-2uu^T) = I - 2uu^T - 2uu^T + 4u\underbrace{(u^Tu)}_{=1}u^T$

$= I - 4uu^T + 4uu^T = I$

(Note $u^Tu = \|u\|^2 = 1$)

* $Hu = (I - 2uu^T)u = Iu - 2u\underbrace{u^Tu}_{=1} = u - 2u = -u.$

So $u$ is an eigenvector for the eigenvalue $-1$

* Pick an ONB $\{v_1, \dots, v_{n-1}\}$ for $\langle u\rangle^\perp$

Then $Hv_j = (I - 2uu^T)v_j = v_j - 2u\underbrace{u^Tv_j}_{=0} = v_j$

So each $v_j$ is an eigenvector for the eigenvalue $1$

i.e. $1$ has multiplicity $n-1$

* $\det(H) = (-1)(1)^{n-1} = -1$

**Problem 3** [10 points] Let $A$ be a symmetric $n$-by-$n$ matrix with positive elements. Show that if $A$ is strictly row-wise diagonally dominant, then $A$ is positive definite.

$A^T = A$ & $A(i,j) > 0 \ \forall i,j$

Also $A(i,i) > \sum_{\substack{j \neq i \\ j=1}}^{n} A(i,j)$ (strictly row-wise diagonally dominant)

1) Gershgorin Theorem $\Rightarrow$ For every eigenvalue $\lambda$ we have $|\lambda - A(i,i)| < \sum_{j \neq i} A(i,j)$ for some $i$

2) $A$ symmetric $\Rightarrow$ Every $\lambda$ is real.

1) & 2) $\Rightarrow$ Every $\lambda$ is positive.

Spectral Thm $\Rightarrow$ $A = V \Lambda V^T$ with $V^T V = I$

So $x^T A x = x^T V \Lambda V^T x$

$= x^T V \Lambda^{1/2} \Lambda^{1/2} V^T x$

$= \| \Lambda^{1/2} V^T x \|_2^2 > 0$ iff $x \neq 0$.

$\Rightarrow A$ is positive definite