HW 6

Problem 1. Suppose A is an $n \times n$ matrix and $D = \mathrm{Diag}(d)$ is a diagonal matrix made from the $n \times 1$ vector d. What does left-mutiplication by D do to A? (Think elementary operations.) What does right-multiplication do?

Consider the matrix

$$\begin{bmatrix} 3 & 4 & 0 \\ 1 & -1 & 1 \\ 0 & 9 & 2 \end{bmatrix}$$

and let $D = \mathrm{Diag}(a, b, c)$. Compute $\tilde{A} := D^{-1}AD$ and then solve for a, b, c so that $\tilde{A}$ becomes a symmetric matrix.

Problem 2. Consider the tridiagonal s.p.d:

$$A = \begin{bmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & a_3 & b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & & b_{n-1} & a_n \end{bmatrix}$$

Then A has the Cholesky factorization $A = LL^T$ for some matrix

$$L = \begin{bmatrix} x_1 & & & & \\ y_1 & x_2 & & & \\ & y_2 & x_3 & & \\ & & \ddots & \ddots & \\ & & & y_{n-1} & x_n \end{bmatrix}$$

Find an algorithm for computing L. First, formally compute the first few rows and columns of LL^T in terms of the unknowns x_i and y_j. Then, match them to the entries of A and solve repeatedly. You should find that the algorithm requires n square-root operations and $3n - 3$ other arithmetic operations.

Problem 3. Draw an $n \times n$ square grid in the plane, meaning that each side has n nodes, and so that $(0, 0)$ is the bottom left corner and all the edges have length 1.

Start by assuming $n = 3$.

(a) Enumerate the nodes using the lexicographic order given by the coordinates of the nodes and write down the Laplacian matrix L.

(b) Assume that the following Dirichlet boundary values are given:

$$\begin{aligned} g_{\text{left}} &= 0 \\ g_{\text{bottom}} &= 0 \\ g_{\text{right}} &= 2 \\ g_{\text{top}} &= 3 \end{aligned}$$

Solve the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{on the interior nodes} \\ u = g & \text{on the boundary nodes} \end{cases}$$

How many equations and how many unknowns are there? Does it matter that the definition of the boundary data at the corners is ambiguous?

Now repeat (a) and (b) with $n = 4$.