1. (10 pts)

The standard five-point stencil approximation of the Laplacian on a square grid using row-by-row ordering takes the block form

$$L = \frac{1}{h^2}\begin{bmatrix} T & I & & & \\ I & T & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & T & I \\ & & & I & T \end{bmatrix},$$

where h is the space between grid points, I is the identity matrix, and T is the matrix

$$T = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix}.$$

(a) Show that L is negative semidefinite.

(b) Prove that there exists a constant $c > 0$ (independent of h) such that every eigenvalue λ of L satisfies $|\lambda| \leq c/h^2$.

2. (10 pts)

Let A be the product

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

(a) Find the four spaces $R(A)$, $N(A)$, $R(A^T)$, $N(A^T)$ (the range and nullspace of A and A^T).

(b) Find the singular values of A.

3. (10 pts)

Let A be a full-rank $m \times n$ real matrix with $m \geq n$.

(a) Show that the system

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

has a unique solution $(r^T, x^T)^T$.

(b) Show that the x from (a) minimizes $\|Ax - b\|_2$.

4. (10 pts)

(a) State the definition of complete orthonormal set in a Hilbert space.

(b) Provide an example of a complete orthonormal set in $L^2((0,1) \times (0,1))$.

5. (10 pts)

Show that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, where

$$f_n(x) = \frac{n}{2}\, e^{-n|x|}, \quad x \in \mathbb{R},$$

is a (Dirac) delta sequence.

6. (10 pts)

Let $K : L^2(0,1) \to L^2(0,1)$ be the following operator:

$$Ku(x) = \int_0^1 u(y)\, dy.$$

(a) Show that K is a self-adjoint compact operator. Provide direct proofs based on the definitions of self-adjointness and compactness of an operator.

Consider the integral equation

$$u(x) - \lambda \int_0^1 u(y)\, dy = f(x),$$

where λ is a real number.

(b) Derive necessary and sufficient conditions on λ and $f \in L^2(0,1)$ for the existence of solutions for this equation.

(c) Derive necessary and sufficient conditions on λ and $f \in L^2(0,1)$ for the uniqueness of solutions for this equation.