Math 715 | Leidong Xu | HW2

if $0 \neq s \in R$ and $E \in R^{n \times n}$

prove $\|E(I - \frac{ss^T}{s^Ts})\|_F^2 = \|E\|_F^2 - \frac{\|Es\|_2^2}{s^Ts}$

Pf: since $\|A\|_F^2 = tr(A^TA)$

$$LHS = tr\left[(E - E\frac{ss^T}{s^Ts})^T(E - E\frac{ss^T}{s^Ts})\right]$$

scalar

$$= tr\left[(E^T - \frac{ss^TE^T}{s^Ts})(E - E\frac{ss^T}{s^Ts})\right]$$

$$= tr(E^TE) - tr(\frac{ss^TE^TE}{s^Ts}) - tr(\frac{E^TEss^T}{s^Ts}) + tr(\frac{s\cancel{s^Ts}s^TE^TE}{\cancel{(s^Ts)^2}})$$

$$= \|E\|_F^2 - tr(\frac{ss^TE^TE}{s^Ts} + \frac{E^TEss^T}{s^Ts} - \frac{ss^TE^TE}{s^Ts})$$

$$= \|E\|_F^2 - \frac{1}{s^Ts}tr(E^TEss^T)$$

$$= \|E\|_F^2 - \frac{\|Es\|_2^2}{s^Ts}$$

Lemma 1.5

(1) to prove $||x||_2 \le ||x||_1 \le \sqrt{n}\,||x||_2$

$$||x||_2 = (\textstyle\sum_n |x_n|^2)^{\frac{1}{2}}$$

$$= (x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)^{\frac{1}{2}}$$

$$\le (x_1^2)^{\frac{1}{2}} + (x_2^2)^{\frac{1}{2}} + \cdots + (x_n^2)^{\frac{1}{2}}$$

$$= |x_1| + |x_2| + \cdots |x_n|$$

$$\Downarrow$$

$$= ||x||_1$$

$\therefore$ $\boxed{||x||_2 \le ||x||_1}$

$\sqrt{n}\;||x||_2$

$$= (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}\ \overbrace{(1+1+\cancel{\cdots}+1)^{\frac{1}{2}}}^{\sqrt{n}} \quad 4$$

from Cauchy - schwartz inequality

$$\sum_{n=1}^{N} a_n b_n \le (\textstyle\sum a_n^2)^{\frac{1}{2}} (\sum b_n^2)^{\frac{1}{2}}$$

$\therefore$ $||x||_1 = |x_1| + |x_2| + \cdots + |x_n| \le 4 = \sqrt{n}\,||x_2|\,)$

(2) prove $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$

$\|x\|_\infty^2 = (\max|x_i|)^2$

$\|x\|_2^2 = \left[(x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2)^{\frac{1}{2}}\right]^2$

$= x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2$

which include $(\max|x_i|)^2$

$\therefore \|x\|_\infty^2 \le \|x\|_2^2$ and all positive

$\therefore \boxed{\|x\|_\infty \le \|x_2\|}$

$(\sqrt{n}\,\|x\|_\infty)^2 = n\cdot(\max|x_i|)^2 \ge x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2$

$\therefore \boxed{\|x\|_2 \le \sqrt{n}\,\|x\|_\infty}$

(3) $\because \|x_\infty\| \le \|x\|_2$ and $\|x\|_2 \le \|x_1\|$

$\therefore \boxed{\|x\|_\infty \le \|x\|_1}$

$\|x_1\| = |x_1| + |x_2| + |x_3| + \dots + |x_n|$ ①

$n\|x\|_\infty = n\,|x|_{max}$

$= \underbrace{|x|_{max} + |x|_{max} + \dots + |x|_{max}}_{n}$ ②

$\therefore$ ① $\le$ ②

$\therefore \boxed{\|x\|_1 \le n\|x\|_\infty}$

Lemma 1.6.

prove An operator norm is a matrix norm.

pf: operator norm is a matrix norm if:

(1) $||A|| \geq 0$ and $||A|| = 0$ if and only if $A = 0$

(2) $||\alpha A|| = |\alpha| \cdot ||A||$

(3) $||A+B|| \leq ||A|| + ||B||$

(4) $$||A||_{\hat{m}\hat{n}} \equiv \max_{\substack{x \neq 0 \\ x \in R^n}} \frac{||Ax||_{\hat{m}}}{||x||_{\hat{n}}}$$

$\because x \neq 0, \Rightarrow ||x||_{\hat{n}} \neq 0$

This could be approved if $||A||_{\hat{m}} = 0 \rightarrow ||Ax||_{\hat{m}} = 0 \Rightarrow ||Ax||_{\hat{m}\hat{n}}$

$||Ax||_{\hat{m}}$ is matrix norm and have the property

that $||A||_{\hat{m}\hat{n}} \geq 0$ and $||A|| = 0$ if and only if $A = 0$

(2) $$||\alpha A||_{\hat{m}\hat{n}} \equiv \max_{\substack{x \neq 0 \\ x \in R^n}} \frac{||\alpha Ax||_{\hat{m}\hat{n}}}{||x||_{\hat{n}}} \quad ①$$

again. because the top part is a matrix norm

$$① = \frac{|\alpha| ||Ax||}{||x||_{\hat{n}}} = |\alpha| \cdot ||A||_{\hat{m}\hat{n}}$$

(3) $\|A+B\|_{\hat{m}\hat{n}} \equiv \max \frac{\|(A+B)x\|_{\hat{m}}}{\|x\|_{\hat{n}}}$

② $\leq \max \frac{\|Ax\|_{\hat{m}} + \|Bx\|_{\hat{m}}}{\|x\|_{\hat{n}}}$

(triangular inequality)

③ $= \max\left(\frac{\|Ax\|_{\hat{m}}}{\|x\|_{\hat{n}}}\right) + \max\left(\frac{\|Bx\|_{\hat{m}}}{\|x\|_{\hat{n}}}\right)$

$= \|A\|_{\hat{m}} + \|B\|_{\hat{n}}$