HW8 | Math 715 | Leodong Xu

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In order to minimize $||AX - I_m||_F^2$

where X is define as $\begin{bmatrix} | & & | \\ x_1 & & x_m \\ | & & | \end{bmatrix}$

$$F(x) = Tr((AX - I_m)^T (AX - I_m)) = ||AX - I_m||_F^2$$

$$= \underbrace{Tr((AX)^T(AX))}_{①} - 2\,Tr(\underset{②}{AX}) + m$$

$$① = \begin{bmatrix} -(Ax_1)^T- \\ \vdots \\ -(Ax_m)^T- \end{bmatrix} \begin{bmatrix} | & & | \\ Ax_1 & \cdots & Ax_m \\ | & & | \end{bmatrix}$$

$$Tr((AX)^T(AX)) = \sum_{j=1}^{m} (Ax_j)^T (Ax_j)$$

$$= \sum_{j=1}^{m} x^T A^T A x_j$$

$$AX = \begin{bmatrix} -e_1^T A \\ \vdots \\ -e_m^T A \end{bmatrix} \begin{bmatrix} | & & | \\ x_1 & \cdots & x_m \\ | & & | \end{bmatrix}$$

$$Tr(AX) = \sum_{j=1}^{m} e_j^T A x_j$$

$$\nabla F = \begin{bmatrix} & | & \\ & 2A^TAx_j - 2A^Te_j \cdots & \\ & | & \end{bmatrix}$$

if $\nabla F = 0 \Rightarrow A^TAx_j = A^Te_j$

$\Rightarrow A^TAX = A^TI_m = A^T$ ③

Since $A = (U_r \Sigma_r V_r^T)_{[m \times r]}$

$A^TA = V_r \Sigma_r^2 V^T$

$A^T = V_r \Sigma_r U_r^T$

③ $\Longrightarrow V_r \Sigma_r^2 V_r^2 X = V_r \Sigma_r U_r^T$

$\Rightarrow \Sigma_r^2 V_r^T X = \Sigma_r U_r^T$

$\Rightarrow V_r^T X = \Sigma_r^{-1} U_r^T$

$\Rightarrow \underbrace{V_r V_r^T}_{④} X = \underbrace{V_r \Sigma_r^{-1} U_r^T}_{⑤}$

⑤ $= A^+$

④ $= \begin{bmatrix} 0 & \\ 0 & 0 \end{bmatrix}$

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show that

(1) $AA^{+}A = A$

$$AA^{+}A = A(A^{T}A)^{-1}A^{T}A$$
$$= A A^{-1}(A^{T})^{-1}A^{T}A$$
$$= A$$

(2) $A^{+}AA^{+} = A^{+}$

$$A^{+}AA^{+} = R^{-1}Q^{T}QR\,R^{-1}Q^{T}$$
$$= R^{-1}Q^{T} = A^{+}$$

(3) $A^{+}A = (A^{+}A)^{T}$

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(1) $AA^{+}A = A$

Is A full rank? We don't know

A is $m \times n$

AA^{+} need to be $m \times m$ Identity matrix, which should be the pseudo invers

(2) Same as (1)

I'm reading Penrose's paper, hopefully it will give me the answer.

(3) $A^+A = (A^+A)^T$

since A^+A is Hermtian?

(4) $AA^+ = (AA^+)^T$

since $(AA^+)^T$ is Hermtian?

All these four are Moore-Penrose conditions, which is required for Moore-Penrose inverse. I'm so confused about the logic in this question.

problem 3

A is $m \times n$ matrix with singular value $\sigma_1 \ge \cdots \ge \sigma_n \ge n$. show that the singular value of $\begin{bmatrix} I_n \\ A \end{bmatrix}$ are $\sqrt{1+\sigma_j^2}$ for $i \le j \le n$

$$M = \begin{bmatrix} I_n \\ A \end{bmatrix}_{(n+m)\times n}$$

$$= \underbrace{\begin{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \\ \begin{bmatrix} \quad A \quad \end{bmatrix} \end{bmatrix}}_{n} \begin{matrix} \big] n \\ \big] m \end{matrix}$$

$$A = U \Sigma V^T$$

$$MM^T = [\, m+n, n \,] \, [\, n, m+n \,] = [\quad]_{m+n, m+n}$$

$$= \begin{bmatrix} I_{n \times n} & A^T_{n \times m} \\ A_{m \times n} & AA^T_{m \times m} \end{bmatrix}_{m+n, m+n}$$ which is symmetric

$\det(MM^T - \lambda I) = 0$

$(I - \lambda)(AA^T - \lambda) - AA^T = 0 \Rightarrow \lambda_i = I - \lambda(AA^T_i) = 1 + \sigma_i^2$

$\sigma_n = \sqrt{\lambda_i} = \sqrt{1 + \sigma_i^2}$