HW 5

Problem 1. Do Question 2.11 in the text (you're allowed to use results proved in Chapter 2 of the text).

Problem 2. (The Sherman-Morrison Formula) Let A be a nonsingular $n \times n$ matrix and let u and v be vectors such that $A + uv^T$ is nonsingular. In view of Question 1.2, any rank one matrix can be written as an outer product uv^T. Hence, $A + uv^T$ sometimes goes under the name of "rank-one perturbation" of A. The Sherman-Morrison formula gives a very useful way of computing the inverse of the rank-one perturbation:

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}.$$

Prove the Sherman-Morrison formula. In particular, you will need to show that the denominator in the right-hand side is nonzero.

Problem 3. (Solution operator) In this problem we will practice interpreting A^{-1} as a "solution operator" instead of a matrix. In practice, it's almost never advisable to invert a matrix to solve a system. For instance, in Matlab, to solve $Ax = b$ for x it is usually better to use

```
x =A\b;
```

rather than

```
x=inv(A)*b;
```

See for instance this discussion:

`https://blogs.mathworks.com/loren/2007/05/16/purpose-of-inv/`

Again assume that A is a nonsingular $n \times n$ matrix and that u and v are vectors such that $A + uv^T$ is nonsingular. Assume that you have access to a fast solver for the system $Ax = b$. That is, assume that for any vector b, you can compute $x = A^{-1}b$ rapidly, without explicitly computing A^{-1}. Explain how you can use the Sherman-Morrison formula to produce a fast solver for the system $(A+uv^T)x = b$.

(Hint: You should be able to solve the modified system by using the fast solver for $Ax = b$ two times, a couple of scalar products, and a few other relatively cheap operations. In particular, you should not need to perform any matrix-matrix multiplications or matrix inversions.)