Math 715 | Leidong Xu | HW3

Problem !

Computer the operator norm and F-norm

of $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ by finding its singula

value

$$A^TA = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\lambda I - A^TA = \begin{bmatrix} \lambda - 5 & -3 \\ -3 & \lambda - 5 \end{bmatrix}$$

$$\det(\lambda I - A^TA) = (\lambda-5)^2 - 9 = 0 \Rightarrow \lambda = 8 \text{ or } 2$$

$$\Sigma_1 = \sqrt{8} = 2\sqrt{2} \qquad \Sigma_2 = \sqrt{2}$$

$$\|A\| = \sigma_1 = \sigma_{max} = 2\sqrt{2} \approx 2.8284$$

$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2}$$

$$= \sqrt{(2\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4 \times 2 + 2}$$

$$= \sqrt{10}$$

Problem 2.

(1) $\|A\|_2 = \sigma_1$, which is the max stretch

A^TA from SVD

$= (V^T\Sigma U)^T(V^T\Sigma U)$

$= U^T\Sigma^T V V^T\Sigma U$

$= U^T\Sigma^T\Sigma U = U^T\Sigma^2 U$

$\therefore \|A^TA\|_2 = \sigma_1 \cdot \sigma_1 = \sigma^2 = \|A\|_2^2$

(2) To prove $k_2(A^TA) = k_2(A)^2$

$k_2(A) = \|A\|\|A^{-1}\| = \frac{\sigma_1}{\sigma_n}$

$A^{-1} = U^{-1}\Sigma^{-1}(V^T)^{-1}$

$= U^T\underbrace{\Sigma^{-1}}_{diag(\frac{1}{\sigma_1}, \dots \frac{1}{\sigma_n})}V$

from (1) $A^TA = U^T\Sigma^2 U$

then $k_2(A^TA)$

$= \|U^T\underbrace{\Sigma^2}_{diag(\sigma_1^2 \dots \sigma_n^2)}U\|\,\|U\underbrace{(\Sigma^2)^{-1}}_{diag(\frac{1}{\sigma_1^n}\dots\frac{1}{\sigma_n^n})}U\|$

$= \|A\|\|A\|\|A^{-1}\|\|A^{-1}\|$

$= k_2(A)^2$

I think this can be approved easier without using SVD from problem 1.

we know $\Sigma = \begin{bmatrix} 2\sqrt{2} & \\ & \sqrt{2} \end{bmatrix}$

we can easily get

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and $U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

and $A^{-1} = (UDV^T)^{-1}$

$$= V \cdot \mathrm{diag}(D^{-1})\, U^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \\ & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

(2) $\|A^k\| = \|A \cdot A \cdot A \cdot A \cdot A\|$

$\leq \|A\| \cdot \|A\| \cdot \|A\| \cdot \|A\|$ ①

sonce $\|A\| \leq 1$

$\therefore$ ① is ganreteed to converge.

(3) $\left\| \begin{bmatrix} 0 & 249.8 \\ 0 & 0 \end{bmatrix} \right\|_2 = 249.8 > 1$

$\|A^k\| = 0$

we just need $\|A \cdot A \cdot A \cdots A\|$

get converge