Math 715 | Leidong Xu | HW 7

3.3.1.

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \cdot \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$\Rightarrow$ $r + Ax = b$ ①

$A^T r = 0$ ②

$A^T(①) \Rightarrow \underbrace{\cancel{A^T r}}_{0} + A^T A x - A^T b = 0$

$\Rightarrow A^T A x = A^T b$

$\Rightarrow x = (A^T A^{-1}) A^T b$

Which is the normal equation

which can minnimizer of $\|Ax - b\|_2^2$

$$M = \begin{array}{c} m \\ n \end{array} \left[\begin{array}{c|c} I & A \\ \hline A^T & 0 \end{array}\right]$$

I still don't know why coefficent matrix cab be written in this form.

$$\hat{M} = \left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right]$$

$A \sim SVD$ $\{v_1 \cdots v_n\}$

ONB of $\mathbb{R}^n$

$\{u_1, \cdots u_n, \hat{u}_{n+1} \cdots \hat{u}_m\}$

$\sigma_1 \geq \cdots \geq \sigma_n \geq 0$

$A v_1 = \sigma_1 u_1$

$$\Rightarrow \left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right] \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A u_1 \\ A^T v \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 \\ \sigma_1 u v_1 \end{bmatrix} = \sigma_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$

And $A = \sum_{j=1} \sigma_j u_j v_j^T$

$$A^T = \sum_{j=1}^{n} \sigma_j v_j u_j^T$$

$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$ is an eigenvector for M with σ_1

$\vdots$

$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$ is an eigenvetor for with σ_n

for $m-n$

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \text{ ------- } \sigma_n$$

$$\left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right] \begin{bmatrix} \hat{u}_{n+1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ A^T \hat{u}_{n+1} \end{bmatrix} \text{ —— } 0$$

$$\begin{bmatrix} \hat{u}_m \\ 0 \end{bmatrix}$$

$$\begin{array}{c} m \\ n \end{array} \left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right] \begin{bmatrix} u_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} -Av_1 \\ A^T u_1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sigma_1 u_1 \\ \sigma_1 v_1 \end{bmatrix}$$

$\begin{bmatrix} u_1 \\ -v_1 \end{bmatrix}$ eigenvector with $-\sigma_1$

$\begin{bmatrix} u_n \\ -v_{nn} \end{bmatrix}$ —————— $-\sigma_n$

honestly, I'm still confused about this question.

$$\kappa(A) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)}$$

3.9.1 $(A^TA)^{-1}$

$$= (V\Sigma^T U^T U\Sigma V^T)^{-1}$$
$$= (V\Sigma\Sigma V^T)^{-1}$$
$$= (V\Sigma^2 V^T)^{-1}$$
$$= (V^T)^{-1}(\Sigma^2)^{-1}(V)^{-1}$$
$$= V\Sigma^{-2}V^T \quad ①$$

2. $(A^TA)^{-1}A^T$

$$= ① A^T = V\Sigma^{-2}V^T V\Sigma U^T$$
$$= V\Sigma^{-1}U^T$$

3. $A(A^TA)^{-1}$

$$= U\Sigma V^T \; V\Sigma^{-2}V^T$$
$$= U\Sigma^{-1}V^T$$

4. $A(A^{-1}A)^{-1}A^{T}$

$$= U\Sigma^{-1}V^T \cdot (V^T)^{-1}(\Sigma)^{-1}(U)^{-1}$$
$$= U\Sigma^{-1}V^T \cdot V \cdot \Sigma U^T$$
$$= I$$