HW10 | Math 715 | Leidong Xu

problem 1

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1b} \\ & A_{22} & \cdots & A_{2b} \\ & & \ddots & \vdots \\ & & & A_{bb} \end{bmatrix}$$

If A is singular, then $\det(A)=0$

If A is nonsingular, A can be decompose as:

$$A = \underset{①}{\begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}} \underset{②}{\begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix}} \underset{③}{\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}}$$

where $A_{11} B = A_{12}$

$$\det(A) = \det(①)\det(②)\det(③)$$

$$= \det(A_{11}) \cdot \det(A_{22}) \cdot 1$$

$$= \Pi_{i=1}^{b} \det(A_{ii}) \quad ①$$

(2)

$$\det(A - \lambda I_n)$$

$$= \det\left(\begin{pmatrix} A_{11}, A_{12} \\ 0, A_{22} \end{pmatrix} - \lambda I_n \right)$$

$$= \det \begin{pmatrix} A_{11} - \lambda I_k & A_{12} \\ 0 & A_{22} - \lambda I_{b-k} \end{pmatrix}$$

$$= \det(A_{11} - \lambda I_k) \times \det(A_{22} - \lambda I_{b-k})$$

?

I wonder if it's ok to use $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B)$

$$\det(A-\lambda I) = \prod_{i=1}^{b} \underbrace{\det(A_{ii}-\lambda I)}_{\text{roots}}$$

$\{\text{roots}\} = \text{spe } A$

$$\text{spec}A = \bigcup_{i=1}^{b} \text{spec}A_{ii}$$

if $\lambda \in \text{spec}A \Rightarrow \det(A-\lambda I)=0$

$\Rightarrow \exists i$ st $\det(A_{ii}-\lambda I)=0$

$\Rightarrow \lambda \in \text{spec}A_{ii}$

$\Rightarrow \lambda \in \bigcup_{j=1}^{b} \text{spec}A_{jj}$

$\supset$ if $\lambda \in \bigcup_{j=1}^{b}$ spec A_{jj}

problem 2.

μ

prove left eigenvector y and right eigenvector x are orthongnal.

λ

for left: $yA = \mu y$ ①

for right $Ax = \lambda x$ ②

based on ① $yAx = \mu yx$

$$\lambda yx = \mu yx$$

plug in ② $\Rightarrow (\lambda y - \mu y)x = 0$

$$\Rightarrow (\lambda - \mu) yx = 0$$

since $\lambda - \mu \neq 0$

$\therefore$ $y \cdot x = 0$

which means y and x are orthongnal

4.2.

$$A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & A_{22} \end{bmatrix}$$

$$A^* = \begin{bmatrix} \overline{a_{11}} & a_{21}^* \\ & A_{22}^* \end{bmatrix}$$

$$AA^* = A^*A$$

$$\Rightarrow AA^* - A^*A = 0$$

$$= \begin{bmatrix} a_{11} & 0 \\ a_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \overline{a_{11}} & a_{21}^* \\ 0 & A_{22}^* \end{bmatrix} - \begin{bmatrix} \overline{a_{11}} & a_{21}^* \\ 0 & A_{22}^* \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\overline{a_{11}} & a_{11}a_{21}^* \\ a_{21}\overline{a_{11}} & a_{21}a_{21}^* + A_{22}A_{22}^* \end{bmatrix} - \begin{bmatrix} \overline{a_{11}}a_{11} + a_{21}^*a_{21}, & a_{21}^*A_{22} \\ A_{22}^*a_{21}, & A_{22}^*A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a_{21}^*a_{21} = 0$$

$$\Rightarrow a_{21}^* = a_{21} = 0$$

$\Rightarrow$ A and A^* are diagonal

2) if A has n orthognal eigenvectors

then $A = QUQ^{+}$

and $A^{+} = QU^{+}Q^{+}$

$AA^{+} = QUQ^{+}QU^{+}Q^{+}$ ①

since U is triangular,

U and U^{+} are diagnoal only if they are normal

As long as U is diagnal

① $= A^{+}A$