HW 1

Problem 1. Consider evaluating the function $y = \ln(x)$ on a computer. Assume for the moment that the logarithm can be computed exactly (not actually true in practice). Due to roundoff errors, the exact value of x will typically not be stored in the computer. Instead, the computer will store an approximation $\tilde{x} = x + \Delta x$. Because of this roundoff error in the input, the result will also contain error; rather than y, the computer will return the value $\tilde{y} = \ln(\tilde{x}) = y + \Delta y$.

Forward error. Forward error is simply a way of quantifying the difference between the value we wanted to get, y, and the value we actually got, $\tilde{y}$. Frequently, this is done through the relative forward error $|\delta y/y|$. For our problem, Taylor's theorem gives

$$y + \Delta y = \ln(x + \Delta x) = \ln(x) + \frac{\Delta x}{x} + O((\Delta x)^2) \asymp y + \frac{\Delta x}{x}.$$

Using this last approximation, approximate the relative forward error $|\Delta y/y|$. (Write the answer so it depends only on x and Δx.)

Backward error. The difference between forward and backward error can seem a bit subtle at first. From the backward error viewpoint, $\tilde{y}$ is viewed as the result of applying the correct function to slightly incorrect data. For the current problem, the goal is to find the smallest Δx so that $\tilde{y} = \ln(x + \Delta x)$. Using the same Taylor polynomial approximation as before, approximate the relative backward error $|\Delta x/x|$. (Write the answer so it depends only on y and Δy.)

Condition number. The condition number, κ, of a problem relates the forward and backward errors:

$$|\Delta y/y| \asymp \kappa |\Delta x/x|.$$

Find the condition number for this problem. (It will depend on x.) What does the size of κ indicate about the accuracy of the logarithm when errors are present in the input? How does the size of κ depend on x?

Problem 2. Do 1.1, 1.2, and 1.3 in the text.