HW 4

Problem 1. (Operator norm and Frobenius norm as optimization) In the last HW set we computed the operator norm and the Frobenius norm of the matrix A below, by first finding its singular values (i.e., solving $\det(\lambda I - A^T A) = 0$ for λ):

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

In this problem, we will compute them using the Calc III technique of Lagrangian multipliers (you might want to consult your old Calc book to review this).

Recall that the (Euclidean) operator norm is defined as the maximum stretch:

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

For simplicity, we can first maximize $\|Ax\|^2 = |Ax(1)|^2 + |Ax(2)|^2$, and then take a square-root later. Likewise $\{x : \|x\| = 1\} = \{x : \|x\|^2 = 1\}$. In other words, we can instead solve the following constrained optimization problem:

$$\text{maximize} \quad (2x_1 + 2x_2)^2 + (-x_1 + x_2)^2$$

$$\text{subject to} \quad x_1^2 + x_2^2 = 1$$

(a) Find the vector x that solves the maximization.
(b) Deduce σ_1 from part (a).
(c) Use the two-dimensionality to find σ_2.

Problem 2. Let $\{u_1, \ldots, u_n\}$ be a basis for $\mathbb{R}^n$, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ be positive numbers. We want to construct an n-by-n matrix A with the u's as eigenvectors and the λ's as eigenvalues.

(a) Build a matrix A such that

$$Au_k = \lambda_k u_k \qquad \text{for all } k = 1, \ldots, n;$$

(b) Use the same construction to compute A^{-1};
(c) Compute $\kappa_2(A)$ in terms of the data of the problem.

Problem 3. Let $\kappa(A)$ be the condition number with respect to an operator norm or the Frobenius norm. Show that

$$\kappa(A) \geq 1.$$