



### 1. (10 pts)

The standard five-point stencil approximation of the Laplacian on a square grid using row-by-row ordering takes the block form

$$L = \frac{1}{h^2} \begin{bmatrix} T & I & & & \\ I & T & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & T & I \\ & & & I & T \end{bmatrix},$$

where  $h$  is the space between grid points,  $I$  is the identity matrix, and  $T$  is the matrix

$$T = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix}.$$

- (a) Show that  $L$  is negative semidefinite.
- (b) Prove that there exists a constant  $c > 0$  (independent of  $h$ ) such that every eigenvalue  $\lambda$  of  $L$  satisfies  $|\lambda| \leq c/h^2$ .



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**2. (10 pts)**

Let  $A$  be the product

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

- (a) Find the four spaces  $R(A)$ ,  $N(A)$ ,  $R(A^T)$ ,  $N(A^T)$  (the range and nullspace of  $A$  and  $A^T$ ).
- (b) Find the singular values of  $A$ .



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**3. (10 pts)**

Let  $A$  be a full-rank  $m \times n$  real matrix with  $m \geq n$ .

(a) Show that the system

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

has a unique solution  $(r^T, x^T)^T$ .

(b) Show that the  $x$  from (a) minimizes  $\|Ax - b\|_2$ .



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**4. (10 pts)**

(a) State the definition of complete orthonormal set in a Hilbert space.

(b) Provide an example of a complete orthonormal set in  $L^2((0, 1) \times (0, 1))$ .



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**5. (10 pts)**

Show that the sequence of functions  $\{f_n\}_{n \in \mathbb{N}}$ , where

$$f_n(x) = \frac{n}{2} e^{-n|x|}, \quad x \in \mathbb{R},$$

is a (Dirac) delta sequence.



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**6. (10 pts)**

Let  $K : L^2(0, 1) \rightarrow L^2(0, 1)$  be the following operator:

$$Ku(x) = \int_0^1 u(y) \, dy.$$

(a) Show that  $K$  is a self-adjoint compact operator. Provide direct proofs based on the definitions of self-adjointness and compactness of an operator.

Consider the integral equation

$$u(x) - \lambda \int_0^1 u(y) \, dy = f(x),$$

where  $\lambda$  is a real number.

(b) Derive necessary and sufficient conditions on  $\lambda$  and  $f \in L^2(0, 1)$  for the existence of solutions for this equation.

(c) Derive necessary and sufficient conditions on  $\lambda$  and  $f \in L^2(0, 1)$  for the uniqueness of solutions for this equation.



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