Math 715 | HW4 | Leidong Xu

Problem 1.

(a) Find the vector x and solve the maximization

$$(2x_1+2x_2)^2+(-x_1+x_2)^2$$

$$= 5x_1^2+5x^2+6x_1x_2+\lambda(1-x_1^2-x_2^2)$$

$$\mathcal{L}(x_1, x_2, \lambda)$$

$$= 5x_1^2+5x_2^2+6x_1x_2+\lambda(1-x_1^2-x_2^2)$$

$$\nabla_x \mathcal{L}=0 \Rightarrow \begin{cases} 10x_1+6x_2-2\lambda x_1=0 \\ 10x_2+6x_1-2\lambda x_2=0 \\ x_1^2+x_2^2=1 \end{cases}$$

$$\Rightarrow \text{when } k=2 \Rightarrow \begin{cases} x_1=-\frac{1}{\sqrt{2}} \\ x_2=\frac{1}{\sqrt{2}} \end{cases} \text{ or } \begin{cases} x_1=\frac{1}{\sqrt{2}} \\ x_2=-\frac{1}{\sqrt{2}} \end{cases}$$

$$\text{When } k=8 \Rightarrow \begin{cases} x_1=-\frac{1}{\sqrt{2}} \\ x_2=-\frac{1}{\sqrt{2}} \end{cases} \text{ or } \begin{cases} x_1=\frac{1}{\sqrt{2}} \\ x_2=\frac{1}{\sqrt{2}} \end{cases}$$

$$\Rightarrow \max (2x_1+2x_2)^2+(-x_1+x_2)^2$$

$$= 8$$

(2) Reduce σ_1 from part (a)

$\mathcal{L}(x_1, x_2, \lambda)$ could be written as

$$\mathcal{L}(\lambda, v) = v^T A^T A v + \mu(1 - v^T v)$$

$$\|A\|_2^2 = \sigma_1^2 = \max v^T A^T A v$$

$$\sigma_1 = \sqrt{8} = 2\sqrt{2}$$

(3)

$$\min (2x_1 + 2x_2)^2 + (-x_1 + x_2)^2$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

$$\sigma_2 = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

Problem2 Build a matrix by $\{u_1 \cdots u_n\}$ as its eigenvector and $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n > 0$ as its eigenvalue.

(a) build A

① $M = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$S = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$

$A = SMS^{-1}$

or $A = \lambda_1 \frac{u_1 u_1^T}{u_1^T u_1} + \lambda_2 \frac{u_2 u_2^T}{u_2^T u_1} + \cdots + \lambda_n \frac{u_n u_n^T}{u_n^T u_n}$

(b) use same constuction for A^{-1}

$A^{-1} = SM^{-1}S^{-1}$

where $M^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n} \end{bmatrix}$

$A^{-1} = \frac{1}{\lambda_1} \frac{u_1 u_1^T}{u_1^T u_1} + \frac{1}{\lambda_2} \frac{u_2 u_2^T}{u_2^T u_1} + \cdots + \frac{1}{\lambda_n} \frac{u_n u_n^T}{u_n^T u_n}$

(c) computer $K_2(A)$ in terms of data

$$K_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

from Lemma 1.7.6

$$\|A\|_2 = \sqrt{\lambda_{max}(A^+ A)}$$

$$= \sqrt{\lambda_{max}(SMS^+)}$$

since S is all real, $S^+ = S^T$

$$K_2(A) = \sqrt{\lambda_{max}(SMS^T)}\ \sqrt{\lambda_{max}(SM^{-1}S^T)}$$

Problem 3

let use F noom as example

$$k_2(A) = \|A^{-1}\|_F \|A\|_F$$

$$\geqslant \|A^{-1} \cdot A\|_F = \|I\|_F = 1$$