HW 3

Problem 1. (Operator norm vs. Frobenius norm) For a vector $x \in \mathbb{R}^n$, the notation $\|x\|$ will denote the usual Euclidean 2-norm. Recall, that given a matrix $A \in \mathbb{R}^{m\times n}$, the operator norm (induced by $\|\cdot\|$) is defined as

$$\|A\| := \sup_{\|x\|=1} \|Ax\|;$$

while the Frobenius norm is

$$\|A\|_F := \left(\sum_{i,j} |A(i,j)|^2\right)^{1/2} = \left(\mathrm{Tr}\left(A^T A\right)\right)^{1/2}.$$

Recall also that the operator norm is the smallest constant C such that

$$\|Ax\| \le C\|x\| \qquad \forall x \in \mathbb{R}^n, \tag{1}$$

so in particular, the operator norm satisfies the submultiplicative property:

$$\|AB\| \le \|A\|\|B\|.$$

In class, we showed that the Frobenius norm also satisfies

$$\|Ax\| \le \|A\|_F\|x\|, \tag{2}$$

and hence it also has the submultiplicative property.

An alternative way of showing all this would to simply establish the fact that

$$\|A\| \le \|A\|_F, \tag{3}$$

because then (1) would hold for $C = \|A||_F$, which is (2).

There is fairly simple way to compare the operator norm and the Frobenius norm, namely, by writing both in terms of the singular values $\sigma_1 \ge \cdots \ge \sigma_r > 0$ of the matrix A. Recall that

$$\|A\| = \sigma_1 = \sigma_{\max}(A) \qquad \text{and} \qquad \|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}.$$

In particular,

$$\|A\|^2 = \sigma_1^2 \le \sigma^2 + \cdots + \sigma_r^2 = \|A\|_F^2.$$

Compute the operator norm and the Frobenius norm of the matrix A below, by first finding its singular values (i.e., solve $\det(\lambda I - A^T A) = 0$ for λ):

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

Problem 2. Consider $A \in \mathbb{R}^{n\times n}$. Use the Singular Value Decomposition of A to show that

$$\|A^T A\|_2 = \|A\|_2^2$$

where $\|\cdot\|_2$ is the operator norm induced by the 2-norm.

Conclude that

$$\kappa_2(A^T A) = \kappa_2(A)^2.$$

Finally, let A be the 2-by-2 matrix from Problem 1, determine its singular value decomposition and deduce its inverse from it.

Problem 3. A matrix A is called *power bounded* if there exists a number $C > 0$ so that

$$\|A^k\| \leq C \qquad \text{for all } k = 0, 1, 2, ...$$

(Note that the norm is not specified.)

(a) Prove that the definition of power boundedness is independent of the norm chosen. Namely, show that power boundedness in one norm implies power boundedness in all norms.

(b) Let $\|\cdot\|$ be a matrix norm satisfying $\|AB\| \leq \|A\|\|B\|$, for instance the operator $\|\cdot\|_2$, or the Frobenius norm). Show that if $\|A\| \leq 1$, then A is power bounded.

(c) Part (b) shows that $\|A\|_2 \leq 1$ is a sufficient condition for power boundedness. However, show that it is not a necessary condition, by finding an example of a 2×2 power bounded matrix that satisfies $\|A\|_2 > 1$.