Problem 1

$y = \ln(x)$

$\hat{y} = \ln(\hat{x}) = y + \Delta y$

$= \ln(x+\Delta x) = \ln x + \frac{\Delta x}{x} + O((\Delta x))^2$

forward error

$$\left|\frac{\Delta y}{y}\right| = \left|\frac{\hat{y}-y}{y}\right| = \left|\frac{\ln x + \frac{\Delta x}{x} + O((\Delta x)^2) - \ln x}{\ln(x)}\right|$$

$$= \frac{\left|\frac{\Delta x}{x} + O((\Delta x)^2)\right|}{|\ln(x)|} \approx \frac{\left|\frac{\Delta x}{x}\right|}{|\ln(x)|}$$

Backward error

$$\left|\frac{\Delta x}{x}\right| = \left|\frac{e^{\hat{y}}-e^{y}}{e^{y}}\right| = \left|\frac{e^{y} + e^{y}(\Delta y) + O((\Delta y)^2) - e^{y}}{e^{y}}\right|$$

$$= \left|\frac{e^{y}(\Delta y) + O((\Delta y^2))}{e^{y}}\right|$$

$$\approx |\Delta y|$$

Condition number

$$K = \frac{|\Delta y / y|}{|\Delta x / x|} = \frac{\left|\frac{\Delta x}{x}\right|}{|\ln(x)|} \cdot \frac{|x|}{|\Delta x|}$$

$$= \frac{1}{|\ln(x)|}$$

The condition number k shows how the output error change with input error change. For this specific problem, the ratio between forward error and backward error is $\frac{1}{|\ln(x)|}$, which means the same size input error Δx will play less effects on the results when x gets big and eventually very minimum effets due to the property of nature logarithm.

Problem 1.1

Let A be an orthogonal matrix.

show (1) $\det(A) = \pm 1$

(2) if B also orthogonal and $\det(A) = -\det(B)$ then $A+B$ is singular.

(1)

$$\det(A^T) = \det(A)$$

$$\det(I) = \det(A^T \cdot A) = \det(A^T)\det(A)$$

$$= \det(A)\det(A) = 1 = [\det(A)]^2$$

$$\therefore \det A = \pm 1$$

(2) $\det(A) = -\det(B) = 1 \text{ or } -1$

$$\det(AB) = \det(A)\det(B) = -1$$

$\therefore$ singular $\leftarrow$ $\det(A+B) = 0$

$$\Downarrow$$

$$\det(A+B) = \underline{-\det(A+B)^T}$$

$$\det(A+B)^T = \det(A^T + B^T)$$

$$= \det(A^{-1} + B^{-1})\det(A)\cdot(-\det(B))$$

$$= -\det(A^{-1}AB + B^{-1}AB)$$

$$= -\det(B+A)$$

$\therefore$ $\det(B+A) = -\det(A+B)^T = 0$

$\therefore$ $A+B$ is singular.

1.2 show that A has rank one $\boxed{\text{if and only if}}$ $A = ab^T$ for some column vector a and b

$$\text{Rank}(A) = 1 \underset{②}{\overset{①}{\rightleftarrows}} A = ab^T$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} \qquad b^T = [b_1, b_2, b_3, b_4 \cdots b_n]$$

① :

A matrix has rank 1 could be written as

$[b_1\vec{a}, b_2\vec{a}, \cdots b_n\vec{a}]$ ① because the spanned column only create a one-dimensional space.

And ① could be written as $\underset{n\times 1}{a}\,\underset{1\times n}{b^T}$

② A matrix from ab^T can be wrote as

$$\begin{bmatrix} a_1b_1, & a_1b_2 & \cdots \\ a_2b_1, & a_2b_2 & \cdots \\ \vdots & & \\ a_nb_1 & \cdots & \end{bmatrix}$$

The whole matrix is linear depend with each other. It could be compressed as only one-dimensional, which is rank 1

And a and b should be nonzero vectors

1.3 A is orthogonal and tran-gular, then it is diagnonal.

Pf: $\begin{bmatrix} - u_1^T - \\ - u_2^T - \\ \vdots \\ - u_n^T - \end{bmatrix} \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} u_1^Tu_1, & u_1^Tu_2 & \cdots \\ \vdots & & \end{bmatrix}$

if A is orthogonal $= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

then $U_m^T U_n = \delta_{mn}$

$\therefore$ U_m^T or $U_n = 0$ when $m \neq n$ ①

if A is also upper triangular:

then A^{-1} must also be upper triangular ②

but A^T must be low triangular ③

① ② ③ $\Rightarrow$ A and A^T must be diangnonal matrix

since $U_m^T U_n = 1$ if $m = n$

so the diagonal elements should be ± 1.