Gauge Choices in Numerical Relativity

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Contents

1 What is a Gauge Choice in Relativity?

In general relativity, the Einstein field equations are covariant under coordinate transformations, which introduces *gauge freedom*. A *gauge choice* refers to selecting specific coordinate conditions or auxiliary equations to simplify the computation and achieve stability in numerical simulations. Gauge choices are fundamental in ensuring that the physical interpretation of a simulation aligns with the observer's frame of reference.

For an in-depth discussion on gauge freedom, refer to the foundational works in general relativity [8, 11].

2 How is a Gauge Choice Implemented?

In the context of the *3+1 formalism* [1,8], the spacetime is decomposed into a three-dimensional spatial hypersurface evolving over time. Gauge choices are implemented through:

- The *lapse function*, α, controlling the rate of proper time progression.

- The *shift vector*, β^i, determining the coordinate system on the spatial hypersurface.

Specific gauge conditions are applied to reduce the complexity of Einstein's equations and to manage the evolution of the metric components and extrinsic curvature [1, 5].

3 Different Gauge Options

Gauge conditions can be divided into two main categories: slicing conditions and spatial gauge conditions. Below, we explore key options and their applications in numerical relativity.

3.1 Slicing Conditions

3.1.1 Geodesic Slicing

Geodesic slicing sets $\alpha = 1$ and $\beta^i = 0$, leading to straightforward coordinate systems. However, this choice often results in the formation of coordinate singularities, such as caustics [9].

3.1.2 Harmonic Slicing

Harmonic slicing enforces the condition $\Box t = 0$, which helps to maintain a smooth evolution and avoids singularities. This condition is frequently used in conjunction with harmonic coordinates [6].

3.1.3 1+log Slicing

The 1+log slicing condition is a widely used option for its stability in black hole simulations. It evolves the lapse function as:
$$\partial_t \alpha = -\alpha^2 K,$$

where K is the trace of the extrinsic curvature. This condition has been shown to prevent the collapse of the lapse function and coordinate singularities [2].

3.1.4 Maximal Slicing

Maximal slicing enforces the condition that the trace of the extrinsic curvature, K, vanishes at each timestep:
$$K = 0.$$

This choice minimizes the distortion of the spatial hypersurface and effectively delays the development of coordinate singularities. It is particularly useful in simulations of compact objects such as black holes. However, solving the elliptic equation for the lapse function under maximal slicing can be computationally expensive [5].

3.2 Spatial Gauge Conditions

3.2.1 Minimal Distortion Shift

The minimal distortion condition minimizes changes in the metric components during evolution. This choice is effective in preserving the geometric properties of the spatial hypersurface [10].

3.2.2 Gamma-Driver Shift Condition

The Gamma-driver condition dynamically evolves the shift vector β^i to reduce distortions in the coordinates:
$$\partial_t \beta^i = B^i, \quad \partial_t B^i = \eta \partial_t \Gamma^i - \beta^j \partial_j B^i,$$

where Γ^i are the Christoffel symbols, and η is a damping parameter [3].

3.3 Combined Gauge Choices

A commonly used combination in numerical simulations, especially for black hole spacetimes, is the 1+log slicing with the Gamma-driver shift condition. This combination is particularly effective for maintaining stability and avoiding coordinate pathologies in simulations involving black holes [4, 7].

4 Conclusion

Gauge choices play a critical role in numerical relativity by shaping the stability and interpretability of the simulations. The selection of appropriate gauge conditions, such as 1+log slicing and Gamma-driver shift, is essential for tackling the complexities of Einstein's equations and extracting meaningful physical results.

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