```
> restart:with(plots):with(plottools):
```

Newtonian

Kinetic Energy, Potential Energy and the Lagrangian

```
> T:=1/2*(diff(r(t), t)^2+r(t)^2*diff(phi(t), t)^2):
  V:=-G*M/r(t):
  L:=T-V;
```

$$L := \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t} r(t)\right)^2 + \frac{1}{2} r(t)^2 \left(\frac{\mathrm{d}}{\mathrm{d}t} \phi(t)\right)^2 + \frac{G M}{r(t)} \tag{1.1}$$

Substitute the variable for 'var#' in oder to perform derivatives properly.

Once derivatives are complete substitue back to original variable names

```
> LSubIn:=subs({r(t)=var1, diff(r(t), t)=var2, phi(t)=var3, diff
  (phi(t), t)=var4}, L):
```

```
> EL1:=diff(LSubIn, var1):
  EL2:=diff(LSubIn, var2):
  EL3:=diff(LSubIn, var3):
  EL4:=diff(LSubIn, var4):

  EL11:=subs({var1=r(t), var2=diff(r(t), t), var3=phi(t), var4=
  diff(phi(t), t)}, {EL1, EL2, EL3, EL4}):

  ELR:=diff(EL11[4], t):
```

ELC0 is the angular momentum term which must be conserved. Therefore, we set this to parameter 'h' to set later with the initial conditions.

```
> ELC0:=EL11[2]=h;
  ELC:=isolate(ELC0, diff(phi(t), t));
```

$$ELC0 := r(t)^2 \left(\frac{\mathrm{d}}{\mathrm{d}t} \phi(t)\right) = h$$

$$ELC := \frac{\mathrm{d}}{\mathrm{d}t} \phi(t) = \frac{h}{r(t)^2} \tag{1.2}$$

The final form of the Euler-Lagrange Equaiton of Motion

```
> ELF0:=ELR-EL11[3]=0:
  ELF:=eval(ELF0, ELC);
```

$$ELF := \frac{\mathrm{d}^2}{\mathrm{d}t^2} r(t) - \frac{h^2}{r(t)^3} + \frac{G M}{r(t)^2} = 0 \tag{1.3}$$

set intial conditions

```
> G:=1:
  M:=1:
  I1:=r(0)=26:
  I2:=D(r)(0)=0:
  I3:=phi(0)=0:
  I4:=D(phi)(0)=0.0071;

  ini1:=I1, I2, I3;
```

$$I4 := \mathrm{D}(\phi)(0) = 0.0071$$

$$ini1 := r(0) = 26, \mathrm{D}(r)(0) = 0, \phi(0) = 0 \tag{1.4}$$

solve for parameter 'h' from the initial conditions

```
> h := eval(lhs(ELC0), {r(t)=rhs(I1), diff(phi(t),t)=rhs(I4)});
```

$$h := 4.7996 \tag{1.5}$$

Calculate the eccentricity. Zero:=Circle, 0<e<1:=Ellipse, e=1:=Parabola, e>1:=hyperbola, e==>infinity is a line

e=sqrt(1+(2*E*L^2)/(mred*alpha^2))

E Total Orbital Energy
L Angular Momentum
mred Reduced mass (mred=1)
alpha Coefficient of Inverse-Square Law

```
> En := eval(T + V, {r(t)=rhs(I1), diff(r(t),t)=rhs(I2), diff(phi(t),t)=rhs(I4)});
  l := eval(lhs(ELC0), {r(t)=rhs(I1), diff(phi(t),t)=rhs(I4)});
  epsilon := sqrt(1 + 2*En*l^2/((G*M)^2));
```

$$En := -0.02142295846$$

$$l := 4.7996$$

$$\varepsilon := 0.1139938402 \tag{1.6}$$

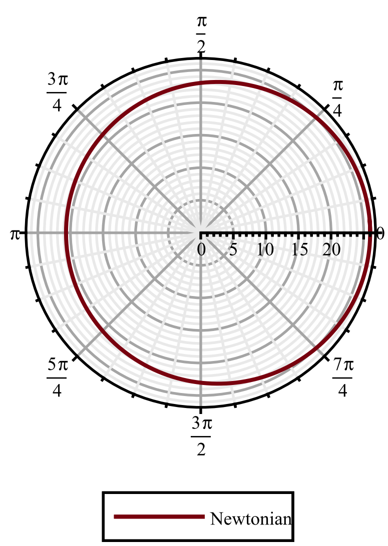
Numerically solve for the differential equaiton

```
> EQS:=dsolve({ELC, ELF, ini1}, {r(t), phi(t)}, numeric, output=listprocedure);
```

$$EQS := \left[t = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}, \phi(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}, r(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc}, \frac{\mathrm{d}}{\mathrm{d}t} r(t) = \mathbf{proc}(t)\ \ldots\ \mathbf{end\ proc} \right] \tag{1.7}$$

Plot the orbit solution

```
> polarplot([rhs(EQS(t)[3]), rhs(EQS(t)[2]), t=0..720], scaling=constrained, legend="Newtonian");
```



Name the plot to be displayed later

```
> pnewton := polarplot([rhs(EQS(t)[3]), rhs(EQS(t)[2]), t=0.
  .720], scaling=constrained, color=black, legend="Newtonian",
  linestyle=3, thickness=2):
```

Schwarzschild

Reset Parameters without a restart

Define the Lagrangian and substitute variable names to perform derivatives

```
> G:='G';
  M:='M';

  f:=1/2*(-(c^2-2*G*M/r(tau))*diff(t(tau), tau)^2+1/(1-2*G*M/
  (c^2*r(tau)))*diff(r(tau), tau)^2+r(tau)^2*diff(phi(tau), tau)
  ^2);

  fs:=subs({t(tau)=var1, diff(t(tau), tau)=var2, r(tau)=var3,
  diff(r(tau), tau)=var4, phi(tau)=var5, diff(phi(tau), tau)=
  var6}, f);
```

$$G := G$$

$$M := M$$

$$f := -\frac{1}{2}\left(c^2 - \frac{2\,G\,M}{r(\tau)}\right)\left(\frac{d}{d\tau}\,t(\tau)\right)^2 + \frac{1}{2}\,\frac{\left(\frac{d}{d\tau}\,r(\tau)\right)^2}{1 - \frac{2\,G\,M}{c^2\,r(\tau)}} + \frac{1}{2}\,r(\tau)^2\left(\frac{d}{d\tau}\,\phi(\tau)\right)^2$$

$$fs := -\frac{1}{2}\left(c^2 - \frac{2\,G\,M}{var3}\right) var2^2 + \frac{1}{2}\,\frac{var4^2}{1 - \frac{2\,G\,M}{c^2\,var3}} + \frac{1}{2}\,var3^2\,var6^2 \qquad \textbf{(2.1)}$$

Perform derivatives and subsitute the names back

```
> EL1:=diff(fs, var1):
  EL2:=diff(fs, var2):
  EL3:=diff(fs, var3):
  EL4:=diff(fs, var4):
  EL5:=diff(fs, var5):
  EL6:=diff(fs, var6):

  ELF1:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL1);
  ELF2:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL2);
  ELF3:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL3);
  ELF4:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL4);
  ELF5:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
```

```
}, EL5);
ELF6:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
}, EL6);
```

$$ELF1 := 0$$

$$ELF2 := -\left(c^2 - \frac{2\,G\,M}{r(\tau)}\right)\left(\frac{d}{d\tau}\,t(\tau)\right)$$

$$ELF3 := -\frac{G\,M\left(\frac{d}{d\tau}\,t(\tau)\right)^2}{r(\tau)^2} - \frac{\left(\frac{d}{d\tau}\,r(\tau)\right)^2 G\,M}{\left(1 - \frac{2\,G\,M}{c^2\,r(\tau)}\right)^2 c^2\,r(\tau)^2} + r(\tau)\left(\frac{d}{d\tau}\,\phi(\tau)\right)^2$$

$$ELF4 := \frac{\frac{d}{d\tau}\,r(\tau)}{1 - \frac{2\,G\,M}{c^2\,r(\tau)}}$$

$$ELF5 := 0$$

$$ELF6 := r(\tau)^2\left(\frac{d}{d\tau}\,\phi(\tau)\right) \tag{2.2}$$

Isolate Angular Momentum [r^2*diff(phi(t), t)] and the other term which does not depend on t or phi to be constants of motion

ELC1 represents the total energy of the motion

```
> ELC0:=ELF6=i:
ELC:=isolate(ELC0, diff(phi(tau), tau));
ELC2:=ELF2=-b:
ELC1:=isolate(ELC2, diff(t(tau), tau));
```

$$ELC := \frac{d}{d\tau}\,\phi(\tau) = \frac{i}{r(\tau)^2}$$

$$ELC1 := \frac{d}{d\tau}\,t(\tau) = -\frac{b}{-c^2 + \frac{2\,G\,M}{r(\tau)}} \tag{2.3}$$

Final setup of the Euler-Lagrange Equations of motion

```
> ELR:=diff(ELF4, tau):
ELF0:=ELR-ELF3=0:
ELF:=subs({ELC, ELC1}, ELF0);
```

$$ELF := -\frac{\left(\frac{d}{d\tau}\,r(\tau)\right)^2 G\,M}{\left(1 - \frac{2\,G\,M}{c^2\,r(\tau)}\right)^2 c^2\,r(\tau)^2} + \frac{\frac{d^2}{d\tau^2}\,r(\tau)}{1 - \frac{2\,G\,M}{c^2\,r(\tau)}} + \frac{G\,M\,b^2}{r(\tau)^2\left(-c^2 + \frac{2\,G\,M}{r(\tau)}\right)^2} - \frac{i^2}{r(\tau)^3} = 0 \tag{2.4}$$

Set initial conditions, parameters for the angular momentum and the total energy, calculate eccentricity

```
> G:=1:
  M:=1:
  c:=1:
  IS1:=r(0)=26:
  IS2:=D(r)(0)=0:
  IS3:=phi(0)=0:
  IS4:=D(phi)(0)=0.0071:

  ini2:=I1, I2, I3;

  i := eval(lhs(ELC0), {r(tau)=rhs(IS1), diff(phi(tau),tau)=rhs
  (IS4)});
  b := sqrt((1 - 2/rhs(IS1))*(1 + i^2/rhs(IS1)^2));

  En1 := eval(T + V, {r(t)=rhs(IS1), diff(r(t),t)=rhs(IS2), diff
  (phi(t),t)=rhs(IS4)});
  Eccentricity1 := sqrt(1 + 2*En1*i^2/((G*M)^2));
```

$$ini2 := r(0) = 26, \mathrm{D}(r)(0) = 0, \phi(0) = 0$$

$$i := 4.7996$$

$$b := 0.9770019258$$

$$En1 := -0.02142295846$$

$$Eccentricity1 := 0.1139938402 \tag{2.5}$$

Solve the differential equation numerically

```
> EQSS:=dsolve({ELC, ELF, ini2}, {r(tau), phi(tau)}, numeric,
  output=listprocedure);
```

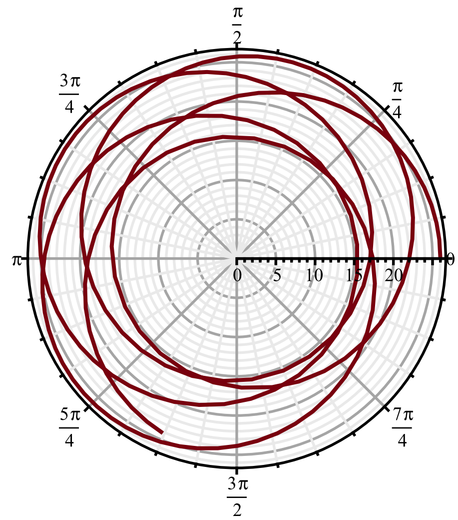
$$EQSS := \left[\tau = \mathbf{proc}(\tau)\ \ldots\ \mathbf{end\ proc}, \phi(\tau) = \mathbf{proc}(\tau)\ \ldots\ \mathbf{end\ proc}, r(\tau) = \mathbf{proc}(\tau) \right. \tag{2.6}$$

$$\ldots$$

$$\left. \mathbf{end\ proc}, \frac{\mathrm{d}}{\mathrm{d}\tau} r(\tau) = \mathbf{proc}(\tau)\ \ldots\ \mathbf{end\ proc} \right]$$

Polarplot the orbits

```
> polarplot([rhs(EQSS(tau)[3]), rhs(EQSS(tau)[2]), tau=0..2500],
  scaling=constrained, axesfont=[TIMES, ROMAN, 12]);
```

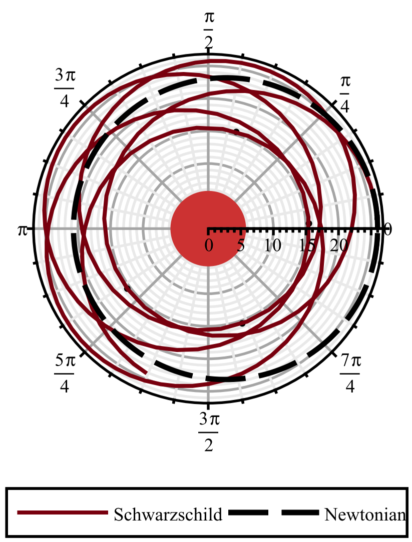


Overlap the Newtonian and Schwarzschild solutions to show procession of the Schwarzschild orbit versus the stationary Newtonian orbit

```
> psch1 := polarplot([rhs(EQSS(tau)[3]), rhs(EQSS(tau)[2]), tau=
  0..2500],
            scaling=constrained, legend="Schwarzschild"):

  psch2 := disk([15.47*cos(3.7797), 15.47*sin(3.7797)], 0.5,
  color=black):
  psch3 := disk([15.47*cos(11.3399), 15.47*sin(11.3399)], 0.5,
  color=black):
  psch4 := disk([15.47*cos(18.9008), 15.47*sin(18.9008)], 0.5,
  color=black):
  psch5 := disk([15.47*cos(26.4221), 15.47*sin(26.4221)], 0.5,
  color=black):
  pns := disk([0,0],5.8, color=orange):


  display([psch1, psch2, psch3, psch4, psch5, pns, pnewton]);
```



Name the Schwarzschild orbit to be used later

```
> pschwarz := polarplot([rhs(EQSS(tau)[3]), rhs(EQSS(tau)[2]),
  tau=0..1000],
              scaling=constrained, color=blue, legend=
  Schwarzschild):
```

Kerr

Reset parameters

```
> M:='M':
  l:='l':
```

J is the angular momentum of the Black Hole gravitational source, M is mass and 'a' Kerr Parameter

Delta for convenience

```
> a:=J/M;
  Delta:=r(tau)^2-2*M*r(tau)+a^2;
```

$$a := \frac{J}{M}$$

$$\Delta := r(\tau)^2 - 2\,M\,r(\tau) + \frac{J^2}{M^2} \tag{3.1}$$

The Lagrangian

```
> L:=1/2*(-(1-2*M/r(tau))*diff(t(tau), tau)^2+r(tau)^2/(Delta)*
  diff(r(tau), tau)^2+(r(tau)^2+a^2+2*M*a^2/r(tau))*diff(phi
  (tau), tau)^2-4*a*M/r(tau)*diff(t(tau), tau)*diff(phi(tau),
  tau));
```

$$L := -\frac{1}{2}\left(1 - \frac{2\,M}{r(\tau)}\right)\left(\frac{d}{d\tau}\,t(\tau)\right)^2 + \frac{1}{2}\,\frac{r(\tau)^2\left(\frac{d}{d\tau}\,r(\tau)\right)^2}{r(\tau)^2 - 2\,M\,r(\tau) + \frac{J^2}{M^2}} + \frac{1}{2}\left(r(\tau)^2 + \frac{J^2}{M^2} + \frac{2\,J^2}{M\,r(\tau)}\right)\left(\frac{d}{d\tau}\,\phi(\tau)\right)^2 - \frac{2\,J\left(\frac{d}{d\tau}\,t(\tau)\right)\left(\frac{d}{d\tau}\,\phi(\tau)\right)}{r(\tau)} \tag{3.2}$$

Substitue variable names to perform derivatives properly

```
> L1:=subs({t(tau)=var1, diff(t(tau), tau)=var2, r(tau)=var3,
  diff(r(tau), tau)=var4, phi(tau)=var5, diff(phi(tau), tau)=
  var6}, L);
```

$$L1 := -\frac{1}{2}\left(1-\frac{2M}{var3}\right)var2^2+\frac{1}{2}\frac{var3^2\,var4^2}{var3^2-2\,M\,var3+\frac{J^2}{M^2}}+\frac{1}{2}\left(var3^2+\frac{J^2}{M^2}+\frac{2\,J^2}{M\,var3}\right)var6^2-\frac{2\,J\,var2\,var6}{var3} \tag{3.3}$$

Take derivatives of all terms and then substitue the names of variables back to origianl

```
> EL11:=diff(L1, var1):
  EL21:=diff(L1, var2):
  EL31:=diff(L1, var3):
  EL41:=diff(L1, var4):
  EL51:=diff(L1, var5):
  EL61:=diff(L1, var6):

  EL1:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL11);
  EL2:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL21);
  EL3:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL31);
  EL4:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL41);
  EL5:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL51);
  EL6:=subs({var1=t(tau), var2=diff(t(tau), tau), var3=r(tau),
  var4=diff(r(tau), tau), var5=phi(tau), var6=diff(phi(tau), tau)
  }, EL61);
```

$$EL1 := 0$$

$$EL2 := -\left(1-\frac{2M}{r(\tau)}\right)\left(\frac{d}{d\tau}t(\tau)\right)-\frac{2J\left(\frac{d}{d\tau}\phi(\tau)\right)}{r(\tau)}$$

$$EL3 := -\frac{M\left(\frac{d}{d\tau}t(\tau)\right)^2}{r(\tau)^2}+\frac{r(\tau)\left(\frac{d}{d\tau}r(\tau)\right)^2}{r(\tau)^2-2\,M\,r(\tau)+\frac{J^2}{M^2}}-\frac{1}{2}\frac{r(\tau)^2\left(\frac{d}{d\tau}r(\tau)\right)^2(2\,r(\tau)-2\,M)}{\left(r(\tau)^2-2\,M\,r(\tau)+\frac{J^2}{M^2}\right)^2}+\frac{1}{2}\left(2\,r(\tau)-\frac{2\,J^2}{M\,r(\tau)^2}\right)\left(\frac{d}{d\tau}\phi(\tau)\right)^2$$

$$+\frac{2J\left(\frac{d}{d\tau}t(\tau)\right)\left(\frac{d}{d\tau}\phi(\tau)\right)}{r(\tau)^2}$$

$$EL4 := \frac{r(\tau)^2\left(\frac{d}{d\tau}r(\tau)\right)}{r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}}$$

$$EL5 := 0$$

$$EL6 := \left(r(\tau)^2+\frac{J^2}{M^2}+\frac{2J^2}{Mr(\tau)}\right)\left(\frac{d}{d\tau}\phi(\tau)\right)-\frac{2J\left(\frac{d}{d\tau}t(\tau)\right)}{r(\tau)} \quad \textbf{(3.4)}$$

Conserved quantities like the total energy and the angular momentum are set to constants to be set later using the initial conditions

```
> ELC20:=EL2=-o:
  ELC60:=EL6=l;
  ELR:=diff(EL4, tau):
```

$$ELC60 := \left(r(\tau)^2+\frac{J^2}{M^2}+\frac{2J^2}{Mr(\tau)}\right)\left(\frac{d}{d\tau}\phi(\tau)\right)-\frac{2J\left(\frac{d}{d\tau}t(\tau)\right)}{r(\tau)}=l \quad \textbf{(3.5)}$$

Final form of the Euler-Lagrange Equations of Motion

```
> ELF0:=ELR-EL3=0;
```

$$ELF0 := \frac{r(\tau)\left(\frac{d}{d\tau}r(\tau)\right)^2}{r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}} \quad \textbf{(3.6)}$$

$$-\frac{r(\tau)^2\left(\frac{d}{d\tau}r(\tau)\right)\left(2r(\tau)\left(\frac{d}{d\tau}r(\tau)\right)-2M\left(\frac{d}{d\tau}r(\tau)\right)\right)}{\left(r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}\right)^2}$$

$$+\frac{r(\tau)^2\left(\frac{d^2}{d\tau^2}r(\tau)\right)}{r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}}+\frac{M\left(\frac{d}{d\tau}t(\tau)\right)^2}{r(\tau)^2}$$

$$+\frac{1}{2}\frac{r(\tau)^2\left(\frac{d}{d\tau}r(\tau)\right)^2(2r(\tau)-2M)}{\left(r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}\right)^2}-\frac{1}{2}\left(2r(\tau)-\frac{2J^2}{Mr(\tau)^2}\right)\left(\frac{d}{d\tau}\phi(\tau)\right)^2$$

$$-\frac{2J\left(\frac{d}{d\tau}t(\tau)\right)\left(\frac{d}{d\tau}\phi(\tau)\right)}{r(\tau)^2}=0$$

Decouple the differential equaitons in order to solve

```
> ELC2:=isolate(ELC20, diff(t(tau), tau)):

  ELC6:=subs(ELC2, ELC60):
  ELC61:=isolate(ELC6, diff(phi(tau), tau));

  ELC20:=subs(ELC61, ELC2):

  ELF:=subs({ELC20, ELC61}, ELF0);
```

$$ELC61 := \frac{d}{d\tau}\phi(\tau)=\frac{-l M^2(-r(\tau)+2M)+2JM^2 o}{r(\tau)^3M^2-2r(\tau)^2M^3+J^2r(\tau)}$$

$$ELF := \frac{r(\tau)\left(\frac{d}{d\tau}r(\tau)\right)^2}{r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}} - \frac{r(\tau)^2\left(\frac{d}{d\tau}r(\tau)\right)\left(2r(\tau)\left(\frac{d}{d\tau}r(\tau)\right)-2M\left(\frac{d}{d\tau}r(\tau)\right)\right)}{\left(r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}\right)^2} + \frac{r(\tau)^2\left(\frac{d^2}{d\tau^2}r(\tau)\right)}{r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}} + \frac{M\left(-o+\frac{2J\left(-lM^2(-r(\tau)+2M)+2JM^2o\right)}{r(\tau)\left(r(\tau)^3M^2-2r(\tau)^2M^3+J^2r(\tau)\right)}\right)^2}{r(\tau)^2\left(-1+\frac{2M}{r(\tau)}\right)^2} + \frac{1}{2}\frac{r(\tau)^2\left(\frac{d}{d\tau}r(\tau)\right)^2(2r(\tau)-2M)}{\left(r(\tau)^2-2Mr(\tau)+\frac{J^2}{M^2}\right)^2} - \frac{1}{2}\frac{\left(2r(\tau)-\frac{2J^2}{Mr(\tau)^2}\right)\left(-lM^2(-r(\tau)+2M)+2JM^2o\right)^2}{\left(r(\tau)^3M^2-2r(\tau)^2M^3+J^2r(\tau)\right)^2} - \left(2J\left(-o\right.\right. \tag{3.7}$$

$$+\frac{2J\left(-l M^2(-r(\tau)+2M)+2JM^2 o\right)}{r(\tau)\left(r(\tau)^3 M^2-2r(\tau)^2 M^3+J^2 r(\tau)\right)}\Bigg)\left(-l M^2(-r(\tau)+2M)+2JM^2 o\right)\Bigg) \Big/ \left(r(\tau)^2\left(-1+\frac{2M}{r(\tau)}\right)\left(r(\tau)^3 M^2-2r(\tau)^2 M^3+J^2 r(\tau)\right)\right)=0$$

Set initial conditions so that the Balck hole is spinning with the particles trajectory called "Kerr Direct"

```
> M:=1; J:=0.37;
  l:=4.7996; o:=0.9772;

  IK1:=r(0)=26:
  IK2:=D(r)(0)=0:
  IK3:=phi(0)=0:
  IK4:=D(phi)(0)=eval(rhs(ELC61), {r(tau)=rhs(IK1)});

  iniK:=IK1, IK2, IK3;
```

$$M := 1$$

$$J := 0.37$$

$$l := 4.7996$$

$$o := 0.9772$$

$$IK4 := \mathrm{D}(\phi)(0) = 0.007143004388$$

$$iniK := r(0) = 26, \mathrm{D}(r)(0) = 0, \phi(0) = 0 \quad \textbf{(3.8)}$$

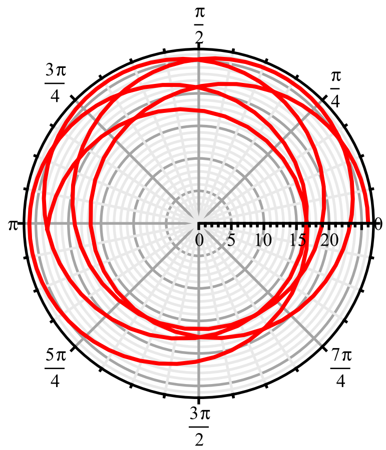
Solve the differential equation numerically

```
> ELKF:=dsolve({ELC61, ELF, iniK}, {r(tau), phi(tau)}, numeric,
  output=listprocedure);
```

$$ELKF := \left[\tau = \textbf{proc}(\tau)\ \ldots\ \textbf{end proc}, \phi(\tau) = \textbf{proc}(\tau)\ \ldots\ \textbf{end proc}, r(\tau) = \textbf{proc}(\tau)\ \ldots\ \textbf{end proc}, \frac{\mathrm{d}}{\mathrm{d}\tau} r(\tau) = \textbf{proc}(\tau)\ \ldots\ \textbf{end proc}\right] \quad \textbf{(3.9)}$$

Polarplot of the Kerr solution

```
> polarplot([rhs(ELKF(tau)[3]), rhs(ELKF(tau)[2]), tau=0..2500],
  scaling=constrained, color=red, legend="Kerr Direct");
```



Name plot to be used later

```
> pkerrd:=polarplot([rhs(ELKF(tau)[3]), rhs(ELKF(tau)[2]), tau=0.
  .2500], scaling=constrained, color=red, legend="Kerr Direct"):
```

Reset the initial conditions so the Black Hole is spinning the opposite direction to the particles motion called "Kerr Indirect"

```
> M:=1;
  J:=-0.37;
  l:=4.7996;
  o:=0.976797;

  IK10:=r(0)=26:
  IK20:=D(r)(0)=0:
  IK30:=phi(0)=0:
  IK40:=D(phi)(0)=eval(rhs(ELC61), {r(tau)=rhs(IK10)});

  iniK0:=IK10, IK20, IK30;
```

$$M := 1$$

$$J := -0.37$$

$$l := 4.7996$$

$$o := 0.976797$$

$$IK40 := \mathrm{D}(\phi)(0) = 0.007053899319$$

$$iniK0 := r(0) = 26, \mathrm{D}(r)(0) = 0, \phi(0) = 0 \tag{3.10}$$

solve the differential equation numerically

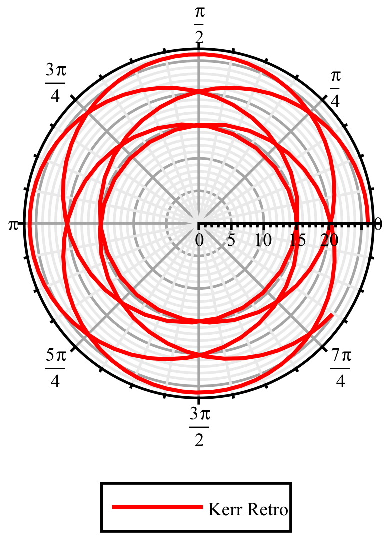
```
> ELKF0:=dsolve({ELC61, ELF, iniK0}, {r(tau), phi(tau)}, numeric,
  output=listprocedure);
```

$$ELKF0 := \left[\tau = \mathbf{proc}(\tau) \; \ldots \; \mathbf{end\ proc}, \phi(\tau) = \mathbf{proc}(\tau) \; \ldots \; \mathbf{end\ proc}, r(\tau) = \mathbf{proc}(\tau) \right. \tag{3.11}$$

...

$$\left. \mathbf{end\ proc}, \frac{\mathrm{d}}{\mathrm{d}\tau} r(\tau) = \mathbf{proc}(\tau) \; \ldots \; \mathbf{end\ proc} \right]$$

```
> polarplot([rhs(ELKF0(tau)[3]), rhs(ELKF0(tau)[2]), tau=0.
  .2500], scaling=constrained, color=red, legend="Kerr Retro");
```



Name the Kerr Indirect plot to be used later along with some useful points along the curve

```
> pkerr0:=polarplot([rhs(ELKF0(tau)[3]), rhs(ELKF0(tau)[2]), tau=
  0..2500], scaling=constrained, color=green, legend="Kerr
  Retro"):

  pkerr1:=disk([16.31*cos(3.63467), 16.31*sin(3.63467)], 0.5,
  color=black):
  pkerr2 := disk([14.53*cos(3.923), 14.53*sin(3.923)], 0.5,
  color=black):
  pkerr3 := disk([16.31*cos(10.9765), 16.31*sin(10.9765)], 0.5,
  color=black):
  pkerr4 := disk([14.53*cos(11.7778), 14.53*sin(11.778)], 0.5,
  color=black):
```

All Orbits Displayed

A plot of the Newtonian, Schwarzschild, Kerr Direct and Kerr Indirect orbits

```
> display([pnewton,pschwarz,pkerrd,pkerr0,pkerr1,pkerr2,psch2,
  pkerr3,pkerr4,psch3,pns]);
```