Spherical Harmonic Solutions - A Study

DLH

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1 Introduction

I wanted to start a reasonable amount of effort into answering the questions and misgivings I had with the spherical harmonics, gauge theory, and numerical analysis.

I start with the derivation of the Spherical Harmonic equations.

Spherical harmonics are a set of mathematical functions that arise in problems with spherical symmetry, such as those in quantum mechanics, electromagnetism, and other fields. These functions are used to represent the angular part of solutions to partial differential equations in spherical coordinates. The derivation of spherical harmonics involves several mathematical steps, and I'll provide an overview of the process.

1. Start with the Laplace equation in spherical coordinates:

   $$\nabla^2\Psi = 0 \tag{1}$$

   Scalar field $\Psi(r, \theta, \phi)$ is a function of the radial field r, polar angle θ and the azimuthal angle ϕ.

2. Separation of Variables:

   Assume that the function Ψ is really three separate functions in each variable

   $$\Psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

   which allows for an easy solution to the Laplace equation on the scalar field.

3. Solve for Radial Equation: Substitution of Ψ will produce

   $$1 = 2 \tag{2}$$

   Here, "l" is the azimuthal quantum number, and "m" is the magnetic quantum number.

2 Radial Solutions

I would imagine this is for a simple Newtonian solution or something.

The radial part of the solution, $R(r)$, is given by the following equation:

$$R(r) = \frac{C_1}{r} + \frac{C_2}{r^2} \tag{3}$$

Where C_1 and C_2 are constants.

2.1 Radial Solution from Python

From my Python calculation for any l.

$$\frac{\sqrt{2}\left(C_1 \sin\left(r\sqrt{-l\left(l+1\right)}\right) - C_2 \cos\left(r\sqrt{-l\left(l+1\right)}\right)\right)}{\sqrt{\pi} r \sqrt[4]{-l\left(l+1\right)}} \tag{4}$$

3 Polar Solutions

The polar part of the solution, $\Theta(\theta)$, is expressed as follows:

$$\Theta(\theta) = C_1 P_l^m(\cos\theta) + C_2 Q_l^m(\cos\theta) \tag{5}$$

Where $P_l^m(\cos\theta)$ and $Q_l^m(\cos\theta)$ are associated Legendre functions, and C_1 and C_2 are constants.

3.1 Polar Solution from Python

Polar solutions:

$$R(r) = C_2 - \frac{C_2 l \theta^2}{2} + \frac{C_2 l \theta^3 \sin\left(\theta\right)}{6} - \frac{C_2 l^2 \theta^2}{2} + \frac{C_2 l^2 \theta^3 \sin\left(\theta\right)}{6} + \frac{C_2 l^2 \theta^4}{24} + \frac{C_2 l^3 \theta^4}{12}$$
$$+ \frac{C_2 l^4 \theta^4}{24} + C_1 \theta - \frac{C_1 \theta^2 \sin\left(\theta\right)}{2} + \frac{C_1 \theta^3 \sin^2\left(\theta\right)}{6} - \frac{C_1 l \theta^3}{6} + \frac{C_1 l \theta^4 \sin\left(\theta\right)}{12}$$
$$- \frac{C_1 l^2 \theta^3}{6} + \frac{C_1 l^2 \theta^4 \sin\left(\theta\right)}{12} + O\left(\theta^6\right)$$

4 Azimuthal Solutions

The azimuthal part of the solution, $\Phi(\phi)$, can be written as:

$$\Phi(\phi) = C_1 \sin(m\phi) + C_2 \cos(m\phi) \tag{6}$$

Where C_1 and C_2 are constants, and m is the magnetic quantum number.

4.1 Azimuthal Solution from Python

Azimuthal solutions:

$$C_1 \sin(\phi |m|) + C_2 \cos(m\phi) \tag{7}$$

5 Python Animation Explanation

The following code generates a 3D animation that visualizes a spherical harmonic function for a specific set of quantum numbers (l and m). Here's a breakdown of the code and its interpretation:

1.

- `numpy`: For numerical operations and array handling.
- `matplotlib.pyplot`: For creating 2D and 3D plots.
- `matplotlib.animation`: For creating animations.
- `mpl_toolkits.mplot3d.Axes3D`: To enable 3D plotting in matplotlib.
- `scipy.special.sph_harm`: For computing spherical harmonics.

The animation visualizes the real part of the spherical harmonic function Y_{lm} as a 3D surface on a unit sphere. The color of the surface is determined by the values of the spherical harmonics, and the animation progresses through frames, updating the spherical harmonic's appearance.

The title of the 3D plot changes dynamically to indicate the quantum numbers l and m. You can change the values of l and m to visualize different spherical harmonics, and you can modify the `update` function to add time-dependent effects or animations if desired.

6 Conclusion

We have presented the solutions to the spherical harmonics using the Laplace equation. These solutions are essential in various fields, including quantum mechanics and electromagnetism, where problems exhibit spherical symmetry.