Chapter 1

Mathematical models in economics

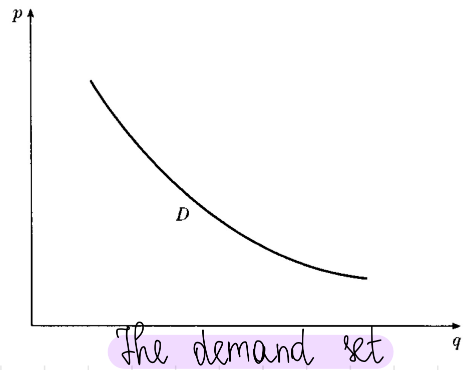
1.2 A model of the market

- in economics: horizontal - q-axis, vertical - p-axis (q, p)

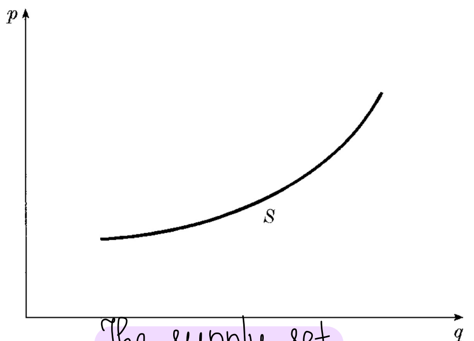
Demand - if p - selling price, then q is a demand - the quantity which would be sold to consumers at that price.

$q^D(p)$ - the quantity which would be sold if the price were p (demand func)

$p^D(q)$ - the inverse demand func



The demand set



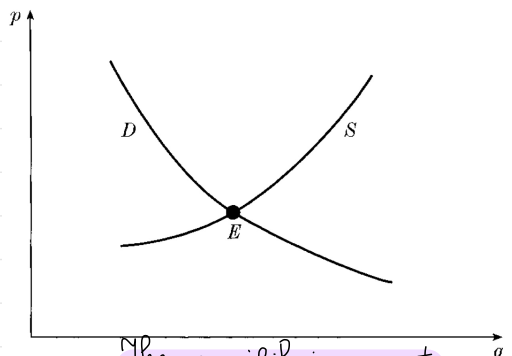
The supply set

- supply function q^S
- inverse supply function p^S

1.3 Market equilibrium

- Equilibrium set $E = S \cap D$ - intersection of the sets S and D

$p^*: \; q^D(p) = q^S(p)$



The equilibrium set
$E = S \cap D$

1.4 Excise tax

Excise tax - legislated tax on specific goods or services at purchase. (fuel, tobacco, alcohol...)

Tax theorem - not all of an excise tax is passed on to the consumer (if S and D are straight lines X_D^S slopes)

Money gov. makes from tax $= q^T \cdot T$ equilibrium q by T

not all of the tax transfers to consumer.

Example:

$6q + 8p = 125$

- we can determine q: $q^D(p) = \frac{125 - 8p}{6}$
- if $p = 4$, then: $q = q^D(4) = \frac{125 - 32}{6} = \frac{93}{6}$

Example: (cont.)

$p^D(q) = \frac{125 - 6q}{8}$

Example:

$q + 5p = 40$
$2q - 15p = -20$

1. $\begin{cases} q + 5p = 40 & \mid \cdot 2 \\ 2q - 15p = -20 \end{cases}$

2. $\begin{cases} 2q - 15p = -20 \\ 2q + 10p = 80 \end{cases} \; -$

$-25p = -100$

$p^* = 4$

3. $q + 20 = 40$

$q^* = 20$

Example:

$2q - 5p = -12$

- $q^S(p) = \frac{5p - 12}{2}$ – supply function
- $p^S(q) = \frac{2q + 12}{5}$ – inverse supply function

Answer: the equilibrium set is a single point $(20; 4)$.

Example:

$q^D(p) = 40 - 5p$

$q^S(p) = \frac{15}{2}p - 10$

$p^* = 4$

T per unit

1. for supplier price $= p - T$, and the new supply func. is:

$q^{ST}(p) = q^S(p - T) = \frac{15}{2}(p - T) - 10$

2. the demand func. remains the same. The new eq. values q^T and p^T satisfy the equations

$q^T = 40 - 5p^T$

$q^T = q^{ST}(p^T) = \frac{15}{2}(p^T - T) - 10$

$\Rightarrow 40 - 5p^T = \frac{15}{2}(p^T - T) - 10$

$40 - 5p^T = \frac{15}{2}p^T - \frac{15}{2}T - 10$

$-5p^T - \frac{15}{2}p^T = -\frac{15}{2}T - 10 - 40$

$p^T\left(5 + \frac{15}{2}\right) = \frac{15}{2}T + 50$

$p^T = \frac{\frac{15}{2}T + 50}{5 + \frac{15}{2}}$

$p^T = \frac{\frac{15T + 100}{2}}{\frac{10 + 15}{2}}$

$p^T = \frac{15T + 100}{\cancel{2}} \cdot \frac{\cancel{2}}{10 + 15} = \frac{15T + 100}{25}$

$p^T = \frac{\cancel{5}(3T + 20)}{\cancel{25}} = \frac{3T + 20}{5}$

$p^T = \frac{3T}{5} + 4$

$q^T = 40 - 5p^T = 20 - 3T$

if $T = 1$, then:

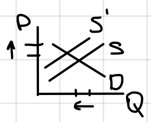
1. $p^* = \frac{3T}{5} + 4 = \frac{3}{5} + 4 = 4.6$

$4 \rightarrow 4.6$

2. $q^* = 40 - 5p^T = 20 - 3T$

$q^* = 40 - 5 \cdot 4.6 = 17$

$20 \rightarrow 17$

$\Rightarrow$ 

$P\uparrow \; Q\downarrow$

- $P\uparrow$ not by T but $3/5\,T \Rightarrow$ not all of the tax is passed on to the consumer

Algorithms 1.

$q^S(p) = a - bp$ - func

$p^S(q) = a - bq$ - inverse func

Equilibrium: substract, add or equalize 2 equations => find p => use p, find q

Tax: 1) $S: (p-T)$ D: p^T remains the same

2) equate right parts of equations (without q)

3) find $p^T = a \pm T$

Chapter 2

Mathematical terms and notations

2.1 Sets

Set - collection of objects defined in a precise way, so that any given object is either in the set or not in the set.
denoted by large letters $X, Y, S, D\ldots$

Members or **elements** of set - numbers listed within the parentheses.

1) $X = \{2, 3, 4, 5, 6, 7, 8, 9\}$

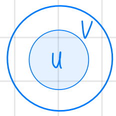
$2 \in X$ — 2 belongs to X; 2 is a member of X

2) $Z = \{n : n \text{ is a positive whole number less than } 5\}$

Z is the set of n such that n is a positive whole number less than 5
$\{1, 2, 3, 4\}$

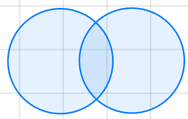
Empty set - set with no members.
- denoted by $\emptyset$

U is a **subset** of V if every member of U is a member of V. $U \subseteq V$
$U \subseteq V$ if and only if $x \in U$ implies $x \in V$.



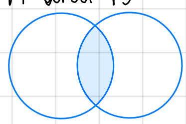
$\subseteq$ ← U can be equal V
$\subset$ definitely not equals (no possibility)

union $A \cup B$ - the set whose members belong to A or B (or both A and B):



$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

We define **intersection** $A \cap B$ to be the set whose members belong to both A and B:



$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Example:

$X = \{2, 3, 5, 7, 8, 9\}$
$Y = \{1, 4, 5, 7, 9\}$
$Z = \{1, 2, 3, 4\}$

$X \cup Y = \{1, 2, 3, 4, 5, 7, 8, 9\}$

$X \cap Y = \{5, 7, 9\}$

$Y \cup Z = \{1, 2, 3, 4, 5, 7, 9\}$

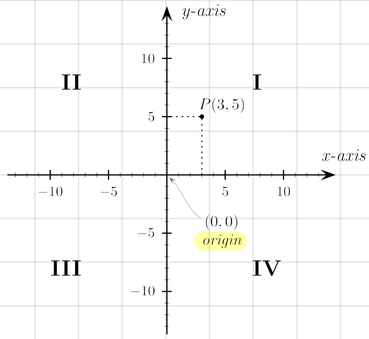
$X \cap (Y \cap Z) = (X \cap Y) \cap Z = \emptyset$

$9 \in X \cap Y$

$Z \subseteq X \cup Y$

Real numbers $\mathbb{R}$ – points on a line (like a), decimal (can be described as) 5834.623496..., may or may not terminate.

subsets of $\mathbb{R}$:



Nonnegative real numbers (reals) $\mathbb{R}_+$ $\{x \mid x \geq 0\}$.

Integers (целые) $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3...\}$.

Natural numbers $\mathbb{N}$ – positive integers $\mathbb{N} = \{1, 2, 3...\}$.

$\mathbb{R}^2$ – set of ordered pairs (x, y) of real numbers. Usually set of points in a plane, x and y being the coordinates of a point with respect to a pair of axes.

$\mathbb{R}^2_+$ – x and y are both nonnegative, first quadrant.

$\mathbb{Q}$ **Rational numbers** – can be written in the form m/n, can be expressed as ratios of integers.
m/n

2.2 Functions

given 2 arbitrary sets A and B,

Function from A to B is a rule which assigns precisely one member of B to each member of A.

Function – is a black box which converts an input into an output. For each input there is a unique, well-defined output.

f – name
x – input
$f(x)$ – output

$$x \longrightarrow \boxed{f} \longrightarrow f(x)$$

In general we can't reverse the arrows

If f is a function for which is possible to reverse the arrows, the resulting "reverse" function is called the **inverse function** for f and is denoted by f^{-1}.

f^{-1} is the undoing of f
(разворачиваем стрелки)

$$x \longrightarrow \boxed{f} \longrightarrow y \qquad\qquad x \longleftarrow \boxed{f^{-1}} \longleftarrow y$$

if $y = f(x)$ then $x = f^{-1}(y)$

Example:

$f(x) = 2x + 7$

x – input
$y = 2x + 7$ – output
f – name

$x \longrightarrow f \longrightarrow y = 2x + 7$

inverse function:

$x = \frac{y-7}{2}$

$f^{-1}(y) = (y-7)/2$

- p^S is the inverse function for q^S
  p^D is the inverse function for q^D
- $q^S(p) = \sim$
  $p^S(q) = \sim$
- $f(x) = y$
  $f^{-1}(y) = x$

$f(x) = x^2$ has no inverse
(cos has more than 1 option: $\pm x \to x^2$)

2.3 Composite functions

Example

$f(x) = 2x+7$

$f^{-1}(y) = \frac{y-7}{2}$

$$f^{-1}(f(x)) = f^{-1}(y) = f^{-1}(2x+7) = \frac{(2x+7)-7}{2} = \frac{2x+7-7}{2} = \frac{2x}{2} = x$$

For any x, $f^{-1}(f(x)) = x$

$k(x) = s(r(x))$

$k = sr$

3 func: f, g, h

$f(gh)$ - composite func.

But r is applied first

$$x \longrightarrow \boxed{r} \longrightarrow r(x) \longrightarrow \boxed{s} \longrightarrow s(r(x)) = (sr)(x)$$

- composite func (func. of a func.) $f \circ g$

Example

$r(x) = x^2+2$

$s(x) = x^3$

$k = sr$

$$k(x) = s(r(x)) = s(x^2+2) = (x^2+2)^3 \text{ - composite func.}$$

$(f^{-1}f)(x) = x$

$(ff^{-1})(x) = x$

Example

$k = sr = f(x) = \sqrt{x^4+x^2+5}$

$s(x) = \sqrt{x}$

$r(x) = x^4+x^2+5$

Example

$f(x) = 1/x$

$g(x) = x^{3/2}$

$h(x) = x^2+2x+3$

$(fgh)(x)$

$$(gh)(x) = g(x^2+2x+3) = (x^2+2x+3)^{3/2}$$

$$(fgh)(x) = f\left((x^2+2x+3)^{3/2}\right) = \frac{1}{(x^2+2x+3)^{3/2}}$$

Identity func. – $i(x) = x$ for all x

$f^{-1}f = i$

$ff^{-1} = i$

- $(fg)^{-1}(y) = (g^{-1}f^{-1})(y)$

$(fg)^{-1} = g^{-1}f^{-1}$

2.4 Graphs and equations

Equations:

linear $Ax+B=0$, $A \neq 0$ $\quad \{x \mid Ax+B=0\}$ has 1 member $-(B/A)$ $\quad f(x) = mx+c$ $\quad m$ - gradient, slope

quadratic $= ax^2+bx+c=0$, $a \neq 0$

Factorisation: $x^2-5x+6 = (x-2)(x-3)$, then

$\{x \mid x^2-5x+6=0\}$ has two members 2 and 3

Completing the square:

$ax^2+bx+c=0 \Rightarrow a(x+d)^2+e=0$

two x'es $\Rightarrow$ one x

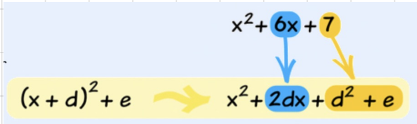
$d=\frac{b}{2a}$

$e=c-\frac{b^2}{4a}$

Example $x^2+6x+7=(x+3)^2-2$

$(x+d)^2+e=x^2+2dx+d^2+e$ → result

try to fit x^2+6x+7 into $x^2+2dx+d^2+e$



$6x=2dx \Rightarrow d=3$

$7=d^2+e \Rightarrow e=-2$

$\Rightarrow (x+3)^2-2$

For $ax^2+bx+c=0$, $a\neq 0$

$b^2-4ac<0$, no solutions

$b^2-4ac=0$, 1 solution $x=-\frac{b}{2a}$

$b^2-4ac>0$, 2 solutions $x_1, x_2=\frac{-b\pm\sqrt{D}}{2a}$

Steps:

$ax^2+bx+c=0 \quad |:a$

$x^2+\frac{b}{a}x=-\frac{c}{a}$

$\left(x+\frac{b}{2a}\right)^2=x^2+\frac{b}{a}x+\frac{b^2}{4a^2}$

$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}=-\frac{c}{a}$ → in the end we have $x=\frac{-b\pm D}{2a}$

To solve equations simultaneously means finding the intersection of the solution sets of the individual equations (like in market equilibrium).

Example:

$G=\{(x,y) \mid y=x^2-5x+6\}$

$L=\{(x,y) \mid y=2x-6\}$

$G\cap L=$

= points of intersection of 2 graphs

$y=x^2-5x+6$

$y=2x-6$

$x^2-5x+6=2x-6$

$x^2-7x+12=0$

$x^2-7x+12=(x-4)(x-3)$

$x_1=4 \quad x_2=3$

$y_1=16-20+6=2$

$y_2=6-6=0$

Answer: $G\cap L$ consists of two points $(4;2)$ and $(3;0)$.

Algorithms 2.

- $f^{-1}(y)=x$ 1. $f(x)=y=\cdots x$

2. express x through y $x=\cdots y$

3. substitute x with $f^{-1}(y)$

- intersection ∩ for equation means finding roots
- in comp. func r becomes my new x which I insert into s

Sequences, recurrences, limits

Sequence of numbers y0, y1.. – is a description of how a variable quantity y evolves with respect to time.
– is an infinite and ordered list of numbers with one term yt, corresponding to each non-negative integer.
Solution – explicit formula for yt.
Recurrence (difference) equation – an equation which defines yt in terms of yt–1, yt–2..
First-order recurrence – in which yt depends on yt–1 but no other previous values. Linear recurrence with constant coefficients:
Time-independent solution – solution which does not vary with t.

$y_t = a \cdot y_{t-1} + b$ $\qquad y^* = \frac{b}{1-a}$

Arithmetic progressions: first term a, common difference d.

$y_t = a + dt$

Arithm. series:

$y_t = y_{t-1} + d \qquad S_t = \frac{t(2a+(t-1)d)}{2}$

Geometric progression: common ratio x.

$y_t = ax^t \qquad S_t = \frac{a(1-x^t)}{1-x} \quad x \neq 1$

$y_t = x\,y_{t-1}$ – for successive terms.

Sum to infinity: finite inly if x between –1 and 1

$S_t \to \frac{a}{1-x}$ as $t \to \infty$

Limiting behaviour

$y_t = y^* + (y_0 - y^*)a^t$

Oscillatory approach to zero – terms are alternatively positive and negative, when a is negative.
Oscillates increasingly – terms behave the same way, but a doesn't approach any limit.

for a^t

Value of a	*Behaviour of a^t*	*Behaviour of y_t*
$a > 1$	$a^t \to \infty$	$y_t \to \infty$
$1 > a \geq 0$	$a^t \to 0$ (decreasing)	$y_t \to y^*$
$0 > a > -1$	$a^t \to 0$ (oscillating)	$y_t \to y^*$
$-1 > a$	oscillates increasingly	oscillates increasingly

If a=1 or =–1, each term is obtained by adding b to the previous one:

$y_t = y_{t-1} + b$

$y_t = y_0 + tb$ – general solution

Algorithm:

Finding an expression for the salary/income...

P - basic money
M - additional payment

1) $y_0 = P$
$y_1 = P(1+i) + M$
$y_2 = y_1(1+i) + M = P(1+i)^2 + M(1+i) + M$
$y_3 = y_2(1+i) + M = P(1+i)^3 + M(1+i)^2 + M(1+i) + M$
$y_4 = y_3(1+i) + M = P(1+i)^4 + M(1+i)^3 + M(1+i)^2 + M(1+i) + M$

2) $y_N = P(1+i)^N + M\big(\underbrace{(1+i)^{N-1} + (1+i)^{N-2} + \ldots + 1}_{\text{geometric progression, use } S_t}\big)$

3) $a = 1$
$x = (1+i) \qquad S_t = \frac{1(1-(1+i)^N)}{1-(1+i)}$
$t = N$

4) substitute S_t inside 2): $y_N = P(1+i)^N + M\frac{1(1-(1+i)^N)}{1-(1+i)}$

The elements of finance

$C = P(1+r)^t$ – capital C after t years

$r = R/100$ R-%

$P = \dfrac{C}{(1+r)^t}$

$C = P\left(1+ r/m\right)^m$ - after one year, year is divided into m equal periods

Compounding (compound interest) – interest is paid on interest previously added to the account.

4% twice a year is better than 8% once a year.
The quantity increases steadily as m increases.

$I(P) = \left(\dfrac{r(1+r)^t}{(1+r)^t - 1}\right) P$ - annual income (annuity)

$P(I) = \dfrac{I}{r}\left(1 - \dfrac{1}{(1+r)^t}\right)$

Limit

$f(y) \to 0$ as $y \to \infty$

same as

$\lim_{y\to\infty} f(y) = 0$

$e = \lim_{y\to\infty}\left(1+ 1/y\right)^y$

$e^x = \lim_{y\to\infty}\left(1+ x/y\right)^y$

Continuous compounding – m tends to infinity:

$e^r = \lim_{m\to\infty}\left(1+ r/m\right)^m$

$C = Pe^{rt}$

Finite series – sum of the form $y_0 + y_2 + \dots + y_{t-1}$

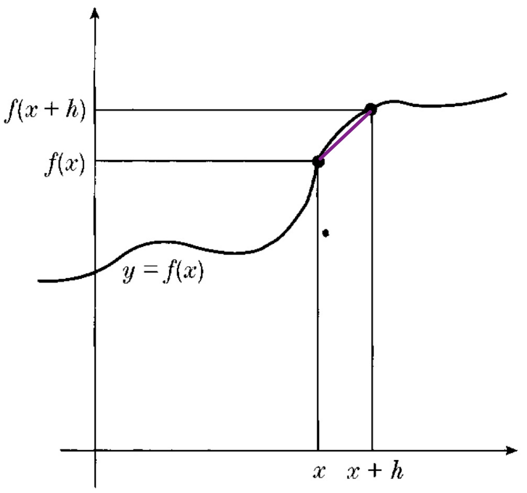
Chapter 6

Introduction to calculus

derivative - speed of changing

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

f' - how small change of input x changes output $f(x)$.



$f'(x) \approx \frac{\Delta f}{\Delta x}$

$\Delta f \approx f'(x) \cdot \Delta x$

Derivatives

power of x	$p(x) = x^k$	$p'(x) = k \cdot x^{k-1}$ (for all k)
the sum rule	$h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$
the product rule	$h(x) = f(x) \cdot g(x)$	$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
the quotient rule	$h(x) = f(x) / g(x)$	$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$ (for $g(x) \neq 0$)
composite func	$k(x) = s(r(x))$	$(sr)'(x) = s'(r(x)) \cdot r'(x)$
inverse func.	$y = f(x)$, $x = g(y)$	$g'(y) = \frac{1}{f'(x)}$ $f'(x) = \frac{\Delta y}{\Delta x}$ $g(y) = x$
	$e^{k \cdot x}$	$k \cdot e^x$
	$\ln x$	$1/x$
	$\sin x$	$\cos x$
	$\cos x$	$-\sin x$

Derivative of an inverse func: $g'(f(x)) = \frac{1}{f'(x)}$ when $g = f^{-1}$

Marginals are derivatives.

$(x)' = 1$

$\text{const}' = 0$

$(ax)' = a$

$(\exp x)' = \exp x$

$a^{x\,\prime} = \ln a \cdot a^x$

$\sqrt{x}\,' = \frac{1}{2\sqrt{x}}$

$\ln(g(x))' = \frac{g'(x)}{g(x)}$

Algorithms:

Derivative of a composite func:
1. $r'(x)$
2. $s'(x)$
3. insert $r(x)$ into $s'(x) = s'(r(x))$
4. multiply $s'(r(x)) \cdot r'(x)$

Some special functions

7.1 Powers

When n is a positive integer the nth power of a (a^n) is the product of n copies of a.

Power rules:

$a^x \cdot a^y = a^{x+y}$

$(a^x)^y = a^{x \cdot y}$

$a^0 = 1 \qquad a^{m+0} = a^m$

$a^{-n} = \frac{1}{a^n} \qquad a^n \cdot a^{-n} = a^0 = 1$

$a^m \cdot a^{-n} \ (m>n) = a^{m-n} = a^{m+(-n)}$

$(a^{1/n})^n = a$

$a^{1/n} = \sqrt[n]{a}$

$a^{m/n} = (a^{1/n})^m$

$f^{-1}(y) = y^{1/n} \qquad f(y) = y^n$

Examples:

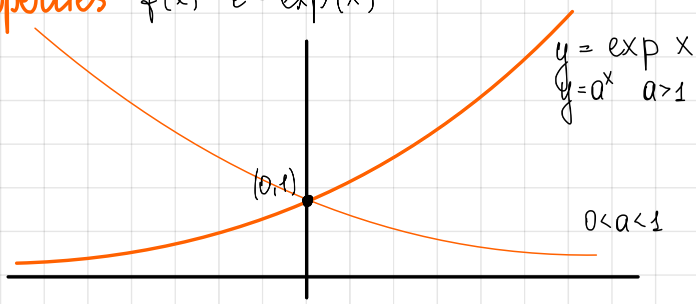
$8^{2/3} = (8^{1/3})^2 = 2^2 = 4$

7.2 The exponential function and its properties

$f(x) = e^x = \exp(x)$

PV = investing \$ 100
for 1 year
r - annual interest
m - equal periods of interest compound

$PV \cdot \left(1+\frac{r}{m}\right)^t$



$\exp(x) = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m$ – the exponential function

$e = \exp(1) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828..$

e.g. $\exp(3/2) = e^{3/2} = \sqrt{(2.71828)^3} = \sqrt{20.086} = 4.48169$

$\exp(x) = e^x$

$P(\exp(r))^t = P\exp(rt)$

$\exp(0) = 1$

$\exp(x) \to \infty$ as $x \to \infty$

$e^{-x} = \frac{1}{e^x}$

$\exp(x) \to 0$ as $x \to -\infty$

$\exp(x) \cdot \exp(y) = \exp(x+y)$

$\exp(x)' = \exp(x)$

Exponential-type func.:
$f(x) = a^x$
always positive
crosses y only at (0,1)

7.3 Continuous compounding of interest

$C = P\exp(r)$ - capital after one year FV

$C = P\exp(rt)$ - for t years

$P = C\exp(-rt) = Ce^{-rt}$ - present value PV

t need not to be an integer

7.4 The logarithm function

The exponential -type? function has an inverse – the logarithm function.

ln y is the natural logarithm of y (y is positive).
Natural logarithm – log to the base e. $\ln = \log_e$

$\log_{10}(y) = 0.4343\ldots \cdot \ln(y)$

x in $\log(x)$ is always positive

Properties:

$\ln(ab) = \ln a + \ln b$

$a = \exp(\ln a)$ a positive

$a^x = (\exp(\ln a))^x = (e^{\ln a})^x = e^{x \ln a} = \exp(x \ln a)$

$a^x = \exp(x \ln a)$ $a > 0$ |log

$\ln(a^x) = x \ln(a)$

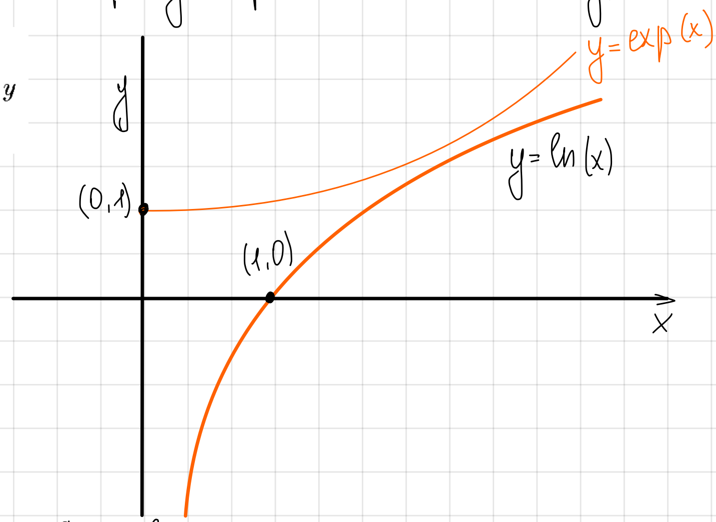
$\ln' y = \frac{1}{y}$ $\ln 1 = 0$

$\ln(a) - \ln(b) = \ln(a/b)$

$\ln(ab) = \ln(a) + \ln(b)$

$\ln(e) = 1$ $\log_e e = 1$

$\log_{10}$ is inverse for 10^x
$\log_e = \ln$ is inverse for e^x
if $y = \exp(x)$ then $x = \ln(y)$



Example:

$2^{\sqrt{2}} = \exp(\sqrt{2} \cdot \ln 2)$ – the easiest way
× multiply 2 √2 times

Example:

Find the derivative

$g(x) = \ln \ln x$

$g'(x) = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{1}{x \ln x}$

What is the number y such that $a^y = x$?
Answ.: $\log_a x$ $a^y = x$

7.5 Trigonometrical functions

- angle is measured in radians, not degrees

$\tan x = \frac{\sin x}{\cos x}$

$\cos^2 x = (\cos x)^2$

$\cos x = \sin\left(x + \frac{\pi}{2}\right)$ shift

$\cos^2 x + \sin^2 x = 1$

$\sin(2x) = 2 \sin x \cdot \cos x$

$\cos(2x) = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

double – angle: $\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$
$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$

180 degrees = π radians

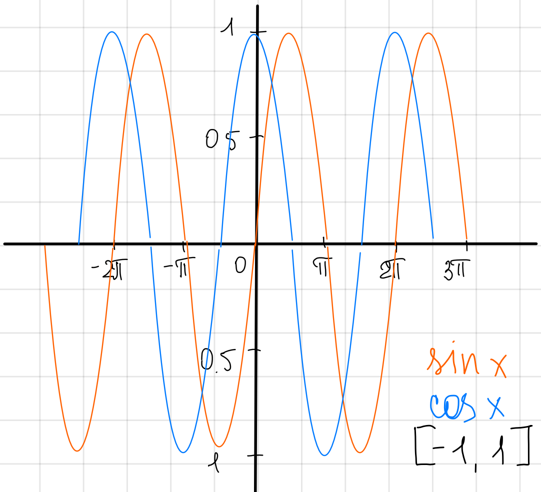
α degrees → β radians

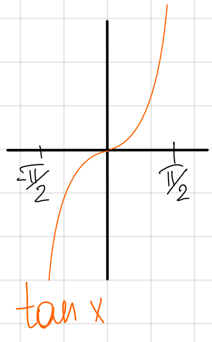
$\alpha \cdot \frac{\pi}{180} = \beta$

$\sin'(x) = \cos(x)$

$\cos'(x) = -\sin(x)$

$\tan'(x) = \frac{1}{(\cos x)^2}$





$TR = (ax+b)x = ax^2 + bx$
if $p = ax+b$
$q = x$

$TC = F + Vx$
x - production level

Break-even output:
$TC = TR$ can be several

Example:

Find the derivative:

$f(x) = \tan(x^2+1)$

composition sr, where

$s(x) = \tan x$

$r(x) = x^2+1$

$r'(x) = 2x$

$f'(x) = \dfrac{2x}{(\cos(x^2+1))^2}$

2 bap:

$(\tan x)' \cdot (x^2+1)'$ $\qquad g = x^2+1$

$\tan' x = \dfrac{1}{(\cos g)^2}$

$= \dfrac{1}{(\cos g)^2} \cdot 2x = \dfrac{2x}{(\cos(x^2+1))^2}$

Algorithm:

- Once each year at the end of the year $C = PV(1+i)^t$
- several years simple r, then compounding r $C = P \cdot (1 + r/m)^t \cdot \exp(rt)$

composite func:
1. solve big one
2. inner one
3. multiply

or

1. substitute inner one with g
2. solve big with g
3. solve g
4. insert g into big one

$(x \ln a)' = \ln a$
same as:
$5x' = x5' = 5$

$e^{x^2} = \exp(x^2)$

$\exp'(x^2) = \exp(x^2) \cdot 2x = 2x \cdot e^{x^2}$

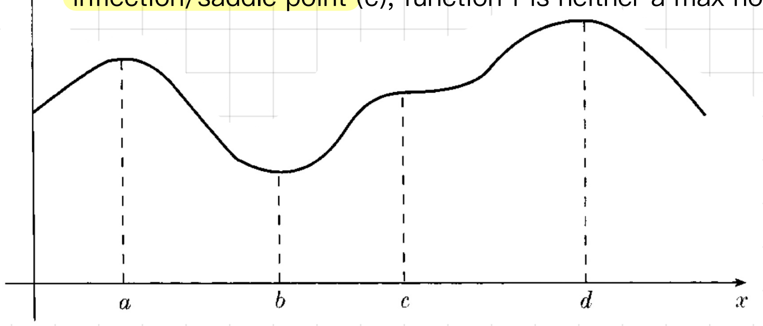
Introduction to maximisation

Optimisation (in math) – finding the "best" value of some quantity. In max/min points graph is horizontal so it's derivative = 0 (f'(x) = 0).

f'(x) > 0	f is increasing at x
f'(x) < 0	f is decreasing at x
f'(x) = 0	horizontal graph

– critical/stationary point, neither increasing nor decreasing

– maximum points (a and d)
– minimum point (b)
– inflection/saddle point (c), function f is neither a max nor a min



$$f''(x) = \frac{d^2 f}{dx^2}$$ second derivative

f'(a)=0 and f''(a)<0	point a is a loc. maximum of f
f'(a)=0 and f''(a)>0	point a is a loc. minimum of f
f'(a)=0 and f''(a)=0	a may be a maximum/minimum/ an inflexion point (we cannot conclude which exactly)

Revenue $R(q) = q \cdot P(q)$

Profit $\Pi(q) = R(q) - C(q) = q \cdot P(q) - C(q)$

8.3 Optimisation in an interval

Crit. points can be local or global

- point a is a local maxima but not max for the whole graph

• *Suppose f is a function for which f' and f'' both exist in the interval $[u, v]$. Then f has a global maximum point m in $[u, v]$, and either (i) $u < m < v$, $f'(m) = 0$ and $f''(m) \leq 0$, or (ii) m is one of u, v.*

- if there's only one local max, then it's global max.

Global optimisation – what is the largest (or smallest) value of f(x) when x is in [u,v]?

I – to find crit. points
II – to classify it

Crit points on an interval [a,b]
can occur at a, b or c(crit point in between)
find crit. points and compare f(x) for a, b and c (even for (a,b))

Curve sketching

1. crosses x-axis $f(x) = 0$ $(x, 0)$
2. crosses y $x = 0$ $(0, y)$
3. crit. points $f'(x) = 0$
4. nature of crit. points $f''(x)$
5. limiting behaviour – what happens to $f(x)$ when x tends to $\pm\infty$
   - examine the highest power n:
     - чётн. • even: $x^n \to \infty$ as $x \to \pm\infty$
     - нечётн. • odd: $f(x) \to \infty$ as $x \to \infty$
       $f(x) \to -\infty$ as $x \to -\infty$
   - in product exp dominates:
     $x^2 \cdot e^{-x} \to 0$ as $x \to \infty$

8.4 Infinite intervals

infinite interval $[u, \infty)$

• *Suppose f is a function for which f' and f'' both exist in the interval $[u, \infty)$. If f has a global maximum point m in $[u, \infty)$, then either (i) $m > u$, $f'(m) = 0$ and $f''(m) \leq 0$, or (ii) $m = u$. A similar statement holds with 'maximum' replaced by 'minimum'.*

SG 3.7

$\Pi(q) = 0$ breakeven point –
$TR = TC$ point where Π just starts to become positive

C - cost func
TC - total cost
FC - fixed cost
VC - variable cost
AC - average cost
AVC - average variable cost
MC - marginal cost
MR - marginal revenue
TR - total revenue
Π - profit

cost of producing +1 item

$AC = TC/q$
$AVC = VC/q$

$TC = FC + VC$
$TC(0) = FC$ > 0

$TR = p \cdot q$
$TR'(q) = MR$

$\Pi = TR - TC$

$MC = TC'$
$TC = \int MC + c$
$TC(0) = \int MC + c = FC$

$\Pi'(q) = 0$ profit maximisation

$MR = MC$
$TR' = TC'$

monopoly:
$p^* = p^D(q)$ same price as cons. wish to pay cos they have nothing to choose from

$TR(q) = q p^D(q)$ – revenue

Marginals are derivatives.

Algorithms:

Crit points: 1. $f'(x)=0$ => critical points are roots
2. $f''(x)$
3. insert 1. roots into 2. 2d derivative
4. classify crit. points

or

1. –"–
2. find $f'(x)$ in the vicinity of roots
3. decide the nature of points

$\searrow \cdot \nearrow$ min $\nearrow \cdot \searrow$ max

Crit points in the interval: 1. –"–
2. which roots are inside of the given interval?
3. insert relevant roots into f''

Global max/min in the interv.: 4. find the values of $f(x)$ for ends of interv. and crit. points:

$[0,2]$ $4/3$ - crit. p. make a table

x	0	$4/3$	2
$f(x)$	-1	$229/27$	7

$229/27 > 7$ => $4/3$ max
-1 min

or: 1. find all local max
2. compare corresponding values of $f(x)$ to find the largest

- if there's only one local max, then it's global max.

Crit. points in infinite intervals: same as previous but may not have a crit. point at all if a func. tends to infinity ($f(x)=x^2$, $x \to \infty$)

Π maximization: $\Pi = TR - TC$
Π'
$\Pi' = 0$ root is max
Π'' to check optional

Examples:

p.84.1 $f(x) = x^3 - 12x^2 + 21x + 100$

$f'(x) = 0$ - crit. point

$f'(x) = 3x^2 - 24x + 21 = 3(x^2 - 8x + 7) = 0$

$D = 64 - 4 \cdot 7 = 36$

$x_1 = \frac{8+6}{2} = 7$

$x_2 = \frac{8-6}{2} = 1$

crit. points

$f''(x) = 6x - 24$

$f''(1) = 6 - 24 < 0$ max

$f''(7) = 42 - 24 > 0$ min

p. 84.2 $R(T) = 20T - 3T^2$

$R'(T) = 20 - 6T = 0$

$T = \frac{20}{6} = \frac{10}{3}$ crit. point

$R''(T) = -6 < 0 \Rightarrow$

$T_m = \frac{10}{3}$ max

$R(T_m) = \frac{200}{3} - \left(\frac{30}{3}\right)^2 = \frac{100}{3}$ - max revenue

p.85 $f(x) = x^3 - 8x^2 + 16x - 1$ in $[0;2]$

$f'(x) = 3x^2 - 16x + 16 = 0$

$D = 16^2 - 4 \cdot 3 \cdot 16 = 64$

$x_1 = \frac{16+8}{6} = 4$

$x_2 = \frac{16-8}{6} = 8/6 = 4/3$

$f'(x) = 3(x - 4/3)(x-4) = (3x-4)(x-4)$

$f''(x) = 6x - 16$

$f''(4/3) = \frac{6 \cdot 4}{3} - 16 < 0$ max

x	0	4/3	2
$f(x)$	-1	229/27	7

$229/27 > 7 \Rightarrow 4/3$ - max

0 - min

p.88 $P(t) = (2000 + 500t) \cdot e^{-(0.1) \cdot t}$

$P'(t) = 500e^{-0.1t} + (2000 + 500t) \cdot e^{-0.1t} \cdot (-0.1) = 500e^{-0.1t} - 200e^{-0.1t} - 50t \cdot e^{-0.1t} = 300e^{-0.1t} - 50t \cdot e^{-0.1t}$

$= e^{-0.1t}(300 - 50t) = 0$

$P'(t) = 0$ when $t = 6$ - crit. point

$P''(t) = e^{-0.1t} \cdot (-0.1) \cdot (300 - 50t) + e^{-0.1t} \cdot (-50) = -30e^{-0.1t} + 5te^{-0.1t} - 50e^{-0.1t} = e^{-0.1t}(-80 + 5t)$

$P''(6) = e^{-50/0.6} < 0 \Rightarrow t = 6$ max.

Partial derivatives

$x, y \rightarrow \boxed{} \rightarrow f(x,y)$ - func of several variables

- Partial derivative of f (with respect to the 1st variable) is obtained by treating the second variable as a constant.

$Z = f(x,y)$ than partial derivative with respect to x is denoted by: $\frac{\partial f}{\partial x} = f_x = \frac{\partial Z}{\partial x} = Z_x = f_1$

Second-order partial derivatives

$\frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{11}$ $f_{xy} = f_{yx}$

- the change in f(x) resulting from a small change in x is approximated by $\Delta f \simeq f'(x) \cdot \Delta x$

Cobb–Douglas production function: $q(k,\ell) = Ak^{\alpha}\ell^{\beta}$

marginal product of capital $q_1(k,\ell) = A\alpha k^{\alpha-1} \cdot \ell^{\beta}$

marginal product of labour $q_2(k,\ell) = A\beta k^{\alpha}\ell^{\beta-1}$

$q_{11}(k,\ell) = A\alpha(\alpha-1)k^{\alpha-2} \cdot \ell^{\beta}$

k - capital
ℓ - labour

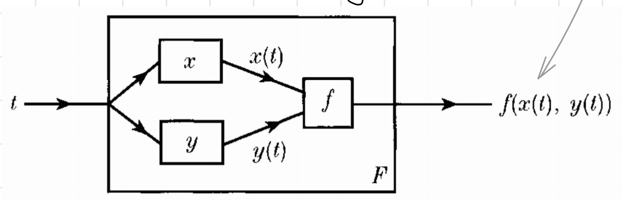
$q_{21}(k,\ell) = q_{12}(k,\ell) = A\alpha\beta k^{\alpha-1} \cdot \ell^{\beta-1}$

$q_{22}(k,\ell) = A\beta(\beta-1)k^{\alpha}\ell^{\beta-2}$

Chain rule

$F'(t) = f_x \cdot x'(t) + f_y \cdot y'(t)$ for $F(t) = f(x(t), y(t))$

$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$ - different notation of chain rule



$F'(t)$ - total derivative of F with resp. to t.

for func $F(u,v) = f(x(u,v), y(u,v))$ part. d. are:

$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$

$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$

Algorithms:

Second order derivative: f_x f_{xx} $f_{xy} = f_{yx}$ x changes, y = const

f_y f_{yy}

Derivative of a func. $F(t) = f(x(t), y(t))$:

option 1:

1. insert x(t) and y(t) into original func. F(t)
2. find derivative of resulted func.

option 2: The chain rule

$F'(t) = f_x \cdot x'(t) + f_y \cdot y'(t)$

Optimisation in two variables

$\Pi(x,y) = R(x,y) - C(x,y) = xp^x(x,y) + yp^y(x,y) - C(x,y)$ – profit, x y - units

Critical point – one where both partial derivatives are zero (surface is horizontal).

Maximum : peak ∩
Minimum : pit ∪
None : saddle point

$Q(x,y) = Ax^2 + Bxy + Cy^2 + Lx + My + N$

$\Delta Q = Ax^2 + \left(\frac{D}{A}\right)y^2$

$Q_x(x,y) = 2Ax + By + L = 0$
$Q_y(x,y) = Bx + 2Cy + M = 0$

$\Delta Q < 0$ maximum
$\Delta Q > 0$ minimum
either signs - saddle point

$x_0 = \frac{BM - 2CL}{D}$ $\quad y_0 = \frac{BL - 2AM}{D}$

Implicit part dif.

$\frac{dy}{dx} = -\frac{g'_x}{g_y} = -\frac{\partial g}{\partial x} : \frac{\partial g}{\partial y}$

- Extended for $g(x,y,z) = c$

$\frac{\partial z}{\partial x} = -\frac{\partial g}{\partial x} : \frac{\partial g}{\partial z}$

$\frac{\partial z}{\partial y} = -\frac{\partial g}{\partial y} : \frac{\partial g}{\partial z}$

Optimisation

(maximising or min. a func)

Critical (stationary) point - where f_x and f_y both $= 0$

Saddle point - crit. point which is neither max or min.

$f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$	and	$f_{xx} < 0$	- max
$f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$	and	$f_{xx} > 0$	- min
$f_{xx} \cdot f_{yy} - f_{xy}^2 < 0$	saddle point		
$f_{xx} \cdot f_{yy} - f_{xy}^2 = 0$	can't classify		

$TR(x,y) = x p_x + y p_y$

$\Pi(x,y) = TR(x,y) - TC(x,y) = x \cdot p_x + y \cdot p_y - TC(x,y)$

Meaning of Lagrange mult.

$\lambda^* = \frac{\partial V}{\partial M} = V_M$

- utility func. V - indirect util. func.

$u(x^*) = u\left(q_1(p_1,p_2,M), q_2(p_1,p_2,M)\right) = V(p_1,p_2,M)$

- change in max. utility: $\Delta V \simeq \lambda^* \cdot \Delta M$

budget constraint
$M = p_1 \cdot x_1 + p_2 \cdot x_2$ M - income

Constrained optimisation

$L(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y)$

L Lagrangean (-gian)
λ Lagrange multiplier

First - order conditions:

$\begin{cases} L_x = 0 \\ L_y = 0 \\ L_\lambda = 0 \end{cases}$

if they're satisfied, then the theory assert that the required points of f, subject to constraint, should be found among these crit. points.

or $g(x,y) = 0$

Algorithms:

Finding crit. points and defining their nature:

1. f_x f_y
2. $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ => roots are crit. points
3. find f_{xx}, f_{yy}, f_{xy} to determine the nature
4. if they're numbers → $f_{xx} \cdot f_{yy} - f_{xy}^2$
4. if they're func's → insert x and y → $f_{xx} \cdot f_{yy} - f_{xy}^2$

optional 5. find $f(x,y)$: insert crit. point into original func

Profit maximisation:

1. find $p_x =$ $p_y =$
2. use $\Pi = x \cdot p_x + y \cdot p_y - TC$ to find func $\Pi(x,y)$
3. $\begin{cases} \Pi_x = 0 \\ \Pi_y = 0 \end{cases}$ => crit. point

optional 4. check if it's a max/min by $\Pi_{xx} \cdot \Pi_{yy} - \Pi_{xy}^2$

5. insert crit. p. into Π

Constrained optimisation (Lagrange):

1. determine what is $f(x)$ - being constrained (max/minimized) and $g(x)$ - constraint itself
2. use $f(x) - \lambda \cdot g(x)$
3. $\begin{cases} L_x = 0 \\ L_y = 0 \\ L_\lambda = 0 \end{cases}$
4. from L_x and L_y find $\lambda =$ and equalize right parts
5. find $y = ..x$
6. substitute it into L_λ to find (x,y) - crit. point

optional 7. find $f(x,y)$ by inserting crit. p. into original func

Vectors, preferences, and convexity

An n–vector v is a list of numbers, written as a row–vector x or a column–vector. y
Numbers v1, v2.. are components/entries/coordinates of v.
Zero vector – vector with all of its entries equal to 0.

$x = (v_1, v_2, v_3)$ $\quad y = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$(0, 0, 0)$

addition:
$$(w_1, w_2, w_3) + (v_1, v_2, v_3) = (w_1 + v_1,\ w_2 + v_2,\ w_3 + v_3)$$

multiply by a scalar:
$$\alpha(v_1, v_2, v_3) = (\alpha v_1, \alpha v_2, \alpha v_3)$$

additional rules:
$$(w + v) + x \qquad (\alpha + \beta) \cdot v = \alpha v + \beta v$$

dot/inner/scalar product:
$$v \cdot w = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3$$
– result is a number
– special case of 'matrix multiplication'

Budget constraint:

cost of bundle $\leq M$
general case: $p \cdot x \leq M$

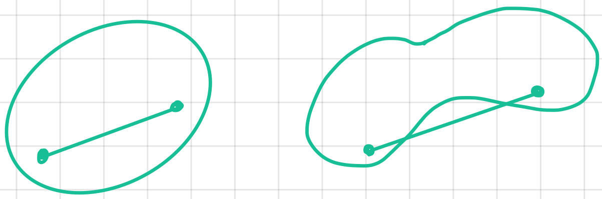
M - budget limit
$p \cdot x = p_1 \cdot x_1 + p_2 \cdot x_2 + p_3 \cdot x_3$

Utility and indifference curves

- consumers prefer more to less
- there's no ec meaning in numerical values of u (how much u(x) is bigger than u(y)), only preference ordering (prefer u(x) on u(y)) if u(x) is twice u(y) it doesn't mean that x is twice better than y
- graph of a utility func u(x,y) is an indifference curve all the points on the curve have the same utility => consumer is indifferent among them
- the further from the origin the better for the consumer

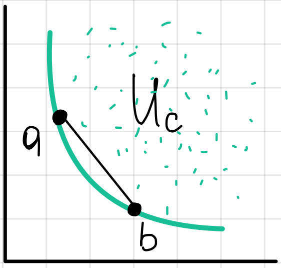
Linear and convex combinations.

Set X is convex if the straight line segment joining any 2 points in X lies wholly in X.



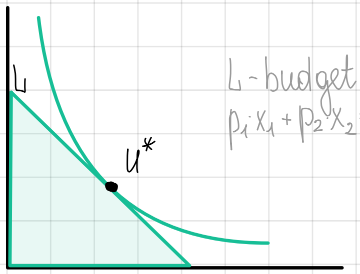
convex non - convex

$U_c = \{x \mid u(x) \geq c\}$ - set for $u \geq c$, is convex



- consumers prefer convex combination (mixture) of and b to either of the extreme a or b bundles (usually true for consumers with large consumption (e. g. manufacturer)).

- optimal level of utility u* – budget line is a tangent to an indifference curve:



L - budget line:
$p_1 x_1 + p_2 x_2 = M$

+slopes of the curves

Matrix algebra

A matrix is an array of numbers, denoted by single letter A or by (aij). Has m rows and n columns (m x n matrix).

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Square matrix – where m=n.
Entry of A – number aij.
- Vector has commas and the matrix does not.

A+B

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

C·A

$$c\cdot\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} c\cdot a_{11} & c\cdot a_{12} \\ c\cdot a_{21} & c\cdot a_{22} \end{pmatrix}$$

$A(BC) = (AB)C$

$A(B+C) = AB + AC$

Matrix multiplication A·B:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 11 & 16 \\ 4 & 7 & 7 \end{pmatrix}$$

$c_{11} = 1\cdot1+3\cdot2=7$
$c_{12} = 1\cdot2+3\cdot3=11$
$c_{13} = 1\cdot1+3\cdot5=16$
$c_{21} = 2\cdot1+1\cdot2=4$
$c_{22} = 2\cdot2+1\cdot3=7$
$c_{23} = 2\cdot1+1\cdot5=7$

$$c_{ij} = a_{i1}\cdot b_{1j} + a_{i2}\cdot b_{2j} + \dots$$

- The entry cij is the dot product of row i of A and column j of B.
- The number of columns of A should be equal to the number or rows of B. (if A has n columns then B must have n rows). In any other case, the product is not defined.
- A mxn and B nxp => AB mxp
  AB and BA aren't equal.

The identity matrix:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Identity matrix:
- 1 in each of the main diagonal positions, 0 elsewhere.
- Is a square matrix.
- There's an identity matrix of any size nxn.
- If A is an nxn => IA=AI=A

15.3 Book

Bond – yield a fixed return.
Stock – yield an uncertain return.
Portfolio – pack of assets, can be expressed as vector.
Riskless portfolio – has the same value whatever happens.
Arbitrage portfolio – costs nothing, cannot lose and in at least one state yields a profit (500 –1000 500).

Linear equations

A system of m linear (simultaneous) equations in n unknowns x1, x2.. is a set of m equations of the form:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$ aij – coefficients of the system.

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$

Elementary row operations:

Gaussian Elimination (Gauss–Jordan/Gaussian) method: reduce the augmented matrix to an echelon form.

- works even when the number of equations and unknowns are different.

Solutions are unaltered when using elementary operations:

- multiply both sides of an equation by a non zero constant
- add a multiple of one equation to another
- interchange two equations

original system: $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_1 + 5x_2 + 4x_3 = 1 \end{cases}$

To transpose – interchanging rows and columns.

augmented matrix

$$(Ab) = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 5 & 4 & 1 \end{pmatrix}$$

echelon (reduced) form

$$Cd = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

Algorithms:

Solving a system using a matrix method:

1 rewrite the coefficients of the system as an augmented matrix
2. use elementary operations to reduce it to an echelon form
3. write back as a system (right column is after =)
4. solve the system
optional: 5 check by substituting the roots back into the original system.

Areas and integrals

- Finding the area under a curve is the inverse of finding the derivative (=integrating)

Indefinite integrals = anti–derivative + C

anti derivative of f: $t^3 + \frac{t^2}{2} + t + C$ – (indefinite) integral

integrand

$$\int f(t)\,dt = F(t) + C$$

F(t) - anti-der. of f

$$\int t^n\,dt = \frac{1}{n+1} t^{n+1} + C \qquad (n \neq -1)$$

$$\int 1\,dx = 1x + C$$

$$\int x\,dt = xt$$

$$\int \big(f(t) \pm g(t)\big)\,dt = \int f(t)\,dt \pm \int g(t)\,dt$$ – „sum rule"

$$\int k \cdot f(t)\,dt = k \cdot \int f(t)\,dt$$ – „product rule", k-const

$x = y'$

$y = \int x$

Definite integrals finds area under the graph between a and b

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(\,)]_a^b$$ – integral of the func f over the interval [a,b]

- the area enclosed by the curve y = f(t), the t–axis and the vertical lines t = a and t = b is equal to $\int_a^b f(t)\,dt$

Standard integrals

$$\int x^n = \frac{x^{n+1}}{(n+1)} + C$$

$$\int e^{kt}\,dt = 1/k \cdot e^{kt} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int 1/x = \ln|x| + C$$

$$\int 1/x^n = -\frac{1}{(n-1)\,x^{n-1}}$$

$$\int e^x = e^x + C$$

$$\int \ln(x) = x \ln x - x + C$$

by the integration by parts rule

Consumer surplus:

$$CS = A - R = \int_0^{q^*} p^D(q)\,dq - p^* \cdot q^*$$

equilibr. P and Q

inverse demand func

A - area under D curve from 0 to q^*

$R = p^* \cdot q^*$

Algorithms:

Definite: 1. calculate the integral

2. insert $\int_a^b$ a and b into variable

3. calculate F(b) - F(a)

Techniques of integration

26.1 Integration by substitution („change of variable")

1. substite func with t
2. $dx \rightarrow du$ $\quad du = t'$ $\quad dx = \frac{1}{u'} \cdot dt$
3. substitute t back with x

26.2 Definite integrals by substitution

$$\int_{x=a}^{x=b} \rightarrow \int_{u=\alpha}^{u=\beta}$$

insert a into u to get α

- no need for back substitution
- no + C

Factorisation:

$\begin{cases} N_1 + N_2 = b \\ N_1 \cdot N_2 = ac \end{cases}$ N-aren't roots!

rewrite as $ax^2 + N_1 x + N_2 x + C$

group $ax(x \pm \cdot) + \cdot(\cdot \pm \cdot) \Rightarrow (\cdot \pm \cdot)(\cdot \pm \cdot)$

or

$N_1 + N_2 = -b/a$ $\quad ax^2 + bx + c = a(x - N_1)(x - N_2)$

$N_1 \cdot N_2 = c/a$ $\quad$ Vieta's theorem $\quad$ N-roots

26.3 Integration by parts

$$\int u' \cdot v \, dx + \int u \cdot v' \, dx = u \cdot v$$

$$\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx$$

or $$\int f \, dg = fg - \int g \, df$$

26.4 Partial fractions

func of the form $\frac{p(t)}{q(t)}$ (linear / quadrat.) – rewrite this expression (p and q are polynomials) into a sum of simpler terms called partial fractions

- numerator should have a smaller degree than denominator

$$\int \frac{p(x)}{q(x)} dx = \int \left(\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} \right) dx = A_1 \ln|x - a_1| + A_2 \ln|x - a_2| + c$$

Algorithms:

$MC = TC'$ (SG 4.7)

$TC = \int MC + c$

$TC(0) = \int MC + c = FC$ (to find c)

Πmax: 1. $\Pi = TR - TC$ or 1. $TR' = TC'$
2. $\Pi' = 0$

$TC(4) - TC(2) = \int_2^4 MC$

Substitutions: $\int (ax+b)^{12}$

1. let (smth) be u
2. calculate $u = smth$ and $x =$ (optional)
   $u' = smth'$
   $du = smth' \, dx$ $\quad du = f'(x) \, dx$
   $dx = \frac{1}{smth'} du$
3. substitute f(x) with u and dx with $\frac{1}{smth} du$
4. calculate the integral
5. substitute u with smth, add +C

Substitution for definite

1.-3. –"–

4. substitute $\int_a^b$ with $\int_\alpha^\beta$ $\quad \alpha = u(a)$, $\beta = u(b)$
5. calculate the integr.
6. insert α and β
7. calculate $F(\beta) - F(\alpha)$

! no back substitution of u

! no +C

Partial fractions: $\int \frac{1x}{x^2 + x + C}$

1. factorise denominator:
2. rewrite as $\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2}$
3. multiply criss-cross, $A_1(\ ) + A_2(\ ) =$ numerator
4. calculate $A_{1,2}$ by inserting $x_{1,2}$
4. 'cover-up rule' $\frac{x}{[(\ )] \cdot (\ )}$ insert $a_{1,2}$ into
5. substitute $A_{1,2}$ with numbers, calc. the integral

By parts: $\int u \cdot v$

1. take one factor as u' and other as v
2. $u' =$ $\quad u = \int u'$
   $v =$ $\quad v' =$
3. insert everything using formula
4. calculate as usual

! if doesn't work try swapping u and v