Probability theory

Probability theory – how likely various events are to occur.
Probability – quantifiable measure of one's degree of belief in a particular event (set) of interest.
– the likelihood of a certain event occuring out of a total possible number of events.
Experiment – e.g. toss of a coin / roll of a die.
Sample space S – set of all possible outcomes of an experiment.
Elements (members) – are themselves events (outcomes). Coin toss: $S = \{Heads, Tails\}$ where H, T – elements
Event – collection of elementary outcomes from the sample space S of an experiment wich is a subset of S.

? | Die score = $\{1,2,3,4,5,6\}$ – elementary outcomes
| Score > 4 = $\{5,6\}$ – event

- events are denoted by letters A = 'an even score', B = 'score greater than 4'
- probability P(A) lie on a scale from 0 to 1 inclusive $0 \le P(A) \le 1$
  - $P(A) = 0$ => A – is an impossible event
  - $P(A) = 1$ => A – is a certain event
  - $P(A) < P(B)$ => A is less likely to occur than B => we can rank events

N – total number of outcomes
n – number of outcomes wich are favourable to our event of interest

$P(A) = \frac{n}{N}$ – event probability for equally likely elementary outcomes (e.g. toss of a fair coin)

Die roll:
$A = \{2,4,6\}$ $P(A) = 3/6 = 1/2 = 0.5$
$B = \{5,6\}$ $P(A) = 2/6 = 1/3$

Relative frequency

Absolute frequency – number of times a particular response was observed.
Relative frequency – absolute frequency in terms of the total number of responses gathered. = experimental probability

$P(A) = \frac{f}{F}, \quad F \to \infty$

A – experiment wich does or doesn't occur
experiment is conducted independently F times
event A occurs f times

Business wants to successfully implement a new management methodology.
they implemented it 500 times
it was successful 85 times

$P(A) = \frac{85}{500} = 0.17$

Randomness

Even if we repeat an experiment under identical (seems like) conditions → we get different results.
- conditions could only be as identical as we're able to control them to be
- there'll always be a lot of uncontrollable and unknown variables
- they'll have a cumulative effect on the result of the experiment => will cause variation in results

Randomness – variation in results.

Properties of probability

Axiom – self-evident truth.

Axioms of probability:
1) for any event A, $P(A) \ge 0$
2) for the sample space, S $P(S) = 1$
3) if $\{A_i\}$, $i = 1, 2, \dots, n$ are mutually exclusive events => probability of their 'union' is the sum of their respective probabilities

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Mutually exclusive events – they cannot both occur simultaneously. Occurrence of 1 event prevents the occurrence of the other.
A = 'obtain an even score', B = 'obtain an odd score'
*consumer won't choose 2 products simultaneously if they're substitutes

Pairwise mutually exclusive events – at most one of the events can occur (no 2 events can occur simultaneously).
3 events: A, B, C   A and B can't occur together, B and C, and A and C too

Collectively exhaustive – at least one of events must occur. All possible experimental outcomes are included among the collection of events.

Notational vocabulary

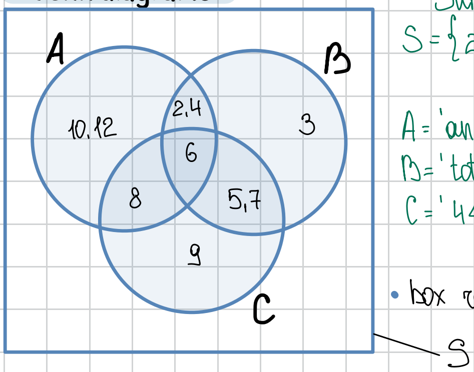
Symbol	Informal version	Formal version	Example
$\cup$	or	union	$A \cup B$ = 'A union B'
$\cap$	and	intersect	$A \cap B$ = 'A intersect B'
^{c}	not	complement of	A^c = 'complement of A'
$\mid$	given	conditional on	$A \mid B$ = 'A conditional on B'

Set – collection of elementary outcomes from S.
A = 'an even die score'

Probability of a set – number on the unit interval [0,1].
$P(A) = 0.5$

$P(A)$ – probability of event A occurring
$P(A^c)$ – probability of event A not occurring

Venn diagrams



Sum of rolling a die twice:
$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

A = 'an even total' $= \{2, 4, 6, 8, 10, 12\}$
B = 'total less than 8' $= \{2, 3, 4, 5, 6, 7\}$
C = '4 < total < 10' $= \{5, 6, 7, 8, 9\}$

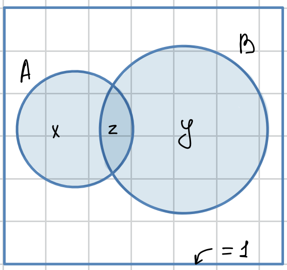
$A \cap B = \{2, 4, 6\}$
$A \cap B \cap C = \{6\}$
$A \cap B \cap C^c = \{2, 4\}$
$(A \cup C)^c \cap B = \{3\}$

$A \cap C = \{6, 8\}$
$(A \cup B \cup C)^c = \{11\}$
$A^c \cap B = \{3, 5, 7\}$
$A \mid C = \{6, 8\}$

• box represents S – every possible outcome of the experiment appears within the box

Additive law

1. A and B – any 2 events
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ – additive law
$P(A \cup B)$ – probability that at least 1 of A and B occurs
$P(A \cap B)$ – both A and B occur

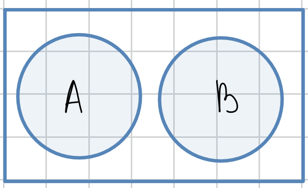


3.11) $P(A) = x + z$
$P(B) = y + z$
$P(A \cap B) = z$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= (x + z) + (y + \cancel{z}) - \cancel{z} = x + y + z$

3.12) Machine can be defective in 2 ways:
$P(\text{defective in 1st way}) = P(D_1) = 0.01$
$P(\text{defective in 2nd way}) = P(D_2) = 0.05$
$P(\text{defective in both ways}) = P(D_1 \cap D_2) = 0.001$

Probability that the component is defective:
$P(D_1 \cup D_2) = P(D_1) + P(D_2) - P(D_1 \cap D_2)$
$= 0.01 + 0.05 - 0.001 = 0.059$

2. A and B are mutually exclusive events, $P(A \cap B) = 0$
$P(A \cup B) = P(A) + P(B)$



3. event A not occurring, the complement A^c
$P(A^c) = 1 - P(A)$

Multiplicative law

Independence - 2 events are independent if one has no influence on the other.

The multiplicative (product) law for independent events: events A and B are independent if the probability of their intersect is the product of their individual probabilities.

$P(A \cap B) = P(A) \cdot P(B)$

- the law doesn't hold for dependent events
- independent $\neq$ mutually exclusive

Rolling 2 fair dice (score on 1 die has no influence on the other):

$P(2 \text{ sixes}) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

Conditional probability. Bayes' formula

Unconditional (marginal) probability - probability that an event occurs without considering any other preceding events.

- "stand-alone" events
- probability that it will rain on a given day (without considering any patterns of the area)

Conditional probability - probability of an event A given that another event B has already occured.

- what is the prob. of one event occuring if another event has already taken place?
- $P(A|B)$ - probability of A given B
- probability that it will rain on a given day considering that there have been raining for the past 3 days

Bayes' formula

For any 2 events A and B, conditional probabilities are:

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ and $P(B|A) = \frac{P(A \cap B)}{P(A)}$, $P(A) > 0$ and $P(B) > 0$

$\Rightarrow P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$

Bayes' formula:

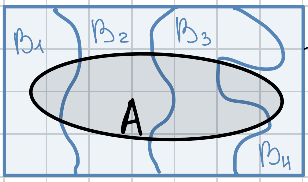
$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

General form of Bayes' formula

for a general partition of the S into $B_1, B_2, \dots, B_n$ and for some event A:

$$P(B_k|A) = \frac{P(A|B_k) \cdot P(B_k)}{\sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)}$$

Total probability



sample space S

- event A within a partitioned sample space
- total probability - a prob. within the partitioned space?

$P(A) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)$ - total probability

- $B_1, B_2, \dots B_n$ - complete group of mutually exclusive events
  - each B_i is mutually exclusive with others $B_i \cap B_j = \emptyset$ for $i \neq j$
  - their union covers the entire sample space (they form a complete partition of the sample space) $\bigcup_{i=1}^{m} B_i = \Omega$
- B_1, B_2, B_3, B_4 partitions the sample space S into n pairwise mutually exclusive (at most 1) and collectively exhaustive (at least 1) events

When to use total probability f-la?

- сложное событие, кот. состоит из неск. разбиений
- сначала происходит один случ. выбор (коробка..) а затем 2ой в рамках уже выбранной группы
- зависит ли итоговая P от того, какой сценарий произошел?

x простая вероятность, нет разбиение, P считается в рамках одного опред. пространства S

$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$

$P(C) = P(C|A) \cdot P(A) + P(C|B) \cdot P(B)$

special cases of total prob. f-la

- B and B^c are mutually exclusive
- B and B^c are collectively exhaustive
- B and B^c - is a partition of sample space

Independent events

(In)dependent – probability of an event is changed when another event is known to occur | only if there's some dependence between them.

- there's a dependence $\Rightarrow P(A|B) \neq P(A)$
- events A and B are independent only if: $P(A|B) = P(A)$
- under independence $P(A \cap B) = P(A) \cdot P(B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A), \text{ if } P(B) > 0$$

Probability trees (tree diagram)

- are used for visualisation (same as Venn diagrams)

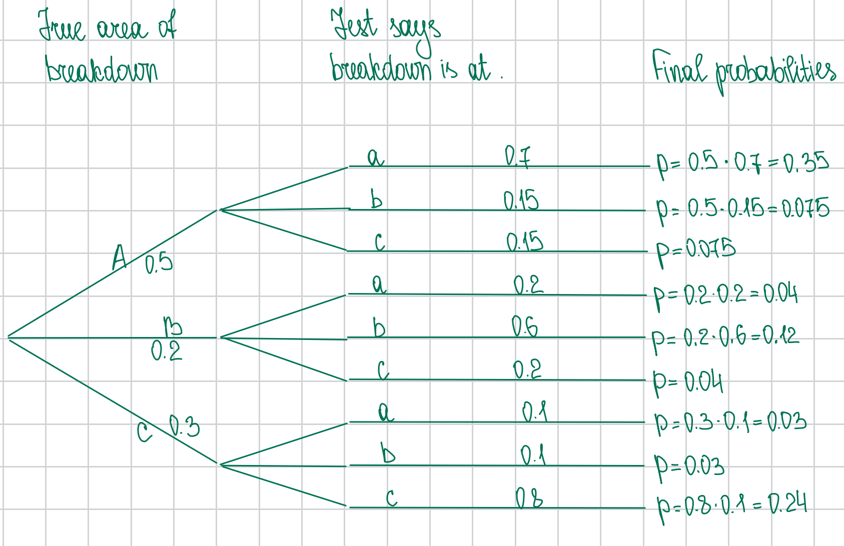
3.18) Factory has a breakdown in 1 of 3 areas. They did some tests and came up with an approximate probabilities:

Suspect area	A	B	C
Probability of a breakdown being in it	0.5	0.2	0.3

Then Co did some additional tests and came up with:
- if the breakdown is at A → probability that the test will say the same way is 70%
- B → 60%
- C → 80%

*in all 3 cases probabilities of other 2 areas are equal

Probability tree



⇒ Sum of branches is basically a total prob. f-la

From the tree we get:

A = "breakdown is at A" a = "test says breakdown is at A"
B = "breakdown is at B" b = "test says breakdown is at B"
C = "breakdown is at C" c = "test says breakdown is at C"

1) What is the probability that the new test will identify A as the area with breakdown? (doesn't matter if the test is right or wrong)

$P(a) = 0.35 + 0.04 + 0.03 = 0.42$ (branches with a)

2) If the new test identifies A as the area with breakdown..

2.1) what's the new prob. that C is the area with breakdown?

conditional probability

$$P(C|a) = \frac{\overbrace{C \cap a}^{\text{branch}}}{P(a)} = \frac{0.03}{0.42} = 0.071$$

2.2) what's the new prob. that B is not the area?

conditional prob.

$$P(B^c|a) = \frac{A \cap a + C \cap a}{P(a)} = \frac{0.35 + 0.03}{0.42} = 0.905$$

or $P(B^c|a) = 1 - \dfrac{B \cap a}{P(a)} = 1 - \dfrac{0.04}{0.42} = 0.905$