Homework 7

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1 Introduction

This project is inspired by Prof. Thomas Sargent's book, **Recursive Macroeconomic Theory** and also his YouTube Talks on Computational Challenges in Macroeconomics. In this talk he tries to show the complex computational aspect of Dynamic Programming Squared Models. These types of problems arose from interaction of government(principal) policies with the actions of the private agents. For Financial Crises we want to solve Dynamic Programming Squared (DP^2) problems. They have Bellman equations inside Bellman Equations. The interior Bellman Equation ($v(x)$) captures incentive effects of people whose incentives are affected by government policy. The outer Bellman Equation ($W[v(x), x]$) puts structure on government incentives subject to agent value maximization constraint. The basic structure is as follows:

$$W[v(x), x] = \max_{d, v(x')} \left\{ R(x, d) + \beta \int W \left[v\left(x'\right), x' \right] dF\left(x' | x\right) \right\} \tag{1}$$

where the maximization is subject to

$$v(x) = \max_{c} \left\{ u(x, c) + \beta \int v\left(x'\right) dF\left(x' | x, c\right) \right\} \tag{2}$$

in this project I am looking into stochastic economic behavior of principal and identical agents in a one-good, pure exchange economy. Each agent hold shares of assets. An asset(land) empowers the agent to claim the output(rent) generated by it. The output of an asset is affected by investment decision of agents and the principle. For example, localities with better public infrastructure (principal investments) and well furnished houses (agents investments) leads to higher rents generation. However, agents will only cooperate with the principal if they are at least as well off as they are without the principal involvement. That is, the principal(government) formulates a contract consisting of tax rate, private and public investment decision sequences such that an agent utility level is unchanged with principal involvement but maximizes the welfare of principals.

2 Economy

1. Agents

The preferences of a representative agent over the random consumption sequences are:

$$\mathbb{E}_0 \left\{ \sum_{0}^{\infty} \beta^t U\left(c_t\right) \right\} \tag{3}$$

where c_t follows stochastic process, depending on investment decisions of the consumers (agents) and government (principal), represent consumption of a single good, β is a discount factor, $U$is currentperiod utility function, and $\mathbb{E}\{.\}$ is expectation operator. $U : \mathbf{R}_+ \to \mathbf{R}$ is bounded, continously differentiable, strictly increasing, and stricly concave, with $U(0) = 0$ and where $\beta \in (0, 1)$.

Let Y be compact subset of $\mathbf{R}^l_+$, with its Borel subsets Y. Rents in any period are denoted by the vector $y = (y_1, \ldots, y_l) \in Y$, here y_i denotes the rent by asset i. Similary, let K be compact subset of $\mathbf{R}^l_+$, with its Borel subsets K. Private investment decisions in any period are denoted by the vector $k = (k_1, \ldots, k_l) \in K$, here k_i denotes the rent by asset i. We will assume that the rents endowments are Markovian, following the exogenous process

$$y^i_{t+1} = G^i\left(y^i_t, x_t, k^i_t, z_t\right) \tag{4}$$

where G^i is a bounded continuous function and $\{z_t\}$ is an iid shock sequence with known distribution ϕ, x_t denotes the public investment decision at time t, and $x_t, k_t \geq 0$ for all t. Thus at time period t, given the government contract $G\left(\tau, x_t\right)$ and the state of the economy y_t, the agent chooses c_t, k_{t+1} so as to maximize the present discounted expected utility eq (3) subject to constraints:

$$c_t + k_t * 1 \leq (1 - \tau) * y_t * 1, \text{ all } z^t, \text{ all } t \tag{5}$$

2. Government

Let the $\{g_t\}_{t=0}^{\infty}$ denotes the consumption sequence of the government (principal). The government problem can also be model as stochastic analogue of one-sector model of optimal growth as follows

$$\sup \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \gamma^t W\left(g_t\right) \right\} \tag{6}$$

Such that

$$g_t + x_t \leq \tau * y_t \tag{7}$$

where $G = \left(G^1, \ldots, G^l\right), G^{i\mathrm{s}}$ are defined in eq (4), g_t follows a stochastic process, representing consumption of a single good by principal, γ is a discount factor, W is a current period utility function, $\mathbb{E}_0$ is the expectations operator which indicates the expected value with respect to the probability distribution of the random variables g_t, x_t, z_t over all t based on information available in period $t = 0.W : \mathbf{R}_+ \to \mathbf{R}$ is bounded, continuously differentiable, strictly increasing, and strictly concave, with $W(0) = 0$ and where $\gamma \in (0, 1)$.

3 Equilibrium

Let $v^*(y)$ be the value of the objective(3) for an agent who begins in state y with current period shock as z and does not participate in government contract and follows an optimum consumption investment policy thereafter.

$$\begin{array}{l} v\left(y_t\right) = \max_{c_t, k_t} \left\{ U\left(c_t\right) + \beta \int v\left[G_t\left(y_t, x_t = 0, k_t, z_t\right)\right] Q\left(z_{t-1}, dz_t\right) \right\} \\ \text{subject to } c_t + k_t \leq y_t = G_{t-1}\left(y_{t-1}, x_{t-1}, z_{t-1}\right); c_t \geq 0; k_t \geq 0 \text{ for all } t \end{array} \tag{8}$$

Let $v^*\left(y_t\right)$ satisfy eq (8). However eq (8) can be simplified by changing the state variable to $y_t = y$ as follows

$$\begin{array}{l} v(y) = \max_k \left\{ U(y-k) + \beta \int v(G(y, x=0, k, z)) Q\left(z', dz\right) \right\} \\ \text{where } y = y_t k = k_t; x = x_t; z = z_t; z' = z_{t-1}; y - k \geq 0; y_{t+1} = G(y, x, k, z) \end{array} \tag{9}$$

Eq(9) can also be written in expectation form, as it will be useful later on to solve the

$$\begin{array}{l} v\left(y_t\right) = \max_{c_t, k_t} \left[U\left(c_t\right) + \beta v\left[G\left(y_t, x_t = 0, k_t, z_t\right)\right] \right\} \\ v\left(y_t\right) = \max_{c_t, k_t} \left\{ U\left(c_t\right) + \beta \mathbb{E}_t\left[v\left(y_{t+1}\right)\right] \right\} \end{array} \tag{10}$$

Thus given the state y, principal has to formulate a contract in such a way the agent is willing to participate. This implies that agent's welfare is atleast $v^*(y)$ from the contract $(\tau, \{x_t\}_{t=0}^{\infty})$ for each $y \in Y$.

An equilibrium is set of continuous functions $w\left(y, v^*(y)\right) : Y \times \mathbf{R}_+ \to \mathbf{R}_+, v(y) : Y \to \mathbf{R}_+$ and $v^*(y) : Y \to \mathbf{R}_+$ such that:

$$\begin{array}{l} w\left(y, v^*(y)\right) = \max_{x,k} \left\{ W(g) + \gamma \int w\left(y', v^*\left(y'\right)\right) Q\left(y, dy'\right) \right\} \\ \text{subject to } g + x \leq \tau y; y' = G(y, x, k, z); g \geq 0; x \geq 0 \\ v(y) = \left\{ U[(1-\tau)y - k] + \beta \int v\left(y'\right) Q\left(y, dy'\right) \right\} \geq v^*(y) \end{array} \tag{11}$$

where $v^*(y)$ defines the agent utility without any principal role, as defined in eq (9). Equation (3.4a) say that given the state y and $v^*(y)$, the principal allocates resources τy optimally in current consumption g and end-of-period investment x'. Equation (11) states the budget constraint for the government. this equation says that principal allocates allocates agent's resources $(1-\tau)y$ in agent current consumption c and agent end-of-period investment k' such that it it is as well off as it is without any principal role. Thus equation set (11) can stated as problem of dynamic programmming squared model.

4 Mathematical Solution

1. Without Government Contract

Objective

$$U\left(y_0, k\right) = \sum_{t=0}^{T} \beta^t * h_t\left(y_t, k_t\right) + W_{T+1}\left(y_{T+1}\right) \tag{12}$$

Equation

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t h_t(y_t, k_t) + W_{T+1}(y_{T+1}) + \sum_{t=0}^{T} \lambda_t \left[G_t(y_t, x_t, k_t) - y_{t+1}\right] \tag{13}$$

First Order Conditions

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial h_t}{\partial k_t} + \lambda_t * \frac{\partial y_{t+1}}{\partial k_t} = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial y_t} = \beta^t \frac{\partial h_t}{\partial y_t} + \lambda_t \frac{\partial y_{t+1}}{\partial y_t} - \lambda_{t-1} = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial y_{T+1}} = W'_{T+1}(y_{T+1}) - \lambda_T = 0 \tag{16}$$

Euler Equation

$$\lambda_t = -\beta^t * \frac{\frac{\partial h_t}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial k_t}}; \lambda_{t-1} = \beta^t \frac{\partial h_t}{\partial y_t} - \beta^t \frac{\frac{\partial h_t}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial k_t}} * \frac{\partial y_{t+1}}{\partial y_t} \tag{17}$$

$$\lambda_t = \beta^{t+1} \frac{\partial h_{t+1}}{\partial y_{t+1}} - \beta^{t+1} \frac{\frac{\partial h_{t+1}}{\partial k_{t+1}}}{\frac{\partial y_{t+2}}{\partial k_{t+1}}} * \frac{\partial y_{t+2}}{\partial y_{t+1}} \tag{18}$$

$$\beta^t \frac{\partial h_t}{\partial k_t} + \left[\beta^{t+1} \frac{\partial h_{t+1}}{\partial y_{t+1}} - \beta^{t+1} \frac{\frac{\partial h_{t+1}}{\partial k_{t+1}}}{\frac{\partial y_{t+2}}{\partial k_{t+1}}} * \frac{\partial y_{t+2}}{\partial y_{t+1}}\right] * \frac{\partial y_{t+1}}{\partial k_t} = 0 \tag{19}$$

$$\frac{\partial h_t}{\partial k_t} = \beta \left[-\frac{\partial h_{t+1}}{\partial y_{t+1}} + \frac{\frac{\partial h_{t+1}}{\partial k_{t+1}}}{\frac{\partial y_{t+2}}{\partial k_{t+1}}} * \frac{\partial y_{t+2}}{\partial y_{t+1}}\right] * \frac{\partial y_{t+1}}{\partial k_t} \tag{20}$$

Thus, the equation (20) could be used to relate k_t and k_{t+1}. Now under the assumption that the agent dies at time T thus $W_{T+1} = 0$ for all y_{T+1} At time period T, the agent solves $h_T\left(y_T, k_T\right)$ such that $k_T \in \Gamma_T\left(y_T\right)$, as $\frac{\partial h_t}{\partial k_t} < 0$ thus agent chooses $k_T = 0$. The equation (20) can also be used to do policy iteration using Coleman Operator as shown in.

2. With Government Contract

Objective

$$w_t(y_t) = \max_{x_t,k_t} \{f_t(y_t, x_t) + \beta w_{t+1}(y_{t+1})\} \tag{21}$$

$$\text{such that } v(y_t) = h(y_t, k_t) + \beta * v_{t+1}(y_{t+1}) \geq v_t^*(y_t) \tag{22}$$

$$y_{t+1} = G_t(y_t, x_t, k_t) \tag{23}$$

Lagrangian Equation

$$\mathcal{L} = w_t(y_t) = f_t(y_t, x_t) + \beta w_{t+1}(y_{t+1}) + \lambda_t \left[h_t(y_t, k_t) + \beta * v_{t+1}(y_{t+1}) - v_t^*(y_t)\right] \tag{24}$$

First Order Conditions

$$\begin{aligned} \tfrac{\partial \mathcal{L}}{\partial x_t} &= \tfrac{\partial f_t}{\partial k_t} + \beta * \tfrac{\partial w_{t+1}}{\partial y_{t+1}} * \tfrac{\partial y_{t+1}}{\partial x_t} + \lambda_t * \beta \tfrac{\partial v_{t+1}}{\partial y_{t+1}} * \tfrac{\partial y_{t+1}}{\partial x_t} = 0 \\ \tfrac{\partial \mathcal{L}}{\partial k_t} &= \beta * \tfrac{\partial w_{t+1}}{\partial y_{t+1}} * \tfrac{\partial y_{t+1}}{\partial k_t} + \lambda_t \left[\tfrac{\partial h_t}{\partial k_t} + \beta * \tfrac{\partial v_{t+1}}{\partial y_{t+1}} * \tfrac{\partial y_{t+1}}{\partial k_t}\right] = 0 \end{aligned} \tag{25}$$

Solutions

$$\text{For } \lambda_t, \frac{\partial w_{t+1}}{\partial y_{t+1}}, \beta \frac{\partial v\left(y_{t+1}\right)}{\partial y_{t+1}} \text{ and } \frac{\partial v_t^*}{\partial y_t} \tag{26}$$

$$\lambda_t = \frac{\frac{\partial f_t}{\partial x_t} * \frac{\partial y_{t+1}}{\partial k_t}}{\frac{\partial y_{t+1}}{\partial x_t} * \frac{\partial h_t}{\partial k_t}}; w'_{t+1}\left(y_{t+1}\right) = \frac{\partial w_{t+1}}{\partial y_{t+1}} = \frac{\partial f_{t+1}}{\partial y_{t+1}} - \frac{\frac{\partial y_{t+2}}{\partial y_{t+1}}}{\frac{\partial y_{t+2}}{\partial x_{t+1}}} + \frac{\frac{\partial f_{t+1}}{\partial x_{t+1}} * \frac{\partial y_{t+2}}{\partial k_{t+1}}}{\frac{\partial y_{t+1}}{\partial x_{t+1}}} \left[\frac{\partial h_{t+1}}{\partial y_{t+1}} - v'^*\left(y_{t+1}\right)\right] \tag{27}$$

$$\beta \frac{\partial v\left(y_{t+1}\right)}{\partial y_{t+1}} = \frac{1}{\frac{\partial y_{t+1}}{\partial y_t}} \left[\frac{\partial v_t^*\left(y_t\right)}{\partial y_t} - \frac{\partial h_t\left(y_t, k_t\right)}{\partial y_t}\right]; \frac{\partial v_t^*}{\partial y_t} = \frac{\partial h_t}{\partial y_t} - \frac{\partial h_t}{\partial k_t} * \frac{\frac{\partial y_{t+1}}{\partial y_t}}{\frac{\partial y_{t+1}}{\partial k_t}} \tag{28}$$

Finally the equation system (20) can be solved by substituting for $\frac{\partial w_{t+1}}{\partial y_{t+1}}$ from equation (27), $\beta \frac{\partial v_{t+1}}{\partial y_{t+1}}$ from equation (28) and finally for $\frac{\partial v_i^*}{\partial y_t}$ from equation (28).

5 Stochastic Dynamic Programming squared Model

Objective

$$\begin{array}{l} w\left(y_t\right)=\max_{k_t, x_t}\left\{f\left(y_t, x_t\right)+\beta \int w\left[G\left(y_t, x_t, k_t\right) z\right] \phi(d z)\right\} \\ \text{ such that } v\left(y_t\right)=h\left(y_t, k_t\right)+\beta \int v\left[G\left(y_t, x_t, k_t\right) * z\right] \phi(d z) \geq v^*\left(y_t\right) \\ \text{ where } v^*\left(y_t\right)=\max_{k_t}\left\{h\left(y_t, k_t\right)+\beta \int v^*\left[G\left(y_t, x_t=0, k_t\right) * z\right] \phi(d z)\right\} \end{array} \tag{29}$$

Lagrangian Equation

$$\begin{aligned} \mathcal{L}=w\left(y_t\right)=f\left(y_t, x_t\right)+\beta \int w\left[G\left(y_t, x_t, k_t\right) * z\right] \phi(d z) \\ +\lambda_t\left[h\left(y_t, k_t\right)+\beta \int v\left[G\left(y_t, x_t, k_t\right) * z\right] \phi(d z)-v^*\left(y_t\right)\right] \end{aligned} \tag{30}$$

First Order Conditions

$$\begin{array}{l} x: \frac{\partial f}{\partial x_t}+\beta \frac{\partial G}{\partial x_t} \int \frac{\partial w(y_{t+1})}{\partial y_{t+1}} z \phi(d z)+\lambda_t \beta \frac{\partial G}{\partial x_t} \int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z \phi(d z)=0 \\ k: \beta \frac{\partial G}{\partial k_t} \int \frac{\partial w(y_{t+1})}{\partial y_{t+1}} z \phi(d z)+\lambda_t\left[\frac{\partial h}{\partial k_t}+\beta \frac{\partial G}{\partial k_t} \int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z \phi(d z)\right]=0 \end{array} \tag{31}$$

** Solutions**

$$\begin{aligned} \lambda_t=\frac{\frac{\partial f}{\partial x_t} *}{\frac{\partial G}{\partial x_t}} * \frac{\partial G}{\partial k_t} \\ \frac{\partial G}{\partial x_t} * \frac{\partial h_t}{\partial k_t} \\ \frac{\partial w\left(y_{t+1}\right)}{\partial y_{t+1}}=\frac{\partial f}{\partial y_{t+1}}-\frac{\partial f_{t+1}}{\partial x_{t+1}}+\frac{\partial f_{t+1}}{\partial x_{t+1}} * \frac{\partial G}{\partial y_{t+1}}-\frac{\partial v^*}{\partial y_{t+1}} \end{aligned} \tag{32}$$

$$\int \frac{\partial v\left(y_{t+1}\right)}{\partial y_{t+1}} z \phi(d z)=\frac{1}{\beta * \frac{\partial G}{\partial y_t}}\left[\frac{\partial v^*}{\partial y_t}-\frac{\partial h}{\partial y_t}\right] \tag{33}$$

Thus equation set (32) can be solved by replacing values of λ_t, $\frac{\partial w(y_{t+1})}{\partial y_{t+1}}$ and $\int \frac{\partial v(y_{t+1})}{\partial y_{t+1}} z \phi(d z)$ from above equations.