Convergence and Divergence Test

#series #Calculus_3 #year2 #limits

Overview

In the previous section, we learned how to nd the sum of a convergent series by finding a formula for the nth partial sum of the series and taking the limit as $n \to \infty$. Unfortunately, most of the time, finding a formula for the nth term is not easy and usually require a tedious amount of computation. For practical reason, it is better to find another way of proving that a series is convergent, and then approximate the sum by its partial sum using as many terms as needed until a desired accuracy is achieved. The basis for this is the next theorem.

Theorem 2 (Convergence Test)**.**

If the series $\sum a_n$ *is convergent, then* $lim_{n\to\infty} a_n = 0$.

$$\sum a_n \text{ is convergent, } \implies \lim_{n\to\infty} a_n = 0$$

Theorem 3 (Divergence Test)**.**

If $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum a_n$ *is divergent.*

$$\lim_{n\to\infty} a_n \neq 0, \implies \sum a_n \text{ is divergent}$$

Example

Example 1.

$\sum_{n=1}^{\infty} \frac{n^2}{n^2-2n+5}$

Solution

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2 - 2n + 5} = \lim_{n \to \infty} \frac{1}{1 - \frac{2}{n} + \frac{5}{n^2}}$$
$$= \frac{1}{1 - 0 + 0}$$
$$= 1 \neq 0$$

Example 2.

$\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

Solution

we find the limit of a_n *as* $n \to \infty$ *as follow*

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} = \lim_{n \to \infty} \frac{3^n}{n^2} \to \frac{\infty}{\infty}$$
$$= \lim_{n \to \infty} \frac{d(3^x)}{d(x^2)} = \lim_{n \to > \infty} \frac{3^x \ln 3}{2x} \to \frac{\infty}{\infty}$$
$$= \lim_{n \to \infty} \frac{3^x (\ln 3)^2}{2} = \infty$$

since a_n *does not converge to 0, the series must be divergent.*

Example 11.

$$\sum_{n=1}^{\infty} \frac{n}{2n + 3}$$

Solution

Divergence test

$$= \lim_{n \to \infty} \frac{n}{2n + 3}$$

apply L'Hopital's Rule

$$= \lim_{n \to \infty} \frac{1}{2}$$

$$= \frac{1}{2} \neq 0 \therefore \text{divergent}$$

Example 12.

$$\sum_{n=1}^{\infty} 4(-1.05)^{n-1}$$

Solution

By divergence test

$$= \lim_{n\to\infty} 4(-1,05)^{n-1}$$

$$= 4(0)$$

by the Properties of Infinity $(n > 1)^{\infty} = \infty$

$$\infty \neq 0 \therefore \text{divergent}$$

Example 13.

$$\sum_{n=1}^{\infty} \frac{(n+1)}{5n!}$$

Solution

$$= \lim_{n\to\infty} \frac{n+1}{5n!} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{5\cancel{n}!}$$

expand n, Permutation

$$= \frac{1}{5} \lim_{n\to\infty} (n+1)$$

$$= \frac{1}{5}(\infty + 1)$$

$$\infty \therefore \text{divergent}$$

Example 14.

$$\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^2+1}}$$

by Divergence test

$$= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}} = \frac{\infty}{\infty}$$

Divide with n

$$= \lim_{n\to\infty} \frac{\frac{2n}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{1 + \frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{1 + \cancel{\frac{1}{n^2}}}}$$

$$\lim_{n\to\infty} \frac{2}{\sqrt{1}} = \frac{2}{1}$$

$$2 \neq 0 \therefore \text{divergent}$$

Proof

Proof. Convergence Test (Theorem 2)

$$\text{if } \sum a_n \text{ is convergent } \implies \lim_{n\to\infty} a_n = 0$$

Recall that

$$\boxed{\sum_{n-1}^{\infty} a_n = \lim_{n\to\infty} a_n = L}$$

$$\lim_{n\to\infty} s_n = L$$

$$\lim_{n\to\infty} s_{n+1} = L$$

$$s_n = s_{n-1} + a_n$$

It follows that

$$\lim_{l\to\infty} s_n = \lim_{n\to\infty} s_{n-1} + \lim_{n\to\infty} a_n$$

$$L - L + \lim_{n\to\infty} a_n$$

$$0 = \lim_{n\to\infty} a_n$$

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

□

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see also

Infinite Series