Limit Comparison Test

#Calculus_3 #series #year2 #limits

Overview

The Limit Comparison Test is a technique used to determine the convergence or divergence of an infinite series. It is often used when dealing with series that are difficult to directly evaluate or compare. The test is based on the idea that if the ratio of two series approaches a finite value as you take more and more terms, then both series have the same convergence behavior.

Theorem

Theorem 9 (Limit Comparison Test)**.**

Let $\sum a_n$ *and* $\sum b_n$ *be two series of positive forms*

$$\text{(i) if } \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0, \implies \text{ series either both converges or diverges}$$

$$\text{(ii) if } \lim_{n\to\infty} \frac{a_n}{b_n} = 0 \text{ and} \sum b_n \text{ converge}, \implies \sum a_n \text{ also converges}$$

$$\text{(iii) if } \lim_{n\to\infty} \frac{a_n}{b_n} = \infty \text{ and } \sum b_n \text{ diverges}, \implies \sum a_n \text{also diverges}$$

To use the limit comparison test, we initially guess the convergence or divergence of the given series and then find a know series to compare the given series with. Usually it does not matter which ever of the two series is put in the numerator or the denominator of the ratio $\frac{a_n}{b_n}$. For as long as the limit is a finite positive constant, the conclusion of the first part holds. But there are special case, which happens very rarely, that the quotient $\frac{a_n}{b_n}$ either tend to 0 or ∞ as $n \to \infty$. Never the less, when this happens, the first part of the test is rendered useless. In such cases, we use the 2nd or 3rd variation of the test, which ever is applicable if you guess that the series to be tested is convergent, put it at the numerator. Otherwise, if you guessed that the series to be tested is divergent, put it in the denominator.

Examples

Example 1.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1}+2}$$

Solution.

We initially guess that this series is divergent being similar to the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$*. We confirm this by taking* $a_n = \frac{1}{\sqrt{n}+2}$ *and* $b_n = \frac{1}{\sqrt{n}}$ *and finding the limit of* $\frac{a_n}{b_n}$ *as* $n \to \infty$*. We have*

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+2}}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+2}\left(\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{2}{\sqrt{n}}}$$

$$= \frac{1}{1+\cancel{\frac{2}{\sqrt{\infty}}}}$$

$$c = 1$$

Since $c > 0$ *and* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *is divergent, by the limit comparison test, the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2}$ *is also divergent.*

Example 3.

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$$

Solution.

We can disregard the insignificant terms

$$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

by p-series we know that the second series will converge
then do the limit comparison test

$$= \lim_{n\to\infty} \frac{\frac{2n^2-1}{3n^5+2n+1}}{\frac{1}{n^3}}$$
$$= \lim_{n\to\infty} \frac{2n^2-1}{3n^5+2n+1}\left(\frac{n^3}{1}\right)$$
$$= \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1}$$
$$= \lim_{n\to\infty} \frac{\frac{2n^5}{n^5}-\frac{n^3}{n^5}}{\frac{3n^5}{n^5}+\frac{2n}{n^5}+\frac{1}{n^5}}$$

cancel the terms, and notice that if we plugin ∞ most of the terms will cancel out, leaving us with

$$c = \frac{2}{3}$$

$c > 0$, then it is either convergent or divergent, since the second series is convergent, therefore $\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$ also converges.

Example 4.

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$$

Solution.

$$b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

by the p-series we know that b_n will converge, let see it n limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2+1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$$
$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}}$$

divide them by n, but since there is a square root in the denominator, we have to divide it by n^2 instead.

$$\frac{1}{1+\cancel{\frac{1}{\infty}}}$$
$$c = 1$$

$c > 0$, then it is either convergent or divergent, since b_n is convergent. Therefore, a_n is also convergent.

see also

Direct Comparison test