Cartesian Product

#set_theory #year2

Overview

The Cartesian product is a set formed from two or more given sets and contains all ordered pairs of elements such that the first element of the pair is from the first set and the second is from the second set, and so on.

Definition

Definition 1 (Cartesian Product of set)**.**

Given 2 sets A and B, the cartesian product of A and B is denoted as, $A \times B$ is set of all ordered pair

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

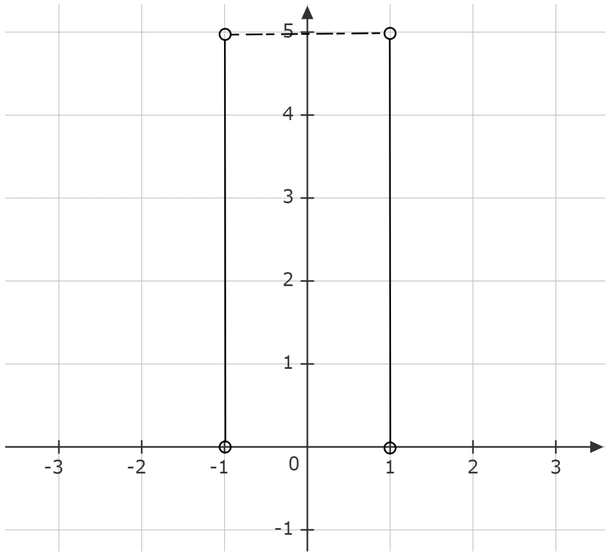
Example

1. $\mathbb{R} \times \mathbb{R} = \{(x, y) \in \mathbb{R}\} = \mathbb{R}^2$
   fun fact: $\mathbb{R}^2$ is also known as the cartesian plane
2. Let A = {1, 2, 3} and B = {a, b, c}
   $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

   > **Tip**
   > The cardinality of A and B is equal to the number of cartesian product

3. A = $\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$
   B = $\{y \in \mathbb{R} \mid 0 < y < 5\}$
   $A \times B = \{(x, y) - 1 \leq x \leq 1 \wedge 0 < y < 5\}$

   $A \times B$ will be looks like this



it will never touch 5 and 0 because there is no equality

Definition 2.

If $A_1, A_2, A_3, \ldots, A_n$ is a finite set the cartesian product of A_i id defined as

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{(a_1, a_2, a_3, \ldots, a_n) \mid a_i \in A, \forall i = 1, 2, 3, ..n\}$$

Example

$\mathbb{R}^3 = \{(x, y, z) \mid x, y, x \in \mathbb{R}\}$
this is also known as 3 dimensional coordinate system

Theorem

Theorem 3 (Properties of Cartesian Product).

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

Proof

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Premise	**Reason**
let $(x, y) \in A \times (B \cup C)$	assumption
$\iff x \in A \wedge y \in B \cup C$	definition of cartesian product
$\iff x \in A \wedge (y \in B \vee y \in C)$	definition of Union
$\iff (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$	distributive property
$\iff ((x, y) \in A \times B) \vee ((x, y) \in A \times C)$	definition of cartesian product
$\iff (x, y) \in (A \times B) \cup (A \times C)$	definition of Union
$\therefore A \times (B \cup C) = (A \times B) \cup (A \times C)$	set equality

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Premise	**Reason**
let $(x, y) \in A \times (B \cap C)$	assumption
$\iff x \in A \wedge y \in (B \cap C)$	definition of cartesian product
$\iff x \in A \wedge (y \in B \wedge y \in C)$	definition of intersection
$\iff (x \in A \wedge x \in A) \wedge (y \in B \wedge y \in C)$	Idempotent property
$\iff (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C)$	Associative
$\iff ((x, y) \in A \times B) \wedge ((x, y) \in A \times C)$	definition of cartesian product
$\iff (x, y) \in (A \times B) \cap (A \times C)$	definition of intersection
$\therefore A \times (B \cap C) = (A \times B) \cap (A \times C)$	set equality

$(iii)\ A \times (B - C) = (A \times B) - (A \times C)$

Premise	**Reason**
let $(x, y) \in A \times (B - C)$	assumption
$\iff x \in A \wedge y \in (B - C)$	definition of cartesian product
$\iff x \in A \wedge (y \in B \wedge y \notin C)$	definition of complement
$\iff (x \in A \wedge x \in A) \wedge (y \in B \wedge y \notin C)$	idempotent property
$\iff (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C)$	associative
$\iff ((x, y) \in A \times B) \wedge ((x, y) \notin A \times C)$	definition of $\times$ and contradiction
$\iff (x, y) \in (A \times B) - (A \times C)$	definition of complement
$\therefore A \times (B - C) = (A \times B) - (A \times C)$	set equality