Direct Comparison test

#DCT #series #Calculus_3 #year2

Overview

The direct comparison test is a technique used to determine the convergence or divergence of an infinite series. The test is based on the idea of comparing the given series to a known series whose behavior is already understood. If the know series converges or diverges, then the direct comparison test can be used to make conclusions about the original series.

Given the pair of series

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \text{ and } \sum_{n=1}^{\infty} \frac{n}{3^n}$$

They are pretty similar. The first is a geometric series with $r = \frac{1}{3}$. So we know right away that the series is convergent since $r < 1$. However, the same conclusion cannot be applied to the second because that is not a geometric series.

Theorem

Theorem 8 (The Comparison Test)**.**

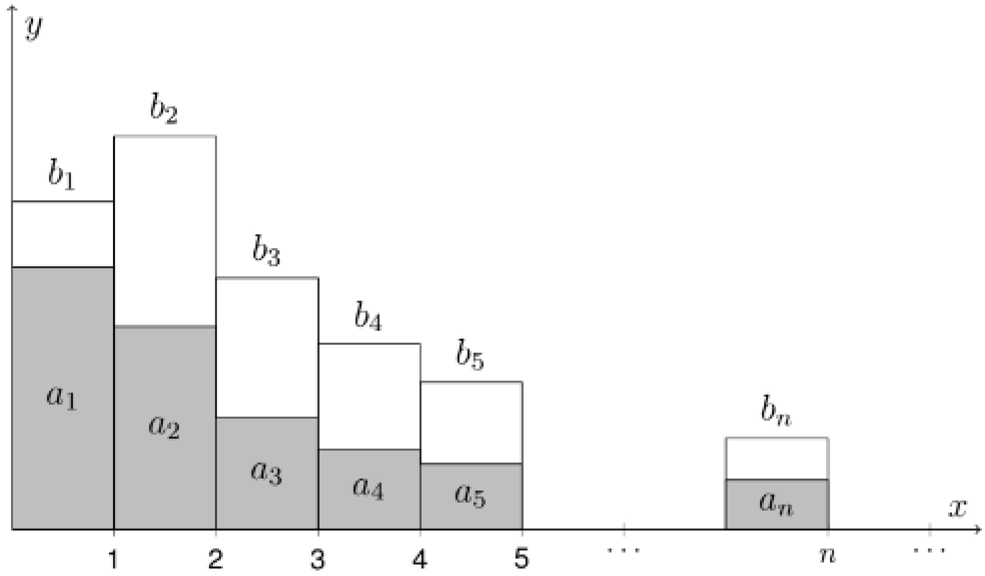
Suppose that $\sum a_n$ *and* $\sum b_n$ *are series with nonnegative terms such that* $0 < a_n \leq b_n, \forall n$.

$$\text{(i)} \quad \text{if } \sum b_n \text{ converges} \implies \sum a_n \text{ converges}$$

$$\text{(ii)} \quad \text{if } \sum a_n \text{ diverges} \implies \sum b_n \text{ diverges}$$

This test applies only to series whose terms are positive. One is always bigger and the other one is always smaller.

The theorem states that if the total area $\sum b_k$ of the |bigger| blank rectangles is finite (convergent), then the total area $\sum a_k$ of the |smaller| shaded rectangles is also finite. On the other hand, if the total area $\sum a_k$ of the shaded rectangles is infinite (divergent), then the total area $\sum b_k$ is also infinite.



Determining the convergence or divergence

Series	$\sum a_n$	**convergence**	**divergence**
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$
geometric series	$\sum_{n=1}^{\infty} r^n$	$r < 1$	$r \geq 1$
integrable function $f(x)$	$\sum_{n=1}^{\infty} f(n)$	$\int_1^{\infty} f(x)\,dx < \infty$	$\int_1^{\infty} f(x)\,dx = \infty$

Examples

Example 1.

$$\sum_{n=1}^{\infty} \frac{2}{1+3^n}$$

Solution.
As $n \to \infty$, *the constant term 1 in the denominator is negligible compared to the value of* 3^n, *so that the given series is comparable to the geometric series*
Infinite Series > Geometric Series (Theorem 1)

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = 2\sum_{n=1}^{\infty} \frac{1}{3^n}$$

which is convergent since $r = \frac{1}{3}$ *and* $|\frac{1}{3} < 1|$. *Our guess that the given series is also convergent. So we compare its terms to that of the convergent series. Recall that the bigger the denominator, the smaller the fraction. Hence*

$$\frac{2}{1+3^n} < \frac{2}{3^n} \implies \sum_{n=1}^{\infty} \frac{2}{1+3^n} + \frac{2}{3^n}$$

by the direct comparison test, $\sum_{n=1}^{\infty} \frac{2}{1+3^n}$ *is also convergent.*

Example 2.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-2}$$

Solution.
As $n \to \infty$, *we can disregard -2 in the denominator*
P-Series > p-Series (Theorem 6)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

notice that the p is equal to $\frac{1}{2}$, *by definition of p-series (p* ≤ 1*) is divergent. Therefore the series is divergent*

$$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{2}-2}$$

Divergent by direct comparison test.

Example 3.

$$\sum_{n=1}^{\infty} \frac{n^5 + 9n^4 + 18n^2 + 100}{4n^6 + 2n^3}$$

Solution.
the leading terms n^5 *and* $4n^6$ *dominate the rest of the terms in both numerator and*

denominator, thus we can disregard the other terms.

$$\sum_{n=1}^{\infty} \frac{n^5}{4n^6} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n}$$

by p-series this is divergent (p = 1).

$$\frac{1}{4n} < \frac{n^5 + 9n^4 + 18n^2 + 100}{4n^6 + 2n^3}$$

therefore, the series is divergent by direct comparison test.

Example 8.

$$\sum_{n-0}^{\infty} e^{-n^2}$$

Solution.

$$\sum_{n=0}^{\infty} e^{-n^2} = \sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{e^n} \implies \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$$

this form a geometric series, where $|r| = |\frac{1}{e}|$ *and* $\frac{1}{e} < 1$ *which is convergent*

$$\frac{1}{e^{n^2}} < \frac{1}{e^n}$$

this is convergent by direct comparison test.

Example 7.

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$$

Solution.

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\frac{\sin^2 n}{n^2} < \frac{1}{n^3}; n > 1$$

this forms a p-series, where $p = 3$*. therefore it is convergent by P-Series > p-Series (Theorem 6)*

Example 6.

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

Solution.

$$\sum_{n=1}^{\infty} \frac{1}{n!} \implies \sum_{n=1}^{\infty} \frac{1}{n^2}$$

if n = 1 both series will be equal to 1. from n = 1, 2, 3 you'll notice that the second series is bigger than the first series. But, when n = 4 above, the first series skyrocket its value faster than the second series. Don't forget that this is a fraction meaning the bigger denominator has the smaller value, Therefore the first series is less than the second series.

$$\frac{1}{n!} < \frac{1}{n^2}; n \geq 4$$

notice the form of second series is p-series, where p = 2 therefore the series is convergent at n=4

Example 4.

$$\sum_{n=1}^{\infty} \frac{6^n + n}{5^n - 1}$$

Solution.
Disregard the insignificant terms in the long run

$$\sum_{n=1}^{\infty} \frac{6^n}{5^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)\left(\frac{6}{5}\right)^{n-1}$$

now this form a geometric series. Infinite Series > Geometric Series (Theorem 1)

$$r = \frac{6}{5} > 1 \therefore \text{divergent}$$

$$\frac{6^n}{5^n} < \frac{6^n + n}{5^n + 1}$$

the smaller series diverges, ∴ the bigger series also diverges by the direct comparison test.

Remark

⚠ Warning

It is important to note taht in using the comparison test, the terms of the series being tested must either be smaller than those of a convergent series or largee than thos of a divergent series. If the terms of a series is larger than the terms of a convergent series, or likewise, if the terms are smaller than a divergent seris, then the direct comparison test in not applicable.

see also

Infinite Series
P-Series