Iterated Integral

Overview

An iterated integral is the result of applying integrals to a function of more than one variable is such a way that each of the integrals consider some of the variables as given constants.

Definition

Definition 2 (Iterated Integral)**.**

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

Important

The outside limits must be a constant

The inside limits can be whatever.

Examples

Example 1.

$$\int_0^{\frac{\pi}{4}} \int_0^1 y\cos x\,dy\,dx$$

Solution.

$$\int_0^{\frac{\pi}{4}} \left[\int_0^1 y\cos x\, dy\right] dx = \int_0^{\frac{\pi}{4}} \left[\cos x \int_0^1 y\, dy\right] dx \tag{1}$$

$$= \int_0^{\frac{\pi}{4}} \left[\cos x \left(\frac{y^2}{2}\right)\right]_0^1 dx \tag{2}$$

$$= \int_0^{\frac{\pi}{4}} \cos x \cdot \frac{1}{2}\, dx \tag{3}$$

$$= \frac{1}{2}[-\sin x]\Big|_0^{\frac{\pi}{4}} \tag{4}$$

$$= \frac{1}{2}\left[-\sin\frac{\pi}{4} - \cancel{\sin 0}\right] \tag{5}$$

$$= \frac{\sqrt{2}}{4} \tag{6}$$

Example 2.

$$\int_0^{\frac{\pi}{4}} \int_0^{\cos\theta} 3r^2 \sin\theta\, dr\, d\theta$$

Solution.

$$\int_0^{\frac{\pi}{4}} \left[\int_0^{\cos\theta} 3r^2 \sin\theta \, dr\right] d\theta = \int_0^{\frac{\pi}{4}} \left[\sin\theta \int_0^{\cos\theta} 3r^2 \, dr\right] d\theta \tag{1}$$

$$= \int_0^{\frac{\pi}{4}} \left[\sin\theta \cdot r^3\right]_0^{\cos\theta} d\theta \tag{2}$$

$$= \int_0^{\frac{\pi}{4}} \sin\theta(\cos^3\theta - 0)d\theta \tag{3}$$

$$= \int_0^{\frac{\pi}{4}} \sin\theta \cos^3\theta \, d\theta \tag{4}$$

$$= \int_0^{\frac{\pi}{4}} u^3 \, dx \tag{5}$$

$$= \left.\frac{u^4}{4}\right|_0^{\frac{\pi}{4}} \tag{6}$$

$$= \left.\frac{(\cos\theta)^4}{4}\right|_0^{\frac{\pi}{4}} \tag{7}$$

$$= \left(\frac{\left(\cos\frac{\pi}{4}\right)^4}{4}\right) - \left(\frac{(\cos 0)^4}{4}\right) \tag{8}$$

$$= \frac{\left(\frac{\sqrt{2}}{2}\right)^4}{4} + \frac{1}{4} \tag{9}$$

$$= \frac{3}{16} \tag{10}$$

apply U-substitution at (5).

Example 3.

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \frac{2}{4-y^2} \, dx \, dy$$

Solution.

$$\int_0^2 \left[\int_0^{\sqrt{4-y^2}} \frac{2}{4-y^2}\,dx\right] dy \tag{1}$$

$$= \int_0^2 \left[\left.\frac{2x}{4-y^2}\right|_0^{\sqrt{4-y^2}}\right] dy \tag{2}$$

$$= \int_0^2 \frac{2\sqrt{4-y^2}}{4-y^2}\,dy \tag{3}$$

$$= \int_0^2 \frac{2}{\sqrt{4-y^2}}\,dy \tag{4}$$

$$= 2\left(\arcsin\frac{2}{2} - \arcsin 0\right) \tag{5}$$

$$= 2(\arcsin 1) \tag{6}$$

$$= \cancel{2} \cdot \frac{\pi}{\cancel{2}} \tag{7}$$

$$= \pi \tag{8}$$

Example 4.

$$\int_{-2}^{2}\int_{-1}^{1} y^2 - x^2\,dx\,dy$$

Solution.

$$\int_{-2}^{2}\left[\int_{-1}^{1} y^2 - x^2\,dx\right]dy = \int_{-2}^{2}\left[xy^2 - \frac{x^3}{3}\Bigg|_{-1}^{1}\right]dy \tag{1}$$

$$= \int_{-2}^{2}\left[y^2 - \frac{1}{3} - \left(-y^2 - \frac{1}{3}\right)\right]dy \tag{2}$$

$$= \int_{-2}^{2} 2y^2 - \frac{2}{3}\,dy \tag{3}$$

$$= \frac{2y^3}{3} - \frac{2}{3}y\Bigg|_{-2}^{2} \tag{4}$$

$$= \left(\frac{2(2)^3}{3} - \frac{2}{3}(2)\right) - \left(\frac{2(-2)^3}{3} - \frac{2}{3}(-2)\right) \tag{5}$$

$$= 8 \tag{6}$$