Paper

1 Introduction

Insulated tumblers have gained widespread popularity among consumers who want to maintain the temperature of their beverages over extend period of time. The popularity of insulated tumblers is influenced by various factors such as product performance, marketing strategy, branding, and alignment with current consumer demands for sustainability and aesthetics. A study by Ramirez2024 that utilized a Marketing Mix, stated that promotion is the most influential factor on purchasing behavior of insulated water flask. The second most influential factor is the product, this because it is positioned as a desirable, cost-saving, eco-friendly, portable, and aesthetically appealing alternative to conventional bottles. Lastly, price is the least influential suggesting that consumers are willing to invest in insulated water flasks.

While marketing and consumer perception play a crucial role in the popularity of insulated tumblers, their success ultimately depends on actual product performance. According to Anyanwu2022, insulated water flask or commonly known as a vacuum flask or thermal flask, is designed to keep beverages hot or cold for extend periods. The core mechanism involves creating a near-vacuum in the gap between two walls(vessels). This partial vacuum serves as an insulating barrier, significantly reducing heat transmission by convection and conduction. Additionally, vacuum flask often incorporate a silver covering on the inner bottle to prevent heat transfer via radiation.

One of popular brand of insulated tumbler is AquaFlask, as mentioned by Ramirez2024. AquaFlask claims that their product ensures iced beverages remain cold for up to 24 hours, while hot drinks stay warm for up to 12 hours. However, no empirical evidence has been publicly provided support their claim. Additionally, there is little to none research available that support the thermal retention performance of AquaFlask products. Thus, this research aims to verify or refute their claim about heat retention through experimental measurement and mathematical modeling based on Newton's Law of Cooling.

Newton's Law of Cooling first appeared in a verbal definition by Sir Isaac Newton in 1701 in a short article titled *'Scala Graduum Caloris'*, his statement was that the rate of loss of heat of a hot object is directly proportional to the difference between the object's temperature and the overall temperature of the system or, in other words, the ambient room temperature (Jain2024). According to Wintertoon2010, Newton did not write his law down in the form of an equation, the equation only becomes know as Newton's Law when the further assumption is made that the heat transfer coefficient is constant. Mathematically, Newton's law can be expressed as a differential equation

$$\frac{dT}{dt} = -k(T - T_{amb})$$

where T is the temperature of the object, T_{amb} is the surrounding temperature, and k is the cooling constant. The solution of this differential equation yields an exponential decay function the models how temperature approaches equilibrium over time. However, Newton's model assume that convection is the only heat transfer mechanism and didn't account for conduction and radiation (Jain2024). Fortunately, Awyanwu2022 stated that the engineering of insulated water flask minimize the heat transfer of convection, conduction, and radiation. Thus, insulated water flask satisfies Newton's law of cooling assumption and taking conduction as the heat transfer mechanism.

The solution to Newton's law of cooling shows the relationship between the temperature of an object and its surrounding, i.e., the temperature of an object will eventually approach the temperature of its surrounding over time. This long-term behavior is tied to the concept of limit, meaning the temperature $T(t)$ converges to the ambient temperature T_{amb} (Sardjito2021). This convergence demonstrate the fundamental principles of real analysis, particularly the study of continuity, limits, and exponential decay.

Therefore, this study seeks to apply Newton's Law of Cooling to model the temperature variation of the liquid contained in an AquaFlask and to analyze its behavior through the lens of real analysis. By examining the function that describes temperature overtime, the study aims to explore analytical concepts such as limit of a sequence and convergence. In doing so, it not only evaluates the validity of AquaFlask's thermal retention claims but also demonstrate how abstract mathematical principles can effectively describe observable physical behavior.

2 Mathematical Formulation and Problem Modeling

In this section, the mathematical concepts required for the objective are defined. This includes Newton's Law of Cooling as the main model for modeling the temperature of the liquid contained in an AquaFlask tumbler, which then be used for verifying their 12 hours thermal retention claim. Real analysis concepts such as limits of a sequence along with convergence are discussed in this section.

2.1 Newton Law of Cooling

According to Jain2024, the rate of temperature change of an object is directly proportional to the instantaneous temperature difference between the object and its surrounding environment, provided that this temperature difference is not overly large. Mathematically it is expressed as

$$\frac{dT}{dt} = -k(T - T_{amb}) \qquad \text{(eq 1)}$$

Where T is the temperature of the object, T_{amb} is the ambient or the surrounding temperature, k is the proportionality constant, the negative coefficient is for cooling process, and $\frac{dT}{dt}$ is the rate of change of the object's temperature with respect to time.

Solving $(\text{eq } 1)$ involves separating of variables, in this case, isolate T and t in order to integrate both sides.

$$\frac{dT}{(T - T_{amb})} = -k\,dt$$
$$\int \frac{dT}{T - T_{amb}} = \int -k\,dt$$

This gives:

$$\ln|T - T_{amb}| = -kt + C \qquad \text{(eq 2)}$$

where C is the constant of integration. Exponentiating both sides of $(\text{eq } 2)$ gives:

$$T - T_{amb} = e^{-kt+C} = e^{C}e^{-kt}$$

Let $e^{C} = (T_0 - T_{amb})$, where T_0 is the initial temperature at $t = 0$

Then:

$$T(t) - T_{amb} = (T_0 - T_{amb})e^{-kt}$$
$$T(t) = T_{amb} + (T_0 - T_{amb})e^{-kt} \qquad \text{(eq 3)}$$

The solution is given by $(\text{eq } 3)$, where $T(t)$ is the temperature of the object at time t, T_{amb} is the ambient temperature, T_0 is the initial temperature of the object, k is the cooling constant that characterized how quickly the object approaches the ambient temperature. To find the value of k, starting from equation 3, rearrange and take the logarithms:

$$T(t) = T_{amb} + (T_0 - T_{amb})e^{-kt}$$
$$\frac{T(t) - T_{amb}}{T_0 - T_{amb}} = e^{-kt}$$
$$\ln\left(\frac{T(t) - T_{amb}}{T_0 - T_{amb}}\right) = -kt$$

Given two measurements at times t_1 and t_2 with temperature $T(t_1)$ and $T(t_2)$, solve for k:

$$k = \frac{-1}{t_2 - t_1}\ln\left(\frac{T(t_2) - T_{amb}}{T(t_1) - T_{amb}}\right)$$

A larger k value indicates faster cooling, while smaller k indicates slower cooling. In this study, Equation 3 serves as the theoretical model for predicting the temperature variation of water inside the AquaFlask tumbler. And the parameters T_0, , and k were determined from the experiment.

2.2 Real Analysis Framing

Newton's Law of Cooling does not only describe a physical process, it also exhibits mathematical behavior that aligns closely with foundational concepts in real analysis, particularly sequences, limits, and convergence.

The foundational nature of Newton's Law of cooling describes the temperature $T(t)$ of an object approaching the ambient temperature T_{amb} as time t tends to infinity (Sardjito2021). This is can be expressed as

$$\lim_{t \to \infty} T(t) = T_{amb}$$

which is the continuous analog of the limit of a sequence. If one were to observe the temperature at discrete, order time intervals $t_1,\ t_2,\ t_3, \ldots t_n$, the resulting temperature measurements $\{T(t_n)\}$ would form a sequence of real numbers $\{a_n\}$ (Bloch2011).

The sequence $\{T(t_n)\}$ represents the temperature values recorded at successive time intervals. As time progresses, each term in the sequence approaches a common value, i.e., the ambient temperature T_{amb}. This behavior demonstrate the concept of convergence, where the terms of a sequence get arbitrarily close to a specific real number as $n \to \infty$. The convergence of $\{T(t_n)\}$ can be rigorously analyzed through two essential properties in real analysis, i.e., boundedness and monotonicity. For a cooling process the sequence of temperatures is bounded, since each measured temperatures lies between the initial temperature T_0 and the ambient temperature T_{amb}. That is,

$$T_{amb} \leq T(t_n) \leq T_0 \quad \forall n \in \mathbb{N}$$

Furthermore, the sequence is monotonically decreasing, because as time increases, the exponential term $e^{-kt_n} \to 0$, which leads to progressively smaller temperature difference from T_{amb}. Conversely, in a heating process where $T_0 < T_{amb}$, the sequence would be monotonically increasing.

A sequence that is both bounded and monotonic satisfies the Monotone Convergence Theorem. The fundamental statement is that a sequence of real numbers that is monotonic is convergent if and only if it is bounded (Bartle2011), This is often paraphrased simply as: Every bounded, monotonic sequence is convergent (Nicolaescu2025). Since the sequence $\{T(t_n)\}$ is both bounded and monotonic, Monotone Convergence Theorem guarantees that it must converge to a real limit. This limit corresponds physically to the final equilibrium temperature, i.e., T_{amb}. Thus, the cooling process governed by Newton's Law can be viewed not only as a physical phenomenon but also as a concrete realization of a convergent sequence in real analysis.

To formalize the convergence of $T(t) \to T_{amb}$, the ε-N definition of limit is used to prove it analytically. It is stated as follows, a sequence a_n converges to L if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, the inequality $|a_n - L| < \varepsilon$ holds (Bloch2011). In this study it is defined as follows, $|T(t) - T_{amb}| < \varepsilon$.

Proof.

Recall from (eq 3) that

$$T(t) = T_{amb} + (T_0 - T_{amb})e^{-kt}$$

where T_0, T_{amb}, and $k > 0$ are constants. The goal is to show that

$$\lim_{t \to \infty} T(t) = T_{amb}$$

Let $\varepsilon > 0$ be given. Find $N > 0$ such that $\forall t > N$, $|T(t) - T_{amb}| < \varepsilon$.

For a cooling process, where $T_0 > T_{amb}$, it follows that $T(t) > T_{amb}$ for all $t > 0$. Hence,

$$|T(t) - T_{amb}| = T(t) - T_{amb}$$

Starting with the definition of $T(t)$,

$$\begin{aligned} T(t) - T_{amb} &= T_{amb} + (T_0 - T_{amb})e^{-kt} - T_{amb} < \varepsilon \\ &= (T_0 - T_{amb})e^{-kt} < \varepsilon \end{aligned}$$

Since $T_0 - T_{amb}$ is constant, dividing both side by $T_0 - T_{amb}$ gives

$$e^{-kt} < \frac{\varepsilon}{T_0 - T_{amb}}$$

Taking the natural logarithm of both sides

$$-kt < \ln\left(\frac{\varepsilon}{T_0 - T_{amb}}\right)$$

Multiplying both side by -1 flips the inequality:

$$\begin{aligned} kt > -\ln\left(\frac{\varepsilon}{T_0 - T_{amb}}\right) &\implies kt > \ln\left(\frac{T_0 - T_{amb}}{\varepsilon}\right) \\ &\implies t > \frac{1}{k}\ln\left(\frac{T_0 - T_{amb}}{\varepsilon}\right) \end{aligned}$$

Notice that it satisfies $t > N$, it follows:

$$N = \frac{1}{k}\ln\left(\frac{T_0 - T_{amb}}{\varepsilon}\right)$$

Then for all $t > N$, the inequality $|T(t) - T_{amb}| < \varepsilon$ holds true.
Therefore,

$$\lim_{t\to\infty} T(t) = T_{amb}$$

This confirms that the temperature function $T(t)$ converges to the ambient temperature, satisfying the ε-N definition of limit.

The proof confirms that analytically a physical model predicts that the temperature of the liquid inside the insulated tumbler will eventually approach the ambient temperature over time.

3 Methodology

Having established the theoretical and analytical foundation of Newton's Law of Cooling and its connection to real analysis, the next step is to validate the model through empirical observation. This section outlines the experimental procedures and computational methods employed to measure the temperature variation of liquid contained in an AquaFlask tumbler over time. By comparing the observed data to the predicted behavior derived from the mathematical model, this study aims to determine the accuracy of AquaFlask's thermal retention claims and assess how well the cooling process conforms to Newton's law.

3.1 Experimental Material and Equipment

The following materials were used to get the necessary data for the study: 22-ounce AquaFlask Tumbler, kitchen thermometer, timer, thermometer for ambient temperature, and notebook for logging the time, internal temperature of AquaFlask, and ambient temperature. These materials were chosen in order to get the required variable for Newton's Law of Cooling model while minimizing the cost due to budget constraints.

3.2 Experimental Setup and Data Collection

The set up for the experiment happens it a small room with an ambient temperature averaging 29 degrees Celsius. The AquaFlask was filled with water around 95 percent of it total volume with initial temperature of 97 degrees Celsius. To measure the internal temperature, the kitchen thermometer was placed on the lid, while minimizing lid opening and heat loss. A total 12 readings was recorded with 15-minute intervals from 0 to 2.75 hours. This interval was chosen to capture sufficient cooling behavior while maintaining manageable measurement frequency. For each readings, the time, internal temperature, and ambient temperature was logged in the notebook. After that, the logs from the notebook was transferred to a data sheet or a CSV file for model fitting.

3.3 Model Fitting

The model fitting was done with Python programming language along side with Jupyter notebook and the following libraries: `pandas`, `matplotlib`, `numpy`, and `scipy` . The CSV file was loaded in Jupyter notebook for processing. A function for Newton's Law of Cooling was created, taking the ambient temperature T_{amb}, initial internal temperature T_0, and the cooling constant k. Nonlinear least squares was used to find the optimal value of k based on the given data points, this was done using the `curve_fit` function from `scipy` library. This approach minimize the squared error between observed and modeled temperature as oppose to taking only 2 temperature reading for finding k.

3.4 Model Validation and Extrapolation

After determining the optimal value of k, a 12-13 hours prediction from the fitted model was generated for later comparison. A late observation at 13th-hour mark was obtained and used for comparing the model's 13-hour prediction after initial measurement. The model was then refitted with the late 13th-hour data point to assess the stability of k and prediction error.

4 Analysis and Results

This section presents the results of the experimental measurements, model fitting, and validation of cooling behavior of water in an AquaFlask tumbler. The fitted model is based on Newton's Law of Cooling, and the cooling rate were determined using nonlinear least-squares.

4.1 Model Fitting and Parameter Estimation

Table of Observations

The experimental data collected over a period of 2.75 hours were fitted to the Newton's Law of Cooling model:

$$T(t) = T_{amb} + (T_0 - T_{amb})e^{-kt}$$

A nonlinear least-squares method was used to find the optimal value of the cooling constant k through `curve_fit` function in Python's SciPy library. The estimated optimal value of k was

$$k = 0.123\ /\mathrm{hr}$$

The mean ambient temperature during the experiment was $T_{amb} = 29$ degrees Celsius, and the initial temperature of the liquid was $T_0 = 97.3$ degrees Celsius. The fitted model closely matched the observed temperature readings, with residuals between -0.59 and +0.58 degrees Celsius, suggesting the theoretical model captures the experimental data very well.

Show table with residuals

4.2 Long-Term Prediction and Model Validation

After parameter estimation, the model was extrapolated to predict the temperature behavior beyond the observed period. The 12-hour temperature prediction gives:

$$T(12) = 44.64^{\circ}\mathrm{C}$$

The 12 hours prediction of the model shows a 45.84 percent decrease to the temperature from the initial temperature of 97.3 degrees Celsius.

5 Conclusion and Reflection