Some important keys to make calculation easy

1. To Evaluate the given determinant use the properties of determinant which are applicable for that determinant.
2. To verify LHS equal to RHS move according to RHS.
3. Factor theorem is an important/easy method to evaluate those determinants whose answers can already be guessed. For example:

   Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

   Here, if we put $a = b$, $b = c$ & $c = a$ in Δ then Δ becomes 0. In this case $(a - b)$, $(b - c)$ & $(c - a)$ are the factors of the answer of Δ. Since the degree of the product of the diagonal elements is 3 the of number of factors can not be more than three but some non zero constant may occur. Let k be that costant. Therefore we can write

   $\Delta = k(a-b)(b-c)(c-a)$

   To find k, which is independent of a, b, c, put any simple values of a, b, c in such a way that Δ and RHS do not become zero.

4. Cramer's rule $x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{\Delta_2}{\Delta}$, $z = \dfrac{\Delta_3}{\Delta}$

   (i) If $\Delta \neq 0$, the system is consistent and has unique Solution:
   (ii) If $\Delta = 0$ and no $\Delta_1, \Delta_2, \Delta_3$ is equal to zero then the system is inconsistent and has no Solution:
   (iii) If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$ then the system is consistent and has infinite number of solutions. In this case we can not use cramor's rule.

For Homogeneous system

5. The Homogeneous system $\Delta X = 0$ possess only zero or trivial solution if $|A| \neq = 0$
   and possess infinite number of solution or non-trivial solution if $|A| = 0$.
6. Rank can be determined can the basis of our text books) by the following methods.
   1. Determining non zero minor of highest order of the determent of the given matrix.
   2. The order of the highest order non zero minor is the rank of the matrix.

3. Reducing the given matrix into an Echelon
non zero rows in that form is the rank of the matrix.

4. Reducing the given matrix into an upper triangular matrix. The no. of non zero rows in that triangular matrix gives the rank of the matrix.

5. Rank determining the linearly independent rows or columns in the matrix. The no of linearly independent rows or columns gives the rank of the matrix.

6. Reducing into cannomical from:
$[I_r]$ or $[I_r : 0]$ or $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ or $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, r denotes the order of the identify matrix this order gives the rank of the given matrix.

7. The system of linear equations,
$AX = b$
is said to be consistent if $\rho(A) = \rho\,[A : b]$, ρ – rank and inconsistent if $\rho(A) \neq \rho\,[A : b]$
For inconsistent system solution does not exist. For consistent system solution exists and unique Solution:
$\rho(A) = = \rho\,[A : b] =$ no of unknowns/variables
If $\rho\,(A) = \rho\,[A : b] <$ no of unknown / variables
the constant system has infinite solutions.

8. Inverse method : $X = A^{-1}b = \dfrac{Ad_j(A)}{|A|}\,b$, $|A| \neq 0$.

9. To calculate eigen values λ
Solve the characteristic equation of the given matrix
A, $|A - \lambda I| = 0$, for λ
λ are the eigenvalues of A.
To check answer add all $\lambda$and match with the sum of the diagonal elements of A. If both are equal you are correct.

10. To calculate eigen vectors make the matrix $[A - \lambda I]$
Take some $X \neq 0$ and make $[A - \lambda I]\,X = 0$
Putting different λ calculated before find different corresponding $X \neq 0$, there are the eigenvectors of A.
Note : To calculate eigen vector X use trial or cross multiplication method.

11. Diagonalization of a matrix A
Step- 1 Find eigen values of A
Step - 2 Find eigenvectors of A
Step - 3 make a matrix C of the eigenvectors, the matrix is called modal matrix
Step - 4 Calculate C^{-1}
Step - 5 Calculate C^{-1} AC this is the diagonal matrix with diagonal elements the eigen values of A.

Exercise - 1

Find the value of the following determinants.

1. (i) $\begin{vmatrix} 1^2 & 2^2 & 3^3 & 4^2 \\ 2^2 & 3^2 & 4^2 & 1^2 \\ 3^2 & 4^2 & 1^2 & 2^2 \\ 4^2 & 1^2 & 2^2 & 3^2 \end{vmatrix}$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1^2 & 2^2 & 3^3 & 4^2 \\ 2^2 & 3^2 & 4^2 & 1^2 \\ 3^2 & 4^2 & 1^2 & 2^2 \\ 4^2 & 1^2 & 2^2 & 3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 1 \\ 9 & 16 & 1 & 4 \\ 16 & 1 & 4 & 9 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \begin{vmatrix} 30 & 4 & 9 & 16 \\ 30 & 9 & 16 & 1 \\ 30 & 16 & 1 & 4 \\ 30 & 1 & 4 & 9 \end{vmatrix}$$

Taking 30 common form C_1

$$= 30 \begin{vmatrix} 1 & 4 & 9 & 16 \\ 1 & 9 & 16 & 1 \\ 1 & 16 & 1 & 4 \\ 1 & 1 & 4 & 9 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - R_1$ respectively. We get

$$= 30 \begin{vmatrix} 1 & 4 & 9 & 16 \\ 0 & 4 & 7 & -15 \\ 0 & 12 & -8 & -12 \\ 0 & -3 & -5 & -7 \end{vmatrix}$$

Expanding along C_1

$$= 30 \begin{vmatrix} 5 & 7 & -15 \\ 12 & -8 & -12 \\ -3 & -5 & -7 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$

$$= 30 \begin{vmatrix} -10 & 7 & -15 \\ 0 & -8 & -12 \\ -10 & -5 & -7 \end{vmatrix}$$

Taking 10 common from C_1

$$= 300 \begin{vmatrix} -1 & 7 & -15 \\ 0 & -8 & -12 \\ -1 & -5 & -7 \end{vmatrix}$$

= Applying $R_3 \to R_3 - R_1$

$$= 300 \begin{vmatrix} -1 & 7 & -15 \\ 0 & -8 & -12 \\ 0 & -12 & 8 \end{vmatrix}$$

Expanding along C_1

$$= 300(-1) \begin{vmatrix} -8 & -12 \\ -12 & -8 \end{vmatrix}$$

$$= 300(-1)(-64 - 144)$$

$$= 300 \times 208$$

$\Delta = 62400$ Ans.

(ii) $\begin{vmatrix} 1^2 & 2^2 & 3^3 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1^2 & 2^2 & 3^3 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_3$ respectively. We get

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_3$ respectively. We get

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{vmatrix}$$

Since $R_3 = R_4$

= 0 Ans.

2. (i) $\begin{vmatrix} a & b & c & 0 \\ a & b & 0 & d \\ a & 0 & c & d \\ 0 & b & c & d \end{vmatrix}$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} a & b & c & 0 \\ a & b & 0 & d \\ a & 0 & c & d \\ 0 & b & c & d \end{vmatrix}$$

Taking common a from C_1 and b from C_2, c from C_3 and d from C_4

$$= abcd \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ resp. We get

$$= abcd \begin{vmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

Expanding along C_1

$$= abcd \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$ we get

$$= abcd \begin{vmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding along C_1

$$= abcd \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= abcd(-2 - 1)$$

$\Delta = -3abcd$ Ans.

(ii) $\begin{vmatrix} 5 & 7 & 10 & 14 \\ 2 & 3 & 7 & 6 \\ 3 & 3 & 6 & 9 \\ 5 & 6 & 11 & 20 \end{vmatrix}$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 5 & 7 & 10 & 14 \\ 2 & 3 & 7 & 6 \\ 3 & 3 & 6 & 9 \\ 5 & 6 & 11 & 20 \end{vmatrix}$$

By interchanging R_1 and R_3

$$= -\begin{vmatrix} 3 & 3 & 6 & 9 \\ 2 & 3 & 7 & 6 \\ 5 & 7 & 10 & 14 \\ 5 & 6 & 11 & 20 \end{vmatrix}$$

Appling $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$ & $C_4 \to C_4 - 3C_1$ respectively. We get

$$= -\begin{vmatrix} 3 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 5 & 2 & 0 & -1 \\ 5 & 1 & 1 & 5 \end{vmatrix}$$

Expanding along R_1

$$= -3\begin{vmatrix} 1 & 3 & 0 \\ 2 & 0 & -1 \\ 1 & 1 & 5 \end{vmatrix}$$

Applying $C_2 \to C_2 - 3C_1$ we get,

$$= -3\begin{vmatrix} 1 & 0 & 0 \\ 2 & -6 & -1 \\ 1 & -2 & 5 \end{vmatrix}$$

Expanding along R_1

$$= -3\begin{vmatrix} -6 & -1 \\ -2 & 5 \end{vmatrix}$$

$$= -3(-30-2)$$

$$= -3 \times -32$$

$$= 96 \text{ Ans.}$$

3. (i) $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

Solution:

Let $\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$

Applying $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 3R_2$ we get,

$$= \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & -8 & 2 \end{vmatrix}$$

Expanding along C_1,

$$= -1\begin{vmatrix} 1 & 2 & 3 \\ 3 & -6 & 1 \\ 0 & -8 & 2 \end{vmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$= -1\begin{vmatrix} 1 & 2 & 3 \\ 0 & -12 & -8 \\ 0 & -8 & 2 \end{vmatrix}$$

Expanding along C_1,

$$= -1\begin{vmatrix} -12 & -8 \\ -8 & 2 \end{vmatrix} = (-1)(-24-64) = 88 \text{ Ans.}$$

(ii) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$

Solution: Let $\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$

Applying, $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ & $R_3 \to R_3 - R_4$ respectively. We get

$$= \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 2 & 3 & 5 \end{vmatrix}$$

Expanding along C_1

$$= -1\begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

Expanding along R_1,

$$= (-1)(-1)(1-0) = 1 \text{ Ans.}$$

4. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$

Solution:

Let $\Delta = \begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$

Applying $C_1 \to C_1 - 3C_3$, $C_2 \to C_2 - 2C_3$, $C_4 \to C_4 - 4C_3$

$$= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 9 & 25 & 2 & 6 \\ 7 & 13 & 3 & 5 \\ 9 & 23 & 8 & 6 \end{vmatrix}$$

Expanding along R_1,

$$= \begin{vmatrix} 9 & 25 & 6 \\ 7 & 13 & 5 \\ 9 & 23 & 6 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$

$$= \begin{vmatrix} 0 & 2 & 0 \\ 7 & 13 & 5 \\ 9 & 23 & 6 \end{vmatrix}$$

Expanding along R_1,

$$= -2 \begin{vmatrix} 7 & 5 \\ 9 & 6 \end{vmatrix}$$

$= -2\,(42 - 45)$

$= -2\,(-3)$

$= 6$

$\therefore \Delta = 6$ Ans.

5. $\begin{vmatrix} 1 & a & b & c \\ 1 & b & c & a \\ 1 & c & a & b \\ 1 & d & d & d \end{vmatrix}$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1 & a & b & c \\ 1 & b & c & a \\ 1 & c & a & b \\ 1 & d & d & d \end{vmatrix}$$

Applying, $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ & $R_3 \to R_3 - R_1$ respectively. We get

$$= \begin{vmatrix} 0 & a-b & b-c & c-a \\ 0 & b-c & c-a & a-b \\ 0 & c-a & a-b & b-c \\ 1 & d & d & d \end{vmatrix}$$

Expanding along C_1

$$= \begin{vmatrix} a-b & b-c & c-d \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ we get,

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & d-b & b-c \end{vmatrix}$$

$= 0$ [$\because$ Column one is zero]

$\therefore \Delta = 0$ Ans.

6. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$ **where ω be cube roots of unity.**

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ we get,

$$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \qquad \because 1 + \omega + \omega^2 = 0$$

$\therefore \Delta = 0$ Ans.

7. **Prove that following.**

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2ab\,(a+b+c)^3$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ resp. we get,

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

= Taking $(a + b + c)$ common from C_1 and C_2

$$= (a+b+c)\,(a+b+c) \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2 - R_1$ we get,

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

Applying $C_1 \to C_1 + \frac{1}{a} C_3$ and $C_2 \to C_2 + \frac{1}{b} C_3$ resp. we get,

$$= (a+b+c)^2 \begin{vmatrix} (b+c) & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

Expanding along R_1

$$= (a+b+c)^2\, 2ab \left[(b+c)(c+a) - \frac{b^2}{a}\cdot\frac{a^2}{b}\right]$$

$$= 2ab\,(a+b+c)^2\,(bc+ab+c^2+ac-ab)$$

$$= 2ab\,(a+b+c)^2\,(bc+c^2+ac)$$

$$= 2ab\,(a+b+c)^2\,(a+b+c)$$

$$= 2abc\,(a+b+c)^3 \text{ Ans.}$$

$$\Delta = 2abc\,(a+b+c)^3$$

8. $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ac \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2)$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ac \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

Applying $C_2 \to C_2 - 2C_1 - 2C_3$ we get,

$$= \begin{vmatrix} a^2 & -a^2-b^2-c^2 & bc \\ b^2 & -a^2-b^2-c^2 & ac \\ c^2 & -a^2-b^2-c^2 & ab \end{vmatrix}$$

Taking common $-(a^2+b^2+c^2)$ from C_2

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_1$

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2-b^2 & 0 & c(b-a) \\ b^2 & 1 & ca \\ c^2-a^2 & 0 & b(a-c) \end{vmatrix}$$

Taking common $(a-b)$ from R_1 and $(c-a)$ from R_3

$$= -(a^2+b^2+c^2)(a-b)(c-a)\begin{vmatrix} a+b & 0 & -c \\ b^2 & 1 & ca \\ c+a & 0 & -b \end{vmatrix}$$

Expanding along C_2

$$= -(a^2+b^2+c^2)(a-b)(c-a)\begin{vmatrix} a+b & -c \\ c+a & -b \end{vmatrix}$$

$$= -(a^2+b^2+c^2)(a-b)(c-a)(-ab-b^2+c^2+ac)$$

$$= -(a^2+b^2+c^2)(a-b)(c-a)(c-b)(a+b+c)$$

$$= (a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a).$$

$$\therefore \Delta = (b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2) \text{ Ans.}$$

9. $\begin{vmatrix} a+1 & 1 & 1 & 1 \\ 1 & a+1 & 1 & 1 \\ 1 & 1 & a+1 & 1 \\ 1 & 1 & 1 & a+1 \end{vmatrix} = a^3(a+4)$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} a+1 & 1 & 1 & 1 \\ 1 & a+1 & 1 & 1 \\ 1 & 1 & a+1 & 1 \\ 1 & 1 & 1 & a+1 \end{vmatrix}$$

Taking a common from R_1, R_2, R_3 and R_4 we get

$$= a^4\begin{vmatrix} \frac{a+1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{a} & \frac{a+1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{a} & \frac{1}{a} & \frac{a+1}{a} & \frac{1}{a} \\ \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{a+1}{a} \end{vmatrix} = a^4\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{a} & 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{a} & \frac{1}{a} & 1+\frac{1}{a} & \frac{1}{a} \\ \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 1+\frac{1}{a} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3 + C_4$ and taking $\left(1+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}\right)$ common from C_1

$$= a^4\left(1+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}+\frac{1}{a}\right)\begin{vmatrix} 1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ 1 & 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ 1 & \frac{1}{a} & 1+\frac{1}{a} & \frac{1}{a} \\ 1 & \frac{1}{a} & \frac{1}{a} & 1+\frac{1}{a} \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 - R_3 - R_1$ and $R_4 \to R_4 - R_1$

$$= a^4\,\frac{(a+4)}{a}\begin{vmatrix} 1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1

$$= a^3(a+4)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_1,

$$= a^3(a+4)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\therefore \Delta = a^3(a+4) \text{ Ans.}$$

10. $\begin{vmatrix} x & 1 & y & 1 \\ 1 & y & 1 & x \\ 1 & x & 1 & y \\ y & 1 & x & 1 \end{vmatrix} = (x+y+2)(x-y)^2(x+y-2)$ (BE 2063)

Solution:

Let $\Delta = \begin{vmatrix} x & 1 & y & 1 \\ 1 & y & 1 & x \\ 1 & x & 1 & y \\ y & 1 & x & 1 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$ and taking $(x + y + 2)$ common from R_1

$$= (x+y+2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & y & 1 & x \\ 1 & x & 1 & y \\ y & 1 & x & 1 \end{vmatrix}$$

Applying $C_3 \to C_3 - C_1$ and $C_4 \to C_4 - C_2$

$$= (x+y+2)\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & y & 0 & x-y \\ 1 & x & 0 & y-x \\ y & 1 & x-y & 0 \end{vmatrix}$$

Taking $(x - y)$ and $(x - y)$ common from C_3 and C_4 respectively. We get

$$= (x+y+2)(x-y)(x-y)\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & y & 0 & 1 \\ 1 & x & 0 & -1 \\ y & 1 & 1 & 0 \end{vmatrix}$$

Expanding along C_3

$$= (x+y+2)(x-y)^2(-1)\begin{vmatrix} 1 & 1 & 0 \\ 1 & y & 1 \\ 1 & x & -1 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$

$$= -(x+y+2)(x-y)^2\begin{vmatrix} 1 & 1 & 0 \\ 2 & x+y & 0 \\ 1 & x & -1 \end{vmatrix}$$

Expanding along C_3

$$= -(x+y+2)(x-y)^2(-1)\begin{vmatrix} 1 & 1 \\ 2 & x+y \end{vmatrix}$$

$$= (x+y+2)(x-y)^2(x+y-2)$$

$\Delta = (x+y+2)(x-y)^2(x+y-2)$ Ans.

11. $\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(ab+bc+ca)^2$

Solution: $\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix}$

Applying $a \to a^2 \times c_1$, $c_2 \to b^2 \times c_2$ and $c_3 \to c^2 \times c_3$

$$= \frac{1}{(abc)^2}\begin{vmatrix} (ab+ac)^2 & b^2c^2 & b^2c^2 \\ a^2c^2 & (bc+ab)^2 & a^2c^2 \\ a^2b^2 & a^2b^2 & (ac+bc)^2 \end{vmatrix}$$

Applying $c_1 \to c_1 - c_3$ & $c_2 \to c_2 - c_3$ and taking common $(ab + bc + ca)$ from c_1 & c_2

$$= \frac{(ab+bc+ca)^2}{(abc)^2}\begin{vmatrix} ab+ac-bc & 0 & b^2c^2 \\ 0 & bc+ab-ac & a^2c^2 \\ ab-ac-bc & ab-ac-bc & (ac+bc)^2 \end{vmatrix}$$

Taking common c^2 from c_3

$$= \frac{(ab+bc+ca)^2}{(ab)^2}\begin{vmatrix} ab+ac-bc & 0 & b^2 \\ 0 & bc+ab-ac & a^2 \\ ab-ac-bc & ab-ac-bc & (a+b)^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - (R_1 + R_2)$

$$= \frac{(ab+bc+ca)^2}{(ab)^2}\begin{vmatrix} ab+ac-bc & 0 & b^2 \\ 0 & bc+ab-ac & a^2 \\ -2ac & -2bc & 2ab \end{vmatrix}$$

$$= 2\frac{(ab+bc+ca)^2}{(ab)^2}\begin{vmatrix} ab+ac-bc & 0 & b^2 \\ 0 & bc+ab-ac & a^2 \\ -ac & -bc & ab \end{vmatrix}$$

Expanding along R_1

$$= 2\frac{(ab+bc+ca)^2}{(ab)^2}[(ab+ac-bc)(ab^2c + a^2b^2 - a^2bc + a^2bc) + b^2(abc^2 + a^2bc - a^2c^2)]$$

$$= 2\frac{(ab+bc+ca)^2}{(ab)^2}[a^2b^3c + a^3b^3 - a^3b^2c + a^2b^2c^2 + a^3b^2c - a^3bc^2 - ab^3c^2 - a^2b^3c + a^2b^2c^2 - a^2 + a^2bc + ab^3c^2 + a^2b^3c - a^2b^2c^2]$$

$$= 2\frac{(ab+bc+ca)^2}{(ab)^2}[(ab+ac-bc)(ab^2c + a^2b^2 - a^2bc + a^2bc) + b^2(abc^2 + a^2bc - a^2c^2)]$$

$$= 2\frac{(ab+bc+ca)^2}{(ab)^2}[a^2b^3c + a^3b^3 + a^2b^2c^2 + a^3b^2c - ab^3c^2 - a^2b^3c + ab^3c^2 + a^2b^3c - a^2b^2c^2)]$$

$$= 2\frac{(ab+bc+ca)^2}{(ab)^2} \times a^2b^2(bc + ab + ac)$$

$$= 2(ab+bc+ca)^3$$

12. $\begin{vmatrix} x & 0 & z & y \\ y & z & 0 & x \\ 0 & x & y & z \\ z & y & x & 0 \end{vmatrix}$

Solution:

Let $\Delta = \begin{vmatrix} x & 0 & z & y \\ y & z & 0 & x \\ 0 & x & y & z \\ z & y & x & 0 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3 + C_4$ we get,

$$= \begin{vmatrix} x+y+z & 0 & z & y \\ x+y+z & z & 0 & x \\ x+y+z & x & y & z \\ x+y+z & y & x & 0 \end{vmatrix}$$

= Taking $(x + y + z)$ common from C_1

$$= (x+y+z)\begin{vmatrix} 1 & 0 & z & y \\ 1 & z & 0 & x \\ 1 & x & y & z \\ 1 & y & x & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ & $R_3 \to R_3 - R_4$ resp. we get,

$$= (x+y+z)\begin{vmatrix} 0 & -z & z & y-x \\ 0 & z-x & -y & x-z \\ 0 & x-y & y-x & z \\ 1 & y & x & 0 \end{vmatrix}$$

Expanding along C_1,

$$= (x+y+z)(-1)\begin{vmatrix} -z & z & y-x \\ z-x & -y & x-z \\ x-y & y-x & z \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ we get,

$$= (x+y+z)(-1)\begin{vmatrix} 0 & z & y-x \\ z-x-y & -y & x-z \\ 0 & y-x & z \end{vmatrix}$$

Expanding along C_1,

$= (x+y+z)(-1)\{-(z-x-y)\}\{z^2-(y-x)^2\}$

$= (x+y+z)(z-y+x)(z+y-x)(z-x-y)$

$= (x+y+z)(x-y+z) - \{(x-y-z)\}\{-(x+y-z)\}$

$= (x+y+z)(x-y-z)(x-y+z)(x+y-z)$

$\therefore \Delta = (x+y+z)(x-y-z)(x-y+z)(x+y-z)$ Ans.

13. $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & b & d & -b \\ -d & c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$

Solution: $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$

Applying $C_1 \to C_1 \times a$, $C_2 \to C_2 \times b$, $C_3 \to C_3 \times c$, $C_4 \to C_4 \times d$ and $C_1 \to C_1 + C_2 + C_3 + C_4$

$$= \frac{1}{abcd}\begin{vmatrix} a^2+b^2+c^2+d^2 & b^2 & c^2 & d^2 \\ 0 & ab & -dc & cd \\ 0 & bd & ac & -bd \\ 0 & -bc & bc & ad \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2+d^2)}{abcd}\begin{vmatrix} 1 & b^2 & c^2 & d^2 \\ 0 & ab & -dc & cd \\ 0 & bd & ac & -bd \\ 0 & -bc & bc & ad \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2+d^2)}{abcd}\begin{vmatrix} ab & -dc & cd \\ bd & ac & -bd \\ -bc & bc & ad \end{vmatrix}$$

Taking common b from C_1, c from C_2 & d from C_3

$$= \frac{(a^2+b^2+c^2+d^2)}{a}\begin{vmatrix} a & -d & c \\ b & a & -b \\ -c & b & a \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2+d^2)}{a}[a(a^2+b^2) + d(ad-bc) + c(bd+ac)]$$

$$= \frac{(a^2+b^2+c^2+d^2)}{a}[a(a^2+b^2) + ad^2 - bcd + bcd + ac^2]$$

$= (a^2+b^2+c^2+d^2)^2$

14. $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

Solution:

Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

Put $a = -b$ $\begin{vmatrix} 2b & 0 & c-b \\ 0 & -2b & c+b \\ c-b & c+b & -2c \end{vmatrix}$

why and how.

Applying $C_3 \to C_3 + C_1$, we get,

$$= \begin{vmatrix} 2b & 0 & c+b \\ 0 & -2b & c+b \\ c-b & c+b & -(c+b) \end{vmatrix}$$

Taking (c + b) common from C_3,

$$= (b+c) \begin{vmatrix} 2b & 0 & 1 \\ 0 & -2b & 1 \\ c-b & c+b & -1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$, $R_2 \to R_2 + R_3$ resp. we get,

$$= (b+c) \begin{vmatrix} b+c & b+c & 0 \\ c-b & c-b & 0 \\ c-b & c+b & -1 \end{vmatrix}$$

Taking (b + c) & (c – b) common from R_1 & R_2,

$$= (b+c)^2 (c-b) \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ c-b & c+b & -1 \end{vmatrix} = 0$$

$\therefore$ a + b is a factor of Δ.

Similarly (b + c) & (c + a) are factor of Δ, since Δ is of third degree, any other factor of Δ must be a numerical constant.

Let, $\begin{vmatrix} -2b & a+b & b+c \\ b+d & -2b & b+c \\ c+d & c+b & -2c \end{vmatrix} = \lambda (a+b)(b+c)(c+a)$

Putting a = 0, b = 1 & c = 1 on both sides we get,

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{vmatrix} = \lambda (1)(2)(1)$$

Applying $C_2 \to C_2 - C_3$ we get,

or, $\begin{vmatrix} 0 & 0 & 1 \\ 1 & -4 & 2 \\ 1 & 4 & -2 \end{vmatrix} = 2\lambda$

Expanding along R_1,

or, $1 \begin{vmatrix} 1 & -4 \\ 1 & 4 \end{vmatrix} = 2\lambda$

or, $1(4 + 4) = 2\lambda$

$\therefore \lambda = 4$

Hence, $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

$= 4(a+b)(b+c)(c+a)$ Ans.

15. $\begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix} = x^2(6+x)$

Solution:

Let $\Delta = \begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 6+x & 2 & 3 \\ 6+x & 2+x & 3 \\ 6+x & 2 & 3+x \end{vmatrix}$$

Taking (6 + x) common from C_1,

$$= (6+x) \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ we get,

$$= (6+x) \begin{vmatrix} 0 & -x & 0 \\ 0 & x & -x \\ 1 & 2 & 3+x \end{vmatrix}$$

Expanding along C_1,

$= (6+x)(1)(x^2)$

$= x^2(6+x)$

$\Delta = x^2(6+x)$ Ans.

16. $\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix} = 4abc$

Solution:

Let $\Delta = \begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix}$

Applying $C_1 \to C_1 - C_2 - C_3$ we get,

$$= \begin{vmatrix} 0 & a & b \\ -2c & a+c & c \\ -2c & c & b+c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$ we get,

$$= \begin{vmatrix} 0 & a & b \\ 0 & a & -b \\ -2c & c & b+c \end{vmatrix}$$

Expanding along C_1,

$= -2c(-ab - ab)$

$= -2c(-2ab)$

$= 4abc$

$\Delta = 4abc$ Ans.

17. $$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -2\,(a^3+b^3+c^3-3abc)$$

Solution. L.H.S. $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$

$$= \begin{vmatrix} a & b+c & c+a \\ b & c+a & a+b \\ c & a+b & b+c \end{vmatrix} + \begin{vmatrix} b & b+c & c+a \\ c & c+a & a+b \\ a & a+b & b+c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c+a \\ b & c & a+b \\ c & a & b+c \end{vmatrix} + \begin{vmatrix} a & c & c+a \\ b & a & a+b \\ c & b & b+c \end{vmatrix} + \begin{vmatrix} b & b & c+a \\ c & c & a+b \\ a & a & b+c \end{vmatrix} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} + \begin{vmatrix} a & c & c \\ b & a & a \\ c & b & b \end{vmatrix} + \begin{vmatrix} a & c & a \\ b & a & b \\ c & b & c \end{vmatrix}$$

$$+ 0 + \begin{vmatrix} b & c & c \\ c & a & a \\ a & b & b \end{vmatrix} + \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad \text{Also, } 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ and taking common $(a+b+c)$ from R_1

$$= 2\,(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $c_3 \to c_3 - c_1$ & $c_2 \to c_2 - c_1$

$$= 2\,(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$$

$$= 2\,(a+b+c)\,[(c-b)\,(b-c) - (a-b)\,(a-c)]$$
$$= 2\,(a+b+c)\,[bc - c^2 - b^2 + bc - a^2 + ac + ab - bc]$$
$$= -2\,(a+b+c)\,(a^2+b^2+c^2-bc-ca-ab)$$
$$= -2\,(a^3+b^3+c^3-3abc)$$

18. $$\begin{vmatrix} 1+a_1 & a_2 & a_3 \\ a_1 & 1+a_2 & a_3 \\ a_1 & a_2 & 1+a_3 \end{vmatrix} = 1 + a_1 + a_2 + a_3$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1+a_1 & a_2 & a_3 \\ a_1 & 1+a_2 & a_3 \\ a_1 & a_2 & 1+a_3 \end{vmatrix}$$

Applying, $C_1 \to C_1 + C_2 + C_3$ we get,

$$= \begin{vmatrix} 1+a_1+a_2+a_3 & a_2 & a_3 \\ 1+a_1+a_2+a_3 & 1+a_2 & a_3 \\ 1+a_1+a_2+a_3 & a_2 & 1+a_3 \end{vmatrix}$$

Taking $(1 + a_1 + a_2 + a_3)$ common from C_1

$$= 1 + a_1 + a_2 + a_3 \begin{vmatrix} 1 & a_2 & a_3 \\ 1 & 1+a_2 & a_3 \\ 1 & a_2 & 1+a_3 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ we get,

$$= (1 + a_1 + a_2 + a_3)\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & a_2 & 1+a_3 \end{vmatrix}$$

Expanding along C_1,

$$= (1 + a_1 + a_2 + a_3)\ 1\ (1)$$

$\therefore\ \Delta = 1 + a_1 + a_2 + a_3$ Ans.

19. $$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

Dividing by x to C_1, y to C_2 & z to C_3

$$= xyz\begin{vmatrix} \frac{1}{x}+1 & \frac{1}{y} & \frac{1}{z} \\ \frac{1}{x} & \frac{1}{y}+1 & \frac{1}{z} \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z}+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{y} & \frac{1}{z} \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{y}+1 & \frac{1}{z} \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1 & \frac{1}{y} & \frac{1}{z}+1 \end{vmatrix}$$

Taking $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)$ common from C_1 we get,

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix}1 & \frac{1}{y} & \frac{1}{z}\\ 1 & \frac{1}{y}+1 & \frac{1}{z}\\ 1 & \frac{1}{y} & \frac{1}{z}+1\end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$ we get,

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right)\begin{vmatrix}0 & -1 & 0\\ 0 & 1 & -1\\ 1 & \frac{1}{y} & \frac{1}{z}+1\end{vmatrix}$$

Expanding along C_1 we get,

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right) . 1\,(1)$$

$$\therefore \Delta = xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+1\right) \text{ Ans.}$$

20. $\begin{vmatrix}1 & a & a^2 & a^3+bcd\\ 1 & b & b^2 & b^3+cba\\ 1 & c & c^2 & c^3+dab\\ 1 & d & d^2 & d^3+abc\end{vmatrix} = 0$ **(BE 2062)**

Solution: Let, $\Delta = \begin{vmatrix}1 & a & a^2 & a^3+bcd\\ 1 & b & b^2 & b^3+cda\\ 1 & c & c^2 & c^3+abd\\ 1 & d & d^2 & d^3+abc\end{vmatrix}$

$$= \begin{vmatrix}1 & a & a^2 & a^3\\ 1 & b & b^2 & b^3\\ 1 & c & c^2 & c^3\\ 1 & d & d^2 & d^3\end{vmatrix} + \begin{vmatrix}1 & a & a^2 & bcd\\ 1 & b & b^2 & cda\\ 1 & c & c^2 & abd\\ 1 & d & d^2 & abc\end{vmatrix}$$

$$= \Delta_1 + \Delta_2$$

Applying $R_1 \rightarrow R_1 \times a$, $R_2 \rightarrow R_2 \times b$, $R_3 \rightarrow R_3 \times c$ and $R_1 \rightarrow R_4 \times d$ in Δ_2

$$\Delta_2 = \frac{1}{abcd}\begin{vmatrix}a & a^2 & a^3 & abcd\\ b & b^2 & b^3 & abcd\\ c & c^2 & c^3 & abcd\\ d & d^2 & d^3 & abcd\end{vmatrix}$$

Taking common abcd from c_4

$$= \begin{vmatrix}a & a^2 & a^3 & 1\\ b & b^2 & b^3 & 1\\ c & c^2 & c^3 & 1\\ d & d^2 & d^3 & 1\end{vmatrix}$$

Applying $c_3 \leftrightarrow c_4$

$$= (-1)\begin{vmatrix}a & a^2 & 1 & a^3\\ b & b^2 & 1 & b^3\\ c & c^2 & 1 & c^3\\ d & d^2 & 1 & d^3\end{vmatrix}$$

Applying $c_2 \leftrightarrow c_3$ and then $c_1 \leftrightarrow c_2$

$$= (-1)\begin{vmatrix}1 & a & a^2 & a^3\\ 1 & b & b^2 & b^3\\ 1 & c & c^2 & c^3\\ 1 & d & d^2 & d^3\end{vmatrix}$$

$$= (-1)\,\Delta_1 = -\Delta_1$$

$$\therefore \Delta = \Delta_1 + \Delta_2 = \Delta_1 - \Delta_1 = 0$$

21. $\begin{vmatrix}y^2z^2 & yz & y+z\\ z^2x^2 & zx & z+x\\ x^2y^2 & xy & x+y\end{vmatrix} = 0$ **(BE 2056, 058)**

Solution: $\begin{vmatrix}y^2z^2 & yz & y+z\\ z^2x^2 & zx & z+x\\ x^2y^2 & xy & x+y\end{vmatrix}$

Applying $R_1 \rightarrow R_1 \times x^2$, $R_2 \rightarrow R_2 \times y^2$ & $R_3 \rightarrow R_3 \times z^2$

$$= \frac{1}{(xyz)^2}\begin{vmatrix}x^2y^2z^2 & x^2yz & x^2(y+z)\\ x^2y^2z^2 & y^2xz & y^2(z+x)\\ x^2y^2z^2 & xyz^2 & z^2(x+y)\end{vmatrix}$$

Taking common $x^2y^2z^2$ from C_1.

$$= \begin{vmatrix}1 & x^2yz & x^2(y+z)\\ 1 & y^2xz & y^2(x+z)\\ 1 & xyz^2 & z^2(x+y)\end{vmatrix}$$

Taking xyz from C_2

$$= xyz\begin{vmatrix}1 & x & x^2y+x^2z\\ 1 & y & xy^2+y^2z\\ 1 & z & xz^2+yz^2\end{vmatrix}$$

Applying $R_2 \rightarrow R_1$ & $R_3 \rightarrow R_3 - R_1$

$$= xyz\begin{vmatrix}1 & x & x^2y+x^2z\\ 0 & y-x & xy^2+y^2z-x^2y-x^2z\\ 0 & z-x & xz^2+yz^2-x^2y-x^2z\end{vmatrix}$$

$$= xyz\begin{vmatrix}1 & x & x^2y+x^2z\\ 0 & y-x & xy(y-x)+z(y^2-x^2)\\ 0 & z-x & xz(z-x)+y(z^2-x^2)\end{vmatrix}$$

Taking common $(y-x)$ from R_2 and $(z-x)$ from R_3 resp. we get

$$= xyz\,(y-x)\,(z-x)\begin{vmatrix}1 & x & x^2y+x^2z\\ 0 & 1 & xy+z(y+x)\\ 0 & 1 & xz+y(z+x)\end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$= xyz\,(y-x)(z-x)\begin{vmatrix} 1 & x & x^2y+x^2z \\ 0 & 1 & xy+z(y+x) \\ 0 & 0 & 0 \end{vmatrix} = 0$$

22. $\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ a & a & b & a \\ b & b & b & a \end{vmatrix}$ (BE 2056)

Solution:

$$\text{Let, } \Delta = \begin{vmatrix} a & b & b & b \\ a & b & a & a \\ a & a & b & a \\ b & b & b & a \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ we get,

$$= \begin{vmatrix} a-b & b & b & b \\ a-b & b & a & a \\ 0 & a & b & a \\ 0 & b & b & a \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ we get,

$$= \begin{vmatrix} a-b & b & b & b \\ 0 & 0 & a-b & a-b \\ 0 & a & b & a \\ 0 & b & b & a \end{vmatrix}$$

Expanding along C_1,

$$= (a-b)\begin{vmatrix} 0 & a-b & a-b \\ a & b & a \\ b & b & a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$

$$= (a-b)\begin{vmatrix} 0 & 0 & a-b \\ a & b-a & a \\ b & b-a & a \end{vmatrix}$$

Expanding along R_1 we get,

$$= (a-b)^2\{a(b-a) - b(b-a)\}$$
$$= (a-b)^3(b-a)$$

$\therefore \Delta = -(b-a)^4$ Ans.

23. $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$ (BE 2065)

Solution: Let $\Delta = \begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 - R_4$, $R_2 \to R_2 - R_4$ and $R_3 \to R_3 - R_4$ resp. we get

$$\begin{vmatrix} (a-1)(a^2+a+1) & 3(a+1)(a-1) & 3(a-1) & 0 \\ (a+1)(a-1) & (a+3)(a-1) & 2(a-1) & 0 \\ (a-1) & 2(a-1) & (a-1) & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

Taking $(a-1)$ common from R_1, R_2 & R_3 and expanding along C_4

$$= (a-1)^3\begin{vmatrix} a^2+a+1 & 3(a+1) & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$

$$= (a-1)^3\begin{vmatrix} a^2+a-2 & 3(a+1) & 3 \\ a-1 & a+3 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

Taking $(a-1)$ common from C_1

$$= (a-1)^3(a-1)\begin{vmatrix} a+2 & 3(a+1) & 3 \\ 1 & (a+3) & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

Expanding along C_1

$$= (a-1)^4[(a+2)(a+3-4) - 1(3a+3-6)]$$
$$= (a-1)^4[(a+2)(a-1) - 3(a-1)]$$
$$= (a-1)^4(a-1)(a+2-3)$$
$$= (a-1)^5(a-1)$$
$$= (a-1)^6$$

$\therefore \Delta = (a-1)^6$

24. Express $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$ as the square of a determinant and hence find its value.

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

Rewrite Δ in the from,

$$\begin{vmatrix} -a.a+b.c+c.b & -a.b+b.a+c.c & -a.c+b.b+c.a \\ -b.a+c.c+a.b & -b.b+c.a+a.c & -b.c+c.b+a.a \\ -c.a+a.c+b.b & -c.b+a.a+b.c & -c.c+a.b+b.c \end{vmatrix}$$

$$= \begin{vmatrix} -a & b & c \\ -b & c & a \\ -c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

Taking common (-1) from C_1 and interchange R_2 & R_3

$$(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Hence, $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

Again for the value of Δ

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

Applying $C_1 \to C_1 + C_2 + C_3$ we get,

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}^2$$

Taking $(a + b + c)$ common from C_1

$$= \left[(a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \right]^2$$

Applying $R_1 \to R_1 - R_2$, $R_2 - R_3$ we get

$$\left[(a+b+c) \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix} \right]^2$$

Expanding along C_1,

$= [(a + b + c) \{(b - c)(a - b) - (c - a)^2\}]^2$

$= [(a + b + c)(ab - b^2 - ac + bc - c^2 + 2ab - a^2)]^2$

$= [(a + b + c)(- a^2 - b^2 - c^2 + ab + bc + ac)]^2$

$= (- a^3 - ab^2 - ac^2 - a^2b + abc + a^2c - a^2b - b^3 - bc^2 + ab^2 + b^2c + abc - a^2c - b^2c - c^3 + abc + bc^2 + ac^2)^2$

$\therefore \Delta = (a^3 + b^3 + c^3 - 3abc)^2$

25. Express $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix}$ **as a perfect square & find its value.**

Solution: L. H. S. = $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2 & R_3 resp. We get

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Again taking a, b, c common from C_1, C_2 & C_3 resp.

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ we get

$$= a^2b^2c^2 \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Expanding along C_1

$= a^2b^2c^3\, 2 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$

$= 2a^2b^2c^2 (1 + 1)$

$= 4a^2b^2c^2$

$$= \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2$$

Again,

$$\text{R. H. S.} = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ cd & bc & a^2+b^2 \end{vmatrix}$$

Rewrite R. H. S. in the form,

$$= \begin{vmatrix} 0.0+c.c+b.b & 0.c+c.0+b.a & a.b+c.a+b.0 \\ c.0+0.c+a.b & c.c+0.0+a.a & c.b+0.a+a.0 \\ b.0+a.c+0.b & b.c+a.0+0.a & b.b+a.a+0.0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

$= \{- c\,(ab) + b\,(ac)\}^2$

$= (2abc)^2 = 4a^2b^2c^2$

$\therefore$ L. H. S. = R. H. S. Ans.

26. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$

where $r^2 = a^2 + b^2 + c^2$, $u^2 = ab + bc + ca$

Solution:

$$\text{Let, } \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ac+ab+bc \\ ab+bc+ca & b^2+c^2+a^2 & bc+ca+ab \\ ac+ba+bc & bc+ac+ab & c^2+a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$$

Where, $r^2 = a^2 + b^2 + c^2$, $u^2 = ab + bc + cd$.

27. Show $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$ as the product of two determinants.

Solution:

Let, $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking a, b, c common from C_1, C_2 and C_3 resp. We get

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}.$$

Applying $C_1 \to C_1 + C_2 - C_3$

$$= abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_3$

$$= abc \begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 2b & b+c & c \end{vmatrix}$$

Expanding along C_1

$$= abc\,(ab)\,(ac - c^2 + ac + c^2)$$
$$= 2ab^2c\,(2ac)$$
$$= 4\,a^2b^2c^2$$
$$= \begin{vmatrix} 2b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \begin{vmatrix} 2b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} 2b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2$$

28. Express $\begin{vmatrix} b^2+c^2+1 & c^2+1 & b^2+1 & b+c \\ c^2+1 & c^2+a^2+1 & a^2+1 & c+a \\ b^2+1 & a^2+1 & a^2+b^2+1 & a+b \\ b+c & c+a & a+b & 3 \end{vmatrix}$

as the square of a determinant and hence find its value.

Solution. $\begin{vmatrix} b^2+c^2+1 & c^2+1 & b^2+1 & b+c \\ c^2+1 & c^2+a^2+1 & a^2+1 & c+a \\ b^2+1 & a^2+1 & a^2+b^2+1 & a+b \\ b+c & c+a & a+b & 3 \end{vmatrix}$

$$= \begin{vmatrix} b & c & 1 & 0 \\ 0 & c & 1 & a \\ b & 0 & 1 & a \\ 1 & 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} b & c & 1 & 0 \\ 0 & c & 1 & a \\ b & 0 & 1 & a \\ 1 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} b & c & 1 & 0 \\ 0 & c & 1 & a \\ b & 0 & 1 & a \\ 1 & 1 & 0 & 1 \end{vmatrix}^2$$

To find value

Applying $C_1 \to C_1 - C_4$, $C_2 \to C_2 - C_4$

$$\begin{vmatrix} b & c & 1 & 0 \\ 0 & c & 1 & a \\ b & 0 & 1 & a \\ 1 & 1 & 0 & 1 \end{vmatrix}^2 = \begin{vmatrix} b & c & 1 & 0 \\ -a & c-a & 1 & a \\ b-a & -a & 1 & a \\ 0 & 0 & 0 & 1 \end{vmatrix}^2$$

Expanding along R_4

$$= \begin{vmatrix} b & c & 1 \\ -a & c-a & 1 \\ b-a & -a & 1 \end{vmatrix}^2$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} b & c & 1 \\ -a-b & -a & 0 \\ -a & -a-c & 0 \end{vmatrix}^2$$

Expanding along R_1

$$= \begin{vmatrix} -a-b & -a \\ -a & -a-c \end{vmatrix}^2$$

Taking common (–) from R_1 & R_2

$$= \begin{vmatrix} a+b & a \\ a & a+c \end{vmatrix}^2$$
$$= [(a+b)(a+c) - a^2]^2$$
$$= [a^2 + ac + ab + bc - a^2]$$
$$= [bc + ca + ab]^2$$

29. Solve the equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

Solution: Given,

$$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix}$$

Applying, $R_1 \to R_1 + R_2 + R_3$

or, $\begin{vmatrix} 3x-2 & 3x-2 & 3x-2 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

Taking $(3x-2)$ common from R_1,

or, $(3x-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$

or, $(3x-2)\begin{vmatrix} 1 & 0 & 1 \\ 3 & 3x-8-3 & 3 \\ 3 & 0 & 3x-8 \end{vmatrix} = 0$

Expanding along C_2,

or, $(3x-2)(3x-8-3)(3x-8-3) = 0$

or, $(3x-2)(3x-11)(3x-11) = 0$

Either $3x - 2 = 0$, $\Rightarrow x = \frac{2}{3}$

or, $(3x-11)^2 = 0$

or, $3x = 11$

$\therefore$ $x = \frac{2}{3}, \frac{11}{3}$ Ans.

30. If $a + b + c = 0$, then solve the equation,

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}$$

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Solution: Given,

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \;\&\; a+b+c=0$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\text{or, } \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix} = 0$$

$$\text{or, } \begin{vmatrix} -x & c & b \\ -x & b-x & a \\ -x & a & c-x \end{vmatrix} = 0$$

Taking $(-x)$ common from C_1,

$$\text{or, } (-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\text{or, } (-x)\begin{vmatrix} 1 & c & b \\ 0 & b-c-x & a-b \\ 0 & a-c & c-b-x \end{vmatrix} = 0$$

Expanding along C_1

or, $(-x)\,[(b-c-x)(c-b-x) - (a-c)(a-b)] = 0$

or, $(-x)\,[bc - b^2 - bx - c^2 + bc + cx - cx + bx + x^2 - a^2 + ab + ac - bc] = 0$

or, $-x\,(x^2 - a^2 - b^2 - c^2 + ab + bc + ac) = 0$

Either,

$-x = 0$

$\therefore$ $x = 0$

or, $x^2 - a^2 - b^2 - c^2 + ab + bc + ac = 0$

or, $x^2 = a^2 + b^2 + c^2 - ab - bc - ac$

$\therefore$ $x = \pm\sqrt{a^2 + b^2 + c^2 - ab - bc - ac}$

$\therefore$ $x = 0, \pm\sqrt{a^2 + b^2 + c^2 - ab - bc - ac}$ Ans.

Exercise - 2

1. Solve the following equations by Cramer's rule.

(i) $x + 2y + 3z = 14$

$3x + y + 2z = 11$

$2x + 3y + z = 11$

Solution:

Here, $x + 2y + 3z = 14$

$3x + y + 2z = 11$

& $2x + 3y + z = 11$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	2	3	14
3	1	2	11
2	3	1	11

So,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$= 1(1-6) - 2(3-4) + 3(9-2)$

$= -5 + 2 + 21$

$= 18 \neq 0$

$$\Delta_1 = \begin{vmatrix} 14 & 2 & 3 \\ 11 & 1 & 2 \\ 11 & 3 & 1 \end{vmatrix}$$

$= 14(1-6) - 2(11-22) + 3(33-11)$

$= -70 + 22 + 66$

$= 18$

$$\Delta_2 = \begin{vmatrix} 1 & 14 & 3 \\ 3 & 11 & 2 \\ 2 & 11 & 1 \end{vmatrix}$$

$= 1(11-22) - 14(3-4) + 3(33-22)$

$= -11 + 14 + 33$

$= 36$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 14 \\ 3 & 1 & 11 \\ 2 & 3 & 11 \end{vmatrix}$$

$= 1(11-33) - 2(33-22) + 14(9-2)$

$= -22 - 22 + 98$

$= 54$

We know that, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{18}{18} = 1$

$y = \dfrac{\Delta_2}{\Delta} = \dfrac{36}{18} = 2$

& $z = \dfrac{\Delta_3}{\Delta} = \dfrac{54}{18} = 3$

$\therefore x = 1, y = 2$ & $z = 3$ Ans.

(ii) $x + y + z = 2$
$x + 2y + 3z = 1$
$3x + y - 5z = 4$

Solution:

Here, $x + y + z = 2$
$x + 2y + 3z = 1$
& $3x + y - 5z = 4$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	1	1	2
1	2	3	1
3	1	–5	4

So,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -5 \end{vmatrix}$$

$= 1(-10-3) - 1(-5-9) + 1(1-6)$
$= -13 + 14 - 5$
$= -4 \neq 0$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & -5 \end{vmatrix}$$

$= 2(-10-3) - 1(-5-12) + 1(1-8)$
$= -26 + 17 - 7$
$= -16$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 4 & -5 \end{vmatrix}$$

$= 1(-5-12) - 2(-5-9) + 1(4-3)$
$= -17 + 28 + 1$
$= 12$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix}$$

$= 1(8-1) - 1(4-3) + 2(1-6)$
$= 7 - 1 - 10$

$= -4$

We know that, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{-4}{-16} = 4$

$y = \dfrac{\Delta_2}{\Delta} = \dfrac{12}{-4} = -3$

& $z = \dfrac{\Delta_3}{\Delta} = \dfrac{-4}{4} = 1$

$\therefore x = 4, y = -3$ & $z = 1$ Ans.

(iii) $x + 2y + 3z = 0$
$2x + 4y + z - 7 = 0$
$3x + 2y + 9z - 14 = 0$

Solution:

Here, $x + 2y + 3z - 6 = 0$ i.e. $x + 2y + 3z = 6$
$2x + 4y + z - 7 = 0$ $2x + 4y + z = 7$
$3x + 2y + 9z - 14 = 0$ $3x + 2y + 9z = 14$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	2	3	6
2	4	1	7
3	2	9	14

So,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}$$

$= 1(36-2) - 2(18-3) + 3(4-12)$
$= 34 - 30 - 24$
$= -20 \neq 0$

$$\Delta_1 = \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix}$$

$= 6(36-2) - 2(63-14) + 3(14-56)$
$= 204 - 98 - 126$
$= -20$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix}$$

$= 1(63-14) - 2(54-42) + 3(6-21)$
$= 49 - 24 - 45$
$= -20$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix}$$

$= 1(56-14) - 2(28-12) + 3(14-24)$

$= 42 - 32 - 30$
$= -20$

We know that, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{-20}{-20} = 1$

$y = \dfrac{\Delta_2}{\Delta} = \dfrac{-20}{-20} = 1$

$\& \; z = \dfrac{\Delta_3}{\Delta} = \dfrac{-20}{-20} = 1$

$\therefore x = 1, y = 1 \;\&\; z = 1$ Ans.

(iv) $x + 3y + 5z - 22 = 0$
$5x - 3y + 2z - 5 = 0$
$9x + 8y - 3z - 16 = 0$

Solution:

Here, $x + 3y + 5z = 22$
$5x - 3y + 2z = 5$
$9x + 8y - 3z = 16$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	3	5	22
5	-3	2	5
9	8	-3	16

So,

$$\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 5 & -3 & 2 \\ 9 & 8 & -3 \end{vmatrix}$$

$= 1(9 - 16) - 3(-15 - 18) + 5(40 + 27)$
$= -7 + 99 + 335$
$= 427 \neq 0$

$$\Delta_1 = \begin{vmatrix} 22 & 3 & 5 \\ 5 & -3 & 2 \\ 16 & 8 & -3 \end{vmatrix}$$

$= 22(9 - 16) - 3(-15 - 32) + 5(40 + 48)$
$= -154 + 141 + 440$
$= 427$

$$\Delta_2 = \begin{vmatrix} 1 & 22 & 5 \\ 5 & 5 & 2 \\ 9 & 16 & -3 \end{vmatrix}$$

$= 1(-15 - 32) - 5(-66 - 80) + 9(44 - 25)$
$= -47 + 730 + 171$
$= 854$

$$\Delta_3 = \begin{vmatrix} 1 & 3 & 22 \\ 5 & -3 & 5 \\ 9 & 8 & 16 \end{vmatrix}$$

$= 1(-48 - 40) - 5(48 - 176) + 9(15 + 66)$
$= -88 + 640 + 729$
$= 1281$

We know that, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{427}{427} = 1$

$y = \dfrac{\Delta_2}{\Delta} = \dfrac{854}{427} = 2$

$\& \; z = \dfrac{\Delta_3}{\Delta} = \dfrac{1281}{427} = 3$

$\therefore x = 1, y = 2 \;\&\; z = 3$ Ans.

(v) $x + 3z = 10$
$2x + y = 4$
$5y - 4z = -2$

Solution:

Here, $x + + 0.y + 3z = 10$
$2x + y + 0.z = 4$
$0.x + 5y - 4z = -2$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	0	3	10
2	1	0	4
0	5	-4	-2

So,

$$\Delta = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 5 & -4 \end{vmatrix}$$

$= 1(-4 - 0) - 2(0 - 15)$
$= -4 + 30$
$= 26 \neq 0$

$$\Delta_1 = \begin{vmatrix} 10 & 0 & 3 \\ 4 & 1 & 0 \\ -2 & 5 & -4 \end{vmatrix}$$

$= 10(-4 - 0) + 3(20 + 2)$
$= -40 + 66$
$= 26$

$$\Delta_2 = \begin{vmatrix} 1 & 10 & 3 \\ 2 & 4 & 0 \\ 0 & -2 & -4 \end{vmatrix}$$

$= 3(-4 - 0) - 4(4 - 20)$
$= -12 + 64$
$= 52$

$$\Delta_3 = \begin{vmatrix} 1 & 0 & 10 \\ 2 & 1 & 4 \\ 0 & 5 & -2 \end{vmatrix}$$

$= 1(-2-20) + 10(10-0)$

$= -22 + 100$

$= 78$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{26}{26} = 1$

$y = \frac{\Delta_2}{\Delta} = \frac{52}{26} = 2$

& $z = \frac{\Delta_3}{\Delta} = \frac{78}{26} = 3$

$\therefore x = 1, y = 2$ & $z = 3$ Ans.

(vi) $\mathbf{x + 4y - 2z = 3}$

$\mathbf{3x + y + 5z = 7}$

$\mathbf{2x + 3y + z = 5}$

Solution:

Here, $x + 4y - 2z = 3$

$3x + y + 5z = 7$

$2x + 3y + z = 5$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	4	−2	3
3	1	5	7
2	3	1	5

So,

$$\Delta = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 1 \end{vmatrix}$$

$= 1(1-15) - 3(4+6) + 2(20+2)$

$= -14 - 30 + 44 = 0$

Here, $\Delta = 0$ so it has no Solution:

(vii) $\mathbf{x + y + z = 3}$

$\mathbf{x + 2y + 3z = 4}$

$\mathbf{x + 4y + 9z = 16}$

Solution:

Here, $x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + 9z = 6$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	1	1	3
1	2	3	4
1	4	9	6

So,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$= 1(18-12) - 1(9-4) + 1(3-2)$

$= 6 - 5 + 1$

$= 2$

$$\Delta_1 = \begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{vmatrix}$$

$= 3(18-12) - 1(36-18) + 1(16-12)$

$= 18 - 18 + 4$

$= 4$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{vmatrix}$$

$= 1(36-18) - 1(27-6) + 1(9-4)$

$= 18 - 21 + 5$

$= 2$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{vmatrix}$$

$= 1(12-16) - 1(6-12) + 1(4-6)$

$= -4 + 6 - 2$

$= 0$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{4}{2} = 2$

$y = \frac{\Delta_2}{\Delta} = \frac{2}{2} = 1$ & $z = \frac{\Delta_3}{\Delta} = \frac{0}{2} = 0$

$\therefore x = 2, y = 1$ & $z = 0$ Ans.

(viii) $\mathbf{x + 3y + 6z = 2}$

$\mathbf{3x - y + 4z = 9}$

$\mathbf{x - 4y + 2z = 7}$

Solution:

Here, $x + 3y + 6z = 2$

$3x - y + 4z = 9$

$x - 4y + 2z = 7$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	3	6	2
3	−1	4	9
1	−4	2	7

So,

$$\Delta = \begin{vmatrix} 1 & 3 & 6 \\ 3 & -1 & 4 \\ 1 & -4 & 2 \end{vmatrix}$$

$= 1(-2+16) - 3(6+24) + 1(12+6)$
$= 14 - 90 + 18$
$= -58$

$$\Delta_1 = \begin{vmatrix} 2 & 3 & 6 \\ 9 & -1 & 4 \\ 7 & -4 & 2 \end{vmatrix}$$

$= 2(-2+16) - 3(18-28) + 6(-36+7)$
$= 28 + 30 + 174$
$= -116$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & 9 & 4 \\ 1 & 7 & 2 \end{vmatrix}$$

$= 1(18-28) - 2(6-4) + 6(21-9)$
$= -10 - 4 + 72$
$= 58$

$$\Delta_3 = \begin{vmatrix} 1 & 3 & 2 \\ 3 & -1 & 9 \\ 1 & -4 & 7 \end{vmatrix}$$

$= 1(-7+36) - 3(21+8) + 1(27+2)$
$= 29 - 87 + 29$
$= -29$

We know that, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{-116}{-58} = 2,\ y = \dfrac{\Delta_2}{\Delta} = \dfrac{58}{-58} = -1$

$\& \ z = \dfrac{\Delta_3}{\Delta} = \dfrac{-29}{-58} = \dfrac{1}{2}$

$\therefore x = 2, y = -1 \ \& \ z = \dfrac{1}{2}$ Ans.

(ix) $\mathbf{x + y + z = 6}$
$\mathbf{x - y + z = 2}$
$\mathbf{2x + y - z = 1}$

Solution:

Here, $x + y + z = 6$
$x - y + z = 2$
$2x + y - z = 1$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	1	1	6
1	-1	1	2
2	1	-1	1

So,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(1-1) - (-1-2) + 1(1+2)$
$= 0 + 3 + 3$
$= 6$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 6(1-1) - 1(-2-1) + 1(2+1)$
$= 0 + 3 + 3$
$= 6$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(-2-1) - 1(-6-1) + 2(6-2)$
$= -3 + 7 + 8$
$= 12$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$$

$= 1(-1-2) - 1(1-6) + 2(2+6)$
$= -3 + 5 + 16$
$= 18$

We know that, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{6}{6} = 1,\ y = \dfrac{\Delta_2}{\Delta} = \dfrac{12}{6} = 2$

$\& \ z = \dfrac{\Delta_3}{\Delta} = \dfrac{18}{6} = 3$

$\therefore x = 1, y = 2 \ \& \ z = 3$ Ans.

(x) $\mathbf{x + y + z = 9}$
$\mathbf{2x + 5y + 7z = 52}$
$\mathbf{2x + y - z = 0}$

Solution:

Here, $x + y + z = 9$
$2x + 5y + 7z = 52$
$2x + y - z = 0$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	1	1	9
2	5	7	52
2	1	-1	0

So,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(-5-7) - 1(-2-14) + 1(2-10)$
$= -12 + 16 - 8$
$= -4$

$$\Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 9(-5-7) - 1(-52-0) + 1(52-0)$
$= -108 + 52 + 52$
$= -4$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$$

$= 1(-52-0) - 9(-2-14) + 1(0-104)$
$= -52 + 144 - 104$
$= -12$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

$= 1(0-52) - 1(0-104) + 9(2-10)$
$= -52 + 104 - 72$
$= -20$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{-12}{4} = 3$

& $z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$

$\therefore x = 1, y = 3$ & $z = 5$ Ans.

(xi) $\mathbf{x + 2y - z = 1}$
$\mathbf{x + y + 2z = 9}$
$\mathbf{2x + y - z = 2}$

Solution:

Here, $x + 2y - z = 1$
$x + y + 2z = 9$
$2x + y - z = 2$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	2	–1	1
1	1	2	9
2	1	–1	2

So,

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(-1-2) - 2(-1-4) - 1(1-2)$
$= -3 + 10 + 1$
$= 8$

$$\Delta_1 = \begin{vmatrix} 1 & 2 & -1 \\ 9 & 1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(-1-2) - 2(-9-4) - 1(9-2)$
$= -3 + 26 + 1$
$= 16$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 9 & 2 \\ 2 & 2 & -1 \end{vmatrix}$$

$= 1(-9-4) - 1(-1-4) - 1(2-18)$
$= -13 + 5 + 16$
$= 8$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 9 \\ 2 & 1 & 2 \end{vmatrix}$$

$= 1(2-9) - 2(2-18) + 1(1-2)$
$= -7 + 32 - 1$
$= 24$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{16}{8} = 2$, $y = \frac{\Delta_2}{\Delta} = \frac{8}{8} = 1$

& $z = \frac{\Delta_3}{\Delta} = \frac{24}{8} = 3$

$\therefore x = 2, y = 1$ & $z = 3$ Ans.

(xii) $\mathbf{x + y - z = 2}$
$\mathbf{x - y + z = 1}$
$\mathbf{x + y + z = 4}$

Solution:

Here, $x + y - z = 2$
$x - y + z = 1$
$x + y + z = 4$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	1	–1	2
1	–1	1	1
1	1	1	4

So,

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= 1(-1-1) - 1(1-1) - 1(1+1)$

$= -2 - 0 - 2$

$= -4$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix}$$

$= 2(-1-1) - 1(1-4) - 1(1+4)$

$= -4 + 3 - 5$

$= -6$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix}$$

$= 1(1-4) - 2(1-1) - 1(4-1)$

$= -3 - 0 - 3$

$= -6$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$= 1(-4-1) - 1(4-1) + 2(1+1)$

$= -5 - 3 + 4$

$= -4$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{-6}{-4} = \frac{3}{2}$, $y = \frac{\Delta_2}{\Delta} = \frac{-6}{-4} = \frac{3}{2}$

$\& \ z = \frac{\Delta_3}{\Delta} = \frac{-4}{-4} = 1$

$\therefore x = \frac{3}{2}, y = \frac{3}{2} \ \& \ z = 1$ Ans.

(xiii) $\mathbf{x + 2y + 3z = 14}$

$\mathbf{3x + 4y + 2z = 17}$

$\mathbf{2x + 3y + z = 11}$

Solution:

Here, $x + 2y + 3z = 14$

$3x + 4y + 2z = 17$

$2x + 3y + z = 11$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	2	3	14
3	4	2	17
2	3	1	11

So,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$= 1(4-6) - 2(3-4) + 3(9-8)$

$= -2 + 2 + 3$

$= 3$

$$\Delta_1 = \begin{vmatrix} 14 & 2 & 3 \\ 17 & 4 & 2 \\ 11 & 3 & 1 \end{vmatrix}$$

$= 14(4-6) - 2(17-22) + 3(51-44)$

$= -28 + 10 + 21$

$= 3$

$$\Delta_2 = \begin{vmatrix} 1 & 14 & 3 \\ 3 & 17 & 2 \\ 2 & 11 & 1 \end{vmatrix}$$

$= 1(17-22) - 14(3-4) + 3(33-34)$

$= -5 + 14 - 3$

$= 6$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 14 \\ 3 & 4 & 17 \\ 2 & 3 & 11 \end{vmatrix}$$

$= 1(44-51) - 2(33-34) + 14(9-8)$

$= -7 + 2 + 14$

$= 9$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{3}{3} = 1$, $y = \frac{\Delta_2}{\Delta} = \frac{6}{3} = 2$

$\& \ z = \frac{\Delta_3}{\Delta} = \frac{9}{3} = 3$

$\therefore x = 1, y = 2 \ \& \ z = 3$ Ans.

(xiv) $\mathbf{x - 2y + 3z = 1}$

$\mathbf{3x - y + 4z = 3}$

$\mathbf{2x + y - 2z = -1}$

Solution:

Here, $x - 2y + 3z = 1$

$3x - y + 4z = 3$

$2x + y - 2z = -1$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	−2	3	1
3	−1	4	3
2	1	−2	−1

So,

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{vmatrix}$$

$= 1(2-4) + 2(-6-8) + 3(3+2)$
$= -2 - 28 + 15$
$= -15$

$$\Delta_1 = \begin{vmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ -1 & 1 & -2 \end{vmatrix}$$

$= 1(2-4) + 2(-6+4) + 3(3-1)$
$= -2 - 4 + 6$
$= 0$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 3 & 4 \\ 2 & -1 & -2 \end{vmatrix}$$

$= 1(-6+4) - 1(-6-8) + 3(-3-6)$
$= -2 + 14 - 27$
$= -15$

$$\Delta_3 = \begin{vmatrix} 1 & -2 & 1 \\ 3 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(1-3) + 2(-3-6) + 1(3+2)$
$= -2 - 18 + 5$
$= -15$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{0}{-15} = 0$, $y = \frac{\Delta_2}{\Delta} = \frac{-15}{-15} = 1$

& $z = \frac{\Delta_3}{\Delta} = \frac{-15}{-15} = 1$

$\therefore x = 0, y = 1$ & $z = 1$ Ans.

(xv) $\mathbf{3x + y + z = 10}$
$\mathbf{x + y - z = 0}$
$\mathbf{5x - 9y = 1}$

Solution:

Here, $3x + y + z = 10$
$x + y - z = 0$
$5x - 9y + 0.z = 1$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
3	1	1	10
1	1	–1	0
5	–9	0	1

So,

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 5 & -9 & 0 \end{vmatrix}$$

$= 3(0-9) - 1(0+5) + 1(-9-5)$
$= -27 - 5 - 14$
$= -46$

$$\Delta_1 = \begin{vmatrix} 10 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -9 & 0 \end{vmatrix}$$

$= 10(0-9) - 1(0+1) + 1(0-1)$
$= -90 - 1 - 1$
$= -92$

$$\Delta_2 = \begin{vmatrix} 3 & 10 & 1 \\ 1 & 0 & -1 \\ 5 & 1 & 0 \end{vmatrix}$$

$= 3(0+1) - 10(0+5) + 1(1-0)$
$= 3 - 50 + 1$
$= -46$

$$\Delta_3 = \begin{vmatrix} 3 & 1 & 10 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{vmatrix}$$

$= 3(1-0) - 1(1-0) + 10(-9-5)$
$= 3 - 1 - 140$
$= -138$

We know that, $x = \frac{\Delta_1}{\Delta} = \frac{-92}{-46} = 2$, $y = \frac{\Delta_2}{\Delta} = \frac{-46}{-46} = 1$

& $z = \frac{\Delta_3}{\Delta} = \frac{-138}{-45} = 3$

$\therefore x = 2, y = 1$ & $z = 3$ Ans.

(xvi) $\mathbf{x + y + z = 3}$
$\mathbf{x + 2y + 3z = 4}$
$\mathbf{x + 4y + 9z = 6}$

Solution:

Here, $x + y + z = 3$
$x + 2y + 3z = 4$
$x + 4y + 9z = 6$

Now,

Coeff. of x	Coeff. of y	Coeff. of z	constant
1	1	1	3
1	2	3	4
1	4	9	6

So,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$= 1(18-12) - 1(9-3) + 1(4-2)$
$= 6 - 6 + 2$
$= 2$

$$\Delta_1 = \begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{vmatrix}$$

$= 3(18-12) - 1(36-18) + 1(16-12)$
$= 18 - 18 + 4$
$= 4$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{vmatrix}$$

$= 1(36-18) - 3(9-3) + 1(6-4)$
$= 18 - 18 + 2$
$= 2$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{vmatrix}$$

$= 1(12-16) - 1(6-4) + 3(4-2)$
$= -4 - 2 + 6$
$= 0$

We know that, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{4}{2} = 2$

$y = \dfrac{\Delta_2}{\Delta} = \dfrac{2}{2} = 1$

& $z = \dfrac{\Delta_3}{\Delta} = \dfrac{0}{2} = 0$

$\therefore x = 2, y = 1$ & $z = 0$ Ans.

2. Show that the following equations have only the trivial Solution:

$x + 2y + z + w = 0$
$2x - y - z + 3w = 0$
$3x + y + 4z + w = 0$
$x - 7y + z - 3w = 0$

Solution:

Here, $x + 2y + z + w = 0$
$2x - y - z + 3w = 0$
$3x + y + 4z + w = 0$
$x - 7y + z - 3w = 0$

Coeff. of x	Coeff. of y	Coeff. of z	Coeff. of w	constant
1	2	1	1	0
2	−1	−1	3	0
3	1	4	1	0
1	−7	1	−3	0

So,

Let, $\Delta = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & -1 & 3 \\ 3 & 1 & 4 & 1 \\ 1 & -7 & 1 & -3 \end{vmatrix}$

Applying $C_4 \to C_4 - C_1$, $C_3 \to C_3 - C_1$ & $C_2 \to C_2 - 2C_1$ respectively. We get

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -5 & -3 & 1 \\ 3 & -5 & 1 & -2 \\ 1 & -9 & 0 & -4 \end{vmatrix}$$

Expanding along R_1,

$$= \begin{vmatrix} -5 & -3 & 1 \\ -5 & 1 & -2 \\ -9 & 0 & -4 \end{vmatrix}$$

Expanding along R_3,

$= -9(6-1) - 4(-5-15) = -45 + 80 = 35 \neq 0$

Here $\Delta \neq 0$

Hence equation (i) have only the trivial Solution:

3. Show that the systems
$x + y - 3z = 0$
$3x - y - z = 0$
$2x + y - 4z = 0$ have non-trivial Solution: Also find its Solution:

Solution: Here,

$x + y - 3z = 0$
$3x - y - z = 0$
$2x + y - 4z = 0$

The determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 1 & -3 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{vmatrix}$$

$= 1(4+1) - 1(-12+2) - 3(3+2)$
$= 5 + 10 - 15 = 0.$

$\therefore$ The system has non trivial solution

Taking 1[st] two equations and using cross-multiplication we get,

$$\frac{x}{-1-3} = \frac{y}{-9+1} = \frac{z}{-1-3}$$

or, $\frac{x}{-4} = \frac{y}{-8} = \frac{z}{-4}$

or, $\frac{x}{1} = \frac{y}{2} = \frac{z}{1} = k$ (say)

$x = k$

$y = 2k$

$z = k$

for $k \neq 0$

Which is the non-trivial solution of the system.

Exercise - 3

1. If $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ **&** $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$, **then find.**

(i) $A + B$

Solution: Here $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$

$$\therefore A + B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 0-1 \\ 2+1 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \text{ Ans.}$$

(ii) $B + A$

Solution: Here $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$

$$\therefore B + A = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & -1+0 \\ 1+2 & -2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \text{ Ans.}$$

(iii) $A - B$

Solution: Here $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$

$$\therefore A - B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & 0+1 \\ 2-1 & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \text{ Ans.}$$

(iv) $B - A$

Solution: Here $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$

$$\therefore B - A = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1-2 & -1-0 \\ 1-2 & -2-2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 \\ -1 & -4 \end{bmatrix} \text{ Ans.}$$

(v) $4A - 3B$

Solution: Here $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$

$$\therefore 4A - 3B = 4\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} - 3\begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 8 & 8 \end{bmatrix} - \begin{bmatrix} -3 & -3 \\ 3 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 8+3 & 0+3 \\ 8-3 & 8+6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 3 \\ 5 & 14 \end{bmatrix} \text{ Ans.}$$

(vi) $3B - 2A$

Solution: Here $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$

$$\therefore 3B - 2A = 3\begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3-4 & -3-0 \\ 3-4 & -6-4 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -3 \\ -1 & -10 \end{bmatrix} \text{ Ans.}$$

2. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 0 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 0 & 0 \\ -4 & 3 & 2 \end{bmatrix}$ find,

(i) $2A - 3B$

Solution: Here,

$$2A - 3B = 2\begin{bmatrix} 2 & 3 & 1 \\ 5 & 0 & 0 \\ -4 & 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & 1 \\ 6 & 0 & 0 \\ -4 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 2 \\ 10 & 0 & 0 \\ -8 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 3 \\ 18 & 0 & 0 \\ -12 & 9 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 6-6 & 2-3 \\ 10-18 & 0-0 & 0-0 \\ -8+12 & 4-9 & 2-6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ -8 & 0 & 0 \\ 4 & -5 & -4 \end{bmatrix} \text{ Ans.}$$

(ii) $A - B$

Solution: Here,

$$A - B = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 0 & 0 \\ -4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 6 & 0 & 0 \\ -4 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 3-2 & 1-1 \\ 5-6 & 0-0 & 0-0 \\ -4+4 & 2-3 & 1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \text{ Ans.}$$

(iii) $B - A$

Solution: Here,

$$B - A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 0 & 0 \\ -4 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 1 \\ 5 & 0 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 2-3 & 1-1 \\ 6-5 & 0-0 & 0-0 \\ -4+4 & 3-2 & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ Ans.}$$

(iv) $3B - 2A$

Solution: Here,

$$3B - 2A = 3\begin{bmatrix} 1 & 2 & 1 \\ 6 & 0 & 0 \\ -4 & 3 & 2 \end{bmatrix} - 2\begin{bmatrix} 2 & 3 & 1 \\ 5 & 0 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 3 \\ 18 & 0 & 0 \\ -12 & 9 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 2 \\ 10 & 0 & 0 \\ -8 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & 6-6 & 3-2 \\ 18-10 & 0-0 & 0-0 \\ -12+8 & 9-4 & 6-2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 8 & 0 & 0 \\ -4 & 5 & 4 \end{bmatrix} \text{ Ans.}$$

(v) A + B

Solution: Here,

$$A+B=\begin{bmatrix}2&3&1\\5&0&0\\-4&2&1\end{bmatrix}+\begin{bmatrix}1&2&1\\6&0&0\\-4&3&2\end{bmatrix}$$

$$=\begin{bmatrix}2+1&3+2&1+1\\5+6&0+0&0+0\\-4-4&2+3&1+2\end{bmatrix}$$

$$=\begin{bmatrix}3&5&2\\11&0&0\\-8&5&3\end{bmatrix}\text{ Ans.}$$

3. If $A+B=\begin{bmatrix}1&-1\\3&0\end{bmatrix}$, $A-B\begin{bmatrix}3&1\\1&4\end{bmatrix}$ then calculate the product AB.

Solution: $(A+B)+(A-B)=\begin{bmatrix}1&-1\\3&0\end{bmatrix}+\begin{bmatrix}3&1\\1&4\end{bmatrix}$

or, $A+B+A-B=\begin{bmatrix}1+3&-1+1\\3+1&0+4\end{bmatrix}$

$$=\begin{bmatrix}4&0\\4&4\end{bmatrix}$$

$$\therefore A=\begin{bmatrix}2&0\\2&2\end{bmatrix}$$

Again, $(A+B)-(A-B)=\begin{bmatrix}1&-1\\3&0\end{bmatrix}-\begin{bmatrix}3&1\\1&4\end{bmatrix}$

or, $A+B-A+B=\begin{bmatrix}1-3&-1-1\\3-1&0-4\end{bmatrix}$

$$=\begin{bmatrix}-2&-2\\2&-4\end{bmatrix}$$

or, $2B=\begin{bmatrix}-2&-2\\2&-4\end{bmatrix}$

$$\therefore B=\begin{bmatrix}-1&-1\\1&-2\end{bmatrix}$$

Now, $AB=\begin{bmatrix}2&0\\2&2\end{bmatrix}\begin{bmatrix}-1&-1\\1&-2\end{bmatrix}$

$$=\begin{bmatrix}-2+0&-2+0\\-2+2&-2-4\end{bmatrix}$$

$$=\begin{bmatrix}-2&-2\\0&-6\end{bmatrix}$$

$$\therefore AB=-2\begin{bmatrix}1&1\\0&3\end{bmatrix}\text{ Ans.}$$

4. If $A=\begin{bmatrix}1&2&3&4\\2&0&1&2\\3&1&0&5\end{bmatrix}$ & $B=\begin{bmatrix}2&1&0\\3&2&1\\1&0&1\end{bmatrix}$, then find AB or BA whichever exists.

Solution: Here the no. of column of A 4 no. of row of B is not equal. So, AB doesn't exist. But no. of column of B = no. of row of A. So,

$$BA=\begin{bmatrix}2&1&0\\3&2&1\\1&0&1\end{bmatrix}\begin{bmatrix}1&2&3&4\\2&0&1&2\\3&1&0&5\end{bmatrix}$$

$$=\begin{bmatrix}2+2+0&4+0+0&6+1+0&8+2+0\\3+4+3&6+0+1&9+2+0&12+4+5\\1+0+3&2+0+1&3+0+0&4+0+5\end{bmatrix}$$

$$\therefore BA=\begin{bmatrix}4&4&7&10\\10&7&11&21\\4&3&3&9\end{bmatrix}\text{ Ans.}$$

5. If $A=\begin{bmatrix}3&-3\\4&-4\end{bmatrix}$ & $B=\begin{bmatrix}1&2\\1&2\end{bmatrix}$, then show AB is null matrix through neither A or B zero.

Solution: Here, $A=\begin{bmatrix}3&-3\\4&-4\end{bmatrix}$ & $B=\begin{bmatrix}1&2\\1&2\end{bmatrix}$

$$\therefore AB=\begin{bmatrix}3&-3\\4&-4\end{bmatrix}\begin{bmatrix}1&2\\1&2\end{bmatrix}$$

$$=\begin{bmatrix}3-3&6-6\\4-4&8-8\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

Hence, AB is null matrix through neither A or B zero.

6. If $A=\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}$ & $B=\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}$, from the product of AB. Is BA defined ?

Solution: $AB=\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}$

$$=\begin{bmatrix}0-1+4&0+0-2\\1-2+6&-2+0-3\\2-3+8&-4+0-4\end{bmatrix}$$

$$\therefore AB=\begin{bmatrix}3&-2\\5&-5\\7&0\end{bmatrix}$$

BA doesn't define, because the no. of column of B of the no. of row of A.

7. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ & $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, verify that (AB) C = A (BC) & A (B + C) = AB + AC.

Solution: Here,

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \& C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2+4 & 1+6 \\ -4+6 & -2+9 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -18+14 & 6+0 \\ -6+14 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

$$\& BC = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -6+2 & 2+0 \\ -6+6 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$$A\,(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4+0 & 2+4 \\ 8+0 & -4+6 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

Hence, (AB) C = A (BC)

$$\text{Again, } B + C = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2-3 & 1+1 \\ 2+2 & 3+0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$A\,(B + C) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -1+8 & 2+6 \\ 2+12 & -4+9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -3+4 & 1+0 \\ 6+6 & -2+0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 12 & -2 \end{bmatrix}$$

Now,

$$AB + AC = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix} = \begin{bmatrix} 6+1 & 7+1 \\ 2+12 & 7-2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix}$$

Hence, A (B + C) = AB + AC Ans.

8. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, then find AB & show that AB ≠ BA.

Solution: Given, $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & -1+2 \\ 6+0 & -3+2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & -1 \end{bmatrix}$$

$$\& BA = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2-3 & 4-2 \\ 0+3 & 0+2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix}$$

9. Find the products AB & BA when, $A = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 5 \end{bmatrix}$ & $B = [3 \ 2 \ 5 \ 1]$.

Solution:

$$AB = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 5 \end{bmatrix} [3 \ 2 \ 5 \ 1]$$

$= [9+4+5+5]$
$= [23]$

$\& \; BA = [3 \;\; 2 \;\; 5 \;\; 1]\begin{bmatrix}3\\2\\1\\5\end{bmatrix}$

$= [9+4+5+5]$
$= [23]$

11. Evaluate,

(i) $\begin{bmatrix}2 & 1 & -1\\4 & -5 & 6\\-3 & 7 & 3\end{bmatrix} \times \begin{bmatrix}1 & 3\\4 & -6\\-2 & 5\end{bmatrix} \times \begin{bmatrix}5 & 3\\-2 & 1\end{bmatrix}$

Solution: $\begin{bmatrix}2 & 1 & -1\\4 & -5 & 6\\-3 & 7 & 3\end{bmatrix} \times \begin{bmatrix}1 & 3\\4 & -6\\-2 & 5\end{bmatrix}$

$$= \begin{bmatrix}2+4+2 & 6-5-5\\4-20-12 & 12+30+30\\-3+28-6 & -9-42+15\end{bmatrix}$$

$$= \begin{bmatrix}8 & -5\\-28 & 72\\19 & -36\end{bmatrix}$$

$$\& \begin{bmatrix}2 & 1 & -1\\4 & -5 & 6\\-3 & 7 & 3\end{bmatrix} \times \begin{bmatrix}1 & 3\\4 & -6\\-2 & 5\end{bmatrix} \times \begin{bmatrix}5 & 3\\-2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}8 & -5\\-28 & 72\\19 & -36\end{bmatrix} \times \begin{bmatrix}5 & 3\\-2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}40+10 & 24-5\\-140-144 & -84+72\\95+72 & 57-36\end{bmatrix}$$

$$= \begin{bmatrix}50 & 19\\-284 & -12\\167 & 21\end{bmatrix}$$

(ii) $\begin{bmatrix}1\\-2\\3\end{bmatrix} \times [4 \;\; 5 \;\; 2] \times \begin{bmatrix}2\\-3\\5\end{bmatrix} \times [3 \;\; 2]$

Solution:

$$\begin{bmatrix}1\\-2\\3\end{bmatrix}[4 \;\; 5 \;\; 2] = \begin{bmatrix}4 & 5 & 2\\-8 & -10 & -4\\12 & 15 & 6\end{bmatrix}$$

$$\begin{bmatrix}1\\-2\\3\end{bmatrix} \times [4 \;\; 5 \;\; 2] \times \begin{bmatrix}2\\-3\\5\end{bmatrix} = \begin{bmatrix}4 & 5 & 2\\-8 & -10 & -4\\12 & 15 & 6\end{bmatrix} \times \begin{bmatrix}2\\-3\\5\end{bmatrix}$$

$$= \begin{bmatrix}8-15+10\\-16+30-20\\24-45+30\end{bmatrix} = \begin{bmatrix}3\\-6\\9\end{bmatrix}$$

$$\therefore \begin{bmatrix}1\\-2\\3\end{bmatrix} \times [4 \;\; 5 \;\; 2] \times \begin{bmatrix}2\\-3\\5\end{bmatrix} \times [3 \;\; 2] = \begin{bmatrix}3\\-6\\9\end{bmatrix} \times [3 \;\; 2]$$

$$= \begin{bmatrix}9 & 6\\-18 & -12\\27 & 18\end{bmatrix} \text{ Ans.}$$

(iii) $[x \;\; y \;\; z] \times \begin{bmatrix}a & h & g\\h & b & f\\g & f & c\end{bmatrix} \times \begin{bmatrix}x\\y\\z\end{bmatrix}$

Solution:

$$[x \; y \; z] \times \begin{bmatrix}a & h & g\\h & b & f\\g & f & c\end{bmatrix} = [ax+hy+gz \quad hx+by+fz \quad gx+fy+cz]$$

$$\& \; [ax+hy+gz \quad hx+by+fz \quad gx+fy+cz] \times \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$= [ax^2+hxy+gxz+hxy+by^2+fyz+gxz+fyz+cz^2]$
$= [ax^2+by^2+cz^2+2hxy+2fyz+2gzx]$ Ans.

12. **If** $A = \begin{bmatrix}2 & 1 & 2 & 1\\1 & 1 & 1 & 1\end{bmatrix}$ & $B = \begin{bmatrix}2 & -1 & 0\\0 & 4 & 1\\-2 & 1 & 0\\1 & -3 & 2\end{bmatrix}$ **then find AB & BA.**

Solution:

$$AB = \begin{bmatrix}2 & 1 & 2 & 1\\1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\0 & 4 & 1\\-2 & 1 & 0\\1 & -3 & 2\end{bmatrix}$$

$$= \begin{bmatrix}4+0-4+1 & -2+4+2-3 & 0+1+0+2\\2+0-2+1 & -1+4+1-3 & 0+1+0+2\end{bmatrix} = \begin{bmatrix}1 & 1 & 3\\1 & 1 & 3\end{bmatrix}$$

& BA doesn't exist.

13. **If** $A = \begin{bmatrix}1 & 1 & -1\\2 & -3 & 4\\3 & -2 & 3\end{bmatrix}$ & $B = \begin{bmatrix}-1 & -2 & -1\\6 & 12 & 6\\5 & 10 & 15\end{bmatrix}$ **then show that AB is null matrix but BA is not null matrix.**

Solution:

$$AB = \begin{bmatrix}1 & 1 & -1\\2 & -3 & 4\\3 & -2 & 3\end{bmatrix}\begin{bmatrix}-1 & -2 & -1\\6 & 12 & 6\\5 & 10 & 15\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Hence AB is null matrix.

$$BA = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 15 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1-4-3 & -1+6+2 & 1-8-3 \\ 6+24+18 & 6-36-12 & -6+48+18 \\ 5+20+45 & 5-30-30 & -5+40+45 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 8 & 10 \\ 48 & -42 & 60 \\ 70 & -55 & 80 \end{bmatrix}$$

Hence BA is not null matrix.

14. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ from the product AB & BA & show that $AB \neq BA$.

Solution:

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6-0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$\& \ BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-6 & -2+0+2 & 3+0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

Hence $AB \neq BA$.

15. If A & B are matrices, then under what conditions does the rule $A^2 - B^2 = (A+B)(A-B)$ hold ?

Solution: $A^2 - B^2 = (A+B)(A-B)$ hold only if $AB = BA$ (commutative property)

16. Evaluate $A^2 + 2A + I$, where I is 2×2 unit matrix & $A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$.

Solution:

$$A^2 = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25+2 & 10+6 \\ 5+3 & 2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 16 \\ 8 & 11 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 4 \\ 2 & 6 \end{bmatrix} \ \& \ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^2 + 2A + I = \begin{bmatrix} 27 & 16 \\ 8 & 11 \end{bmatrix} + \begin{bmatrix} 10 & 4 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 27+10+1 & 16+4+0 \\ 8+2+0 & 11+6+1 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & 20 \\ 10 & 18 \end{bmatrix} \text{ Ans.}$$

17. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$, verify that $(A+B)^2 = A^2 + BA + AB + B^2$.

Solution:

$$A + B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 4 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 16+2+10 & 4+0+0 & 0+5+0 \\ 8+0+20 & 2+0-10 & 0+0+20 \\ 16-4+16 & 4+0-8 & 0-10+16 \end{bmatrix} = \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}$$

Again,

$$A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+4+0 & 2+0-1 & -1+6-2 \\ 2+0+0 & 4+0+3 & -2+0+6 \\ 0+2+0 & 0+0+2 & 0+3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 & 3 \\ 2 & 7 & 4 \\ 2 & 2 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3-2+0 & 0+0+1 & -3-3+2 \\ 0+0+0 & 0+0+2 & 0+0+4 \\ 4-6+0 & 8+0+2 & -4-9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 & -4 \\ 0 & 2 & 4 \\ -2 & 10 & -9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0-4 & -1+0+3 & 1+4-2 \\ 6+0+12 & -2+0-9 & 2+0+6 \\ 0+0+8 & 0+0-6 & 0+2+4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 18 & -11 & 8 \\ 8 & -6 & 6 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9+0+4 & -3+0-3 & 3-2+2 \\ 0+0+8 & 0+0-6 & 0+0+4 \\ 12+0+8 & -4+0-6 & 4-6+4 \end{bmatrix} = \begin{bmatrix} 13 & -6 & 3 \\ 8 & -6 & 4 \\ 20 & -10 & 2 \end{bmatrix}$$

$$A^2 + BA + AB + B^2 = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 7 & 4 \\ 2 & 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 7 & -4 \\ 0 & 2 & 4 \\ -2 & 10 & -9 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & 2 & 3 \\ 18 & -11 & 8 \\ 8 & -6 & 6 \end{bmatrix} + \begin{bmatrix} 13 & -6 & 3 \\ 8 & -6 & 4 \\ 20 & -10 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5+1-1+13 & 1+7+2-6 & 3-4+3+3 \\ 2+0+18+8 & 7+2-11-6 & 4+4+8+4 \\ 2-2+8+20 & 2+10-6-10 & 7-9+6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & 5 \\ 28 & -8 & 20 \\ 28 & -4 & 6 \end{bmatrix}$$

Hence, $(A + B)^2 = A^2 + BA + AB + B^2$ Ans.

18. **If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ & I is the unit matrix of order 3, evaluate $A^2 - 3A + 9I$.**

Solution:

$$A^2 = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4-9 & -2-6+3 & 3+2+6 \\ 2+6+3 & -4+9-1 & 6-3-2 \\ -3+2-6 & 6+3+2 & -9-1+4 \end{bmatrix} = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix}$$

$$\& \; 9I = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\therefore A^2 - 3A + 9I = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -12-3+9 & -5+6+0 & 11-9+0 \\ 11-6+0 & 4-9+9 & 1+3+0 \\ -7+9+0 & 11-3+0 & -6-6+9 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$$

19. **If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$, where I is square unit matrix.**

Solution:

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} \; \& \; 7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A^2 - 5A + 7I = 0$ Ans.

20. **Find the value of x, y & z if,**

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 0 \\ -1 & 2 & 3 \end{bmatrix}, \; x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; \& \; AX = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

Solution:

$$AX = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 5 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} 2x+3y- & z \\ x+5y+ & 0.z \\ -x+2y+ & 3z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$$

Equating the corresponding coefficients we get,

$2x + 3y - z = 4$ .... (i)

$x + 5y = 6$ .... (ii)

$-x + 2y + 3z = 4$ .... (iii)

Solving (ii) & (iii),

$$\begin{array}{r} x + 5y = 6 \\ -x + 2y + 3z = 4 \\ \hline 7y + 3z = 10 \end{array} \quad \text{..... (iv)}$$

Solving (i) & (iii),

$$\begin{array}{r} 2x + 3y - z = 4 \\ -2x + 4y + 6z = 8 \\ \hline 7y + 5z = 12 \end{array} \quad \text{...... (v)}$$

Solving (iv) & (v),

$$\begin{array}{r} 7y + 3z = 10 \\ 7y + 5z = 12 \\ - \quad - \quad\quad - \\ \hline -2z = -2 \end{array} \qquad \therefore z = 1$$

Putting the value of y in (iv),

$7y + 3.1 = 10$

$\therefore y = \dfrac{10-3}{7} = 1$

Putting the value of y in (ii),

$x + 5.1 = 6$

$x = 6 - 5 = 1$

$\therefore x = 1, y = 1$ & $z = 1$ Ans.

Exercise - 4

1. **If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA^T = I = A^TA$ where I is the unit matrix.**

Solution:

Given, $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

So, $A^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\& \ A^TA = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore AA^T = I = A^TA$ Ans.

2. **If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)^T = B^TA^T$.**

Solution:

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+4+0 & 0+2-1 & 0+0-3 \\ 3+0+0 & 0+0+2 & 0+0+6 \\ 4+10+0 & 0+5+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix}$$

$$\therefore (AB)^T = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix}$$

$$\& \ A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 3+0+0 & 4+10+0 \\ 0+2-1 & 0+0+2 & 0+5+0 \\ 0+0-3 & 0+0+6 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix}$$

Hence, $(AB)^T = B^TA^T$ Ans.

3. **If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$, compute Adj (A) & prove that A Adj (A) = Adj (A) A = |A| I.**

Solution:

Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}$$

$= 1(36-25) - 1(24-15) + 1(10-9)$

$= 11 - 9 + 1$

$= 3$

$A_{11} = \text{Cofactor of } a_{11} = \begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} = 36 - 25 = 1$

$A_{12} = \text{Cofactor of } a_{12} = -\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -(12-5) = -7$

$A_{13} = \text{Cofactor of } a_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = (5-3) = 2$

$A_{21} = \text{Cofactor of } a_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} = -(24-15) = -9$

$A_{22} = \text{Cofactor of } a_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 12 - 3 = 9$

$A_{23} = \text{Cofactor of } a_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -(5-2) = -3$

$A_{31} = \text{Cofactor of } a_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1$

$A_{32} = \text{Cofactor of } a_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -(5-3) = -2$

$A_{33} = \text{Cofactor of } a_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$

$\therefore$ Cofactor of $A = \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{bmatrix}$ & $\text{Adj}(A) = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

$$A\,\text{Adj}(A) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11-14+6 & -9+18-9 & 1-4+3 \\ 11-21+10 & -9+27-15 & 1-6+5 \\ 11-35+24 & -9+45-36 & 1-10+12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Again,

$$\text{Adj}(A)\,A = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 11-9+1 & 22-27+5 & 33-45+12 \\ -7+9-2 & -14+27-10 & -21+45-24 \\ 2-3+1 & 4-9+5 & 6-15+12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I$$

Hence, A Adj (A) = Adj (A) A = |A| I Ans.

4. **Find the adjoint of,**

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

$A_{11} = \text{Cofactor of } a_{11} = \begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} = 15 - 0 = 15$

$A_{12} = \text{Cofactor of } a_{12} = -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = -(0+0) = 0$

$A_{13} = \text{Cofactor of } a_{13} = \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = 0 - 10 = -10$

$A_{22} = \text{Cofactor of } a_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3 - 6 = -3$

$A_{23} = \text{Cofactor of } a_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = -(4-4) = 0$

A_{31} = Cofactor of $a_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = 0 - 15 = -15$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5 - 0 = 5$

$\therefore$ Cofactor of $A = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$

& Adj $(A) = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

A_{11} = Cofactor of $a_{11} = \begin{vmatrix} 1 & 1 \\ -5 & 2 \end{vmatrix} = 2 + 5 = 7$

A_{12} = Cofactor of $a_{12} = -\begin{vmatrix} -2 & 1 \\ -4 & 2 \end{vmatrix} = -(-4+4) = 0$

A_{13} = Cofactor of $a_{13} = \begin{vmatrix} -2 & 1 \\ -4 & -5 \end{vmatrix} = 10 + 4 = 14$

A_{21} = Cofactor of $a_{21} = -\begin{vmatrix} -2 & 3 \\ -5 & 2 \end{vmatrix} = -(-4+15) = -11$

A_{22} = Cofactor of $a_{22} = \begin{vmatrix} -1 & 3 \\ -4 & 2 \end{vmatrix} = -2 + 12 = 10$

A_{23} = Cofactor of $a_{23} = -\begin{vmatrix} -1 & -2 \\ -4 & -5 \end{vmatrix} = -(5-8) = 3$

A_{31} = Cofactor of $a_{31} = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = -2 - 3 = -5$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} -1 & 3 \\ -2 & 1 \end{vmatrix} = -(-1+6) = -5$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix} = -1 - 4 = -5$

$\therefore$ Cofactor of $A = \begin{bmatrix} 7 & 0 & 14 \\ -11 & 10 & 3 \\ -5 & -5 & -5 \end{bmatrix}$ & Adj $(A) = \begin{bmatrix} 7 & 11 & -5 \\ 0 & 10 & -5 \\ 14 & 3 & -5 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

A_{11} = Cofactor of $a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

A_{12} = Cofactor of $a_{12} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -(1-0) = 1$

A_{13} = Cofactor of $a_{13} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

A_{21} = Cofactor of $a_{21} = -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = -(0-0) = 0$

A_{22} = Cofactor of $a_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$

A_{23} = Cofactor of $a_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = -(0-0) = 0$

A_{31} = Cofactor of $a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = -(0-0) = 0$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1 - 0 = 1$

$\therefore$ Cofactor of $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ & Adj $(A) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ Ans.

(iv) $\begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 2 \\ -2 & 1 & 4 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 2 \\ -2 & 1 & 4 \end{bmatrix}$

A_{11} = Cofactor of $a_{11} = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 8 - 2 = 6$

A_{12} = Cofactor of $a_{12} = -\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = -(4+4) = -8$

A_{13} = Cofactor of $a_{13} = \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = 1 \cdot 4 = 5$

A_{21} = Cofactor of $a_{21} = -\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} = -(12 + 1) = -13$

A_{22} = Cofactor of $a_{22} = \begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix} = 4 - 2 = 2$

A_{23} = Cofactor of $a_{23} = -\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = -(1 + 6) = -7$

A_{31} = Cofactor of $a_{31} = \begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} = 6 + 2 = 8$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = -(2 + 1) = -3$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1$

$\therefore$ Cofactor of A = $\begin{bmatrix} 6 & -8 & 5 \\ -13 & 2 & -7 \\ 8 & -3 & -1 \end{bmatrix}$

& Adj (A) = $\begin{bmatrix} 6 & -13 & 8 \\ -8 & 2 & -3 \\ 5 & -7 & -1 \end{bmatrix}$ Ans.

5. Find the inverse of,

(i) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

Solution: Let, A = $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{vmatrix} = 1(-1 - 8) - 2(-8 + 3) = -9 + 10 = 1$

A_{11} = Cofactor of $a_{11} = \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} = -1 - 8 = -9$

A_{12} = Cofactor of $a_{12} = -\begin{vmatrix} 0 & 4 \\ -2 & 1 \end{vmatrix} = -(0 + 8) = -8$

A_{13} = Cofactor of $a_{13} = \begin{vmatrix} 0 & -1 \\ -2 & 2 \end{vmatrix} = 0 - 2 = -2$

A_{21} = Cofactor of $a_{21} = -\begin{vmatrix} -2 & 3 \\ 2 & 2 \end{vmatrix} = -(-2 - 6) = 8$

A_{22} = Cofactor of $a_{22} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 + 6 = 7$

A_{23} = Cofactor of $a_{23} = -\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = -(2 - 4) = 2$

A_{31} = Cofactor of $a_{31} = \begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix} = -8 + 3 = -5$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -(4 - 0) = -4$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} = -1 + 0 = -1$

$\therefore$ Cofactor of A = $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

& Adj (A) = $\begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$

We know that,

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{1}{1}\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix} = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix} \text{ Ans.}$$

(ii) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Solution:

Let, A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$

$= 1(18 - 12) - 1(18 - 3) + 1(8 - 2)$
$= 6 - 15 + 6$
$= -3$

A_{11} = Cofactor of $a_{11} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$

A_{12} = Cofactor of $a_{12} = -\begin{vmatrix} 2 & 3 \\ 1 & 9 \end{vmatrix} = -(18 - 3) = -15$

A_{13} = Cofactor of $a_{13} = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 8 - 2 = 6$

A_{21} = Cofactor of $a_{21} = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9 - 4) = -5$

A_{22} = Cofactor of $a_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9 - 1 = 8$

A_{23} = Cofactor of $a_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$

A_{31} = Cofactor of $a_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -(3-2) = -1$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$

$\therefore$ Cofactor of A $= \begin{bmatrix} -6 & -15 & -6 \\ -5 & 8 & -3 \\ 1 & -1 & 0 \end{bmatrix}$

& Adj (A) $= \begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}$

We know that,

$A^{-1} = \frac{\text{Adj}(A)}{|A|} = -\frac{1}{3}\begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & 5 & -1 \\ 15 & -8 & 1 \\ -6 & 3 & 0 \end{bmatrix}$ Ans.

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solution:

Let A $= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(1-0) = 1$

A_{11} = Cofactor of $a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

A_{12} = Cofactor of $a_{12} = -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -(1-0) = 1$

A_{13} = Cofactor of $a_{13} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

A_{21} = Cofactor of $a_{21} = -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = -(0-0) = 0$

A_{22} = Cofactor of $a_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$

A_{23} = Cofactor of $a_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$

A_{31} = Cofactor of $a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$

$\therefore$ Cofactor of A $= \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

& Adj (A) $= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

We know that,

$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ Ans.

(iv) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution:

Let A $= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix}$

$= 1(-12-12) - 1(-4+12) - 2(-3-9)$
$= -24 - 8 + 24$
$= -8$

A_{11} = Cofactor of $a_{11} = \begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix} = -12 - 12 = -24$

A_{12} = Cofactor of $a_{12} = -\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} = -(-4-6) = 10$

A_{13} = Cofactor of $a_{13} = \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = (-4+6) = 2$

A_{21} = Cofactor of $a_{21} = \begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -(-4+12) = -8$

A_{22} = Cofactor of $a_{22} = \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = -4 + 6 = 2$

A_{23} = Cofactor of $a_{23} = -\begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = -(-4+2) = 2$

$$A_{31} = \text{Cofactor of } a_{31} = \begin{vmatrix} 1 & 3 \\ 3 & -3 \end{vmatrix} = -3-9 = -12$$

$$A_{32} = \text{Cofactor of } a_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = -(-3-3) = 6$$

$$A_{33} = \text{Cofactor of } a_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 3-1 = 2$$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$$

$$\& \text{ Adj}(A) = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

We know that,

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{-1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & 1 & -1 \end{bmatrix} \text{ Ans.}$$

(v) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 2(1-4) - 3(5-6) + 1(10-3)$$

$$= -6 + 3 + 7 = 4$$

$$A_{11} = \text{Cofactor of } a_{11} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3$$

$$A_{12} = \text{Cofactor of } a_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -(3-2) = -1$$

$$A_{13} = \text{Cofactor of } a_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 6-1 = 5$$

$$A_{21} = \text{Cofactor of } a_{21} = -\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = -(5-6) = 1$$

$$A_{22} = \text{Cofactor of } a_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = -1$$

$$A_{23} = \text{Cofactor of } a_{23} = -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = -(4-5) = 1$$

$$A_{31} = \text{Cofactor of } a_{31} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10-3 = 7$$

$$A_{32} = \text{Cofactor of } a_{32} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -(4-9) = 5$$

$$A_{33} = \text{Cofactor of } a_{33} = \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = 2-15 = -13$$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$$

$$\& \text{ Adj}(A) = \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

We know that,

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix} \text{ Ans.}$$

(vi) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 0 - 1(1-9) + 2(1-6) = 8 - 10 = -2$$

$$A_{11} = \text{Cofactor of } a_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = -1$$

$$A_{12} = \text{Cofactor of } a_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -(1-9) = 8$$

$$A_{13} = \text{Cofactor of } a_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = -5$$

$$A_{21} = \text{Cofactor of } a_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1-2) = 1$$

$$A_{22} = \text{Cofactor of } a_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = 0-6 = -6$$

$$A_{23} = \text{Cofactor of } a_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -(0-3) = 3$$

$$A_{31} = \text{Cofactor of } a_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3-4 = -1$$

A_{32} = Cofactor of $a_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(0-2) = 2$

A_{33} = Cofactor of $a_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 0 - 1 = -1$

$\therefore$ Cofactor of A = $\begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

& Adj (A) = $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$

We know that,

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{-1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & -1 \end{bmatrix} \text{ Ans.}$$

6. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, **verify that** $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:

$$AB = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+2 & 2+3-2 & 0-1+6 \\ 1+18+3 & 2+27-3 & 0-9+9 \\ 1+8+2 & 2+12-2 & 0-4+6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & 5 \\ 22 & 26 & 0 \\ 11 & 12 & 2 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 5 & 3 & 5 \\ 22 & 26 & 0 \\ 11 & 12 & 2 \end{vmatrix}$$

$= 5(52-0) - 3(44-0) + 5(264-286)$

$= 260 - 132 - 110$

$= 18$

For cofactor of AB,

A_{11} = Cofactor of $a_{11} = 52 - 0 = 52$

A_{12} = Cofactor of $a_{12} = -(44-0) = -44$

A_{13} = Cofactor of $a_{13} = 264 - 286 = -22$

A_{21} = Cofactor of $a_{21} = -(6-60) = 54$

A_{22} = Cofactor of $a_{22} = 10 - 55 = -45$

A_{23} = Cofactor of $a_{23} = -(60-33) = -27$

A_{31} = Cofactor of $a_{31} = 0 - 130 = -130$

A_{32} = Cofactor of $a_{32} = -(0-110) = 110$

A_{33} = Cofactor of $a_{33} = 130 - 66 = 64$

$\therefore$ Cofactor of AB = $\begin{bmatrix} 52 & -44 & -22 \\ 54 & -45 & -27 \\ -130 & 110 & 64 \end{bmatrix}$

& Adj (AB) = $\begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 64 \end{bmatrix}$

We know that,

$$(AB)^{-1} = \frac{\text{Adj}(AB)}{|AB|} = \frac{1}{18}\begin{bmatrix} 52 & 54 & -130 \\ 44 & -45 & 110 \\ -22 & -27 & 64 \end{bmatrix}$$

For cofactor of A,

A_{11} = Cofactor of $a_{11} = 18 - 12 = 6$

A_{12} = Cofactor of $a_{12} = -(2-3) = 1$

A_{13} = Cofactor of $a_{13} = 4 - 9 = -5$

A_{21} = Cofactor of $a_{21} = -(2-8) = 6$

A_{22} = Cofactor of $a_{22} = 2 - 2 = 0$

A_{23} = Cofactor of $a_{23} = -(4-1) = -3$

A_{31} = Cofactor of $a_{31} = 3 - 18 = -15$

A_{32} = Cofactor of $a_{32} = -(3-2) = -1$

A_{33} = Cofactor of $a_{33} = 9 - 1 = 8$

$\therefore$ Cofactor of A = $\begin{bmatrix} 6 & 1 & -5 \\ 6 & 0 & -3 \\ -15 & -1 & 8 \end{bmatrix}$

& Adj (A) = $\begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{vmatrix}$$

$= 1(18-12) - 1(2-8) + 1(3-18)$

$= 6 + 6 - 15$

$= -3 \neq 0$

$$\therefore A^{-1} = \frac{\text{Ad}_j(A)}{|A|}$$

$$= -\frac{1}{3}\begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

Again, for cofactor of B,

B_{11} = Cofactor of $b_{11} = 9 - 1 = 8$

B_{12} = Cofactor of $b_{12} = -(6+1) = -7$

B_{13} = Cofactor of $b_{13} = -2 - 3 = -5$

B_{21} = Cofactor of $b_{21} = -(6-0) = -6$

B_{22} = Cofactor of $b_{22} = 3 - 0 = 3$

B_{23} = Cofactor of $b_{23} = -(-1-2) = 3$
B_{31} = Cofactor of $b_{31} = -2-0 = -2$
B_{32} = Cofactor of $b_{32} = -(-1-0) = 1$
B_{33} = Cofactor of $b_{33} = 3-4 = -1$

$$\therefore \text{Cofactor of } B = \begin{bmatrix} 8 & -7 & -5 \\ -6 & 3 & 3 \\ -2 & 1 & -1 \end{bmatrix}$$

$$\& \text{ Adj}(B) = \begin{bmatrix} 8 & -6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 1(9-1) - 2(6+1) = 8 - 14 = -6 \neq 0$$

$$\therefore B^{-1} = \frac{\text{Adj}(B)}{|B|} = -\frac{1}{6}\begin{bmatrix} 8 & -6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}$$

And,

$$B^{-1}A^{-1} = -\frac{1}{6}\begin{bmatrix} 8 & -6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix} \times -\frac{1}{3}\begin{bmatrix} 6 & 6 & -15 \\ 1 & 0 & -1 \\ -5 & -3 & 8 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 48-6+10 & 48-0+6 & -120+6-16 \\ -42+3-5 & -42+0-3 & 105-3+8 \\ -30+3+5 & -30+0+3 & 75-3-8 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 52 & 54 & -130 \\ -44 & -45 & 110 \\ -22 & -27 & 64 \end{bmatrix}$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$ Hence verify.

7. **Find the inverse of the following matrices & also verify that $AA^{-1} = I$**

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix}$$

$= 1(36-25) - 1(24-15) + 1(10-9)$
$= 11 - 9 + 1$
$= 3$

A_{11} = Cofactor of $a_{11} = 36 - 25 = 11$

A_{12} = Cofactor of $a_{12} = -(12-5) = -7$
A_{13} = Cofactor of $a_{13} = 5-3 = 2$
A_{21} = Cofactor of $a_{21} = -(24-15) = -9$
A_{22} = Cofactor of $a_{22} = 12-3 = 9$
A_{23} = Cofactor of $a_{23} = -(5-2) = -3$
A_{31} = Cofactor of $a_{31} = 10-9 = 1$
A_{32} = Cofactor of $a_{32} = -(5-3) = -2$
A_{33} = Cofactor of $a_{33} = 3-2 = 1$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\& \text{ Adj}(A) = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\text{Now, } AA^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 11-14+6 & -9+18-9 & 1-2+3 \\ 11-21+10 & -9+27-15 & 1-6+5 \\ 11-35+24 & -9+45-36 & 1-10+12 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{-1} = I$ Hence verify.

(ii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\therefore |A| = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$= 1(16-9) - 1(12-9) + 1(9-12)$
$= 7 - 3 - 3 = 1$

A_{11} = Cofactor of $a_{11} = 16-9 = 7$
A_{12} = Cofactor of $a_{12} = -(4-3) = -1$
A_{13} = Cofactor of $a_{13} = 3-4 = -1$
A_{21} = Cofactor of $a_{21} = -(12-9) = -3$

A_{22} = Cofactor of $a_{22} = 4-3 = 1$
A_{23} = Cofactor of $a_{23} = -(3-0) = 0$
A_{31} = Cofactor of $a_{31} = 9-12 = -3$
A_{32} = Cofactor of $a_{32} = -(3-3) = 0$
A_{33} = Cofactor of $a_{33} = 4-3 = 1$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\& \text{ Adj}(A) = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore AA^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{-1} = I$ Hence verify

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= 1(-2-2) - 2(3-4) + 3(6+8)$$

$$= -4 + 2 + 42 = 40$$

A_{11} = Cofactor of $a_{11} = -2-2 = -4$
A_{12} = Cofactor of $a_{12} = -(3-4) = 1$
A_{13} = Cofactor of $a_{13} = 6+8 = 14$
A_{21} = Cofactor of $a_{21} = -(2-6) = 4$
A_{22} = Cofactor of $a_{22} = 1-12 = -11$
A_{23} = Cofactor of $a_{23} = -(2-8) = 6$

A_{31} = Cofactor of $a_{31} = 2+6 = 8$
A_{32} = Cofactor of $a_{32} = -(1-9) = 8$
A_{33} = Cofactor of $a_{33} = -2-6 = -8$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} -4 & 1 & 14 \\ 4 & -11 & 6 \\ 8 & 8 & -8 \end{bmatrix}$$

$$\& \text{ Adj}(A) = \begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

$$AA^{T} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \times \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} -4+2+42 & 4-22+18 & 8+16-24 \\ -12-2+14 & 12+22+6 & 24-16-8 \\ -16+2+14 & 16-22+6 & 32+16-8 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \text{ Ans.}$$

8. **If** $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$**, find** A^{-1} **& show that** $A^{3} = A^{-1}$**.**

Solution:

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) - 2(-3+4) = 3-2 = 1$$

A_{11} = Cofactor of $a_{11} = -3+4 = 1$
A_{12} = Cofactor of $a_{12} = -(2-0) = -2$
A_{13} = Cofactor of $a_{13} = -2+0 = -2$
A_{21} = Cofactor of $a_{21} = -(-3+4) = -1$
A_{22} = Cofactor of $a_{22} = 3-0 = 3$
A_{23} = Cofactor of $a_{23} = -(-3+0) = 3$
A_{31} = Cofactor of $a_{31} = -12+12 = 0$
A_{32} = Cofactor of $a_{32} = -(12-8) = -4$
A_{33} = Cofactor of $a_{33} = -9+6 = -3$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$\& \text{ Adj}(A) = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{1}{1}\begin{bmatrix}1&-1&0\\-2&3&-4\\-2&3&-3\end{bmatrix} = \begin{bmatrix}1&-1&0\\-2&3&-4\\-2&3&-3\end{bmatrix}$$

Again,

$$A^2 = \begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$$

$$= \begin{bmatrix}9-6+0 & -9+9-4 & 12-12+4\\6-6+0 & -6+9-4 & 8-12+4\\0-2+0 & 0+3-1 & 0-4+1\end{bmatrix} = \begin{bmatrix}3&-4&4\\0&-1&0\\-2&2&-3\end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix}3&-4&4\\0&-1&0\\-2&2&-3\end{bmatrix}\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$$

$$= \begin{bmatrix}9-8+0 & -9+12-4 & 12-16+4\\0-2+0 & 0+3+0 & 0-4+0\\-6+4+0 & 6-6+3 & -8+8-3\end{bmatrix}$$

$$= \begin{bmatrix}1&-1&0\\-2&3&-4\\-2&3&-3\end{bmatrix} = A^{-1}$$

Hence $A^3 = A^{-1}$ Ans.

9. **Express each of the following matrices as the sum of symmetric & skew-symmetric matrix.**

(i) $\begin{bmatrix}3&-2&6\\2&7&-1\\5&4&0\end{bmatrix}$

Solution:

Let $A = \begin{bmatrix}3&-2&6\\2&7&-1\\5&4&0\end{bmatrix}$

So that, $A^T = \begin{bmatrix}3&2&5\\-2&7&4\\6&-1&0\end{bmatrix}$ then,

$$A + A^T = \begin{bmatrix}3&-2&6\\2&7&-1\\5&4&0\end{bmatrix} + \begin{bmatrix}3&2&5\\-2&7&4\\6&-1&0\end{bmatrix} = \begin{bmatrix}8&-4&1\\4&0&-5\\-1&5&0\end{bmatrix}$$

Which is symmetric.

And,

$$A - A^T = \begin{bmatrix}3&-2&6\\2&7&-1\\5&4&0\end{bmatrix} - \begin{bmatrix}3&2&5\\-2&7&4\\6&-1&0\end{bmatrix} = \begin{bmatrix}0&-4&1\\4&0&-5\\-1&5&0\end{bmatrix}$$

Which is skew symmetric.

$$\therefore\ A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\begin{bmatrix}6&0&11\\0&14&3\\11&3&0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&-4&1\\4&0&-5\\-1&5&0\end{bmatrix}$$

$$= \begin{bmatrix}3&0&5.5\\0&7&1.5\\5.5&1.5&0\end{bmatrix} + \begin{bmatrix}0&-2&0.5\\2&0&-2.5\\-0.5&2.5&0\end{bmatrix} \text{ Ans.}$$

(ii) $\begin{bmatrix}a&a&b\\c&b&b\\c&a&c\end{bmatrix}$

Solution:

Let $A = \begin{bmatrix}a&a&b\\c&b&b\\c&a&c\end{bmatrix}$ So that $A^T = \begin{bmatrix}a&c&c\\a&b&a\\b&c&c\end{bmatrix}$

$$A + A^T = \begin{bmatrix}a&a&b\\c&b&b\\c&a&c\end{bmatrix} + \begin{bmatrix}a&c&c\\a&b&a\\b&b&c\end{bmatrix} = \begin{bmatrix}2a&a+c&b+c\\c+a&2b&b+a\\c+b&a+b&2c\end{bmatrix}$$

Which is symmetric.

$$\& \ A - A^T = \begin{bmatrix}a&a&b\\c&b&b\\c&a&c\end{bmatrix} - \begin{bmatrix}a&c&c\\a&b&a\\b&b&c\end{bmatrix} = \begin{bmatrix}0&a-c&b-c\\c-a&0&b-a\\c-b&a-b&0\end{bmatrix}$$

Which is skew symmetric.

$$\therefore\ A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\begin{bmatrix}2a&a+c&b+c\\c+a&2b&b+a\\c+b&a+b&2c\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0&a-c&b-c\\c-a&0&b-a\\c-b&a-b&0\end{bmatrix} \text{ Ans.}$$

10. **If $A = \frac{1}{3}\begin{bmatrix}1&2&2\\2&1&-2\\-2&2&1\end{bmatrix}$ prove that A is orthogonal matrix.**

Solution:

$A = \frac{1}{3}\begin{bmatrix}1&2&2\\2&1&-2\\-2&2&1\end{bmatrix}$. So that $A^T = \frac{1}{3}\begin{bmatrix}1&2&-2\\2&1&2\\2&-2&-1\end{bmatrix}$

$$AA^T = \frac{1}{3}\begin{bmatrix}1&2&2\\2&1&-2\\-2&2&1\end{bmatrix} \times \frac{1}{3}\begin{bmatrix}1&2&-2\\2&1&-2\\2&-2&-1\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}1+4+4 & 2+2-4 & -2+4-2\\2+2-4 & 4+1+4 & -4+2+2\\-2+4-2 & -4+2+2 & 4+4+1\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Hence, A is an orthogonal matrix.

11. **Square matrix A is ... the matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.**

Solution:

Let $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+8 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, $A^2 = A$, So A is idempotent.

12. **If A is square matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 5 \end{bmatrix}$, then prove that $A + A^T$ is symmetric & $A - A^T$ is skew symmetric.**

Solution:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 5 \end{bmatrix}$ So that $A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 3 & 5 & 5 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 3 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 0 & 8 \\ 4 & 8 & 10 \end{bmatrix}$$

Which is skew-symmetric matrix.

13. **Prove that the matrix $\begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ is hermitian.**

Solution:

Let $A = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ then $\bar{A} \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{bmatrix}$

Now the hermitian of A is,

$$A^* = (\bar{A})^T = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix} = A$$

$\therefore A^* = A$. Hence, the matrix A is hermitian.

14. **If A is any square matrix, prove that $A + A^*$ is Hermitian & $A - A^*$ is skew Hermitian.**

Solution: Given A is any square matrix. Then,

$(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*$

$\therefore (A + A^*) = A + A^*$

Hence, $A + A^*$ is Hermitian.

Again, $(A - A^*)^* = A^* - (A^*)^* = A^* - A = -(A - A^*)$

$\therefore (A - A^*)^* = -(A - A^*)$

Hence, $A - A^*$ is skew Hermitian.

15. **If A is Hermitian matrix, then show that i A is a skew - Hermitian matrix.**

Solution: Given A is Hermitian matrix. So $A^* = A$.

Now, $(i A)^* = -\bar{i} A^* = -i A$

$\therefore (i A)^* = -A$

Hence, i A is a skew Hermitian.

16. **Given that $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I - A)(I + A)^{-1}$ is a unitdr matrix.**

Solution: $I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$ and $|I + A| = 1 - (2i + 1)(2i - 1)$

$= 1 + 4 + 1 = 6$

Adj (I + A) = Transpose of the matrix of cofactors,

$= \begin{bmatrix} 1 & 1-2i \\ -1-2i & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1-2i \\ -1-2i & 1 \end{bmatrix}$

$\therefore (I + A)^{-1} = \dfrac{\text{Adj}(I+A)}{|A|}$

17. **Show that the matrix $B^* AB$ is Hermitian or skew Hermitian according as A is Hermitian or skew Hermitian.**

Solution: $(B^* AB)^* = B^* (B^*A)^* = B^*A^* (B^*)^* = B^*AB$

If A is Hermitian then $A^* = A$

If A is Hermitian then $A^* = A$

So, $(B^* AB)^* = B^* AB$

Hence, B^*AB is Hermitian according as A is Hermitian.

Again, $(B^* AB)^* = B^* (B^*A)^* = B^*A^*(B^*)^* = B^*A^*B$

If A is skew - symmetric then $A^* = -A$

So, $(B^* AB)^* = B^* (-A) B = -B^* AB$

Hence, B^*AB is skew Hermitian according as A is skew Hermitian.

Exercise - 5

Find the rank of the following matrices.

1. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Minor of order 1 are, $|1| = 1$ & $|1| = 1$
Which are non-equal to zero. He ce rank of A = 1

2. $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

The only one minor of order 2 is $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = 3 - 4 = -1 \neq 0$

Hence, rank of A = 2

3. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

The only one minor of order 3 is, $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{vmatrix}$

$= 1\,(20 - 12) - 1\,(10 - 18) + 2\,(4 - 12)$
$= 8 + 8 - 16 = 0$

The minor of A of order 2 is $\begin{vmatrix} 4 & 2 \\ 6 & 5 \end{vmatrix} = 20 - 12 = 8 \neq 0$

Hence, rank of A = 2

4. $\begin{bmatrix} 3 & -1 & -2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} 3 & -1 & -2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

The only one minor of A of order 3 is,

$\begin{vmatrix} 3 & -1 & -2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix} = 3\,(4 - 4) + 1\,(-12 + 12) - 2\,(-6 + 6) = 0$

The minor of order 2 is, $\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4 - 4 = 0$

Similarly we can show all minor of order 2 are zero.
Hence the rank of A = 1.

5. $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

The minor of order 3 is, $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 5 & 2 & 4 \end{vmatrix}$

$= 2\,(16 + 4) - 1\,(-4 - 6) + 5\,(2 - 12) = 40 + 10 - 50 = 0$

Similarly we can show other minor of order 3 are zero.

Again, the minor of order 2 is, $\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 8 + 1 = 9 \neq 0$

Hence, the rank of A = 2

6. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(BE 2056)**

Solution:

Let A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

The only one minor of order 4 is,

$\begin{vmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{vmatrix}$

Applying $R_3 \to R_3 - R_4$ & $R_4 \to R_4 - R_1$

$= \begin{vmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{vmatrix}$

Expanding C_2,

$$= (-1)\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} = (-1) \quad [\because \text{two rows are same}]$$

$$= 0$$

Again, the minor of order 3 is,

$$\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} = -1\,(0 + 3) + 3\,(1 - 0) = -3 + 3 = 0$$

Similarly, we can show that orther minor of order 3 are zero.
Now, the minor of order 2 is,

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

Hence, rank of A = 2

7. $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$$

The only one minor of order 4 is,

$$\begin{vmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{vmatrix}$$

Applying, $C_1 \to C_1 - C_2$

$$= \begin{vmatrix} 1 & 6 & 7 & 8 \\ -1 & 7 & 8 & 9 \\ -1 & 12 & 13 & 14 \\ -1 & 17 & 18 & 19 \end{vmatrix}$$

Applying, $R_2 \to R_2 - R_3$, $R_3 \to R_3 - R_4$, $R_4 \to R_4 + R_1$

$$= \begin{vmatrix} 1 & 6 & 7 & 8 \\ 0 & -5 & -5 & -5 \\ 0 & -5 & -5 & -5 \\ 0 & 23 & 25 & 27 \end{vmatrix}$$

Expanding C_1,

$$= \begin{vmatrix} 6 & 7 & 8 \\ -5 & -5 & -5 \\ -5 & -5 & -5 \end{vmatrix} = 0 \quad [\because \text{two rows are same}]$$

Now, the minor of A of order 3 is,

$$\begin{vmatrix} 5 & 6 & 7 \\ 6 & 7 & 8 \\ 11 & 12 & 13 \end{vmatrix}$$

$= 5\,(91 - 96) - 6\,(78 - 88) + 7\,(72 - 77)$
$= -25 + 60 - 35 = 0$

Similarly we can know all mnor of order 3 of A are zero.

Again, the minor of A of order 2 is $\begin{vmatrix} 5 & 6 \\ 6 & 7 \end{vmatrix} = 35 - 36 = -1 \neq 0$

Hence, the rank of A = 2

8. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

The only one minor of order 4 is,

$$\begin{vmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$

$$= \begin{vmatrix} -1 & 3 & -1 & -1 \\ 2 & -1 & -2 & -4 \\ 2 & 1 & 3 & -2 \\ 3 & 3 & 0 & -7 \end{vmatrix}$$

Applying, $C_1 \to C_1 - C_3$, $C_3 \to C_3 - C_4$ & $C_4 \to C_4 - C_1$

$$= \begin{vmatrix} 0 & 3 & 0 & 0 \\ 4 & -1 & 2 & -6 \\ -1 & 1 & 5 & -4 \\ -3 & 3 & 7 & -10 \end{vmatrix}$$

Expanding R_1,

$$= -3\begin{vmatrix} 4 & 2 & -6 \\ -1 & 5 & -4 \\ 3 & 7 & -10 \end{vmatrix}$$

$= -3\,\{4\,(-50 + 28) + 1\,(-20 + 42) + 3\,(-8 + 30)\}$
$= -3\,\{(-88 + 22 + 66)\}$
$= 0$

Again, the minor of A of order 3 is $\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix}$

$= 2\,(-3 + 2) - 1\,(9 + 1) + 3\,(-6 - 1)$

$= -2-10-21$
$= -33 \neq 0$
Hence, the rank of $A = 3$

9. $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

The only one minor of order 4 is,

$$\begin{vmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{vmatrix}$$

Applying, $R_1 \to R_1 - R_2$, $R_2 \to R_2 + R_3$, $R_3 \to R_3 + R_4$ we get,

$$= \begin{vmatrix} -3 & -3 & -6 & 0 \\ 2 & 2 & 4 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \end{vmatrix}$$

Expanding C_4,

$$= -(-1) \begin{vmatrix} -3 & -3 & -6 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{vmatrix} = 0 \qquad [\because \text{ two rows are same.}]$$

Again, the minor of order 3 is,

$$\begin{vmatrix} -2 & -1 & -3 \\ 1 & 2 & 3 \\ 1 & 0 & 1 \end{vmatrix}$$

$= -2(2-0) - 1(-1-0) + 1(-3+6) = -4+1+3 = 0$

Similarly, we can show that all minor of order three are zero.

Now, the minor of order 2 is $\begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} = -4+1 = -3 \neq 0$

Hence, the rank of $A = 2$

Find the rank of the following matrices by reducing either to normal form or the echelon form whichever possible.

10. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Solution:

Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_3$

$$\sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is upper triangular matrix & it has two non zero rows.
Hence rank of $A = 2$

11. $\begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & 2 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & 2 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 4 & 2 \\ 2 & -3 & 4 \\ -1 & 8 & 2 \end{bmatrix}$$

Applying, $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -11 & 0 \\ 0 & 12 & 4 \end{bmatrix}$$

Applying $C_2 \to C_2 - 3C_3$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & -11 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Which is the triangular matrix & it has 3 non zero rows. Hence the rank of $A = 3$

12. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

Applying, $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3$ ...

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $C_2 \rightarrow C_2 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying, $R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$ & $R_4 \rightarrow R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_4 \rightarrow C_4 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

The order of identity matrix I is 3.

Hence, Rank of A = 3

13. $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

Applying, $R_2 \rightarrow R_2 - 4R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Applying, $R_3 \rightarrow R_3 - R_2$ & $R_4 \rightarrow R_4 - R_3$

$$\sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & -2 & 8 & -10 \\ 0 & -2 & 0 & -2 \end{bmatrix}$$

Applying, $C_2 \rightarrow C_2 - C_4$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -3 \\ 0 & 9 & -8 & 14 \\ 0 & 8 & 8 & -10 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Applying, $C_2 \rightarrow C_2 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 17 & -8 & 14 \\ 0 & 0 & 8 & -10 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Which is triangular matrix & it has four non zero rows.

Hence rank of A = 4

14. $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ **(BE 2057)**

Solution:

Let $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

Applying, $C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & 1 & 4 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 4 & 1 & 4 & 2 \\ 0 & 2 & 3 & 5 \end{bmatrix}$$

Applying, $R_3 \rightarrow R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

$C_3 \rightarrow C_3 - 2C_4$

$$\sim \begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -10 & 5 \end{bmatrix}$$

Applying $\frac{C_3}{2}$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Applying, $C_3 \rightarrow C_3 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Applying, $C_4 \to C_4 - 2C_2 + C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

Applying $\frac{C_3}{-5}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [I_3, 0]$$

Hence rank of A = 3

15. $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Solution:

Let, $A = \begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

$C_1 \leftrightarrow C_4$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 1 & 2 & 2 & 0 \\ 2 & -2 & -3 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Applying, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ & $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 4 & 2 & -3 \\ 0 & 2 & -3 & -5 \\ 0 & 3 & 2 & -3 \end{bmatrix}$$

Applying $C_2 \to C_2 - 2C_3$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 2 & -3 \\ 0 & 8 & -3 & -5 \\ 0 & -1 & 2 & -3 \end{bmatrix}$$

Applying, $R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 2 & -3 \\ 0 & 8 & -3 & -5 \\ 0 & -1 & 2 & -6 \end{bmatrix}$$

Applying, $C_2 \to C_2 + C_3 + C_4$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -3 & -5 \\ 0 & -7 & 0 & -6 \end{bmatrix}$$

Applying, $C_3 \to C_3 + 3C_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -3 & -5 \\ 0 & -7 & 0 & -6 \end{bmatrix}$$

Applying, $R_4 \to R_4 - 7R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & -15 \end{bmatrix}$$

Which is the triangular matrix & it has four non-zero rows.
Hence, rank of A = 4

16. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$

Solution:

Let, $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 7 & 4 & 13 \\ -3 & 4 & -1 \end{bmatrix}$$

Applying, $R_2 \to R_2 - 7R_1$ & $R_3 \to R_3 + 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -10 & -8 \\ 0 & 10 & 8 \end{bmatrix}$$

Applying, $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -10 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is the triangular matrix & it has two non zero rows.
Hence, rank of A = 2

17. $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$ **(BE 2063)**

Solution:

Let, $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$

Applying, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ & $R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & -6 & 1 & 0 \\ 0 & -6 & 3 & 0 \end{bmatrix}$$

Applying, $C_2 \to C_2 + C_3$, $C_3 \to C_3 + C_4$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix}$$

Applying, $C_2 \to C_2 + C_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 4 & 3 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Applying, $R_2 \to R_2 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Applying, $R_2 \to R_2 - R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Applying, $R_4 \to R_4 - 3R_2$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying, $C_2 \to C_2 + C_3$ & $C_4 \to C_4 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $\frac{C_2}{-2}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence of A = 3

18. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$ (BE 2064)

Solution:

Let, $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

Applying, $R_3 \to R_3 - R_1$ & $R_4 \to R_4 + 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Applying, $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Applying, $C_3 \to C_3 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Applying, $C_4 \to C_4 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 12 & 2 \end{bmatrix}$$

Applying $C_4 \to C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 12 & 0 \end{bmatrix}$$

Applying, $C_3 \to C_3 + 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 12 & 0 \end{bmatrix}$$

$\frac{C_3}{12}$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

Applying, $R_3 \to R_3 + R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Applying, $R_4 \to R_3$ & $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, rank of A = 3

19. $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

Applying, $R_4 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 2 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 0 & 6 \end{bmatrix}$$

Applying $C_1 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 2 & 4 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & -2 & 2 & 6 \end{bmatrix}$$

Applying $C_2 \to C_2 + C_3$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 6 & 4 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Applying, $R_4 \to R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 6 & 4 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is triangular matrix & it has 3 non zero rows.
Hence, rank of A = 3

20. Test for consistency & solve:

(i) $3x - 4y = 2$
$5x + 2y = 12$
$x - 3y = -1$

Solution: Given, equation are,
$3x - 4y = 2$
$5x + 2y = 12$
$x - 3y = -1$

Then, we have, $\begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$

Applying, $5R_1$, $3R_2$

$$\begin{bmatrix} 15 & -20 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 36 \end{bmatrix}$$

Applying, $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 15 & -20 \\ 0 & 26 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \end{bmatrix}$$

The rank of the coefficient matrix & argumented matrix are both 2.
Hence equation are consistent & the system is equivalent to,
$15x - 20y = 10$
$26y = 26$ which given $y = 1$ &
$15x - 20 . 1 = 10$
or, $x = \dfrac{10 + 20}{2} = 2$ $\quad \therefore x = 2$ & $y = 1$ Ans.

(ii) $x + y + z = 4$
$x + y + 2z = 2$
$2x + 2y + z = 5$

Solution: Given, equation are,
$x + y + z = 4$
$x + y + 2z = 2$
$2x + 2y + z = 5$

Then we have, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

Applying $2R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$$

Applying, $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -8 \end{bmatrix}$$

The rank of coefficient matrix ≠ the rank of argumented matrix.
Hence, the equation are inconsistent.

(iii) $x + 2y - z = 3$
$2x - 2y + 3z = 2$
$3x - y + 2z = 1$

Solution: Given equation are,

$x + 2y - z = 3$
$2x - 2y + 3z = 2$
$3x - y + 2z = 1$

Then, we have

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 3 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Applying, $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -6 & 5 \\ 0 & -7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -8 \end{bmatrix}$$

Applying, $7R_2$, $6R_3$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -42 & 35 \\ 0 & -42 & 30 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -28 \\ -48 \end{bmatrix}$$

Applying, $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -42 & 35 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -28 \\ -20 \end{bmatrix}$$

The rank of the coefficient matrix & augmented matrix for the last set of equation both 3. Hence the equations are consistant. The system of equation can be written as,

$x + 2y - z = 3, -42y + 35z = -28$ & $-5z = -20$

$\therefore z = \frac{-20}{-5} = 4$. Then, $-42y + 35 \cdot 4 = -28$

$y = \frac{-28-140}{-42} = \frac{-168}{-42} = 4$

& $x + 2\times4 - 4 = 3$

or, $x = 3 - 4$

$\therefore x = -1$

Hence, $x = -1$, $y = 4$ & $z = 4$

(iv) $2x - 3y + 7z = 5$
$3x + y - 3z = 13$
$2x + 19y - 47z = 32$

Solution: Given equation are,

$2x - 3y + 7z = 5$
$3x + y - 3z = 13$
& $2x + 19y - 47z = 32$

Then we have,

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Applying, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 27 \end{bmatrix}$$

Applying, $3R_1$, $2R_2$

$$\begin{bmatrix} 6 & -9 & 21 \\ 6 & 2 & -6 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 26 \\ 27 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 6 & -9 & 21 \\ 0 & 11 & -27 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 11 \\ 27 \end{bmatrix}$$

Applying, $2R_2$

$$\begin{bmatrix} 6 & -9 & 21 \\ 0 & 22 & -54 \\ 0 & 22 & -54 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \\ 27 \end{bmatrix}$$

Applying, $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 6 & -9 & 21 \\ 0 & 22 & -54 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \\ 27 \end{bmatrix}$$

The rank of coefficient matrix the $\neq$ rank of argumented matrix. Hence the equation are inconsistent.

(v) $3x + y - 4z = 0$
$2x + 5y + 6z = 13$
$x - 3y - 8z = -10$

Solution: Given equation are,

$3x + y - 4z = 0$
$2x + 5y + 6z = 13$
$x - 3y - 8z = -10$

Then we have,

$$\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & -3 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ -10 \end{bmatrix}$$

Applying $2R_1$, $3R_2$ & $6R_3$

$$\begin{bmatrix} 6 & 2 & -8 \\ 6 & 15 & 18 \\ 6 & -18 & -48 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 39 \\ -60 \end{bmatrix}$$

Applying , $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 6 & 2 & -8 \\ 0 & 13 & 26 \\ 0 & -20 & -40 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 39 \\ -60 \end{bmatrix}$$

Applying $\frac{R_1}{2}\ \frac{R_2}{13}\ \frac{R_3}{20}$

$$\begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

The rank of coefficient matrix & angumented for the last set equation are both 2. Hence the equation are consistent. Also equation can be written as, $3x + y - 4z = 0$, $y + 2z = 3$

$\therefore y = 3 - 2z$, $x = -1 + 2z$

Hence, $x = -1$, $y = 3$ & $z = 0$ is particular Solution:

(viii) $\mathbf{x + 2y + z = 3}$

$\mathbf{2x + 3y + 2z = 5}$

$\mathbf{3x - 5y + 5z = 2}$

Solution: Given equation are,

$x + 2y + z = 3$

$2x + 3y + 2z = 5$

$3x - 5y + 5z = 2$

Then we have,

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

Applying, $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -11 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -7 \end{bmatrix}$$

Applying $R_3 \to R_3 - 11R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

The rank of coefficient matrix & dugmented matrix for the last set of equation are both 3.

Hence the equation are consistent. Also the equations can be written as,

$x + 2y + z = 3$, $-1y + 0.z = -1$ & $2z = 4$

Which gives, $z = 2$, $y = 1$ &

$x + 2.1 + 2 = 3$

$x = -1$

Hence, $x = -1$, $y = 1$ & $z = 2$

(ix) $\mathbf{x + y + z = 6}$

$\mathbf{x + 2y + 3z = 10}$

$\mathbf{x = 2y + 4z = 1}$

Solution: Given equation are, **(BE 2056)**

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + 4z = 1$

Then we have,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 1 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -5 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -9 \end{bmatrix}$$

The rank of coefficient matrix & augmented matrix for the last set of equation are both 3.

Hence the equation are consistent. Also the equation can be written as,

$x + y + z = 6$

$y + 2z = 4$

& $z = -9$

Which gives, $z = -9$

$y + 2.-9 = 4$

$y = 4 + 18 = 22$

& $x + 22 - 9 = 6$

$x = 6 - 22 + 9 = -7$

Hence, $x = -7$, $y = 22$ & $z = -9$ Ans.

21. Solve the following equation by using inverse of matrix.

(i) $\mathbf{x + y + 2z = 4}$

$\mathbf{2x - y + 3z = 9}$

$\mathbf{3x - y - z = 2}$

Solution:

The system of the equations,

$x + y + 2z = 4$

$2x - y + 3z = 9$

$3x - y - z = 2$ can be written as in matrix from,

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

or, $AX = C$

Where, $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $C = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}$$

$= 1(1+3) - 1(-2-9) + 2(-2+3) = 4 + 11 + 2$

$= 17 \neq 0$, then A^{-1} exist so that

$X = A^{-1}C$

We know that, $A^{-1} = \dfrac{\text{Adj}(A)}{|A|}$

In R_1

Cofactor of $1 = (1+3) = 4$

Cofactor of $1 = -(-2-9) = 11$

Cofactor of $2 = (-2+3) = 1$

In R_2,

Cofactor of $2 = -(-1+2) = -1$

Cofactor of $-1 = -1-6 = -7$

Cofactor of $3 = -(-1-3) = 4$

In R_3

Cofactor of $3 = 3+2 = 5$

Cofactor of $-1 = -(3-4) = 1$

Cofactor of $-1 = -1-2 = -3$

$$\therefore \text{ Cofactor of } A = \begin{bmatrix} 4 & 11 & 1 \\ -1 & -7 & 4 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 16-9+10 \\ 44-63+2 \\ 4+36-6 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 17 \\ -17 \\ 34 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Hence, $x = 1, y = -1$ & $z = 2$

(ii) $x + 3y + 3z = 1$
$x + 4y + 3z = 0$
$x + 3y + 4z = 2$

Solution: The system of equations,

$x + 3y + 3z = 1$

$x + 4y + 3z = 0$

$x + 3y + 4z = 2$ can be written as in the matrix.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

or, $AX = C$

Where, $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $C = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$= 1(16-9) - 3(4-3) + 3(3-4)$

$= 7 - 3 - 3$

$= 1 \neq 0$ then A^{-1} exist so that

$X = A^{-1}C$

We know that $A^{-1} = \dfrac{\text{Adj}(A)}{|A|}$

in R_1

Cofactor of $1 = (16-9) = 7$

Cofactor of $3 = -(4-3) = -1$

Cofactor of $3 = (3-4) = -1$

in R_2

Cofactor of $1 = -(12-9) = -3$

Cofactor of $4 = (4-3) = 1$

Cofactor of $3 = -(3-3) = 0$

in R_3

Cofactor of $1 = (9-12) = 3$

Cofactor of $3 = -(3-3) = 0$

Cofactor of 4 $(4-3) = 1$

$$\therefore \text{ Cofactor of } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} 7 & -3 & 3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -3 & 3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7-0+6 \\ -1+0+0 \\ -1+0+2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $x = 1, y = -1$ & $z = 1$ Ans.

(iii) $\mathbf{x - 2y + 3z = 11}$

$\mathbf{3x + y - z = 2}$

$\mathbf{5x + 3y + 2z = 3}$

Solution: The system of equations,

$x - 2y + 3z = 11$

$3x + y - z = 2$

$5x + 3y + 2z = 3$ can be written as in the matrix form,

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \\ 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 3 \end{bmatrix}$$

or, $AX = C$

Where, $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \\ 5 & 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $C = \begin{bmatrix} 11 \\ 2 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \\ 5 & 3 & 2 \end{vmatrix}$$

$= 1(2+3) + 2(6+5) + 3(9-5)$

$= 5 + 22 + 12$

$= 39 \neq 0$ then A^{-1} exist so that $X = A^{-1}C$

in R_1

Cofactor of $1 = 2 + 3 = 5$

Cofactor of $-2 = -(6+5) = -11$

Cofactor of $3 = 9 - 5 = 4$

in R_2

Cofactor of $3 = -(-4-9) = 13$

Cofactor of $1 = 2 - 15 = -13$

Cofactor of $-1 = -(3+10) = -13$

in R_3

Cofactor of $5 = 2 - 3 = -1$

Cofactor of $3 = -(-1-9) = 10$

Cofactor of $2 = 1 + 6 = 7$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} 5 & -11 & 4 \\ 13 & -13 & -13 \\ -1 & 10 & 7 \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} 5 & 13 & -1 \\ -11 & -13 & 10 \\ 4 & -13 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{39} \begin{bmatrix} 5 & 13 & -1 \\ -11 & -13 & 10 \\ 4 & -13 & 7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{39} \begin{bmatrix} 5 & 13 & -1 \\ -11 & -13 & 10 \\ 4 & -13 & 7 \end{bmatrix} \begin{bmatrix} 11 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{39} \begin{bmatrix} 55+26-3 \\ -121-26+30 \\ 44-26+21 \end{bmatrix}$$

$$= \frac{1}{39} \begin{bmatrix} 78 \\ -117 \\ 39 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Hence, $x = 2, y = -3, z = 1$

(iv) $\mathbf{4y + 3z = 13}$

$\mathbf{x - 2y + z = 3}$

$\mathbf{3x + 5y = 11}$

Solution: The system of equations,

$4y + 3z = 13$

$x - 2y + z = 3$

$3x + 5y = 11$ can be written as in the matrix form.

$$\begin{bmatrix} 0 & 4 & 3 \\ 1 & -2 & 1 \\ 3 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \\ 11 \end{bmatrix}$$

or, $AX = C$ where,

$A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -2 & 1 \\ 3 & 5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $C = \begin{bmatrix} 13 \\ 3 \\ 11 \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & 4 & 3 \\ 1 & -2 & 1 \\ 3 & 5 & 0 \end{vmatrix}$$

$= -4(0-3) + 3(5+6)$

$= 12 + 33$

$= 45 \neq 0$ then A^{-1} exist so that $X = A^{-1}C$.

in R_1

Cofactor of $0 = 0 - 5 = -5$

Cofactor of $4 = -(0-3) = 3$

Cofactor of $5 + 6 = 11$

in R_2

Cofactor of $1 = -(0-15) = 15$

Cofactor of $-2 = 0 - 9 = -9$

Cofactor of $1 = -(0-12) = 12$

in R_3

Cofactor of $3 = 4 + 6 = 10$

Cofactor of $5 = -(0-3) = 3$

Cofactor of $0 = 0-4) = -4$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} -5 & 15 & 10 \\ 15 & -9 & 12 \\ 10 & 3 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{45}\begin{bmatrix} -5 & 15 & 10 \\ 3 & -9 & 3 \\ 11 & 12 & -4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{45}\begin{bmatrix} -5 & 15 & 10 \\ 3 & -9 & 3 \\ 11 & 12 & -4 \end{bmatrix}\begin{bmatrix} 13 \\ 3 \\ 11 \end{bmatrix} = \frac{1}{45}\begin{bmatrix} -65+45+110 \\ 39-27+33 \\ 143+36-44 \end{bmatrix}$$

$$= \frac{1}{45}\begin{bmatrix} 90 \\ 45 \\ 135 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2, y = 1$ & $z = 3$ Ans.

22. Solve the following equation by Gauss climination method.

(i) $2x + 3y + 4z = 20$

$3x + 4y + 5z = 26$

$3x + 5y + 6z = 31$

Solution: Here, the matrix form of system of equation,

$2x + 3y + 4z = 20$

$3x + 4y + 5z = 26$

$3x + 5y + 6z = 31$ is,

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 26 \\ 31 \end{bmatrix}$$

Applying $3R_1$, $2R_2$ & $2R_3$

$$\begin{bmatrix} 6 & 9 & 12 \\ 6 & 8 & 10 \\ 6 & 10 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 52 \\ 62 \end{bmatrix}$$

First step :

Elimination of x from the second & third equation & matrix form applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\left| \begin{array}{l} 6x + 9y + 12z = 60 \\ -y - 2z = -8 \\ y = 2 \end{array} \right| \begin{bmatrix} 6 & 9 & 12 \\ 0 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ -8 \\ 2 \end{bmatrix}$$

Back substitution:

From last equation, we get

$y = 2$ then,

$-2 - 2z = -8$

$$\therefore z = \frac{-6}{-2} = 3$$

& $6x + 9 . 2 + 12 . 3 = 60$

$$x = \frac{60-18-36}{6} = 1$$

Hence, $x = 1, y = 2$ & $z = 3$ Ans.

(ii) $2x - 3y + 2z = 5$

$x + 2y - 3z = 4$

$3x + y - 4z = 7$

Solution: Here, the matrix form of system of equation,

$2x - 3y + 2z = 5$

$x + 2y - 3z = 4$

$3x + y - 4z = 7$ is,

$$\begin{bmatrix} 2 & -3 & 2 \\ 1 & 2 & -3 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}$$

Applying, $R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 2 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}$$

1st step : Elimination of x from the second & third equation & matrix form applying $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$x + 2y - 3z = 4$

$-7y + 8z = -3$

$-5y + 5z = -5$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 8 \\ 0 & -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -5 \end{bmatrix}$$

Applying $5R_2$ & $7R_3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -35 & 40 \\ 0 & -35 & 35 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -15 \\ -35 \end{bmatrix}$$

2nd step : Elimination of y from third equation & matrix form applying $R_3 \to R_3 - R_2$

$x + 2y - 3z = 4$

$-35y + 40z = -15$

$-5z = -20$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -35 & 40 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -15 \\ -20 \end{bmatrix}$$

Back substitution:

From last equation we get $z = \dfrac{-20}{-5} = 4$

then,

$-35y + 40 . 4 = -15$

$$y = \frac{-15-160}{-35} = 5$$

$\& \; x + 2.5 - 3.4 = 4$

$x = 4 - 10 + 12 = 6$

Hence, $x = 6, y = 5 \;\&\; z = 4$ Ans.

(iii) $\mathbf{x - 2y + 3z = 2}$

$\mathbf{2x - 3y + z = 1}$

$\mathbf{3x - y + 2z = 9}$

Solution: Here, the matrix form of system of equations,

$x - 2y + 3z = 2$

$2x - 3y + z = 1$

$3x - y + 2z = 9$ is,

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}$$

First step : Elimination x from the second & third equation & matrix form applying,

$R_2 \to R_2 - 2R_1 \;\&\; R_3 \to R_3 - 3R_1$

$x - 2y + 3z = 2$

$y - 5z = -3$

$5y - 7z = 3$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

Second step

Elimination x from the third equation & matrix form applying

$R_3 \to R_3 - 5R_2,$

$x - 2y + 3z = 2$

$y - 5z = -3$

$18z = 18$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 18 \end{bmatrix}$$

Back substitution :

From the last equation we get,

$z = 1$ then,

$y - 5.1 = -3$

$y = -3 + 5 = 2$

$\& \; x - 2.2 + 3.1 = 2$

$x = 2 + 4 - 3 = 3$

Hence, $x = 3, y = 2 \;\&\; z = 1$ Ans.

(iv) $\mathbf{x + y - z = 3}$

$\mathbf{2x - 3y + 9z = 60}$

$\mathbf{7x + 3y + 3z = 69}$

Solution: Here, the matrix form of system of equation,

$x + y - z = 3$

$2x - 3y + 9z = 60$

$7x + 3y + 3z = 69$ is,

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 9 \\ 7 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 60 \\ 69 \end{bmatrix}$$

First step

Elimination x from the second & third equation & matrix from applying

$R_2 \to R_2 - 2R_1 \;\&\; R_3 \to R_3 - 7R_1$

$x + y - z = 3$

$-5y + 11y = 54$

$-4y + 10z = 48$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 11 \\ 0 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 54 \\ 48 \end{bmatrix}$$

Applying $4R_2 + 5R_3$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -20 & 44 \\ 0 & -20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 216 \\ 240 \end{bmatrix}$$

2nd step :

Elimination y from the third equation & matrix from applying

$R_3 \to R_3 - R_2$

$x + y - z = 3$

$-20y + 44z = 216$

$6z = 24$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -20 & 44 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 216 \\ 24 \end{bmatrix}$$

Back substitution

From the last equation we get,

$z = \frac{24}{6} = 4$ then,

$-20y + 44.4 = 216$

$y = \frac{216 - 176}{-20} = -2$

$\& \; x - 2 - 4 = 3$

$x = 3 + 4 + 2 = 9$

Hence, $x = 9, y = -2 \;\&\; z = 4$ Ans.

(vi) $\mathbf{3x - y + z = -2}$

$\mathbf{x + 5y + 2z = 6}$

$\mathbf{2x + 3y + z = 0}$

Solution: Here, the matrix form of system of equations,

$3x - y + z = -2$
$x + 5y + 2z = 6$
$2x + 3y + z = 0$

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & 5 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 5 & 2 \\ 3 & -1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 0 \end{bmatrix}$$

First step:
Elimination x from the second & third equation & matrix form applying $R_2 \to R_2 - 3R_1$ & $R_3 \to R_3 - 2R_1$

$x + 5y + 2z = 6$
$-16y - 5z = -20$
$-7y - 3z = -12$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & -16 & -5 \\ 0 & -7 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -20 \\ 12 \end{bmatrix}$$

Applying $7R_2$ & $16R_3$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & -112 & -35 \\ 0 & -112 & -48 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -140 \\ 192 \end{bmatrix}$$

2nd step
Elimination y from third equation & matrix form applying $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & -112 & -35 \\ 0 & 0 & -13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -140 \\ 52 \end{bmatrix}$$

$x + 5y + 2z = 6$
$-112y - 35z = -140$
$-13z = -52$

Back substitution :
From the last equation we get,

$z = \dfrac{-52}{-13} = 4$ then,

$-112y - 35 \cdot 4 = -140$

$y = \dfrac{-140 + 140}{-112} = 0$

& $x + 5 \cdot 0 + 2 \cdot 4 = 6$
$x = 6 - 8 = -2$

Hence, $x = -2, y = 0$ & $z = 4$ Ans.

(vii) $x + y + z = -1$
$4y + 6z = 6$
$y + z = 1$

Solution: Here, the matrix from of system of equations,

$x + y + z = -1$
$4y + 6z = 6$
$y + z = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 1 \end{bmatrix}$$

First step
Here x is already eliminated from 2nd & third equation.
Applying $4R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 6 \\ 0 & 4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

2nd step, eliminating y from 3rd equation and matrix from applying $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 6 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 2 \end{bmatrix}$$

$x + y + z = -1$
$4y + 6z = 6$
$-2z = -2$

Back substitution
From the last equation we get,

$z = \dfrac{-2}{-2} = 1$ then,

$4y + 6.1 = 6$

$y = \dfrac{6 - 6}{4} = 0$

& $x + 0 + 1 = -1$
$x = -1 - 1 = -2$

Hence, $x = -2, y = 0$ & $z = 1$ Ans.

Exercise - 6

1. Are the following vectors linearly dependent or independent

(i) $x_1 = (1, 2, 4), x_2 = (2, -1, 3), x_3 = (0, 1, 2), x_4 = (-3, 7, 2)$

Solution: Let C_1, C_2, C_3 and C_4 are four real numbers. So that

$C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 = 0$

i.e. $C_1 (1, 2, 4) + C_2 (2, -1, 3) + C_3 (0, 1, 2) + C_4 (-3, 7, 2) = 0$

Which is equivalent to,

$C_1 + 2C_1 + 0.C_3 - 3C_4 = 0$

$2C_1 - C_2 + C_3 + 7C_4 = 0$

$4C_1 + 3C_2 + 2C_3 + 2C_4 = 0$

Now, to check rank of the coefficient matrix, we have

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2xR_1$

$R_3 \to R_3 - 4xR_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Here no. of non zero rows 3

$\therefore$ Rank of the matrix $= 3 < 4 =$ no. of unknowns $C_1, C_2, C_3 \& C_4$

Hence, there are infinite values of $C_1, C_2, C_3 \& C_4$ & the vectors are linearly dependent.

(ii) $x_1 = (3, 2, 7), x_2 = (2, 4, 1), x_3 = (1, -2, 6)$

Solution: Let $C_1, C_2 \& C_3$ are three real numbers.

So that,

$C_1x_1 + C_2x_2 + C_3x_3 = 0$

i.e. $C_1 (3, 2, 7) + C_2 (2, 4, 1) + C_3 (1, -2, 6) = 0$

Which is equivalent to,

$3C_1 + 2C_2 + C_3 = 0$

$2C_1 + 4C_2 - 2C_3 = 0$

$7C_1 + C_2 + 6C_3 = 0$

Now, to check rank of the coefficient matrix, we have

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & -2 \\ 7 & 1 & 6 \end{bmatrix}$$

Applying $C_1 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & 7 \end{bmatrix}$$

Applying $R_2 \to R_2 + 2R_1$ & $R_3 \to R_3 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 8 \\ 0 & -11 & -11 \end{bmatrix}$$

Applying $C_2 \to C_2 - C_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & -11 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2 \times \frac{11}{8}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Here no. of non-zero rows 2.

$\therefore$ Rank of the matrix $= 2 \angle$ no. of un known $C_1, C_2 \& C_3$

Hence, there are infinite values of $C_1, C_2 \& C_3$ & the vectors are linedrly dependent.

(iii) $x_1 = (2, 3, 5), x_2 = (4, 9, 11)$

Solution: Let $C_1 \& C_2$ are two real numbers. So that,

$C_1x_1 + C_2x_2 = 0$

i.e. $C_1 (2, 3, 5) + C_2 (4, 9, 11) = 0$

Which is equivalent to,

$2C_1 + 4C_2 = 0$

$3C_1 + 9C_2 = 0$

$5C_1 + 11C_2 = 0$

Now, the check rank of the coefficient matrix, we have,

$$\begin{bmatrix} 2 & 4 \\ 3 & 9 \\ 5 & 11 \end{bmatrix}$$

Applying $R_2 \to R_2 - \frac{3}{2} \times R_1$

$R_3 \to R_3 - \frac{5}{2} \times R_1$

$$\sim \begin{bmatrix} 2 & 4 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - \frac{1}{3} \times R_2$

$$\sim \begin{bmatrix} 2 & 4 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

The no. of non-zero rows is 2.

Hence, rank of the matrix = 2 = no. of unknowns C_1 & C_2.

$\therefore$ The vectors are linearly independent.

(iv) (1, 1, 1), (1, 1, 0), (0, 1, 1)

Solution: Let $x_1 = (1, 1, 1)$, $x_2 = (1, 1, 0)$ and $x_3 = (0, 1, 1)$ are three vectors and also C_1, C_2 & C_3 are three real numbers.

So that,

$C_1x_1 + C_2x_2 + C_3x_3 = 0$

i.e. $C_1(1, 1, 1) + C_2(1, 1, 0) + C_3(0, 1, 1) = 0$

Which is equivalent to,

$$C_1 + C_2 + 0C_3 = 0$$
$$C_1 + C_2 + C_3 = 0$$
$$C_1 + 0.C_2 + C_3 = 0$$

Now, to check rank of the coefficient matrix, we have,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Applying, $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Here no. of non-zero rows 3 = 3 = no. of unknowns C_1, C_2 & C_3. Hence there are only zero values of C_1, C_2 & C_3 & vectors are linedrly independent.

(v) (1, 2, – 1), (2, 3, 0), (0, 0, 0)

Solution: Let $x_1 = (1, 2, -1)$, $x_2 = (2, 3, 0)$ & $x_3 = (0, 0, 0)$ are three. Vectors and also, C_1, C_2 & C_3 are three real numbers.

So that,

$C_1x_1 + C_2x_2 + C_3x_3 = 0$

i.e. $C_1(1, 2, -1) + C_2(2, 3, 0) + C_3(0, 0, 0) = 0$

Which is equivalent to,

$$C_1 + 2C_2 + 0.\ C_3 = 0$$
$$2C_1 + 3C_2 + 0.C_3 = 0$$
$$-C_1 + 0.\ C_2 + 0.\ C_3 = 0$$

Now, to check rank of the coefficient matrix, we have,

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The only one minor of 3 is,

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 0$$

The minor of 2 is $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

The rank of matrix = 2 $\angle$ 3 = no. of unknown C_1, C_2 & C_3.

Hence there are infinite solution of C_1, C_2 & C_3 & the vectors are linearly dependent.

(vi) (1, 2, – 1, 0), (1, 3, 1, 2), (6, 1, 0, 1) and (4, 1, 2, 0)

Solution: Let $x_1 = (1, 2, -1, 0)$, $x_2 = (1, 3, 1, 2)$, $x_3 = (6, 1, 0, 1)$ and $x_4 = (4, 1, 2, 0)$ are four vectors.

Also, Let C_1, C_2 C_3 and C_4 are three real numbers. So that

$C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 = 0$

i.e. $C_1(1, 2, -1, 0) + C_2(1, 3, 1, 2) + C_3(6, 1, 0, 1) + C_4(4, 1, 2, 0)$

Which is equivalent to,

$$C_1 + C_2 + 6C_3 + 4C_4 = 0$$
$$2C_1 + 3C_2 + C_3 + C_4 = 0$$
$$-C_1 + C_2 + 0.\ C_3 + 2C_4 = 0$$
$$0.C_1 + 2C_2 + C_3 + 0.C_4 = 0$$

Now, to check the rank of the coefficient matrix, We have,

$$\begin{bmatrix} 1 & 1 & 6 & 4 \\ 2 & 3 & 1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2 \times R_1$

$R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 1 & 6 & 4 \\ 0 & 1 & -11 & -7 \\ 0 & 2 & 6 & 6 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2 \times R_2$

$R_4 \to R_4 - 2 \times R_2$

$$\sim \begin{bmatrix} 1 & 1 & 6 & 4 \\ 0 & 1 & -11 & -7 \\ 0 & 0 & 28 & 20 \\ 0 & 0 & 23 & 14 \end{bmatrix}$$

Applying $R_4 \to R_4 - \frac{23}{28} \times R_3$

$$\sim\begin{bmatrix}1 & 1 & 6 & 4\\ 0 & 1 & -11 & -7\\ 0 & 0 & 28 & 20\\ 0 & 0 & 0 & -\frac{17}{7}\end{bmatrix}$$

Here, the no. of non-zero is 4 and the rank of the matrix = 4 = no. of unknowns C_1, C_2, C_3 & C_4.

$\therefore$ Hence the vectors are linerly independents.

(2) Find the point $x = (x_1, x_2, x_3)$ such that the matrix $A = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ transforms the point $x = (x_1, x_2, x_3)$ into (2, – 1, 2).

Solution: Given matrix $A = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$

We have, $Y = AX$

or, $\begin{bmatrix}2\\ -1\\ 2\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

or, $\begin{bmatrix}2\\ -1\\ 2\end{bmatrix} = \begin{bmatrix}-x_1\\ -x_2\\ -x_3\end{bmatrix}$

$\therefore x_1 = -2, x_2 = 1, x_3 = -2$

Hence the point is $X = (-2, 1, -2)$

(3) Represent each of the transformations $x_1 = 3y_1 + 2y_2$, $y_1 = z_1 + 2z_2$, $x_2 = -y_1 + 4y_2$, $y_2 = 3z_1$ by the use of matrices and find composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

Solution: Given relation are,

$x_1 = 3y_1 + 2y_2$ and $y_1 = z_1 + 2z_2$

$x_2 = -y_1 + 4y_2$ $\qquad y_2 = 3z_1$

$\Rightarrow \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}3 & 2\\ -1 & 4\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}$

and $\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$

$\therefore \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}3 & 2\\ -1 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$

$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}9 & 6\\ 11 & -2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$

Which shows that x_1, x_2 interms of z_1, z_2.

(7) Find the Eigen value and Eigen vectors of the matrices.

(i) $\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}$

Solution: Let $A = \begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}$

The characteristics equation of the matrix A is,

$$|A - \lambda I| = 0$$

i.e. $\begin{vmatrix}1-\lambda & 4\\ 3 & 2-\lambda\end{vmatrix} = 0$

or, $(1 - \lambda)(2 - \lambda) - 12 = 0$

or, $2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$

or, $\lambda^2 - 3\lambda - 10 = 0$

or, $\lambda(\lambda + 2) - 5(\lambda + 2) = 0$

$\therefore \lambda = -2, 5$

Thus the eigen values of A are $\lambda = -2, 5$

If x, y be the components of eigen vector corresponding to the eigen value λ, then the matrix equation is,

$$(A - \lambda I)X = 0$$

or, $\begin{vmatrix}1-\lambda & 4\\ 3 & 2-\lambda\end{vmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

Putting $\lambda = -2$ we have,

$$\begin{bmatrix}3 & 4\\ 3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$3x + 4y = 0$

$3x + 4y = 0$

Clearly $x = 1, y = 1$

Thus the eigen-vectors corresponding to $\lambda = -2$ is (1, – 1). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = -2$.

Similarly putting $\lambda = 5$, we have,

$$\begin{bmatrix}-4 & 4\\ 3 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$-4x + 4y = 0$ & $3x - 3y = 0$

Clearly $x = 1, y = 1$

Thus the eigen-vector corresponding to $\lambda = 5$ is (1, 1). Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = 5$.

Therefore the eigen vectors are,

$a_1(1, -1), a_1 \neq 0$ & $a_2(1, 1), a_2 \neq 0$

(ii) $\begin{bmatrix}1 & -2\\ -5 & 4\end{bmatrix}$

Solution: Let $A = \begin{bmatrix}1 & -2\\ -5 & 4\end{bmatrix}$

The characteristic equation of the matrix A is,

$$|A - \lambda I) = 0$$

i.e. $\begin{vmatrix}1-\lambda & -2\\ -5 & 4-\lambda\end{vmatrix} = 0$

or, $(1-\lambda)(4-\lambda) - 10 = 0$

or, $4 - \lambda - 4\lambda + \lambda^2 - 10 = 0$

or, $\lambda^2 - 6\lambda + \lambda - 6 = 0$

or, $\lambda(\lambda - 6) + 1(\lambda - 6) = 0$

$\therefore \lambda = -1, 6$

Thus the eigen values of A are $\lambda = -1, 6$.

If x, y be components of eigen vector corresponding to the eig values λ, then the matrix equation is,

$(A - \lambda I) X = 0$

i.e. $\begin{vmatrix}1-\lambda & -2\\ -5 & 4-\lambda\end{vmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

Putting $\lambda = -1$

$$\begin{bmatrix}2 & -2\\ -5 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$2x - 2y = 0$

$-5x + 5y = 0$

Clearly $x = 1$ & $y = 1$

Thus the eigen vectors corresponding to $\lambda = -1$ is (1, 1). Al every non-zero multiple of this vector corresponding to $\lambda = -2$

Similarly putting $\lambda = 6$

$$\begin{bmatrix}-5 & -2\\ -5 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$-5x - 2y = 0$

$-5x - 2y = 0$

Clearly $x = 2, y = -5$

Thus the eigen vectors corresponding to $\lambda = 6$ is $(2, -5)$.

Also every non-zero multiple of this vector is eigen vec corresponding to $\lambda = 6$.

Therefore the eigen vector are,

$a_1(1, 1), a_1 \neq$ - & $a_2(2, -5), a_2 \neq 0$

(iii) $\begin{bmatrix}2 & 0 & 1\\ 1 & 1 & 1\\ 1 & -1 & 3\end{bmatrix}$ (BE 205

Solution: Let, $A = \begin{bmatrix}2 & 0 & 1\\ 1 & 1 & 1\\ 1 & -1 & 3\end{bmatrix}$

The characteristics equation of the matrix A is,

$$|A - \lambda I| = 0$$

i.e. $\begin{bmatrix}2-\lambda & 0 & 1\\ 1 & 1-\lambda & 1\\ 1 & -1 & 3-\lambda\end{bmatrix} = 0$

or, $(2-\lambda)\begin{vmatrix}1-\lambda & 1\\ -1 & 3-\lambda\end{vmatrix} + 1\begin{vmatrix}1 & 1-\lambda\\ 1 & -1\end{vmatrix} = 0$

or, $(2-\lambda)\{(1-\lambda)(3-\lambda) + 1\} + 1\{-1 - 1(1-\lambda)\} = 0$

or, $(2-\lambda)(3 - 3\lambda - \lambda + \lambda^2 + 1) + (-1 - 1 + \lambda) = 0$

or, $(2-\lambda)(\lambda^2 - 4\lambda + 4) + (-2 + \lambda) = 0$

or, $(2-\lambda)(\lambda - 2)(\lambda - 2) + (\lambda - 2) = 0$

or, $(\lambda - 2)\{(2-\lambda)(\lambda - 2) + 1\} = 0$

or, $(\lambda - 2)(2\lambda - \lambda^2 + 4 + 2\lambda + 1) = 0$

or, $(\lambda - 2)(-\lambda^2 + 4\lambda - 3) = 0$

or, $(\lambda - 2)(-\lambda^2 + 3\lambda + \lambda - 3) = 0$

or, $(\lambda - 2)\{-\lambda(\lambda - 3) + 1(\lambda - 3)\} = 0$

or, $(\lambda - 2)(-\lambda + 1)(\lambda - 3) = 0$

$\therefore \lambda = 1, 2, 3$.

Thus the eigen values of A are, $\lambda = 1, 2, 3$. If x, y, z be components of eigen vector corresponding to the eigen values λ, then the matrix equation. is,

$$(A - \lambda I) X = 0$$

i.e. $\begin{bmatrix}2-\lambda & 0 & 1\\ 1 & 1-\lambda & 1\\ 1 & -1 & 3-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

Putting $\lambda = 1$,

$$\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$x + z = 0$

$x + z = 0$

$x - y + 2z = 0$

Clearly, $x = -1, y = 1, z = 1$

Thus the eigen vector corresponding to $\lambda = 1$ is $(-1, 1, 1)$.

Also every non zero multiple of this vector is eigen vector corresponding to $\lambda = 1$.

Similarly, putting $\lambda = 2$

$$\begin{bmatrix}0 & 0 & 1\\ 1 & -1 & 1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$z = 0$$
$$x - y + z = 0$$
$$x - y + z = 0$$

Clearly, $x = 1, y = 1, z = 0$

Thus, the eigen vector corresponding to $\lambda = 2$ is $(1, 1, 0)$.

Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 2$.

Similarly, putting $\lambda = 3$.

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + z = 0$$
$$x - 2y + z = 0$$
$$x - y = 0$$

Clearly, $x = 1, y = 1, z = 1$

Thus the eigen vector corresponding to $\lambda = 3$ is $(1, 1, 1)$.

Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 3$.

$\therefore$ $a_1'(-1, 1, 1), a_1 \neq 0$,

$a_2\,(1, 1, 0), a_2 \neq 0$

& $a_3\,(1, 1, 1), a_3 \neq 0$, are eigen vectors.

(iv) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of the matrix A is,

$$|A - \lambda I| = 0$$

i.e. $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

or, $(8-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} + 6\begin{vmatrix} -6 & -4 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & 7-\lambda \\ 2 & -4 \end{vmatrix} = 0$

or, $(8-\lambda)\{(7-\lambda)(3-\lambda) - 16\} + 6\{-6(3-\lambda) + 8\} + 2\{24 - 2(7-\lambda)\} = 0$

or, $(8-\lambda)(21 - 3\lambda - 7\lambda + \lambda^2 - 16) + 6(-18 + 6\lambda + 8) + 2(24 - 14 + 2\lambda) = 0$

or, $(8-\lambda)(\lambda^2 - 10\lambda + 5) + 36\lambda - 60 + 4\lambda + 20 = 0$

or, $(8-\lambda)(\lambda^2 - 10 + 5) + 40\lambda - 40 = 0$

or, $8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 40\lambda - 40 = 0 = 0$

or, $-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

or, $-\lambda(\lambda^2 - 18\lambda + 45) = 0$

or, $-\lambda\{\lambda(\lambda - 15) - 3(\lambda - 15)\} = 0$

or, $-\lambda(\lambda - 3)(\lambda - 15) = 0$

$\therefore \lambda = 0, 3, 15$

Thus the eigen values of A are $\lambda = 0, 3, 15$.

If x, y, z be components of eigen vector corresponding to the eigen values λ, then the matrix equation is,

$$(A - \lambda I)X = 0$$

i.e. $\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Putting $\lambda = 0$,

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x - 6y + 2z = 0$$
$$-6x + 7y - 4z = 0$$
$$2x - 4y + 3z = 0$$

Solving first two equation,

$$\frac{x}{24 - 14} = \frac{y}{-12 + 32} = \frac{z}{56 - 36}$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$\therefore x = 1, y = 2, \& z = 2$

Thus the eigen vectors corresponding to $\lambda = 0$ is $(1, 2, 2)$.

Also every non-zero multiple of this vectors is eigen vector corresponding to $\lambda = 0$.

Similarly, putting $\lambda = 3$.

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x - 6y + 2z = 0$$
$$-6x + 4y - 4z = 0$$
$$2x - 4y - 0.z = 0$$

Solving first two eqatuion,

$$\frac{x}{24 - 8} = \frac{y}{-12 + 20} = \frac{z}{20 - 36}$$

$$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$\therefore x = 2, y = 1, z = -2$

Thus the eigen vectors corresponding to $\lambda = 3$ is $(2, 1, -2)$.
Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 3$.

Similarly putting $\lambda = 15$,

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x - 6y + 2z = 0$$
$$-6x - 8y - 4z = 0$$
$$2x - 4y - 12z = 0$$

Solving first two equation,

$$\frac{x}{24+16} = \frac{y}{-12-28} = \frac{z}{56-36}$$

$$\frac{x}{40} = \frac{y}{-40} = \frac{z}{20}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$\therefore x = 2, y = -2, z = 1$

Thus the eigen vectors corresponding to $\lambda = 15$ is $(2, -2, 1)$.
Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 15$.

Therefore the eigen vectors corresponding to $\lambda = 0, 3, 15$ are, $a_1 = (1, 2, 2)$, $a_1 \neq 0$
$a_2\,(2, 1, -2)$, $a_2 \neq 0$
& $a_3\,(2, -2, 1)$, $a_3 \neq 0$

(v) $\begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ **(BE 2057)**

Solution: Let, $A = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

The characteristic equation of the matrix A is, $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 5-\lambda & 2 & 2 \\ 2 & 2-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)\{(2-\lambda)^2 - 1\} - 2\{2(2-\lambda) - 2\} + 2\{2 - 2(2-\lambda)\} = 0$
or, $(5-\lambda)(4 - 4\lambda + \lambda^2 - 1) - 4(2-\lambda) + 4 + 4 - 4(2-\lambda) = 0$
or, $(5-\lambda)(\lambda^2 - 4\lambda + 3) + 8 - 8(2-\lambda) = 0$
or, $(5-\lambda)(\lambda^2 - 3\lambda - \lambda + 3) + 8(\lambda - 1) = 0$
or, $(5-\lambda)\{\lambda(\lambda-3) - 1(\lambda-3)\} + 8(\lambda-1) = 0$
or, $(5-\lambda)(\lambda-1)(\lambda-3) + 8(\lambda-1) = 0$
or, $(\lambda-1)\{(5-\lambda)(\lambda-3) + 8\} = 0$
or, $(\lambda-1)(5\lambda - 15 - \lambda^2 + 3\lambda + 8) = 0$

or, $(\lambda-1)(-\lambda^2 + 8\lambda - 7) = 0$
or, $(\lambda-1)(-\lambda^2 + 7\lambda + \lambda - 7) = 0$
or, $(\lambda-1)\{-\lambda(\lambda-7) + 1(\lambda-7)\} = 0$
or, $(\lambda-1)(-\lambda+1)(\lambda-7) = 0$
$\therefore \lambda = 1, 1, 7$

Thus the eigen values of A are $\lambda = 1, 1, 7$
If x, y, z be components eigen vector corresponding to the eigen values λ, then the matrix equation is $(A - \lambda I)X = 0$

i.e. $\begin{bmatrix} 5-\lambda & 2 & 2 \\ 2 & 2-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Putting, $\lambda = 1$

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x + 2y + 2z = 0$$
$$2x + y + z = 0$$
$$2x + y + z = 0$$

Clearly $x = -1, y = 1$ & $z = 1$

Thus the eigen vectors corresponding to $\lambda = 1$ is $(-1, 1, 1)$.
Also every non zero multiple of this vector is eigen vector corresponding to $\lambda = 1$.

Similarly, putting $\lambda = 7$

$$\begin{bmatrix} -2 & 2 & 2 \\ 2 & -5 & 1 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + 2y + 2z = 0$ i.e. $x - y - z = 0$
$2x - 5y + z = 0$ $2x - 5y + z = 0$
$2x + y - 5z = 0$ $2x + y - 5z = 0$

Clearly $x = 2, y = 1, z = 1$

Thus the eigen vectors corresponding to $\lambda = 7$ is $(2, 1, 1)$.
Also every non zero multiple of this vector is eigen vector corresponding to $\lambda = 7$.

Therefore the eigen vectors corresponding to eigen value $\lambda = 1, 1, 7$ are,

$a_1\,(-1, 1, 1)$, $a_1 \neq 0$
& $a_2\,(2, 1, 1)$, $a_2 \neq 0$

(vi) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

The characteristics equation of the matrix A is,

$$|A - \lambda I| = 0$$

i.e. $\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = 0$

or, $-\lambda(\lambda^2 - 1) - 1(-\lambda + 2) + 2(-1 + 2\lambda) = 0$

or, $-\lambda^3 + \lambda + \lambda - 2 - 2 + 4\lambda = 0$

or, $-\lambda^3 + 6\lambda - 4 = 0$

or, $\lambda^3 - 6\lambda + 4 = 0$

or, $\lambda^3 - 2\lambda^2 + 2\lambda^2 - 4\lambda - 2\lambda + 4 = 0$

or, $\lambda^2(\lambda - 2) + 2\lambda(\lambda - 2) - 2(\lambda - 2) = 0$

or, $(\lambda - 2)(\lambda^2 + 2\lambda - 2) = 0$

$\therefore \lambda = 2$

Thus the eigen value of A is 2.

If x, y, z are components eigen vector corresponding to eigen values $\lambda = 2$. Then the matrix equation is,

$$(A - \lambda I)X = 0$$

i.e. $\begin{bmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Putting $\lambda = 2$

$$\begin{bmatrix} -2 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-2x + y + 2z = 0$

$x - 2y - z = 0$

$2x - y - 2z = 0$

Solving first two equation,

$$\frac{x}{-1+4} = \frac{y}{2-2} = \frac{z}{4-1}$$

$$\frac{x}{3} = \frac{y}{0} = \frac{z}{3}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

$\therefore x = 1, y = 0$ & $z = 1$

Thus the eigen vectors corresponding to $\lambda = 2$ is (1, 0, 1).

Also, every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 2$.

Therefore eigen vector corresponding to eigen value $\lambda = 2$ is, a (1, 0, 1), $a \neq 0$

(vii) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (BE 2067)

Solution: $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 3 & 2-\lambda \end{vmatrix} = 0$

$(2 - \lambda)\{(3 - \lambda)(2 - \lambda) - 2\} - 2(2 - \lambda - 1) + 1(2 - 3 - \lambda) = 0$

or, $(2 - \lambda)(6 - 2\lambda - 3\lambda + \lambda^2 - 2) - 2(-\lambda + 1) + 1(\lambda - 1) = 0$

or, $(2 - \lambda)(\lambda^2 - 5\lambda + 4) + 2(\lambda - 1) + (\lambda - 1) = 0$

or, $(2 - \lambda)\{\lambda^2 - 4\lambda - \lambda + 4\} + 2(\lambda - 1) + (\lambda - 1) = 0$

or, $(2 - \lambda)\{\lambda(\lambda - 4) - 1(\lambda - 4)\} + 2(\lambda - 1) + (\lambda - 1) = 0$

or, $(2 - \lambda)(\lambda - 4)(\lambda - 1) + 2(\lambda - 1) + (\lambda - 1) = 0$

or, $(\lambda - 1)\{(2 - \lambda)(\lambda - 4) + 2 + 1\} = 0$

or, $(\lambda - 1)(2\lambda - \lambda^2 - 8 + 4\lambda + 3) = 0$

or, $(\lambda - 1)(-\lambda^2 + 6\lambda - 5) = 0$

or, $(\lambda - 1)(-\lambda^2 + 5\lambda + \lambda - 5) = 0$

or, $(\lambda - 1)\{-\lambda(\lambda - 5) + 1(\lambda - 5)\} = 0$

or, $(\lambda - 1)(\lambda - 5)(-\lambda + 1) = 0$

$\therefore \lambda = 1, 1, 5$

Thus the eigen values of A are $\lambda = 1, 1, 5$.

If x, y, z are components eigen vector corresponding to eigen value $\lambda = 1, 1, 5$. Then the matrix equation is,

$(A - \lambda I)x = 0$

i.e. $\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Putting $\lambda = 1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y + z = 0$

$x + 2y + z = 0$

$x + 2y + z = 0$

Clearly, $x = 1, y = 0, z = -1$

Thus the eige vector corresponding to $\lambda = 1$ is (1, 0, −1). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 1$.

Similarly, putting $\lambda = 5$

$$\begin{bmatrix}-3 & 2 & 1\\ 1 & -2 & 1\\ 1 & 2 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$-3x+2y+z=0$

$x-2y+z=0$

$x+2y-3z=0$

Solving first two equation,

$$\frac{x}{2+2}=\frac{y}{1+3}=\frac{z}{6-2}$$

$$\frac{x}{4}=\frac{y}{4}=\frac{z}{4}$$

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$$

$\therefore x=1, y=1 \ \& \ z=1$

Thus the eigen vector corresponding to the eigen value $\lambda = 5$ is (1, 1, 1)

Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 5$.

Therefore eigen vectors corresponding to $\lambda = 1, 1, 5$ are,

$a_1(1, 0, -1), a_1 \neq 0$

$a_2(1, 1, 1), a_2 \neq 0$

(viii) $\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$

Solution: Let $A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$

The characteristic equation of the matrix A is,

$|A-\lambda I|=0$

i.e. $\begin{bmatrix}2-\lambda & 0 & 0\\ 0 & 2-\lambda & 1\\ 0 & 0 & 2-\lambda\end{bmatrix}=0$

$(2-\lambda)(2-\lambda)(2-\lambda)=0$

$\lambda = 2, 2, 2$

Thus the eigen values of A are $\lambda = 2, 2, 2$.

If x, y, z are components eigen vector corresponding to eigen value $\lambda = 2, 2, 2$. Then the matrix equation is,

$(A-\lambda I)x=0$

i.e. $\begin{bmatrix}2-\lambda & 0 & 0\\ 0 & 2-\lambda & 1\\ 0 & 0 & 2-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

Putting $\lambda = 2$,

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$0.x+y+0.z=0$

$0.x+0.y+z=0$

$0.x+0.y+0.z=0$

Clearly,

$x=1, y=0, z=0$

Thus the eigen vector corresponding to the eigen value $\lambda = 2$ is (1, 0, 0). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 2$. Therefore eigen vector corresponding to $\lambda = 2$ is,

$a(1, 0, 0), a \neq 0$

(ix) $\begin{bmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 3\end{bmatrix}$

Solution: Let $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 3\end{bmatrix}$

The characteristic equation of the matrix A is,

$|A-\lambda I|=0$

i.e. $\begin{bmatrix}2-\lambda & 0 & 0\\ 0 & 4-\lambda & 0\\ 0 & 0 & 3-\lambda\end{bmatrix}=0$

or, $(2-\lambda)(4-\lambda)(3-\lambda)=0$

$\therefore \lambda = 2, 4, 3$

Thus the eigen values of A are λ 2, 4, 3.

If x, y, z are components eigen vector corresponding to eigen value $\lambda = 2, 4, 3$. Then the matrix equation is,

$(A-\lambda I)x=0$

i.e. $\begin{bmatrix}2-\lambda & 0 & 0\\ 0 & 4-\lambda & 0\\ 0 & 0 & 3-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

Putting $\lambda = 2$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$0.x+0.y+0.z=0$

$0.x+2y+0.z=0$

$0.x+0.y+z=0$

Clearly, $x=1, y=0, z=0$

Thus the eigen vector corresponding to the eigen value $\lambda = 2$ is

(1, 0, 0). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 2$.

Similarly, putting $\lambda = 4$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + 0.y + 0.z = 0$
$0.x + 0.y + 0.z = 0$
$0.x + 0.y - z = 0$

Clearly, $x = 0, y = 1, z = 0$

Thus the eigen vector corresponding to the eigen value $\lambda = 4$ is (0, 1, 0). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 4$.

Again, putting $\lambda = 3$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + 0.y + 0.z = 0$
$0.x + y + 0.z = 0$
$0.x + 0.y + 0.z = 0$

Clearly, $x = 0, y = 0, z = 1$

Thus the eigen vector corresponding to the eigen value $\lambda = 3$ is (0, 0, 1). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 3$.

Therefore the eigen vector corresponding to the eigen values λ = 2, 4, 3 are,

a_1 (1, 0, 0), $a_1 \neq 0$
a_2 (0, 1, 0), $a_2 \neq 0$
& a_3 (0, 0, 1), $a_3 \neq 0$

(x) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$

or, $(3 - \lambda)(2 - \lambda)(5 - \lambda) = 0$

$\therefore \lambda = 3, 2, 5$

Thus the eigen values of A are, $\lambda = 2, 3, 5$.

If x, y, z be components eigen vectors corresponding to the eigen value λ. Then the matrix equation is,

$(A - \lambda I)\, x = 0$

i.e. $\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Putting, $\lambda = 2$,

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + 4z = 0$
$0.x + 0.y + 6z = 0$
$0.x + 0.y + 3z = 0$

Solving first two equation,

$$\frac{x}{6} = \frac{y}{-6} = \frac{z}{0}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$$

$\therefore x = 1, y = -1, z = 0$

Thus the eigen vector corresponding to the eigen value $\lambda = 2$ is (1, – 1, 0). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 2$.

Similarly, putting $\lambda = 3$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0.x + y + 4z = 0$
$0.x - y + 6z = 0$
$0.x + 0.y + 2z = 0$

Solving first two equation,

$$\frac{x}{6+4} = \frac{y}{0-0} = \frac{z}{0-0}$$

$$\frac{x}{10} = \frac{y}{0} = \frac{z}{0}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

$\therefore x = 1, y = 0, z = 0$

Thus the eigen vector corresponding to the eigen value $\lambda = 3$ is (1, 0, 0). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 3$.

Similarly, putting $\lambda = 5$,

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + 4z = 0$
$0.x - 3y + 6z = 0$
$0.x + 0.y + 0.z = 0$
Solving first two equation,

$$\frac{x}{6+12} = \frac{y}{0+12} = \frac{z}{6-0}$$

$$\frac{x}{18} = \frac{y}{12} = \frac{z}{6}$$

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{1}$$

$\therefore x = 3, y = 2, z = 1$

Thus the eigen vector corresponding to the eigen value $\lambda = 5$ is (3, 2, 1). Also every non-zero multiple of this vector is eigen vector corresponding to $\lambda = 5$.

Therefore the eigen vecror corresponding to the eigen value λ = 3, 3, 5 are,

$a_1 (1, -1, 0), a_1 \neq 0$
$a_2 (1, 0, 0), a_2 \neq 0$
$a_3 (3, 2, 1), a_3 \neq 0$

(xi) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ (BE 2...)

Solution: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

The characteristic equation of the matrix A is,
$|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

or, $(1 - \lambda)(2 - \lambda)(2 - \lambda) = 0$
$\therefore \lambda = 1, 2, 2$

Thus the eigen value of A are, $\lambda = 1, 2, 2$.

If x, y, z be components eigen vectors error esponding to the eigen value λ. Then the matrix equation is,

$(A - \lambda I)\, x = 0$

i.e. $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Putting $\lambda = 1$,

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0.x + 2y + 3z = 0$
$0.x + y + 3z = 0$
$0.x + 0.y + z = 0$
Solving first two equation,

$$\frac{x}{6-3} = \frac{y}{0-0} = \frac{z}{0-0}$$

$$\frac{x}{3} = \frac{y}{0} = \frac{z}{0}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

$\therefore x = 1, y = 0, z = 0$

Thus the eigen vector corresponding to the eigen value $\lambda = 1$ is (1, 0, 0). Also every non-zero multiple of this vector is eigen vector corresponding to eigen value $\lambda = 1$

Similarly, putting $\lambda = 2$.

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$-x + 2y + 3z = 0$
$0.x + 0.y + 3z = 0$
$0.x + 0.y + 0.z = 0$
Solving first two equation,

$$\frac{x}{6-0} = \frac{y}{0+3} = \frac{z}{0-0}$$

$$\frac{x}{6} = \frac{y}{3} = \frac{z}{0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{0}$$

$\therefore x = 2, y = 1, z = 0$

Thus the eigen vector corresponding to the eiven value $\lambda = 2$ is (2, 1, 0). Also every non-zero multiple of this vecotor eigen vector corresponding to $\lambda = 2$.

Therefore the eigen vectors corresponding to eigen value $\lambda = 1, 2$, are,

$a_1 (1, 0, 0) = a_1 \neq 0$
& $a_2 (2, 1, 0), a_2 \neq 0$

(xii) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix} = 0$

or, $(3-\lambda)\{(-3-\lambda)(7-\lambda)+20\} - 10\{-2(7-\lambda)+12\} + 5\{-10-3(-3-\lambda)\}$

or, $(3-\lambda)(-21-7\lambda+3\lambda+\lambda^2+20) - 10(-14+2\lambda+12) + 5(-10+9+3\lambda)$

or, $(3-\lambda)(\lambda^2-4\lambda-1) + 140 - 20\lambda - 120 - 50 + 45 + 15\lambda = 0$

or, $(3-\lambda)(\lambda^2-4\lambda-1) - 5\lambda + 15 = 0$

or, $3\lambda^2 - \lambda^3 - 12\lambda + 4\lambda^2 - 3 + \lambda - 5\lambda + 15 = 0$

or, $-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$

or, $\lambda^3 + 7\lambda^2 + 16\lambda - 12 = 0$

or, $\lambda^3 - 2\lambda^2 - 5\lambda^2 + 10\lambda + 6\lambda - 12 = 0$

or, $\lambda^2(\lambda-2) - 5\lambda(\lambda-2) + 6(\lambda-2) = 0$

or, $(\lambda-2)(\lambda^2-5\lambda+6) = 0$

or, $(\lambda-2)(\lambda^2-3\lambda-2\lambda+6) = 0$

or, $(\lambda-2)\{\lambda(\lambda-3)-2(\lambda-3)\} = 0$

or, $(\lambda-2)(\lambda-2)(\lambda-3) = 0$

$\therefore \lambda = 2, 2, 3$

Thus the eigen values of A are $\lambda = 2, 2, 3$.

If x, y, z be components eigen vector corresponding to the eigen value λ. Then the matrix equation is,

$(A - \lambda I)x = 0$

$$\begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 2$,

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 10y + 5z = 0$

$-2x - 5y - 4z = 0$

$3x + 5y + 5z = 0$

Solving first two equation,

$$\frac{x}{-40+25} = \frac{y}{-10+4} = \frac{z}{-5+10}$$

$$\frac{x}{-15} = \frac{y}{-6} = \frac{z}{5}$$

$$\frac{x}{5} = \frac{y}{2} = \frac{z}{-5}$$

$\therefore x = 5, y = 2, z = -5$

Thus the eigen vector corresponding to the eigen value $\lambda = 2$ is (5, 2, − 5). Also every non-zero multiple of this vector is eigen vector corresponding to eigen value $\lambda = 2$.

Similarly putting $\lambda = 3$,

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0.x + 10y + 5z = 0$

$-2x - 6y - 4z = 0$ i.e. $2x + 6y + 48 = 0$

$3x + 5z + 4z = 0$

Solving first two equation,

$$\frac{x}{-40+30} = \frac{y}{-10+0} = \frac{z}{-0+20}$$

$$\frac{x}{-10} = \frac{y}{-10} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-2}$$

$\therefore x = 1, y = 1, z = -2$

Thus the eigen vector corresponding to the eigen value $\lambda = 3$ is (1, 1, − 2). Also every non-zero multiple of this vector is eigen value corresponding to eigen value $\lambda = 3$.

Therefore the eigen vector corresponding to the eigen value $\lambda = 2$, 3 are,

$a_1(5, 2, -5), a_1 \neq 0$

& $a_2(1, 1, -2), a_2 \neq 0$

(xiii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **(BE 2062)**

Solution: Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$

or, $(-2-\lambda)(-\lambda+\lambda^2-12) - 2(-2\lambda-6) - 3(-4+1-\lambda) = 0$

or, $2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$

or, $-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

or, $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

or, $\lambda^3 + 3\lambda^2 - 2\lambda^2 - 6\lambda - 15\lambda - 45 = 0$

or, $\lambda^2(\lambda + 3) - 2\lambda(\lambda + 3) - 15(\lambda + 3) = 0$

or, $(\lambda + 3)(\lambda^2 - 5\lambda + 3\lambda - 15) = 0$

or, $(\lambda + 3)\{\lambda(\lambda - 5) + 3(\lambda - 5)\} = 0$

or, $(\lambda + 3)(\lambda - 5)(\lambda + 3) = 0$

$\therefore \lambda = -3, -3, 5$

Thus the eigen vector of A are $\lambda = -3, -3, 5$.

If x, y, z be component eigen vector corresponding to the eigen value λ. Then the matrix equation, is,

$(A - \lambda I)x = 0$

i.e. $\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Putting $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y - 3z = 0$

$2x + 4y - 6z = 0$

$-x - 2y + 3z = 0$

Clearly $x = 1, y = 1, z = 1$

Thus eigen vector corresponding to the eigen value $\lambda = -3$ is (1, 1, 1). Also every non-zero multiple of this vector is eigen vector corresponding to the eigen value $\lambda = -3$.

Similarly, putting $\lambda = 5$,

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x + 2y - 3z = 0$

$2x - 4y - 6z = 0$

$-x - 2y - 5z = 0$

Solving first two equation,

$$\frac{x}{-12-12} = \frac{y}{-6-42} = \frac{z}{28-4}$$

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$\therefore x = 1, y = 2, z = -1$

Thus eigen vector corresponding to the eigen value $\lambda = 5$ is (1, 2, -1). Also every non-zero multiple of this vector is eigen vecor corresponding to the eigen value $\lambda = 5$.

Therefore the eigen vector corresponding to the eigen value $\lambda = -3, 5$ are,

$a_1(1, 1, 1), a_1 \neq 0$

$a_2(1, 2, -1)\ a_2 \neq 0$

8. Verify cayley-Hamilton theorem for the matrix A & find it's inverst.

(i) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ y - 2x + 1

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 3 \end{bmatrix} = 0$

or, $(2 - \lambda)(4 - 4\lambda + \lambda^2 - 1) + (-2 + \lambda + 1) + (1 - 2 + \lambda) = 0$

or, $(2 - \lambda)(3 - 4\lambda + \lambda^2) - 2 + \lambda + 1 + 1 - 2 + \lambda = 0$

or, $6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 - 2 + 2\lambda = 0$

or, $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$

i.e. $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Now, we have to verify that, $A^3 - 6A^2 + 9A - 4I = 0$ ..... (i)

For this,

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\& \ A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies caley - Hamilton theorem.

To find the inverse, multiplying both sides of (i) by A^{-1} we get,

$A^2 - 6A + 9I - 4A^{-1} = 0$

or, $4A^{-1} = 6A - A^2 - 9I$

$$= -6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12+6+9 & 6-5-0 & -6+5-0 \\ 6-5-0 & -12+6-9 & 6-5+0 \\ -6+5-0 & 6-5-0 & -12+6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ **(BE 2063)**

Solution: Let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & -2-\lambda & 1 \\ 4 & 2 & 1-\lambda \end{bmatrix} = 0$

or, $(1-\lambda)\{(-2-\lambda)(1-\lambda) - 2\} - 2\{3(1-\lambda) - 4\} + 3\{6 - 4(-2-\lambda)\} = 0$

or, $(1-\lambda)(-2+2\lambda-\lambda+\lambda^2-2) - 2(3-3\lambda-4) + 3(6+8+4\lambda) = 0$

or, $(1-\lambda)(\lambda^2+\lambda-4) + 2(3\lambda+1) + 3(4\lambda+14) = 0$

or, $\lambda^2+\lambda-4-\lambda^3-\lambda^2+4\lambda+6\lambda+2+12\lambda+42 = 0$

or, $-\lambda^3 + 23\lambda - 40 = 0$

i.e. $\lambda^3 - 23\lambda - 40 = 0$

Now we have to verify that,

$A^3 - 23A - 40I = 0$ .... (i)

For this,

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

$\therefore A^3 - 23A - 40I$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 60-23-40 & 46-46-0 & 69-69-0 \\ 69-69-0 & -6+46-40 & 23-23+0 \\ 92-92-0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies caley-Hemilton theorem.

To find the inverse, multiplying both sides of (i) by A^{-1}, we get,

$A^2 - 23I - 40A^{-1}$

$40A^{-1} = A^2 - 23I$

$$= \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} - 23\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

(iii) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The characteristic equation of the matrix is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$

or, $(1-\lambda)\{(2-\lambda)(1-\lambda) - 6\} - 3\{4(1-\lambda) - 3\} + 7\{8 - 1(2-\lambda)\} = 0$

or, $(1-\lambda)(2-2\lambda-\lambda+\lambda^2-6) - 3(4-4\lambda-3) + 7(8-2+\lambda) = 0$

or, $(1-\lambda)(-4-3\lambda+\lambda^2) - 3(1-4\lambda) + 7(6+\lambda) = 0$

or, $-4-3\lambda+\lambda^2+4\lambda+3\lambda^2-\lambda^3-3+12\lambda+42+7\lambda = 0$

or, $-\lambda^3 + 4\lambda^2 + 20\lambda + 35 = 0$

or, $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

Now, we have to verify that,

$A^3 - 4A^2 - 20A - 35I = 0$ .... (i)

For this,

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$\therefore A^3 - 4A^2 - 20A - 35I$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135-80-20-35 & 152-92-60-0 & 232-92-140-0 \\ 140-60-80-0 & 163-88-40-35 & 208-148-60-0 \\ 60-40-20-0 & 76-36-40 & 111-56-20-35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies caley-Hemilton theorem.

To find the inverse, multiplying both sides of (i) by A^{-1},

We get,

$A^2 - 4A - 20I - 35A^{-1} = 0$

$\therefore 35A^{-1} = A^2 - 4A - 20I$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

(iv) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ (BE 2064)

Solution: Let, $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{bmatrix} = 0$

or, $(1-\lambda)\{(2-\lambda)(3-\lambda) - 0\} + 2\{0 - 2(2-\lambda)\} = 0$

or, $(1-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2) + 2(-4 + 2\lambda) = 0$

or, $(1-\lambda)(6 - 5\lambda + \lambda^2) - 8 + 4\lambda = 0$

or, $6 - 5\lambda + \lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 - 8 + 4\lambda = 0$

or, $-\lambda^3 + 6\lambda^2 - 7\lambda - 2 = 0$

or, $\lambda^3 - 6\lambda^2 - 7\lambda + 2 = 0$

Now, We have to verify that,

$A^3 - 6A^2 - 7A + 2I = 0$ .... (i)

For this,

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 7A + 2I$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0+0+0+0 & 34-48+14+0 \\ 12-12-0-0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0-0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies caley-Hemilton theorem.

To find the inverse, multiplying both sides of (i) by A^{-1}, we get,

$A^2 - 6A + 7I + 2A^{-1} = 0$

$\therefore -2A^{-1} = A^2 - 6A + 7I$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & -4 \\ 2 & -1 & -1 \\ -4 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = -\frac{1}{2}\begin{bmatrix} 6 & 0 & -4 \\ 2 & -1 & -1 \\ -4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$$

(v) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

A characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{bmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix} = 0$

or, $(1-\lambda)(1-2\lambda+\lambda^2-1)+3(-2-1+\lambda)=0$

or, $(1-\lambda)(\lambda^2-2\lambda)+3(\lambda-3)=0$

or, $\lambda^2-2\lambda-\lambda^3+2\lambda^2+3\lambda-9=0$

or, $-\lambda^3+3\lambda^2+\lambda-9=0$

or, $\lambda^3-3\lambda^2-\lambda+9=0$

Now, we have to verify that,

$A^3-3A^2-A+9I=0$ .... (i)

For this,

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\& \ A^3 = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$$

$\therefore A^3-3A^2-A+9I$

$$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} - 3 \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-12-1+9 & -9+9-0+0 & 21-18-3+0 \\ 11-9-2+0 & -2-6-1+9 & 11-12+1+0 \\ 1=0-1+0 & -7+6+1+0 & 7-15-1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This verifies caley-Hemilto theorem.

To find the inverse, multiplying both sides of (i) by A^{-1} we get,

$A^2-3A-I+9A^{-1}=0$

$\therefore -9A^{-1}=A^2-3A-I$

$$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3-1 & -3-0-0 & 6-9-0 \\ 3-6-0 & 2-3-1 & 4+3-0 \\ 0-3-0 & -2+3-0 & 5-3-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{9} \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

9. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then show that $A^2-4A-5I=0$. Where I is the identity matrix.

Solution: Given, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$\therefore A^2-4A-5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore A^2-4A-5I=0$ Where I is the identity matrix.

(10) Diagonalise the following matrces.

(i) $\begin{bmatrix} 3 & 5 \\ 0 & 2 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 3 & 5 \\ 0 & 2 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A-\lambda I|=0$

i.e. $\begin{vmatrix} 3-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)(2-\lambda)-0=0$

or, $(2-\lambda)(3-\lambda)=0$

$\therefore \lambda = 2, 3$

$\therefore \lambda = 2, 3$ are characteristic values of A. Characteristic vectors for $\lambda = 2$ is,

$$\begin{bmatrix} 3-2 & 5 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to the equations,

$x_1+5x_2=0$

$0.x_1+0.x_2=0$

Clearly, $x_1=5, x_2=-1$

So the characteristic vector is, $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

Again the characteristic vector for $\lambda = 3$ is,

$$\begin{bmatrix} 3-3 & 5 \\ 0 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to the equations,

$0.x_1 + 5x_2 = 0$

$0.x_1 - x_2 = 0$

Clearly, $x_1 = 1, x_2 = 0$

So the characteristic vector is, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Hence the model matrix is,

$$C = \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix}$$

$$|C| = \begin{vmatrix} 5 & 1 \\ -1 & 0 \end{vmatrix} = 0 + 1 = 1$$

$$C^{-1} = \frac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix}$$

We kn w that diagonal of the given matrix A is,

$$C^{-1}AC = \begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0-2 \\ 3+0 & 5+10 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \\ 3 & 15 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0+2 & 0+0 \\ 15-15 & 3+0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

(ii) $\begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$

The characteristic equation of the matrix A is,

$|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 1 \\ 5 & 6-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)(6-\lambda) - 5 = 0$

or, $12 - 2\lambda - 6\lambda + \lambda^2 - 5 = 0$

or, $\lambda^2 - 8\lambda + 7 = 0$

or, $\lambda^2 - 7\lambda - \lambda + 7 = 0$

or, $\lambda(\lambda - 7) - 1(\lambda - 7) = 0$

or, $(\lambda - 7)(\lambda - 1) = 0$

$\therefore \lambda = 7, 1$ are the characteristic value of A.

Characteristic vector for $\lambda = 1$ is,

$$\begin{bmatrix} 2-1 & 1 \\ 5 & 6-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to the equations,

$x_1 + x_2 = 0$

$5x_1 + 5x_2 = 0$

$\therefore x_1 = 1, x_2 = -1$

So the characteristic vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Characteristic vector for $\lambda = 7$ is,

$$\begin{bmatrix} 2-7 & 1 \\ 5 & 6-7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to the equations,

$-5x_1 + x_2 = 0$

$5x_1 - x_2 = 0$

$\therefore x_1 = 1, x_2 = 5$

So the characteristic vector is, $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Hence the model matrix is,

$$C = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$$

$$|C| = \begin{vmatrix} 1 & 1 \\ -1 & 5 \end{vmatrix} = 5 + 1 = 6$$

$$C^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}$$

We know the diagonal of the given matrix A is,

$$C^{-1}AC = \frac{1}{6} \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 10-5 & 5-6 \\ 2+5 & 1+6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & -1 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5+1 & 5-5 \\ 7-7 & 7+35 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 42 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

(iv) $\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$

Solution: Let, $A = \begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$

The characteristic equation of the matrix A is,

i.e. $\begin{vmatrix}1-\lambda & 0 & -1\\ 1 & 2-\lambda & 1\\ 2 & 2 & 3-\lambda\end{vmatrix} = 0$

or, $(1-\lambda)\{(2-\lambda)(3-\lambda)-2\} - 1\{2-2(2-\lambda)\}$

or, $(1-\lambda)(6-2\lambda-3\lambda+\lambda^2-2) - (2-4+2\lambda)$

or, $(1-\lambda)(\lambda^2-5\lambda+4) - 2(-2+2\lambda)$

or, $\lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda+2-2\lambda$

or, $-\lambda^3+6\lambda^2-11\lambda+6=0$

or, $\lambda^3-6\lambda^2+11\lambda-6=0$

or, $\lambda^3-\lambda^2-5\lambda^2+5\lambda+6\lambda-6=0$

or, $\lambda^2(\lambda-1)-5\lambda(\lambda-1)+6(\lambda-1)=0$

or, $(\lambda-1)(\lambda^2- \quad \lambda+6)=0$

or, $(\lambda-1)(\lambda^2-2\lambda-3\lambda+6)=0$

or, $(\lambda-1)\{\lambda(\lambda-2)-3(\lambda-2)\}=0$

or, $(\lambda-1)(\lambda-2)(\lambda-3)=0$

$\therefore \lambda = 1, 2, 3$

So the characteristic values of A are, $\lambda = 1, 2, 3$

Characteristic vector for $\lambda = 1$ is,

$$\begin{bmatrix}1-1 & 0 & -1\\ 1 & 2-1 & 1\\ 2 & 2 & 3-1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & -1\\ 1 & 1 & 1\\ 2 & 2 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This reduces to the equations,

$0.x_1 + 0.x_2 - x_3 = 0$

$x_1 + x_2 + x_3 = 0$

$2x_1 + 2x_2 + 2x_3 = 0$

Clearly $x_1 = 1, x_2 = -1, x_3 = 0$

So the characteristic vector is, $\begin{bmatrix}1\\-1\\0\end{bmatrix}$

Characteristic vector for $\lambda = 2$ is,

$$\begin{bmatrix}1-2 & 0 & -1\\ 1 & 2-2 & 1\\ 2 & 2 & 3-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0 & -1\\ 1 & 0 & 1\\ 2 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This reduces to the equations,

$-x_1 + 0.x_2 - x_3 = 0$

$x_1 + 0.x_2 + x_3 = 0$

$2x_1 + 2x_2 + x_3 = 0$

Solving last two equation,

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore x_1 = -2, x_2 = 1, x_3 = 2$

So the characteristic vector is, $\begin{bmatrix}-2\\1\\2\end{bmatrix}$

Characteristic vector for $\lambda = 3$,

$$\begin{bmatrix}1-3 & 0 & -1\\ 1 & 2-3 & 1\\ 2 & 2 & 3-3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This reduces to the equations,

$-2x_1 + 0.x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$2x_1 + 2x_2 + 0.x_3 = 0$

Solving first two equation,

$$\frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore x_1 = -1, x_2 = 1, x_3 = 2$

So the characteristic vector is, $\begin{bmatrix}-1\\1\\2\end{bmatrix}$

Hence the model matrix is,

$C = \begin{bmatrix}1 & -2 & -1\\ -1 & 1 & 1\\ 0 & 2 & 2\end{bmatrix}$

$|C| = \begin{bmatrix}1 & -2 & -1\\ -1 & 1 & 1\\ 0 & 2 & 2\end{bmatrix}$

$= 1(2-2) + 1(-4+2)$

$= -2$

For cofactor of C,

First row, cofactor of $1 = 2 - 2 = 0$

cofactor of $-2 = -(-2 - 0) = 2$

cofactor of $-1 = (-2 - 0) = -2$

2nd row, cofactor of $-1 = -(-4 + 2) = 2$

cofactor of $1 = (2 - 0) = 2$

cofactor of $1 = -(2 - 0) = -2$

3rd row, cofactor of $0 = -2 + 1 = -1$

cofactor of $2 = -(1 - 1) = 0$

cofactor of $2 = 1 - 2 = -1$

$$\therefore \text{Cofactor of } C = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\text{Adj}(C) = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$C^{-1} = \frac{\text{Adj}(C)}{|C|} = \frac{1}{-2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

We know the diagonal of given matrix A is,

$$C^{-1}AC = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 4 & 4 & 0 \\ -6 & -6 & -3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(v) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: Let, $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation of the matrix A is,

i.e. $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

or, $(-2-\lambda)\{(1-\lambda)(-\lambda) - 12\} - 2(-2\lambda - 6) - 3\{-4 + 1(1-\lambda)\} = 0$

or, $(-2-\lambda)(-\lambda + \lambda^2 - 12) + 4\lambda + 12 + 12 - 3 + 3\lambda = 0$

or, $2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 7\lambda + 21 = 0$

or, $-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

or, $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

or, $\lambda^3 + 3\lambda^2 - 2\lambda^2 - 6\lambda - 15\lambda - 45 = 0$

or, $\lambda^2(\lambda + 3) - 2\lambda(\lambda + 3) - 15(\lambda + 3) = 0$

or, $(\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$

or, $(\lambda + 3)(\lambda^2 - 5\lambda + 3\lambda - 15) = 0$

or, $(\lambda + 3)\{\lambda(\lambda - 5) + 3(\lambda - 5)\} = 0$

or, $(\lambda + 3)(\lambda - 5)(\lambda + 3) = 0$

$\therefore \lambda = -3, -3, 5$ are characteristic vectors of A.

Characteristic vector for $\lambda = -3$ is,

$$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This reduces to the equations,

$x_1 + 2x_2 - 3x_3 = 0$

$2x_1 + 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 + 3x_3 = 0$

Clearly, $x_1 = 1, x_2 = 1, x_3 = 1$

So that characteristic vector is, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Also above equations satisfy for, $x_1 = 4, x_2 = 1, x_3 = 2$

So the characteristic vector is, $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Characteristic vector for $\lambda = 5$ is,

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This reduces to the equations,

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 - 5x_1 = 0$

Clearly, $x_1 = -1, x_2 = -2, x_3 = 1$

So the characteristic vector is, $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Hence the model matrix is,

$$C = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$|C| = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= 1(1+4) - 1(4+2) + 1(-8+1)$$

$$= 5 - 6 - 7 = -8$$

For the cofactor of C,

Ist row, cofactor of $1 = 1 + 4 = 5$

cofactor of $4 = -(1+1) = -3$

cofactor of $-1 = 2 - 1 = 1$

2nd row, cofactor of $1 = -(4+2) = -6$

cofactor of $1 = 1 + 1 = 2$

cofactor of $-2 = -(2-4) = 2$

3rd row, cofactor of $1 = -8 + 1 = -7$

cofactor of $2 = -(-2+1) = 1$

cofactor of $1 = 1 - 4 = -3$

$$\therefore \text{Cofactor of } C = \begin{bmatrix} 5 & -3 & 1 \\ -6 & 2 & 2 \\ -7 & 1 & -3 \end{bmatrix}$$

$$\text{Adj}(C) = \begin{bmatrix} 5 & -6 & -7 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$C^{-1} = -\frac{1}{8}\begin{bmatrix} 5 & -6 & -7 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

We know the diagonal of given matrix A is,

$$C^{-1}AC = -\frac{1}{8}\begin{bmatrix} 5 & -6 & -7 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} 15 & 18 & 21 \\ 9 & -6 & -3 \\ 5 & 10 & -15 \end{bmatrix}\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -15+18+21 & -60+18+42 & 15-36+21 \\ 9-6-3 & 36-6-6 & -9+12-3 \\ 5+10-15 & 20+10-30 & -5-20-15 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & -40 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Line Surface and Volume Integrals

2

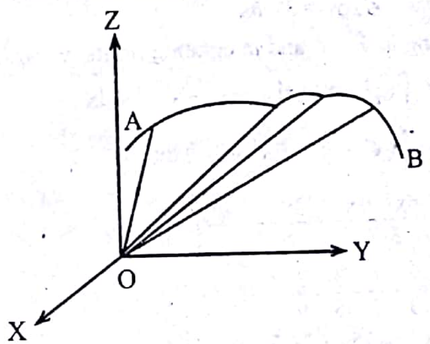
Definitions

Smooth curve : If the curve $\bar{r} = f(t)$ is continuous and differentiable for all values of t in the interval $a \le t \le b$, then the curve is called smooth in that interval.

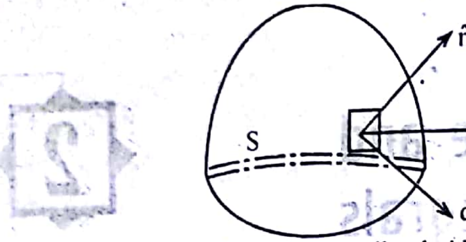
Closed curve : If the initial point and the final point of the curve are same then the curve is called closed curve. If the closed curve doesn't interect itself anywhere, then is said to be closed curve.

Line integral : Any integral which is to be evaluated along a curve is called a line integral.

Surface integral : Any integral evaluated over a surface is called a surface integral which is generalization of double integral. So, the fundamental concept of definite integral and double integrals are pre-requisite to study the surface integral.



Volume integral : Any integral which is evaluated over a volume is called volume integral. Volume integrals are defined and evaluate as surface integrals.



If the volume V is divided in to small cuboids with sides parallel to co-ordinates axes so that $dv = dx\ dy\ dz$, then the volume integral can be written as

$\iiint dv = \iiint \phi dx\, dy\, dz$

Exercise - 7

Evaluate :

(i) **$\int_C \bar{F}.d\bar{r}$ where $\bar{F} = (x^2+y^2)\bar{i} + (x^2-y^2)\bar{j}$ and C is the curve $y = x^3$ in the xy plane from (0,0) to (1, 1).**

Solution: Let $\bar{r} = x\bar{i} + y\bar{j}$

$d\bar{r} = \bar{i}\,dx + \bar{j}\,dy$

$\therefore \bar{F}.d\bar{r} = (x^2+y^2)\ \bar{i} + (x^2-y^2)\ \bar{j}\ .(\bar{i}\,dx + \bar{j}\,dy)$

$= (x^2+y^2)\,dx + (x^2-y^2)\,dy$

$\therefore$ Curve, $y = x^3$, $dy = 3x^2dx$

Substituting in $\bar{F}.d\bar{r}$ and integrating limits, $y = 0$ to $y = 1$

$$\int_C \bar{F}.d\bar{r} = \int_0^1 (x^2+x^6)\,dx + (x^2-x^6)\,3x^2dx$$

$$= \int_0^1 (x^2+x^6+3x^4-3x^8)\,dx$$

$$= \left[\frac{x^3}{3}+\frac{x^7}{7}+\frac{3x^5}{5}-\frac{3x^9}{9}\right]_0^1$$

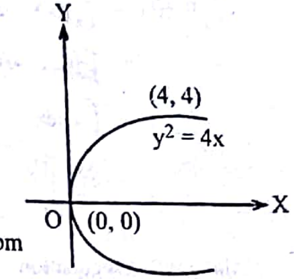
$$= \frac{1}{3}+\frac{1}{7}+\frac{3}{5}-\frac{3}{9}$$

$$= \frac{35+15+63-35}{105}$$

$$= \frac{78}{105}$$

$$= \frac{26}{35}$$

(ii) **$\int_C \bar{F}.\,d\bar{r}$ where $\bar{F} = x^2y^2\ \bar{i} + y\bar{j}$ and C is the curve $y^2 = 4x$ from (0, 0) to (4, 4).**



Solution: Here,

$\bar{F} = x^2y^2\bar{i} + y\bar{j}$ and C is the curve of the parabola,

$y^2 = 4x$, $2ydy = 4dx$

$\therefore dx = \frac{1}{2}\,y\,dy$:

From (0, 0) to (4, 4), so y varies from $y = 0$ to $y = 4$

$\therefore \bar{F}\,.\,d\bar{r} = (x^2y^2\bar{i} + y\bar{j})\,.\,(\bar{i}\,dx + \bar{j}\,dy)$

$= x^2y^2dx + y\,dy$

$$\int_C \bar{F}\,.\,d\bar{r} = \int_C x^2y^2dx + ydy$$

$$= \int_0^4 \left[\frac{y^4.y^2}{16}\times\frac{y}{2}dy + ydy\right]$$

$$= \int_0^4 \left(\frac{y^7}{32}+y\right)dy$$

$$= \left[\frac{y^8}{32\times 8}+\frac{y^2}{2}\right]_0^4$$

$$= \left[\frac{256\times 256}{32\times 8}+\frac{16}{2}-0\right]$$

$= 256 + 8$

$= 264$

$\therefore \int_C \bar{F}\,.\,d\bar{r} = 264$ Ans.

(iii) **$\int_C \bar{F}.\,d\bar{r}$ where $\bar{F} = (2x+y)\ \bar{i} + (3y-x)\ \bar{j}$ and C is the curve in xy – plane along the path which is straight lines from (0, 0) to (3, 0) and then to (4, 2).**

Solution: Here,

$\bar{F} = (2x+y)\ \bar{i} + (3y-x)\ \bar{j}$

$\therefore \bar{F}\,.\,d\bar{r} = \{(2x+y)\ \bar{i} + (3y-x)\ \bar{j}\}\,(\bar{i}\,dx + \bar{j}\,dy)$

$= (2x+y)\,dx + (3y-x)\,dy$

and C is the curve in the xy plane from 0 (0, 0) to A (2, 0) and then B (3, 2)

$$\int_C \bar{F}\,.\,d\bar{r} = \int_{0A} \bar{F}\,.\,d\bar{r} + \int_{AB} \bar{F}\,.\,d\bar{r}$$

$$= \int_{0A}[(2x+y)dx+(3y-x)dy] + \int_{AB}[(2x+y)dx+(3y-x)dy]$$

along OA, $y = 0$, $dy = 0$ and x varies from $x = 0$ to $x = 2$

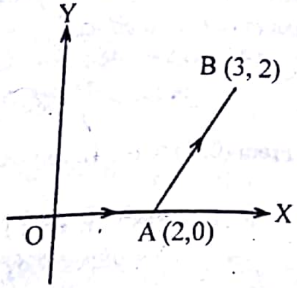
$$\int_{OA} \bar{F} \cdot d\bar{r} = \int_{OA} [(2x+y)dx + (3y-x)dy]$$

$$= \int_0^2 [(2x+0)dx + 0]$$

$$= \int_0^2 2x dx$$

$$= \left[\frac{2x^2}{2}\right]_0^2$$

$$= 4$$



Along AB, the equation of the line AB is,

$$y - 0 = \frac{2-0}{3-2}(x-2)$$

or, $y = 2(x-2) = 2x - 4$

$dy = 2dx$ as x varies from 2 to 3.

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_{AB} (2x+y)\,dx + (3y-x)\,dy$$

$$= \int_2^3 [(2x+2x-4)dx + \{3(2x-4)-x\}2dx]$$

$$= \int_2^3 [4x-4+12x-24-2x]\,dx$$

$$= \int_2^3 [4x-4+10x-24]\,dx$$

$$= \int_2^3 [14x-28]\,dx$$

$$= \left[14\frac{x^2}{2} - 28x\right]_2^3$$

$$= 7 \times 9 - 28 \times 3 - 7 \times 4 + 56$$

$$= 63 - 84 - 28 + 56$$

$$= 119 - 112 = 7$$

$$\therefore \int_{AB} \bar{F} \cdot d\bar{r} = 7$$

Hence, $\int_C \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r}$

$$= 4 + 7$$

$$= 11$$

$$\because \int_C \bar{F} \cdot d\bar{r} = 11 \text{ Ans.}$$

(iv) $\int_C \bar{F} \cdot d\bar{r}$ **where** $\bar{F} = x^2\bar{i} + y^3\bar{j}$ **and C is the arc of the parabola** $y = x^2$.

Solution: Here,

$$\bar{F} = x^2\bar{i} + y^3\bar{j},\ r = x\bar{i} + y\bar{j}$$

$$d\bar{r} = \bar{i}\,dx + \bar{j}\,dy$$

$$\bar{F} \cdot d\bar{r} = (x^2\bar{i} + y^3\bar{j}) \cdot (\bar{i}\,dx + \bar{j}\,dy)$$

$$= x^2dx + y^3dy$$

Also, $dy = 2xdx$ since $y = x^2$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_C x^2\,dx + y^3dy$$

$$= \int_0^1 (x^2dx + x^6(2xdx))$$

$$= \int_0^1 (x^2 + 2x^7)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^8}{4}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{4}$$

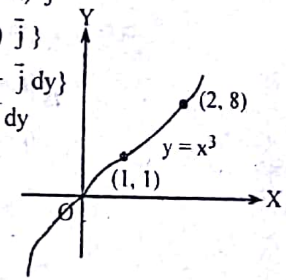
$$= \frac{7}{12} \text{ Ans.}$$

(vi) $\int_C \bar{F} \cdot d\bar{r}$ **where** $\bar{F} = (5xy - 6x^2)\,\bar{i} + (2y - 4x)\,\bar{j}$ **along the curve C in xy plane** $y = x^3$ **from the point (1, 1) to (2, 8).**

Solution: Here, $\bar{F} = (5xy - 6x^2)\,\bar{i} + (2y - 4x)\,\bar{j}$

$$\therefore \bar{F} \cdot d\bar{r} = \{(5xy - 6x^2)\,\bar{i} + (2y-4x)\,\bar{j}\} \cdot \{\bar{i}\,dx + \bar{j}\,dy\}$$

$$= (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$



Here, The given curve is,

$$y = x^3,\ dy = 3x^2dx$$

and x curies from $x = 1$ to $x = 2$

$$\int_C \bar{F} \cdot d\bar{r} = \int_1^2 [(5x \times x^3 - 6x^2)\,dx + (2 \times x^3 - 4x) \times x3x^2dx]$$

$$= \int_1^2 [(5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

$$= \left[\frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4}\right]_1^2$$

$$= (32-1) - 2(8-1) + (64-1) - 3(16-1)$$

$$= 31 - 14 + 63 - 45$$

$$= 94 - 59 = 35$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = 35 \text{ Ans.}$$

(vii) $\int_C \bar{F} \cdot d\bar{r}$ **where** $\bar{F} = (x-y)\,\bar{i} + (x+y)\,\bar{j}$ **along the curve bounded by** $y = \sqrt{x}$ **and** $y = x^2$. **(BE 2061)**

Solution: Here, $\bar{F} = (x-y)\,\bar{i} + (x+y)\,\bar{j}$

$$\bar{F} \cdot d\bar{r} = \{(x-y)\,\bar{i} + (x+y)\,\bar{j}\} \cdot (\bar{i}\,dx + \bar{j}\,dy)$$

$$= (x - y)\,dx + (x + y)\,.\,dy$$

and C is the curve bounded by $y^2 = x$ and $x^2 = y$

$$\int_C \bar{F}.d\bar{r} = \int_{0A} \bar{F}.d\bar{r} + \int_{A0} \bar{F}.d\bar{r}$$

Along 0A, $y = x^2$, $dy = 2xdx$ and x varies from $x = 0$ to $x = 1$

$$\int_{0A} \bar{F}.d\bar{r} = \int_{0A}[(x-y)dx+(x+y)dy]$$

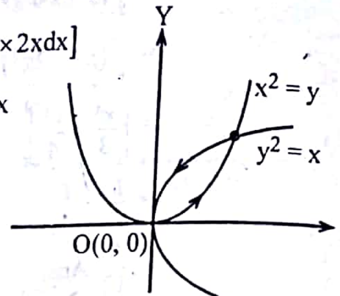
$$= \int_0^1[(x-x^2)dx+(x+x^2)\times 2xdx]$$

$$= \int_0^1(x - x^2 + 2x^2 + 2x^3)\,dx$$

$$= \int_0^1(2x^3 + x^2 + x)\,dx$$

$$= \left[\frac{2x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}+\frac{1}{3}+\frac{1}{2} = 1+\frac{1}{3} = \frac{4}{3}$$



Along A0,
$y^2 = x$, $2ydy = dx$ and y varies form $y = 1$ to $y = 0$
So,

$$\int_{A0} \bar{F}.d\bar{r} = \int_{A0}[(x-y)dx+(x+y)dy]$$

$$= \int_1^0[(y^2-y)\times 2ydy+(y^2+y)\,dy]$$

$$= \int_1^0[(2y^3-2y^2+y^2+y)\,dy$$

$$= \int_1^0[(y^3-y^2+y)\,dy$$

$$= \left[\frac{2y^4}{4}-\frac{y^3}{3}+\frac{y^2}{2}\right]_1^0$$

$$= 0-\frac{1}{2}+\frac{1}{3}-\frac{1}{2}$$

$$= \frac{1}{3}-1$$

$$= -\frac{2}{3}$$

Thus, $\int_C \bar{F}.d\bar{r} = \int_{0A} \bar{F}.d\bar{r} + \int_{A0} \bar{F}.d\bar{r}$

$$= \frac{4}{3}-\frac{2}{3}$$

$$= \frac{2}{3}$$

$\therefore\ \int_C \bar{F}.d\bar{r} = \frac{2}{3}$ Ans.

2. Evaluate :

(i) $\int_C \bar{F}.d\bar{r}$ **where** $\bar{F} = y^2\,\bar{i} + x^2\,\bar{j}$ **and C is the curve of the parabola** $y = x^2$ **from (0, 0) to (1, 1)**

Solution: The equation of the straight line joining (0, 0) to (1, 1) is $y = x$

$$\bar{F} = y^2\bar{i} + x^2\bar{j},\ \bar{r} = x\bar{i} + y\bar{j}$$

$$\therefore d\bar{r} = dx\,\bar{i} + dy\,\bar{j},\ dy = dx \text{ as } y = x$$

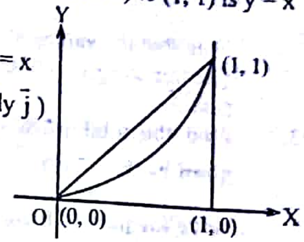
$$\therefore \int_C \bar{F}.d\bar{r} = \int_C(y^2\bar{i}+x^2\bar{j}).(dx\,\bar{i}+dy\,\bar{j})$$

$$= \int_C(y^2dx + x^2dy)$$

$$= \int_0^1(x^2dx + x^2dx)$$

$$= 2\int_0^1 x^2\,dx$$

$$= \frac{2}{3} \text{ Ans.}$$



(ii) $\int_C \bar{F}.d\bar{r}$ **where** $\bar{F} = y^2\,\bar{i} + x^2\,\bar{j}$ **and C is the straight line from (0, 0) to (1, 1)**

Solution: The curve is $y = x^2$, hence $dy = 2xdx$

$$\int_C \bar{F}.d\bar{r} = \int_C(y^2\bar{i}+x^2\bar{j}).(\bar{i}\,dx + \bar{j}\,dy)$$

$$= \int_C(y^2dx + x^2dy)$$

$$= \int_0^1(x^4dx + x^2.2xdx)$$

$$= \left[\frac{x^5}{5}+\frac{x^4}{2}\right]_0^1$$

$$= \frac{1}{5}+\frac{1}{2}$$

$$= \frac{7}{10} \text{ Ans.}$$

(iii) $\int_C \bar{F}.d\bar{r}$ **where** $\bar{F} = y^2\,\bar{i} + x^2\,\bar{j}$ **C is the straight line from (0, 0) to (1, 0) and then from (1, 0) to (1, 1).**

Solution: Let the straight line from (0, 0) to (1, 0) be C_1 and that from (1, 0) to (1, 1) be C_2, so that the path C consists of C_1 and C_2.

For C_1, $y = 0$, $\bar{F} = x^2\bar{j}$, $\bar{r} = x\bar{i}$

$$\therefore \int_{C_1} \bar{F}.d\bar{r} = \int_{C_1}(x^2\bar{j}).(dx\,\bar{i}) = 0$$

For C_2, $x = 1$, $dx = 0$

$$\bar{F} = y^2\bar{i} + \bar{j},\ \bar{r} = x\,\bar{i} + y\bar{j},\ d\bar{r} = dy\,\bar{j}$$

$$\therefore \int_{C_2} \bar{F}.d\bar{r} = \int_{C_2}(y^2\bar{i}+\bar{j}).(dy\,\bar{j})$$

$$= \int_0^1 dy = 1$$

$\therefore$ Hence, $\int_C \bar{F} \cdot d\bar{r} = \int_{C_1} \bar{F} \cdot d\bar{r} + \int_{C_2} \bar{F} \cdot d\bar{r}$

$$= 0 + 1$$
$$= 1$$

Note that the values of integral along different paths are not same. The line integral of the given vector function is not independent of path.

3. **Find the total work done in moving particle in a force field given by $\bar{F} = 2xyz^2\,\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\,\bar{k}$ along the path C from (0, 0, 1) to $\left(1, \frac{\pi}{4}, 2\right)$.**

Solution: Here, $\bar{F} = 2xyz^2\,\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\,\bar{k}$

$$\bar{F} \cdot d\bar{r} = \{2xyz^2\,\bar{i} + (x^2z^2 + z\cos yz)\,\bar{j} + (2x^2yz + y\cos yz)\,\bar{k}\} \cdot (\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz)$$
$$= 2xyz^2dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$$
$$= 2xyz^2dx + x^2z^2dy + 2x^2yzdz + z\cos(yz)\,dy + y\cos(yz)\,dz$$
$$= d(x^2z^2y) + \cos yz\,d(yz)$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_{(0,0,1)}^{\left(1,\frac{\pi}{4},2\right)} [d(x^2z^2y) + \cos yz\,d(yz)]$$
$$= [x^2z^2y + \sin yz]_{(0,0,1)}^{\left(1,\frac{\pi}{4},2\right)}$$
$$= \left[1 \times 4 \times \frac{\pi}{4} + \sin\frac{\pi}{2} - 0 - \sin 0\right]$$
$$= \pi + 1 - 0$$
$$= \pi + 1$$

$\therefore \int_C \bar{F} \cdot d\bar{r} = \pi + 1$ Ans.

4. **Show that $\int_C \bar{F} \cdot d\bar{r} = 2\pi$ where $\bar{F} = \frac{-y}{x^2 + y^2}\,\bar{i} + \frac{x}{x^2 + y^2}\,\bar{j}$ and C is the circle $x^2 + y^2 = 1$ in the z-plane described in the anti-clockwise sense.**

Solution: Let $\bar{F} = \frac{-y}{x^2 + y^2}\,\bar{i} + \frac{x}{x^2 + y^2}\,\bar{j}$.

$$\therefore \bar{F} \cdot d\bar{r} = \left[\left(\frac{-y}{x^2 + y^2}\right)\bar{i} + \left(\frac{x}{x^2 + y^2}\right)\bar{j}\right] \cdot [\bar{i}\,dx + \bar{j}\,dy]$$
$$= \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \;\;\ldots\ldots\text{(i)}$$

The equation of the cirle $x^2 + y^2 = 1$, $z = 0$ in parametric from is $x = \cos\theta$, $y = \sin\theta$, $z = 0$

$dx = -\sin\theta d\theta$, $dy = \cos\theta d\theta$, $dz = 0$

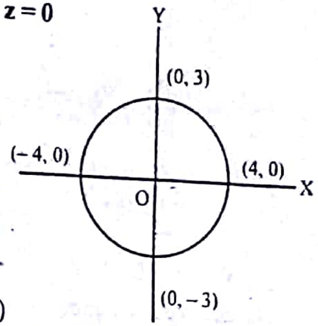
and θ varies from $\theta = 0$ to $\theta = 2\pi$

$$\therefore \bar{F} \cdot d\bar{r} = -\sin\theta(-\sin\theta)\,d\theta + \cos\theta \cdot \cos\theta \cdot d\theta$$
$$= (\sin^2\theta + \cos^2\theta)\,d\theta$$
$$= d\theta$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} d\theta$$
$$= [\theta]_0^{2\pi}$$
$$= 2\pi - 0$$
$$= 2\pi$$

$\therefore \int_C \bar{F} \cdot d\bar{r} = 2\pi$ Ans.

5. **Find the work-done in moving particle in the field $\bar{F} = (3x - 4y + 2z)\,\bar{i} + (4x + 2y - 3z^2)\,\bar{j} + (2xz - 4y^2 + z^3)\,\bar{k}$ along one round of the ellipse $\frac{x^2}{10} + \frac{y^2}{9} = 1, z = 0$**



Solution: Here, $\bar{F} = (3x - 4y + 2z)\,\bar{i} + (4x + 2y - 3z^2)\,\bar{j} + (2xz - 4y^2 + z^3)\,\bar{k}$

$$\therefore \bar{F} \cdot d\bar{r} = [(3x - 4y + 2z)\,\bar{i} + (4x + 2y - 3z^2)\,\bar{j} + (2xz - 4y^2 + z^3)\,\bar{k}] \cdot (\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz)$$
$$= (3x - 4y + 2z)\,dx + (4x + 2y - 3z^2)\,dy + (2xz - 4y^2 + z^3)\,dy$$

And C is the cuve of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1, z = 0$$

Its parametric equation are,

$x = 4\cos\theta$, $y = 3\sin\theta$, $z = 0$

$dx = -4\sin\theta\,d\theta$, $dy = 3\cos\theta\,d\theta$, $dz = 0$

and θ varies from $\theta = 0$ to $\theta = 2\pi$

Thus,

$$\bar{F} \cdot d\bar{r} = (3x - 4y)\,dx + (4x + 2y)\,dy$$
$$= (3 \times 4\cos\theta - 4 \times 3\sin\theta) \times (-4\sin\theta\,d\theta) + (4 \times 4\cos\theta + 2 \times 3\sin\theta) \times (3\cos\theta\,d\theta)$$
$$= [(12\cos\theta - 12\sin\theta)(-4\sin\theta) + (48\cos^2\theta + 18\sin\theta\cos\theta)\,d\theta$$
$$= [48(\sin^2\theta + \cos^2\theta) - 30\sin\theta\cos\theta]\,d\theta$$
$$= (48 - 15\sin 2\theta)\,d\theta$$

Thus, $\bar{F} \cdot d\bar{r} = \int_0^{2\pi} (48 - 15\sin 2\theta)\,d\theta$

$$= \left[48\theta + \frac{15\cos 2\theta}{2} \right]_0^{2\pi}$$

$$= 48 \times 2\pi + \frac{15}{2}(1-1)$$

$$= 96\pi + 0$$

$$= 96\pi$$

$\therefore \int_c \vec{F}.d\vec{r} = 96\pi$ Ans.

6. **Show that $\vec{F} = (x^2 - yz)\,\bar{i} + (y^2 - 2x)\,\bar{j} + (z^2 - xy)\,\bar{k}$ is irrotational and also find its scalar potential function. (BE 2062)**

Solution: Here, $\bar{F} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$

$$\nabla \times \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial y}\right) \times (yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k})$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \bar{i}\,(x-x) - \bar{j}\,(y-y) + \bar{k}\,(z-z)$$

$$= 0 + 0 + 0$$

$$= 0$$

$\therefore \nabla \times \bar{F} = 0$

Hence, vector dunction $\bar{F}$ is irrotational

Also, we have,

$\bar{F} = \nabla\phi$ or, $\bar{F}\,.\,d\bar{r} = \nabla\phi\,.\,d\bar{r}$

or, $(yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k})\,.\,(\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz)$

or, $(yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k})\,.\,(\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz)$

$$= \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial y}\right).(\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz)$$

$$= yzdx + zxdy + xydz$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial y}dz$$

$$= d\,(xyz)$$

$$= d\phi$$

On integration,

$\phi = xyz + C$

$\therefore$ Scalar potential function is $\phi = xyz + C$ Ans.

7. **Show that $\vec{F} = (2xz^3 + 6y)\,\bar{i} + (6x - 2yz)\,\bar{j} + (3x^2z^2 - y^2)\,\bar{k}$ is conscrleative vector field.**

Solution: Here, $\bar{F} = (2xz^3 + 6y)\,\bar{i} + (6x - 2yz)\,\bar{j} + (3x^2z^2 - y^2)\,\bar{k}$

Curl $\bar{F} = \nabla \times \bar{F}$

$$\therefore \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial y}\right) \times [(2xz^3 + 6y)\,\bar{i} + (6x - 2yz)\,\bar{j} + (3x^2z^2 - y^2)\,\bar{k}]$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^3 & 6x - 2yz & 3x^2z^2 - y^2 \end{vmatrix}$$

$$= \bar{i}\,(-2y + 2y) - \bar{j}\,(6xz^2 - 6xz^2) + \bar{k}\,(6-6)$$

$$= 0 + 0 + 0 = 0$$

$\therefore$ Curl $\bar{F} = 0$

Hence, $\bar{F}$ is a conservative field.

8. **If $\bar{F} = (2xy - z)\,\bar{i} + yz\bar{j} + x\bar{k}$ then evaluate $\int_c \bar{F}.d\bar{r}$ along the following curve C.**

(i) **C is the curve $x = t$, $y = 2t$ $z = t^2 - 1$ with t incresing from $t = 0$ to $t = 1$.**

Solution: Here $x = t$, $y = 2t$, $z = t^2 - 1$

$\therefore dx = dt$, $dy = 2dt$, $dz = 2tdt$

Also, we have, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = \bar{i}dx + \bar{j}dy + \bar{k}dz$

$$\int_c \bar{F}.d\bar{r} = \int_c \{(2xy - z)\bar{i} + yz\bar{j} + x\bar{k}\}.(\bar{i}dx + \bar{j}dy + \bar{k}dz)$$

$$= \int_c [(2xy - z)dx + yzdy + xdz]$$

$$= \int_0^1 [(2xy - z)\,dx + yzdy + xdz]$$

$$= \int_0^1 (4t^2 - t^2 + 1)\,dt + 2t\,(t^2 - 1)\,2dt + t.\,2tdt$$

$$= \int_0^1 (3t^2 + 1 + 4t^3 - 4t + 2t^2)\,dt$$

$$= \int_0^1 (4t^3 + 5t^2 - 4t + 1)\,dt$$

$$= \left[t^4 + \frac{5}{3}t^3 - 2t^2 + t \right]_0^1$$

$$= 1 + \frac{5}{3} - 2 + 1 = \frac{5}{3}$$

(ii) C consists of two straight lines from (0, 0,0) to the point (1, 0, – 1) and from (1, 0, – 1) to the point (2, 3, – 3)

Solution: Let C_1 donote the line segment from (0, 0, 0) to (1, 0, –1) and C_2 denotes the line segment from (1, 0, –1) to (2, 3, – 3). The equation of line through (0, 0, 0) and (1, 0, – 1) is $\frac{x-1}{1-0} = \frac{y-0}{0-0} = \frac{z+1}{-1+0} = t$

i.e. $x = t + 1, y = 0, z = -t - 1$

$\therefore dx = dt, dy = 0, dz = -dt$

$\therefore$ The integral along the line segment C_1 is

$$\int_C \bar{F}.d\bar{r} = \int_C [(2xy - z)\, dx + yzdy + xdz]$$

$$= \int [(t+1)\, dt - (t+1)\, dt]$$

The equation of the line through (1, 0, – 1) and (2, 3, – 3) is $\frac{x-1}{2-1} = \frac{y}{3} = \frac{z+1}{-3+1} = t\,(\text{say})$

i.e. $x = t + 1, y = 3t, z = -2t - 1$

and $dx = dt, dy = 3dt, dz = -2dt$

$\therefore$ The integral along the line segment C_2 is

$$\int_C \bar{F}.d\bar{r} = \int_{C_2} (6t^2 + 8t + 1)\, dt + (-18t^2 - 9t)\, dt + (-2t - 2)\, dt$$

$$= \int_{C_2} (-12t^2 - 3t - 1)\, dt$$

In the curve, $x = t + 1, y = 3t, z = -2t - 1$

$x = 1, y = 0, z = -1 \Rightarrow t = 0$ and $x = 2, y = 3, z = -3 \Rightarrow t = 1$

$$\therefore \int_{C_2} \bar{F}.d\bar{r} = \int_0^1 (-12t^2 - 3t - 1)\, dt$$

$$= \left[-\frac{12t^3}{3} - \frac{3t^2}{2} - t\right]_0^1$$

$$= -4 - \frac{3}{2} - 1$$

$$= \frac{-8-3-2}{2}$$

$$= -\frac{13}{2}$$

$$\therefore \int_C \bar{F}.d\bar{r} = \int_{C_1} \bar{F}.d\bar{r} + \int_{C_2} \bar{F}.d\bar{r}$$

$$= 0 + \left(-\frac{13}{2}\right) = -\frac{13}{2}$$

9. **Find the work-done in moving particle in the force field $\bar{F} = 3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\,\bar{k}$ along.**

(i) **The straight line from (0, 0, 0) to (2, 1, 3)**

(ii) **The curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.**

Solution: $\bar{F} = 3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\bar{k}$

$$\bar{F}.d\bar{r} = [3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\bar{k}].[\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz]$$

$$= 3x^2dx + (2xz - y)\,dy + zdz$$

The equation of the line from (0, 0, 0) to (2, 1, 3) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say), $x = 2t, y = t, z = 3t, dx = 2dt, dy = dt, dz = 3dt$

and t varies from t = 0 to t = 1

$$\int_C \bar{F}.d\bar{r} = \int_C [3x^2.dx + (2xz - y)\, dy + zdz]$$

$$= \int_0^1 [3 \times 4t^2 \times 2dt + (2 \times 2t \times 3t - t)\, dt + (3t)\, 3dt]$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\, dt$$

$$= \int_0^1 (36t^2 + 8t)\, dt$$

$$= \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$= 12 + 4$$

$$= 16$$

$\therefore \int_C \bar{F}.d\bar{r} = 16$ Ans.

(ii)

$\bar{F} = 3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\bar{k}$

$$\bar{F}.d\bar{r} = [3x^2\,\bar{i} + (2xz - y)\,\bar{j} + z\bar{k}].(\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz)$$

$$= 3x^2dx + (2xz - y)\,dy + zdz$$

and the curve is,

$x^2 = 4y, 3x^3 = 8z$

$2xdx = 4dy, 9x^2dx = 8dz$

$\Rightarrow dy = \frac{1}{2}\,xdx,\ dz = \frac{9}{8}\,x^2\,dx$

and x varies from x = 0 to x = 2

$$\int_C \bar{F}.d\bar{r} = \int_C [3x^2\,dx + (2xz - y)\,dy + zdz]$$

$$= \int_0^2 \left[3x^2dx + \left(2x \times \frac{3x^3}{8} - \frac{x^2}{4}\right)\frac{1}{2}xdx + \frac{3x^3}{8} \times \frac{9}{8}x^2\right] dx$$

$$= \left[\frac{3x^3}{3} + \frac{3x^6}{8\times6} - \frac{x^4}{8\times4} + \frac{27}{64}\times\frac{x^6}{6}\right]_0^2$$

$$= 8 + \frac{64}{16} - \frac{16}{32} + \frac{27}{64}\times\frac{64}{6} - 0$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2}$$

$$= 12 + 4$$

$$= 16$$

$$\therefore \int_C \bar{F}\,.\,d\bar{r} = 16 \text{ Ans.}$$

10. Using the line integral, compute the work done by the force, $\bar{F} = (2x - y + 2z)\,\bar{i} + (x + y - z)\,\bar{j} + (3x - 2y - 5z)\,\bar{k}$ when it moves once around a circle $x^2 + y^2 = 4,\ z = 0$

Solution: Parametric equations of the circle are,

$x = 2\cos\theta,\ y = 2\sin\theta$ so that,

$dx = -\sin\theta\, d\theta,\ \ dy = 2\cos\theta d\theta$

$$\int_C \bar{F}\,.\,d\bar{r} = \int_C [\,(2x - y)\,\bar{i} + (x + y)\,\bar{j} + (3x - 2y)\,\bar{k}\,]\,.\,(\bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz)$$

$$= \int_C (2x - y)\,dx + (x + y)\,dy$$

$$= \int_0^{2\pi} (2.2\cos\theta - 2\sin\theta)\,(-2\sin\theta\, d\theta) + (2\cos\theta + 2\sin\theta)\,2\cos\theta\, d\theta$$

$$= 4\int_0^{2\pi} (1 - \cos\theta\,\sin\theta)\,d\theta$$

$$= 4\left[\theta - \frac{\cos 2\theta}{4}\right]_0^{2\pi}$$

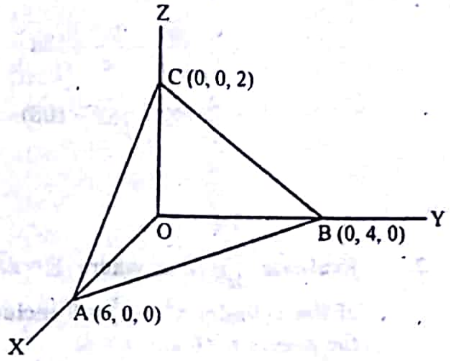
$$= 8\pi \text{ Ans.}$$

Exercise - 8

1. Evaluate $\iint_S \bar{F}\,.\,\hat{n}\,ds$ where $\bar{F} = 6z\bar{i} - 4\bar{j} + y\bar{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.

Solution: The given surface is the plane $2x + 3y + 6z = 12$ in the first octant.

Let $\phi = 2x + 3y + 6z$



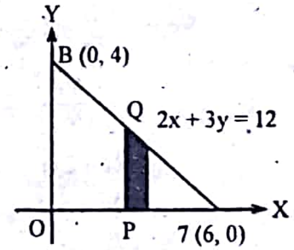
$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{\sqrt{4 + 9 + 36}}$$

$$= \frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{7}$$

Let R be the projection of the plane $2x + 3y + 6z = 12$ on xy – plane which is a triangle OAB bounded by the lines $y = 0$, $x = 0$ and $2x + 3y = 12$

$$ds = \frac{dxdy}{|\hat{n}.\bar{k}|} = \frac{7}{6}\,dx\,dy$$



Along the vertical strip PQ, y varies from 0 to $\frac{12 - 2x}{3}$ and in the region R, x varies from 0 to 6.

$$\iint_S \bar{F}.\hat{n}\,ds = \iint_R (6z\bar{i} - 4\bar{j} + y\bar{k})\,.\left(\frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{7}\right).\,\frac{7}{6}\,dx\,dy$$

$$= \frac{1}{6}\iint_R (12z - 12 + 6y)\,dx\,dy$$

$$= \iint_R (2z - 2 + y)dxdy$$

$$= \iint_R \left[2\left(\frac{12 - 2x - 3y}{6}\right) - 2 + y\right] dx\,dy$$

$$= \frac{1}{3}\iint_R (12 - 2x - 3y - 6 + 3y)\,dx\,dy$$

$$= \frac{1}{3}\iint_R (6 - 2x)\,dxdy$$

$$= \frac{1}{3}\left[\int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\right] dx\, dy$$

$$= \frac{2}{3}\int_0^6 (3-x)\,[y]_0^{\frac{12-2x}{3}}\, dx$$

$$= \frac{2}{3}\int_0^6 (3-x)\left(\frac{12-2x}{3}\right) dx$$

$$= \frac{4}{9}\int_0^6 (x^2-9x+18)\, dx$$

$$= \frac{4}{9}\left[\frac{x^3}{3}-\frac{9x^2}{2}+18x\right]_0^6$$

$$= \frac{4}{9}(72-162+108)$$

$$= \frac{4}{9}\times 18$$

$$= 8$$

2. **Evaluate $\int_S \bar{F}.\hat{n}\, ds$ where $\bar{F} = z\bar{i} + x\bar{j} - yz\bar{k}$ and s is the surface of the cylinder $x^2 + y^2 = 9$ included in the first octant between the planes $z = 0$ and $z = 4$.**

Solution: Here, the surface is $\phi = x^2 + y^2 - 9$. So, vector normal to the surface is

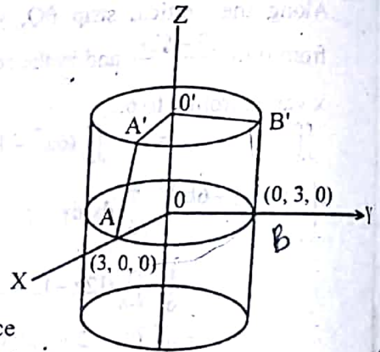
$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2-9) = 2x\bar{i} + 2y\bar{j}$$

$$\therefore\ \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}}$$

$$= \frac{x\bar{i}+y\bar{j}}{\sqrt{x^2+y^2}}$$

$$= \frac{x\bar{i}+y\bar{j}}{3}\quad [\because x^2+y^2 = 9]$$



The projection of the surface on the first octant on yz–plane is OBB'0'. Then

$$\iint_S \bar{F}.\hat{n}\, ds = \iint_{OBB'0'} \bar{F}.\hat{n}\frac{dydz}{|\hat{n}.\bar{i}|}$$

Now, $\hat{n}.\bar{i} = \frac{1}{3}(x\bar{i} + y\bar{j}).\bar{i} = \frac{1}{3}x$

$$\therefore\ \bar{F}.\hat{n} = (z\bar{i} + x\bar{j} - yz\bar{k}).\frac{1}{3}(x\bar{i} + y\bar{j})$$

$$= \frac{1}{3}xz + \frac{1}{3}xy$$

$$= \frac{1}{3}x(y+z)$$

The region 0BB'0' is bounded by $y = 0$ to $y = 4$ and $z = 0$ to $z = 5$.

Here, $\iint_S \bar{F}.\hat{n}\, ds = \iint_S \frac{1}{3}x(y+z)\dfrac{dydz}{\frac{1}{3}x}$

$$= \int_0^4\int_0^3 (y+z)\, dy\, dz$$

$$= \int_0^4\left[\frac{y^2}{2}+yz\right]_0^3 dz$$

$$= \int_0^4\left[\frac{9}{2}+3z\right] dz$$

$$= \left[\frac{9z}{2}+\frac{3z^2}{2}\right]_0^4$$

$$= \frac{9\times 4}{2}+\frac{3\times 4\times 4}{2}$$

$$= 18 + 24$$

$$= 42$$

$$\therefore\ \iint_S \bar{F}.\hat{n}\, ds = 42$$

3. **Evaluate $\iint_S \bar{F}.\hat{n}\, ds$ where $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and (i) S is the surface of the cube bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$.** **(BE 2056, 058)**

Solution: Let ADBECF is a cube which consists. Six surfaces, then,

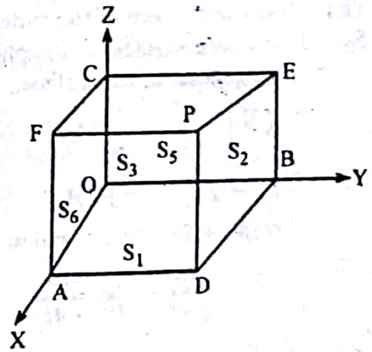
$$\iint_S \bar{F}.\hat{n}\, ds = \iint_{S_1}\bar{F}.\hat{n}\, ds + \iint_{S_2}\bar{F}.\hat{n}\, ds + \iint_{S_3}\bar{F}.\hat{n}\, ds + \iint_{S_4}\bar{F}.\hat{n}\, ds + \iint_{S_5}\bar{F}.\hat{n}\, ds$$

For the surface OADB, $\hat{n} = -\bar{k}$, and $z = 0$

$$\bar{F}.\hat{n} = (x\bar{i}+y\bar{j}+z\bar{k}).(-\bar{k})$$

$$= 0$$

$$\therefore\ \iint_{S_1}\bar{F}.\hat{n}\, ds = 0$$



For the surface DBEP

$\hat{n} = \hat{j}$, $y = a$

$$\bar{F}.\hat{n} = (x\bar{i}+y\bar{j}+z\bar{k}).\,\hat{j},\ y = a$$

$$\iint_S \bar{F}.\hat{n}\, ds = \iint_{S_2} a\, dzdx$$

$$= a\int_0^a\int_0^a dy\,dx$$

$$= a\int_0^a [x]_0^a\,dy$$

$$= a^2\,[y]_0^a = a^3$$

$$\therefore \iint_{S_2} \bar{F}.\hat{n}\,ds = a^3$$

For the surface EPFC, $\hat{n} = \bar{k}, z = a$

$$\bar{F}.\hat{n} = \left(x\bar{i}+y\bar{j}+z\bar{k}\right).\ \bar{k} \Rightarrow z = a$$

$$\iint_{S_4} \bar{F}.\hat{n}\,ds = \int_0^a\int_0^a a\,dx\,dy = a\int_0^a\int_0^a dx\,dy = a^3$$

$$\therefore \iint_{S_4} \bar{F}.\hat{n}\,ds = a^3$$

For the surface AOCF, $\hat{n} = -\hat{j}, y = 0$

$$\bar{F}.\hat{n} = \left(x\bar{i}+y\bar{j}+z\bar{k}\right)\left(-\hat{j}\right) = -y = 0$$

$$\therefore \iint_{S_6} \bar{F}.\hat{n}\,ds = 0$$

For the surface ADPF, $\hat{n} = \hat{i}, x = a$

$$\bar{F}.\hat{n} = \left(x\bar{i}+y\bar{j}+z\bar{k}\right).\ \hat{i} = x \Rightarrow x = a$$

$$\bar{F}.\hat{n} = \int_0^a\int_0^a a\,dy\,dz = a^3$$

$$\therefore \iint_{S_3} \bar{F}.\hat{n}\,ds = a^3$$

For the surface OBEC, $\hat{n} = -\hat{i}, x = 0$

$$\iint_{S_5} \bar{F}.\hat{n}\,ds\ 0$$

$\therefore$ Therefore, surface integral

$$\iint_S \bar{F}.\hat{n}\,ds = 0 + a^3 + a^3 + 0 + a^3 + 0 = 3a^3$$

$$\therefore \iint_S \bar{F}.\hat{n}\,ds = 3a^3$$

(ii) S is the surface of the sphere $x^2+y^2+z^2 = 9$.

Solution: Given surface of the sphere is $\phi = x^2+y^2+z^2-9 = 0$

Any normal to the surface,

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

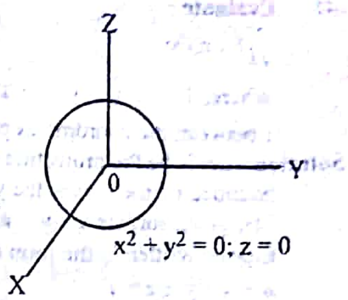
$$= 2x\bar{i}+2y\bar{j}+2z\bar{k}.$$

Let $\hat{n}$ be the unit normal to the surface.

$$\therefore \hat{n} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{2.3}$$

$$= \frac{x\bar{i}+y\bar{j}+z\bar{k}}{3}$$



Given, $\bar{F} = x.\bar{i}+y\bar{j}+z\bar{k}$

$$\therefore \bar{F}.\hat{n} = x.\bar{i}+y\bar{j}+z\bar{k}\ .\ \frac{(x\bar{i}+y\bar{j}+z\bar{k})}{3}$$

$$= \frac{x^2+y^2+z^2}{3}$$

$$= \frac{9}{3}$$

$$= 3$$

Let R be the projection of the surface on xy-plane. The value of the surface integrals is 8 times the projection of surface on xy-plane. Then,

$$\iint \bar{F}.\hat{n}\,ds = 8\iint_R \bar{F}.\hat{n}\frac{dxdy}{|\hat{n}.\bar{k}|}$$

$$\therefore \hat{n}.\bar{k} = \left(\frac{x\bar{i}+y\bar{j}+z\bar{k}}{3}\right).\ \bar{k} = \frac{3}{3}$$

$$= 8\iint_R 3\frac{dxdy}{\frac{z}{3}}$$

$$= 8\iint_R 9\frac{1}{z}\,dxdy$$

$$= 72\int_0^3\int_0^{\sqrt{9-x^2}} \frac{1}{\sqrt{9-x^2-y^2}}\,dx\,dy \qquad \because z^2 = 9-x^2-y^2$$

Put $y = \sqrt{9-x^2}\,\sin\theta$

$dy = \sqrt{9-x^2}\,\cos\theta d\theta$

So that θ varies from 0 to $\frac{\pi}{2}$

$$= 72\int_0^{\frac{\pi}{2}} \frac{\sqrt{9-x^2}}{\sqrt{9-x^2-(9-x^2)\sin^2\theta}}\,d\theta\,dx$$

$$= 72\int_0^3\int_0^{\frac{\pi}{2}} \frac{\sqrt{9-x^2}}{\sqrt{9-x^2}}\,\frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta\,dx$$

$$= 72\int_0^3 [\theta]_0^{\frac{\pi}{2}}\,dx = 72\,\frac{\pi}{2}\int_0^3 dx = 36\pi\,[x]_0^3 = 108\pi$$

$$\therefore \iint \bar{F}.\hat{n}\,ds = 108\pi$$

(4) Evaluate : **(BE 2062)**

$$\iint \bar{F} . n \, ds$$

Where, $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ and s is the finite plane $x + y + z = 1$ between the coordinates planes.

Solution: Let R be the projection of s in the xy plance. Clearly R is bounded by the x-axis, the y-axis and the line $x + y = 1$.

The given surface $x + y + z = 1$

may be written in the form $\phi (x, y, z) = C$ so,

$\phi = x + y + z = 1$

The normal vector of s is given by $\Delta\phi$.

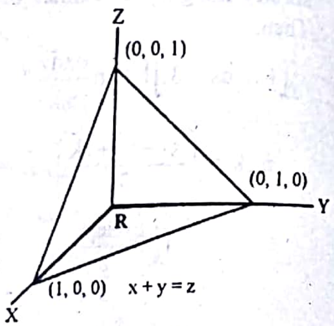
$$\Delta\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x + y + z)$$

$$= \bar{i} + \bar{j} + \bar{k}$$

Hence, the unit normal vector $\bar{n}$ is given by

$$\bar{n} = \frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{(1^2+1^2+1^2}}$$

$$= \frac{1}{\sqrt{3}}\left(\bar{i}+\bar{j}+\bar{k}\right)$$



Thus, $\bar{n}.\bar{k} = \frac{1}{\sqrt{3}}\left(\bar{i}+\bar{j}+\bar{k}\right) . \bar{k}$

$$= \frac{1}{\sqrt{3}}$$

and,

$$\bar{F} . \bar{n} = (x^2\bar{i} + y^2\bar{j} + z^2\bar{k}) . \frac{1}{\sqrt{3}}\left(\bar{i}+\bar{j}+\bar{k}\right)$$

$$= \frac{1}{\sqrt{3}}(x^2 + y^2 + z^2)$$

Hence, $\bar{F} . \bar{n} = \frac{1}{\sqrt{3}}[x^2 + y^2 + (1 - x - y)^2]$

$$\therefore \iint \bar{F} . \bar{n} \, ds = \iint_R \bar{F} . \bar{n} \frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}[x^2 + y^2 + (1 - x - y)^2]\,dx\,dy$$

$$= \int_0^1\left[x^2y + \frac{1}{3}y^3 - \frac{(1-x-y)^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1\left[x^2(1-x) + \frac{1}{3}(1-x)^3 + \frac{1}{3}(1-x)^3\right] dx$$

$$= \int_0^1\left[x^2 - x^3 + \frac{2}{3}(1-x)^3\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{2}{3}\frac{(1-x)^4}{4}\right]_0^1$$

$$= \frac{1}{4} \text{ Ans.}$$

(5) Evaluate $\iint_S \bar{F} . \bar{n}\, ds$ where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and s is the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Solution: The projection of the surface on the xy–plane is the region R bounded by the x-axis y-axis and the are of the circle $x^2 + y^2 = 1$.

The flux of $\bar{F}$ through s

$$= \iint_S \bar{F} . \bar{n}\, ds = \iint_R \bar{F} . \bar{n}\frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

Writing the equation of the surface s as $\phi (x, y, z) = C$, when $\phi = x^2 + y^2 + z^2 = 1$, the normal vector s is,

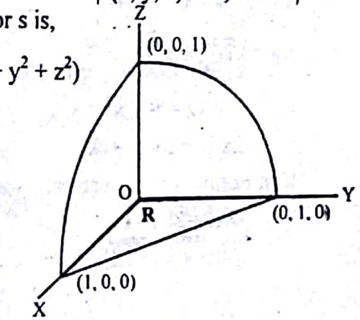
$$\Delta\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$= 2(x\bar{i} + y\bar{j} + z\bar{k})$$

$\therefore$ The unit normal vector

$$\bar{n} = \frac{2\left(x\bar{i}+y\bar{j}+z\bar{k}\right)}{\sqrt{4(x^2+y^2+z^2)}}$$

$$= x\bar{i} + y\bar{j} + z\bar{k}$$



Hence, $\bar{n} . \bar{k} = (x\bar{i} + y\bar{j} + z\bar{k}) . \bar{k}$

$$= z$$

and $\bar{F} . \bar{n} = (yz\bar{i} + zx\bar{j} + xy\bar{k}) . (x\bar{i} + y\bar{j} + z\bar{k})$

$$= 3xyz$$

Now, $\iint_S \bar{F} . \bar{n}\, ds = \iint_R [\bar{F} . \bar{n}]\frac{dx\,dy}{|\bar{n}.\bar{k}|}$

$$= \iint_R (3xyz)\frac{dx\,dy}{z}$$

$$= 3\iint_R xy\;dx\,dy$$

$$= 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\;dx\,dy$$

$$= 3\int_0^1\left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 (x - x^3)\, dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{2}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$= \frac{3}{8}$$

(6) Find the flux of $\bar{F}$ through the part of the surface S, $x^2 + y^2 + z = 9$ with Z > and $\bar{F} = 3x\,\bar{i} + 3y\,\bar{j} + z\bar{k}$.

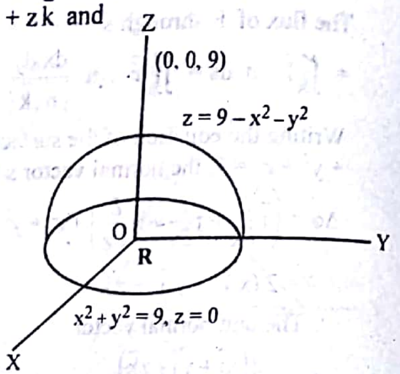
Solution: Here, $\bar{F} = 3x\,\bar{i} + 3y\,\bar{j} + z\bar{k}$ and

S is the surface

$z = 9 - x^2 - y^2$ with $z \geq 0$,

Normal to the surface is

$\phi = z - 9 - x^2 - y^2$ is,



$$\Delta\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(z - 9 + x^2 + y^2)$$

$$= 2x\,\bar{i} + 2y\,\bar{j} + \bar{k}$$

Unit normal to the surface,

$$\bar{n} = \frac{2x\bar{i} + 2y\bar{j} + \bar{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\therefore\ \bar{F} \cdot \bar{n} = (3x\,\bar{i} + 3y\,\bar{j} + z\bar{k}) \cdot \frac{(2x\bar{i} + y\bar{j} + \bar{k})}{\sqrt{4x^2 + y^2 + 1}}$$

$$= \frac{6x^2 + 6y^2 + z}{\sqrt{4x^2 + 4y^2 + 1}}$$

Let R be the projection of the surface on the xy plane then,

$$\bar{n} \cdot \bar{k} = \frac{2x\bar{i} + 2y\bar{j} + \bar{k}}{\sqrt{4x^2 + 4y^2 + 1}} \cdot \bar{k}$$

$$= \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\therefore\ \iint_S \bar{A} \cdot \bar{n}\ ds = \iint_R \bar{F} \cdot \bar{n}\ \frac{dx\,dy}{|\bar{n} \cdot \bar{k}|}$$

$$= \iint_R \frac{6x^2 + 6y^2 + z}{\sqrt{4x^2 + 4y^2 + 1}}\ \frac{dx\,dy}{\frac{1}{\sqrt{4x^2 + 4y^2 + 1}}}$$

$$= \iint_R (6x^2 + 6y^2 + z)\, dy\, dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (6x^2 + 6y^2 - 9 - x^2 - y^2)\, dy\, dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5x^2 + 5y^2 + 9)\, dy\, dx$$

$$= 2\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} (5x^2 + 5y^2 + 9)\, dy\, dx$$

$$= 4\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}} [5(x^2 + y^2) + 9]\, dy\, dx$$

To change it polar form put $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$, $dx\,dy = r\,dr d\theta$

and r varies from $r = 0$ to $r = 3$ and $\theta = 0$ to $\theta = \frac{\pi}{2}$

So,

$$\iint_S \bar{A} \cdot \bar{n}\ ds = 4\int_0^{\frac{\pi}{2}}\int_0^3 (5r^2 + 9r)\, r\,dr d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^3 (5r^3 + 9r^2)\, dr\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left[\frac{5r^4}{4} + \frac{9r^2}{2}\right]_0^3 d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left(\frac{5\times 81}{4} + \frac{81}{2} - 0\right) d\theta$$

$$= 4\left[\left(\frac{405}{4} + \frac{81}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= 4 \times \frac{(405 + 162)}{4}\ \frac{\pi}{2}$$

$$= \frac{567\pi}{2}$$

$$\therefore\ \iint_S \bar{A} \cdot \bar{n}\ ds = \frac{567\pi}{2}\ \text{Ans.}$$

(7) Evaluate $\iint_S \bar{F} \cdot \bar{n}\ ds$ where, $\bar{F} = (2x - z)\,\bar{i} + x^2y\,\bar{j} - xz^2\bar{k}$ and S is the surface of the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution: Here, $\bar{A} = \{(2x - z)\,\bar{i} + x^2y\,\bar{j} - xz^2\bar{k}\}$ and S is the surface of planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

We have,

$$\iint_S \bar{A} \cdot \bar{n}\ ds = \iint_{S_1} \bar{A} \cdot \bar{n}\,ds + \iint_{S_2} \bar{A} \cdot \bar{n}\,ds + \iint_{S_3} \bar{A} \cdot \bar{n}\ ds$$
$$+ \iint_{S_4} \bar{A} \cdot \bar{n}\ ds + \iint_{S_5} \bar{A} \cdot \bar{n}\ ds + \iint_{S_6} \bar{A} \cdot \bar{n}\ ds$$

For the surface S_1, $z = 0$,

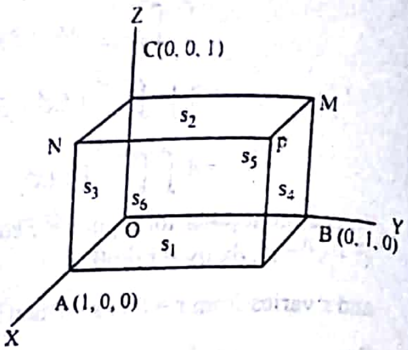
$\bar{n} = -\bar{k}$

$\bar{A} \cdot \bar{n} = [(2x - z)\ \bar{i} + x^2y\ \bar{j} - xz^2\bar{k}] \cdot (-\bar{k})$

$= x^2 \times 0$

$= 0$

$\therefore \iint_{S_1} \bar{A} \cdot \bar{n}\, ds = 0$



For the surface S_2, $z = 1$, $\bar{n} = \bar{k}$

$\bar{A} \cdot \bar{n} = [(2x - z)\ \bar{i} + x^2y\ \bar{j} - xz^2\bar{k}] \cdot (\bar{k})$

$= -x \times 1 = -x$

$\therefore \iint_{S_2} \bar{A} \cdot \bar{n}\, ds = \int_0^1 \int_0^1 (-x)\ dx\, dy$

$= \int_0^1 \left[\frac{-x^2}{2}\right]_0^1 dy = -\int_0^1 \frac{dy}{2}$

$= -\frac{1}{2} \int_0^1 dy = -\left[\frac{y}{2}\right]_0^1 = -\frac{1}{2}$

For the surface S_3, $y = 0$, $\bar{n} = -\bar{j}$

$\bar{A} \cdot \bar{n} = [(2x - z)\ \bar{i} + x^2y\ \bar{j} - xz^2\bar{k}] \cdot (-\bar{j})$

$= -x^2 \times 0 = 0$

$\therefore \iint_{S_3} \bar{A} \cdot \bar{n}\, ds = 0$

For the surface S_4, $y = 1$, $\bar{n} = \bar{j}$

$\bar{A} \cdot \bar{n} = [(2x - z)\ \bar{i} + x^2y\ \bar{j} - xz^2\bar{k}] \cdot (\bar{j})$

$= x^2 \times 1 = x^2$

$\therefore \iint_{S_4} \bar{A} \cdot \bar{n}\, ds = \int_0^1 \int_0^1 x^2\ dz\, dx$

$= \int_0^1 x^2 [z]_0^1\ dx$

$= \int_0^1 x^2\ dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

For the surface S_5, $x = 0$, $\bar{n} = -\bar{i}$

$\bar{A} \cdot \bar{n} = [(2x - z)\ \bar{i} + x^2y\ \bar{j} - xz^2\bar{k}] \cdot (-\bar{i})$

$= (0 - z)(-1)$

$= z$

$\therefore \iint_{S_5} \bar{A} \cdot \bar{n}\, dy\, dz = \int_0^1 \int_0^1 z dy\ dz$

$= \int_0^1 z[y]_0^1\ dz$

$= \int_0^1 z dz = \left[\frac{z^2}{z}\right]_0^1 = \frac{1}{2}$

For the surface S_6, $x = 1$, $\bar{n} = \bar{i}$

$\bar{A} \cdot \bar{n} = [(2x - z)\ \bar{i} + x^2y\ \bar{j} - xz^2\bar{k}] \cdot (\bar{i})$

$= (2 - z)$

$\therefore \iint_{S_6} \bar{A} \cdot \bar{n}\, dy\, dz = \int_0^1 \int_0^1 (2 - z)\ dy\, dz$

$= \int_0^1 (2 - z)\ [y]_0^1\ dz$

$= \int_0^1 (2 - z)\ dz$

$= \left[2z - \frac{z^2}{2}\right]_0^1$

$= 2 - \frac{1}{2}$

$= \frac{3}{2}$

Thus,

$$\iint_S \bar{A} \cdot \bar{n}\ ds = \iint_{S_1} \bar{A} \cdot \bar{n}\, ds + \iint_{S_1} \bar{A} \cdot \bar{n}\, ds + \iint_{S_3} \bar{A} \cdot \bar{n}\ ds$$

$$+ \iint_{S_4} \bar{A} \cdot \bar{n}\ ds + \iint_{S_5} \bar{A} \cdot \bar{n}\ ds + \iint_{S_6} \bar{A} \cdot \bar{n}\ ds$$

$= 0 - \frac{1}{2} + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{2}$

$= \frac{1}{3} + \frac{3}{2}$

$= \frac{2 + 9}{6}$

$= \frac{11}{6}$

$\iint_S \bar{A} \cdot \bar{n}\ ds = \frac{11}{6}$ Ans.

Exercise - 9

1. **Verify Green's theorem for $\oint_C [(xy + y^2)\, dx + x^2\, dy]$ where C is bounded by $y = x$ and $y = x^2$.** [BE 2062]

Solution: The curves $y = x$ and $y = x^2$ meet as (0, 0) and (1, 1). Along $y = x^2$ from (0, 0) t0 (1, 1), the line integral is,

$$\int_{(0,0)}^{(1,1)} (xy + y^2)\, dx + x^2\, dy$$

$$= \int_0^1 (x \,.\, x^2 + x^4 + x^2 \,.\, 2x)\, dx$$

$$= \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3}{4}x^4 + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5}$$

$$= \frac{19}{20}$$

Along $y = x$ form (1, 1) to (0, 0), the line integral,

$$= \int_{(1,1)}^{(0,0)} (xy + y^2)\, dx + x^2\, dy \quad \text{where } y = x \text{ and } dy = dx$$

$$= \int_1^0 (x \,.\, x + x^2)\, dx + x^2\, dx$$

$$= \int_1^0 3x^2\, dx$$

$$= [x^3]_1^0$$

$$= -1$$

$$\therefore \oint_C (xy + y^2)\, dx + x^2 dy = \frac{19}{20} - 1 = -\frac{1}{20}$$

Also, $\iint_R \left[\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y}(xy + y^2)\right] dx\, dy$

$$= \iint_R (2x - x - 2y)\, dx\, dy$$

$$= \iint_R (x - 2y)\, dx\, dy$$

$$= \int_0^1 \left[\int_{y=x^2}^{y=x} (x - 2y)dy\right] dx$$

$$= \int_0^1 [xy - y^2]_{y=x^2}^{y=x}\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= -\frac{1}{20}$$

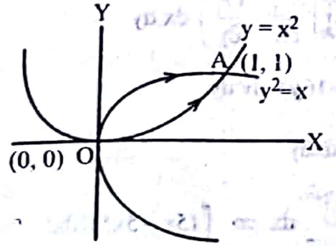
Thus, theorem is verified.

2. **Verify Green's theorem in the plane for $\oint_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)]\, dx$ where C is the boundry of the region defined by $y = \sqrt{x}$, $y = x^2$**

Solution: Here, the line integral is,

$$\oint_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)]\, dy$$

Where, C is the closed curve given by $y = x^2$ and $y^2 = x$



$$\oint [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$$

$$= \int_{OA} (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy + \int_{AO} (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$$

Along 0A, $y = x^2$ so that $dy = 2x\, dx$

as x varies from x = 0 to x = 1

$$\int_{OA} (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$$

$$= \int_0^1 (3x^2 - 8x^4)\, dx + (4x^2 - 6x^3)\, 2x\, dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\, dx$$

$$= \left[\frac{3x^3}{3} - \frac{8x^5}{5} - \frac{12x^5}{5} + \frac{8x^4}{4}\right]_0^1 \Rightarrow 1 - 4 + 2 + 1 = -1$$

Along, A0, $y^2 = x$

$2y\, dy = dx$

$$\int_{AO} (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$$

$$= \int_1^0 (3y^4 - 8y^2)\, 2y\, dy + (4y - 6y^3)\, dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\, dy$$

$$= \left[\frac{6y^6}{6} - \frac{16y^4}{4} + \frac{4y^2}{2} - \frac{6y^4}{4}\right]$$

$$= -1 + 4 - 2 + \frac{3}{2}$$

$$= 1 + \frac{3}{2}$$

$$= \frac{5}{2}$$

$$\therefore \int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \frac{5}{2} - 1 = \frac{3}{2}$$

Also, $F_1 = 3x^2 - 8y^2$ and $F_2 = 4y - 6xy$

So that,

$$\frac{\partial F_1}{\partial y} = -16y \qquad \frac{\partial F_2}{\partial x} = -6y$$

We have, $\iint_E \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$ =

$$\int_0^1 \int_{x^2}^{\sqrt{x}} (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (10y)\,dxdy$$

$$\Rightarrow \int_0^1 \left[\frac{10y^2}{2}\right]_{x^2}^{\sqrt{x}} dx \Rightarrow \int_0^1 (5x - 5x^4)\,dx$$

$$\Rightarrow \left[\frac{5x^2}{2} - \frac{5x^5}{5}\right]_0^1$$

$$= \frac{5}{2} - 1$$

$$= \frac{3}{2}$$

$$\therefore \iint_E \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \frac{3}{2}$$

Hence, Green's theorem is verified.

(3) Verify Green's theorem for $\int_C [(x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy]$ where C is a square with verfices at (0, 0), (2, 0), (2, 2), (0, 2).

Solution: Here, $\int_C [(x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy]$

Let, $F_1 = x^2 - xy^3$, $F_2 = y^2 - 2xy$

$$\frac{\partial F_1}{\partial y} = -3xy^2, \qquad \frac{\partial F_2}{\partial y} = -2y$$

Now, we have,

$$\iint_E \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (-2y + 3xy^2)\,dx\,dy$$

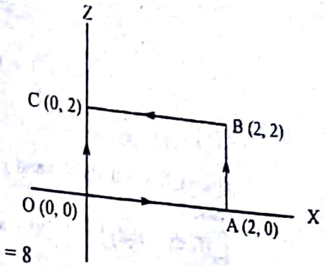
$$= \int_0^2 [-y^2 + xy^3]_0^2\,dx$$

$$= \int_0^2 [-4 + 8x]\,dx$$

$$= \left[-4x + \frac{8x^2}{2}\right]_0^2$$

$$= -8 + 16$$

$$= 8$$



$$\therefore \iint_E \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = 8$$

Also,

$$\int_C [F_1 dx + F_2 dy] = \int_{OA} [F_1\,dx + F_2\,dy] + \int_{AB} [F_1\,dx + F_2\,dy] + \int_{BC} [F_1\,dx + F_2\,dy) + \int_{CO} [F_1\,dx + F_2\,dy]$$

Along OA, $y = 0$, $dy = 0$

and x varies from $x = 0$ to $x = 2$

$$\int_{OA} [F_1\,dx + F_2\,dy] = \int_{OA} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$= \int_0^2 (x^2 dx + 0)$$

$$= \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{8}{3}$$

Along AB, $x = 2$, $dx = 0$ and y varies from $y = 0$ to $y = 2$

Thus,

$$\int_{AB} [F_1\,dx + F_2\,dy] = \int_{AB} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$= \int_0^2 [0 + (y^2 - 4y)\,dy]$$

$$= \left[\frac{y^3}{3} - \frac{4y^2}{2}\right]_0^2$$

$$= \frac{8}{3} - 8$$

$$= -\frac{16}{3}$$

Along BC, $y = 2$, $dy = 0$ and x varies from $x = 2$ to $x = 0$

Thus,

$$\int_{BC}[F_1\,dx + F_2\,dy] = \int_{BC}(x^2 - xy^3)\,dx - (y^2 - 2xy)\,dy$$

$$= \int_2^0 [x^2 - 8x)\,dx + 0]$$

$$= \left[\frac{x^3}{3} - \frac{8x^2}{2}\right]_2^0$$

$$= 0 - \frac{8}{3} + 16$$

$$= \frac{40}{3}$$

Along C0, $x = 0$, $dx = 0$ and y varies from $y = 2$ to $y = 0$

Thus,

$$\int_{C0}[F_1\,dx + F_2 dy] = \int_{C0}[x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$= \int_2^0 [0 + y^2\,dy]$$

$$= \left[\frac{y^3}{3}\right]_2^0$$

$$= 0 - \frac{8}{3}$$

$$= -\frac{8}{3}$$

$$\therefore \int_C [F_1 dx + F_2\,dy] = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3}$$

$$= \frac{24}{3}$$

$$= 8$$

$$\therefore \int_C [F_1 dx + F_2\,dy] = 8$$

Hence, Green's theorem,

$$\therefore \int_C [F_1 dx + F_2\,dy] = \iint_E \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \text{ is verified.}$$

(6) Apply Green's theorem to evaluate $\int_C[(2x - y + 4)\,dx + (5y + 3x - 6)\,dy]$ around a triangle in the xy- plane with vertices at (0, 0), (3, 0), (3, 2)

Solution: Here, $\int_C[(2x - y + 4)\,dx + (5y + 3x - 6)\,dy]$

Let, $F_1 = 2x - y + 4$, $F_2 = 5y + 3x - 6$

$$\frac{\partial F_1}{\partial y} = -1, \qquad \frac{\partial F_2}{\partial x} = 3$$

Thus,

We have by Green's theorem,

$$\int_C[(2x - y + 4)\,dx + (5y + 3x - 6)\,dy]$$

$$= \iint_E \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_E (3 + 1)\,dx\,dy$$

$$= 4\iint_E dx\,dy$$

Where, E is the region bounded by the triangle in the xy-plane with vertices (0, 0), (3, 0), (3, 2). The equation of the line joining 0 (0, 0) and B (3, 2) is,

$$y - 0 = \frac{2-0}{3-0}(x - 0) \qquad \text{i.e. } y = \frac{2}{3}x$$

So, y varies from $y = 0$, to $y = \frac{2}{3}x$ and x varies from $x = 0$ to $x = 3$.

$$\therefore \int_C[(2x - y + 4)\,dx + (5y + 3x - 6)\,dy]$$

$$= 4\int_0^3\int_0^{\frac{2x}{3}} dx\,dy = 4\int_0^3 \frac{2x}{3}\,dx = \frac{8}{3}\int_0^3 x\,dx = \frac{8}{3}\left[\frac{x^2}{2}\right]_0^3 = \frac{8}{3} \times \frac{9}{2}$$

$$= 4 \times 3$$

$$= 12$$

$$\therefore \int_C[(2x - y + 4)\,dx + (5y + 3x - 6)\,dy] = 12 \text{ Ans.}$$

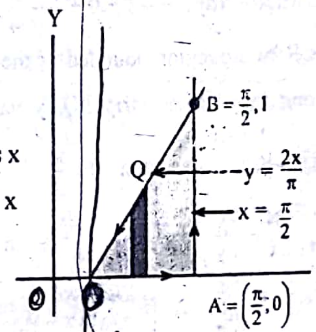
(7) Apply Green's theorem to evaluate $\int_C (y - \sin x)\,dx + \cos x\,dy$ where C is the plane triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ ad $y = \frac{2x}{\pi}$.

Solution: The point of intersection of the line $y = \frac{2x}{\pi}$ ad $x = \frac{\pi}{2}$ is obtained as $y = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$

Hence, B : $\left(\frac{\pi}{2}, 1\right)$

Now, $M = y - \sin x$, $N = \cos x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$



$$\int_C (Mdx + Ndy) = \int_{OA}(Mdx + Ndy) + \int_{AB}(Mdx + Ndy)$$

$$+ \int_{BO}(Mdx + Ndy) \quad \text{........ (i)}$$

Along OA : y = 0

x varies from 0 to $\frac{\pi}{2}$

$$\int_{OA}(Mdx + Ndy) = \int_{OA}[(y - \sin x)dx + \cos x dy]$$

$$= \int_0^{\frac{\pi}{2}}(-\sin x)\, dx = |\cos x|_0^{\frac{\pi}{2}} = -1$$

Along AB : x = $\frac{\pi}{2}$, dx = 0

y varies from 0 to 1.

$$\int_{AB}(Mdx + Ndy) = \int[(y - \sin x)dx + \cos x dy]$$

$$= \int_0^1 \cos\frac{\pi}{2}dy = 0$$

Along BO = y = $\frac{2x}{\pi}$, dy = $\frac{2}{\pi}$ dx

x varies from $\frac{\pi}{2}$ to 0

$$\int_{BO}(Mdx + Ndy) = \int_{BO}[(y - \sin x)dx + \cos x dy]$$

$$= \int_{\frac{\pi}{2}}^0\left[\left(\frac{2x}{\pi} - \sin x\right)dx + \cos x.\frac{2}{\pi}dx\right]$$

$$= \left[\frac{2}{\pi}.\frac{x^2}{2} + \cos x + \frac{2}{\pi}\sin x\right]_{\frac{\pi}{2}}^0$$

$$= \cos - \frac{1}{\pi}\cdot\frac{\pi^2}{4} - \cos\frac{\pi}{2} - \frac{2}{\pi}\sin\frac{\pi}{2}$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Then form equation (i)

$$\int_C(Mdx + Ndy) = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\left(\frac{\pi^2 + 8}{4\pi}\right) \quad \ldots \text{(ii)}$$

Let R be the region bounded by the triangle OAB .

Along the vertical strip PQ, y varies from 0 to $\frac{2x}{\pi}$ and in the region R, x varies from o to $\frac{\pi}{2}$.

$$\iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\frac{2x}{\pi}}(-\sin x - 1)\ dxdy$$

$$= \int_0^{\frac{\pi}{2}}|-y\sin x - y|_0^{\frac{2x}{\pi}}\ dx$$

$$= \int_0^{\frac{\pi}{2}}\left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right)dx$$

$$= -\frac{2}{\pi}\left[x(-\cos x) - (-\sin x) - \frac{x^2}{2}\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi}\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \frac{\pi^2}{8} - 0\right)$$

$$= -\frac{2}{\pi}\left(1 + \frac{\pi^2}{8}\right) = -\left(\frac{\pi^2 + 8}{4\pi}\right) \quad \ldots\ldots \text{(iii)}$$

Then from equation (ii) and (iii)

$$\int_C(Mdx + Ndy) = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = -\left(\frac{\pi^2 + 8}{4\pi}\right)$$

Hence, Green's theorem is verified.

8. **Evaluate by Green's theorem, $\int_C(e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$, C is the rectangle with vertices (0, 0) (π, 0), $\left(\pi, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$.**

Solution: By Green's theorem,

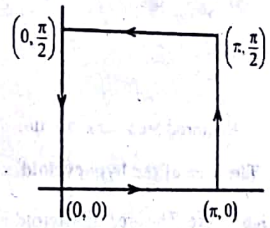
$$\int_C(F_1\,dx + F_2\,dy) = \iint_R\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}}\left[\frac{\partial}{\partial x}(e^{-x}\sin y) - \frac{\partial}{\partial y}e^{-x}\cos y\right]dx\,dy$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}}(-e^{-x}\cos y - e^{-x}\cos y)\,dy$$

$$= -2\int_0^{\pi}\int_0^{\frac{\pi}{2}}(e^{-x}\cos y\,dx\,dy$$

$$= -2\int_0^{\pi}[e^{-x}\sin y]_0^{\frac{\pi}{2}}\,dx$$



$$= -2\int_0^{\pi}e^{-x}, 1\,dx$$

$$= -2\left[\frac{e^{-x}}{-1}\right]_0^{\pi}$$

$$= 2e^{-\pi} - 2$$

$$= 2(e^{-\pi} - 1)$$

(9) Using Green's theorem find

(i) area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: In parametric from equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be written as,

$x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi$

Hence, the required area,

$$= \frac{1}{2} \oint (x\,dx - y\,dx)$$

$$= \frac{1}{2} \int_0^{2\pi} a\cos\theta\,(b\cos\theta)\,d\theta - b\sin\theta\,(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} a\,b\,d\theta$$

$$= \frac{1}{2} [ab\theta]_0^{2\pi}$$

$$= \pi ab$$

(ii) The area of the circle $x^2 + y^2 = a^2$

Solution: Here, the equation of the circle is;

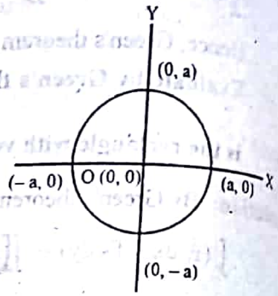
$x^2 + y^2 = a^2$

Also, its parametric equation is,

$x = a\cos\theta,\ y = a\sin\theta$

$dx = -a\sin\theta\,d\theta, \quad dy = a\cos\theta\,d\theta$

and θ varies from $\theta = 0$ to $\theta = 2\pi$



We have, the required area of the circle is given by,

$$A = \frac{1}{2} \oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2} \int_0^{2\pi} [a\cos\theta \times (a\cos\theta) - a\sin\theta\,(-a\sin\theta\,d\theta)]$$

$$= \frac{1}{2} \int_0^{2\pi} a^2\,(\sin^2\theta + \cos^2\theta)\,d\theta$$

$$= \frac{a^2}{2} \int_0^{2\pi} d\theta$$

$$= \frac{a^2}{2} \times 2\pi$$

$$= \pi a^2$$

$\therefore$ Required area $= \pi a^2$ sq. unit Ans.

(iii) The area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Solution: Here, The area of asteroid is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Its parametric equation is,

$x = a\cos^3\theta,\ y = a\sin^3\theta$

$dx = -3a\cos^2\theta\sin\theta\,d\theta$

$dy = 3a\sin^2\theta\cos\theta\,d\theta$

and θ varies from 0 to 2π

We have he required area of asteroid is given by,

$$A = \frac{1}{2} \oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2} \int_0^{2\pi} [a\cos^3\theta \times (3a\sin^2\theta\cos\theta\,d\theta) - a\sin^3\theta\,(-3a\cos^2\theta\sin\theta\,d\theta)]$$

$$= \frac{1}{2} \int_0^{2\pi} 3a^2\sin^2\theta\cos^2\theta\,(\sin^2\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} 3a^2\sin^2\theta\cos^2\theta\,d\theta = \frac{3a^2}{8} \int_0^{2\pi} \sin^2 2\theta\,d\theta$$

$$= \frac{3a^2}{16} \int_0^{2\pi} (1 - \cos 4\theta)\,d\theta$$

$$= \frac{3a^2}{16} \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{3a^2}{16} \times 2\pi$$

$$= \frac{3\pi a^2}{8}$$

$\therefore$ Required area $\frac{3\pi a^2}{8}$ sq. unit Ans.

(10) Verify stoke's theorem for the vector field. $\vec{F} = (x^2 - y^2)\,\vec{i} + 2xy\,\vec{j}$ integrated round the rectangle. in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$; $x = a$ and $y = b$.

Solution: Here, $\vec{F} = (x^2 - y^2)\,\vec{i} + 2xy\,\vec{j}$. i.e., $\hat{n} = \vec{k}$

$$= \Delta \times \vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times [(x^2 - y^2)\,\vec{i} + 2xy\,\vec{j}\,]$$

$$= \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{bmatrix}$$

$$= \vec{i}\,(0 - 0) - \vec{j}\,(0 - 0) + \vec{k}\,(2y + 2y)$$

$$= (2y + 2y)\,\vec{k}$$

$$= 4y\,\vec{k}$$

So,

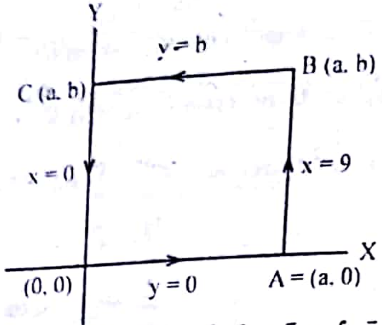
$$\iint_S \text{curl}\,\vec{F}\,.\,\vec{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{b} (4y\vec{k})\,.\,\vec{k}\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dy\,dx$$

$$= \int_0^a \left[\frac{4y^2}{2}\right]_0^b dx$$

$$\therefore \iint_S \text{curl}\,\vec{F}\,.\,\vec{n}\,ds = 2ab^2 \ \ldots\ldots\ldots\ldots \text{(i)}$$

Let OABC is the rectangle as shown as figure,



$$\therefore \int_{OABC} \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r}$$

and $\bar{F}\cdot d\bar{r} = [(x^2-y^2)\,\bar{i} + 2xy\,\bar{j}]\cdot(\bar{i}\,dx + \bar{j}\,dy)$

$= (x^2-y^2)\,dx + 2xy\,dy$

Along OA, $y = 0$, $dy = 0$ and x varies from $x = 0$ to $x = a$

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{OA} [(x^2-0)\,dx + 0]$$

$$= \int_{OA} x^2\,dx$$

$$= \int_0^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

Along AB, $x = a$, $dx = 0$ and y varies from $y = 0$ to $y = b$

$$\int_{AB} \bar{F}\cdot d\bar{r} = \int_{AB} [0 + 2ay\,dy]$$

$$= \int_0^b (2ay)\,dy$$

$$= \left[\frac{2ay^2}{2}\right]_0^b$$

$$= ab^2$$

Along BC, $y = b$, $dy = 0$, x varies from $x = a$ to $x = 0$

$$\int_{BC} \bar{F}\cdot d\bar{r} = \int_{BC} [(x^2-b^2)\,dx + 0]$$

$$= \int_a^0 (x^2-b^2)\,dx$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0$$

$$= -\frac{a^3}{3} + ab^2$$

Along CO, $x = 0$, $dx = 0$

y caries from $y = b$ to $y = 0$

$$\int_{CO} \bar{F}\cdot d\bar{r} = 0$$

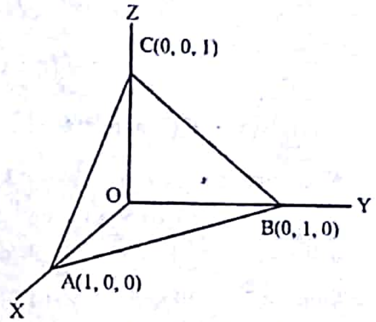
So, $\int_{OABC} \bar{F}\cdot d\bar{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} - ab^2 + 0 = 2ab^2$

$\therefore \int_{OABC} \bar{F}\cdot d\bar{r} = 2ab^2$ .... (ii)

From (i) and (ii), we concluded that Stoke's theorem is verified.

(11) **Verify Stoke's theorem for the function $\bar{F} = x\,\bar{i} + z^2\,\bar{j} + y^2\,\bar{k}$ over the plane surface $x + y + z = 1$ lying in the first octant.**

Solution: Here, $\bar{F} = x\,\bar{i} + z^2\,\bar{j} + y^2\,\bar{k}$ and the surface is the plane surface $x + y + z = 1$ with vertices (1, 0, 0), B (0, 1, 0) and C (0, 0, 1).



$\nabla\phi = \nabla(x + y + z - 1)$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x + y + z - 1)$$

$$= \bar{i} + \bar{j} + \bar{k}$$

The unit normal to the surface is

$$\bar{n} = \frac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$$

Curl $\bar{F} = \nabla \times \bar{F}$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (x\,\bar{i} + z^2\,\bar{j} + y^2\,\bar{k})$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z^2 & y^2 \end{vmatrix}$$

$$= \bar{i}\,(2y - 2z) - \bar{j}\,(0 - 0) + \bar{k}\,(0 - 0)$$

$$= \bar{i}\,(2y - 2z)$$

$$\nabla \times \bar{F}\cdot\bar{n} = \bar{i}\,(2y - 2z)\cdot\frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$$

$$= \frac{1}{\sqrt{3}}(2y - 2z)$$

and $\bar{n}\cdot\bar{k} = \frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})\cdot\bar{k} = \frac{1}{\sqrt{3}}$

$$\iint_S (\nabla\times\bar{F}).\bar{n}\,ds = \int_0^1\int_0^{1-x}\frac{1}{\sqrt{3}}(2y-2x)\frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

$$= \int_0^1\int_0^{1-x}\frac{1}{\sqrt{3}}[2y-2(1-x-y)]\frac{dx\,dy}{\left(\frac{1}{\sqrt{3}}\right)}$$

$$= 2\int_0^1\int_0^{1-x}(y-1+x+y)\,dx\,dy$$

$$= 2\int_0^1\left[xy+\frac{2y^2}{2}-y\right]_0^{1-x}dx$$

$$= 2\int_0^1[x(1-x)+(1-x)^2-(1-x)]\,dx$$

$$= 2\int_0^1[(1-x)^2-(1-x)^2]\,dx$$

$$\therefore \iint_S (\nabla\times\bar{F}).\bar{n}\,ds = 0 \text{ .... (i)}$$

Also, $\bar{F}.d\bar{r} = (x\bar{i}+z^2\bar{j}+y^2\bar{k}).(\bar{i}\,dx+\bar{j}\,dy+\bar{k}\,dz)$

$= x\,dx + z^2dy + y^2\,dz$

$$\int_C\bar{F}.d\bar{r} = \int_{AB}\bar{F}.d\bar{r} + \int_{BC}\bar{F}.d\bar{r} + \int_{CA}\bar{F}.d\bar{r}$$

Along AB, $z = 0$ and $x + y = 1$

Its parametric equation is,

$x = t,\ y = 1-t,\ z = 0$

$dx = dt,\ dy = -dt,\ dz = 0$ and t varies,

From $t = 1$ to $t = 0$

$$\int_{AB}\bar{F}.d\bar{r} = \int_{AB}(x\,dx+z^2dy+y^2\,dz)$$

$$= \int_1^0(t\,dt+0+(1-t)^2.0)$$

$$= \int_1^0(t\,dt = \left[\frac{t^2}{2}\right]_1^0 = 0-\frac{1}{2} = -\frac{1}{2}$$

Along BC, $x = 0,\ y + z = 1$,

its parametric equation is,

$x = 0,\ y = t,\ z = 1-t$

$dx = 0,\ dy = dt,\ dz = -dt$ and t varies from $t = 1$ to $t = 0$

$$\int_{BC}\bar{F}.d\bar{r} = \int_{BC}(x\,dx+z^2dy+y^2dz)$$

$$= \int_1^0[0+(1-t)^2\,dt+t^2(-dt)]$$

$$= \int_1^0 (1-2t+t^2-t^2)\,dt$$

$$= \int_1^0 (1-2t)\,dt$$

$$= \left[1-\frac{2t^2}{2}\right]_1^0 = -1-1=0$$

Along CA, $y = 0,\ x + z = 1$, Its parametric equation, is.

$x = t.\ z = 1-t,\ y = 0$

$dx = dt,\ dz = -dt,\ dy = 0$ and t varies $t = 0$ to $t = 1$

$$\int_{CA}\bar{F}.d\bar{r} = \int_{CA}(x\,dx+z^2dy+y^2\,dz)$$

$$= \int_0^1[t\,dt+(1-t)^2.0+0]$$

$$= \int_0^1 t\,dt$$

$$= \left[\frac{t^2}{2}\right]_1^0 = \frac{1}{2}$$

$$\therefore \int_C\bar{F}.d\bar{r} = -\frac{1}{2}+\frac{1}{2} = 0 \text{ .... (ii)}$$

From (i) and (ii), we conclude that stoke's theorem is verified.

(12) Verify Stoke's theorem for the function $\bar{F} = x^2\bar{i}+xy\bar{j}$ integrated round the square in the plane $z = 0$ and bounded by the line $x = 0, y = 0, x = a$ and $y = a$.

Solution: Unit-normal vector $\bar{n}$ to be the surface is $\bar{k}$.

Curl $\bar{F} = \nabla\times(x^2\bar{i}+xy\bar{j})$

$$= \begin{bmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & xy & 0\end{bmatrix} = \bar{k}\,y$$

$$\iint_S \text{curl } \bar{F}.\bar{n}\,ds = \int_0^a\int_0^a\bar{k}\,y.\bar{k}\,dx\,dy$$

$$= \int_0^a\left[\int_0^a y\,dy\right]dx = \int_0^a\frac{a^2}{2}\,dx = \left[\frac{a^2x}{2}\right]_0^a = \frac{a^3}{2}$$

Denoting the square by OABC where the vertices are 0 (0, 0), A (a, 0), B (a, a), C (0, a), the line integral in the closed curcle OABC is the sum of the four line integrals from 0 to A, A to B, B to C and C to O.

$$\int_{OA}\bar{F}.d\bar{r} = \int_0^a(x^2\bar{i}+xy\bar{j}).(\bar{i}\,dx+\bar{j}\,dy)$$

$$= \int_{(0,0)}^{(a,0)}x^2\,dx+xy\,dy$$

$$= \int_0^a x^2\,dx \quad [\because \text{ Along OA, } y = 0, dy = 0 \text{ and } x \text{ varies from 0 to a}]$$

$$= \frac{a^3}{3}$$

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_{(a,0)}^{(a,a)} (x^2\, dx + xy\, dy)$$

$$= \int_0^a ay\, dy \qquad [\because \text{ Along AB, } x = a,\ dx = 0,\ y \text{ varies from 0 to a}]$$

$$= \left[a \times \frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{(a,a)}^{(0,a)} (x^2\, dx + xy\, dy)$$

$$= \int_a^0 x^2\, dx \qquad [\because \text{ Along BC, } y = a,\ dy = 0\ x \text{ varies from a to 0}]$$

$$= -\frac{a^3}{3}$$

$$\int_{CO} \bar{F} \cdot d\bar{r} = \int_{(0,a)}^{(0,0)} (x^2\, dx + xy\, dy)$$

$$= 0 \qquad [\because \text{ Along CO, } x = 0,\ dx = 0]$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$

$$= \frac{a^3}{2}$$

Hence, $\iint_S \text{curl } \bar{F} \cdot \bar{n}\ ds = \int_C \bar{F} \cdot d\bar{r}$

Thus, Stoke's theorem is verified.

(13) Verify Stoke's theorem for $\bar{F} = 2y\,\bar{i} + 3x\,\bar{j} - z^2\bar{k}$ where S is upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is boundary.

Solution: Here, $\bar{F} = 2y\,\bar{i} + 3x\,\bar{j} - z^2\bar{k}$

By stoke's theorem, we have to verify that,

$$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\ ds$$

$$\text{Now, } (\nabla \times \bar{F}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix}$$

$$= \bar{i}\ (0 - 0) - \bar{j}\ (0 - 0) + \bar{k}\ (3 - 2) = \bar{k}$$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\ ds = \iint_S \bar{k} \cdot \bar{n}\ ds = \iint_R dx\ dy$$

Since $\bar{k} \cdot \bar{n}\ ds = dx\,dy$ and R, the projection of the surface S of the sphere $x^2 + y^2 + z^2 = 9$ on xy-plane, is circle $x^2 + y^2 = 9$, $z = 0$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\ ds = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\, dx = 4 \int_0^3 \sqrt{9 - x^2}\, dx$$

$$= 4\left[\frac{x\sqrt{9-x^2}}{2} - \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= 4 \times \frac{9}{2} \times \frac{\pi}{2} = 9\pi$$

Also, the surface S on xy – plane is bounded by the circle $x^2 + y^2 = 9$, $z = 0$

Its parametric equation is,

$x = 3\cos\theta,\ y = 3\sin\theta,\ z = 0$

$dx = -3\sin\theta\, d\theta,\ \ dy = 3\cos\theta d\theta,\ dz = 0$

and θ varies from 0 to 2π

Therefore,

$$\bar{F} \cdot d\bar{r} = \{2y\,\bar{i} + 3x\,\bar{j} - z^2\bar{k}\} \cdot (\bar{i}\, dx + \bar{j}\, dy + \bar{k}\, dz)$$

$$= 2y\, dx + 3x\, dy - z^2\, dz$$

$$= [(2 \times 3\sin\theta(\,(-3\sin\theta d\theta) + 3 \times 3\cos\theta\ (3\cos\theta\ d\theta) - 0]$$

$$= [-18\sin^2\theta + 27\cos^2\theta]\ d\theta.$$

$$= \left[\frac{27(1 + \cos 2\theta)}{2} - 9(1 - \cos 2\theta)\right] d\theta$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^{2\pi} \left[\frac{27(1 + \cos 2\theta)}{2} - 9(1 - \cos 2\theta)\right] d\theta$$

$$= \left[\frac{27\theta}{2} + \frac{27\sin 2\theta)}{4} - 9\theta + 9\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= [27\pi + 0 - 18\pi + 0]$$

$$= 9\pi$$

Thus, Stoke's thorem is verified.

(14) Evaluate by Stoke's theorem $\int_C (e^x\ dx + 2ydy - dz)$ where C is the curve $x^2 + y^2 = 4$, $z = 2$

Solution: Here, $\int_C (e^x\ dx + 2ydy - dz)$

So that $F_1 = e^x$, $F_2 = 2y$, $F_3 = -1$

$\bar{F} = F_1\,\bar{i} + F_2\,\bar{j} + F_3\,\bar{k}$

$= e^x \cdot \bar{i} + 2y\,\bar{j} - \bar{k}$

By Stoke's theorem, we have,

$$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

$$(\nabla \times \bar{F}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \bar{i}\ (0 - 0) - \bar{j}\ (0 - 0) + \bar{k}\ (0 - 0)$$

$$= 0$$

Thus, by Stoke's theorem,

$$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$$

$$= 0$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = 0 \text{ Ans.}$$

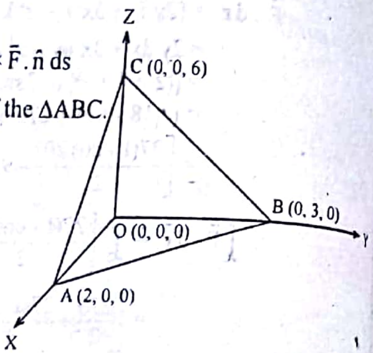
15. Apply stoke's theorem to evaluate $\int_C [(x + y)\, dx + (2x - z)\, dy + (y + z)\, dz]$ where C is the boundary of the triangle with vertices (2, 0, 0) (0, 3, 0) and (0, 0, 6).

Solution: Work done $= \int_C \bar{F} \cdot d\bar{r}$

By stoke's theorem,

$$\int_C \bar{F} \cdot d\bar{r} = \iint \nabla \times \bar{F} \cdot \hat{n}\, ds$$

Thus, work done $= \iint \nabla \times \bar{F} \cdot \hat{n}\, ds$



Where, S is the surface of the ΔABC.

Equation of the ΔABC is,

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$3x + 2y + z = 6$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & 2x - z & y + z \end{vmatrix}$$

$$= \bar{i}(1 + 1) - \bar{j}(0 - 0) + \bar{k}(2 - 1)$$

$$= 2\bar{i} + \bar{k}$$

Let $\phi = 3x + 2y + z$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{9 + 4 + 1}} = \frac{3\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{14}}$$

Projection of ΔABC on xy –plane is the ΔOAB bounded by the lines $y = 0$, $3x + 2y = 6$, $x = 0$

$$ds = \frac{dxdy}{|\hat{n}.\bar{k}|} = \sqrt{14}\ dxdy$$

Let R be the region bounded by the ΔOAB. Along the vertical strip PQ, y varies from 0 to $\frac{6 - 3x}{2}$ and in the region R, x varies from 0 to 2.

$$\iint_S \nabla \times \bar{F} \cdot \hat{n}\, ds = \iint_R (2\bar{i} + \bar{k}) \cdot \left(\frac{3\bar{i} + 2\bar{j} + \bar{k}}{\sqrt{14}}\right) \sqrt{14}\ dx\, dy$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} 7\, dy\, dx = 7 \int_0^2 |y|_0^{\frac{6-3x}{2}} dx$$

$$= 7 \int_0^2 \left(\frac{6 - 3x}{2}\right) dx = 7 \left[3x - \frac{3x^2}{4}\right]_0^2 = 21$$

or, $\iint_S \nabla \times \bar{F} \cdot \hat{n}\, ds = 7 \iint_R dxdy$

$$= 7 \text{ (Area of } \Delta OAB)$$

$$= 7 \times \frac{1}{2} \times 2 \times 3$$

$$= 21$$

(16) Prove of the following from Stoke's theorem.

(i) $\int_C d\bar{r} \cdot \bar{F} = \iint_S (\hat{n} \times \nabla) \cdot \bar{F}\, ds$

Solution: Stoke's theorem for a vector function,

$\bar{F}$ is $\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds$

But $(\nabla \times \bar{F}) \cdot \hat{n} = \hat{n} \cdot (\nabla \times \bar{F})$, (dot product is commutative)

$$= (\hat{n} \times \nabla) \cdot \bar{F}$$

[$\because$ Treating ∇ as a vector and interchanging dot and cross]

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \iint_S (\bar{n} \times \nabla) \cdot \bar{F}\, ds$$

or, $\int_C d\bar{r} \cdot \bar{F} = \iint_S (\bar{n} \times \nabla) \cdot \bar{F}\, ds$

(ii) $\int_C d\bar{r}\, \phi = \iint_S (\bar{n} \times \nabla)\, \phi\, ds$

Solution: Stoke's theorem for a vector function

$\bar{F}$ is $\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\ ds$

Putting $\bar{F} = \bar{a}\,\phi$ where $\bar{a}$ is a constant vector,

We have,

$$\int_C (\bar{a}\phi) \cdot d\bar{r} = \iint_S (\nabla \times \bar{a}\phi) \cdot \bar{n}\ ds$$

Now, $\nabla \times \bar{a}\,\phi = \phi\,(\nabla \times \bar{a}) + (\nabla\phi) \times \bar{a}$

$$= (\nabla\phi) \times \bar{a}$$

$\because \nabla \times \bar{a} = 0$, $\bar{a}$ being a constant vector

$$\therefore \int_C (\bar{a}\phi) \cdot d\bar{r} = \iint_S (\nabla\phi) \times \bar{a} \cdot \bar{n}\ ds$$

$$= -\iint_S (\bar{a} \times \nabla\phi) \cdot \bar{n}\ ds$$

$$= -\iint_S \bar{a} \cdot (\nabla\phi \times \bar{n})\, ds$$

$$= -\iint_S \bar{a} \cdot (\bar{n} \times \nabla\phi)\, ds$$

or, $\bar{a} \cdot \int_C (\phi\, d\bar{r}) = \bar{a} \cdot \iint_S (\bar{n} \times \nabla\phi)\, ds$

or, $\bar{a}\left[\int_C \phi d\bar{r} - \iint_S (\bar{n} \times \nabla\phi) ds\right] = 0$

or, $\int_C \phi d\bar{r} - \iint_S (\bar{n} \times \nabla\phi)\, ds$

$\bar{a}$ being a constant vector

Which not necessarily a null vector.

$\therefore \int_C d\bar{r}\,\phi = \iint_S \bar{n} \times \nabla\phi\, ds$

(iii) $\int_C d\bar{r} \times \bar{F} = \iint_S (\hat{n} \times \nabla) \times \bar{F}\, ds$

Solution: Stoke's theorem for a vector function $\bar{F}$ is,

$\int_C d\bar{r} \times \bar{F} = \iint_S (\nabla \times \bar{F}) . \hat{n}\, ds$

$= \iint_S \hat{n}.(\nabla \times \bar{F})\, ds$

Replacing $\bar{F}$ by $\bar{a} \times \bar{F}$, $\bar{a}$ being a constant vector, in stoke's theorem,

$\int_C \bar{F}.d\bar{r} = \iint_S (\nabla \times \bar{F}).\hat{n}\, ds$, we get

$\int_C (\bar{a} \times \bar{F}) . d\bar{r} = \iint_S \{\nabla \times (\bar{a} \times \bar{F}).\hat{n} ds\}$

or, $\int_C \bar{a} . (\bar{F} \times d\bar{r}) = \iint_S \{(\nabla \times \bar{F})\bar{a} - (\nabla \times \bar{a})\bar{F}\}$

or, $\bar{a} . \int_C \bar{F} \times d\bar{r} = \bar{a} \iint (\hat{n} \times \nabla)\hat{n} - \nabla(\bar{F} \times \hat{n}) ds$

or, $-\bar{a} \int_C d\bar{r} \times \bar{F} = -\bar{a} \iint_S \{(\hat{n} \times \nabla)\bar{F}\} ds$

$\therefore \int_C d\bar{r} \times \bar{F} = \iint_S \{(\hat{n} \times \nabla)\bar{F}\} ds$

(iv) $\int_C \bar{r} \times d\bar{r} = 2 \iint_S \bar{n}$ **ds where C is the boundary of the surface S.**

Solution: We have, by stoke's theorem,

$\int_C \bar{F} . d\bar{r} = \iint_S (\nabla \times \bar{F}) . \bar{n}\, ds$

Let $\bar{F} = \bar{a} \times \bar{r}$ where $\bar{a}$ is constant vector,

$\int_C \bar{a} \times \bar{r} . d\bar{r} = \iint_S \nabla \times (\bar{a} \times \bar{r}) . \bar{n}\, ds$

$\int_C \bar{a} . \bar{r} \times d\bar{r} = \iint_S \nabla \times (\bar{a} \times \bar{r}) . \bar{n}\, ds$ .... (i)

But $\nabla \times (\bar{a} \times \bar{r}) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (\bar{a} \times \bar{r})$

$= \bar{i} \times \frac{\partial}{\partial x}(\bar{a} \times \bar{r}) + \bar{j} \times \frac{\partial}{\partial y}(\bar{a} \times \bar{r}) + \bar{k} \times \frac{\partial}{\partial z}(\bar{a} \times \bar{r})$

$= \bar{i} \times \left(\bar{a} \times \frac{\partial \bar{r}}{\partial x}\right) + \bar{j} \times \left(\bar{a} \times \frac{\partial \bar{r}}{\partial y}\right) + \bar{k} \times \left(\bar{a} \times \frac{\partial \bar{r}}{\partial z}\right)$

$= \bar{i} \times \left(\bar{a} \times \frac{\partial \bar{r}}{\partial x}\right) + \bar{j} \times \left(\bar{a} \times \frac{\partial \bar{r}}{\partial y}\right) + \bar{k} \times \left(\bar{a} \times \frac{\partial \bar{r}}{\partial z}\right)$

Also, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\frac{\partial \bar{r}}{\partial x} = \bar{i},\ \frac{\partial \bar{r}}{\partial y} = \bar{j},\ \frac{\partial \bar{r}}{\partial z} = \bar{k}$

Thus, $\nabla \times (\bar{a} \times \bar{r}) = \bar{i} \times (\bar{a} \times \bar{i}) + \bar{j} \times (\bar{a} \times \bar{j}) + \bar{k} \times (\bar{a} \times \bar{k})$

$= (\bar{i} . \bar{i})\bar{a} - (\bar{i} . \bar{a})\bar{i} + (\bar{j} . \bar{j})\bar{a} - (\bar{j} . \bar{a})\bar{j}$

$+ (\bar{k} . \bar{k})\bar{a} - (\bar{k} . \bar{a})\bar{k}$

$= \bar{a} - (\bar{i} . \bar{a})\bar{i} + \bar{a} - (\bar{j} . \bar{a})\bar{j} + \bar{a} - (\bar{k} . \bar{a})\bar{k}$

and, $a = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{i} . \bar{a} = a_1,\ \bar{j} . \bar{a} = a_2,\ \bar{k} . \bar{a} = a_3$

$\therefore \nabla \times (\bar{a} \times \bar{r}) = 3\bar{a} - (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$

$= 3\bar{a} - \bar{a}$

$= 2\bar{a}$

So, (i) becomes,

$\int_C \bar{a} . \bar{r} \times d\bar{r} = \iint_S 2\bar{a} . \bar{n}\, ds$

or, $\bar{a} \int_C \bar{r} \times d\bar{r} = \bar{a} . 2 \iint_S \bar{n}\, ds$

or, $\bar{a} . \left[\int_C \bar{r} \times d\bar{r} - 2\iint_S \bar{n} ds\right] = 0$

But, $\bar{a} \neq 0$, being a constant vector,

$\therefore \int_C \bar{r} \times d\bar{r} = 2 \iint_S \hat{n}\, ds$ Proved.

Exercise - 10

1. **Evaluate $\iiint_V \phi\, dv$ where V is the region bounded by unit cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ & $\phi = 2x + y$** (BE 2062)

Solution. Given, $\phi = 2x + y$ & v is the unit cube $x = 0, x = 1, y = 0, y = 1,$ $z = 0$ $z = 1$.

$$\iiint_V \phi\, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(2x+y)\; dz\, dy\, dx$$

$$= \int_0^1\int_0^1 (2x+y)\; [z]_0^1\, dy\, dx$$

$$= \int_0^1\int_0^1 (2x+y)\; dydx$$

$$= \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(2x + \frac{1}{2}\right) dx$$

$$= \left[x^2 + \frac{1}{2}x\right]_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

$$\therefore \iiint_V \phi\, dv = \frac{3}{2}$$

2. **Evaluate $\iiint_V \bar{F}\; dv$ where v is the region bounded by $x = 0, y = 0, y = 6, z = x^2$ & $z = 4$ & $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$.** (BE 2061)

Solution. Here, $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ & the volume is enclosed by plane surfaces $x = 0$, $y = 0$, $y = 6$ & $z = 4$ and the curved surface $z = x^2$.

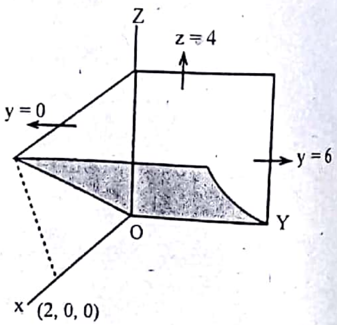
$\therefore 4 = x^2 \Rightarrow x = 2$

Hence the integration shall be carried out in the following order.

From $z = x^2$ to $z = 4$

$y = 0$ to $y = 6$

$x = 0$ to $x = 2$



$$\therefore \iiint_V \bar{F}\; dv = \iiint_V \bar{F}\; (x, y, z)\; dzdydx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}(x\bar{i} + y\bar{j} + z\bar{k})\; dz\, dydx$$

$$= \int_0^2\int_0^6 \left[(x\,\bar{i} + y\bar{j})\,z + \frac{z^2}{2}\,\bar{k}\right]_{x^2}^{4} dy\, dx$$

$$= \int_0^2\int_0^6 (x\bar{i} + y\bar{j})\,4 + \frac{4^2}{2}\,.\bar{k} - (x\bar{i} + y\bar{j})\,x^2 - \frac{(x^2)^2}{2}\,\bar{k}\; dydx$$

$$= \int_0^2\int_0^6 4x\bar{i} + 4y\bar{j} + 8\,\bar{k} - x^3\,\bar{i} - x^2y\,\bar{j} - \frac{x^4}{2}\,\bar{k}\; dy\, dx$$

$$= \int_0^2\int_0^6 (4x - x^3)\bar{i} + (4y - x^2y)\;\bar{j} + \left(8 - \frac{x^4}{x}\right)\bar{k}\; dy\, dx$$

$$= \int_0^2 \left[(4x - x^3)y\bar{i} + \left(2y^2 - x^2\frac{y^2}{2}\right)\bar{j} + \left(8 - \frac{x^4}{x}\right)y\,\bar{k}\right]_0^6 dx$$

$$= \int_0^2 (4x - x^3)6\bar{i} + \left(2.6^2 - x^2\frac{6^2}{2}\right)\bar{j} + \left(8 - \frac{x^4}{x}\right)6\,\bar{k}\; dx$$

$$= \int_0^2 (24x - 6x^3)\bar{i} + (72 - 18x^2)\;\bar{j} + (48 - 3x^4)\;\bar{k}\; dx$$

$$= \left[\left(12x^2 - \frac{3}{2}x^4\right)\bar{i} + (72x - 6x^3)\bar{j} + \left(48x - \frac{3}{5}x^5\right)\bar{k}\right]_0^2$$

$$= \left(12.2^2 - \frac{3}{2}2^4\right)\bar{i} + (72.2 - 6.2^3)\bar{j} + \left(48.2 - \frac{3}{5}2^5\right)\bar{k}$$

$$= (48 - 24)\;\bar{i} + (144 - 48)\;\bar{j} + \left(96 - \frac{96}{5}\right)\bar{k}$$

$$= 24\,\bar{i} + 96\,\bar{j} + \frac{384}{5}\,.\,\bar{k} = 24\,(\bar{i} + 4\,\bar{j} + \frac{16}{5}\,\bar{k})$$

$$\therefore \iiint_V \bar{F}\, dv = 24\,(\bar{i} + 4\,\bar{j} + \frac{16}{5}\,\bar{k})$$

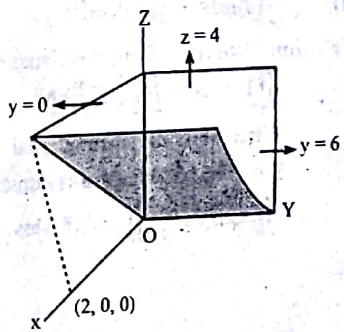
3. **Evaluate $\iiint_V \bar{F}\; dv$ where v is the region bounded by $x = 0$, $y = 0$, $z = x^2$, $y = 6$, $z = 4$ & $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$.**

Solution. Here, $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ & the volume v is enclosed by plane surfaces $x = 0$, $y = 0$, $y = 6$, $z = 4$ & the curved surface $z = x^2$.

$\therefore 4 = x^2 \Rightarrow x^2 \Rightarrow x = 2$

Hence the integration shall be carried out in the following order.

From, $z = x^2$ to $z = 4$



$y = 0$ to $y = 6$
$x = 0$ to $x = 2$

$$\therefore \iiint_V \bar{F}\, dv = \iiint_V \bar{F}(x, y, z)\, dz\, dy\, dx$$

$$= \int_0^2 \int_{y=0}^{6} \int_{z=x^2}^{4} (2xz\bar{i} - x\bar{j} + y^2\bar{k})\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^6 \left[2x\frac{z^2}{2}\bar{i} - xz\bar{j} + y^2 z\bar{k}\right]_{x^2}^{4} dy\, dx$$

$$= \int_0^2 \int_0^6 (x.4^2\bar{i} - x.4\bar{j} + y^2.4\bar{k}) - (x.(x^2)^2\bar{i} - x.x^2\bar{j} + y^2.x^2\bar{k})\, dx dy$$

$$= \int_0^2 \int_0^6 16x\,\bar{i} - 4x\bar{j} + 4y^2\bar{k} - x^5\bar{i} + x^3\bar{j} - x^2y^2\bar{k}\, dy\, dx$$

$$= \int_0^2 \int_0^6 (16x - x^5)\,\bar{i} - (4x - x^3)\,\bar{j} + (4y^2 - x^2y^2\bar{k})\, dy dx$$

$$= \int_0^2 \left[(16x - x^5)y\bar{i} - (4x - x^3)y\bar{j} + \left(\frac{4}{3}y^3 - x^2\frac{y^3}{3}\right)\bar{k}\right]_0^6 dx$$

$$= \int_0^2 (16x - x^5)6\,\bar{i} - (4x - x^3)6y\,\bar{j} + \left(\frac{4}{3}.6^3 - x^2.\frac{6^3}{3}\right)\bar{k}\, dx$$

$$= \int_0^2 (96x - 6x^5)\,\bar{i} - (24x - 6x^3)\,\bar{j} + (288x - 24x^3)\bar{k}\, dx$$

$$= \left[(48x^2 - x^6)\bar{i} - \left(12x^2 - \frac{3}{2}x^4\right)\bar{j} + (288x - 24x^3)\bar{k}\right]_0^2$$

$$= (48.2^2 - 2^6)\,\bar{i} - \left(12.2^2 - \frac{3}{2}.2^4\right)\bar{j} + (288.2 - 24.2^3)\,\bar{k}$$

$$= (192 - 64)\,\bar{i} - (48 - 24)\,\bar{j} + (576 - 192)\,\bar{k}$$

$$= 128\,\bar{i} - 24\,\bar{j} + 384\,\bar{k}$$

$$= 8(16\,\bar{i} - 3\,\bar{j} + 48\,\bar{k})$$

$$\therefore \iiint_V \bar{F}\, dv = 8(16\,\bar{i} - 3\,\bar{j} + 48\,\bar{k})$$

4. Applying gauss' divergence theorem prove that,

(i) $\iint_S \hat{n}\phi ds = \iiint_V \nabla\phi dv$

Solution. We have from the Gauss's divergence theorem,

$$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V (\nabla.\bar{F}) dv$$

Where $\hat{n}$ is the unit external vector to any closed surfaces.

Let, $\bar{F} = \bar{a}\phi$ where $\bar{a}$ is constant vector.

$$\iint_S \bar{a}\phi.\hat{n}\, ds = \iiint_V \nabla.(\bar{a}\,\phi) dv$$

But $\nabla.\bar{a}\phi = \left(\bar{i}.\frac{\partial}{\partial x} - \bar{j}.\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{a}\phi$

$$= \bar{i}.\frac{\partial}{\partial x}(\bar{a}\phi) + \bar{j}.\frac{\partial}{\partial y}(\bar{a}\phi) + \bar{k}.\frac{\partial}{\partial z}(\bar{a}\phi)$$

$$= \bar{i}.\bar{a}\frac{\partial\phi}{\partial x} + \bar{j}.\bar{a}\frac{\partial}{\partial y}\phi + \bar{k}.\bar{a}\frac{\partial\phi}{\partial z}$$

$$= \bar{a}.\bar{i}\frac{\partial\phi}{\partial x} + \bar{a}.\bar{j}\frac{\partial\phi}{\partial y} + \bar{a}.\bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{a}.\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \bar{a}.(\nabla\phi)$$

$$\iint_S \bar{a}\phi.\hat{n}\, ds = \iiint_V \bar{a}.(\nabla\phi) dv$$

or, $\bar{a}\left[\iint_S \hat{n}\,\phi - \iiint_V \nabla\phi\, dv\right] = 0$

Since $\bar{a} \neq 0$

$$\therefore \iint_S \hat{n}\,\phi ds - \iiint_V \nabla\phi\, dv = 0$$

i.e. $\iint_S \hat{n}\phi\, ds = \iiint_V \nabla\phi\, dv$

(ii) $\iint_S \hat{n}.\bar{F} ds = \iiint_V \nabla.\bar{F} dv$

(Hint : Let $\bar{F} = \bar{a}\times\bar{F}$ where $\bar{a}$ is constant vector)

Solution. We have from the Gauss's divergence theorem,

$$\iint_S \bar{f}.\hat{n}\, ds = \iiint_V (\nabla.\bar{F}) dv$$

Where $\hat{n}$ is the unit external vector to any closed surface S.

Let, $\bar{f} = \bar{a}\times\bar{F}$ where $\bar{a}$ is constant vector.

So, $\iint_S \hat{a}\times\bar{F}.\hat{n}\, ds = \iiint_V \nabla(\bar{a}\times\bar{F}) dv$

But, $\nabla.\bar{a}\times\bar{F} = \left(\bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(\bar{a}\times\bar{F})$

$$= \bar{i}.\frac{\partial}{\partial x}(\bar{a}\times\bar{F}) + \bar{j}.\frac{\partial}{\partial y}(\bar{a}\times\bar{F}) + \bar{k}\frac{\partial}{\partial z}(\bar{a}\times\bar{F})$$

$$= \bar{i}.\bar{a}\times\frac{\partial\bar{F}}{\partial x} + \bar{j}.\bar{a}\times\frac{\partial\bar{F}}{\partial y} + \bar{k}.\bar{a}\times\frac{\partial\bar{F}}{\partial z}$$

$$= -\bar{i}.\frac{\partial\bar{F}}{\partial x}\times\bar{a} - \bar{j}.\frac{\partial\bar{F}}{\partial y}\times\bar{a} - \bar{k}.\frac{\partial\bar{F}}{\partial z}\times\bar{a}$$

$$= -\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{F}\times\bar{a}$$

$$= -(\nabla.\bar{F})\times\bar{a} = \bar{a}\times(\nabla.\bar{F})$$

$\therefore \iint_S \bar{a} \times \bar{F}.\hat{n}\, ds = \iiint_V \bar{a} \times (\nabla.\bar{F})\, dv$

or, $\bar{a} \times \left[\iint_S \bar{F}.\hat{n}\, ds - \iiint_V \nabla.\bar{F}\, dv\right] = 0$

Since $\bar{a} \neq 0$

$\therefore \iint_S \bar{F}.\hat{n}\, ds - \iiint_V \nabla \times \bar{F}\, dv$

i.e. $\iint_S \bar{F}.\hat{n}\, ds = \iiint_V \nabla \times \bar{F}\, dv$

(iii) $\iint_S (\hat{n} \times \bar{F})\, ds = \iiint_V (\nabla \times \bar{F})\, dv$

Solution. We have from the Gauss's divergence theorem,

$\iint_S \bar{f}.\hat{n}\, ds = \iiint_V (\nabla.\bar{f})\, dv$, where $\hat{n}$ is the unit external vector to any closed surface S.

Let, $\bar{f} = \bar{a} \times \bar{F}$ where $\bar{a}$ is constant vector.

So, $\iint_S (\bar{a} \times \bar{F}).\hat{n} = \iiint_V \nabla.(\bar{a} \times \bar{F})\, dv$

or, $\iint_S \bar{a}.(\bar{F} \times \hat{n}) = \iiint_V \nabla.(\bar{a} \times \bar{F})\, dv$ by interchanging dot & cross.

But, $\nabla.(\bar{a} \times \bar{F}) = \left(\bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(\bar{a} \times \bar{F})$

$= \bar{i}\,.\,\frac{\partial}{\partial x}(\bar{a} \times \bar{F}) + \bar{j}\,.\,\frac{\partial}{\partial y}(\bar{a} \times \bar{F}) + \bar{k}\,\frac{\partial}{\partial z}(\bar{a} \times \bar{F})$

$= \bar{i}\,.\,\bar{a} \times \frac{\partial \bar{F}}{\partial x} + \bar{j}\,.\,\bar{a} \times \frac{\partial \bar{F}}{\partial y} + \bar{k}\,.\,\bar{a} \times \frac{\partial \bar{F}}{\partial z}$

$= -\bar{i}\,.\,\frac{\partial \bar{F}}{\partial x} \times \bar{a} - \bar{j}\,.\,\frac{\partial \bar{F}}{\partial y} \times \bar{a} - \bar{k}\,.\,\frac{\partial \bar{F}}{\partial z} \times \bar{a}$

$= -\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\ \bar{F} \times \bar{a}$

$= -\nabla.(\bar{F} \times \bar{a})$

$= -(\nabla \times \bar{F}).\bar{a}$ by interchanging dot & cross.

$= -\bar{a}\,.\,(\nabla \times \bar{F})$

$\therefore \iint_S \bar{a}.(\bar{F} \times \hat{n})\, ds = \iiint_V \bar{a}.(\nabla \times \bar{F})\, dv$

or, $\bar{a}\,.\left[\iint_S (\bar{F} \times \hat{n})\, ds = \iiint_V (\nabla \times \bar{F})\, dv\right] = 0$

Since $\bar{a} \neq 0$

$\therefore \iint_S (\bar{F} \times \hat{n})\, ds = \iiint_V (\nabla \times \bar{F})\, dv = 0$

or, $\iint_S (\bar{F} \times \hat{n})\, ds = -\iiint_V (\nabla \times \bar{F})\, dv$

or, $-\iint_S (\hat{n} \times \bar{F})\, ds = -\iiint_V (\nabla \times \bar{F})\, dv$

$\therefore \iint_S (\hat{n} \times \bar{F})\, ds = \iiint_V (\nabla \times \bar{F})\, dv$

(iv) $\iint_S \bar{r}.\hat{n}\, ds = 3V$**, where V is the volume enclosed by a closed surfaces.**

Solution. We have from the Gauss's divergence theorem,

$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V (\nabla.\bar{F})\, dv$

Let, $\bar{F} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ where $\bar{r}$ is the position vector

$\iint_S \bar{r}.\hat{n}\, ds = \iiint_V (\nabla.\bar{r})\, dv$

$= \iiint_V \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(x\bar{i} + y\bar{j} + z\bar{k})\, dv$

$= \iiint_V (1+1+1)\, dv$

$= 3\iiint_V dv$

$\hat{n}\, ds = 3v$ where v is the volume enclosed by a closed surface.

(v) $\iint_S \nabla \times \bar{F}.\hat{n}\, ds = 0$

Solution. We have from the Gauss's divergence theorem,

$\iint_S \bar{f}.\hat{n}\, ds = \iiint_V (\nabla.\bar{f})\, dv$

Let, $\bar{f} = \nabla \times \bar{F}$

So, $\iint_S (\nabla \times \bar{F}).\hat{n}\, ds = \iiint_V \nabla.(\nabla \times \bar{F})\, dv$

$= \iiint_V 0\,.\, dv$

$= 0$

$\iint_S (\nabla \times \bar{F}).\hat{n}\, ds = 0$

(vi) $\iiint_V \nabla.\hat{n}\, dv = S$**, where V is the volume enclosed by a surface S.**

Solution. We have from the Gauss's divergence theorem,

$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V \nabla.\bar{F}\, dV$

Let, $\bar{F} = \hat{n}$ where $\hat{n}$ is the unit external vector to any closed surface S.

So, $\iint_S \hat{n}.\hat{n}\, ds = \iiint_V \nabla.\hat{n}\, ds$

$\therefore \iiint_V \nabla.\hat{n}\, ds = \iint_S 1\, ds = S$

i.e. $\iiint_V \nabla.\hat{n}\, ds = S$

4. **Verify Gauss' divergence theorem for $\bar{F} = x^2\bar{i} + z\bar{j} + yz\bar{k}$ taken over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.**

Solution. Given, $\bar{F} = x^2\bar{i} + z\bar{j} + yz\bar{k}$ & V is the unit cube, $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

We have the Gauss divergence theorem is,

$$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V \nabla.\bar{F}\, dv$$

Now, $\nabla.\ \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\ (x^2\bar{i} + z\bar{j} + yz\bar{k})$

$\nabla.\ \bar{F} = 2x + y$

So, $\iiint_V (\nabla.\bar{F}) = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2x+y)\ x^2\bar{i} - z\bar{j} + yz\bar{k})\, dz\, dy\, dx$

$$= \int_0^1\int_0^1 (2x+y)\ [z]_0^1\ dy\, dx$$

$$= \int_0^1\int_0^1 (2x+y)\ dy\, dx$$

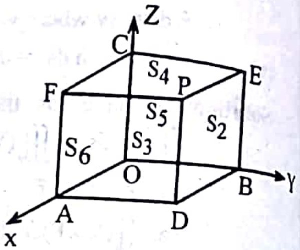
$$= \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(2x + \frac{1}{2}\right) dx$$

$$= \left[x^2 + \frac{1}{2}x\right]_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$



Let ADBECF is a cube which consists six surfaces so that,

$$\iint_S \bar{F}.\hat{n}\, ds = \iint_{S_1} \bar{F}.\hat{n}\, ds + \iint_{S_2} \bar{F}.\hat{n}\, ds + \iint_{S_3} \bar{F}.\hat{n}\, ds + \iint_{S_4} \bar{F}.\hat{n}\, ds + \iint_{S_5} \bar{F}.\hat{n}\, ds + \iint_{S_6} \bar{F}.\hat{n}\, ds.$$

For the surface OADB $\hat{n} = \bar{k}$ and $z = 0$

$\bar{F}.\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k}).(-\bar{k}) = yz = y.\ 0 = 0$

$\therefore \iint_{S_1} \bar{F}.\hat{n}\, ds = 0$

For the surface DBEF $\hat{n} = \bar{j}, y = 1$

$\bar{F}.\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k}).\ \bar{j} = z$

But the surface projection on XZ plane with projection R_1.

$\therefore \iint_{S_2} \bar{F}.\hat{n}\, ds = \iint_{S_2} z\, dz dx = \int_0^1\int_0^1 z\, dz dx$

$$= \int_0^1 \left[\frac{z^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \frac{1}{2}\, dx$$

$$= \left[\frac{1}{2}x\right]_0^1$$

$$= \frac{1}{2}$$

For the surface EPFC, $\hat{n} = \bar{k}, z = 1$

$\bar{F}.\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k}).\ \bar{k}$

$= yz = y\ .\ 1 = y$

But the surface projection on XY plane with projection R_2

$\therefore \iint_{S_4} \bar{F}.\hat{n}\, ds = \int_0^1\int_0^1 y\, dy\, dx$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \frac{1}{2}\, dx$$

$$= \left[\frac{1}{2}x\right]_0^1$$

$$= \frac{1}{2}$$

For the surface AOCF $\hat{n} = -\bar{j}, y = 0$

$\bar{F}.\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k})(-\bar{j}) = -z$

Let the surface projection on XZ plane with projection R_3.

$\therefore \iint_{S_6} \bar{F}.\hat{n}\, ds = \int_0^1\int_0^1 -z\, dz\, dx$

$$= -\int_0^1 \left[\frac{z^2}{2}\right]_0^1 dx$$

$$= -\int_0^1 \frac{1}{2}\, dx$$

$$= -\left[\frac{1}{2}x\right]_0^1$$

$$= -\frac{1}{2}$$

For the surface ADPF, $\hat{n} = \bar{i}, x = 1$

$\bar{F}.\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k}).\ \bar{i} = x^2 = 1$

But the surface projection on yz plane with projection R_3

$\therefore \iint_{S_3} \bar{F}.\hat{n}\, ds = \int_0^1\int_0^1 1\, dy\, dz$

$$= \int [z] \, dy$$

$$= \int dy$$

$$= [y]_0^1$$

$$= 1$$

For the surface OBFC, $\hat{n} = -\bar{i}, x = 0$.

$$\bar{F}.\hat{n} = (x^2\bar{i} + z\bar{j} + yz\bar{k}).(-\bar{i}) = x^2 = 0$$

$$\therefore \iint_{S_5} \bar{F}.\hat{n}\, ds$$

$\therefore$ The surface integral

$$\iint_{S_5} \bar{F}.\hat{n}\, ds = 0 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + 1 + 0$$

$$= \frac{1}{2} + 1 = \frac{3}{2}$$

Hence, $\iint_S \bar{F}.\hat{n}\, ds = \frac{3}{2} = \iiint_V \nabla.\bar{F}\, dV$

Thus Gauss's divergence theorem verified.

5. **Using divergence, theorem evaluate $\iint_S \bar{r}.\hat{n}\, ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.**

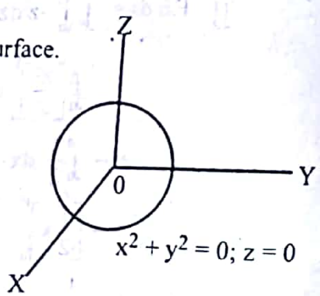
Solution. Given surface of the sphere is $\phi = x^2 + y^2 + z^2 - 9 = 0$

Any normal to the surface,

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)$$

$$= 2x\bar{i} + 2y\bar{j} + 2z\bar{k}.$$

Let $\hat{n}$ be the unit normal to the surface.



$$\therefore \hat{n} = \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{2.3}$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{3}$$

Here, $\bar{r} = x.\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \bar{r}.\hat{n} = x.\bar{i} + y\bar{j} + z\bar{k} . \frac{(x\bar{i} + y\bar{j} + z\bar{k})}{3}$$

$$= \frac{x^2 + y^2 + z^2}{3}$$

$$= \frac{9}{3} = 3$$

Let R be the projection of the surface on xy-plane. The value of the surface integrals is 8 times the projection of surface on xy-plane. Then.

$$\iint_S \bar{r}.\hat{n}\, ds = 8 \iint_R \bar{r}.\hat{n} \frac{dxdy}{|\hat{n}.\bar{k}|}$$

$$\therefore \hat{n}.\bar{k} = \left(\frac{x\bar{i} + y\bar{j} + z\bar{k}}{3}\right).\bar{k} = \frac{3}{3}$$

$$= 8 \iint_R 3 \frac{dxdy}{\frac{z}{3}}$$

$$= 8 \iint_R 9 \frac{1}{z}\, dxdy$$

$$= 72 \int_0^3 \int_0^{\sqrt{9-x^2}} \frac{1}{\sqrt{9 - x^2 - y^2}}\, dx\, dy \qquad \because z^2 = 9 - x^2 - y^2$$

Put $y = \sqrt{9 - x^2}\, \sin\theta$

$dy = \sqrt{9 - x^2}\, \cos\theta d\theta$

So that θ varies from 0 to $\frac{\pi}{2}$

$$= 72 \int_0^{\frac{\pi}{2}} \frac{\sqrt{9 - x^2}}{\sqrt{9 - x^2 - (9 - x^2)\sin^2\theta}}\, d\theta\, dx$$

$$= 72 \int_0^3 \int_0^{\frac{\pi}{2}} \frac{\sqrt{9 - x^2}}{\sqrt{9 - x^2}} \frac{\cos\theta}{\sqrt{1 - \sin^2\theta}}\, d\theta\, dx$$

$$= 72 \int_0^3 [\theta]_0^{\frac{\pi}{2}}\, dx$$

$$= 72 \frac{\pi}{2} \int_0^3 dx$$

$$= 36\pi\, [x]_0^3$$

$$= 108\pi$$

$$\therefore \iint_S \bar{r}.\hat{n}\, ds = 108\pi$$

6. **Prove $\iint_S \bar{F}.\hat{n}\, ds = 6v$ where $\bar{F} = x\bar{i} + 2y\bar{j} + 3z\bar{k}$ & v is the volume enclosed by a surfaces.**

Solution. Here, $\bar{F} = x\bar{i} + 2y\bar{j} + 3z\bar{k}$

We have from the Gauss'divergence theorem,

$$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V \nabla.\bar{F}\, dv$$

$$= \iiint_V \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x\bar{i} + 2\bar{y} + 3z\bar{k})\, dv$$

$$= \iiint (1+2+3)\, dv$$

$$= 6 \iiint dv$$

$$= 6v$$

$$\therefore \iint \bar{F}.\hat{n}\, ds = 6v$$

7. **Verify Gauss' divergence theorem for $\iint \bar{F}.\hat{n}\, ds$ where $\bar{F} = (2xy+z)\,\bar{i} + y^2\bar{j} - (x+3y)\,\bar{k}$ & S is the surface bounded by the plane $2x + 2y + z = 6$, $x = 0$, $y = 0$, $z = 0$.**

Solution. Here, $\bar{F} = (2xy+z)\,\bar{i} + y^2\bar{j} - (x+3y)\,\bar{k}$

$$\nabla\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) . \left\{(2xy+z)\,\bar{i} + y^2\bar{j} - (x+3y)\,\bar{k}\right\}$$

$$= 2y + 2y = 0$$

$$= 4y$$

$$\iiint (\nabla.\bar{F})\, dv = \iiint 4y\, dx\, dy\, dz$$

$$= \int_{x=0}^{3}\int_{y=0}^{3-x}\int_{z=0}^{6-2x-3y} 4y\; dz\, dy\, dx$$

$$= \int_0^3\int_0^{3-x} 4y\;(6-2x-2y)\, dy\, dx$$

$$= 8\int_0^3\int_0^{3-x} y\,(3-x-y)\, dy\, dx$$

$$= 8\int_0^3\int_0^{3-x} [(3-x)\,y - y^2]\, dy\, dx$$

$$= 8\int_0^3 \left[(3-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{3-x} dx$$

$$= 8\int_0^3 \left[(3-x)\frac{(3-x)^2}{2} - \frac{(3-x)^3}{3}\right] dx$$

$$= 8\int_0^3 \left[\frac{(3-x)^3}{2} - \frac{(3-x)^3}{3}\right] dx$$

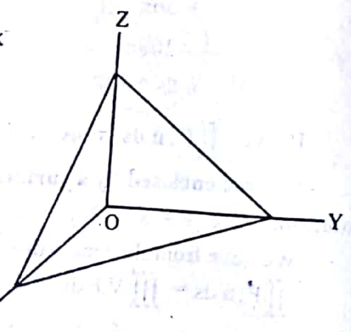
$$= 8\int_0^3 (3-x)^3\left(\frac{1}{2} - \frac{1}{3}\right) dx$$

$$= 8\int_0^3 (3-x)^3\left(\frac{3-2}{6}\right) dx$$

$$= \frac{4}{3}\left[\frac{(3-x)^4}{-4}\right]_0^3$$

$$= -\frac{1}{3}\,[0-(3)^4]$$

$$= -\frac{1}{3}\,(0-81)$$



$$= \frac{81}{3}$$

$$= 27$$

$$\therefore \iiint (\nabla.\bar{F})\, dv = 27$$

8. **Use Gauss divergence theorem to evaluate $\iint \bar{F}.\hat{n}\, ds$ where $\bar{F} = (2x-z)\,\bar{i} + x^2y\bar{j} - xz^2\,\bar{k}$ where S is the surface of the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. (BE 2062)**

Solution. Here, $\bar{F} = \left\{(2x-z)\,\bar{i} + x^2y\bar{j} - xz^2\,\bar{k}\right\}$ & S is the surface of planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

We have, $\iint \bar{F}.\hat{n}\, ds = \iint_{S_1} \bar{F}.\hat{n}\, ds + \iint_{S_2} \bar{F}.\hat{n}\, ds + \iint_{S_3} \bar{F}.\hat{n}\, ds$

$$+ \iint_{S_4} \bar{F}.\hat{n} + \iint_{S_5} \bar{F}.\hat{n}\, ds + \iint_{S_6} \bar{F}.\hat{n}\, ds$$

For the surface $S_1 = z = 0$, $\hat{n} = -\bar{k}$

$$\bar{F}.\hat{n} = \{(2x-z)\,\bar{i} + x^2y\,\bar{j} - xz^2\bar{k}\}.(-\bar{k})$$

$$= x^2 \times 0 = 0$$

$$\therefore \iint_{S_1} \bar{F}.\hat{n}\, ds = 0$$

For the surfaces S_2, $z = 1$, $\hat{n} = \bar{k}$,

$$\bar{F}.\hat{n} = \{(2x-z)\,\bar{i} + x^2y\,\bar{j} - xz^2\bar{k}\}.(\bar{k})$$

$$= -x \times 1 = -x$$

$$\therefore \iint_{S_2} \bar{F}.\hat{n}\, ds = \int_0^1\int_0^1 (-x)\, dx\, dy$$

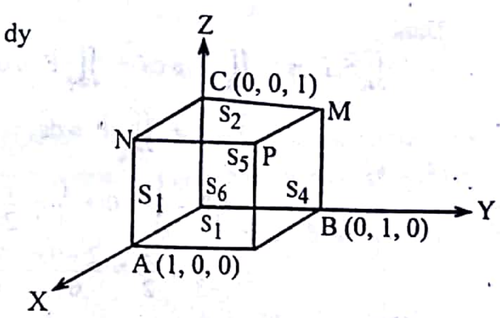
$$= \int_0^1 \left[-\frac{x^2}{2}\right]_0^1 dy$$

$$= -\int_0^1 \frac{dy}{2}$$

$$= -\frac{1}{2}\int_0^1 dy$$

$$= -\left[\frac{y}{2}\right]_0^1$$

$$= -\frac{1}{2}$$



For the surface S_3, $y = 0$, $\hat{n} = -\bar{j}$

$$\bar{F}.\hat{n} = [(2x-z)\,\bar{i} + x^2y\,\bar{j} - xz^2\bar{k}].(-\bar{j})$$

$$= -x^2 \times 0 = 0$$

$$\therefore \iint_{S_3} \bar{F}.\hat{n}\, ds = 0$$

For the surface S_4, $y = 1$, $\hat{n} = \bar{j}$

$$\bar{F}.\hat{n} = [(2x - z)\,\bar{i} - x^2y\,\bar{j} - xz^2\,\bar{k}\,].\,\bar{j}$$
$$= x^2 \times 1 = x^2$$

$$\therefore \iint_{S_4} \bar{F}.\hat{n}\, ds = \int_0^1\int_0^1 x^2\, dz\, dx$$

$$= \int_0^1 x^2\,[z]_0^1\, dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

For the surface S_5, $x = 0$, $\hat{n} = -\,\bar{i}$

$$\bar{F}.\hat{n} = [(2x - z)\,\bar{i} + x^2y\,\bar{j} - xz^2\,\bar{k}\,].(-\,\bar{i}\,)$$
$$= (0 - 2)\,(-1)$$
$$= z$$

$$\therefore \iint_S \bar{F}.\hat{n}\, dy\, dz = \int_0^1\int_0^1 z\, dy\, dz$$

$$= \int_0^1 z[y]_0^1\, dz = \int_0^1 z\, dz = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}$$

For the surface S_6, $x = 1$, $\hat{n} = \bar{i}$

$$\bar{F}.\hat{n} = [(2x - z)\,\bar{i} + x^2y\,\bar{j} - xz^2\,\bar{k}\,].\,\bar{i}$$
$$= (2 - z)$$

$$\therefore \iint_S \bar{F}.\hat{n}\, dy\, dz = \int_0^1\int_0^1 (2 - z)\, dy\, dz$$

$$= \int_0^1 (2 - z)[y]_0^1\, dz$$

$$= \int_0^1 (2 - z)\, dz = \left[2z - \frac{z^2}{2}\right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

Thus,

$$\therefore \iint_S \bar{F}.\hat{n}\, ds = \iint_{S_1} \bar{F}.\hat{n}\, ds + \iint_{S_2} \bar{F}.\hat{n}\, ds + \iint_{S_3} \bar{F}.\hat{n}\, ds$$

$$+ \iint_{S_4} \bar{F}.\hat{n}\, ds + \iint_{S_5} \bar{F}.\hat{n}\, ds + \iint_{S_6} \bar{F}.\hat{n}\, ds$$

$$= 0 - \frac{1}{2} + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{2}$$

$$= \frac{1}{3} + \frac{3}{2} = \frac{2 + 9}{6} = \frac{11}{6}$$

$$\therefore \iint_S \bar{F}.\hat{n}\, ds = \frac{11}{6}$$

9. Evaluate, by Gauss' divergence theorem for $\iint_S \bar{F}.\hat{n}\, ds$ where $\bar{F} = x\bar{i} - y\bar{j} + (z^2 - 1)\bar{k}$ and S is the cylinder formed by the surface $z = 0$, $z = 1$ & $x^2 + y^2 = 4$.

Solution. We have from the Gauss' divergence theorem,

$$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V \nabla.\bar{F}\, dv$$

Here, $\bar{F} = x\bar{i} - y\bar{j} + (z^2 - 1)\,\bar{k}$.

$$\nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) . \{(x\bar{i} - y\bar{j} + (z^2 - 1)\,\bar{k}\}$$
$$= 1 - 1 + 2z$$
$$= 2z$$

$$\therefore \iint_S \bar{F}.\hat{n}\, ds = \iiint_V 2z\, dy\, dx\, dz$$

$$= \int_0^1\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4+x^2}} z\, dy\, dx\, dz$$

$$= \int_0^1\int_{x=-2}^{2} 2z[y]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$$= \int_0^1\int_{-2}^{2} 2z\left(\sqrt{4 - x^2} + \sqrt{4 - x^2}\right) dx\, dz$$

$$= \int_0^1\int_{-2}^{2} 2z \times 2\sqrt{4 - x^2}\, dx\, dz$$

$$= 4\int_0^1\int_{-2}^{2} z\sqrt{4 - x^2}\, dx\, dz$$

$$= 4 \times 2\int_0^1 z\left(\int_0^2 \sqrt{4 - x^2}\,dx\right) dz$$

$$= 8\int_0^1 z\left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 dz$$

$$= 8\int_0^1 z\left(0 + 2\frac{\pi}{2} - 0\right) dz$$

$$= 8\pi\left[\frac{z^2}{2}\right]_0^1 = 8\pi \times \frac{1}{2} = 4\pi$$

$$\therefore \iiint_V \nabla.\bar{F}\, dv = 4\pi$$

10. Verify Gauss' divergence theorem for the function $\bar{F} = y\bar{i} + x\bar{j} + z^2\bar{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$, & $z = 2$.

Solution. Here, $\bar{F} = y\bar{i} + x\bar{j} + z^2\bar{k}$

$$\nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) . \{(y\bar{i} + x\bar{j} + z^2\bar{k}\}$$
$$= 0 + 0 + 2z = 2z$$

$$\iiint_V \nabla.\bar{F}\, dv = \iiint_V 2z\, dx\, dy\, dz$$

Where V is the region bounded by,

$z = 0$, to $z = 2$ $\;\; y = -\sqrt{9 - x^2}$ to $y = \sqrt{9 - x^2}$ & $x = -3$ to $x = 3$

So. $\iiint \nabla.\vec{F}\, dv = \int_{z=0}^{2} \int_{x=-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (2z)\, dy\, dx\, dz$

$= \int_0^2 \int_{-3}^{3} 2z[y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dx\, dz$

$= \int_0^2 \int_{-3}^{3} 2z\left(\sqrt{9-x^2} + \sqrt{9-x^2}\right) dx\, dz$

$= 4 \int_0^2 \int_{-3}^{3} z\left\{\sqrt{9^2 - x^2}\right\} dx\, dz$

$= 4 \times 2 \int_0^2 z\left(\int_0^3 \sqrt{9-x^2} dxdz\right)$

$= 8 \int_0^2 z\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 dz$

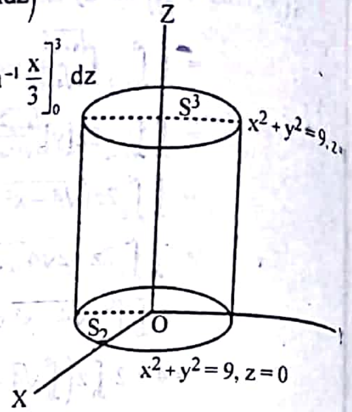
$= 8 \int_0^2 z\left[0 + \frac{9}{2}\cdot\frac{\pi}{2}\right] dz$

$= 2\times 9\pi \int_0^2 z\, dz$

$= 18\pi \left[\frac{z^2}{2}\right]_0^2$

$= 18 \times 2 . \pi$

$= 36\pi$



Also, we have,

$\iint_S \vec{F}.\hat{n}\, ds = \iint_{S_1} \vec{F}.\hat{n}\, ds + \iint_{S_2} \vec{F}.\hat{n}\, ds + \iint_{S_3} \vec{F}.\hat{n}\, ds.$

For the surface S_2, $z = 0$, $\hat{n} = -\vec{k}$

$\vec{F}.\hat{n} = (y\vec{i} + x\vec{j} + z^2\vec{k}).(-\vec{k})$

$= z^2 = 0$

$\therefore \iint_{S_2} \vec{F}.\hat{n}\, ds = 0$

For the surface S_3, $z = 2$, $\hat{n} = \vec{k}$

$\vec{F}.\hat{n} = (y\vec{i} + x\vec{j} + z^2\vec{k}) . \vec{k}$

$= z^2 = 4$ & $\hat{n} = \vec{k} = 1$

$\therefore \iint_{S_3} \vec{F}.\hat{n}\, ds = \iint_R \vec{F}.\hat{n} \frac{dxdy}{|\hat{n}.\vec{k}|} = \iint_R 4dxdy$

$= \int_{-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\, dy\, dx$

$= 4 \int_{-3}^{3} [y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$

$= 4 \int_{-3}^{3} \left(\sqrt{9-x^2} + \sqrt{9-x^2}\right) dx$

$= 4 \times 2 \int_{-3}^{3} \sqrt{9-x^2}\, dx$

$= 8 \times 2 \int_0^3 \sqrt{9-x^2}\, dx$

$= 16 \left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$

$= 16 \left[0 + \frac{9}{2}\cdot\frac{\pi}{2} - 0\right] = 18 \frac{\pi.9}{4} = 36\pi$

For the surface S_1,

The normal to the surface $\phi = x^2 + y^2 - 9 = 0$ is,

$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) (x^2 + y^2 - 9)$

$= 2x\,\vec{i} + 2y\,\vec{j}$

Unit normal to the surface,

$\hat{n} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2(x\vec{i} + y\vec{j})}{\sqrt{4(x^2 + y^2)}} = \frac{2(x\vec{i} + y\vec{j})}{2.3} = \frac{x\vec{i} + y\vec{j}}{3}$.

Let R be the projection of the surface on the yz – Plane then,

$\hat{n} . \vec{i} = (y\vec{i} + x\vec{j} + z^2\vec{k}) . \vec{i} = \frac{x}{3}$

& $\vec{F}.\hat{n} = (y\vec{i} + x\vec{j} + z^2\vec{k}) . \frac{(x\vec{i} + y\vec{j})}{3} = \frac{xy + xy}{3} = \frac{2xy}{3}$

$\therefore \iint_{S_1} \vec{F}.\hat{n}\, ds = \iint_{R_1} 2xy \frac{dydz}{|\hat{n}.\vec{k}|} = \iint_R \frac{2xy}{3} . \frac{dydz}{\frac{x}{3}}$

$= \iint 2y\, dy\, dz = 2 \int_0^2 \int_{-3}^{3} y\, dy\, dz = 2 \int_0^2 \left[\frac{y^2}{2}\right]_{-3}^{3} dz = 0$

$\therefore \iint_S \vec{F}.\hat{n}\, ds = 0 + 36\pi + 0 = 36\pi$

Hence, $\iint_S \vec{F}.\hat{n}\, ds = 36\pi = \iiint_V \nabla.\vec{F}\, dv$

$\therefore$ Gauss' divergence theorem is verified.

(11) Evaluate $\iint_S \vec{F}.\hat{n}\, ds$ where $\vec{F} = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$ & S is the surface of sphere $x^2 + y^2 + z^2 = 1$ by Gauss' divergence theorem.

Solution. Here, $\vec{F} = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$

$\therefore \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) . (2x\vec{i} + 3y\vec{j} + 4z\vec{k})$

$= 2 + 3 + 4 = 9$

We have, by Gauss's divergence theorem,

$$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V \nabla.\bar{F}\, dV$$

$$= \iiint_V 9\, dV$$

$= 9V$ where V is the volume of the sphere, $x^2 + y^2 + z^2$...

∴ The sphere $x^2 + y^2 + z^2 = 1$ is the unit sphere so,

The volume of the unit sphere $x^2 + y^2 + z^2 = 1$ is,

$$V = \frac{4\pi(1)^3}{3} = \frac{4\pi}{3}$$

$$\therefore \iint_S \bar{F}.\hat{n}\, ds = 9 \times \frac{4\pi}{3} = 12\pi^3$$

(12) Evaluate $\iint_S \bar{F}.\hat{n}\, ds$ where $\bar{F} = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$ and S is the surface bounded by x = 0, y = 0, z = 0 and the plane x + y + z = 1 by Gauss's divergence theorem.

Solution. Here, $\bar{F} = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$

$$\therefore \nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(xy\bar{i} + z^2\bar{j} + 2yz\bar{k})$$

$$= y + 2y$$

$$= 3y$$

We have by Gauss's divergence theorem,

$$\iint_S \bar{F}.\hat{n}\, ds = \iiint_V \nabla.\bar{F}$$

$$= \iiint 3y\; dx\, dy\, dz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3y\; dz\, dy\, dx$$

$$= \int_0^1\int_0^{1-x} 3y\; [z]_0^{1-x-y}\; dy\, dx$$

$$= \int_0^1\int_0^{1-x} 3y\,(1 - x - y)\, dy\, dx$$

$$= 3\int_0^1\int_0^{1-x} [(1-x)\,y - y^2]\, dy\, dx$$

$$= 3\int_0^1 \left[(1-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{(1-x)} dx$$

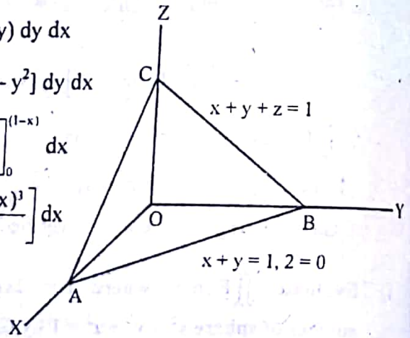
$$= 3\int_0^1 \left[\frac{(1-x)^3}{6} - \frac{(1-x)^3}{3}\right] dx$$

$$= 3\int_0^1 \frac{(1-x)^3}{6}\, dx$$

$$= \frac{3}{6}\left[\frac{(1-x)^4}{4(-1)}\right]_0^1$$

$$= \frac{1}{2}\cdot\frac{1}{4}$$

$$\iint_S \bar{F}.\hat{n}\, ds = \frac{1}{8}$$



3 Laplace transforms

Definitions

Let f(t) be function defined for all $t \geq 0$ then $F(s) = \int_0^\infty e^{-st} f(t)\, dt$.

Provided the integral exist, is called the Laplace transform of f(t). It is denoted by L [f(t)] or F(s).

Laplace transform of some standard formulae

1. $L[1] = \frac{1}{s}$
2. $L[t^n] = \frac{n!}{s^{n+1}}$ when n = 0, 1, 2, 3....
3. $L[e^{at}] = \frac{1}{s-a}$
4. $L[\cos h\, at] = \frac{s}{s^2 - a^2}$
5. $L[\sin h\, at] = \frac{a}{s^2 - a^2}$
6. $L[\sin at] = \frac{a}{s^2 + a^2}$
7. $L[\cos at] = \frac{s}{s^2 + a^2}$ where, $s > a$ and $s^2 > a^2$

Inverse Laplace transform

If $L[f(t)] = F(s)$, then f(t) is called inverse Laplace transform of F(s) and symbolically written as $f(t) = L^{-1}\{F(s)\}$.

Where, L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods

(i) Standard results
(ii) Second suifting theorem
(iii) Differentiation of F(s)
(iv) Partical fraction expansion
(v) Convolution theorem

Exercise - 11

1. **Find the laplace trans forms of the following functions.**

(i) $t^2 + at + b$

Solution. We have, $L = [t^2 + at + b]$

$$= L[t^2] + L[at] + L[b]$$
$$= L[t^2] + aL[t] + bL[b]$$
$$= \frac{2}{S^3} + \frac{a}{S^2} + \frac{b}{S}$$

(ii) e^{at+b}

Solution. We have, $L[e^{at} . e^b]$

$$= e^b L[e^{at}]$$
$$= e^b . \frac{1}{S-a}$$

(iii) $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

Solution. We have, $L[e^{2t} + 4t^3 - 2\sin 3t + 3\cos 2t]$

$$= L[e^{2t}] + 4L[t^3] - 2L[\sin 3t] + 3L[\cos 3t]$$
$$= \frac{1}{S-2} + 4.\frac{3!}{S^4} - 2.\frac{3}{S^2+9} + 3.\frac{S}{S^2+9}$$
$$= \frac{1}{S-2} + \frac{24}{S^4} - \frac{6}{S^2+9} + \frac{3S}{S^2+9}$$

(iv) $t - \sinh 2t$

Solution. We have, $L[t - \sin h\, 2t]$

$$= L[t] - L[\sinh 2t]$$
$$= \frac{1}{S^2} - \frac{2}{S^2-4} = \frac{S^2-4-2S^2}{S^2(S^2-4)} = \frac{4+S^2}{S^2(4-S^2)} \text{ Ans.}$$

(v) $\cos(\omega t + \theta)$

Solution. We have, $L[\cos(\omega t + \theta)]$

$$= L[\cos\omega t \cos\theta - \sin\omega t \sin\theta]$$
$$= \cos\theta\, L[\cos\omega t] - \sin\theta\, [\sin\omega t]$$
$$= \frac{s\cos\theta}{S^2+\omega^2} - \frac{\omega\sin\theta}{S^2+\omega^2}$$
$$= \frac{S\cos\theta - \omega\sin\theta}{S^2+\omega^2} \text{ Ans.}$$

(vi) $\sin^2 3t$

Solution. We have, $L[\sin^2 3t]$

$$= L\left[\frac{1-\cos 6t}{2}\right] = \frac{1}{2}L[1] - \frac{1}{2}L[\cos 6t]$$
$$= \frac{1}{2}.\frac{1}{S} - \frac{1}{2}.\frac{S}{S^2+36}$$
$$= \frac{1}{2}\left[\frac{S^2+36-S^2}{S(S^2+36)}\right] = \frac{18}{S(S^2+36)} \text{ Ans.}$$

(vii) $\cos^3 2t$

Solution. We have, $L[\cos^3 2t]$

$$= L\left[\frac{1}{4}\cos 6t + \frac{3}{4}\cos 2t\right]$$
$$= \frac{1}{4}L[\cos 6t] + \frac{3}{4}L[\cos 2t]$$
$$= \frac{1}{4}\frac{s}{S^2+36} + \frac{3}{4}\frac{S}{S^2+4}$$
$$= \frac{S}{4}\left[\frac{1}{S^2+36} + \frac{3}{S^2+4}\right]$$
$$= \frac{S}{4}\left[\frac{S^2+4+3S^2+108}{(S^2+36)(S^2+4)}\right]$$
$$= \frac{S}{4}\left[\frac{4S^2+112}{(S^2+36)(S^2+4)}\right]$$
$$= S\frac{(S^2+28)}{(S^2+36)(S^2+4)} \text{ Ans.}$$

(viii) $\cosh^3 2t$ **(BE 2065)**

Solution. We have, $L[\cosh^3 2t]$

$$= L\left[\frac{\cosh 6t + 3\cosh 2t}{4}\right]$$
$$= \frac{1}{4}L[\cosh 6t] + \frac{3}{4}L[\cosh 2t]$$
$$= \frac{1}{4}\frac{S}{S^2-36} + \frac{3}{4}.\frac{S}{S^2-4}$$
$$= \frac{S}{4}\left[\frac{S^2-4+3S^2-108}{(S^2-36)(S^2-4)}\right]$$
$$= \frac{S}{4}\left[\frac{4S^2-112}{(S^2-36)(S^2-4)}\right]$$
$$= \frac{S}{4}.4\left[\frac{S^2-28}{(S^2-36)(S^2-4)}\right]$$
$$= \left[\frac{S(S^2-28)}{(S^2-36)(S^2-4)}\right] \text{ Ans.}$$

(ix) $\sinh^3 2t$

Solution. We have, $L[\sinh^3 2t]$

$$= L\left[\frac{3\sinh 2t + \sinh 6t}{4}\right]$$
$$= \frac{3}{4}L[\sinh 2t] + \frac{1}{4}L[\sinh 6t]$$
$$= \frac{3}{4}.\frac{2}{S^2-4} + \frac{1}{4}.\frac{6}{S^2-36}$$

$$= \frac{6}{24}\left[\frac{S^2-36-S^2-4}{(S^2-4)(S^2-36)}\right]$$

$$= \frac{3}{2}\,\frac{-32}{(S^2-4)(S^2-36)}$$

$$= \frac{-48}{(S^2-4)(S^2-36)} \text{ Ans.}$$

(x) sin 2t cos 3t

Solution. We have, L [sin 2t cos 3t]

$$= \frac{1}{2} L\,[2\sin 2t \cos 3t]$$

$$= \frac{1}{2} L\,[\sin 5t] - \frac{1}{2} L\,[\sin t]$$

$$= \frac{1}{2}\,\frac{s}{s^2+25} - \frac{1}{2}\,\frac{1}{s^2+1}$$

$$= \frac{1}{2}\left[\frac{5s^2+5-s^2-25}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2-20}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{1}{2}.4\left[\frac{s^2-5}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{2(s^2-5)}{(s^2+25)(s^2+1)} \text{ Ans.}$$

(xi) sin 3t cos 3t

Solution. We have, L [sin 3t cos 3t]

$$= \frac{1}{2} L\,[\sin 6t] + \frac{1}{2} L\,[\sin 0]$$

$$= \frac{1}{2}\,\frac{6}{s^2+36} + 6 = \frac{3}{s^2+36} \text{ Ans.}$$

(xii) sin 3t sin 2t

Solution. We have, L [sin 3t cos 2t]

$$= \frac{1}{2} L\,[\cos 5t] - \frac{1}{2} L\,[\cos t]$$

$$= \frac{1}{2}\cdot\frac{s}{s^2+25} - \frac{1}{2}\cdot\frac{s}{s^2+1}$$

$$= \frac{s}{2}\left[\frac{s^2+1-s^2-25}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{s}{2}\,\frac{-24}{(s^2+25)(s^2+1)}$$

$$= \frac{-12s}{(s^2+25)(s^2+1)} \text{ Ans.}$$

(xiii) cos 3t cos 2t

Solution. We have, L [cos 3t cos 2t]

$$= \frac{1}{2} L\,[2\cos 3t \cos 2t]$$

$$= \frac{1}{2} L\,[\cos 5t] + \frac{1}{2}\,[\cos t]$$

$$= \frac{1}{2}\,\frac{s}{s^2+25} + \frac{1}{2}\,\frac{s}{s^2+1}$$

$$= \frac{s}{2}\left[\frac{s^2+1+s^2+25}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{s}{2}\left[\frac{s^2+26}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{s}{2}\,.\,2\,\frac{(s^2+13)}{(s^2+25)(s^2+1)}$$

$$= \frac{s(s^2+13)}{(s^2+25)(s^2+1)} \text{ Ans.}$$

(xiv) cos at sinh at

Soluiton. We have, L [cos at sinh at]

$$= \frac{1}{2} L\,[\cos at\,(e^{at}-e^{-at})]$$

$$= \frac{1}{2} L\,[e^{at}\cos at] - \frac{1}{2} L\,[\cos at\, e^{-at}]$$

$$= \frac{1}{2}\cdot\frac{s-a}{(s-a)^2+a^2} - \frac{1}{2}\cdot\frac{s+a}{(s+a)^2+a^2}$$

$$= \frac{1}{2}\left[\frac{s^3+2as^2+2a^2s-as^2-2a^2s-2a^3-s^3+2as^2-2a^2s-as^2+2a^2s-2a^3}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{1}{2}\,\frac{2as^2-4a^3}{(s^2+2a)^2-4a^2s^2}$$

$$= \frac{1}{2}\,.\,2\,\frac{a(s^2-2a^2)}{s^2+4a^2s^2+4a^4-4a^2s^2}$$

$$= \frac{a(s^2-2a^2)}{s^2+4a^4} \text{ Ans.}$$

(xv) sin at cosh at **(BE 2061)**

Solution. We have, L [sin at cosh at]

$$= \frac{1}{2} L\,[\sin at\,(e^{at}+e^{-at})]$$

$$= \frac{1}{2} L\,[e^{at}\sin at] + \frac{1}{2} L\,[e^{-at}\sin at]$$

$$= \frac{1}{2}\,\frac{a}{(s-a)^2+a^2} + \frac{1}{2}\,\frac{a}{(s+a)^2+a^2}$$

$$= \frac{a}{2}\left[\frac{s^2+2as+2a^2+s^2-2as+2a^2}{(s^2-2as+a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{a}{2}\,\frac{(2s^2+4a^2)}{s^4+4a^4}$$

$$= \frac{a}{2}\,.\,2\frac{(s^2+2a^2)}{s^4+4a^4}$$

$$= \frac{a(s^2+2a^2)}{s^4+4a^4} \text{ Ans.}$$

(xvi) cos at cosh at

Solution. We have, L [cos at cosh at]

$$= \frac{1}{2} L\,[\cos at\,(e^{at}+e^{-at})]$$

$$= \frac{1}{2} L\,[e^{at}\cos at] + \frac{1}{2} L\,[e^{-at}\cos at]$$

$$= \frac{1}{2}\,\frac{s-a}{(s-a)^2+a^2} + \frac{1}{2}\,\frac{s+a}{(s+a)^2+a^2}$$

$$= \frac{1}{2}\left[\frac{(s-a)(s^2+2as+2a^2)+(s+a)(s^2-2as+2a^2)}{s^4+4a^4}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+2as^2+2a^2s-as^2-2a^2s-2a^3+s^3-2as^2+2a^2s+as^2-2a^2s+2a^3}{s^4+4a^4}\right]$$

$$= \frac{1}{2}\,\frac{2s^3}{s^4+4a^4}$$

$$= \frac{s^3}{s^4+4a^4} \text{ Ans.}$$

(xvii) sin at sinh at

Solution. We have, L [sin at sinh at]

$$= \frac{1}{2} L\,[\sin at\,(e^{at}-e^{-at}]$$

$$= \frac{1}{2} L\,[e^{at}\sin at] - \frac{1}{2} L\,[e^{-at}\sin at]$$

$$= \frac{1}{2}\,\frac{a}{(s-a)^2+a^2} - \frac{1}{2}\,\frac{a}{(s+a)^2+a^2}$$

$$= \frac{a}{2}\left[\frac{s^2+2as+2a^2-s^2+2as-2a^2}{s^4+4a^4}\right]$$

$$= \frac{a}{2}\,.\,\frac{4as}{s^4+4a^4}$$

$$= \frac{2a^2s}{s^4+4a^4} \text{ Ans.}$$

2. Find the Laplace transforms of the following functions.

(i) e^{-3t} [2 cos 5t − 3 sin 5t]

Solution. We have. $L\,[e^{-3t}(2\cos 5t - 3\sin 5t)]$

$$= 2L\,[e^{-3t}\cos 5t] - 3L\,[e^{-3t}\sin 5t]$$

By using the first shifting theorem of Laplace transform. We get.

$$= \frac{2(s+3)}{(s+3)^2+25} - 3\,.\,\frac{5}{(s-3)^2+25}$$

$$= \frac{2s+6-15}{s^2+6s+9+25}$$

$$= \frac{2s-9}{s^2+6s+34} \text{ Ans.}$$

(ii) $e^{-t}\sin^2 t$

Solution. We have, $L\,[e^{-t}\sin^2 t]$

$$= \frac{1}{2} L\,[e^{-t}(1-\cos 2t]$$

$$= \frac{1}{2} L\,[e^{-t}] - \frac{1}{2} L\,[e^{-t}\cos 2t]$$

By using the first shifting theorem of Laplace transform. We get,

$$= \frac{1}{2}\,.\,\frac{1}{s+1} - \frac{1}{2}\,\frac{s+1}{(s+1)^2+4}$$

$$= \frac{1}{2}\,\frac{(s^2+2s+5-s^2-2s+1}{(s+1)(s^2+2s+s)}$$

$$= \frac{1}{2}\,\frac{4}{(s+1)(s^2+2s+5)}$$

$$= \frac{2}{(s+1)(s^2+2s+5)} \text{ Ans.}$$

(iii) e^{-t} sinh 3t

Solution. We have, $L\,[e^{-t}\sinh 3t]$

By using the first shifting theorem of Laplace transform. We get,

$$= \frac{3}{(s+1)^2-9}$$

$$= \frac{3}{s+2s-8} \text{ Ans.}$$

(iv) $(t+2)^2 e^t$

Solution. We have, $L\,[t+2)^2 e^t]$

$$= L\,[(t^2+4+4)\,e^t]$$

$$= L\,[e^t t^2] + 4\,[e^t t] + 4\,L\,[e^t]$$

By using the first shifting theorem of Laplace transform. We get.

$$= \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1}$$

$$= \frac{2+4(s-1)+4.(s-1)^2}{(s-1)^3}$$

$$= \frac{2+4s-4+4s^2-8s+4}{(s-1)^3}$$

$$= \frac{4s^2-4s+2}{(s-1)^3}$$

$$= \frac{2(2s^2-2s+1)}{(s-1)^3} \text{ Ans.}$$

(v) $e^{-2t}\sin^2 3t$

Solution. We have, $L[e^{-2t}\sin^2 3t]$

$$= \frac{1}{2}L[e^{-2t}] - \frac{1}{2}L[e^{-2t}\cos 6t]$$

$$= \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\,\frac{s+2}{(s+2s^2+36)}$$

$$= \frac{1}{2}\left[\frac{s^2+4s+40-s^2-4s-4}{(s+2)(s^2+4s+40)}\right]$$

$$= \frac{1}{2}\,\frac{36}{(s+2)(s^2+4s+40)}$$

$$= \frac{18}{(s+2)(s^2+4s+40)} \text{ Ans.}$$

(vi) $t\sin^3 3t$ **(BE 2062)**

Solution. We have, $L[t\sin^3 3t]$

Now, $L[\sin^3 3t] = \frac{3}{4}L[\sin 3t] - \frac{1}{4}L[\sin 9t]$

$$= \frac{3}{4}\cdot\frac{3}{s^2+9} - \frac{1}{4}\cdot\frac{9}{s^2+81}$$

$$= \frac{9}{4}\,\frac{1}{s^2+9} - \frac{1}{s^2+81}$$

By using the theorem of L. T. we get,

$$L[t\sin^3 3t] = (-1)\frac{d}{ds}\left[\frac{9}{4}\left(\frac{1}{s^2+9}-\frac{1}{s^2+81}\right)\right]$$

$$= \frac{9}{4}\left[\frac{d}{ds}\left(\frac{1}{s^2+9}\right)-\frac{d}{ds}\left(\frac{1}{s^2+81}\right)\right]$$

$$= -\frac{9}{4}\left[-\frac{2s}{(s^2+9)^2}+\frac{2s}{(s^2+81)^2}\right]$$

$$= \frac{9}{4}\left[\frac{2s}{(s^2+9)^2}-\frac{2s}{(s^2+81)^2}\right]$$

$$= \frac{9}{4}\cdot 2s\left[\frac{s^4+162s^2+6561-s^4-18s^2-81}{(s^2+9)^2(s^2+81)^2}\right]$$

$$= \frac{9s}{2}\left[\frac{144s^2+6480}{(s^2+9)^2(s^2+81)^2}\right]$$

$$= \frac{9s}{2}\times 144\left[\frac{s^2+45}{(s^2-9)^2(s^2+81)^2}\right]$$

$$= 648s\,\frac{(s^2+45)}{(s^2+9)^2(s^2+81)^2} \text{ Ans.}$$

(vii) $t^3 e^{-3t}$ **(BE 2063)**

Solution. We have, $L[t^3 e^{-3t}]$

$$= \frac{3!}{(s+3)^4}$$

$$= \frac{6}{(s+3)^4} \text{ Ans.}$$

(viii) $t\cosh at$

Solution. We have, Now $L[\cosh at] = \frac{s}{s^2-a^2}$

$L[t\cosh at]$ By using the theorem of L. T.

$$= (-1)^1\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)$$

$$= -\left[\frac{(s^2-a^2).1-s.2s}{(s^2-a^2)^2}\right]$$

$$= -\frac{(s^2-a^2-2.s^2)}{(s^2-a^2)^2}$$

$$= \frac{(s^2+a^2)}{(s^2-a^2)^2} \text{ Ans.}$$

(ix) $t\sin h\,at$ **(BE 2062)**

Solution. We have, $L[t\sinh at]$

Now, $L[\sinh at] = \frac{a}{(s^2-a^2)}$

By using the theorem of L. T. we get,

$$L[t\sinh at] = (-1)'\frac{d}{ds}\frac{a}{(s^2-a^2)}$$

$$= \frac{2as}{(s^2-a^2)^2} \text{ Ans.}$$

(x) $t\sin at$

Solution. We have, $L[t\sin at]$

Now, $L(t\sin at] = \frac{a}{(s^2+a^2)}$

By using the theorem of L. T. we get,

$$L[t\sin at] = (-1)'\frac{d}{ds}\frac{a}{(s^2+a^2)} = \frac{2as}{(s^2+a^2)} \text{ Ans.}$$

(xi) $t\cos at$

Solution. We have, $L[t\cos at]$

Now, L [cos at] = $\frac{s}{(s^2+a^2)}$

By using the theorem of L . T. we get.

$$L[t\cos at] = (-1)^1 \frac{d}{ds}\frac{s}{(s^2+a^2)}$$

$$= -\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right]$$

$$= -\frac{(a^2-s^2)}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2} \text{ Ans.}$$

(xii) t^2 cos at **(BE 2060, 2061)**

Solution. We have, L [t^2 cos at]

Now, L [cos at] = $\frac{s}{(s^2-a^2)}$

By using the theorem of L. T. we get,

$$L[t^2\cos at] = (-1)^2 \frac{d^2}{ds^2}\frac{s}{(s^2-a^2)}$$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(a^2-s^2)}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2 - 2s - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= (s^2+a^2)\left[\frac{-2s(s^2+a^2) - (a^2-s^2)4s}{(s^2+a^2)^4}\right]$$

$$= \frac{-2s^3 - 2sa^2 - 4sa^2 + 4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6sa^2}{(s^2+a^2)^3} = \frac{2(s^3 - 3sa^2)}{(s^2+a^2)^3} \text{ Ans.}$$

(xiii) t^2 sin at

Solution. We have, L [t^2 sin at]

Now, L [sin at] = $\frac{a}{(s^2+a^2)}$

By using the theorem of L. T. we get.

$$L[t^2\sin at] = (-1)^2 \frac{d^2}{ds^2}\frac{a}{(s^2+a^2)}$$

$$= -a\frac{d}{ds}\left[\frac{2s}{(s^2+a^2)^2}\right]$$

$$= -2a\left[\frac{(s^2+a^2)^2.1 - s.2(s^2+a^2).2s}{(s^2-a^2)^4}\right]$$

$$= -\frac{2a(s^2+a^2)(s^2+a^2-4s^2)}{(s^2+a^2)^4}$$

$$= -\frac{2a(a^2-3s^2)}{(s^2+a^2)^3}$$

$$= \frac{6s^2a - 2a^3}{(s^2+a^2)^3}$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

(xiv) $t\,e^{-t}$ cosh t

Solution. We have, L [t e^{-t} cosh t]

Now, L [e^{-t} cosht] = $\frac{s+1}{(s+1)^2-1} = \frac{s+1}{s^2+2s}$

By using the theorem of L. T. we get,

$$L[te^{-t}\cosh t] = (-1)\frac{d}{ds}\left[\frac{s+1}{s^2+2s}\right]$$

$$= -\left[\frac{(s^2+2s).1 - (s+1)(2s+2)}{(s^2+2s)^2}\right]$$

$$= -\left[\frac{s^2+2s-2s^2-2s-2}{(s^2+2s)^2}\right]$$

$$= -\frac{(-s^2-2s-2)}{(s^2+2s)^2}$$

$$= \frac{(s^2+2s+2)}{(s^2+2s)^2} \text{ Ans.}$$

(xv) te^{-3t} cos 2t **(BE 2060)**

Solution. We hae, L [te^{-3t} cos 2t]

Now, L [e^{-3t} cos 2t] = $\frac{s+3}{(s+3)^2+4} = \frac{s+3}{s^2+6s+13}$

By using the theorem of L. T. we get,

$$L[t\,e^{-3t}\cos 2t] = (-1)^1\frac{d}{ds}\left[\frac{s+3}{s^2+6s+13}\right]$$

$$= -\left[\frac{(s^2+6s+13).1 - (s+3)(2s+6)}{(s^2+6s+13)^2}\right]$$

$$= -\frac{s^2+6s+13-2s^2-6s-6s-18}{(s^2+6s+13)^2}$$

$$= -\frac{(-s^2-6s-5)}{(s^2+6s+13)^2} = \frac{s^2+6s+5}{(s^2+6s+13)^2} \text{ Ans.}$$

(xvi) $t\,e^{-t}\sin t$ (BE 2062)

Solution. We have, $L\,[t\,e^{-t}\sin t]$

Now, $L\,[e^{-t}\sin t] = \dfrac{1}{(s+1)^2+1} = \dfrac{1}{s^2+2s+2}$

By using the theorem of L. T. we get,

$$L\,[t\,e^{-t}\sin t] = (-1)^1\frac{d}{ds}\left\{\frac{1}{(s^2+2s+2)}\right\}$$

$$= +\frac{2s+2}{(s^2+2s+2)^2}$$

$$= \frac{2(s+1)}{(s^2+2s+2)^2}\ \text{Ans.}$$

? **(xvii)** $t^2 e^{-t}\sin t$ (BE 2063)

Solution. We have $L\,[t^2 e^{-t}\sin t]$

Now, $L\,[e^{-t}\sin t] = \dfrac{1}{(s-1)^2+1} = \dfrac{1}{s^2-2s+2}$

By using the theorem of L. T. we get,

$$L\,[t^2 e^{-t}\sin t] = (-1)^2\frac{d}{ds}\left[\frac{1}{s^2-2s+2}\right] = \frac{d}{ds}\left[\frac{1}{(s^2-2s+2)}\right]$$

$$= \frac{(s^2-2s+2)^2.2-2(s-1).2(s^2-2s+2).2(s-1)}{(s^2-2s+2)^2}$$

$$= \frac{(s^2-2s+2)[2s^2-4s+4-8s^2+16s-18]}{(s^2-2s+2)^4}$$

$$= -\frac{6s^2+12s-4}{(s^2-2s+2)^3} = \frac{-2(3s^2-6s+2)}{(s^2-2s+2)^3}\ \text{Ans.}$$

3. Find the Laplace transforms of the following functions.

(i) $\dfrac{e^{-at}-e^{-bt}}{t}$ (BE 2060, 064)

Solution. We have, $L\left[\dfrac{e^{-at}-e^{-bt}}{t}\right]$

Now, $L\,[e^{-at}-e^{-bt}] = L\,(e^{-at}) - L\,(e^{-bt})$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

By using the theorem of L. T. we get,

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty\left[\frac{1}{s+a}-\frac{1}{s+b}\right]ds$$

$$= [\log(s+a)-\log(s+b)]_s^\infty$$

$$= -\log(s+a)+\log(s+b)$$

$$= \log\frac{(s+b)}{(s+a)}\ \text{Ans.}$$

(ii) $\dfrac{\cos 2t-\cos 3t}{t}$

Solution. We have, $L\left[\dfrac{\cos 2t-\cos 3t}{t}\right]$

Now, $L\,[\cos 2t-\cos 3t] = L\,[\cos 2t] - L\,[\cos 3t]$

$$= \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

By using the theorem of L. T. we get,

$$\therefore L\left[\frac{\cos 2t-\cos 3t}{t}\right] = \int_s^\infty\left[\frac{s}{(s^2+4)}-\frac{s}{(s^2+9)}\right]ds$$

$$= \frac{1}{2}\,[\log(s^2+4)-\log(s^2+9)]_s^\infty$$

$$= \frac{1}{2}\,[-\log(s^2+4)+\log(s^2+9)]$$

$$= \frac{1}{2}\log\frac{s^2+9}{s^2+4}\ \text{Ans.}$$

(iii) $\dfrac{1-\cos t}{t}$

Solution. We have, $L\left[\dfrac{1-\cos t}{t}\right]$

Now, $L\,[1-\cos t] - L\,[1] - L\,[\cos t] = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

By using the theorem of L. T. we get,

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+1}\right]ds$$

$$= \left[\log s-\frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\log s+\frac{1}{2}\log(s^2+1) = \log\frac{\sqrt{s^2+1}}{s}\ \text{Ans.}$$

(iv) $\dfrac{e^{at}-\cos 6t}{t}$

Solution. We have $L\left[\dfrac{e^{at}-\cos 6t}{t}\right]$

Now, $L\,[e^{at}-\cos 6t] = L\,[e^{at}] - L\,[\cos 6t]$

$$= \frac{1}{s-a} - \frac{s}{s^2+36}$$

By using the theorem of L. T. we get,

$$L\left[\frac{e^{at}-\cos 6t}{t}\right] = \int_s^\infty\left[\frac{1}{s-a}-\frac{s}{s^2+36}\right]ds = \left[\log(s-a)-\frac{1}{2}\log(s^2+36)\right]_s^\infty$$

$$= -\log(s-a)+\frac{1}{2}\log(s^2+36) = \log\frac{\sqrt{s^2+36}}{s-a}\ \text{Ans.}$$

(v) $\frac{\sin ht}{t}$

Solution. We have $L\left[\frac{\sin ht}{t}\right]$

Now, $L[\sin ht] = \frac{1}{s^2-1}$

By using the theorem of L. T. we get,

$\Rightarrow L[\sin ht] = \int_s^\infty \frac{1}{s^2-1} ds$

$$L\left[\frac{\sin ht}{t}\right] = \int_s^\infty \frac{1}{2}\left[\frac{1}{(s-1)} - \frac{1}{(s-1)}\right] ds$$

$$= \frac{1}{2}\left[\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty = -\frac{1}{2}\log\left(\frac{s-1}{s+1}\right) = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right) \text{ Ans.}$$

(vi) $\frac{e^{-t}\sin t}{t}$

Solution. We have, $L\left[\frac{e^{-t}\sin t}{t}\right]$

Now, $L[e^{-t}\sin t] = \frac{1}{(s+1)^2+1}$

By using the theorem of L. T. we get,

$$L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty \left[\frac{1}{(s+1)^2+1}\right] ds$$

$$= [\tan^{-1}(s+1)]_s^\infty \quad [\text{Note: } \cot^{-1}x + \tan^{-1}x = \frac{\pi}{2}]$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1) \text{ Ans.}$$

(vii) $\frac{\sin t \sin 5t}{t}$

Solution. We have, $L\left[\frac{\sin t \sin 5t}{t}\right]$

Now, $L[\sin t \sin 5t]\left[\frac{1}{2}(2\sin t \sin 5t)\right]$

$$\Rightarrow \frac{1}{2}[\cos(t-5t) - \cos(t+5t)]$$

$$= \frac{1}{2} L[\cos 4t] - \frac{1}{2} . L(\cos 6t]$$

$$= \frac{1}{2} . \frac{s}{s^2+16} - \frac{1}{2} . \frac{s}{s^2+36}$$

By using theorem of L. T. we get,

$$L\left[\frac{\sin t \sin 5t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{s}{s^2+16} - \frac{s}{s^2+36}\right] ds$$

$$\frac{1}{4}[\log(s^2+16) - \log(s^2+36)]_s^\infty$$

$$= \frac{1}{4}[-\log(s^2+16) + \log(s^2+36)]$$

$$= \frac{1}{4}\log\left(\frac{s^2+36}{s^2+16}\right) \text{ Ans.}$$

4. **Find the Laplace transform of the following integral.**

(i) $\int_0^t \frac{\sin t}{t} dt$

Solution. We have $L\left[\int_0^t \frac{\sin t}{t} dt\right]$

Now, $L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1} ds = [\tan^{-1}s]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

By using the theorem of L. T. of integral transform we get,

$$L\left[\int_0^t \frac{\sin t}{t} dt\right] = \frac{1}{s}\cot^{-1}(s)$$

(iv) $\int_0^t t^2 e^{-t} dt$

Solution.

$$L[t^2 e^{-t}] = \frac{2!}{(s+1)^2} = \frac{2}{(s+1)^3}$$

By using the theorem of integral L. T. we get,

$$L\int_0^t [t^2 e^{-t} dt] = \frac{2}{s(s+1)^3} \text{ Ans.}$$

7. **Evaluate the following integrals by using L. T.**

(i) $\int_0^\infty t\, e^{-2t} \cos t\, dt$

Solution. We have, $\int_0^\infty t\, e^{-2t}\cos t\, dt = \int_0^\infty e^{-2t}(t\cos t)\, dt$

$$= \int_0^\infty e^{-st}(t\cos t)\, dt \text{ when } s = 2$$

$$= L[t\cos t] \text{ when } s = 2$$

But we know that,

$L[\cos t] = \frac{s}{s^2+1}$ when $s = 2$

By using the theorem of L. T. we get,

$$\int_0^\infty t\, e^{-2t}\cos t\, dt = (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1).1 - s.2s}{(s^2+1)^2}\right]$$

$$= -\frac{(-s^2+1)}{(s^2+1)^2} = \frac{(s^2-1)}{(s^2+1)^2} \quad \text{when } s = 2$$

On putting s = 2, we get,

$$\int_0^\infty t\, e^{-2t} \cos t\, dt = \frac{(2^2-1)}{(2^2+1)^2} = \frac{3}{25} \quad \text{Ans.}$$

(ii) $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\, dt$

Solution. We have, $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\, dt = \int_0^\infty e^{-t} \frac{(1-e^{-2t})}{t}\, dt$

$$= \int_0^\infty e^{-st} \left[\frac{1-e^{-2t}}{t}\right] dt \quad \text{when } s = 1$$

$$= L\left[\frac{1-e^{-2t}}{t}\right] \quad \text{when } s = 1$$

Now, $L[1-e^{-2t}] = L[1] - 1[e^{-2t}]$

$$= \frac{1}{s} - \frac{1}{s+2} \quad \text{when } s = 1$$

By using the theorem of L. T.

$$L\left[\frac{1-e^{2t}}{t}\right] = \int_s^\infty \left[\frac{1}{s} - \frac{1}{s+2}\right] ds \quad \text{when } s = 1$$

$$= [\log s - \log s+2]_s^\infty \quad \text{when } s = 1$$

$$= -\log s + \log(s+2) \quad \text{when } s = 1$$

$$= \log\frac{(s+2)}{s} \quad \text{when } s = 1$$

$$\therefore \int_0^\infty \frac{e^{-t}-e^{-3t}}{t}.\, dt = \log\frac{(s+2)}{s} \quad \text{when } s = 1$$

$$= \log\frac{(1+2)}{1}$$

$$= \log 3 - \log 1 \quad [\log 1 = 0] = \log 3 \quad \text{Ans.}$$

(iii) $\int_0^\infty te^{-2t} \sin t\, dt$

Solution. $\int_0^\infty te^{-2t} \sin t\, dt = \int_0^\infty e^{-2t}\, t \sin t\, dt$

$$= \int_0^\infty e^{-st}\, t \sin t\, dt \quad \text{when } s = 2$$

$$= L[t \sin t] \quad \text{when } s = 2$$

Now, $L[\sin t] = \frac{1}{s^2+1}$ when s = 2

By using the theorem of L. T.

$$L[t \sin t] = (-1)^2 \frac{d}{ds}\left(\frac{1}{s^2+1}\right) \quad \text{when } s = 2$$

$$= \frac{2s}{(s^2+1)^2} \quad \text{when } s = 2$$

$$\therefore \int_0^\infty te^{-2t} \sin t\, dt = \frac{2s}{(s^2+1)^2} \quad \text{when } s = 2$$

$$= \frac{2.2}{(s^2+1)^2} = \frac{4}{25} \quad \text{Ans.}$$

(iv) $\int_0^\infty t^3 e^{-t} \sin t\, dt$

Solution. We have, $\int_0^\infty t^3 e^{-t}.\sin t\, dt = \int_0^\infty e^{-t}\, t^3 \sin t\, dt$

$$= \int_0^\infty e^{-st}\, t^3 \sin t\, dt \quad \text{when } s = 1$$

$$= L[t^3 \sin t] \quad \text{when } s = 1$$

Now, $L[\sin t] = \frac{1}{s^2+1}$ when s = 1

By using the theorem of L. T. we get,

$$L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3} \frac{1}{(s^2+1)^2} \quad \text{when } s = 1$$

$$= \frac{d^2}{ds^2} \frac{-2s}{(s^2+1)^2} \quad \text{when } s = 1$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)^2.2 - 2s.2(s^2+1).2s}{(s^2+1)^4}\right] \quad \text{when } s = 1$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)^2(2s^2+2-8s^2)}{(s^2+1)^4}\right] \quad \text{when } s = 1$$

$$= \frac{d}{ds} \frac{(2-6s^2)}{(s^2+1)^3}$$

$$= \frac{(s^2+1)^3.-12s - (2-6s^2).3(s^2+1)^2.2s}{(s^2+1)^6} \quad \text{when } s = 1$$

$$= \frac{(s^2+1)^2[-12s^3 - 12s - 12s + 36s^3]}{(s^2+1)^6} \quad \text{when } s = 1$$

$$= \frac{24s^3 - 24s}{(s^2+1)^4} \quad \text{when } s = 1$$

$$\therefore \int_0^\infty t^3 e^{-t} \sin t\, dt = \frac{24s^3 - 24s}{(s^2+1)^4} \quad \text{when } s = 1$$

$$= \frac{24(1)^3 - 24(1)}{(1^2+1)^4} = \frac{0}{24} \quad \text{Ans.}$$

Exercise - 12

Find the inverse Lapcues transform of :

1. $\dfrac{5}{s+3}$

Solution. We have, $L^{-1}\left[\dfrac{5}{s+3}\right] = 5\,L^{-1}\left[\dfrac{1}{s+3}\right] = 5e^{-3t}$

2. $\dfrac{1}{s(s+1)}$

Solution. We have, $L^{-1}\left[\dfrac{1}{s(s+1)}\right]$

By partial fraction

$$= L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = L^{1}\left[\frac{1}{s}\right] - L^{1}\left[\frac{1}{s+1}\right]$$

$$= 1 - e^{-t} \text{ Ans.}$$

3. $\dfrac{3s}{s^2+2s-8}$

Solution. We have, $L^{-1}\left[\dfrac{3s}{s^2+2s-8}\right]$

Now, $\dfrac{3s}{s^2+2s-8} = \dfrac{3s}{s^2-4s+2s-8} = \dfrac{3s}{s(s-4)+2(s-4)}$

$$= \frac{3s}{(s-4)(s+2)}$$

$\therefore$ By partial fraction

Let, $\dfrac{3s}{s^2+2s-8} = \dfrac{A}{s-4} + \dfrac{B}{s+2}$

$3s = A(s+2) + b(s-4)$

Put, $s = 4$, $12 = 6A$

$A = 2$

Put $s = -2$

$-6 = -6B$

$B = 1$

$\therefore \dfrac{3s}{(s-4)(s+2)} = \dfrac{2}{s-4} + \dfrac{1}{s+2}$

$\therefore L^{-1}\left[\dfrac{3s}{s^2+2s-8}\right] = L^{-1}\left[\dfrac{2}{s-4} + \dfrac{1}{s+2}\right]$

$$= 2\,L^{-1}\left[\frac{1}{s-4}\right] + L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 2e^{4t} + e^{-2t} \text{ Ans.}$$

4. $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$.

Solution. We have, $L^{-1}\left[\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}\right]$

Now, $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$

$$= \frac{2\left(s-\frac{5}{2}\right)}{4\left\{s^2+\left(\frac{5}{2}\right)^2\right\}} - \frac{4(s-18)}{s^2-9}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+\left(\frac{5}{2}\right)^2} - \frac{5}{2}\times\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right] - 4\cdot\frac{s}{s^2-9} + 18\,\frac{1}{s^2-9}$$

$\therefore L^{-1}\left[\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}\right]$

$$= \frac{1}{2}\,L^{-1}\left[\frac{s}{s^2+\left(\frac{5}{2}\right)^2}\right] - \frac{5}{4}\,L^{-1}\left[\frac{1}{s^2+\left(\frac{5}{2}\right)^2}\right] - 4L^{-1}\left[\frac{s}{s^2-3^2}\right]$$

$$+ 18\,L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{5}{4}\cdot\frac{2}{5}\sin\frac{5}{2}t - 4\cosh 3t + 18\cdot\frac{1}{3}\sinh 3t$$

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin\frac{5}{2}t - 4\cosh 3t + 6\sinh 3t \text{ Ans.}$$

5. $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$

Solution. We have, $L^{-1}\left[\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right]$

Now, $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)} = \dfrac{s^2-10s+13}{(s-7)(s-2)(s-3)}$

Let, $\dfrac{s^2-10s+13}{(s-7)(s-2)(s-3)} = \dfrac{A}{(s-7)} + \dfrac{B}{(s-2)} + \dfrac{C}{(s-3)}$

$\therefore s^2 - 10s + 13 = A(S-2)(s-3) + B(s-3)(s-7) + C(s-2)(s-7)$

put $s = 2 \Rightarrow -3 = 5B \Rightarrow B = -\dfrac{3}{5}$

put $s = 3 \Rightarrow -8 = -4C \Rightarrow C = 2$

put $s = 7 \Rightarrow -8 = 20A \Rightarrow A = -\frac{2}{5}$

$$\therefore \frac{A}{s-7} + \frac{B}{s-2} + \frac{c}{s-3}$$

$$\therefore L^{-1}\left[\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right] =$$

$$= -2\,L^{-1}\left[\frac{1}{s-3}\right] - \frac{3}{5}\,L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{5}\,L^{-1}\left[\frac{1}{s-7}\right]$$

$$= 2e^{3t} - \frac{3}{5}\,e^{2t} - \frac{2}{5}\,e^{7t} \text{ Ans.}$$

6. $\dfrac{s^2}{(s-1)^3}$ (BE 2058)

Solution. We have $L^{-1}\dfrac{s^2}{(s-1)^3}$

Now, Let $\dfrac{s^2}{(s-1)^3} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-1)^2} + \dfrac{6}{(s-1)^3}$

$\therefore s^2 = A(s-1)^2 + B(s-1) + C$

put $s = 1 \Rightarrow C = 1$

Equating the coefficient of s^2, $A = 1$

Equating the constant term

$A - B + C = 0 \Rightarrow B = 2.$

$$\therefore \frac{s^2}{(s-1)^3} = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$\therefore L^{-1}\left[\frac{s^2}{(s-1)^3}\right] = L^{-1}\left[\frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] + 2\,L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$= e^t + 2t\,e^t + \frac{t^2}{2!}\,e^t$$

$$= e^t\left[1 + 2t + \frac{t^2}{2}\right] \text{ Ans.}$$

7. $\dfrac{2s+3}{s^2+5s-6}$ (BE 2062)

Solution. We have, $L^{-1}\left[\dfrac{2s+3}{s^2+5s-6}\right]$

$$\text{Now, } \frac{2s+3}{s^2+5s-6} = \frac{2s+3}{s^2+6s-s-6} = \frac{2s+3}{s(s+6)-1(s+6)}$$

$$= \frac{2s+3}{(s+6)(s-1)}$$

Let, $\dfrac{2s+3}{s^2+5s-6} = \dfrac{A}{s-6} + \dfrac{B}{s-1}$

put $s = 1 \Rightarrow B = \frac{5}{7}$

$s = -6 \Rightarrow A = \frac{9}{7}$

$$\therefore \frac{2s+3}{(s+6)(s-1)} = \frac{9}{7}\,\frac{1}{s+6} + \frac{5}{7}\,\frac{1}{s-1}$$

$$\therefore L^{-1}\left[\frac{2s+3}{(s+6)(s-1)}\right] = L^{-1}\left[\frac{9}{7}\times\frac{1}{s+6} + \frac{5}{7}\times\frac{1}{s-1}\right]$$

$$= \frac{9}{7}\,L^{-1}\left[\frac{1}{s+6}\right] + \frac{5}{7}\,L^{-1}\left[\frac{1}{s-1}\right]$$

$$= \frac{9}{7}\,e^{-6t} + \frac{5}{7}\,e^t \text{ Ans.}$$

8. $\dfrac{s^2+6}{(s^2+1)(s^2+4)}$

Solution. We have, $L^{-1}\left[\dfrac{s^2+6}{(s^2+1)(s^2+4)}\right]$

Now, lat $\left[\dfrac{s^2+6}{(s^2+1)(s^2+4)}\right] = \dfrac{A}{s^2+1} + \dfrac{B}{s^2+4}$

$\therefore s^2 + 6 = A(s^2+4) + B(s^2+1)$

put $s^2 = -4 \Rightarrow 2 = -2B \Rightarrow B = -\frac{2}{3}$

$s^2 = -1 \Rightarrow 5 = 3A \Rightarrow A = \frac{5}{3}$

Taking inverse

$$L^{-1}\left[\frac{s^2+6}{(s^2+1)(s^2+4)}\right]$$

$$= L^{-1}\left[\frac{5}{3}\,\frac{1}{s^2+1} - \frac{2}{3}\,\frac{1}{s^2+4}\right]$$

$$= \frac{5}{3}\,L^{-1}\left[\frac{1}{s^2+1}\right] - \frac{2}{3}\,L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

$$= \frac{1}{3}\,[5\sin t - \sin 2t]$$

9. $\dfrac{1}{s^2(s^2+4)}$

Solution. We have, $L^{-1}\left[\dfrac{1}{s^2(s^2+4)}\right]$

Now, Let $\frac{1}{s^2(s^2+4)} = \frac{A}{s^2}+\frac{B}{s^2+4}$

$1 = A(s^2+4)+Bs^2$

put $s^2=-4, \Rightarrow B = \frac{-1}{4}$

put $s=0 \Rightarrow A = \frac{1}{4}$

$\therefore \frac{1}{s^2(s^2+4)} = \frac{1}{4} \cdot \frac{1}{s^2} - \frac{1}{4}\,\frac{1}{s^2+4}$

$\therefore L^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s^2-4}\right] = L^{-1}\left[\frac{A}{s^2}+\frac{B}{s^2+4}\right]$

$= \frac{1}{4} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{4} L^{-1}\left[\frac{1}{(s^2+2^2)}\right]$

$= \frac{1}{4}\, t - \frac{1}{4}\cdot\frac{1}{2}\sin 2t$

$= \frac{1}{4}\left[t-\frac{1}{2}\sin 2t\right]$ Ans.

10. $\frac{3s+1}{(s+1)^4}$

Solution. We have, $L^{-1}\left[\frac{3s+1}{(s+1)^4}\right]$

Now, $\frac{3s+1}{(s+1)^4} = \frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}+\frac{D}{(s+1)^4}$

$\therefore 3s+1 = A(s+1)^3 + B(s+1)^2 + C(s+1) + D$

put $s=-1, D=-2$

Equating the coefficient of s^3; $A=0$

Equating the coefficient of s^2; $3A+B=0 \Rightarrow B=0$

Equating the constant term $A+B+C+D=1$

$\Rightarrow C-2=1 \Rightarrow C=3$

$\therefore \frac{3s+1}{(s+1)^4} = \frac{3}{(s+1)^3} - \frac{2}{(s+1)^4}$

$\therefore L^{-1}\left[\frac{3s+1}{(s+1)^4}\right] = L^{-1}\left[\frac{3}{(s+1)^3} - \frac{2}{(s+1)^4}\right]$

$= 3\,L^{-1}\left[\frac{1}{(s+1)3}\right] - 2\,L^{-1}\left[\frac{1}{(s+1)^4}\right]$

$= \frac{3}{2!}\, e^{-t} t^2 - 2\,\frac{e^{-t}t^3}{3!}$

$= \frac{3}{2}\, e^{t} t^2 - \frac{1}{3}\, e^{-t} t^3$ Ans.

11. $\frac{s+1}{s^2(s^2+1)}$

Solution. We have, $L^{-1}\left[\frac{s+1}{s^2(s^2+1)}\right]$ (BE 2061)

Now, $\frac{s+1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$

$\therefore s+1 = (As+B)(s^2+1) + (Cs+D)s^2$

put $s=0 \Rightarrow B=1$

Equating the coefficient of s; $A=1$

Equating the coefficient s^2; $B+D=0 \Rightarrow D=-1$

Equating the coefficient s^3 $A+C=0 \Rightarrow C=-1$

$\therefore \frac{s+1}{s^2(s^2+1)} = \frac{s+1}{s^2} + \frac{-s-1}{s^2+1}$

$\therefore L^{-1}\,\frac{s+1}{s^2(s^2+1)} = L^{-1}\left[\frac{As+B}{s^2}\right] + L^{-1}\left[\frac{CS+D}{s^2+1}\right]$

$= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2}\right] - L^{-1}\left[\frac{s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s^2+1}\right]$

$= 1 + t^2 - \cos t - \sin t$

12. $\frac{2s-3}{s^2+4s+13}$

Solution. We have, $L^{-1}\left[\frac{2s-3}{s^2+4s+13}\right]$

$= L^{-1}\left[\frac{2s-3}{(s+2)^2+9}\right]$

$= L^{-1}\left[\frac{2(s+2)-4-3}{(s+2)^2+3^2}\right]$

$= 2\,L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - 7\,L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$

$= 2\,e^{-2t}\cos 3t - \frac{7}{3}\,e^{-2t}\sin 3t$

$= e^{-2t}\left(2\cos 3t - \frac{7}{3}\sin 3t\right)$ Ans.

13. $\frac{s}{(s+3)^2+4}$

Solution. We have, $L^{-1}\,\frac{s}{(s+3)^2+4}$

$= L^{-1}\left[\frac{s+3-3}{(s+3)^2+4}\right]$

$$= L^{-1}\left[\frac{(s+3)}{(s+3)^2+4}\right] - 3\,L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$

$$= e^{-3t}\cos 2t - \frac{3}{2}\,e^{-3t}\sin 2t$$

$$= e^{-3t}\left(\cos 2t - \frac{3}{2}\sin 2t\right) \text{ Ans.}$$

14. $\dfrac{s+2}{s^2-4s+13}$

Solution. We have, $L^{-1}\left[\dfrac{s+2}{s^2-4s+13}\right]$

$$= L^{-1}\left[\frac{s-2+2+2}{s^2-4s+13}\right]$$

$$= L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]$$

$$= e^{2t}\cos 3t + \frac{4}{3}\,e^{2t}\sin 3t$$

$$= e^{2t}\left(\cos 3t + \frac{4}{3}\sin 3t\right) \text{ Ans.}$$

15. $\dfrac{s}{(s+1)^2(s^2+1)}$

Solution. We have $L^{-1}\left[\dfrac{s}{(s+1)^2(s^2+1)}\right]$

Now, $\dfrac{s}{(s+1)^2(s^2+1)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2} + \dfrac{Cs+D}{(s^2+1)}$

$\therefore s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$

put $s = -1$ then $B = -\dfrac{1}{2}$

put $s = 0$ then $A + B + D = 0$ ....(i)

Equating the coefficient s^3; $A + C = 0 \Rightarrow A = -C$ ....(ii)

Equating the coefficient s^2; $A + B + 2C + D = 0$ ....(iii)

From (i), (ii) and (iii)

$2c = 0 \Rightarrow C = 0$

$\therefore A = 0$

$D = \dfrac{1}{2}$

$$\therefore \frac{s}{(s+1)^2(s^2+1)} = -\frac{1}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s^2+1)}$$

$$\therefore L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] = L^{-1} - \frac{1}{2}\left[\frac{1}{(s+1)^2} \mp \frac{1}{2}\times\frac{1}{(s^2+1)}\right]$$

$$= -\frac{1}{2}\left[L^{-1}\left[\frac{1}{(s+1)^2}\right] \mp \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$= -\frac{1}{2}e^{-t}.t + \frac{1}{2}\sin t$$

$$= \frac{1}{2}(\sin t - te^{-t}) \text{ Ans.}$$

16. $\dfrac{2s+1}{(s-1)^2(s+2)^2}$

Solution. We have, $L^{-1}\left[\dfrac{2s+1}{(s-1)^2(s+2)^2}\right]$

Now, $\dfrac{2s+1}{(s-1)^2(s+2)^2} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s+2)} + \dfrac{D}{(s+2)^2}$

$2s + 1 = A(s-1)(s+2)^2 + B(s+2)^2 + C(s-1)^2(s+2) + D(s-1)^2$

put $s = 1 \Rightarrow B = \dfrac{3}{9} = \dfrac{1}{3}$ $\therefore$ Put $s = -2 \Rightarrow D = \dfrac{-3}{9} = -\dfrac{1}{3}$

Equating the coefficient of s^3; $A + C = 0 \Rightarrow A = -C$

Equating the constant term; $-4A + 4B - 2C + D = 1$

$$\therefore -4A + \frac{4}{3} - 2A - \frac{1}{3} = 1$$

$$\Rightarrow -6A + \frac{4}{3} = 1 + \frac{1}{3}$$

$$= -6A + \frac{4}{3} = \frac{4}{3}$$

$$= -6A = 0 \Rightarrow A = 0 \;\&\; C = 0$$

$$\therefore \frac{2s+1}{(s-1)^2(s^2+1)} = \frac{1}{3}\times\frac{1}{(s-1)^2} - \frac{1}{3}\times\frac{1}{(s+2)^2}$$

$$\therefore L^{-1}\frac{2s+1}{(s-1)^2(s+1)} = L^{-1}\left[\frac{1}{3}\times\frac{1}{(s-1)^2} - \frac{1}{3}\times\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{3}L^{-1}\frac{1}{(s-1)^2} - \frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right].$$

$$= \frac{1}{3}te^{t} - \frac{1}{3}te^{-2t}$$

$$= \frac{1}{3}t(e^{t} - e^{-2t}) \text{ Ans.}$$

Find f (t) if F (s) = L [f (t)] is as follows.

17. $\dfrac{s}{(s^2+1)(s^2+4)}$

Solution. We have, $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+4)}\right]$

Now, $\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{s^2+4}$

$s = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

Equating the constant term, $4B + D = 0$ .... (i)

Equating the coefficient of s; $4A + C = 1$ .... (ii)

Equating the coefficient of s^2; $B + D = 0$ .... (iii)

Equating the coefficient of s^3; $A + C = 0$ .... (iv)

From (i) & (iii)

$4B - B = 0 \Rightarrow B = 0$ and $D = 0$

From (ii) and (iv)

$4A - A = 1 \Rightarrow A = \frac{1}{3}$ and $C = -\frac{1}{3}$

$$\therefore \frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3} \times \frac{s}{s^2+1} - \frac{1}{3} \times \frac{s}{s^2+4}$$

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{1}{3} \times \frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}\right]$$

$$= \frac{1}{3} L^{-1}\left[\frac{s}{s^2+1^2}\right] - \frac{1}{3} L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{3} \text{cost} - \frac{1}{3} \cos 2t$$

$$= \frac{1}{3} (\cos t - \cos 2t) \text{ Ans.}$$

18. $\frac{s}{s^4+s^2+1}$

Solution. We have, $L^{-1}\left[\frac{s}{s^4+s^2+1}\right]$

Now, $\frac{s}{s^4+s^2+1} = \frac{s}{(s^2)^2+1^2+s^2} = \frac{s}{(s^2+1)^2-2s^2+s^2}$

$$= \frac{s}{(s^2+1)^2-(s)^2}$$

$$= \frac{s}{(s^2+s+1)(s^2-s+1)}$$

Let, $\frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{(s^2+s+1)} + \frac{Cs+D}{(s^2-s+1)}$

or, $s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$

Equating the constant term; $B + D = 0$ .... (i)

Equating the coefficient of s; $A - B + C + D = 1$ .... (ii)

Equating the coefficient of s^2; $-A + B + C + D = 0$ .... (iii)

Equating the coefficient of s^3; $A + C = 0$ .... (iv)

From (i) & (ii) $-A + C = 0$ .... (v)

From (v) and (iv) $2C = 0 \Rightarrow C = 0$ & $A = 0$

From (i) and (ii)

$0 - B + 0 - B = 1 \Rightarrow B = -\frac{1}{2}$

or, $D = \frac{1}{2}$

$$\therefore \frac{s}{(s^2+s+1)(s^2-s+1)} = -\frac{1}{2} \times \frac{1}{s^2+s+1} + \frac{1}{2} \times \frac{1}{s^2-s+1}$$

$$= -\frac{1}{2} \frac{1}{s^2+2s.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\frac{1}{4}+1} + \frac{1}{2} \frac{1}{s^2-2s.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\frac{1}{4}+1}$$

$$= -\frac{1}{2} \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2} \cdot \frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\therefore L^{-1}\left[\frac{s}{s^4+s^2+1}\right] = L^{-1}\left[-\frac{1}{2}\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{3}} e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t - \frac{1}{2} \cdot \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$$

$$= \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left[e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right]$$

$$= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left[\frac{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}{2}\right]$$

$$= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \sinh \frac{t}{2} \text{ Ans.}$$

19. $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$

Solution. We have, $L^{-1}\left[\frac{s^2+s}{(s^2+1)(s^2+2s+2)}\right]$

Now, $\frac{s^2+s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$

$s^2 + s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$

Equating the constant term; $2B + D = 0 \Rightarrow D = -2B$ ...... (i)

Equating the coefficient of s; $2A + 2B + C = 1$ .... (ii)
Equating the coefficient of s^2: $2A + B + D = 1$ .... (iii)
Equating the coefficient of s^3; $A + C = 0 \Rightarrow A = -C$ ...... (iv)

From (ii) & (iv)
$2A + 3B - A = 1$
$A + 2B = 1$ ...... (v)

From (i) & (iii)
$2A + B - 2B = 1$
$2A - B = 1$ ...... (vi)

From (v) and (vi)
$2A - B = 1$
$2A + 4B = 2$
$-\quad -\quad -$
$-5B = -1$

$\therefore B = \frac{1}{5}$

$\Rightarrow A = \frac{3}{5}, C = -\frac{3}{5}, D = -\frac{2}{5}$

$$\therefore \frac{s^2+s}{(s^2+1)(s^2+2s+2)} = L^{-1}\left[\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}\right]$$

$$= \frac{3}{5}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right] - \frac{3}{5}L^{-1}\left[\frac{s+1-1}{(s+1)^2+1^2}\right] + \frac{2}{5}L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= \frac{3}{5}\cos t + \frac{1}{5}\sin t - \frac{3}{5}e^{-t}\cos t + \frac{3}{5}e^{-t}\sin t + \frac{2}{5}e^{-t}\sin t$$

$$= \frac{3}{5}(1-e^{-t})\cos t + \frac{1}{5}\sin t + \frac{1}{5}e^{-t}\sin t$$

$$= \frac{3}{5}(1-e^{-t})\cos t + \frac{1}{5}(1-e^{-t})\sin t \text{ Ans.}$$

20. $\frac{1}{s^3-a^3}$

Solution. We have, $L^{-1}\left[\frac{1}{s^3-a^3}\right]$

Now, $\frac{1}{s^3-a^3} = \frac{1}{(s-a)(s^2+as+a^2)}$

Let, $\frac{1}{(s-a)(s^2+as+a^2)} = \frac{A}{(s-a)} + \frac{Bs+C}{s^2+as+a^2}$

$\therefore 1 = (s^2+as+a^2)A + (Bs+C)(s-a)$

put $s = a \Rightarrow A = \frac{1}{3}a^2$

Equating the constant term;
$a^2A - ac = 1$

$a^2 \cdot \frac{1}{3a^2} - ac = 1$

$-ac = 1 - \frac{1}{3} = \frac{2}{3}$

$\Rightarrow C = -\frac{2}{3a}$

Equating the coefficient of s^2; $A + B = 0 \Rightarrow A = -B = \frac{1}{-3a^2}$

$$\therefore \frac{1}{(s-a)(s^2+as+a^2)} = \frac{1}{3a^2}\cdot\frac{1}{s-a} - \frac{1}{3a^2}\frac{s}{s^2+as-a^2} - \frac{2}{3a}\cdot\frac{1}{s^2+as-a^2}$$

$$\therefore L^{-1}\left[\frac{1}{(s^3-a^3)}\right] = L^{-1}\left[\frac{1}{3a^2}\cdot\frac{1}{s-a} - \frac{1}{3a^2}\frac{s}{s^2+as+a^2}\right]$$

$$= \frac{1}{3a^2}L^{-1}\left[\frac{1}{s-a}\right] - \frac{1}{3a^2}L^{-1}\left[\frac{s+\frac{a}{2}}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right] + \left(\frac{1}{6a}-\frac{2}{3a}\right)L^{-1}\left[\frac{1}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right]$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-\frac{a}{2}t}\cos\frac{\sqrt{3}}{2}at - \frac{3}{6a}\cdot\frac{2}{\sqrt{3}a}e^{-\frac{a}{2}t}\sin\frac{\sqrt{3}a}{2}t$$

$$= \frac{1}{3a^2}\left[e^{at} - e^{-\frac{a}{2}t}\left\{\cos\frac{\sqrt{3}}{2}at + \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right\}\right] \text{ Ans.}$$

21. $\frac{s}{s^4+4a^4}$

Solution. We have, $L^{-1}\left[\frac{s}{s^4+4a^4}\right]$

Now, $\frac{s}{s^4+4a^4} = \frac{s}{(s^2)^2+(2a)^2} = \frac{s}{(s^2+2a^2)^2-4a^2s^2}$

$$= \frac{s}{(s^2+2a^2)^2-(2as)^2} = \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

$\therefore$ Let, $\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$

$$= \frac{As+B}{(s^2+2as+2a^2)} + \frac{Cs+D}{(s^2-2as+2a^2)}$$

$s = (As + B)(s^2 - 2as + 2a^2) + (Cs + D)(s^2 + 2as + 2a^2)$

Equating the constant term.

$2a^2B + 2a^2D = 0 \Rightarrow B = -D$ ....(i)

$2a^2A - 2aB + 2a^2C + 2aD = 1$ .... (ii)

$-2aA + B + 2aC + D = 0$ .... (iii)

$A + C = 0 \Rightarrow A = -C$ .... (iv)

From (i) & (iii)

$2aA + B + 2aC - B = 0$

$-A + C = 0$ .... (v)

From (iv) & (v) $A = C = 0$

From (i), (ii) & (iv)

$2a^2A - 2aB - 2a^2A - 2aB = 1$

$-4aB = 1$

$\therefore B = \frac{1}{-4a}$ & $D = \frac{1}{4a}$

$$\therefore \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)} = \frac{1}{4a}\cdot\frac{1}{(s^2+2as+2a^2)} + \frac{1}{4a}\cdot\frac{1}{(s^2-2as+2a^2)}$$

$$\therefore L^{-1}\left[\frac{s}{s^4+4a^4}\right] = L^{-1}\left[\frac{1}{4a}\cdot\frac{1}{(s^2+2as+2a^2)} + \frac{1}{4a}\cdot\frac{1}{(s^2-2as+2a^2)}\right]$$

$$= -\frac{1}{4a}L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right] + \frac{1}{4a}L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]$$

$$= -\frac{1}{4a}\cdot\frac{1}{a}e^{-at}\sin at + \frac{1}{4a}\cdot\frac{1}{a}e^{at}\sin at$$

$$= \frac{1}{4a^2}\sin at\,[e^{at} - e^{-at}] \Rightarrow \frac{2}{4a^2}\sin at\left[\frac{e^{at}-e^{-at}}{2}\right]$$

$$= \frac{1}{2a^2}\sin at \sinh at. \text{ Ans.}$$

22. $\frac{5s+3}{(s-1)(s^2+2s+5)}$

Solution. We have, $L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]$

Let, $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}$

$\therefore 5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$

put $s = 1 \Rightarrow A = \frac{8}{8} = 1$

Equating the constant term, $5A - C = 3$

$-C = 3 - 5 \Rightarrow C = 2$

Equating the coefficient of s^2, $A + B = 0 \Rightarrow B = -A = -1$

$$\therefore L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{A}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s}{s^2+2s+5}\right] + 2L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$= e^t - L^{-1}\left[\frac{s}{(s+1)^2+2^2}\right] + 2L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] + 2L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + 2\cdot\frac{1}{2}e^{-t}\sin 2t$$

$$= e^t - e^{-t}\cos t + \frac{3}{2}e^{-t}\sin 2t. \text{ Ans.}$$

23. $\frac{a(s^2-2a^2)}{s^4+4a^4}$

Solution. We have, $L^{-1}\left[\frac{a(s^2-2a^2)}{s^4+4a^4}\right]$

Now, $\frac{a(s^2-2a^2)}{s^4+4a^4} = \frac{a(s^2-2a^2)}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$

Let, $\frac{as^2-2a^3}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$

$$= \frac{As+B}{(s^2-2as+2a^2)} + \frac{Cs+D}{(s^2+2as+2a^2)}$$

$\therefore as^2 - 2a^3 = (As+B)(s^2+2as+2a^2) + (Cs+D)(s^2-2as+2a^2)$

Equating the coefficient of s^3; $A + C = 0 \Rightarrow A = -C$ ....(i)

Equating the coefficient of s^2; $2aA + B - 2aC + D = a$ .... (ii)

Equating the coefficient of s; $2a^2A + 2aB + 2a^2C - 2aD = 0$ .... (iii)

Equating the coustant term, $2a^2B + 2a^2D = -2a^3$

$B + D = -a$

$\Rightarrow B = -a - D$ .... (iv)

From (i), (iii) & (iv)

$2a^2A + 2a(-a - d) - 2a^2A - 2aD = 0$

$-2a^2 - 2aD - 2aD = 0 \Rightarrow -4aD = 2a^2$

or, $D = \frac{2a^2}{-4a} = \frac{a}{-2}$

$\Rightarrow B = -a + \frac{a}{2} = -\frac{a}{2}$

From (ii)

$2aA + B + 2aA - a - B = a$

$4aA = 2a$

$A = \frac{1}{2} \Rightarrow C = -\frac{1}{2}$

$$\therefore \frac{a(s^2-2a^2)}{(s^2-2as+2a^2)(s^2+2as+2a^2)}$$

$$= \frac{1}{2}\,\frac{s-a}{s^2-2as+2a^2} - \frac{1}{2}\,\frac{s+a}{s^2+2as+2a^2}$$

$$\therefore L^{-1}\left[\frac{a(s^2-2a^2)}{s^4+4a^4}\right] = L^{-1}\left[\frac{1}{2}\,\frac{s-a}{s^2-2as+2a^2} - \frac{1}{2}\,\frac{s+a}{s^2+2as+2a^2}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{s-a}{(s-a)^2+a^2}\right] - \frac{1}{2} L^{-1}\left[\frac{s+a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2} e^{at}\cos at - \frac{1}{2} e^{-at}\cos at$$

$$= \cos at\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$= \cos at \sinh at \text{ Ans.}$$

24. $\dfrac{1}{s(s+1)(s+2)}$

Solution. We have, $L^{-1}\left[\dfrac{1}{s(s+1)(s+2)}\right]$

Now, Let $\dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$

$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

put $s = 0 \Rightarrow A = \dfrac{1}{2}$

put $s = -a \Rightarrow B = -1$

put $s = -2 \Rightarrow C = \dfrac{1}{2}$

$$\therefore \frac{1}{s(s+1)(s+2)} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}$$

$$\therefore L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \text{ Ans.}$$

25. $\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$

Solution. We have, $L^{-1}\left[\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right]$

Now, Let, $\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$\therefore 2s^2 - 6s + 5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-$

put $s = 1 \Rightarrow A = \dfrac{1}{2}$

put $s = 2 \Rightarrow B = -1$

put $s = 3 \Rightarrow C = \dfrac{5}{2}$

$$\therefore \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\cdot\frac{1}{s-3}$$

$$\therefore \left[\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right]$$

$$= L^{-1}\left[\frac{1}{2}\times\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\times\frac{1}{s-3}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2} L^{-1}\left[\frac{1}{s-3}\right]$$

$$= \frac{1}{2} e^{t} - e^{2t} + \frac{5}{2} e^{3t} \text{ Ans.}$$

26. $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

Solution. We have, $L^{-1}\left[\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]$

Now, $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

Let, $s^2 + 2s = x$

$\therefore \dfrac{x+3}{(x+5)(x+2)}$

Let, $\dfrac{x+3}{(x+5)(x+2)} = \dfrac{A}{(x+5)} + \dfrac{B}{(x+2)}$

$x + 3 = A(x+2) + B(x+5)$

put $x = -5 \Rightarrow A = \dfrac{2}{3}$

put $x = -2 \Rightarrow B = \dfrac{1}{3}$

$$\therefore \frac{x+3}{(x+5)(x+2)} = \frac{2}{3}\cdot\frac{1}{x+5} + \frac{1}{3}\cdot\frac{1}{x+2}$$

Replacing the value of x.

$$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{2}{3}\cdot\frac{1}{(s^2+2s+5)} + \frac{1}{3}\cdot\frac{1}{(s^2+2s+2)}.$$

$$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = L^{-1}\left[\frac{2}{3}\cdot\frac{1}{(s^2+2s+5)} + \frac{1}{3}\cdot\frac{1}{(s^2+2s+2)}\right]$$

$$= \frac{2}{3} L^{-1} \frac{1}{(s^2+2s+1^2+4)} + \frac{1}{3} L^{-1} \frac{1}{(s^2+2s+1^2+1)}$$

$$= \frac{2}{3} L^{-1} \frac{1}{(s^2+1)^2+2^2} + \frac{1}{3} L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{2}{3} \cdot \frac{1}{2} e^{-t} \sin 2t + \frac{1}{3} e^{-t} \sin t$$

$$= \frac{1}{3} e^{-t} (\sin 2t + \sin t) \text{ Ans.}$$

27. $\dfrac{s+4}{s(s-1)(s^2+4)}$

Solution. We have, $L^{-1}\left[\dfrac{s+4}{s(s-1)(s^2+4)}\right]$

Now, let $\dfrac{s+4}{s(s-1)(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{Cs+D}{s^2+4}$

or, $s+4 = A(s-1)(s^2+4) + B(s^2+4)s + (Cs+D)(s-1)s$

Put $s = 1 \Rightarrow B = 1$

Put $s = 0 \Rightarrow A = -1$

Equating the coefficient of s^3; $A + B + C = 0 \Rightarrow C = 0$

Equating the coefficient of s^2; $-A + D = 0 \Rightarrow D = A = -1$

$$\therefore \frac{s+4}{s(s-1)(s^2+4)} = -\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}$$

$$L^{-1}\left[\frac{s+4}{s(s-1)(s^2-4)}\right] = L^{-1}\left[-\frac{1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}\right]$$

$$= -L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= -1 + e^t - \frac{1}{2} \sin 2t \text{ Ans.}$$

28. $\dfrac{s-1}{s^2-6s+25}$

Solution. We have, $L^{-1}\left[\dfrac{s-1}{s^2-6s+25}\right]$

$$= L^{-1}\left[\frac{s-1}{s^2-2.s.3+3^2+16}\right] = L^{-1}\left[\frac{s-1}{(s-3)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{s-3+3-1}{(s-3)^2+4^2}\right] = L^{-1}\left[\frac{s-3}{(s-3)^2+4^2}\right] + 2L^{-1}\left[\frac{1}{(s-3)^2+4^2}\right]$$

$$= e^{3t}\cos 4t + 2 \cdot \frac{1}{4} e^{3t} \sin 4t$$

$$= e^{3t}\cos 4t + \frac{1}{2} e^{3t} \sin 4t \text{ Ans.}$$

29. $\dfrac{2a^2}{s^4-a^4}$

Solution. We have, $L^{-1}\left[\dfrac{2a^2}{s^4-a^4}\right]$

Now, $\dfrac{2a^2}{s^4-a^4} = \dfrac{2a^2}{(s^2+a^2)(s+a)(s-a)}$

Let, $\dfrac{2a^2}{(s^2+a^2)(s+a)(s-a)} = \dfrac{As+B}{s^2+a^2} + \dfrac{C}{s+a} + \dfrac{D}{(s-a)}$

$2a^2 = (As+B)(s+a)(s-a) + C(s^2+a^2)(s-a)$
$+ D(s^2+a^2)(s+a)$

put $s = a \Rightarrow D = \dfrac{1}{2a}$

put $s = -a \Rightarrow C = -\dfrac{1}{2a}$

Equating the coefficient of s^3; $A + C + D = 0 \Rightarrow A = 0$

Equating the constant term; $-Ba^2 - Ca^3 + Da^3 = 2a^2$

$$-B - Ca + Da = 2$$

$$-B + \frac{1}{2a} . a + \frac{1}{2a} . a = 2$$

$$-B = 2 - \frac{1}{2} - \frac{1}{2}$$

$$\therefore B = -1$$

$$\therefore \frac{2a^2}{(s^2+a^2)(s+a)(s-a)} = -\frac{1}{s^2+a^2} - \frac{1}{2a} \cdot \frac{1}{s+a} + \frac{1}{2a} \cdot \frac{1}{s-a}$$

$$L^{-1}\left[\frac{2a^2}{(s^2+a^2)(s+a)(s-a)}\right] = L^{-1}\left[-\frac{1}{s^2+a^2} - \frac{1}{2a} \cdot \frac{1}{s+a} + \frac{1}{2a} \cdot \frac{1}{s-a}\right]$$

$$= -L^{-1}\left[\frac{1}{s^2+a^2}\right] - \frac{1}{2a} L^{-1}\left[\frac{1}{s+a}\right] + \frac{1}{2a} L^{-1}\left[\frac{1}{s-a}\right]$$

$$= -\frac{1}{a} \sin at - \frac{1}{2a} e^{-at} + \frac{1}{2a} e^{at}$$

$$= \frac{1}{2a} e^{at} - \frac{1}{2a} e^{-at} - \frac{1}{a} \sin at$$

$$= \frac{1}{a}\left[\frac{e^{at}-e^{-at}}{2} - \sin at\right]$$

$$= \frac{1}{a} [\sinh at - \sin at] \text{ Ans.}$$

30. $\dfrac{1}{(s-2)(s^2+1)}$

Solution. We have, $L^{-1}\left[\dfrac{1}{(s-2)(s^2+1)}\right]$

Let $\frac{1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$

$1 = A(s^2+1) + (Bs+C)(s-2)$

Put $s = 2 \Rightarrow A = \frac{1}{5}$

Equating the constant term,

$A - 2s = 1$

$-2s = 1 - A = 1 - \frac{1}{5} = \frac{5-1}{5}$

$\therefore -2s = \frac{4}{5}$

$s = -\frac{4}{2.s} = \frac{-2}{5}$

Equating the coefficient of s^2,

$A + B = 0$

$\Rightarrow B = -\frac{1}{5}$

$\therefore \frac{1}{(s-2)(s^2+1)} = \frac{1}{5}\cdot\frac{1}{s-2} - \frac{1}{5}\cdot\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1}$

$L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right] = L^{-1}\left[\frac{1}{5}\cdot\frac{1}{s-2} - \frac{1}{5}\cdot\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1}\right]$

$= \frac{1}{5}L^{-1}\left[\frac{1}{s-2}\right] - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+1^2}\right] - \frac{2}{5}L^{-1}\left[\frac{1}{s^2+1^2}\right]$

$= \frac{1}{5}e^{2t} - \frac{1}{5}\cos t - \frac{2}{5}\sin t$. Ans.

(BE 2058)

31. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Solution. We have $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Now, $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $s^2 = x$

$\frac{x}{(x+a^2)(x+b^2)}$

Let, $\frac{x}{(x+a^2)(x+b^2)} = \frac{A}{(x+a^2)} + \frac{B}{(x+b^2)}$

$x = A(x+b^2) + B(x+a^2)$

put $x = -a^2 \Rightarrow A = \frac{a^2}{a^2-b^2}$

put $x = -b^2 \Rightarrow B = \frac{b^2}{a^2-b^2}$

$\therefore \frac{x}{(x+a^2)(x+b^2)} = \frac{a^2}{a^2-b^2}\cdot\frac{1}{(x+a^2)} + \frac{b^2}{a^2-b^2}\cdot\frac{1}{(x+b^2)}$

Replacing the value of x,

$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\cdot\left[\frac{a^2}{s^2+a^2} + \frac{b^2}{s^2+b^2}\right]$

$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{1}{a^2-b^2}\cdot\left[\frac{a^2}{s^2+a^2} + \frac{b^2}{s^2+b^2}\right]\right]$

$= \frac{1}{a^2-b^2}\left\{a^2L^{-1}\left[\frac{1}{s^2+a^2}\right] + b^2L^{-1}\left[\frac{1}{s^2+b^2}\right]\right\}$

$= \frac{1}{a^2-b^2}\left[a^2\times\frac{1}{a}\sin at + b^2\times\frac{1}{b}\sin bt\right]$

$= \frac{1}{(a^2-b^2)}(a\sin at + b\sin bt)$ Ans.

(BE 2062)

32. $\frac{2s^2-1}{(s^2+1)(s^2+4)}$

Solution. We have, $L^{-1}\left[\frac{2s^2-1}{(s^2+1)(s^2+4)}\right]$

Now, $\frac{2s^2-1}{(s^2+1)(s^2+4)}$

Let, $s^2 = x$

$\frac{2x}{(x+1)(x+4)}$

Let, $\frac{2x-1}{(x+1)(x+4)} = \frac{4}{(x+1)} + \frac{B}{(x+4)}$

$\therefore 2x - 1 = A(x+4) + B(x+1)$

Put $x = -4 \Rightarrow B = 3$

Put $x = -1 \Rightarrow A = -1$

$\therefore \frac{2x-1}{(x+1)(x+4)} = -\frac{1}{x+1} + \frac{3}{x+4}$

Replacing the value of x

$\frac{2s^2-1}{(s^2+1)(s^2+4)} = -\frac{1}{s^2+1} + 3\frac{1}{s^2+4}$

$L^{-1}\left[\frac{2s^2-1}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[-\frac{1}{s^2+1} + 3\frac{1}{s^2+4}\right]$

$= -L^{-1}\left[\frac{1}{s^2+1}\right] 3 + L^{-1}\left[\frac{1}{s^2+2^2}\right]$

$= -\sin t + 3\cdot\frac{1}{2}\sin 2t = \frac{3}{2}\sin 2t - \sin t$ Ans.

Find the inverse transform of

(1) $\log\left(\frac{s+1}{s}\right)$

Solution: Let, $L^{-1}\left[\log\left(\frac{s+1}{s}\right)\right] = f(t)$ (say)

or, $L[f(t)] = \log\frac{s+1}{s}$

By using the theorem of L. T., we have,

$$L[t\,f(t)] = -\frac{d}{ds}\log\left(\frac{s+1}{s}\right)$$

$$= -\frac{d}{ds}[\log(s+1) - \log s]$$

$$= -\left(\frac{1}{s+1} - \frac{1}{s}\right)$$

$$L[t\,f(t)] = \frac{1}{s} - \frac{1}{s+1}$$

$$\therefore t\,f(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 1 - e^{-t}$$

$$f(t) = \frac{1}{t}[1 - e^{-t}]$$

Here, $L^{-1}\left[\log\left(\frac{s+1}{s}\right)\right] = \frac{1}{t}(1 - e^{-t})$ Ans.

(2) $\log\frac{s(s+1)}{s^2+4}$ **(BE 2067)**

Solution: Let $L^{-1}\log\left[\frac{s(s+1)}{s^2+4}\right] = f(t)$ (say)

or, $L[f(t)] = \log\left[\frac{s(s+1)}{s^2+4}\right]$

By using the theorem of L. T. we have,

$$L[t\,f(t)] = -\frac{d}{ds}\log\frac{s(s+1)}{s^2+4}$$

$$= -\frac{d}{ds}[\log s + \log(s+1) - \log(s^2+4)]$$

$$= -\left(\frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+4}\right)$$

$$\therefore t\,f(t) = L^{-1}\left[\frac{2s}{s^2+4} - \frac{1}{s} - \frac{1}{s+1}\right]$$

$$= 2L^{-1}\left[\frac{s}{s^2+2^2}\right] - L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 2\cos 2t - 1 - e^{-t}$$

$$\therefore f(t) = \frac{1}{t}(2\cos 2t - e^{-t} - 1)$$

Hence, $L^{-1}\left[\log\frac{s(s+1)}{s^2+4}\right] = \frac{1}{t}(2\cos 2t - e^{-t} - 1)$

(3) $\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

Solution: Let, $L^{-1}\left[\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}\right] = f(t)$ (say)

$$L[f(t)] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

By using the theorem of L. T. we have,

$$L[t\ \ f(t)] = -\frac{d}{ds}\left\{\frac{1}{2}\log\frac{(s^2+b^2)}{s^2+a^2}\right\}$$

$$= -\frac{1}{2}\frac{d}{ds}[\log(s^2+b^2) - (s^2+b^2)].$$

$$= -\frac{1}{2}\left[\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right]$$

$$L[t\ f(t)] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

or, $t\,f(t) = L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$

$$= \cos at - \cos bt$$

or, $f(t) = \frac{1}{t}(\cos at - \cos bt)$

Hence, $L^{-1}\left[\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}\right] = \frac{1}{t}(\cos at - \cos bt)$

(4) $\cot^{-1}(s+1)$

Solution: Let, $L^{-1}(\cot^{-1}(s+1)) = f(t)$ (say)

or, $L[f(t)] = \cot^{-1}(s+1)$

By using the theorem of L. T. we have,

$$L[t\ \ f(t)] = -\frac{d}{ds}[\cot^{-1}(s+1)]$$

$$= -\frac{-1}{(s+1)^2+1^2}$$

$$\therefore L^{-1}[t\ f(t)] = \frac{1}{(s+1)^2+1^2}$$

$$\text{or, } t\,f(t) = L^{-1}\left[\frac{-1}{(s+1)^2+1^2}\right]$$

$$= e^{-t}\sin t$$

$$\text{or, } f(t) = \frac{1}{t}\,e^{-t}\sin t$$

Hence, $L^{-1}[\cot^{-1}(s+1)] = \frac{1}{t}\,e^{-t}\sin t$. Ans.

(5) $\log\left(1-\frac{a^2}{s^2}\right)$

Solution: Let, $L^{-1}\left[\log\left(1-\frac{a^2}{s^2}\right)\right] = f(t)$ (say)

$$\text{or, } [f(t)] = \log\left[\frac{s^2-a^2}{s^2}\right]$$

By using the theorem of L. T. we have,

$$L[t\,f(t)] = -\frac{d}{ds}\log\left(\frac{s^2-a^2}{s^2}\right)$$

$$= -\frac{d}{ds}[\log(s^2-a^2)-\log s^2]$$

$$= -\left[\frac{2s}{s^2-a^2}-\frac{1}{s^2}.2s\right]$$

$$\text{or, } L[t\,f(t)] = 2\left[\frac{1}{s}-\frac{s}{s^2-a^2}\right]$$

$$\text{or, } t\,f(t) = L^{-1}\left[\frac{1}{s}-\frac{s}{s^2-a^2}\right]$$

$$\text{or, } t\ f(t) = 2\left\{L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{s^2-a^2}\right]\right\}$$

$$= 2(1-\cosh at)$$

$$\text{or, } f(t) = \frac{2(1-\cosh at)}{t}$$

Hence, $L^{-1}\left[\log\left(1-\frac{a^2}{s^2}\right)\right] = 2\,\frac{(1-\cosh at)}{t}$ Ans.

(6) $\tan^{-1}\frac{2}{s}$

Solution: Let $L^{-1}\left[\tan^{-1}\frac{2}{s}\right] = f(t)$ (say)

$$\text{or, } L\ [f(t)] = \tan^{-1}\frac{2}{s}$$

By using the theoream of L. T. we have,

$$L[t\ f(t)] = -\frac{d}{ds}\left[\tan^{-1}\frac{2}{s}\right]$$

$$= -\frac{1}{1+\left(\frac{2}{s}\right)^2}\cdot\left(-\frac{2}{s^2}\right)$$

$$= \frac{2s^2}{(s^2-2^2)}\cdot\frac{1}{s^2}$$

$$= \frac{2}{s^2+2^2}$$

$$\text{or, } t\ f(t) = L^{-1}\left[\frac{2}{s^2+2^2}\right] = \frac{2}{2}.\sin 2t$$

$$\text{or, } f(t) = \frac{1}{t}\sin 2t$$

Hence, $L^{-1}\left[\tan^{-1}\frac{2}{s}\right] = \frac{1}{t}\sin 2t$ Ans.

(7) $\frac{s+2}{(s^2+4s+5)^2}$

Solution: We have, $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$

Now, Let $F(s) = \frac{1}{s^2+4s+5}$

We know,

$$f(t) = L^{-1}[F(s)]$$

$$= L^{-1}\left[\frac{1}{s^2+4s+5}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+1^2}\right]$$

$$= e^{-2t}\sin t.$$

By using the theorem of L. T. we have,

$$L[t\ f(t)] = -\frac{d}{ds}[f(s)]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+4s+5}\right]$$

$$= -\frac{-(2s+4)}{(s^2+4s+5)^2}$$

$$= 2\,\frac{(s+2)}{(s^2+4s+5)^2}$$

$$\therefore \frac{t\ f(t)}{2} = L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$$

$$\text{Hence, } \frac{t e^{-2t} \sin t}{2} = L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \frac{t e^{-2t} \sin t}{2} \text{ Ans.}$$

(8) $\dfrac{s+3}{(s^2+6s+13)^2}$

Solution: We have, $L^{-1}\left[\dfrac{s+3}{(s^2+6s+13)^2}\right]$

Now, Let $f(s) = \dfrac{1}{s^2+6s+13}$

We know, $f(t) = L^{-1}[f(s)]$

$$= L^{-1}\left[\frac{1}{s^2+6s+13}\right]$$

$$= L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$

$f(t) = \dfrac{1}{2} e^{-3t} \sin 2t$

By using the theorem of L. T. we have,

$$L[t\ f(t)] = -\frac{d}{ds}[f(s)]$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+6s+13}\right)$$

$$= -\frac{-(2s+6)}{(s^2+6s+13)^2}$$

$$\therefore\ L[t\ f(t) = 2\frac{(s+3)}{(s^2+6s+13)^2}$$

$$\therefore\ \frac{t\ f(t)}{2} = L^{-1}\frac{(s+3)}{(s^2+6s+13)^2}$$

$$\text{Hence, } \frac{t e^{-3t} \sin 2t}{2\times 2} = L^{-1}\left[\frac{(s+3)}{(s^2+6s+13)^2}\right]$$

$$\therefore\ L^{-1}\left[\frac{(s+3)}{(s^2+6s+13)^2}\right] = \frac{t e^{-3t} \sin 2t}{4} \text{ Ans.}$$

(9) $\dfrac{s}{(s^2-1)^2}$

Solution: We have, $L^{-1}\left[\dfrac{s}{(s^2-1)^2}\right]$

Let $F(s) = \dfrac{1}{s^2-1}$

We know, $f(t) = L^{-1}[F(s)]$

$$= L^{-1}\left[\frac{1}{s^2-1}\right]$$

$f(t) = \sin h\ t$

By using the theorem of L. T. we have,

$$L[t\ f(t)] = -\frac{d}{ds}[f(s)]$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2-1}\right)$$

$$\text{or, } L[t\ f(t) = -\frac{-2s}{(s^2-1)^2}$$

$$\text{or, } L[t\ f(t)] = 2\frac{s}{(s^2-1)^2}$$

$$\therefore\ \frac{t\ f(t)}{2} = L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$$

$$\text{or, } L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = \frac{t \sin h\ t}{2} \text{ Ans.}$$

(10) $\log\left(\dfrac{s+b}{s+a}\right)$

Solution: Let, $L^{-1}\left[\log\dfrac{s+b}{s+a}\right] = f(t)$ (say)

$$\text{or, } L[f(t)] = \log\left(\frac{s+b}{s+a}\right)$$

By using the theorem of L. T. we have,

$$L[t\ f(t)] = -\frac{d}{ds}\left[\log\left(\frac{s+b}{s+a}\right)\right]$$

$$= \frac{d}{ds}[\log(s+b) - \log(s+a)]$$

$$= -\left(\frac{1}{s+b} - \frac{1}{s+a}\right)$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

$$\therefore\ t\ f(t) = L^{-1}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

$$= L^{-1}\left[\frac{1}{s+a}\right] - L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} - e^{-bt}$$

$$\therefore\ f(t) = \frac{1}{t}(e^{-at} - e^{-bt})$$

$$\text{Hence, } L^{-1}\left[\log\left(\frac{s+b}{s+a}\right)\right] = \frac{1}{t}(e^{-at} - e^{-bt}) \text{ Ans.}$$

Find the inverse transforms by using convolution theorem.

(11) $\frac{1}{s(s+2)}$

Solution: We have, $L^{-1}\left[\frac{1}{s(s+2)}\right] = L^{-1}\left[\frac{1}{s} \times \frac{1}{s+2}\right]$

Now, $L^{-1}\left[\frac{1}{s}\right] = 1 = g(t)$ (say)

$L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t} = f(t)$ (say)

By using the convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s} \times \frac{1}{s+2}\right] = \int_0^t f(u)\ g(t-u)\ du$$

$$= \int_0^t e^{-2u}\ .\ 1\ du = \left[-\frac{e^{-2u}}{2}\right]_0^t = -\frac{e^{-2t}+1}{2} = \frac{1-e^{-2t}}{2}$$

(12) $\frac{1}{s^2(s+2)}$

Solution: We have, $L^{-1}\left[\frac{1}{s^2(s+2)}\right] = L^{-1}\left[\frac{1}{s^2} \times \frac{1}{(s+2)}\right]$

Now, $L^{-1}\left[\frac{1}{s^2}\right] = \frac{t}{1!} = t = g(t)$ (say)

or, $L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t} = f(t)$ (say)

By using the convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s^2} \times \frac{1}{(s+2)}\right] = \int_0^t f(u)\ g(t-u)\ du$$

$$= \int_0^t e^{-2u}\ (t-u)\ du$$

$$= \int_0^t t\, e^{-2u}\ du - \int_0^t u\, e^{-2u}\ du$$

$$= \left[\frac{t\, e^{-2u}}{-2}\right]_0^t - \left[\frac{u\, e^{-2u}}{-2}\right]_0^t + \frac{1}{2}\int_0^t e^{-2u}\ du$$

$$= \frac{t\, e^{-2t}}{-2} + \frac{t}{2} + \frac{t\, e^{-2t}}{-2} - \frac{1}{4}\left[e^{-2u}\right]_0^t$$

$$= \frac{t}{2} - \frac{1}{4}e^{-2t} + \frac{1}{4}$$

$$= \frac{1}{4}$$

(13) $\frac{1}{s^2(s^2+a^2)}$

Solution: We have, $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

$$= L^{-1}\left[\frac{1}{s^2} \times \frac{1}{s^2+a^2}\right]$$

Now, $L^{-1}\left[\frac{1}{s^2}\right] = \frac{t}{1!} = t = g(t)$ (say)

$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f(t)$ (say)

By using the convolution theorem of inverse lapulace transform.

We get,

$$L^{-1}\left[\frac{1}{s^2} \times \frac{1}{s^2+a^2}\right] = \int_0^t f(u)\, g(t-u)\ du$$

$$= \int_0^t \frac{1}{a}\ \sin au\ (t-u)\ du$$

$$= \frac{1}{a}\int_0^t (t-u)\ \sin au\ du$$

$$= \frac{1}{a}\left[-(t-u)\frac{\cos au}{a} - \frac{\sin au}{a^2}\right]_0^t$$

$$= \frac{1}{a}\left[0 - \frac{\sin at}{a^2} + \frac{t}{a} + 0\right]$$

$$= \frac{1}{a^3}\ [at - \sin at]$$

$$\therefore L^{-1}\left[\frac{1}{s^2} \times \frac{1}{s^2+a^2}\right] = \frac{1}{a^3}\ [at - \sin at]\ \text{Ans.}$$

(14) $\frac{1}{s(s^2+4)}$

Solution: We have, $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$

$$= L^{-1}\left[\frac{1}{s} \times \frac{1}{s^2+4}\right]$$

Now, $L^{-1}\left[\frac{1}{s}\right] = 1 = g(t)$ (say)

& $L^{-1}\left[\frac{1}{s^2+2^2}\right] = \frac{1}{2}\sin 2t\ f(t)$ (say)

By using the convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s} \times \frac{1}{s^2+4}\right] = \int_0^t f(u)\ g(t-u)\ du$$

$$= \int_0^t \frac{1}{2} \sin 2u \cdot 1\, du$$

$$= \frac{1}{2}\left[\frac{1}{2} - \cos 2u\right]_0^t = \frac{1}{4}\,[-\cos 2t + 1]$$

$$\therefore \left[\frac{1}{s} \times \frac{1}{s^2+4}\right] = \frac{1}{4}\,[1 - \cos 2t] \text{ Ans.}$$

(15) $\dfrac{1}{(s^2+a^2)^2}$

Solution: We have, $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\dfrac{1}{s^2+a^2} \times \dfrac{1}{s^2+a^2}\right]$

Now, $L^{-1}\left[\dfrac{1}{(s^2+a^2)}\right] = \dfrac{1}{a} \sin at = g(t)$ (say)

& $L^{-1}\left[\dfrac{1}{(s^2+a^2)}\right] = \dfrac{1}{a} \sin at = f(t)$ (say)

By using, convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s^2+a^2} \times \frac{1}{s^2+a^2}\right] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \frac{\sin au}{a} \cdot \frac{1}{a} \sin(at - au)\, du$$

$$= \frac{1}{a^2} \cdot \frac{1}{2} \int_0^t 2 \sin au \cdot \sin(at - au)\, du$$

$$= \frac{1}{2a^2} \int_0^t [\cos(-at) + \cos(2au - at)\, du$$

$$= \frac{1}{2a^2} \int_0^t [\cos at + \cos(2au - at)]\, du$$

$$= \frac{1}{2a^2} \left[-\cos at + \frac{\sin(2au - at)}{2a}\right]_0^t$$

$$= \frac{1}{2a^2} \left[-t \cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$L^{-1}\left[\frac{1}{s^2+a^2} \times \frac{1}{s^2+a^2}\right] = \frac{1}{2a^2}\,[\sin at - at \cos at] \text{ Ans.}$$

(16) $\dfrac{1}{(s+1)(s+2)}$

Solution: We have, $L^{-1}\left[\dfrac{1}{(s+1)(s+2)}\right] = L^{-1}\left[\dfrac{1}{s+1} \times \dfrac{1}{s+2}\right]$

Now, $L^{-1}\left[\dfrac{1}{s+1}\right] = e^{-t} = g(t)$ (say)

& $L^{-1}\left[\dfrac{1}{s+2}\right] = e^{-2t} = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s+1} \times \frac{1}{s+2}\right] = \int_0^t f(u)\; g(t-u)\, du$$

$$= \int_0^t e^{-2u} \cdot e^{-(t-u)} \cdot du$$

$$= \int_0^t e^{-2u-t+u} \cdot du$$

$$= \int_0^t e^{-u-t}\, du$$

$$= \int_0^t t\, e^{-(u+t)}\, du$$

$$= \left[\frac{e^{-(u+t)}}{-1}\right]_0^t = [-e^{-2t} + e^{-t}]$$

$$\therefore L^{-1}\left[\frac{1}{s+1} \times \frac{1}{s+2}\right] = e^{-t} - e^{-2t} \text{ Ans.}$$

(17) $\dfrac{2as}{(s^2+a^2)^2}$

Solution: We have, $L^{-1}\left[\dfrac{2as}{(s^2+a^2)^2}\right] = L^{-1}\left[\dfrac{2a}{(s^2+a^2)} \times \dfrac{s}{(s^2+a^2)}\right]$

Now, $L^{-1}\left[\dfrac{2a}{s^2+a^2}\right] = 2a\, L^{-1}\left[\dfrac{1}{s^2+a^2}\right]$

$$= 2a \cdot \frac{1}{a} \sin at = 2 \sin at = g(t) \text{ (say)}$$

& $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at \;\; f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{2a}{(s^2+a^2)} \times \frac{s}{(s^2+a^2)}\right] = \int_0^t f(u)\; g(t-u)\, du$$

$$= \int_0^t \cos au \cdot 2 \sin(at - au)\, du$$

$$= \int_0^t 2 \sin(at - au) \cdot \cos au\, du$$

$$= \int_0^t [\sin at + \sin(at - 2au)]\, du$$

$$= \left[\sin at \cdot U + \frac{\cos(at - 2au)}{-2a}\right]_0^t$$

$$= t \sin at - \frac{\cos at}{2a} + \frac{\cos at}{2a}$$

$$L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right] = t \sin at$$

(18) $\dfrac{s^2}{(s^2+a^2)}$ (BE 20

Solution: We have, $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)}\right] = L^{-1}\left[\dfrac{s}{s^2+a^2}\times\dfrac{s}{s^2+a^2}\right]$

Now, $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = g(t)$ (say)

and $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{s}{s^2+a^2}\times\frac{s}{s^2+a^2}\right] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \cos au \cos(at-au)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos au\cos(at-au)\,du$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au-at)]\,du$$

$$= \frac{1}{2}\left[\cos at\,.\,U + \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)}\right] = \frac{1}{2a}[\sin at + at\cos at]$$

(19) $\dfrac{1}{s(s+1)^3}$ (BE 20

Solution: We have, $L^{-1}\left[\dfrac{1}{s(s+1)^3}\right] = L^{-1}\left[\dfrac{1}{s}\times\dfrac{1}{(s+1)^3}\right]$

Now, $L^{-1}\left[\dfrac{1}{s}\right] = 1 = g(t)$ (say)

and $L^{-1}\left[\dfrac{1}{(s+1)^3}\right] = \dfrac{t^2}{2!}e^{-t} = \dfrac{t^2e^{-t}}{2} = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s}\times\frac{1}{(s+1)^3}\right] = \int_0^t \frac{U^2e^{-u}}{2}\,.\,1\,du$$

$$= \frac{1}{2}\int_0^t u^2e^{-u}\,du$$

$$= \frac{1}{2}\left[\frac{u^2e^{-u}}{-1} - 2ue^{-u} + \frac{2e^{-u}}{-1}\right]_0^t$$

$$\frac{1}{2}\left[\frac{t^2e^{-t}}{-1} - 2te^{-t} - 2e^{-t} + 2\right]$$

$$L^{-1}\left[\frac{1}{s}\times\frac{1}{(s+1)^3}\right] = 1 - \frac{e^{-t}}{2}(t^2+2t+2)\ \text{Ans.}$$

(20) $\dfrac{1}{(s+1)(s+9)^2}$

Solution: We have, $L^{-1}\left[\dfrac{1}{(s+1)(s+9)^2}\right] = L^{-1}\left[\dfrac{1}{s+1}\times\dfrac{1}{(s-9)^2}\right]$

Now, $L^{-1}\left[\dfrac{1}{s+1}\right] = e^{-t} = g(t)$ say.

and $L^{-1}\left[\dfrac{1}{(s+9)^2}\right] = te^{-9t} = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s+1}\times\frac{1}{(s+9)^2}\right] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t u\,e^{-9u}\,e^{-(t-u)}\,du$$

$$= \int_0^t u\,e^{-(8u+t)}\,du$$

$$= \left[\frac{ue^{-(8u+t)}}{-8} - \frac{e^{-(8u+t)}}{64}\right]_0^t$$

$$= \frac{te^{-9t}}{-8} - \frac{e^{-9t}}{64} + \frac{e^{-t}}{64}$$

$$= \frac{e^{-t}}{64}[1 - e^{-8t(1+8t)}]$$

$$\therefore L^{-1}\left[\frac{1}{s+1}\times\frac{1}{(s+9)^2}\right] = \frac{e^{-t}}{64}[1 - e^{-8t(1+8t)}]\ \text{Ans.}$$

(21) $\dfrac{1}{(s-2)(s+2)^2}$

Solution: We have, $\dfrac{1}{(s-2)(s+2)^2} = L^{-1}\left[\dfrac{1}{s-2}\times\dfrac{1}{(s+2)^2}\right]$

Now, $L^{-1}\left[\dfrac{1}{s-2}\right] = e^{2t} = g(t)$ (say)

and $L^{-1}\left[\dfrac{1}{(s+2)^2}\right] = te^{-2t} = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s-2}\times\frac{1}{(s+2)^2}\right] = \int_0^t u\,e^{-2u}\,.\,e^{2t-2u}\,du$$

$$= \int_0^t u e^{2t-4u}\, du$$

$$= \left[\frac{ue^{2t-4u}}{-4} - \frac{e^{2t-4u}}{16}\right]_0^t$$

$$= \frac{te^{-2t}}{-4} - \frac{e^{-2t}}{16} + \frac{e^{2t}}{16}$$

$$= \frac{1}{16}\left[e^{-2t} - e^{-2t} - 4te^{-2t}\right]$$

$$\therefore L^{-1}\left[\frac{1}{s-2} \times \frac{1}{(s+2)^2}\right] = \frac{1}{16}\left[e^{-2t} - e^{-2t} - 4te^{-2t}\right] \text{ Ans.}$$

22. $\frac{s^2}{s^4-16}$

Solution: We have, $L^{-1}\left[\frac{s^2}{s^4-16}\right]$

$$= L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{s}{s^2-4}\right]$$

Now, $L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t = f(t)$

and $L^{-1}\left[\frac{s}{s^2-2^2}\right] = \cosh 2t = g(t)$

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{s}{s^2-4}\right] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos 2u \, \cos(2t-2u)\, du$$

$$= \int_0^t \cos 2u \left[\frac{e^{(2t-2u)} + e^{-(2t-2u)}}{2}\right] du$$

$$= \frac{1}{2}\left[\underbrace{\int_0^t e^{2t}e^{-2u} \cos 2u\, du}_{I_1} + \underbrace{\frac{1}{2}\int_0^t e^{-2t}e^{2u} \cos 2u\, du}_{I_2}\right]$$

For $I_1 = \frac{e^{2t}}{2} \int_0^t e^{-2u} \cos 2u\, du$

$$= \frac{e^{2t}}{2}\left[\frac{-2e^{-2u}.(\cos 2u - \sin 2u)}{2^2+2^2}\right]_0^t$$

$$= \frac{e^{2t}}{2} \cdot -\frac{2}{8}\left[e^{-2t}(\cos 2t - \sin 2t) - 1\right]$$

$$= -\frac{1}{8}(\cos 2t - \sin 2t) + \frac{1}{8}e^{2t}$$

$$I_2 = \frac{1}{2} e^{-2t} \int_0^t e^{2u} \cos 2u\, du$$

$$= \frac{e^{-2t}}{2} \int_0^t e^{2u} \cos 2u\, du$$

$$= \frac{e^{-2t}}{2}\left[2e^{2u}\frac{(\cos 2u + \sin 2u)}{2^2+2^2}\right]_0^t$$

$$= \frac{e^{-2t}}{2} \cdot \frac{2}{8}\left[e^{2t}(\cos 2t + \sin 2t) - 1\right]$$

$$= \frac{1}{8}(\cos 2t + \sin 2t) - \frac{1}{8}e^{-2t}$$

Thus,

$$L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{s}{s^2-4}\right] = I_1 + I_2$$

$$= -\frac{1}{8}\cos 2t + \frac{1}{8}\sin 2t + \frac{1}{8}e^{2t} + \frac{1}{8}\cos 2t + \frac{1}{8}\sin 2t - \frac{1}{8}e^{-2t}$$

$$= \frac{1}{4}\sin 2t - \frac{1}{4}\left[\frac{e^{2t} - e^{-2t}}{2}\right]$$

$$= \frac{1}{4}\left[\sin 2t - \sinh 2t\right]$$

23. $\frac{s^2-9}{(s^2+9)^2}$

Solution: We have $L^{-1}\left[\frac{s^2-9}{(s^2+9)^2}\right]$

$$= L^{-1}\left[\frac{s^2}{(s^2+9)^2} - \frac{9}{(s^2+9)^2}\right]$$

$$= \underbrace{L^{-1}\left[\frac{s}{s^2+9} \cdot \frac{s}{s^2+9}\right]}_{1^{st}\text{ part}} - \underbrace{L^{-1}\left[\frac{3}{s^2+9} \cdot \frac{3}{s^2+9}\right]}_{2^{nd}\text{ part}}$$

For 1^{st} part

$$L^{-1}\left[\frac{s}{s^2+9} \cdot \frac{s}{s^2+9}\right]$$

Now, $L^{-1}\left[\frac{s}{s^2+3^2}\right] = \cos 3t = f(t)$ (say)

and $L^{-1}\left[\frac{s}{s^2+3^2}\right] = \cos 3t = g(t)$ (say)

By using the convolution theorem of inverse L. T.
We have,

$$L^{-1}\left[\frac{s}{s^2+9} \cdot \frac{s}{s^2+9}\right] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos 3u \, \cos(3t-3u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos 3t + \cos(6u - 3t)]\, du$$

$$= \frac{1}{2}\left[\cos 3t.u + \frac{\sin(6u-3t)}{6}\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 3t + \frac{\sin 3t}{6} + \frac{\sin 3t}{6}\right]$$

$$= \frac{1}{2}\left[t\cos 3t + \frac{\sin 3t}{3}\right]$$

For 2[nd] part

$$L^{-1}\left[\frac{3}{s^2+9}\cdot\frac{3}{s^2+9}\right]$$

Now, $L^{-1}\left[\frac{3}{s^2+3^2}\right] = 3.\frac{1}{3}\sin 3t = f(t)$ (say)

and $L^{-1}\left[\frac{3}{s^2+3^2}\right] = 3.\frac{1}{3}\sin 3t = g(t)$ (say)

Now, by using the convolution theorem of inverse L. T., we get,

$$L^{-1}\left[\frac{3}{s^2+9}\cdot\frac{3}{s^2+9}\right] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \sin 3u \sin(3t - 3u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos(6u-3t) - \cos 3t]\, du$$

$$= \frac{1}{2}\left[\frac{\sin(6u-3t)}{6} - \cos 3t.u\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin 3t}{6} - t\cos 3t + \frac{\sin 3t}{6}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

Thus, $L^{-1}\left[\frac{s^2}{(s^2+9)^2} - \frac{9}{(s^2+9)^2}\right]$

= 1[st] part – 2[nd] part

$$= \frac{1}{2}\left[t\cos 3t + \frac{\sin 3t}{3}\right] - \frac{1}{2}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

$$= \frac{1}{2}\left[t\cos 3t + \frac{\sin 3t}{3} - \frac{\sin 3t}{3} + t\cos 3t\right]$$

$$\therefore L^{-1}\left[\frac{s^2-9}{(s^2+9)^2}\right] = t\cos 3t$$

(24) $\frac{s^2}{(s^2+1)(s^2+4)}$

Solution: We have, $L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] = L^{-1}\left[\frac{s}{s^2+1}\times\frac{s}{s^2-4}\right]$

Now, $L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t = g(t)$ (say)

and $L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 3t = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{s}{s^2+1}\times\frac{s}{s^2+4}\right] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos 2u \cos(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos(u+t) + \cos(3u-t)]\, du$$

$$= \frac{1}{2}\left[\sin(u+t) + \frac{\sin(3u-t)}{3}\right]_0^t$$

$$= \frac{1}{2}\left[\sin 2t + \frac{\sin 2t}{3} - \sin t + \frac{\sin t}{3}\right]$$

$$= \frac{1}{2}\left[\frac{4}{3}\sin 2t - \frac{2}{3}\sin t\right]$$

$$= \frac{1}{2}.2\left[\frac{2}{3}\sin 2t - \frac{1}{3}\sin t\right]$$

$$= \frac{1}{3}[2\sin 2t - \sin t]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}[2\sin 2t - \sin t] \text{ Ans.}$$

(25) $\frac{1}{s(s+3)^2}$ **(BE 2062)**

Solution: We have, $L^{-1}\left[\frac{1}{s(s+3)^2}\right]$

Now, $L^{-1}\left[\frac{1}{s}\right] = 1 = g(t)$ (say)

and $L^{-1}\left[\frac{1}{(s+3)^2}\right] = te^{-3t} = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s}\times\frac{1}{(s+3)^2}\right] = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t u\, e^{-3u} \,.\, 1.\, du$$

$$= \int_0^t u\, e^{-3u}\, du$$

$$= \left[\frac{ue^{-3u}}{-3} - \frac{e^{-3u}}{9}\right]_0^t$$

$$= \frac{te^{-3t}}{-3} - \frac{e^{-3t}}{9} + \frac{1}{9}$$

$$= \frac{1}{9}\,[1 - e^{-3t} - 3te^{-3t}]$$

$$\therefore L^{-1}\left[\frac{1}{s}\times\frac{1}{(s+3)^2}\right] = \frac{1}{9}\,[1 - e^{-3t} - 3te^{-3t}] \quad \text{Ans.}$$

(26) $\dfrac{s^2}{(s^2+4)^2}$ **(BE 2058)**

Solution: We have, $L^{-1}\left[\dfrac{s^2}{(s^2+4)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+4}\times\dfrac{s}{s^2+4}\right]$

Now, $L^{-1}\left[\dfrac{s}{s^2+4}\right] = \cos 2t = g(t)$ (say)

and $L^{-1}\left[\dfrac{s}{s^2+4}\right] = \cos 2t = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{s}{s^2+4}\times\frac{s}{s^2+4}\right] = \int_0^t f(u)\; g(t-u)\, du$$

$$= \int_0^t \cos\, 2u \cos\,(2t-2u)\, du$$

$$= \frac{1}{2}\int_0^t 2\cos 2u \cos\,(2t-2u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos\,(4u-2t)]\, du$$

$$= \frac{1}{2}\left[u\cos 2t + \frac{\sin(4u-2t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4}\right]$$

$$= \frac{1}{4}\,[2t\cos 2t + \sin 2t]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \frac{1}{4}\,[2t\cos 2t + \sin 2t] \quad \text{Ans.}$$

(27) $\dfrac{1}{(s-3)\,(s+3)^2}$

Solution: We have, $L^{-1}\left[\dfrac{1}{(s-3)\,(s+3)^2}\right] = L^{-1}\left[\dfrac{1}{s-3}\times\dfrac{1}{(s+3)^2}\right]$

Now, $L^{-1}\left[\dfrac{1}{s-3}\right] = e^{+3t} = g\,(t)$ (say)

and $L^{-1}\left[\dfrac{1}{(s+3)^2}\right] = t\,e^{-3t} = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s-3}\times\frac{1}{(s+3)^2}\right] = \int_0^t u\; e^{-3u}\,.\,e^{3t-3u}\; du$$

$$= \int_0^t u\, e^{-3u+3t-3u}\, du$$

$$= \int_0^t u\, e^{3t-6u}\, du$$

$$= \left[\frac{ue^{3t-6u}}{-6} - \frac{e^{3t-6u}}{36}\right]_0^t$$

$$= \left[\frac{te^{-3t}}{-6} - \frac{e^{-3t}}{36} + \frac{e^{3t}}{36}\right]$$

$$= \frac{1}{36}\,[e^{3t} - e^{-3t} - 6 + e^{-3t}]$$

$$\therefore L^{-1}\left[\frac{1}{s-3}\times\frac{1}{(s+3)^2}\right] = \frac{1}{36}\,[e^{3t} - e^{-3t} - 6\,t\,e^{-3t}] \quad \text{Ans.}$$

(28) $\dfrac{1}{s^3(s^2+1)}$ **(BE 2063)**

Solution: We have, $L^{-1}\left[\dfrac{1}{s^3(s^2+1)}\right] = L^{-1}\left[\dfrac{1}{s^3}\times\dfrac{1}{s^2+1}\right]$

Now, $L^{-1}\left[\dfrac{1}{s^3}\right] = \dfrac{t^2}{2!} = \dfrac{t^2}{2} = g\,(t)$ (say)

and $L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t = f(t)$ (say)

By using convolution theorem of inverse L. T. we get,

$$L^{-1}\left[\frac{1}{s^3}\times\frac{1}{s^2+1}\right] = \int_0^t f(u)\; g\,(t-u)\, du$$

$$= \int_0^t \sin u\; \frac{(t-u)^2}{2}\; du$$

$$= \frac{1}{2}\int_0^t (t-u)^2\; \sin u\, du$$

$$= \frac{1}{2}\left[\frac{(t-u)^2\cos u}{-1} - 2(t-u)\sin u + 2(\cos u)\right]_0^t$$

$$= \frac{1}{2}[2\cos t + t^2 - 2] = \frac{t^2}{2} + \cos t - 1$$

$$\therefore L^{-1}\left[\frac{1}{s^3}\times\frac{1}{s^2+1}\right] = \frac{t^2}{2} + \cos t - 1 \quad \text{Ans.}$$

Find the inverse Laplace transform by using second shifting theorem.

(29) $\frac{e^{-\pi s}}{s+3}$

Solution: We have, $L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right] = L^{-1}\left[e^{-\pi s}\times\frac{1}{s+3}\right]$

Now, $L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t} = f(t)$ (say)

By using the second shifting theorem of inverse laplace transform

$$L^{-1}\left[e^{-\pi s}\times\frac{1}{s+3}\right] = [f(t-a)\,u(t-a)]$$

$$= e^{-3t+3\pi}\,u(t-\pi)$$

$$\therefore \left[e^{-\pi s}\times\frac{1}{s+3}\right] = e^{-3t+3\pi}\,u(t-\pi) \quad \text{Ans.}$$

(30) $\frac{s\,e^{-2s}}{s^2-1}$ **(BE 2062)**

Solution: We have, $L^{-1}\left[\frac{s\,e^{-2s}}{s^2-1}\right] = L^{-1}\left[e^{-2s}\times\frac{s}{s^2-1}\right]$

Now, $L^{-1}\left[\frac{s}{s^2-1}\right] = \cosh t = f(t)$ (say)

By using second shifting theorem of inverse laplace transform

$$L^{-1}\left[e^{-2s}\times\frac{s}{s^2-1}\right] = [f(t-a)\,u(t-a)]$$

$$= \cosh(t-2)\,u(t-2)$$

$$\therefore \left[e^{-2s}\times\frac{s}{s^2-1}\right] = \cosh(t-2)\,u(t-2) \quad \text{Ans.}$$

(31) $\frac{e^{-2s}}{(s+1)(s^2+2s+2)}$

Solution: We have, $L^{-1}\left[\frac{e^{-2s}}{(s+1)(s^2+2s+2)}\right]$

$$= L^{-1}\left[e^{-2s}\times\frac{1}{(s+1)(s^2+2s+2)}\right]$$

Now, $L^{-1}\left[e^{-2s}\times\frac{1}{(s+1)(s^2+2s-2)}\right] = L^{-1}\left[\frac{1}{(s+1)}\times\frac{1}{(s+1)^2-1}\right]$

Here, $L^{-1}\left[\frac{1}{s+1}\right] = e^{-t} = g(t)$ (say)

and $L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t = f(t)$ (say)

By using the convolution theorem of inverse laplace transform

$$L^{-1}\left[\frac{1}{(s+1)}\times\frac{1}{(s+1)^2+1}\right] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t e^{-u}\sin u\,.\,e^{-t+u}\,du$$

$$= \int_0^t e^{-u-t+u}\sin u\,du$$

$$= \int_0^t e^{-t}\sin u\,du$$

$$= \left[\frac{e^{-t}\cos u}{-1}\right]_0^t$$

$$= \frac{e^{-t}\cos t}{-1} + e^{-t}$$

$$= e^{-t}[1-\cos t]$$

$$= f(t) \text{ (say)}$$

By using the second shifting theorem of inverse laplace transform

$$L^{-1}\left[e^{-2s}\times\frac{1}{(s+1)(s+1)^2+1}\right] = [f(t-a)\,u(t-a)]$$

$$= e^{2-t}[1-\cos(t-2)]\,u(t-2)$$

(32) $\frac{e^{-\pi s}(s+1)}{(s^2+2s+2)}$

Solution: We have, $L^{-1}\left[\frac{e^{-\pi s}(s+1)}{(s^2+2s+2)}\right] = L^{-1}\left[e^{-\pi s}\times\frac{(s+1)}{(s^2+2s+2)}\right]$

Now, $L^{-1}\left[\frac{(s+1)}{s^2+2s+2}\right] = L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] = e^{-t}\cos t$

By using the second shifting theorem of inverse laplace transform.

$$L^{-1}\left[e^{-\pi s}\times\frac{(s+1)}{(s^2+2s+2)}\right] = [f(t-a)\,u(t-a)]$$

$$= e^{-t+\pi}\cos(t-\pi)\,u(t-\pi)$$

$$= e^{\pi-t}\cos(\pi-t)\,u(t-\pi)$$

$$= -e^{\pi-t}\cos t\,u(t-\pi)$$

$$\therefore L^{-1}\left[e^{-\pi s}\times\frac{(s+1)}{(s^2+2s+2)}\right] = -e^{\pi-t}\cos t\;u(t-\pi) \quad \text{Ans.}$$

Exercise - 14

Save the following different equation by Laplace transform method.

(1) $y'' + 4y = 0;\ y(0) = 0;\ y'(0) = 1$

Solution: Given differential equation is,

$y'' + 4y = 0$

Taking Laplace transform on both sides we get,

$L[y'' + 4y] = L[0]$

or, $L[y''(t)] + 4L[y(t)] = 0$

or, $s^2 L[y(t)] - s\,y(0) - y'(0) + 4L[y(t)] = 0$

By using given condition $y(0) = 0;\ y'(0) = 1$

or, $s^2 Y(s) - s.0 - 1 + 4Y(s) = 0$

or, $(s^2 + 4)Y(s) = 1$

$$Y(s) = \frac{1}{s^2+4}$$

Taking inverse Laplace transform on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s^2+4}\right]$$

or, $y(t) = \frac{1}{2}\sin 2t$ Ans.

(2) $y'' - 2y' - 3y = 0;\ y(0) = 1,\ y'(0) = 7$

Solution: Given different equation is,

$y'' - 2y' - 3y = 0$

Taking Laplace transform on both sides,

$L[y'' - 2y' - 3y] = L[0]$

or, $L[y''(t)] - 2L(y'(t)] - 3L[y(t)] = 0$

or, $s^2 L[y(t)] - s\,y(0) - y'(0) - 2s\,L[y(t)] + 2y(0) - 3L[y(t)] = 0$

using the given condition,

$y(0) = 1;\ y'(0) = 7$

or, $s^2 Y(s) - s.1 - 7 - 2s\,Y(s) + 2.1 - 3Y(s) = 0$

or, $(s^2 - 2s - 3)Y(s) = s + 5$

$$Y(s) = \frac{s+5}{s^2-2s-3}$$

Taking inverse L. T. on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{s+5}{s^2-2s-3}\right] = L^{-1}\left[\frac{(s+5)}{(s-3)(s+1)}\right]$$

Now, $\frac{(s+5)}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{(s+1)}$

$s + 5 = A(s+1) + B(s-3)$

put $s = -1 \Rightarrow B = \frac{-4}{4} = -1$

put $s = 4 \Rightarrow A = \frac{8}{4} = 2$

$$\frac{s-5}{(s-3)(s+1)} = 2\frac{1}{s-3} - \frac{1}{s+1}$$

$$\therefore L^{-1}\left[\frac{(s+5)}{(s-3)(s+1)}\right] = L^{-1}\left[2\frac{1}{s-3} - \frac{1}{s+1}\right]$$

$$= 2L^{-1}\left[\frac{1}{s-3}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 2e^{3t} - e^{-t}$$

$$= 2e^{3t} - e^{-t}$$

$$\therefore L^{-1}\left[\frac{(s+5)}{(s-3)(s+1)}\right] = 2e^{3t} - e^{-t}$$

$$\therefore L^{-1}[Y(s)] = L^{-1}\left[\frac{(s+5)}{(s-3)(s+1)}\right]$$

or, $y(t) = 2e^{3t} - e^{-t}$ Ans.

(3) $\frac{d^2y}{dx^2} + \omega^2 y = 0;\ y(0) = A,\ \left(\frac{dy}{dx}\right)_{at\ x=0} = B.$

Solution: Givern different equation is,

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

o, $y''(x) + \omega^2 y(x) = 0$

Taking Laplace traustorus on both sides,

$L[y''(x) + \omega^2 y(x)] = L[0]$

or, $L[y''(x)] + \omega^2 L[y(x)] = 0$

or, $s^2 L[y(x)] - s\,y(0) - y'(0) + \omega^2 L[y(x)] = 0$

Using given condition $y(0) = A.\ \left(\frac{dy}{dx}\right)_{at\ x=0} = y'(0) = B$

or, $s^2 Y(s) - S.A - B + \omega^2 Y(s) = 0$

or, $(s^2 + \omega^2)Y(s) = sA + B$

or, $Y(s) = \frac{AS+B}{s^2+\omega^2}$

Taking inverse Lapelace on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{AS+B}{s^2+\omega^2}\right] = L^{-1}\left[\frac{A}{(s^2+\omega^2)}\right] + L^{-1}\left[\frac{B}{(s^2+\omega^2)}\right]$$

$$= A\,L^{-1}\left[\frac{S}{s^2+\omega^2}\right] + B\,L^{-1}\left[\frac{1}{s^2+\omega^2}\right]$$

$y(x) = A\cos\omega x + \frac{B}{\omega}\sin\omega x.$

(4) $y'' + ky' - 2k^2y = 0;\ y(0) = 2,\ y'(0) = 2k$ **(BE 2061)**

Solution: Givern. different equation is,

$y'' + ky' - 2k^2y = 0$

Taking L. T. on both sides,

$L[y'' + ky' - 2k^2y] = L[0]$

or, $L[y''(t)] + kL[y'(t)] - 2k^2 L[y(t)] = 0$

or, $s^2 L[y(t)] - s\,y(0) - y'(0) + ks\,L[y(t)] - k\,y(0) - 2k^2 L[y(t)] = 0$

Using given condition,

$y(0) = 2,\ y'(0) = 2k$

or, $s^2 Y(s) - s.2 - 2k + ks\,Y(s) - k.2 - 2k^2 Y(s) = 0$

or, $(s^2 + ks - 2k^2)\,Ys = 2s + 4k$

$$\text{or, } Y(s) = \frac{2(s+2k)}{(s^2+ks-2k^2)}$$

$$= \frac{2(s+2k)}{s^2+2ks-ks-2k^2}$$

$$= \frac{2(s+2k)}{s(s+2k)-k(s+2k)}$$

$$= \frac{2(s+2k)}{(s-k)(s+2k)}$$

$$Y(s) = \frac{2}{s-k}$$

Taking inverse L. T. on both sides;

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{2}{s-k}\right]$$

$\therefore\ y(t) = 2e^{kt}$

(5) $y'' + y = \sin 3t;\ y(0) = 0,\ y'(0) = 0$

Solution: Given different equation is,

$y'' + y = \sin 3t$

taking L. T. on both sides,

$L[y'' + y] = L(\sin 3t)$

$$\text{or, } L[y''(t)] + L[y(t)] = \frac{3}{s^2+9}$$

$$\text{or, } s^2 L[y(t)] - s\,y(0) - y'(0) + L[y(t)] = \frac{3}{s^2+9}$$

Using given condition,

$y(0) = 0,\ y'(0) = 0$

$$s^2 Y(s) - s.0 - 0 + Y(s) = \frac{3}{s^2+9}$$

$$\text{or, } Y(s)(s^2+1) = \frac{3}{s^2+9}$$

$$\text{or, } Y(s) = \frac{3}{(s^2+1)(s^2+9)}$$

Taking inverse L. T. on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{3}{(s^2+1)(s^2+9)}\right]$$

$$y(t) = 3L^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right] \quad \text{......... (i)}$$

$$\text{Now, } L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = g(t) \quad \text{(say)}$$

$$L^{-1}\left[\frac{1}{s^2+9}\right] = \frac{1}{3}\sin 3t = f(t) \quad \text{(say)}$$

By using the convolution theorem of inverse L. T.

$$L^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right]$$

$$= \int_0^t f(u)\ g(t-u)\,du$$

$$= \int_0^t \frac{1}{3}\sin 3u \sin(t-u)\,du$$

$$= \frac{1}{3}\cdot\frac{1}{2}\int_0^t 2\sin 3u.\sin(t-u)\,du$$

$$= \frac{1}{6}\int_0^t \cos(4u-t) - \cos(2u-t)\,du$$

$$= \frac{1}{6}\left[\frac{\sin(4u-t)}{4} - \frac{\sin(2u+t)}{2}\right]_0^t$$

$$= \frac{1}{6}\left[\frac{\sin 3t}{4} - \frac{\sin 3t}{2} + \frac{\sin t}{4} + \frac{\sin t}{2}\right]$$

$$= \frac{1}{6}\left[\frac{\sin 3t - 2\sin 3t + \sin + 2\sin t}{4}\right]$$

$$= \frac{1}{24}[3\sin t - \sin 3t]$$

Now (1) becomes,

$$y(t) = 3.\frac{1}{24}(3\sin 3t - \sin 3t)$$

$$y(t) = \frac{3}{8}\sin 3t - \frac{1}{8}\sin 3t$$

(6) $y'' + 25y = 10\cos 5t;\ y(0) = 2,\ y'(0) = 0$

Solution: Given diff. equation is,

$y'' + 25y = 10\cos 5t$

Taking L. T. on both sides;

$L[y'' + 25y] = L(10\cos 5t]$

or, $L[y''(t)] + 25\,L[y(t)] = 10 \cdot \dfrac{s}{s^2+25}$

or, $s^2 L[y(t)] - s\,y(0) - y'(0) + 25\,L[y(t)] = \dfrac{10s}{s^2+25}$

Using given condition;

$y(0) = 2,\ y'(0) = 0$

$s^2 Y(s) - S.2 - 0 + 25\,Y(s) = \dfrac{10s}{s^2+25}$

or, $(s^2+25)\,Y(s) = \dfrac{10s}{s^2+25} + 2s$

or, $Y(s) = \dfrac{10s + 2s^3 + 60s}{(s^2+25)(s^2+25)}$

$Y(s) = \dfrac{2s^3+60s}{(s^2+25)^2}$ .... (i)

Now, $\dfrac{2s^3+60s}{(s^2+25)^2} = \dfrac{AS+B}{(s+25)} + \dfrac{Cs+D}{(s+25)^2}$

or, $2s^3 + 60s = (As+B)(s+25) + (Cs+D)$

Equating the coefficient of s^3, s^2, s and s^0 we get,

$A = 2,\ B = 0,\ C = 10$ and $D = 0$

$\therefore \dfrac{2s^3+60s}{(s^2+25)^2} = \dfrac{2s}{s^2+25} + \dfrac{10s}{(s^2+25)^2}$

Now (i) becomes,

$Y(s) = \dfrac{2s}{s^2+25} + \dfrac{10s}{(s^2+25)^2}$

Taking inverse L. T. on both sides,

$L^{-1}(Y(s)] = L^{-1}\left[\dfrac{2s}{s^2+25} + \dfrac{10s}{(s^2+25)^2}\right]$

$= 2\,L^{-1}\left[\dfrac{s}{s^2+5^2}\right] + 10\,L^{-1}\left[\dfrac{s}{(s^2+5^2)^2}\right]$

$\therefore y(t) = 2\cos 5t + 10t\cos 5t.$

(7) $\mathbf{y'' + 9y = 9\,u(t-3);\ y(0) = 0,\ y'(0) = 0}$

Solution: Given different equation is,

$y'' + 9y = 9\,u(t-3)$

Taking Laplace Transform on both sides;

$L[y'' + 9y] = L(9u(t-3)]$

or, $L[y''(t)] + 9\,L[y(t)] = 9\,L[u(t-3)]$

or, $s^2 L[y(t)] - s\,y(0) - y'(0) + 9\,L[y(t)] = 9\,e^{-3s} \cdot \dfrac{1}{s}$

Using given condition;

or, $s^2 Y(s) - 0 - 0 + 9\,Y(s) = 9e^{-3s} \cdot \dfrac{1}{s}$

or, $(s^2+9)\,Y(s) = e^{-3s}\,\dfrac{9}{s}$

$Y(s) = e^{-3s} \cdot \dfrac{9}{s.(s^2+9)}$

Taking inverse L. T. on both sides

$L^{-1}[Y(s)] = L^{-1}\left[e^{-3s} \cdot \dfrac{9}{s(s^2+9)}\right]$

$y(t) = 9\left[e^{-3s} \cdot \dfrac{1}{s(s^2+9)}\right]$ .... (i)

For, $L^{-1}\left[e^{-3s} \cdot \dfrac{9}{s(s^2+9)}\right]$

Now, $L^{-1}\left[\dfrac{1}{s(s^2+9)}\right] = L^{-1}\left[\dfrac{1}{s} \times \dfrac{1}{s^2+9}\right]$

Now, $L^{-1}\left[\dfrac{1}{s}\right] = 1 = g(t)$. (say)

and $L^{-1}\left[\dfrac{1}{s^2+9}\right] = \dfrac{1}{3}\sin 3t = f(t)$ (say)

By using the convolution theorem of inverse L. T.

$L^{-1}\left[\dfrac{1}{s} \times \dfrac{1}{s^2+9}\right] = \dfrac{1}{3}\int_0^t \sin 3u\,du$

$= \dfrac{1}{3}\left[\dfrac{-\cos 3u}{3}\right]_0^t$

$= \dfrac{1}{9}\left[-\cos 3u\right]_0^t$

$= \dfrac{1}{9}\left[-\cos 3t + 1\right]$

$= \dfrac{1}{9}\left[1 - \cos 3t\right] = f(t)$ (say)

Now by using the second suifting theorem of inverse L. T.

$L^{-1}\left[e^{-3s} \cdot \dfrac{1}{s(s^2+9)}\right] = [f(t-a)\,u(t-a)]$

$= \dfrac{1}{9}\left[1 - \cos(3t-9)\,u(t-3)\right]$

Now (i) becomes,

$y(t) = \dfrac{1}{9}\,L^{-1}\left[e^{-3s} \cdot \dfrac{9}{s(s^2+9)}\right]$

$= 9 \cdot \frac{1}{9} [1 - \cos(3t-9)\, u(t-3)]$

$\therefore y(t) = [1 - \cos(3t-9]\, u(t-3)]$

(8) $y'' + 2y' + 2y = 5\sin t;\ y(0), y'(0) = 0$

Solution: Given different equation is,

$y'' + 2y' + 2y = 5\sin t$

Taking L. T. on both sides,

$L[y'' + 2y' + 2y] = L[5\sin t]$

or, $L[y''(t)] + 2L[y'(t)] + 2L[y(t)] = \frac{5}{s^2+1}$

Using given condition,

$s^2 Y(s) - 0 - 0 + 2sY(s) + 0 + 2Y(s) = \frac{5}{(s^2+1)}$

or, $(s^2 + 2s + 2)\, Y(s) = \frac{5}{(s^2+1)}$

or, $Y(s) = \frac{5}{(s^2+1)(s^2+2s+2)}$

Taking inverse L. T. on both sides,

$L^{-1}[Y(s)] = L^{-1}\left[\frac{5}{(s^2+1)(s^2+2s+2)}\right]$ .... (i)

$= 5L^{-1}\left[\frac{1}{s^2+1} \cdot \frac{1}{s^2+2s+2}\right]$

Now, $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = g(t)$ (say)

$L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t = f(t)$ (say)

By using the convolution theorem of inverse laplace transform, we get,

$L^{-1}\left[\frac{1}{s^2+1} \cdot \frac{1}{s^2+2s+2}\right] = \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t e^{-u} \sin u \cdot \sin(t-u)\, du$

$= \frac{1}{2}\int_0^t e^{-u}\, 2\sin u \sin(t-u)\, du$

$= \frac{1}{2}\int_0^t e^{-u}\, [\cos(2u-t) - \cos t]\, du$

$= \frac{1}{2}\int_0^t e^{-u} \cos(2u-t)\, du - \frac{1}{2}\cos t \int_0^t e^{-u}\, du$

$= \frac{1}{2}\left[\frac{-e^{-u}\cos(2u-t) - 2\sin(2u-t)}{4}\right]_0^t - \frac{1}{2}\cos t\, [-e^{-u}]_0^t$

$= \frac{1}{2}\left[\frac{-e^{-u}\cos t - 2\sin t}{4} + \frac{\cos t - 2\sin t}{4}\right] + \frac{1}{2}e^{-t}\cos t - \frac{1}{2}\cos t$

$= \frac{1}{2} \cdot \frac{1}{4}(-e^{-t}\cos t - 2\sin t + \cos t - 2\sin t) + \frac{1}{2}(e^{-t}\cos t - \cos t)$

$= \frac{1}{8}(-e^{-t}\cos t - 2\sin t + \cos t - 2\sin t + 2e^{-t}\cos t - 2\cos t)$

(9) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t;\ x(0) = 0, x'(0) = 1$ **(BE 2057)**

Solution: Given different equation is,

$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t$

or, $x''(t) + 2x'(t) + 5x = e^{-t}\sin t$

Taking L. T. on both sides,

$L^{-1}[x''(t)] + 2x'(t) + 5x(t)] = L[e^{-t}\sin t]$

or, $L^{-1}[x''(t)] + L^{-1}[2x'(t)] + L[5x(t)]$

or, $s^2 L[x(t)] - s\,x(0) - x'(0) + 2s\,L[x(t)] - 2x(0) + 5L[x(t)] = \frac{1}{(s+1)^2+1^2}$

Using given condition,

$s^2 \times (s) - 0 - 1 + 2s \times (s) - 0 + 5 \times (s) = \frac{1}{(s+1)^2+1^2}$

or, $(s^2 + 2s + 5) \times (s) = \frac{1}{(s^2+2s+2)} + 1$

or, $X(s) = \frac{1 + (s^2+2s+2)}{(s^2+2s+2)(s^2+2s+5)}$

or, $X(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

Taking inverse L. T. on both sides we get,

$L^{-1}[X(s)] = L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$ .... (i)

(Exercise = 12 Q. No. 26)

$x(t) = \frac{1}{3}e^{-t}(\sin 2t + \sin t)$

10. $y'' + y' - 2y = t;\ y(0) = 1, y'(0) = 0$ **(BE 2060)**

Solution: Given differential equation is,

$y'' + y' - 2y = t;\ y(0) = 1, y'(0) = 0$

Taking L. T. on both sides we get,

$L[y'' + y' - 2y] = L[t]$

or, $L[y''(t)] + L[y'(t)] - 2[y(t)] = \frac{1}{s^2}$

or, $s^2 L[y(t)] - s\,y(0) - y'(0) + s\,L[y(t)] - y(0) - 2L[y(t)] = \frac{1}{s^2}$

using the given condition; $y(0) = 1,\ y'(0) = 0$

or, $s^2\, Y(s) - s \,.\, 1 - 0 + s\, Y(s) - 1 - 2\, Y(s) = \frac{1}{s^2}$

or, $(s^2 + s - 2)\, Y(s) = \frac{1}{s^2} + s + 1$

$= \frac{1 + s^3 + s^2}{s^2}$

$\therefore\ Y(s) = \frac{(s^3 + s^2 + 1)}{s^2 (s^2 + s - 2)}$

Taking inverse L. T. on both sides,

$L^{-1}[Y(s)] = L^{-1}\left[\frac{(s^3 + s^2 + 1)}{s^2 (s^2 + s - 2)}\right]$ ………(i)

Now, $\frac{(s^3 + s^2 + 1)}{s^2 (s+2)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s-1}$

$s^3 + s^2 + 1 = AS\,(s + 2)(s - 1) + B\,(s + 2)(s - 1) + C\,(s - 1)s^2 +$
$D\,(s + 2)\, s^2$

put $s = 1 \Rightarrow D = 1$

put $s = -2 \Rightarrow C = \frac{1}{4}$

put $s = 0 \Rightarrow B = -\frac{1}{2}$

Equating the coefficient of s^3;

$A + C + D = 1$

$A = 1 - \frac{1}{4} - 1$

$\Rightarrow A = -\frac{1}{4}$

$\therefore\ \frac{(s^3 + s^2 + 1)}{s^2 (s+2)(s-2)} = -\frac{1}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} + \frac{1}{s-1}$

Now (i) becomes

$\therefore\ L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s-1} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{4}\cdot\frac{1}{s}\right]$

$y(t) = L^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{4} L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{4} L^{-1}\left[\frac{1}{s}\right]$

$\therefore\ y(t) = e^t + \frac{1}{4} e^{-2t} - \frac{1}{2} t - \frac{1}{4}$

(11) $y'' + 4y' + 3y = t,\ t > 0;\ y(0) = 0,\ y'(0) = 1$

Solution: Given different eqution is,

$y'' + 4y' + 3y = t$

Taking L. T. on both sides,

$L[y'' + 4y' + 3y] = L[t]$

or, $L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = \frac{1}{s^2}$

or, $s^2 L[y(t)] - s\, y(0) - y'(0) + 4\, sL[y(t)] - 4$ [illegible] $3L[y(t)] = \frac{1}{s^2}$

Using the given condition.

$y(0) = 0,\ y'(0) = 1$

or, $s^2\, Y(s) - 0 - 1 + 4s\, Y(s) - 0 + 3\, Y(s) = \frac{1}{s^2}$

or, $(s^2 + 4s + 3)\, Y(s) = \frac{1}{s^2} + 1$

or, $Y(s) = \frac{(1 + s^2)}{s^2 (s^2 + 4s + 3)}$ ………(i)

Now, $\frac{1 + s^2}{s^2 (s^2 + 4s + 3)} = \frac{1 + s^2}{s^2 (s^2 + 3s + s + 3)} = \frac{1 + s^2}{s^2 \{s(s+3) + 1(s+3)\}}$

$= \frac{1 + s^2}{s^2 (s+3)(s+1)}$

$\therefore\ \frac{1 + s^2}{s^2 (s+3)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+3)} + \frac{D}{(s+1)}$

$1 + s^2 = As\,(s + 3)(s + 1) + B\,(s + 3)(s + 1) + Cs^2\,(s + 1) + Ds^2\,(s$
$+ 3)$

put $s = 0 \Rightarrow B = \frac{1}{3}$

put $s = -1 \Rightarrow D = 1$

put $s = -3 \Rightarrow C = -\frac{5}{9}$

Equating coefficient of s^3, $A + C + D = 0$

$A = \frac{5}{9} - 1 = -\frac{4}{9}$

$\frac{1 + s^2}{s^2 (s+3)(s+1)} = \frac{-4}{9}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s^2} - \frac{5}{9}\cdot\frac{1}{s+3} + \frac{1}{s+1}$

Now (i) becomes,

$Y(s) = -\frac{-4}{9}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s^2} - \frac{5}{9}\cdot\frac{1}{s+3} + \frac{1}{s+1}$

Taking inverse L. T. on both sides

$L^{-1}[Y(s)] = L^{-1}\left[-\frac{4}{9}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s^2} - \frac{5}{9}\cdot\frac{1}{s+3} + \frac{1}{s+1}\right]$

$\therefore\ y(t) = -\frac{4}{9} L^{-1}\left[\frac{1}{s}\right] + \frac{1}{3} L^{-1}\left[\frac{1}{s^2}\right] - \frac{5}{9} L^{-1}\left[\frac{1}{s+3}\right] + 1$

$= -\frac{4}{9} + \frac{1}{3}.t - \frac{5}{9} e^{-3t} + e^{-t}$

or, $y(t) = -\frac{4}{9} + \frac{1}{3}.t - \frac{5}{9}e^{-3t} + e^{-t}$

(12) $\mathbf{x'' - 2x' + x = e^t;\ x(0) = 2 \text{ and } x'(0) = -1}$ **(BE 2062)**

Solution: Given, differential equation is,

$x'' - 2x' + x = e^t$

Taking L. T. on both sides,

$L[x'' - 2x' + x] = L[e^t]$

or, $L[x''(t)] - 2L[x'(t)] + L[x(t)] = \frac{1}{s-1}$

or, $s^2 L[x(t)] - s\,x(0) - x'(0) - 2\,SL[x(t)] + 2x(0) + L[x(t)] = \frac{1}{s-1}$

Using the given condition,

$x(0) = 2,\ x'(0) = 1$

or, $s^2 X(s) - 2s + 1 - 2s\,X(s) + 4 + X(s) = \frac{1}{s-1}$

or, $(s^2 - 2s + 4)\,X(s) = \frac{1}{s-1} + 2s - 5$

or, $X(s) = \frac{1 + 2s^2 - 5s - 2s + 5}{(s-1)(s^2 - 2s + 4)}$

$= \frac{2s^2 - 7s + 6}{(s-1)(s-2)^2} = \frac{2s^2 - 4s - 3s + 6}{(s-1)(s-2)^2}$

$= \frac{2s(s-2) - 3(s-2)}{(s-1)(s-2)^2} = \frac{(2s-3)(s-2)}{(s-1)(s-2)^2} = \frac{2s-3}{(s-1)(s-2)}$ ......(i)

Now, $\frac{2s-3}{(s-1)(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-2)}$

or, $2s - 3 = A(s-2) + B(s-1)$

put $s = 2, \Rightarrow B = 1$

put $s = 1 \Rightarrow A = 1$

$\therefore \frac{2s-3}{(s-1)(s-2)} = \frac{1}{s-1} + \frac{1}{s-2}$

Taking inverse L. T. on both sides

$L^{-1}[x(s)] = L^{-1}\left[\frac{1}{s-1} + \frac{1}{s-2}\right]$

or, $x(t) = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right]$

$x(t) = e^t + e^{2t}$

(13) $\mathbf{\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = \sin t}$ **given that** $\mathbf{y(0) = y'(0) = 0}$

Solution: Given different equation is,

$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = \sin t$

or, $y''(t) + 2\,y'(t) + y(t) = \sin t$

Taking L. T. on both sides,

$L[y''(t) + 2y'(t) + y(t)] = L[\sin t]$

or, $L[y''(t)] + 2\,L[y'(t)] + L[y(t)] = \frac{1}{s^2+1}$

or, $s^2 L[y(t)] - s\,y(0) - y'(0) + 2\,s\,L[y(t)] - 2\,y(0) + L[y(t)] = \frac{1}{s^2+1}$

Using given condition, $y(0) = y'(0) = 0$

or, $s^2\,Y(s) - 0 - 0 + 2s\,Y(s) - 0 + Y(s) = \frac{1}{(s^2+1)}$

or, $(s^2 + 2s + 1)\,Y(s) = \frac{1}{(s^2+1)}$

or, $Y(s) = \frac{1}{(s^2+2s+1)(s^2+1)}$

$Y(s) = \frac{1}{(s^2+2s+1)(s^2+1)}$ .... (i)

Now, $\frac{1}{(s^2+2s+1)(s^2+1)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+1)}$

or, $1 = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$

put $s = -1; \Rightarrow B = \frac{1}{2}$

Put $s = 0,\ A + B + D = 1$, ... (i)

Equating the coefficient of s^3, $A + C = 0$

$\Rightarrow A = -C$ .... (ii)

Equating the coefficient of s^2,

$A + B + 2C + D = 0$ ...... (iii)

From (i) & (ii)

$C = -\frac{1}{2}$ & $A = \frac{1}{2}$

Again from (i) $\frac{1}{2} + \frac{1}{2} + D = 1$

$D = 0$

$\therefore \frac{1}{(s^2+2s+1)(s^2+1)} = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{1}{2}\frac{s}{(s^2+1)}$

Now (i) becomes,

$Y(s) = \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{1}{2}\frac{s}{(s^2+1)}$

Taking inverse L. T. on both sies,

$L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{1}{2}\frac{s}{(s^2+1)}\right]$

$$L^{-1}[Y(s) = \frac{1}{2} L^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{2} L^{-1}\left[\frac{s}{(s^2+1)}\right]$$

$$\therefore y(t) = \frac{1}{2} e^{-t} + \frac{1}{2} te^{-t} - \frac{1}{2} \cos t$$

14. $y'' - 3y' + 2y = 1 - e^{2t}, y(0) = 1, y'(0) = 0$ **(BE 2058)**

Solution: Given differential equation is,

$y'' - 3y' + 2y = 1 - e^{2t}$

Taking L. T. on both sides we get,

$L[y'' - 3y' + 2y] = L[1 - e^{2t}]$

or, $L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[1] - L[e^{2t}]$

or, $s^2 L[y(t)] - sy'(0) - y(0) - 3sL[y(t)] + 3y(0) + 2L[y(t)]$

$$= \frac{1}{s} - \frac{1}{(s-2)}$$

using the given condition $y(0) = 1, y'(0) = 0$

$$\text{or, } s^2 Y(s) - 0 - 1 - 3s\,Y(s) + 3 + 2\,Y(s) = \frac{s-2-s}{s(s-2)}$$

$$\text{or, } (s^2 - 3s + 2)\,Y(s) = -\frac{2}{s(s-2)} - 2$$

$$\text{or, } Y(s) = -2\,\frac{(1+s^2-2s)}{s(s-2)(s^2-3s+2)}$$

$$= -2\,\frac{(s-1)^2}{s(s-2)(s-1)(s-2)}$$

$$= -2\,\frac{(s-1)}{s(s-2)^2}$$

Taking inverse L. T. on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{2(s-1)}{s(s-2)^2}\right]$$

$$L^{-1}[Y(s)] = -2\,L^{-1}\left[\frac{(s-1)}{s(s-2)^2}\right] \quad \ldots\ldots \text{(i)}$$

$$\frac{s-1}{s(s-2)^2} = \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

$\therefore s - 1 = A(s-2)^2 + B(s-2)s + CS$

put $s = 2; \Rightarrow C = \frac{1}{2}$

put $s = 0; \Rightarrow A = -\frac{1}{4}$

Equating the coefficient of s^2

$\therefore A + B = 0$

$\Rightarrow B = \frac{1}{4}$

$$\therefore \frac{s-1}{s(s-2)^2} = -\frac{1}{4}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s-2} + \frac{1}{2}\cdot\frac{1}{(s-2)^2}$$

Now (i) becomes

$$L^{-1}[Y(s)] = -2L^{-1}\left[\frac{1}{4}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s-2} + \frac{1}{2}\frac{1}{(s-2)^2}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{s-2}\right] - L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$y(t) = \frac{1}{2} - \frac{1}{2} e^{2t} - te^{2t}$$

15. $(D^2 + 9)\,y = \cos 3t, t>0$ **given that** $y = 0, Dy = 0$ **at** $t = 0$

Solution: Given differential equation is,

$(D^2 + 9)\,y = \cos 3t, t > 0$, given that $y = 0, Dy = 0$ at $t = 0$

$$\text{or, } \frac{d^2y}{dt^2} + 9y = \cos 3t$$

or, $y''(t) + 9\,y(t) = \cos 3t$

Taking L. T. on both sides,

$L[y''(t) + 9y(t)] = L(\cos 3t)$

$$\text{or, } L[y''(t)] + 9L[y(t)] = \frac{s}{s^2+9}$$

using given condition $y = 0, Dy = 0$ i.e. $y'(0) = 0$ at $t = 0$

$$s^2\,Y(s) - 0 - 0 + 9Y(s) = \frac{s}{(s^2+9)}$$

$$\text{or, } (s^2+9)\,Y(s) = \frac{s}{(s^2+9)}$$

$$\therefore Y(s) = \frac{s}{(s^2+9)^2}$$

Taking inverse L. T. on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{s}{(s^2+9)^2}\right]$$

$$\text{Now, } L^{-1}\left[\frac{s}{(s^2+9)^2}\right] = f(t) \text{ (say)}$$

$$\therefore L[f(t)] = \frac{s}{(s^2+9)^2}$$

By using the theorem of L. T. we get,

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \frac{s}{(s^2+9)^2}\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+9)^2}\,ds$$

Putting $s^2 + 9 = y$ so that

$2s\,ds = dy$

$$= \frac{1}{2}\int \frac{dy}{y^2}$$

$$= \frac{1}{2}\left[-\frac{1}{y}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{s^2+9}\right]_s^{\infty}$$

$$= \frac{1}{2}\,\frac{1}{s^2+9}$$

$$\therefore \frac{f(t)}{t} = \frac{1}{2}L^{-1}\left[\frac{1}{(s^2+9)}\right] = \frac{1}{2}\cdot\frac{1}{3}\sin 3t$$

or, $f(t) = \frac{1}{6}\, t \sin 3t$

Hence, $L^{-1}\left[\frac{s}{s^2+9}\right] = \frac{1}{6}\, t \sin 3t$.

16. **$(D^4 - k^4)\,y = 0;\ y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0$ (BE 2058)**

Solution: Given differential equation is,

$(D^4 - k^4)\,y = 0;\ y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0$

or, $\frac{d^4y}{dt^4} - k^4 y = 0$

or, $y^{iv}(t) - k^4\,y(t) = 0$

Taking L. T. on both sides,

$L[y^{iv}(t) - k^4\,y(t)] = L[0]$

$s^4\,L[y(t)] - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0) - k^4\,L[y(t)] = 0$

Using given condition $y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0$ we get

$s^4\,Y(s) - s^3 - 0 - 0 - 0 - k^4\,Y(s) = 0$

or, $(s^4 - k^4)\,Y(s) = s^3$

or, $Y(s) = \frac{s^3}{(s^2+k^2)(s^2-k^2)}$

Taking inverse L. T. on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{s^3}{(s^2+k^2)(s^2-k^2)}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{s}{s^2+k^2} + \frac{s}{s^2-k^2}\right]$$

$$\therefore y(t) = \frac{1}{2}(\cos kt + \cosh kt)$$

17. **$y'' - 4y' + 3y = 6t - 8:\ y(0) = 9,\ y'(0) = 0$**

Solution: Given differential equation is,

$y'' - 4y' + 3y = 6t - 8$

Taking L. T. on both sides,

$L[y'' - 4y' + 3y] = L(6t - 8)$

or, $L[y''(t)] - 4L[y'(t)] + 3L[y(t)] = 6L[t] - 8L[1]$

or, $s^2\,L[y(t)] - s\,y'(0) - y(0) - 4s\,L[y(t)] + 4y(0) + 3L[y(t)] = \frac{6}{s^2} - \frac{8}{s}$

using the given condition, $y(0) = 0,\ y(0) = 0$

$s^2\,Y(s) - 0 - 0 - 4s\,Y(s) + 0 + 3Y(s)$

$$= \frac{6-8s}{s^2}$$

or, $(s^2 - 4s + 3)\,Y(s) = \frac{6-8s}{s^2}$

$$Y(s) = \frac{(6-8s)}{s^2(s^2-4s+3)}$$

$$Y(s) = \frac{6-8s}{s^2(s-3)(s-1)}$$

Taking inverse L. T. on both sides,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{6-8s}{s^2(s-3)(s-1)}\right] \quad \text{......... (i)}$$

Now, $\frac{6-8s}{s^2(s-3)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1}$

$\therefore 6 - 8s = A\,s(s-3)(s-1) + B(s-1)(s-3) + C(s-1)s^2 + D\,s^2(s-3)$

put $s = 1$, $\Rightarrow D = 1$

put $s = 3$, $\Rightarrow C = -1$

put $s = 0 \Rightarrow B = 2$

Equating the coefficient of s^3

$A + C + D = 0$

$\Rightarrow A = 0$

$$\therefore \frac{6-8s}{s^2(s-3)(s-1)} = \frac{2}{s^2} - \frac{1}{s-3} + \frac{1}{s-1}$$

Now (i) becomes,

$$\therefore L^{-1}[Y(s)] = L^{-1}\left[\frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s-3}\right]$$

$$\therefore y(t) = 2L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s-3}\right]$$

$$y(t) = 2t + e^t - e^{3t}$$

(18) **$y'' - 3y' + 2y = e^{3t};\ y(0) = 0,\ y'(0) = 1$**

Solution: Given different equation is,

$y'' - 3y' + 2y = e^{3t}$

Taking Laplace transform on both sides,

$L[y'' - 3y' + 2y] = L[e^{3t}]$

or, $L[y''(t)] - 3L[y'(t)] - 2L[y(t)] = \frac{1}{s-3}$

or, $s^2\,L[y(t)] - s\,y(0) - y'(0) - 3s\,L[y(t)] + 3y(0) + 2L[y(t)] = \frac{1}{s-3}$

Using given condition,

$y(0) = 0,\ y'(0) = 1$

or. $s^2\,Y(s) - 0 - 1 - 3s\,Y(s) + 0 + 2\,Y(s) = \dfrac{1}{s-3}$

or, $(s^2 - 3s + 2)\,Y(s) = \dfrac{1}{s-3}$

or, $Y(s) = \dfrac{1}{(s-3)(s^2-3s+2)}$

$= \dfrac{1}{(s-3)(s^2-2s-s+2)} = \dfrac{1}{(s-3)\{s.(s-2)-1(s-2)\}}$

$Y(s) = \dfrac{1}{(s-3)(s-1)(s-2)}$ .... (i)

Now, $\dfrac{1}{(s-3)(s-1)(s-2)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-2)} + \dfrac{C}{(s-3)}$

or, $1 = A\,(s-2)\,(s-3) + B\,(s-1)\,(s-3) + C\,(s-1)\,(s-2)$

put $s = 1 \Rightarrow A = \dfrac{1}{2}$

put $s = 2 \Rightarrow B = -1$

put $s = 3 \Rightarrow C = \dfrac{1}{2}$

Now, $\dfrac{1}{(s-1)(s-2)(s-3)} = \dfrac{1}{2}\cdot\dfrac{1}{s-1} - \dfrac{1}{s-2} + \dfrac{1}{2}\cdot\dfrac{1}{s-3}$

Now (i) becomes,

$Y(s) = \dfrac{1}{2}\cdot\dfrac{1}{s-1} - \dfrac{1}{s-2} + \dfrac{1}{2}\cdot\dfrac{1}{s-3}$

Taking inverse L. T. on both sides;

$L^{-1}[Y(s)] = L^{-1}\left[\dfrac{1}{2}\cdot\dfrac{1}{s-1} - \dfrac{1}{s-2} + \dfrac{1}{2}\cdot\dfrac{1}{s-3}\right]$

$= \dfrac{1}{2}\,L^{-1}\left[\dfrac{1}{s-1}\right] - L^{-1}\left[\dfrac{1}{s-2}\right] + \dfrac{1}{2}\,L^{-1}\left[\dfrac{1}{s-3}\right]$

$= \dfrac{1}{2}\,e^{t} - e^{2t} + \dfrac{1}{2}\,e^{3t}$

(19) $y''' + 2y'' - y' - 2y = 2e^{3t}$; $y(0) = 0,\ y'(0) = 0,\ y''(0) = 0$

Solution: Given different equation is,

$y''' + 2y'' - y' - 2y = 2e^{3t}$

Taking laplace transfors on both sides,

$L\,[y''' + 2y'' - y' - 2y] = L\,[2e^{3t}]$

or, $L\,[y'''(t)] + 2\,L\,[y''(t)] - L[y'(t)] - 2L\,[y(t)] = 2L[e^{3t}]$

or, $s^3\,L[y(t)] - s^2\,y(0) - sy'(0) - y''(0) + 2s^2\,L[y(t)] - 2sy(0)$

$- 2y'(0) - s\,L[y(t)] + y(0) - 2L\,[y(t)] = 2\,.\,\dfrac{1}{s-3}$

Using given condition,

$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0$

$s^3\,Y(s) - 0 - 0 - 0 + 2s^2\,Y(s) - 0 - 0 - sY(s) - 0 - 2Y(s) = \dfrac{2}{s-3}$

or. $(s^3 + 2s^2 - s - 2)\,Y(s) = \dfrac{2}{(s-3)}$

or, $Y(s) = \dfrac{2}{\{s^2(s-2) - 1(s-2)\}}\cdot\dfrac{1}{(s-3)}$

$Y(s) = \dfrac{2}{(s-1)(s-1)(s-2)(s-3)}$ .... (i)

Now, $\dfrac{2}{(s+1)(s-1)(s+2)(s-3)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-1)} + \dfrac{C}{(s+2)} + \dfrac{D}{(s-3)}$

$\therefore\ 2 = A\,(s-1)\,(s+2)\,(s-3) + B\,(s+1)\,(s+2)\,(s-3)$
$\quad + C\,(s+1)\,(s-1)\,(s-3) + D\,(s+1)\,(s-1)\,(s+2)$

put $s = 1 \Rightarrow B = -\dfrac{1}{6}$

put $s = -1 \Rightarrow A = \dfrac{1}{4}$

put $s = -2 \Rightarrow C = -\dfrac{2}{15}$

put $s = 3 \Rightarrow D = \dfrac{1}{20}$

$\therefore\ \dfrac{2}{(s+1)(s-1)(s+2)(s-3)} = \dfrac{1}{4}\cdot\dfrac{1}{s+1} - \dfrac{1}{6}\cdot\dfrac{1}{s-1} - \dfrac{2}{15}\cdot\dfrac{1}{s+2} + \dfrac{1}{20}\cdot\dfrac{1}{s-3}$

Now, (i) becomes,

$Y(s) = \dfrac{1}{4}\cdot\dfrac{1}{s+1} - \dfrac{1}{6}\cdot\dfrac{1}{s-1} - \dfrac{2}{15}\cdot\dfrac{1}{s+2} + \dfrac{1}{20}\cdot\dfrac{1}{s-3}$

Taking inverse L. T. on both sides,

$L^{-1}\,[Y(s)] = L^{-1}\left[\dfrac{1}{4}\cdot\dfrac{1}{s+1} - \dfrac{1}{6}\cdot\dfrac{1}{s-1} - \dfrac{2}{15}\cdot\dfrac{1}{s+2} + \dfrac{1}{20}\cdot\dfrac{1}{s-3}\right]$

or, $y(t) = \dfrac{1}{4}\,L^{-1}\left[\dfrac{1}{s+1}\right] - \dfrac{1}{6}\,L^{-1}\left[\dfrac{1}{s-1}\right] - \dfrac{2}{15}\,L^{-1}\left[\dfrac{1}{s+2}\right] + \dfrac{1}{20}\,L^{-1}\left[\dfrac{1}{s-3}\right]$

or, $y(t) = \dfrac{1}{4}\,e^{-t} - \dfrac{1}{6}\,e^{t} - \dfrac{2}{15}\,e^{-2t} + \dfrac{1}{20}\,e^{3t}$

(20) $y''' - y'' - 4y' + 4y = 0$; $y(0) = 0,\ y'(0) = 0,\ y''(0) = 2$

Solution: Given different equation is,

$y''' - y'' - 4y' + 4y = 0$ .... (i)

Taking L. T. on both sides,

$L\,[y''' - y'' - 4y' + 4y] = L\,[0]$

or, $L\,[y'''(t)] - L\,[y''(t)] - 4\,L[y'(t)] + 4L\,[y(t)] = 0$

or, $s^3\,L[y(t)] - s^2y(0) - sy'(0) - y''(0) - s^2\,L[y(t)] + sy(0) + y'(0)$

$- 4sL[y(t)] + 4y(0) + 4L\,[y(t)] = 0$

Using given condition,

$$s^3 Y(s) - 0 - 0 - 2 - s^2 Y(s) + 0 + 0 - 4 s Y(s) + 0 + 4 Y(s) = 0$$

or, $(s^3 - s^2 - 4s + 4) Y(s) = 2$

or, $Y(s) = \dfrac{2}{s^2(s-1) - 4(s-1)} = \dfrac{2}{(s-1)(s^2-4)}$

$= \dfrac{2}{(s-1)(s-2)(s+2)}$ .... (i)

Now, $\dfrac{2}{(s-1)(s-2)(s+2)} = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{s+2}$

or, $2 = A(s-2)(s+2) + B(s-1)(s+2) + C(s-1)(s-2)$

put $s = 2 \Rightarrow B = \dfrac{1}{2}$

put $s = -2 \Rightarrow C = \dfrac{1}{6}$

put $s = 1 \Rightarrow A = -\dfrac{2}{3}$

$\therefore \dfrac{2}{(s-1)(s-2)(s+2)} = -\dfrac{2}{3}\cdot\dfrac{1}{s-1} + \dfrac{1}{2}\cdot\dfrac{1}{s-2} + \dfrac{1}{6}\cdot\dfrac{1}{s+2}$

Now (i) becomes

$Y(s) = -\dfrac{2}{3}\cdot\dfrac{1}{s-1} + \dfrac{1}{2}\cdot\dfrac{1}{s-2} + \dfrac{1}{6}\cdot\dfrac{1}{s+2}$

Taking inverse L. T. on both sides,

$L^{-1}[Y(s)] = L^{-1}\left[-\dfrac{2}{3}\cdot\dfrac{1}{s-1} + \dfrac{1}{2}\cdot\dfrac{1}{s-2} + \dfrac{1}{6}\cdot\dfrac{1}{s+2}\right]$

$= -\dfrac{2}{3} L^{-1}\left[\dfrac{1}{s-1}\right] + \dfrac{1}{2} L^{-1}\left[\dfrac{1}{s-2}\right] + \dfrac{1}{6} L^{-1}\left[\dfrac{1}{s+2}\right]$

orm $y(t) = -\dfrac{2}{3} e^{t} + \dfrac{1}{2} e^{2t} + \dfrac{1}{6} e^{-2t}$

$= \dfrac{1}{2} e^{2t} - \dfrac{2}{3} e^{t} + \dfrac{1}{6} e^{-2t}$

Linear programming problem (LPP) 4

1. **Introduction :** It is the optimization (minimization/ maximization) of a linear function (known as objective function) under a set of linear inequalities (known as constraints). All variables in an LPP are non negative i.e. can not take negative values.

2. **General form of LPP :** A general LPP can be written as,
Maxmimize [minimize], $Z = C_1x_1 + C_2x_2 + C_3x_3$
Subject to, $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le, \ge, = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le, \ge, = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le, \ge, = b_3$
$x_1, x_2, x_3 \ge 0$
i.e. $Z = C^TX$
$AX \le, \ge, = b$
Where, $C = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

3. **Standard LPP :** An LPP of the form
Maximize, $Z = C_1x_1 + C_2x_2 + C_3x_3$
Subject to, $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le b_3$
$x_1, x_2, x_3 \ge 0$, is know as standard LPP

4. **Characteristics of standard LPP :** A standard LPP has following characteristics,
   (i) Objective function should be in "maximize" form
   (ii) All inequalities should be in the form "$\le$".
   (iii) All choise variables $(x_1, x_2, \ldots\ldots x_n)$ must be non-negative.
   (iv) Constants in RHS of all inequalities must be positive.

5. **Mini-max theorem :** If an objective function z takes minimum value W for some value of the variables, then '$-W$' takes maximum value for those values and $W = -Z$.
i.e. Minimum (Z) = – Maximum (W) where $W = -Z$.
This theorem is applied if the objective function is not in the standard form.

6. **Some terminologies**

(i) **Slack variables :** The variables that are added to the inequalities to make LHS & RHS equal with the constant in right hand side, are known as slack variables.

(ii) **Basic variables :** The variables that has only one non-zero element in its column in the simplex table, mark that the number of basic variables in each step of simplex table, are same.

(iii) **Surplus variable :** The variables that are subtracted from LHS to make it equal to RHS are known as surplus variables.

7. **Fundamental principles of simplex method**

(i) The values of basic variables are determined from the constant column of the table.

(ii) The values of non-basic variables in each step are directly set to zero i.e. made zero.

8. **Simplex method for an LPP in standard form**

Algorithms

(i) Cheeck the LPP is in standard form or not.

(ii) Write each of the inequalities in equal form, using suitable (slack, surplus, artificial) variables.

(iii) Make each relations (objective function and inequalities) in complete form (that can be made by using 0 coefficient for the absent variables.)

(iv) Write first simplex table using coefficients of each of the variables.

(v) Select the column with most negative number in first row and mark the column of the number as pivot column.

(vi) Find the ratio of numbers in the constant column written numbers in pivot column the ratio $\frac{b_i}{a_{ij}}$, where b_i is the number in constant column and, a_{ij}, is the number in the corresponding row of b_i along pivot column. The row with smallest non-negative ratio is taken as pivot row.

(vii) Mark the number that lies in pivot row and column as pivot element and apply row operations to make the pivot column variable as basic.

Dual of an LPP

Let the given LPP (primal LPP) be,

Minimize (maximize) $Z = C_1x_1 + C_2x_2 + C_3x_3$

Subject to, $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \geq b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \geq b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \geq b_3$

$x_1, x_2, x_3 \geq 0$

Then an LPP defined by,

Maximize (Minimize), $D = b_1y_1 + b_2y_2 + b_3y_3$

Subject to, $a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \leq c_1$

$a_{12}y_1 + a_{22}y_2 + a_{32}y_3 \leq c_2$

$a_{13}y_1 + a_{23}y_2 + a_{33}y_3 \leq c_3$

$y_1, y_2, y_3 \geq 0$, is known as dual of the given LPP.

Formation of dual : Let the primal LPP (above) be represented in matrix form as,

$$A = \left[\begin{array}{c|ccc} 1 & C_1 & C_2 & C_3 \\ \hline b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{21} & a_{22} & a_{23} \\ b_3 & a_{31} & a_{32} & a_{33} \end{array}\right]$$

Where first column is the other side (not in the side x_1, x_2, x_3) column of the objective function and constrant of the inequalities. Taking the transpose of A, we have,

$$A^T = \left[\begin{array}{c|ccc} 1 & b_1 & b_2 & b_3 \\ \hline c_1 & a_{11} & a_{21} & a_{31} \\ c_2 & a_{12} & a_{22} & a_{32} \\ c_3 & a_{13} & a_{23} & a_{33} \end{array}\right]$$

∴ The dual can be written as,

$D = b_1y_1 + b_2y_2 + b_3y_3$

Subject to,

$a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \leq c_1$

$a_{12}y_1 + a_{22}y_2 + a_{33}y_3 \leq c_2$

$a_{13}y_1 + a_{23}y_2 + a_{33}y_3 \leq c_3, y_1, y_2, y_3 \geq 0.$

9. **Fundamental theorem of dual :** An LPP (primal) has a solution, if and only if, its dual has an optimal Solution: Moreover, the optimal solution of primal LPP is same as optimal solution of its dual.

A consequence of dual theorem is that, the slack variables of dual LPP are same as choice variables of primal LPP (i.e. x_1, x_2, x_3 .....) and their values fall directly in the first row of last simplex table of the dual LPP.

Simplex method for LPP with mixed constraints (Use of artificial variables)

'a_i', – 'Ma_i'

(a) **Big-M method**

For each inequalities containing '≥' or '=' an artificial variable is added. For each artificial variable 'a_i', '–Ma_i' is added to the objective function (know as penalty). The artificial variables are removed from objective function substituting their values from their respective inequalities. We can then apply the usual standa[rd] method to solve it.

(b) Two phase method

In two phase method also, we use artifical variable for the inequalities of the form '≥', '=' and the problem is solved in two phases (steps).

Phase I: In phase I we solve a new objective function, known as auxiliary objective function. Which is formed by using artificial variables only, and is given by,

Maximize, $A = -a_1 - a_2$ .... each artificial

Variables are used in auxiliary function.

It is to be noted that this auxiliary objective function, must have maximum value '0' in order to the given LPP hae an optimal Solution: The step in which maximum value of A is '0' completes the first phase.

Phase II: We rewrite the last table of phase I, for the given objective function (Z), and apply the ordinary process of standard simplex method.

Exercise - 15

1. A company manufacturers of two type of clothes, using three different colors of wool. One yard length of the type A cloth requires 4 oz. of red wool, 5 oz. of green wool and 3 oz of yellow wool. One yard length of type B cloth requires 5 oz. of red wool, 2 oz of green wool and 8 oz of yellow wool. The wool available for manufacturer is 1000 oz. of red wool, 1000 of oz of green wool and 1200 oz. of yellow wool. The manufacturer can make a profit Rs. 5 on one yard of type A cloth and Rs.3 on one yard of type B. Formulate the best combination of quantities of the type A and type B cloth which gives him maximum profit and solve it graphically.

Solution :

Let x and y yards of cloth A and B be produced. Also, profit in one yard of A and B are Rs. 5 and Rs. 3 respectively, then profit function, ρ is given by,

$$\rho = 5x + 3y$$

The other information's can be written in tabular from as,

	Cloth A x yard	Cloth B y yard	Total (oz)
Red wool	4	5	1000
Green wool	5	2	1000
Yellow wool	3	8	1200

The red wool constraint, $4x + 5y \le 1000$

The green wool constraint, $5x + 2y \le 1000$

The yellow wool constraint, $3x + 8y \le 1200$

The above production problem can be represented as an LPP as.

Manimize, $\rho = 5x + 3y$

Subject to, $4x + 5y \le 1000$ .... (i)

$5x + 2y \le 1000$ .... (ii)

$3x + 8y \le 1200$ .... (iii)

The boundary lines for the solution set of the in equations are,

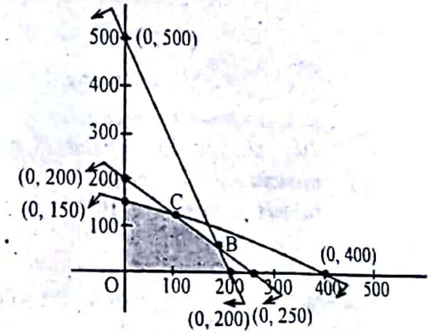
from (i) $4x \quad = 1000$

$$\therefore \frac{x}{250} + \frac{y}{200} = 1$$

From (ii), $\frac{x}{200} + \frac{y}{250} = 01$

and from (iii) $\frac{x}{400} + \frac{y}{150} = 1$

The graph of boundary lines are shown in figure, Taking (0, 0) as testing point for each in equalities, the solutions lie towards origin in each case. The values of the objective (profit) function at different vertices of the convex polygonal (feasible) region are as follows,



$\rho(0, 0) = 0$

$\rho(200, 0) = 5 \times 200 + 3 \times 0 = 1000$

$$\rho\left(\frac{3000}{17}, \frac{1000}{17}\right) = 5 \times \frac{3000}{17} + \frac{3 \times 1500}{17}$$

$$= \frac{14000}{17} + \frac{4500}{17}$$

$$= \frac{18500}{17}$$

$\rho(0, 150) = 0 + 3 \times 150 = 450$

Thus, ρ has maximum value $\frac{18500}{17}$ at $x = \frac{2800}{17}, y = \frac{1500}{17}$

(2) Solve following Lpp by graphical method,

(i) maximize, $z = 5x_1 + 3x_2$

subject to, $x_1 + x_2 \le 6$

$x_1 - x_2 \le 4, \; x_1, x_2 \ge 0$

Solution: The boundary lines of solution set of given Lpp are,

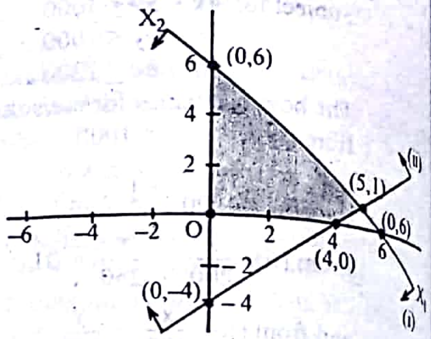
$x_1 + x_2 = 6$ .... (i)

$\therefore \frac{x_1}{6} + \frac{x_2}{6} = 1$

and, $x_1 - x_2 = 4$ .... (ii)

$\therefore \frac{x_1}{4} + \frac{x_2}{-4} = 1$

The graphs of boundary lines are shown is figure. Taking (0, 0) as testing points for each inequalities, the solutions lie towards origin.



The values of objective function at the vertices of feasible region are as follows, (the vertices are obtained by solving the intersecting lines).

$z(0, 0) = 0$

$z(4, 0) = 5 \times 4 + 3 \times 0 = 20$

$z(5, 1) = 5 \times 5 + 3 \times 1 = 28$

$z(0, 6) = 0 + 3 \times 6 = 18$

Thus, z has maximum value 28 at $x_1 = 5$ and $x_2 = 1$

(ii) minimize, $z = 2x_1 + x_2$

subject to, $x_1 + 2x_2 \le 10$

$x_2 \le 4$

$x_1 + x_2 \ge 1$, $x_1, x_2 \ge 0$

Solution: The boundary lines for the solution set of inequalities are,

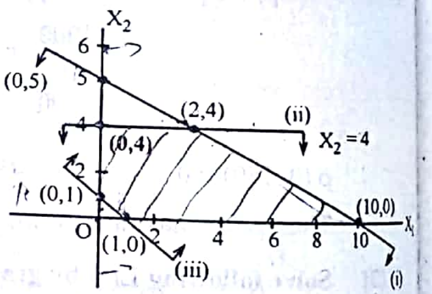
$x_1 + 2x_2 = 10$ .... (i)

$\therefore \frac{x_1}{10} + \frac{x_2}{5} = 1$

$x_2 = 4$ ...... (ii)

and $x_1 + x_2 = 1$ .... (iii)

The graphs of boundary lines are shown in figure. Taking (0, 0) as testing point for inequalities (i) & (ii) the solutions lie towards origine. But the solution for (iii) lies aways from origin. The values of the objective function at different vertices are as follows,



$z(1, 0) = 2 \times 1 + 0 = 2$

$z(10, 0) = 2 \times 10 + 0 = 20$

$z(2, 4) = 2 \cdot 2 + 4 = 8$

$z(0, 4) = 0 + 4 = 4$

$z(0, 1) = 0 + 1 = 1$

Thus, z has minimum value '1' at $x_1 = 0$ and $x_2 = 1$

(iii) maximige, $z = 5x_1 + 3x_2$

subject to, $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$, $x_1, x_2 \ge 0$

Solution: The boundary lines for the solution set are,

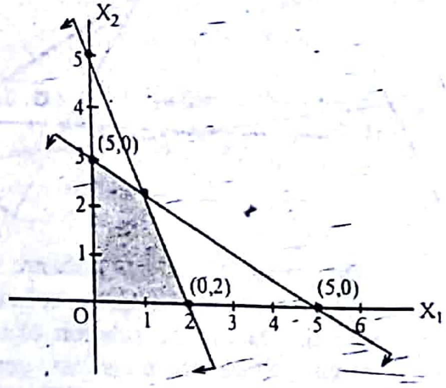
$3x_1 + 5x_2 = 15$

$\therefore \frac{x_1}{5} + \frac{x_2}{3} = 1$

and $5x_1 + 2x_2 = 10$

$\therefore \frac{x_1}{2} + \frac{x_2}{5} = 1$

The graphs of boundary lines are shown in figure. Also, taking (0, 0) as testing point for both the inequalities, the solutions lie, towards origin.



The values of objective function at different vertices are as follows,

$z(0, 0) = 0$

$z(2, 0) = 5 \times 2 + 0 = 10$

$z\left(\frac{20}{19}, \frac{45}{19}\right) = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} = \frac{235}{19}$

Thus z has maximum value $\frac{235}{19}$ at $\left(\frac{20}{19}, \frac{45}{19}\right)$.

(iv) minimize, $z = 20x_1 + 10x_2$

subject to, $x_1 + 2x_2 \le 40$

$3x_1 + x_2 \ge 30$

$4x_1 + 3x_2 \ge 60$, $x_1, x_2 \ge 0$

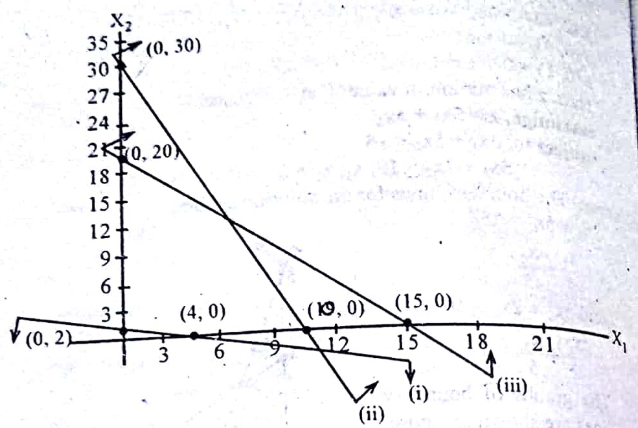
Solution: The boundary lines for the solution set are,

$x_1 + 2x_2 = 4$ .... (i) $\quad \therefore \frac{x_1}{4} + \frac{x_2}{2} = 1$

$3x_1 + x_2 = 30$ .... (ii) $\quad \therefore \frac{x_1}{10} + \frac{x_2}{30} = 1$

and, $4x_1 + 3x_2 = 60$ .... (iii)

$\frac{x_1}{15} + \frac{x_2}{20} = 1$



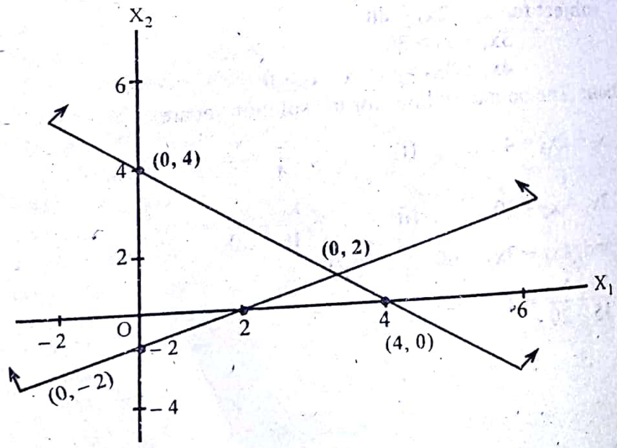
The graphs of boundary lines are shown in figure.
Taking (0, 0) as testing, the solution of first inequality lies towards origin, whereas the solution of second and third lies away from origin. Since there is no common solution region of the inequalities, so it has no Solution:

(v) maximize, $z = 2x_1 + 3x_2$
subject to, $x_1 - x_2 \le 2$
$x_1 + x_2 \ge 4$, $x_1, x_2 \ge 0$

Solution: The boundary lines for the solution set are,

$x_1 - x_2 = 2$ $\quad\therefore \frac{x_1}{2} + \frac{x_2}{2} = 1$ .... (i)

& $x_1 + x_2 = 4$ $\quad\therefore \frac{x_1}{4} + \frac{x_2}{4} = 1$ .... (ii)



The graphs of boundary lines are shown in figure.
Taking (0, 0) as testing point, the solution of first inequality lies towards origin and the solution of second inequality lies away from origin. As the region is not bounded, Its maximum value can not be determined.

(3) Solve the following linear programming problem by simplex method.

(i) Maximize $z = x_1 + 3x_2$
Subject to $x_1 + 2x_2 \le 10$
$x_1 \le 5$
$x_2 \le 4$ $x_1, x_2 \ge 0$

Solution: Using black variables s_1, s_2 and s_3 given Lpp can be written in the normal form as,

$z - x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 = 0$
$x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 10$
$x_1 + 0x_2 + 0s_1 + s_2 + 0s_3 = 5$
$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 4$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	−1	−3	0	0	0	0	
s_1	0	1	2	1	0	0	10	$\frac{10}{2}$
s_2	0	1	0	0	1	0	5	$\frac{5}{10}$
s_3	0	0	$\boxed{1}$	0	0	1	5	$\frac{4}{1}$ → exit

Basic, $s_1 = 10$, $s_2 = 5$, $s_3 = 4$ and non-basic $x_1 = x_2 = 0$ for this $z = 0$.
x_2-column is pivot column and R_4 is pivot row, so 1 is pivot element.
Applying $R_1 \to R_1 + 3R_4$, $R_2 \to R_2 - 2R_4$, $R_3 \to R_3$
The next simplex table is,

Basic	z	x_1 → Extrant	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	−1	0	0	0	3	12	
s_1	0	$\boxed{1}$	0	1	0	−2	2	$\frac{2}{1}$ → exit
s_2	0	1	0	0	1	0	5	$\frac{5}{1}$
x_2	0	0	1	0	0	1	4	$\frac{4}{0}$

Basic, $s_1 = 2$, $s_2 = 5$, $x_2 = 4$ and non-basic $x_2 = s_3 = 0$. For this $z = 12$.

Also, x_1-column is pivot column and R_2 is pivot row, so 1 is pivot element.

Applying, $R_1 \to R_1 + R_2$, $R_3 \to R_3 - R_2$, $R_4 \to R_4$

The next simplex table is,

Basic	z	x_1	x_2	s_1	s_2	s_3	b
	1	0	0	1	0	1	14
	0	1	0	1	0	–2	2
	0	0	0	–1	1	2	3
	0	0	1	0	0	1	4

Since there are no negative numbers in first row, so simplex process is completed, from table max-z = 14 at $x_1 = 2$ and $x_2 = 4$

(ii) Maximize $z = 10x_1 + x_2 + 2x_3$

Subject to, $x_1 + x_2 - 2x_3 \leq 10$

$4x_1 + x_2 + x_3 \leq 20$, $x_1, x_2, x_3 \geq 0$

Solution: Using s_1 and s_2 as slack variables, given LPP can be written in the normal form as,

$z - 10x_1 - x_2 - 2x_3 + 0s_1 + 0s_2 = 0$

$x_1 + x_2 - 2x_3 + s_1 + 0s_2 = 10$

$4x_1 + x_2 + x_3 + 0s_1 + s_2 = 20$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	– 10	– 1	–2	0	0	0	
s_1	0	1	1	–2	1	0	10	$\frac{10}{1}$
s_2	0	$\boxed{4}$	1	1	0	1	20	$\frac{20}{4} \to$ exit

Basic, $s_1 = 10$, $s_2 = 20$

Non-basic $x_2 = x_2 = x_3 = 0$, for this $z = 0$

Also, x_1-column is pivot column and R_3 is pivot row, so '4' is pivot element.

Applying $R_1 \to R_1 + \frac{5}{2} R_3$, $R_2 \to R_2 - \frac{1}{4} R_3$

The next simplex table is,

Basic	z	x_1	x_2	x_3	s_1	s_2	b
	1	0	$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{5}{2}$	50
s_1	0	0	$\frac{3}{4}$	$-\frac{9}{4}$	1	$-\frac{1}{4}$	5
x_1	0	4	1	1	0	1	20

Since there are no negative numbers, so simplex process is completed. Hence from table, max. z = 50 at $x_1 = 5$, $x_2 = 0$, $x_3 = 0$

(iii) Maximize, $z = x_1 + 3x_2 + 5x_3$

Subject to, $x_1 + 2x_2 \leq 10$

$3x_2 + x_3 \leq 24$, $x_1, x_2, x_3 \geq 0$

Solution: Using s_1 and s_2 as black variables, given LPP can be written as.

$z - x_1 - 3x_2 - 5x_3 + 0s_1 + 0s_2 = 0$

$x_1 + 2x_2 + 0x_3 + s_1 + 0s_2 = 10$

$0x_1 + 3x_2 + x_3 + 0s_1 + s_2 = 24$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	x_3	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	–1	–3	–5	0	0	0	
s_1	0	1	2	0	1	0	10	$\frac{10}{0}$
s_2	0	0	3	$\boxed{1}$	0	1	24	$\frac{24}{1} \to$ exit

Basic, $s_1 = 10$, $s_2 = 24$

Non-baisc, $x_1 = x_2 = x_3 = 0$, for this $z = 0$

Also, x_3-column is pivot column and R_3 is pivot row, so '1' is pivot element.

Applying, $R_1 \to R_1 + 5R_3$, $R_2 \to R_2$

The next simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	x_3	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	–1	12	0	0	5	120	
s_1	0	$\boxed{1}$	2	0	1	0	10	$\frac{10}{1} \to$ exit
x_3	0	0	3	1	0	1	24	$\frac{24}{0}$

Basic, $s_1 = 10$, $x_3 = 24$

Non-baisc $x_1 = x_2 = s_2 = 0$, for this $z = 120$

Also, x_1-column is pivot column and R_2 is pivot row, so '1' is pivot element.

Applying, $R_1 \to R_1 + R_2$, $R_3 \to R_3$, the next simplex table is,

Basic	z	x_1	x_2	x_3	s_1	s_2	b
	1	0	14	0	1	5	130
	0	1	2	0	1	0	10
	0	0	3	1	0	1	24

Since there are no negative numbers in first row, so simplex process is completed.

Also, max-z = 130 at $x_1 = 10$, $x_2 = 0$, $x_3 = 24$

(iv) Maximize, $z = 15x_1 + 10x_2$ **(correct the question, $2x_1 + 2x_2 \leq 10$)** **(BE 2057)**

Subject to, $2x_1 + x_2 \leq 10$

$x_1 + 3x_2 \leq 10$, $x_1, x_2 \geq 0$

Solution: Using black variables s_1 and s_2, given Lpp can be written in the normal form as,

$$z - 15x_1 - 10x_2 + 0s_1 + 0s_2 = 0$$
$$2x_1 + x_2 + s_1 + 0s_2 = 10$$
$$x_1 + 3x_2 + 0s_1 + s_2 = 10$$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	–15	–10	0	0	0	
s_1	0	$\boxed{2}$	1	1	0	10	$\frac{10}{2} \to$ exit
s_2	0	1	3	0	1	10	$\frac{10}{1}$

Basic, $s_1 = 10$, $s_2 = 10$

Non-basic, $x_1 = x_2 = 0$, for this $z = 0$

Also, x_1-column is pivot column and R_2 is pivot row, so '2' is pivot element.

Applying $R_1 \to R_1 + \frac{15}{2} R_2$, $R_3 \to R_3 - \frac{1}{2} R_2$

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	0	$-\frac{5}{2}$	$\frac{15}{2}$	0	75	
s_1	0	2	1	1	0	10	$\frac{10}{1}$
s_2	0	0	$\boxed{\frac{5}{2}}$	$-\frac{1}{2}$	1	5	$\frac{5}{2} \to$ exit

Basic, $x_1 = 5$, $s_2 = 5$

Non-baisc, $x_2 = s_1 = 0$, for this $z = 75$

x_2-column is pivot column and R_3 is pivot now, so $\frac{5}{2}$ is pivot element.

Applying $R_1 \to R_1 + R_3$, $R_2 \to R_2 - \frac{5}{2} R_3$

the next simplex table is,

Basic	z	x_1	x_2	s_1	s_2	b
	1	0	0	7	1	80
x_1	0	2	0	$\frac{6}{5}$	$-\frac{2}{5}$	8
x_2	0	0	$\frac{5}{2}$	$-\frac{1}{2}$	1	5

Since, there are no negative terms in first row, so simplex process is completed.

From table, max-z = 80 at $x_1 = 4$ and $x_2 = 2$

(v) **Minimize, $z = 5x_1 - 20x_2$**

Subject to $-2x_1 + 10x_2 \leq 5$

$$2x_1 + 5x_2 \leq 10, \quad x_1, x_2 \geq 0$$

Solution: Using mini-max theorem the objective function can be written as, [min (z) = – max (w), Where w = – z]

Maximize $w = -z = -5x_1 + 20x_2$

Subject to $-2x_1 + 10x_2 \leq 5$

$$2x_1 + 5x_2 \leq 10$$

Using slack variables s_1 and s_2, it can be written in normal form as,

$$w + 5x_1 - 20x_2 + 0s_1 + 0s_2 = 0$$
$$-2x_1 + 10x_2 + s_1 + 0s_2 = 5$$
$$2x_1 + 5x_2 + 0s_1 + s_2 = 10$$

The first simplex table is,

Basic	w	x_1	$\downarrow x_2$ Entrant	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	5	–20	0	0	0	
s_1	0	–2	$\boxed{10}$	1	0	5	$\frac{5}{10} \to$ exit
s_2	0	2	5	0	1	10	$\frac{10}{5}$

Basic, $s_1 = 5$, $s_2 = 10$ and non-basic $x_1 = x_2 = 0$, for this $w = 0$

Also, x_2-column is pivot column and R_2 is pivot row, so '10' is pivot element.

Applying $R_1 \to R_1 + 2R_2$, $R_3 \to R_3 - \frac{1}{2} R_2$

The next simplex table is,

Basic	w	x_1 $\to$ Entrant	x_2	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	1	0	2	0	10	
s_1	0	–2	10	1	0	5	
s_2	0	3	0	$-\frac{1}{2}$	1	$\frac{15}{2}$	

Since there are no negative numbers in first row, so simplex process is completed from table max w = 10, at $x_1 = 0$, $x_2 = \frac{1}{2}$.

Hence by minimax theorem,

min z = – max (w) = – 10 at $x_1 = 0$, $x_2 = \frac{1}{2}$

(vi) **Maximize, $z = 5x_1 + x_2$**

Subject to, $x_1 + x_2 \leq 2$

$$3x_1 + 8x_2 \leq 12, \quad x_1, x_2 \geq 0$$

Solution: Using slack variables s_1, s_2 and s_3, given Lpp can be, written in normal form as,

$$z - 5x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$$
$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$$
$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12$$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	-5	-1	0	0	0	0	
s_1	0	$\boxed{1}$	1	1	0	0	2	$\frac{2}{1} \rightarrow$ exit
s_2	0	5	2	0	1	0	10	$\frac{10}{5}$
s_3	0	3	8	0	0	1	12	$\frac{12}{3}$

Basic, $s_1 = 2$, $s_2 = 10$, $s_3 = 12$ non-basic $x_1 = x_2 = 0$, for this $z = 0$
x_1-column is pivot column and R_2 (we can also take R_3 as well) is pivot row, So '1' is pivot element.

Applying $R_1 \rightarrow R_1 + 5R_2$, $R_3 \rightarrow R_3 - 5R_2$, $R_4 \rightarrow R_4 - 3R_2$

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	0	4	5	0	0	10	
s_1	0	1	1	1	0	0	2	
s_2	0	0	-3	-5	1	0	2	
s_3	0	0	5	-3	0	1	6	

Since, there are no negative numbers in first row, so simplex process is completed. From table, max-z = 10, $x_1 = 2$, $x_2 = 0$

(vii) Maximize, $z = 30x_1 + 40x_2$
Subject to, $6x_1 + 12x_2 \leq 120$
$8x_1 + 5x_2 = 60$, x_1, x_2

Solution: [We will be such problems in next exercise]

(viii) Maximize, $z = 10x_1 + 30x_2 + 20x_3$
Subject to, $2x_1 - 3x_2 - x_3 \leq 8$
$50x_1 + 10x_2 + 20x_3 \leq 400$, $x_1, x_2, x_3 \geq 0$

Solution: Using s_1, s_2, s_3 as slack variables, given Lpp can be written in normal form as,

$$z - 10x_1 - 30x_2 - 20x_3 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$2x_1 + 3x_2 - x_3 + s_1 + 0s_2 + 0s_3 = 8$$
$$5x_1 - 3x_2 + 6x_3 + 0s_1 + s_2 + 0s_3 = 40$$
$$50x_1 + 10x_2 + 20x_3 + 0s_1 + 0s_2 + s_3 = 400$$

The first simplex table is,

Basic	z	x_1	$\downarrow x_2$ Entrant	x_3	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	-10	-30	-20	0	0	0	0	
s_1	0	1	$\boxed{3}$	-1	1	0		8	$\frac{8}{3} \rightarrow$ exit
s_2	0	5	-3	6	0	1	0	40	$\frac{40}{-3}$
s_3	0	50	10	20	0	0	1	400	$\frac{400}{10}$

Basic, $s_1 = 8$, $s_2 = 40$, $s_3 = 400$, non-basic $x_1 = x_2 = x_3 = 0$, for this $z = 0$
x_2-column is pivot column and R_2 is pivot row and so 3 is pivot element.

Applying $R_1 \rightarrow R_1 + 10R_2$, $R_3 \rightarrow R_3 + R_2$, $R_4 \rightarrow R_4 - \frac{10}{3} R_2$

The next simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	x_3	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	10	0	-30	10	0	0	80	
s_1	0	2	3	-1	1	0	0	8	$\frac{8}{-1}$
s_2	0	7	0	$\boxed{5}$	1	1	0	48	$\frac{48}{5} \rightarrow$ exit
s_3	0	$\frac{130}{3}$	0	$\frac{70}{3}$	$-\frac{10}{3}$	0	1	$\frac{1120}{3}$	$\frac{1120}{70}$

Basic, $x_2 = \frac{8}{3}$, $s_2 = 48$, $s_3 = \frac{1120}{3}$ and non-basic $x_1 = x_3 = s_1 = 0$ for this $z = 80$.
x_3-column is pivot column and R_3 is pivot row, so '5' is pivot element.

Applying $R_1 \rightarrow R_1 + 6R_3$, $R_2 \rightarrow R_2 + \frac{1}{5} R_3$, $R_4 \rightarrow R_4 - \frac{14}{3} R_3$

The next simplex table is,

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	52	0	0	16	6	0	368	
	0	$\frac{17}{5}$	3	0	$\frac{1}{3}$	$\frac{1}{5}$	0	$\frac{88}{15}$	
	0	7	0	5	1	1	0	48	
	0	$\frac{44}{15}$	0	0	$-\frac{24}{3}$	$-\frac{14}{3}$	1	147	

Since there are no negative terms in the first row, so simplex process is completed, from table, max-z = 368 at $x_1 = 0$, $x_2 = \frac{88}{15}$, $x_3 = \frac{48}{5}$

(ix) Maximize, $z = 4x_1 + 3x_2$
Subject to, $2x_1 + 3x_2 \leq 6$

$$-x_1 + 2x_2 \leq 3$$
$$2x_2 \leq 5$$
$$2x_1 + x_2 \leq 4,\ x_1, x_2 \geq 0$$

Solution: Using black variables s_1, s_2, s_3 and s_4, given Lpp can be written in the normal form as,

$$z - 4x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 = 0$$
$$2x_1 + 3x_2 + s_1 + 0s_2 + 0s_3 + 0s_4 = 6$$
$$-x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 + 0s_4 = 3$$
$$0x_1 + 2x_2 + 0s_1 + 0s_2 + s_3 + 0s_4 = 5$$
$$2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 + s_4 = 4$$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	s_3	s_4	b	$\frac{b_i}{a_{ij}}$
	1	–4	–3	0	0	0	0	0	
s_1	0	2	3	1	0	0	0	6	$\frac{6}{2}$
s_2	0	–1	2	0	1	0	0	3	$\frac{3}{-1}$
s_3	0	0	2	0	0	1	0	5	$\frac{5}{0}$
s_4	0	$\boxed{2}$	1	0	0	0	1	4	$\frac{4}{2} \to$ exit

Basic, $s_1 = 6$, $s_2 = 3$, $s_3 = 5$, $s_4 = 4$ and non-basic $x_1 = x_2 = 0$, for this $z = 0$. Also, x_1-column is pivot column and R_5 is pivot row and so 2 is pivot element.

Applying $R_1 \to R_1 + 2R_5$, $R_2 \to R_2 - R_5$, $R_3 \to R_3 + \frac{1}{2} R_5$

$R_4 \to R_4$. The next simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	s_3	s_4	b	$\frac{b_i}{a_{ij}}$
	1	0	–1	0	0	0	2	8	
s_1	0	0	$\boxed{2}$	1	0	0	–1	2	$\frac{2}{2} \to$ exit
s_2	0	0	$\frac{5}{2}$	0	1	0	$\frac{1}{2}$	5	2
s_3	0	0	2	0	0	1	0	5	$\frac{5}{2}$
s_4	0	2	1	0	0	0	1	4	$\frac{4}{1}$

Basic, $s_1 = 2$, $s_2 = 5$, $s_3 = 5$, $x_1 = 4$ and non-basic $x_2 = s_4 = 0$, for this $z = 8$. Also, x_2-column is pivot column and R_2 is pivot row, so '2' is pivot element.

Applying $R_1 \to R_1 + \frac{1}{2} R_2$, $R_3 \to R_3 - \frac{4}{5} R_2$, $R_4 \to R_4 - R_2$, $R_5 \to R_5 - \frac{1}{2} R_2$

The next simplex table is,

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	b
	1	0	0	$\frac{1}{2}$	0	0	$\frac{3}{2}$	9
s_1	0	0	2	1	0	0	–1	2
s_2	0	0	0	$-\frac{5}{4}$	1	0	$\frac{7}{4}$	$\frac{5}{2}$
s_3	0	0	0	–1	0	1	1	3
s_4	0	2	0	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	3

Since, there are no negative terms in first row, so simplex process is completed, from table max-z = 9 at $x_1 = \frac{3}{2}$ & $x_2 = 1$.

(x) Maximize, $z = 5x_1 + 2x_2$
Subject to, $x_1 + 4x_2 \leq 8$

$$4x_1 + 2x_2 \leq 12$$
$$2x_1 + 3x_2 \leq 14,\ x_1, x_2 \geq 0$$

Solution: Using s_1, s_2 and s_3 as slack variables, given LPP can be written in the normal form as,

$$z - 5x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$x_1 + 4x_2 + s_1 + 0s_2 + 0s_3 = 8$$
$$4x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 12$$
$$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 14$$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	–5	–2	0	0	0	0	
s_1	0	1	4	1	0	0	8	$\frac{8}{1}$
s_2	0	4	2	0	1	0	12	$\frac{12}{4} \to$ exit
s_3	0	2	3	$\boxed{0}$	0	1	14	$\frac{14}{2}$

Basic, $s_1 = 8$, $s_2 = 12$, $s_3 = 14$ and non-basic $x_1 = x_2 = 0$, for this $z = 0$. Also x_1 column is pivot column and R_3 is pivot row, so '4' is pivot element.

Applying $R_1 \to R_1 + \frac{5}{4} R_3, R_2 \to R_2 - \frac{1}{4} R_3, R_4 \to R_4 - \frac{1}{2} R_3$

The next simplex table is,

Basic	z	x_1	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	0	$\frac{1}{2}$	0	$\frac{5}{4}$	0	15	
	0	0	$\frac{7}{2}$	1	$-\frac{1}{4}$	0	5	
	0	4	2	0	1	0	12	
	0	0	2	0	$-\frac{1}{2}$	1	8	

Since, there are no negative terms in the first row, so simplex process is completed. From table, max-z = 15 at $x_1 = 3$ and $x_2 = 0$

(xi) **Maximize, $z = 9x_1 + 2x_2 + 5x_3$**
Subject to $2x_1 + 3x_2 - 5x_3 \leq 12$
$2x_1 - 3x_2 + 3x_3 \leq 3$
$3x_1 + x_2 - 2x_3 \leq 2,\ x_1, x_2, x_3 \leq 0$

Solution: Using s_1, s_2 and s_3 as slack variables, given Lpp can be written as,

$$z - 9x_1 - 2x_2 - 5x_3 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$2x_1 + 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = 12$$
$$2x_1 - 3x_2 + 3x_3 + 0s_1 + s_2 + 0s_3 = 3$$
$$3x_1 + x_2 - 2x_3 + 0s_1 + 0s_2 + s_3 = 2$$

The first simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	x_3	s_1	s_1	s_3	b	$\frac{b_i}{a_{ij}}$
	1	−9	−2	−5	0	0	0	0	
s_1	0	2	3	−5	1	0	0	12	$\frac{12}{2}$
s_2	0	2	−3	3	0	1	0	3	$\frac{3}{2}$
s_3	0	$\boxed{3}$	1	−2	0	0	1	2	$\frac{2}{3} \to$ exit

Basic, $s_1 = 12$, $s_2 = 3$, $s_3 = 2$ and non-basic $x_1 = x_2 = x_3 = 0$, for this z = 0. Also, x_1-column is pivot column and R_4 is pivot row and so 3 is pivot element.

Applying $R_1 \to R_1 + 3R_4, R_2 \to R_2 - \frac{2}{3} R_4, R_3 \to R_3 - \frac{2}{3} R_4$

The next simplex table is,

Basic	z	x_1	x_2	$x_3 \to$ Extrant	s_1	s_1	s_3	b	$\frac{b_i}{a_{ij}}$
	1	0	1	−11	0	0	3	6	
s_1	0	0	$\frac{7}{3}$	$-\frac{11}{3}$	1	0	$-\frac{2}{3}$	$\frac{32}{3}$	
s_2	0	0	$\frac{11}{3}$	$\boxed{\frac{13}{3}}$	0	1	$-\frac{2}{3}$	$\frac{5}{3} \to$ exit	
x_1	0	3	1	−2	0	0	1	2	

Basic, $s_1 = \frac{32}{3}$, $s_2 = \frac{5}{3}$, $x_1 = \frac{2}{3}$, non-basic $x_2 = x_3 = s_3 = 0$, for this z = 6. Also x_3-column is pivot column and R_3 is pivot row, so $\frac{13}{3}$ is pivot element.

Applying $R_1 \to R_1 + \frac{33}{13} R_3, R_2 \to R_2 + \frac{11}{13} R_3, R_4 \to R_4 + \frac{6}{13} R_3$

The next simplex table is,

asic	z	x_1	x_2	x_3	s_1	s_1	s_3	b
	1	0	$\frac{134}{13}$	0	0	$\frac{33}{13}$	$\frac{17}{13}$	$\frac{133}{13}$
	0	0	$\frac{212}{39}$	0	1	$\frac{11}{3}$	$-\frac{48}{39}$	$\frac{521}{39}$
	0	0	$\frac{11}{3}$	$\frac{13}{3}$	0	1	$-\frac{2}{3}$	$\frac{5}{3}$
	0	3	$\frac{105}{39}$	0	0	$\frac{6}{13}$	$\frac{27}{39}$	$\frac{36}{13}$

Since, there are no negative numbers in first row, so simplex process is completed. From table max.z = $\frac{133}{13}$ at $x_1 = \frac{12}{13}$, $x_2 = 0$, $x_3 = \frac{5}{13}$

(xii) **Maximize, $z = 6x_1 + 10x_2 + 2x_3$**
Subject to, $2x_1 + 4x_2 + 3x_3 \leq 40$
$x_1 + x_2 \leq 10$
$2x_1 + x_3 \leq 12,\quad x_1, x_2, x_3 \geq 0$

Solution: Using slack variables s_1, s_2 and s_3, given Lpp can be written in normal form as,

$$z - 6x_1 - 10x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3 = 0$$
$$2x_1 + 4x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 40$$
$$x_1 + x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 = 10$$
$$2x_1 + 0x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 12$$

The first simplex table is,

Basic	z	x_1	$\downarrow x_2$ Entrant	x_3	s_1	s_1	s_3	b	$\frac{b_i}{a_{ij}}$
	1	−6	−10	−2	0	0	0	0	
s_1	0	2	$\boxed{4}$	3	1	0	0	40	$\frac{40}{4}$ → exit
s_2	0	1	1	0	0	1	0	10	$\frac{10}{1}$
s_3	0	2	0	1	0	0	1	12	$\frac{12}{0}$

Basic, $s_1 = 40$, $s_2 = 10$, $s_3 = 12$ and non-basic $x_1 = {}_2 = x_3 = 0$. For this $z = 0$. Also, x_2-column is pivot column and R_2 is pivot row, so 4 is pivot element.

Applying $R_1 \to R_1 + \frac{5}{2} R_2, R_3 \to R_3 - \frac{1}{4} R_2, R_4 \to R_4$

The next simplex table is,

Basic	z	$\downarrow x_1$ Entrant	x_2	x_3	s_1	s_1	s_3	b	$\frac{b_i}{a_{ij}}$
	1	−1	0	$\frac{11}{12}$	$\frac{5}{2}$	0	0	100	
s_1	0	2	4	3	1	0	0	40	
s_2	0	$\boxed{\frac{1}{2}}$	0	$-\frac{3}{4}$	$-\frac{1}{4}$	1	0	0	0→ exit
s_3	0	2	0	1	0	0	1	12	

Basic, $x_2 = 10$, $s_2 = 0$, $s_3 = 12$, non-basic $x_1 = x_3 = s_1 = 0$, for this $z = 100$. [Such case where basic variable has zero value is called non-degenerate case in Lρρ.] Also, x_1-column is pivot column and R_3 is pivot row, so $\frac{1}{2}$ is pivot element.

Applying, $R_1 \to R_1 + 2R_3$, $R_2 \to R_2 - 4R_3$, $R_4 \to R_4 - 4R_3$

The next simplex table is,

Basic	z	x_1	x_2	x_3	s_1	s_1	s_3	b
	1	0	0	4	2	2	0	100
x_2	0	0	4	6	2	−4	0	40
x_1	0	$\frac{1}{2}$	0	$-\frac{3}{4}$	$-\frac{1}{4}$	1	0	0
s_3	0	0	0	4	1	−4	1	12

Since there are no negative numbers in first row, so simplex process is completed. From table, max. $z = 100$ at $x_1 = 0$, $x_2 = 10$ and $x_3 = 0$

(xiii) Maximize, $z = 16x_1 + 17x_2 + 10x_3$

Subject to, $x_1 + x_2 + 4x_3 \le 2000$

$$2x_1 + x_2 + x_3 \le 3600$$

$$x_1 + 2x_2 + 2x_3 \le 2400, \quad x_1, x_2, x_3 \ge 0$$

Solution: Using slack variables s_1, s_2 and s_3 given system can be written in the normal form as.

$$z - 16x_1 - 17x_2 - 10x_3 + 0s_1 - 0s_2 + 0s_3 = 0$$
$$x_1 + x_2 + 4x_3 + s_1 + 0s_2 + 0s_3 = 2000$$
$$2x_1 + x_2 + x_3 + 0s_1 + s_2 + 0s_3 = 3600$$
$$x_1 + 2x_2 + 2x_3 + 0s_1 + 0s_2 + s_3 = 2400$$

The first simplex table is,

Basic	z	x_1	x_2 Intrant	x_3	s_1	s_1	s_3	b	$\frac{b_i}{a_{ij}}$
	1	−16	7	−10	0	0	0	0	
s_1	0	1	1	4	1	0	0	2000	$\frac{2000}{1}$
s_2	0	2	1	1	0	1	0	3600	$\frac{3600}{1}$
s_3	0	1	$\boxed{2}$	2	0	0	1	2400	$\frac{2400}{2}$ → exit

Basic, $s_1 = 2000$, $s_3 = 3600$, $s_3 = 2400$ and non-basic $x_1 = x_2 = x_3 = 0$ For this $z = 0$. Also, x_2-column is pivot column and R_4 is pivot row, so '2' is pivot element.

Applying $R_1 \to R_1 + \frac{17}{2} R_4$, $R_2 \to R_2 - \frac{1}{2} R_4$, $R_3 \to R_3 - \frac{1}{2} R_4$

The next simplex table is,

Basic	z	x_1	$x_2 \to$ Entrant	x_3	s_1	s_1	s_3	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{15}{2}$	0	7	0	0	$\frac{17}{2}$	20400	
s_1	0	$\boxed{\frac{1}{2}}$	0	3	1	0	$-\frac{1}{2}$	800	→ exit
s_2	0	$\frac{3}{2}$	0	0	0	1	$-\frac{1}{2}$	2400	
s_3	0	1	2	2	0	0	1	2400	

Basic, $s_1 = 800$, $s_2 = 2400$, $x_2 = 1200$ and non-basic $x_1 = x_2 = x_3 = 0$ for this $z = 20400$. Also, x_1-column is pivot column and R_2 is pivot row, so $\frac{1}{2}$ is pivot element.

Applying $R_1 \to R_1 + 15R_2$, $R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - 2R_2$

the next simplex table is,

Basic	z	x_1	x_2	x_3	s_1	s_1	s_3	b
	1	0	0	52	15	0	1	32400
x_1	0	$\frac{1}{2}$	0	3	1	0	$-\frac{1}{2}$	800
s_1	0	0	0	−6	−3	1	1	0
x_2	0	0	2	−4	−2	0	2	800

Since, there are no negative numbers in first row, so simplex process is completed from table max-z = 32400 at $x_1 = 1600$, $x_2 = 400$ and $x_3 = 0$.

Exercise - 16

(1) Construct the dual problem corresponding to each of the following linear programming problems,

(i) Maximize, $z = 3x_1 + 3x_2$

Subject to, $x_1 + x_2 \le 4$

$$3x_1 + 2x_2 \le 9, \quad x_1, x_2 \ge 0$$

Solution: Given Lpp can be written in the matrix from as,

$$A = \left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 1 & 1 \\ 9 & 3 & 2 \end{array}\right]$$

Here, first row is objective row and first column is column of RHS constants of constraints and coefficient of Z (other side of variables x_1, x_2)

$$\therefore A^T = \left[\begin{array}{c|cc} 1 & 4 & 9 \\ \hline 3 & 1 & 3 \\ 2 & 1 & 2 \end{array}\right]$$

Using this matrix (similar to the given Lpp and matrix A.), the dual D of given Lpp is,

Minimize, $D = 4y_1 + 9y_2$

Subject to $y_1 + 3y_2 \ge 3$

$$y_1 + 2y_2 \ge 2, \quad y_1, y_2 \ge 0$$

Where y_1, y_2 are dual variables.

(ii) Minimize, $Z = 8x_1 + 9x_2$

Subject to, $x_1 + x_2 \ge 5$

$$3x_1 + x_2 \ge 21, \quad x_1, x_2 \ge 0$$

Solution: The matrix from of given Lpp is,

$$A = \left[\begin{array}{c|cc} 1 & 8 & 9 \\ \hline 5 & 1 & 1 \\ 21 & 3 & 1 \end{array}\right]$$

Taking transpose of A,

$$\therefore AT = \left[\begin{array}{c|cc} 1 & 5 & 21 \\ \hline 8 & 1 & 3 \\ 9 & 1 & 1 \end{array}\right]$$

$\therefore$ The dual D, of given Lpp is,

Maximize, $D = 5y_1 + 21y_2$

Subject to, $y_1 + 3y_2 \le 8$

$$y_1 + y_2 \le 9, \quad y_1, y_2 \ge 0$$

(iii) Maximize, $Z = 5x_1 + 3x_2 + 14x_3$

Subject to, $2x_1 + x_2 + 3x_2 \le 14$

$$x_1 + 3x_2 + 2x_3 \le 15$$

$$x_1 + x_2 + x_3 \ge 8, \quad x_1, x_2, x_3 \ge 0.$$

Solution: The given Lpp can be written as,

Maximize, $Z = 5x_1 + 3x_2 + 14x_3$

Subject to, $2x_1 + x_2 + 3x_2 \le 14$

$$x_1 + 3x_2 + 2x_3 \le 15$$

$$-x_1 - x_2 - x_3 \le -8$$

It's matrix from is,

$$A = \left[\begin{array}{c|ccc} 1 & 5 & 3 & 14 \\ \hline 14 & 2 & 1 & 3 \\ 15 & 1 & 3 & 2 \\ -8 & & -1 & -1 \end{array}\right]$$

Transposi: we have,

$$A^T = \left[\begin{array}{c|ccc} 1 & 14 & 15 & -8 \\ \hline 5 & 2 & 1 & -1 \\ 3 & 1 & 3 & -1 \\ 14 & 3 & 2 & -1 \end{array}\right]$$

$\therefore$ The dual D of the Lpp is,

Minimize, $D = 14y_1 + 15y_2 - 8y_3$

Subject to, $2y_1 + y_2 - y_3 \ge 5$

$$y_1 + 3y_2 - y_3 \ge 3$$

$$3y_1 + 2y_2 - y_3 \ge 14, \quad y_1, y_2, y_3 \ge 0$$

(iv) Minimize, $z = 2x_1 + x_2$

Subject to, $3x_1 + x_2 \ge 3$

$$4x_1 + 3x_2 \ge 6$$

$$x_1 + 2x_2 \le 3, \quad x_1, x_2 \ge 0$$

Solution: Given Lpp can be written as,

Minimize, $z = 2x_1 + x_2$

Subject to, $3x_1 + x_2 \ge 3$

$$4x_1 + 3x_2 \ge 6$$

$$-x_1 - 2x_2 \ge -3, \quad x_1, x_2 \ge 0$$

The matrix from of the L Lpp is,

$$A = \left[\begin{array}{c|cc} 1 & 2 & 1 \\ \hline 3 & 3 & 1 \\ 6 & 4 & 3 \\ -3 & -1 & -2 \end{array}\right]$$

It's transpose is,

$$\therefore A^T = \left[\begin{array}{c|ccc} 1 & 3 & 6 & -3 \\ \hline 2 & 3 & 4 & -1 \\ 1 & 1 & 3 & -2 \end{array}\right]$$

$\therefore$ The dual D is given,

Maximize, $D = 3y_1 + 6y_2 - 3y_3$

Subject to, $3y_1 + 4y_2 - y_3 \le 2$

$$y_1 + 3y_2 - 2y_3 \le 1, \quad y_1, y_2, y_3 \ge 0$$

(2) Solve the following Lpp by simplex method using duality.

(i) Minimize, $z = 21x_1 + 50x_2$

Subject to $2x_1 + 5x_2 \ge 12$

$$3x_1 + 7x_2 \ge 17, \quad x_1, x_2 \ge 0$$

Solution: The matrix from of given Lpp is.

$$\left[\begin{array}{c|cc} 1 & 21 & 50 \\ \hline 12 & 2 & 5 \\ 17 & 3 & 7 \end{array}\right]$$

Taking transpose,

$$\therefore A^T = \left[\begin{array}{c|cc} 1 & 12 & 17 \\ \hline 21 & 2 & 3 \\ 50 & 5 & 7 \end{array}\right]$$

Hence, the dual D, is given by,

Maximize, $D = 12y_1 + 17y_2$

Subject to $2y_1 + 3y_2 \le 21$

$5y_1 + 7y_2 \le 50,\ y_1, y_2 \ge 0$

Using slack variables s_1, s_2 the dual can be written in normal form (canonical form) as, (In case of dual we can also use x_1, x_2 as slack variables)

$$D - 12y_1 - 17y_2 + 0s_1 + 0s_2 = 0$$
$$2y_1 + 3y_2 + s_1 + 0s_2 = 21$$
$$5y_1 + 7y_2 + 0s_1 + s_2 = 50$$

The first simplex table is,

Basic	D	y_1	Entrant y_2 ↓	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	– 12	– 17	0	0	0	
s_1	0	2	$\boxed{3}$	1	0	21	$\frac{21}{3}$ → exit
s_2	0	5	7	0	1	50	$\frac{50}{7}$

Here, s_1 and s_2 are basic variables and $s_1 = 21$, $s_2 = 17$ (from table) and y_1, y_2 are non-basic variables, so $y_1 = y_2 = 0$ (By simplex algorithem/ rule). For this D = 0

Moreover, the most negative number in first row is in the column of y_2, so y_2 is the pivot column and the smallest non-negative ratio $\frac{b_i}{a_{ij}}$ (b_i from column of b and a_{ij} from column of y_2) is in R_2, so R_2 is pivot row and hence 3 is pivot element.

Applying, $R_1 \to R_1 + \frac{17}{3} R_2$, $R_3 \to R_3 - \frac{7}{3} R_2$

The next simplex table is,

Basic	D	Entrant y_1 ↓	y_2	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{2}{3}$	0	$\frac{17}{3}$	0	119	
s_1	0	2	3	1	0	21	$\frac{21}{2}$ → exit
s_2	0	$\boxed{\frac{1}{3}}$	0	$-\frac{7}{3}$	1	1	3 → exit

Basic, $y_2 = \frac{21}{3} = 7$, $s_1 = 1$

Non-basic, $y_1 = s_1 = 0$, for this D = 119

y_1-column is pivot column and R_3 is pivot row and so $\frac{1}{3}$ is pivot element.

Applying $R_1 \to R_1 + 2R_3$, $R_2 \to R_2 - 6R_3$

Basic	D	y_1	y_2	s_1'	s_2	b
	1	0	0	1	2	121
	0	0	3	$\frac{45}{3}$	–6	15
	0	$\frac{1}{3}$	0	$-\frac{7}{3}$	1	1

Since there are no negative number in R_1, so simplex process is completed. Also, maximum value of D, max D = 121 at $y_1 = 3$, $y_2 = 5$.

Hence, by the theorem of duality

Min z = 121 at $x_1 = s_1 = 1$ and $x_2 = s_2 = 2$

(ii) **Minimize, $z = 3x_1 + 2x_2$**

Subject to, $3x_1 - x_2 \ge -5$

$$-x_1 + x_2 \ge 1$$
$$2x_1 + 4x_2 \ge 12,\ x_1, x_2 \ge 0$$

Solution: The matrix form of given Lpp is,

$$A = \left[\begin{array}{c|cc} 1 & 3 & 2 \\ \hline -5 & 3 & -1 \\ 1 & -1 & 1 \\ 12 & 2 & 4 \end{array}\right]$$

Taking transpose of A,

$$\therefore A^T = \left[\begin{array}{c|ccc} 1 & -5 & 1 & 12 \\ \hline 3 & 3 & -1 & 2 \\ 2 & -1 & 1 & 4 \end{array}\right]$$

∴ The dual of given Lpp is,

Maximize $D = -5y_1 + y_2 + 12y_3$

Subject to $3y_1 - y_2 + 2y_3 \le 3$

$-y_1 + y_2 + 4y_3 \le 2,\ y_1, y_2, y_3 \ge 0$

Using black variables $x_1 (= s_1)$ and $x_2 (= s_2)$, the dual can be written as,

$D + 5y_1 - y_2 - 12y_3 + 0x_1 + 0x_2 = 0$

$3y_1 - y_2 + 2y_3 + x_1 + 0x_2 = 3$

$-y_1 + y_2 + 4y_3 + 0x_1 + x_2 = 2$

The first simplex table is,

Basic	D	y_1	y_2	Entrant y_3 ↓	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	5	−1	−12	0	0	0	
	0	3	−1	2	1	0	3	$\frac{3}{2}$
	0	−1	1	$\boxed{4}$	0	1	2	$\frac{2}{4}$ → exit

Basic, $x_1 = 3$, $x_2 = 2$ and non-basic $y_1 = y_2 = y_3 = 0$ for this $D = 0$

Also, y_3 column is pivot column and R_3 is pivot row and 4 is pivot element.

Applying $R_1 \to R_1 + 3R_3$, $R_2 \to R_2 - \frac{1}{2}R_3$

The next simplex table is,

Basic	D	y_1	y_2	y_3	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	2	2	0	0	3	6	
	0	$\frac{7}{2}$	$-\frac{3}{2}$	0	1	$-\frac{1}{2}$	2	
	0	−1	1	4	0	1	2	

From this table, max $D = 6$ at $y_1 = 0$, $y_2 = 0$, $y_3 = 4$

Hence by duality theorem, min $z = 6$ at $x_1 = 0$, $x_2 = 3$ (from table)

(iii) Minimize, $z = 3x_1 + 2x_2$

Subject to, $2x_1 + 4x_2 \geq 10$

$4x_1 + 2x_2 \geq 10$

$x_2 \geq 4$, $x_1, x_2 \geq 0$

Solution: The matrix from of given Lpp is,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 10 & 2 & 4 \\ 10 & 4 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Taking the transpose of A,

$$\therefore A^T = \begin{bmatrix} 1 & 10 & 10 & 4 \\ 3 & 2 & 4 & 0 \\ 2 & 4 & 2 & 1 \end{bmatrix}$$

The dual of given Lpp is,

max. $D = 10y_1 + 10y_2 + 4y_3$

subject to, $2y_1 + 4y_2 \leq 3$

$4y_1 + 2y_2 + y_3 \leq 2$, $x_1, y_2, y_3 \geq 0$

Using x_1 and x_2 as slack variables, the dual can be written as,

$D - 10y_1 - 10y_2 - 4y_3 + 0x_1 + 0x_2 = 0$

$2y_1 + 4y_2 + 0y_3 + x_1 + 0x_2 = 3$

$4y_1 + 2y_2 + y_3 + 0x_1 + x_2 = 2$

The first simplex table is,

Basic	D	Entrant y_1 ↓	y_2	y_3	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	−10	−10	4	0	0	0	
x_1	0	2	4	0	1	0	3	$\frac{3}{2}$
x_2	0	$\boxed{4}$	2	1	0	1	2	$\frac{2}{4}$ → exit

Basic $x_1 = 3$, $x_2 = 2$ and non-basic $y_1 = y_2 = y_3 = 0$ for this $D = 0$

Also, y_1 column is pivot column and R_3 is pivot row, so 4 is pivot element.

Applying, $R_1 \to R_1 + \frac{5}{2}R_3$, $R_2 \to R_2 - \frac{1}{2}R_3$

Basic	D	Entrant y_1 ↓	y_2	y_3	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	0	−5	$\frac{13}{2}$	0	$\frac{5}{2}$	5	
x_1	0	0	$\boxed{3}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	2	$\frac{2}{2}$ → exit
x_2	0	4	2	1	0	1	2	$\frac{2}{2}$

Basic, $x_1 = 2$, $y_1 = 2$ and non basic $y_2 = y_3 = x_2 = 0$ for this $D = 5$

Also, y_2 column is pivot column and R_2 is pivot row, so 2 is pivot element.

Applying, $R_1 \to R_1 + \frac{5}{3}R_2$, $R_3 \to R_3 - \frac{2}{3}R_2$

The next simplex table is,

Basic	D	y_1	y_2	y_3	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	0	0	$\frac{17}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{25}{3}$	
y_2	0	0	3	$-\frac{1}{2}$	1	$-\frac{1}{2}$	2	
y_1	0	4	0	$\frac{4}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	

Since there are no negative terms in first row, so simplex process is completed. From table max $D = \frac{25}{3}$ at $y_1 = \frac{8}{3}, y_2 = \frac{2}{3}, y_3 = 0$

Hence by duality theorem min $z = \frac{25}{3}$ at $x_1 = \frac{5}{3}, x_2 = \frac{5}{3}$

(iv) Minimize $z = 0.7x_1 + 0.5x_2$
Subject to, $x_1 \geq 4$
$x_2 \geq 6$
$x_1 + 2x_2 \geq 20$
$2x_1 + x_2 \geq 18$, $x_1, x_2 \geq 0$

Solution: The matrix form of given Lpp is,

$$A = \left[\begin{array}{c|cc} 1 & 0.7 & 0.5 \\ \hline 4 & 1 & 0 \\ 6 & 0 & 1 \\ 20 & 1 & 2 \\ 18 & 2 & 1 \end{array}\right]$$

Taking transpose of A,

$$\therefore A^T = \left[\begin{array}{c|cccc} 1 & 4 & 6 & 20 & 18 \\ \hline 0.7 & 1 & 0 & 1 & 2 \\ 0.5 & 0 & 1 & 2 & 1 \end{array}\right]$$

$\therefore$ Dual of given Lpp is,
Maximize $D = 4y_1 + 6y_2 + 20y_3 + 18y_4$
Subject to, $y_1 + y_3 + 2y_4 \leq 0.7$
$y_2 + 2y_3 + y_4 \leq 0.5$, $y_1, y_2, y_3, y_4 \geq 0$

Using x_1 and x_2 as slack variables, the dual can be written as,
$D - 4y_1 - 6y_2 - 20y_3 - 18y_4 + 0x_1 + 0x_2 = 0$

$$y_1 + 0y_2 + y_3 + 2y_4 + x_1 + 0x_2 = 0.7 = \frac{7}{10}$$

$$0y_1 + y_2 + 2y_3 + y_4 + 0x_1 + x_2 = 0.5 = \frac{1}{2}$$

The first simplex table is,

Basic	D	y_1	y_2	Entract y_3 ↓	y_4	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	−4	−6	−20	−18	0	0	0	
x_1	0	1	0	1	2	1	0	$\frac{7}{10}$	$\frac{7}{10}$
x_2	0	0	1	$\boxed{2}$	1	0	1	$\frac{1}{2}$	$\frac{1}{4}$ → exit

Basic, $x_1 = \frac{7}{10}$, $x_2 = \frac{1}{2}$ and non-basic $y_1 = y_2 = y_3 = y_4 = 0$

For this $D = 0$. Also y_3 column is pivot column and R_3 is pivot row. So 2 is pivot element.

Applying $R_1 \to R_1 + 10R_3$, $R_2 \to R_2 - \frac{1}{2} R_3$,

The next simplex table is,

Basic	D	y_1	y_2	y_3	Entract y_4 ↓	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	−4	4	0	−8	0	10	5	
x_1	0	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	$\frac{9}{20}$	→ exit
x_2	0	0	1	2	1	0	1	$\frac{1}{2}$	

Basic, $x_1 = \frac{9}{20}$, $y_3 = \frac{1}{4}$, and non-basic $y_1 = y_2 = x_2 = 0$

For this $D = 5$. Also, y_4 – column is pivot column and R_2 is pivot row so $\frac{3}{2}$ is pivot element.

Applying $R_1 \to R_1 + \frac{16}{3} R_2$, $R_3 \to R_3 - \frac{2}{3} R_2$

Basic	D	y_1	y_2	y_3	y_4	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	$\frac{4}{3}$	$\frac{4}{3}$	0	0	$\frac{16}{3}$	$\frac{22}{3}$	$\frac{111}{15}$	
y_4	0	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	$\frac{9}{20}$	
x_2	0	$-\frac{2}{3}$	$\frac{4}{3}$	2	0	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{11}{40}$	

From table, max $D = \frac{111}{15}$ at $y_1 = 0, y_2 = 0, y_3 = \frac{11}{80}$. Hence by dual theorem, min$z = \frac{111}{15}$ at $x_1 = \frac{16}{3}, x_2 = \frac{22}{3}$.

(v) Minimize, $z = 4x_1 + 3x_2 + x_3$
$x_1 + 2x_2 + 4x_3 \geq 12$
$3x_1 + 2x_2 + x_3 \geq 8$ and $x_1, x_2, x_3 \geq 0$

Solution: The matrix form of given Lpp is,

$$A = \left[\begin{array}{c|ccc} 1 & 4 & 3 & 1 \\ \hline 12 & 1 & 2 & 4 \\ 8 & 3 & 2 & 1 \end{array}\right]$$

$$\therefore A^T = \left[\begin{array}{c|cc} 1 & 12 & 8 \\ \hline 4 & 1 & 3 \\ 3 & 2 & 2 \\ 1 & 4 & 1 \end{array}\right]$$

The dual of given Lpp is,
Maximze, $D = 12y_1 + 8y_2$

Subject to, $y_1 + 3y_2 \le 4$

$2y_1 + 2y_2 \le 3$

$4y_1 + y_2 \le 1, y_1, y_2 \ge 0$

Using x_1, x_2, x_3 as black variables, the dual can be written as;

$D - 12y_1 - 8y_2 + 0x_1 + 0x_2 + 0x_3 = 0$

$y_1 + 3y_2 + x_1 + 0x_2 + 0x_3 = 4$

$2y_1 + 2y_2 + 0x_1 = x_2 + 0x_3 = 3$

$4y_1 + y_2 + 0x_1 + 0x_2 + x_3 = 1$

The first simplex table is,

Basic	D	Entrant y_1 ↓	y_2	x_1	x_2	x_3	b	$\frac{b_i}{a_{ij}}$
	1	−12	−8	0	0	0	0	
x_1	0	1	3	1	0	0	4	$\frac{4}{1}$
x_2	0	2	2	0	1	0	3	$\frac{3}{2}$
x_3	0	$\boxed{4}$	1	0	0	1	1	$\frac{1}{4}$ → exit

Basic, $x_1 = 4$, $x_2 = 3$, $x_3 = 1$ and non-basic $y_1 = y_2 = 0$; for this D = 0. Also y_1 column is pivot column and R_4 is pivot row, so '4' is pivot element.

Applying $R_1 \to R_1 + 3R_4$, $R_2 \to R_2 - \frac{1}{4} R_4$, $R_3 \to R_3 - \frac{1}{2} R_4$

We get,

Basic	D	y_1	Entrant y_2 ↓	x_1	x_2	x_3	b	$\frac{b_i}{a_{ij}}$
	1	0	−5	0	0	3	3	
x_1	0	0	$\boxed{\frac{11}{4}}$	1	0	$-\frac{1}{4}$	$\frac{15}{4}$	$\frac{15}{11}$ → exit
x_2	0	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{3}$
y_1	0	4	1	0	0	1	1	$\frac{1}{1}$

Basic, $x_1 = \frac{1}{4}$, $x_2 = \frac{5}{2}$, $y_1 = \frac{1}{4}$ and non-basic $y_2 = x_3 = 0$. For this D = 3. Also, y_2 column is pivot column and R_2 is pivot row, so $\frac{11}{4}$ is pivot element.

Applying $R_1 \to R_1 + \frac{20}{11} R_2$, $R_3 \to R_3 - \frac{6}{11} R_2$, $R_4 \to R_4 - \frac{4}{11} R_2$

we get,

Basic	D	y_1	y_2	x_1	x_2	x_3	b	$\frac{b_i}{a_{ij}}$
	1	0	0	$\frac{20}{11}$	0	$\frac{28}{11}$	$\frac{108}{11}$	
y_2	0	0	$\frac{11}{4}$	1	0	$-\frac{1}{4}$	$\frac{15}{4}$	
x_2	0	0	0	$-\frac{6}{11}$	0	$-\frac{4}{11}$	$\frac{5}{11}$	
y_1	0	4	0	$-\frac{4}{11}$	0	$\frac{12}{11}$	$-\frac{4}{11}$	

Since, $y_1 = -\frac{1}{11}$ which is not possible, since $y_1 \ge 0$.

So the dual has no solution hence given Lpp has no Solution:

(vi) Minimize, $z = 2x_1 + 5x_2$

Subject to, $3x_1 + 2x_2 \ge 8$

$x_1 + 4x_2 \ge 6, x_1\ x_2 \ge 0$

Solution: The matrix from of given Lpp is,

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 8 & 3 & 2 \\ 6 & 1 & 4 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 1 & 8 & 6 \\ 2 & 3 & 1 \\ 5 & 2 & 4 \end{bmatrix}$$

$\therefore$ The dual of the given Lpp is,

Maximize, $D = 8y_1 + 6y_2$

Subject to, $3y_1 + y_2 \le 2$

$2y_1 + 4y_2 \le 5$

Using slack variables, x_1 and x_2 the normal form of given dual is,

$D - 8y_1 - 6y_2 + 0x_1 + 0x_2 = 0$

$3y_1 + y_2 + x_1 + 0x_2 = 2$

$2y_1 + 4y_2 + 0x_1 + x_2 = 5$.

The simplex table,

Basic	D	Entrant y_1 ↓	y_2	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	−8	−6	0	0	0	
x_1	0	$\boxed{3}$	1	1	0	2	$\frac{2}{3}$ → exit
x_2	0	2	4	0	1	5	$\frac{5}{2}$

Basic, $x_1 = 2$, $x_2 = 5$ and non-basic $y_1 = y_2 = 0$ for this D = 0.

Also, y_1 column is pivot column and R_2 is pivot row, so '3' is pivot element.

Applying $R_1 \to R_1 + \frac{8}{3} R_2, R_3 \to R_3 - \frac{2}{3} R_2$

The next simplex table is,

Basic	D	y_1	y_2	x_1	x_2	b
	1	0	$-\frac{10}{3}$	$\frac{8}{3}$	0	$\frac{16}{3}$
y_1	0	3	1	1	0	2
x_2	0	0	$\boxed{\frac{10}{3}}$	$-\frac{2}{3}$	1	$\frac{11}{3} \to$ exit

Basic, $y_1 = \frac{2}{3}, x_2 = \frac{11}{3}$, non-baisc $y_2 = x_1 = 0$, For this $D = \frac{16}{3}$

Also, y_2 column is pivot column and R_3 is pivot row, so $\frac{10}{3}$ is pivot element.

Applying $R_1 \to R_1 + R_3, R_2 \to R_2 - \frac{3}{10} R_3$

the next simplex table, is

Basic	D	y_1	y_2	x_1	x_2	b
	1	0	0	2	1	9
	0	3	0	$\frac{6}{5}$	$-\frac{3}{10}$	$\frac{9}{10}$
	0	0	$\frac{10}{3}$	$-\frac{2}{3}$	1	$\frac{11}{3}$

Since there are no negative terms in first row, so the simplex process is completed. From table, $D = 9$, $y_1 = \frac{3}{10}$, $y_2 = \frac{11}{10}$.

Hence, by dual theorem, min $Z = 9$ at $x_1 = 2$ and $x_2 = 1$.

(vii) Minimize, $z = 10x_1 + 15x_2$
Subjected to, $x_1 + x_2 \geq 8$
$10x_1 + 6x_2 \geq 60$, $x_1, x_2 \geq 0$

Solution: The matrix form of given Lpp is,

$$A = \begin{bmatrix} 1 & 10 & 15 \\ 8 & 1 & 1 \\ 60 & 10 & 6 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 1 & 8 & 60 \\ 10 & 1 & 10 \\ 15 & 1 & 6 \end{bmatrix}$$

$\therefore$ The dual of given Lpp is,

Maximize, $D = 8y_1 + 60y_2$

Subject to, $y_1 + 10y_2 \leq 10$

$y_1 + 6y_2 \leq 15$, $y_1, y_2 \geq 0$

Using slack variables x_1 and x_2 the dual can be written in the normal form as,

$D - 8y_1 - 60y_2 + 0x_1 + 0x_2 = 0$

$y_1 + 10y_2 + x_1 + 0x_2 = 10$

$y_1 + 6y_2 + 0x_1 + x_2 = 15$

The first simplex table is,

Basic	D	y_1	y_2 ↓ Entrant	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	–8	–60	0	0	0	
x_1	0	1	$\boxed{10}$	1	0	10	$\frac{10}{10} \to$ exit
x_2	0	1	6	0	1	15	$\frac{15}{6}$

Basic $x_1 = 10$, $x_2 = 15$ and non-basic $y_1 = y_2 = 0$, for this $D = 0$.
Also, y_2-column is pivot column and R_2 is pivot row and so '10' is pivot element.

Applying $R_1 \to R_1 + 6R_2, R_3 \to R_3 - \frac{3}{5} R_2$.

The next simplex table is,

Basic	D	y_1 ↓ Entrant	y_2	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	–2	0	6	0	60	
y_2	0	$\boxed{1}$	10	1	0	10	$\frac{10}{10} \to$ exit
x_2	0	$\frac{2}{5}$	0	$-\frac{3}{5}$	1	9	$\frac{45}{2}$

Basic $y_2 = 1$, $x_2 = 9$ and non-basic $y_1 = x_1 = 0$, for this, $D = 90$.
Also, y_1-column is pivot column and R_2 is pivot row, so '1' is pivot element.

Applying, $R_1 \to R_1 + 2R_2, R_3 \to R_3 - \frac{2}{5} R_2$.

the next simplex table is,

Basic	D	y_1	y_2	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	0	20	8	0	80	
	0	1	10	1	0	10	
	0	0	–4	–1	1	5	

Since there are no negative terms, so the simplex process is completed. From table max $D = 80$ at $y_1 = 10$, $y_2 = 0$. Hence by theorem of duality, Minz = 80 at $x_1 = 8$, $x_2 = 0$.

(viii) Minimize $Z = x_1 + x_2$
Subject to, $7x_1 + 5x_2 \geq 40$
$x_1 + 4x_2 \geq 9$, $x_1, x_2 \geq 0$

Solution: The matrix from of given LPP is,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 40 & 7 & 5 \\ 9 & 1 & 4 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 1 & 40 & 9 \\ 1 & 7 & 1 \\ 1 & 5 & 4 \end{bmatrix}$$

$\therefore$ The dual of given LPP is,

Maximize $D = 40y_1 + 9y_2$

Subject to, $7y_1 + y_2 \le 1$

$5y_1 + 4y_2 \le 1,\ y_1, y_2 \ge 0$

Using slack variables x_1, x_2 the dual can be written in the normal form as,

$D - 40y_1 - 9y_2 + 0x_1 + 0x_2 = 0$

$y_1 + y_2 + x_1 + 0x_2 = 1$

$5y_1 + 4y_2 + 0x_1 + x_2 = 1$

This first simplex table is,

Basic	D	y_1 ↓ Entrant	y_2	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	−40	−9	0	0	0	
x_1	0	$\boxed{7}$	1	1	0	1	$\frac{1}{7}$ → exit
x_2	0	5	4	0	1	1	$\frac{1}{5}$

Basic, $x_1 = 1$, $x_2 = 1$ and non-basic $y_1 = y_2 = 0$, for this $D = 0$. Also column is pivot column and R_2 is pivot row so 7 is pivot element.

Applying, $R_1 \to \frac{40}{7} R_2$, $R_3 \to R_3 - \frac{5}{7} R_2$

The next simplex table is,

Basic	D	y_1	y_2 ↓ Entrant	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	0	$-\frac{23}{7}$	$\frac{40}{7}$	0	$\frac{40}{7}$	
y_1	0	7	1	1	0	1	
x_2	0	0	$\boxed{\frac{23}{7}}$	$-\frac{5}{7}$	1	$\frac{2}{7}$	→ exit

Basic, $y_1 = \frac{1}{7}$; $x_2 = \frac{2}{7}$, $y_2 = x_2 = 0$, for this $D = \frac{40}{7}$.

Also, y_2 column is pivot column and R_3 is pivot row, so $\frac{23}{7}$ is pivot element.

Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 \to \frac{7}{23} R_3$

The next simplex table is,

Basic	D	y_1	y_2	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	0	0	5	1	6	
	0	7	0	$\frac{7}{2}$	$-\frac{7}{2}$	$\frac{9}{7}$	
	0	0	$\frac{23}{7}$	$-\frac{5}{7}$	1	$\frac{2}{7}$	

Since there are no negative numbers in first row, so simplex process is completed. From table max. $D = 6$ at $y_1 = \frac{9}{49}$, $y_2 = \frac{2}{23}$.

Hence by dual theorem min $z = 6$ at $x_1 = 5$ and $x_2 = 1$.

x) Minimize, $z = 20x_1 + 30x_2$

Subject to, $x_1 + 4x_2 \ge 8$

$2x_1 + x_2 \ge 7,\ x_1, x_2 \ge 0$

Solution: The matrix from of given Lpp is,

$$A = \begin{bmatrix} 1 & 20 & 30 \\ 8 & 1 & 4 \\ 5 & 1 & 1 \\ 7 & 2 & 1 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 1 & 8 & 5 & 7 \\ 20 & 1 & 1 & 2 \\ 30 & 4 & 1 & 1 \end{bmatrix}$$

The dual of given Lpp is,

Maximize, $D = 8y_1 + 5y_2 + 7y_3$

Subject to, $y_1 + y_2 + 2y_3 \le 20$

$4y_1 + y_2 + y_3 \le 30,\ y_1, y_2, y_3 \ge 0$

Using slack variables x_1 and x_2, the dual can be written in normal from as,

$D - 8y_1 - 5y_2 - 7y_3 + 0x_1 + 0x_2 = 0$

$y_1 + y_2 + 2y_3 + x_1 + 0x_2 = 20$

$4y_1 + y_2 + y_3 + 0x_1 + x_2 = 30$

The first simplex table is,

Basic	D	↓ Entrant y_1	y_2	y_3	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	−8	−5	−7	0	0	0	
x_1	0	1	1	2	1	0	20	$\frac{20}{1}$
x_2	0	$\boxed{4}$	1	1	0	1	30	$\frac{30}{4}$ → exit

Basic, $x_1 = 20$, $x_2 = 30$ and non-basic $y_1 = y_2 = y_3 = 0$, for this $D = 0$.

Also, y_1 column is pivot column and R_1 is pivot row and, so 4 is pivot element.

Applying, $R_1 \to R_1 + 2R_3$, $R_2 \to R_2 - \frac{1}{4} R_3$

The next simplex table is,

Basic	D	y_1	y_2	Entrant y_3 ↓	x_1	x_2	b
	1	0	−3	−5	0	2	0
x_1	0	0	$\boxed{\frac{3}{4}}$	$\frac{7}{4}$	1	$-\frac{1}{4}$	$\frac{25}{2}$ → exit
y_1	0	4	1	1	0	1	30

Basic, $y_1 = \frac{15}{2}$, $x_1 = \frac{25}{2}$, non-basic $y_2 = y_3 = x_2 = 0$, for this D = 40.

Also, y_3 column is pivot column and R_2 is pivot row, so $\frac{3}{4}$ is pivot element.

Applying, $R_1 \to R_1 + 4R_2$, $R_3 \to R_3 - \frac{4}{3} R_2$

The next simplex table is,

Basic	D	y_1	y_2	y_3	x_1	x_2	b
	1	0	0	2	4	1	110
	0	0	$\frac{3}{4}$	$\frac{7}{4}$	1	$-\frac{1}{4}$	$\frac{25}{2}$
	0	4	0	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{4}{3}$	$\frac{65}{3}$

Since there are no negative numbers in first row. So simplex process is completed from table, max. D = 110 at $y_1 = \frac{65}{12}$, $y_2 = \frac{50}{3}$ and $y_3 = 0$.

Hence by the theorem of duality,
min z = 110 at $x_1 = 4$, $x_2 = 1$

(x) **Minimize, $z = 10x_1 + 5x_2 + 20x_3$**
Subject to, $x_1 + x_2 + x_3 \geq 100$
$2x_1 + 3x_2 + 10x_3 \geq 200$

Solution: The matrix form of given Lpp is,

$$A = \left[\begin{array}{c|ccc} 1 & 10 & 5 & 20 \\ \hline 100 & 1 & 1 & 1 \\ 200 & 2 & 3 & 10 \end{array}\right]$$

$$\therefore A^T = \left[\begin{array}{c|cc} 1 & 100 & 200 \\ \hline 10 & 1 & 2 \\ 5 & 1 & 3 \\ 20 & 1 & 10 \end{array}\right]$$

The dual of given Lpp is,
Maximize $D = 100y_1 + 200y_2$
Subject to, $y_1 + 2y_2 \leq 10$
$y_1 + 3y_2 \leq 5$
$y_1 + 10y_2 \leq 20$, $y_1, y_2, y_3 \geq 0$

Using slack variables x_1, x_2, x_3 the dual can be written in the normal form as.
$D - 100y_1 - 200y_2 + 0x_1 + 0x_2 + 0x_3 = 0$
$y_1 + 2y_2 + x_1 + 0x_2 + 0x_3 = 10$
$y_1 + 3y_2 + 0x_1 + x_2 + 0x_3 = 5$
$y_1 + 10y_2 + 0x_1 + 0x_2 + x_3 = 20$

The first simplex table is,

Basic	D	y_1	entrant y_2 ↓	x_1	x_2	x_3	b	$\frac{b_i}{a_{ij}}$
	1	−100	−200	0	0	0	0	
x_1	0	1	2	1	0	0	10	$\frac{10}{2}$
x_2	0	1	$\boxed{3}$	0	1	0	5	$\frac{5}{3}$ → exit
x_3	0	1	10	0	0	1	20	$\frac{20}{10}$

Baisc, $x_1 = 10$, $x_2 = 5$ and $x_3 = 20$ and non-basic $y_1 = y_2 = 0$, for this D = 0. Alos, y_2 column is pivot column and R_3 is pivot row, so 3 is pivot element.

Applying $R_1 \to R_1 + \frac{200}{3} R_3$, $R_2 \to R_2 - \frac{2}{3} R_3$, $R_4 \to R_4 - \frac{10}{3} R_3$

The next simplex table is,

Basic	D	entrant y_1 ↓	y_2	x_1	x_2	x_3	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{100}{3}$	0	0	$\frac{200}{3}$	0	$\frac{100}{3}$	
x_1	0	$\frac{1}{3}$	0	1	$-\frac{2}{3}$	0	$\frac{20}{3}$	20
y_2	0	$\boxed{1}$	3	0	1	0	5	5 → exit
x_3	0	$-\frac{7}{3}$	0	0	$-\frac{10}{3}$	1	$\frac{10}{3}$	

Basic $y_2 = \frac{5}{3}$, $x_3 = \frac{10}{3}$, $x_1 = \frac{5}{3}$ non basic, $y_1 = x_2 = 0$, for [illegible] $= \frac{100}{3}$. Also y_1 column is pivot column and R_3 is.

Applying $R_1 \rightarrow R_1 + \frac{100}{3} R_3$

$R_2 \rightarrow R_2 - \frac{1}{3} R_3$

$R_4 \rightarrow R_4 + \frac{7}{3} R_3$

The next simplex table is,

Basic	D	y_1	y_2	x_1	x_2	x_3	b	[illegible]
	1	0	100	0	100	0	500	
	0	0	–1	1	–1	0	5	
	0	1	3	0	1	0	5	
	0	0	7	0	$-\frac{20}{3}$	1	15	

Since there are no negative numbers in first row, so sim[illegible] process is completed from above table max D = 500 at $y_1 = 5$, [illegible] 0.

Hence by dual theorem min Z = 500 at $x_1 = 0$, $x_2 = 100$ and $x_3 =$ [illegible]

(xi) Minimize, $z = 4x_1 + 3x_2 + 8x_3$

Subject to, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ (BE 20[illegible]

Solution: The given Lpp is,

Minimize, $z = 4x_1 + 3x_2 + 8x_3$

Subject to, $x_1 + x_3 \geq 2$

$x_2 + 2x_3 \geq 5$, $x_1, x_2, x_3 \geq 0$

Writing given Lpp in the matrix form,

$$A = \begin{bmatrix} 1 & 4 & 3 & 8 \\ 2 & 1 & 0 & 1 \\ 5 & 0 & 1 & 2 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \\ 8 & 1 & 2 \end{bmatrix}$$

Hence, the dual is,

Maximize, $D = 2y_1 + 5y_2$

Subject to, $y_1 \leq 4$

$y_2 \leq 3$

$y_1 + 2y_2 \leq 8$, $y_1, y_2 \geq 0$

Using x_1, x_2 and x_3 as slack variables, the dual can be written as,

$D - 2y_1 - 5y_2 - 0x_1 + 0x_2 + 0x_3 = 0$

$y_1 + 0y_2 + x_1 + 0x_2 + 0x_3 = 4$

$0y_1 + y_2 + 0x_1 + x_2 + 0x_3 = 3$

$y_1 + 2y_2 + 0x_1 + 0x_2 + x_3 = 8$

The first simplex table is,

Basic	D	y_1	entrant ↓ y_2	x_1	x_2	x_3	b	$\frac{b}{a}$
	1	–2	–5	0	0	0	0	
x_1	0	1	0	1	0	0	4	$\frac{4}{0}$
x_2	0	0	$\boxed{1}$	0	1	0	3	$\frac{3}{1}$ → exit
x_3	0	1	2	0	0	1	8	$\frac{8}{2}$

Basic, $x_1 = 4$, $x_2 = 3$, $x_3 = 8$ and non-basic $y_1 = y_2 = 0$. For this D = 0.

Also. y_2-column is pivot column and R_3 is pivot row and so '1' is pivot element.

Applying, $R_1 \rightarrow R_1 + 5R_3$, $R_4 \rightarrow R_4 - 2R_3$, $R_2 \rightarrow R_2$.

The next simplex table is.

Basic	D	entrant ↓ y_1	y_2	x_1	x_2	x_3	b	$\frac{b}{a}$
	1	–2	0	0	5	0	15	
x_1	0	1	0	1	0	0	4	$\frac{4}{1}$
x_2	0	0	1	0	1	0	3	$\frac{3}{0}$
x_3	0	$\boxed{1}$	0	0	-2	1	2	$\frac{2}{1}$ → Exit

Basic, $y_2 = 3$, $x_1 = 4$, $x_3 = 2$ and non-basic $y_1 = x_2 = 0$, for this D = 15.

Also, y_1-column is pivot column and r_4 is pivot row, so '1' is pivot element.

Applying $R_1 \rightarrow R_1 + 2R_4$, $R_2 \rightarrow R_2 - R_4$, $R_3 \rightarrow R_3$

Basic	D	y_1	y_2	x_1	x_2	x_3	b
	1	0	0	0	1	2	19
	0	0	0	1	2	–1	2
	0	0	1	0	1	0	3
	0	1	0	0	–2	1	2

Since there are no negative number in first row, so simplex process is completed from the table, max D = 19 at $y_1 = 2$ and $y_2 = 3$. Hence by dual theorem minz = 19 at $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$.

(xii) **Minimize, $z = 3x_1 + 2.25x_2 = 3x_1 + \frac{9}{4}x_2$**

Subject, $2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50,\ x_1, x_2 \geq 0$

Solution: The matrix from of given Lpp is,

$$A = \left[\begin{array}{c|cc} 1 & 3 & \frac{9}{4} \\ \hline 40 & 2 & 4 \\ 50 & 3 & 2 \end{array}\right]$$

$$\therefore A^T = \left[\begin{array}{c|cc} 1 & 40 & 50 \\ \hline 3 & 2 & 3 \\ \frac{9}{4} & 4 & 2 \end{array}\right]$$

The dual is,

Maximize $D = 40y_1 + 50y_2$

Subject to $2y_1 + 3y_2 \leq 3$

$4y_1 + 2y_2 \leq \frac{9}{4}$

Using x_1 and x_2 as slack variable, the normal form is,

$D - 40y_1 - 50y_2 + 0x_1 + 0x_2 = 0$

$2y_1 + 3y_2 + x_1 + 0x_2 = 3$

$4y_1 + 2y_2 + 0x_1 + x_2 = \frac{9}{4}$

The first simplex table is,

Basic	D	y_1	entrant ↓ y_2	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	−40	−50	0	0	0	
x_1	0	2	$\boxed{3}$	1	0	3	$\frac{3}{3}$ → Exit
x_2	0	4	2	0	1	$\frac{9}{4}$	$\frac{9}{8}$

Basic, $x_1 = 3$, $x_2 = \frac{9}{4}$, non-basic $y_1 = y_2 = 0$. For this $D = 0$.

Also, y_2-column is pivot column and r_2 is pivot row and so 3 is pivot element.

Applying $R_1 \to R_1 + \frac{50}{3} R_2$, $R_3 \to R_3 - \frac{2}{3} R_2$

The next simplex table is,

Basic	D	entrant ↓ y_1	y_2	x_1	x_2	b	$\frac{b}{a}$
	1	$-\frac{20}{3}$	0	$\frac{50}{3}$	0	50	
y_2	0	2	3	1	0	3	$\frac{3}{2}$
x_2	0	$\boxed{\frac{8}{3}}$	0	$-\frac{2}{3}$	1	$\frac{1}{4}$	$\frac{3}{32}$ → Exit

Basic, $y_2 = 1$, $x_2 = \frac{1}{4}$ and non-basic $y_1 = x_1 = 0$. For this $D = 50$.

Also y_1-column pivot column and R_3 is pivot row and so $\frac{8}{3}$ is pivot element.

Applying $R_1 \to R_1 + \frac{5}{2} R_3$, $R_2 \to R_2 - \frac{3}{4} R_3$

The next simplex table is,

Basic	D	y_1	y_2	x_1	x_2	b	$\frac{b_i}{a_{ij}}$
	1	0	0	15	$\frac{5}{2}$	$\frac{405}{8}$	
y_2	0	0	3	2	$-\frac{3}{4}$	15	
y_1	0	$\frac{8}{3}$	0	$-\frac{2}{3}$	1	$\frac{1}{4}$	

Since, there are no negative numbers in first row, so, simplex process is completed from table, $D = \frac{405}{8}$ at $y_1 = \frac{3}{32}$, $y_2 = 5$. From dual theorem min $z = \frac{405}{8}$ at $x_1 = 15$ and $x_2 = \frac{5}{2}$.

3. **Solve the following linear programming problem by simplex method using removal of artifical variables: [Big M method]**

(i) **Minimize, $z = 4x_1 + 2x_2$**

Subject to, $3x_1 + x_2 \geq 27$

$-x_1 - x_2 \leq -21$

$x_1 + 2x_2 \geq 30,\ x_1, x_2 \geq 0$

Solution: Using mini-max theorem given Lpp can be written as,

Maximize, $w = -z = -4x_1 - 2x_2$

Subject to $3x_1 + x_2 \geq 27$

$x_1 + x_2 \geq 21$ [multiplied by −'1']

$x_1 + 2x_2 \geq 30$

Using s_1, s_2, s_3 as surplus variables and a_1, a_2, a_3 as artificial variables, the modified objective function for big- M method is given by.

$C = -4x_1 - 2x_2 - Ma_1 - Ma_2 - Ma_3$

and $3x_1 + x_2 - s_1 + 0s_2 + 0s_3 + a_1 + 0a_2 + 0a_3 = 27$ .... (i)

$x_1 + x_2 + 0s_1 - s_2 + 0s_3 + 0a_1 + a_2 + 0a_3 = 21$ .... (ii)

$x_1 + 2x_2 + 0s_1 + 0s_2 - s_3 + 0a_1 + 0a_2 + a_3 = 30$ .... (iii)

Using ( ), (ii), & (iii) in 'C' we get,

$C = -4x_1 - 2x_2 - M(27 - 3x_1 - x_2 + s_1) - M(21 - x_1 - x_2 + s_2) - M(30 - x_1 - 2x_2 + s_3)$

or, $C = (5M-4)x_1 + (4M-2)x_2 - Ms_1 - Ms_2 - Ms_3 = 78M$

$\therefore C - (5M-4)x_1 - (4M-2)x_2 + Ms_1 + Ms_2 + Ms_3 = -78M$ .. (iv)

The simplex table of (iv) with (i), (ii) & (iii) is,

Basic	C	entrant x_1 ↓	x_2	s_1	s_2	s_3	a_1	a_2	a_3	b	$\frac{b_i}{a_{ij}}$
	1	$-(5M-4)$	$-(4M-2)$	M	M	M	0	0	0	$-78M$	
a_1	0	$\boxed{3}$	1	-1	0	0	1	0	0	27	$\frac{27}{3}$ → Exit
a_2	0	1	1	0	-1	0	0	1	0	21	$\frac{21}{1}$
a_3	0	1	2	0	0	-1	0	0	1	30	$\frac{30}{1}$

Basic, $a_1 = 27$, $a_2 = 21$, $a_3 = 30$, non-basic $x_1 = x_2 = s_1 = s_2 = s_3 = 0$, for this $C = -78M$. Also, x_1–column is pivot column and R_2 is pivot row, so '3' is pivot element.

Applying, $R_1 \rightarrow R_1 + \frac{(5M-4)}{3} R_2$, $R_3 \rightarrow R_3 - \frac{1}{3} R_2$, $R_4 \rightarrow R_4 - \frac{1}{3} R_2$

The next simplex table is,

Basic	C	x_1	entrant x_2 ↓	s_1	s_2	s_3	a_1	a_2	a_3	b	$\frac{b_i}{a_{ij}}$
	1	0	$-\frac{(7M-2)}{3}$	$-2\frac{(M-2)}{3}$	M	M	$\frac{(4M-4)}{3}$	0	0	$-3(11M+12)$	
a_1	0	3	1	-1	0	0	1	0	0	27	$\frac{27}{1}$
a_2	0	0	$\frac{2}{3}$	$\frac{1}{3}$	-1	0	$-\frac{1}{3}$	1	0	12	$\frac{36}{2}$
a_3	0	1	$\boxed{\frac{5}{3}}$	$\frac{1}{3}$	0	-1	$-\frac{1}{3}$	0	1	21	$\frac{63}{5}$ → Exit

Basix $x_1 = 9$, $a_2 = 12$, $a_3 = 21$ and non-basic $x_2 = s_1 = s_2 = s_3 = a_1 = 0$ for this $C = -3(11M + 12)$. Also, x_2 – column is pivot column and R_4 is pivot row, so $\frac{5}{3}$ is pivot element.

Basic	C	x_1	x_2	entrant s_1 ↓	s_2	s_3	a_1	a_2	a_3	b	$\frac{b}{a_{ij}}$
	1	0	0	$-\frac{(M-6)}{5}$	M	$\frac{(12M-1)}{5}$	$\frac{6(M-1)}{5}$	0	$\frac{(7M-2)}{5}$	$-3\frac{(18M+222)}{5}$	
x_1	0	3	0	$-\frac{6}{5}$	0	$\frac{3}{5}$	$\frac{6}{5}$	0	$-\frac{3}{5}$	$\frac{72}{5}$	
a_2	0	0	0	$\boxed{\frac{1}{5}}$	-1	$\frac{2}{5}$	$-\frac{1}{5}$	1	$-\frac{2}{5}$	$\frac{18}{5}$	→ Exit
x_2	0	0	$\frac{5}{3}$	$\frac{1}{3}$	0	-1	$-\frac{1}{3}$	0	1	21	

Basic, $x_1 = \frac{24}{5}$, $x_2 = \frac{63}{5}$, $a_1 = \frac{18}{5}$ and non-basic $s_1 = s_2 = s_3 = a_1 = a_3 = 0$

for this $C = -\left(\frac{18M+222}{5}\right)$.

Also, s_1 –column is pivot column and R_3 is pivot row, so $\frac{1}{5}$ is pivot element.

Applying, $R_1 \rightarrow R_1 + (M-6)R_3$, $R_2 \rightarrow R_2 + 6R_3$, $R_4 \rightarrow R_4 - \frac{5}{3}R_3$

The next simplex table is.

Basic	C	x_1	x_2	s_1	s_2	s_3	a_1	a_2	a_3	b
	1	0	0	0	6	$\frac{14(M-1)}{5}$	M	$M-6$	$M+2$	-66
x_1	0	3	0	0	-6	3	0	6	-3	36
s_1	0	0	0	$\frac{1}{5}$	-1	$\frac{2}{5}$	$-\frac{1}{5}$	1	$-\frac{2}{5}$	$\frac{18}{5}$
x_2	0	0	$\frac{5}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{5}{3}$	15

Basic $x_1 = 12$, $x_2 = 9$ and non-basic $s_2 = s_3 = a_1 = a_2 = a_3 = 0$, for this $C = -66$

Since $a_1 = a_2 = a_3 = 0$, so at this step ' C' and 'w' are same so neglecting a_1, a_2, a_3 and rewriting the above table for w, we have,

Basic	w	x_1	x_2	s_1	s_2	s_1	b
	1	0	0	0	6	$\frac{14(M-1}{5}$	– 66
	0	3	0	0	– 6	3	36
	0	0	0	$\frac{1}{5}$	–1	$\frac{2}{5}$	$\frac{18}{5}$
	0	0	$\frac{5}{3}$	0	$\frac{5}{3}$	$-\frac{5}{3}$	15

Since there are no-negative terms in first row, so it completes the simplex process. Also, the max. $w = -66$ at $x_1 = 12$, $x_2 = 9$.

Hence, min $z = -w = 66$ at $x_1 = 12$ & $x_2 = 9$

(ii) **Maximize, $Z = x_1 + 2x_2 + 3x_3 - x_4$**

Subject to, $x_1 + 2x_2 + 3x_3 \leq 15$

$2x_1 + x_2 + 5x_3 \leq 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$, $x_1, x_2, x_3, x_4 \geq 0$

Solution: Using black variables s_1, s_2 and artificial variable, a_1,

$w = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1$ ...... (i)

and $x_1 + 2x_2 + 3x_3 + 0x_3 + s_1 + 0s_2 + 0a_1 = 15$ .... (ii)

$2x_1 + x_2 + 5x_3 + 0x_4 + 0s_1 + s_2 + 0a_1 = 20$ .... (iii)

$x_1 + 2x_2 + x_3 + x_4 + 0s_1 + 0s_2 + a_1 = 10$ .... (iv)

Using (iv) in (i), we have

$w = x_1 + 2x_2 + 3x_3 - x_4 - M(10 - x_1 - 2x_2 - x_3 - x_4)$

or, $w = (M + 1)x_1 + 2(M + 1)x_2 + (M + 3)x_3 + (M - 1)x_4 - 10M$

$\therefore w - (M + 1)x_1 - 2(M + 1)x_2 - (M + 3)x_3 - (M - 1)x_4 - 10M$

The simplex table of 'w' with (ii), (iii) & (iv) is,

Basic	w	x_1	entrant ↓ x_2	x_3	x_4	s_1	s_2	a_1	b	$\frac{b_i}{a_{ij}}$
	1	– (M + 1)	– 2(M + 1)	– (M + 3)	– (M – 1)	0	0	0	– 10M	
s_1	0	1	2	3	0	1	0	0	15	$\frac{15}{2}$
s_2	0	2	1	5	0	0	1	0	20	$\frac{20}{1}$
a_1	0	1	$\boxed{2}$	1	1	0	0	1	10	$\frac{10}{2}$ → Exit

Basic, $s_1 = 15$, $s_2 = 20$, $a_1 = 10$ and non-basic $x_1 = x_2 = x_3 = x_4 = 0$, for this $w = -10M$.

Also, x_2–column is pivot column and R_4 is pivot row, so '2' pivot element.

Applying, $R_1 \to R_1 + (M + 1) R_4$

$R_2 \to R_2 - R_4$

$R_3 \to R_3 - \frac{1}{2} R_4$, the next simplex table is.

Basic	w	x_1	x_2	x_3	x_4	s_1	s_2	a_1	b	$\frac{b}{a_*}$
	1	0	0	–2	2	0	0	M + 1	10	
s_1	0	0	0	2	–1	1	0	–1	5	
s_2	0	$\frac{3}{2}$	0	$\frac{9}{2}$	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	15	
x_2	0	1	2	1	1	0	0	1	10	

Basic, $x_2 = 5$, $s_1 = 5$, $s_2 = 15$ and non-basic $x_1 = x_3 = x_4 = a_1 = 0$, for this $w = 10$.

Since $a_1 = 0$, at this step w and z are same, so rewriting above table for z, we have,

Basic	z	x_1	x_2	x_3	x_4	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	0	0	–2	2	0	0	10	
s_1	0	0	0	$\boxed{2}$	–1	1	0	5	$\frac{5}{2}$ → Exit
s_2	0	$\frac{3}{2}$	0	$\frac{9}{2}$	$-\frac{1}{2}$	0	0	15	$\frac{30}{9}$
x_2	0	1	2	1	1	0	1	10	$\frac{10}{1}$

x_3–column is pivot column and R_2 is pivot row, so '2' is pivot element.

Applying, $R_1 \to R_1 + R_2$

$R_3 \to R_3 - \frac{4}{9} R_2$

$R_4 \to R_4 - \frac{1}{2} R_2$, the next simplex table is.

Basic	z	x_1	x_2	x_3	x_4	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
	1	0	0	0	1	1	0	15	
x_3	0	0	0	2	–1	1	0	5	
s_2	0	$\frac{3}{2}$	0	0	$-\frac{1}{18}$	$-\frac{4}{9}$	0	$\frac{115}{9}$	
x_2	0	0	1	2	$\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{15}{2}$	

There are no negative terms so, simplex process is completed, from table max$z = 15$ at $x_1 = 0$, $x_2 = \frac{15}{4}$, $x_3 = \frac{5}{2}$, $x_4 = 0$.

(iii) **Maximize, $z = 2x_1 + x_2 + 3x_3$**

Subjected to, $x_1 + x_2 + 2x_3 \leq 5$

$$2x_1 + 3x_2 + 4x_3 = 12 \quad x_1, x_2, x_3 \geq 10$$

Solution: Using s_1 as slack variable and a_1 as artificial variable. The modified objective function for big - M method, is

$w = 2x_1 + x_2 + 3x_3 - Ma_1$ .... (i)

$x_1 + x_2 + 2x_3 + s_1 + 0a_1 = 5$ .... (ii)

$2x_1 + 3x_2 + 4x_3 + 0s_1 + a_1 = 12$ .... (iii)

Using (iii) in (i), we have,

$W = 2x_1 + x_2 + 3x_3 - M(12 - 2x_1 - 3x_2 - 4x_3)$

or, $W = 2(M + 1)x_1 + (3x + 1)x_2 + (4x + 3)x_3 - 12M$

$\therefore W - 2(M + 1)x_1 - (3M + 1)x_2 - (4M + 3)x_3 = -12M$

The simplex table of 'W' with (ii) & (iii), is,

Basic	W	x_1	entrant ↓ x_2	x_3	s_1	a_1	b	$\frac{b_i}{a_{ij}}$
	1	$-2(M+1)$	$-(3M+1)$	$-(4M+3)$	0	0	$-12M$	
s_1	0	1	1	$\boxed{2}$	1	0	5	$\frac{5}{2}$ → Exit
a_1	0	2	3	4	0	1	12	$\frac{12}{4}$

Basic, $s_1 = 5$, $a_1 = 12$ and non-basic $x_1 = x_2 = x_3 = 0$, for this $W = -12M$. Also, x_2 column is pivot column and R_2 is pivot row, so '2' is pivot element.

Applying, $R_1 \rightarrow R_1 + \frac{(4m+3)}{2} R_2$

$R_3 \rightarrow R_3 - 2R_2$, the next simplex table is.

Basic	W	x_1	entrant ↓ x_2	x_3	s_1	a_1	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{1}{2}$	$-\frac{(2M-1)}{2}$	0	$\frac{4M+3}{2}$	0	$-\frac{(4M-15)}{2}$	
x_3	0	1	1	2	1	0	5	$\frac{5}{2}$
a_1	0	0	$\boxed{1}$	0	-2	1	2	$\frac{2}{1}$ → Exit

Basic, $x_3 = \frac{5}{2}$, $a_1 = 2$ and non-b asic $x_1 = x_2 = s_1 = 0$ for this

$W = -\frac{(4M-15)}{2}$. Also x_2 column is pivot column and R_3 is pivot row, so '1' is pivot element.

Applying. $R_1 \rightarrow R_2 - R_3$, the next simplex table is,

Basic	W	x_1	x_2	x_3	s_1	a_1	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{2M-1}{2}$	$\frac{13}{2}$	
x_3	0	1	0	2	3	-1	3	
x_2	0	0	1	0	-2	1	2	

Basic, $x_2 = 0$, $x_1 = 3$ and non-basic $x_1 = s_1 = a_1 = 0$, for this $W = \frac{13}{2}$.

Since $a_1 = 0$, so at this step W and Z are same,

So rewriting above table for z we get

Basic	W	entrant ↓ x_1	x_2	x_3	s_1	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{13}{2}$	
x_3	0	1	0	2	3	3	$\frac{3}{1}$ → Exit
x_2	0	0	1	0	-2	2	$\frac{2}{0}$

Basic, $x_2 = 2$, $x_3 = \frac{3}{2}$ and non-basic $x_1 = s_1 = 0$, for this $Z = \frac{13}{2}$.

Also, x_1 column is pivot column and R_2 is pivot row, so '1' is pivot element.

Applying $R_1 \rightarrow R_1 + \frac{1}{2} R_2$

$R_3 \rightarrow R_3$, the next simplex table is,

Basic	Z	x_1	x_2	x_3	s_1	b	$\frac{b_i}{a_{ij}}$
	1	0	0	1	2	8	
x_1	0	1	0	2	3	3	
x_2	0	0	1	0	-2	2	

Since there is no negative terms in first row so, simplex process is completed. From above table, $\max z = 8$ at $x_1 = 3, x_2 = 2$.

(iv) **Minimize, $z = 5x_1 + 3x_2$**

Subject to, $2x_1 + 4x_2 \leq 12$

$2x_1 + 2x_2 = 10$

$5x_1 + 2x_2 \geq 10$, $x_1, x_2 \geq 0$

Solution: Using mini-max theorem, given Lpp can be written as,

Maximise $W = -z = -5x_1 - 3x_2$

Subjected to, $x_1 + 2x_2 \leq 6$

$x_1 + x_2 = 5$

$5x_1 + 2x_2 \geq 10$

Using s_1 as black, s_2 as surplus and a_1 and a_2 as artificial variables, the modified function for big-M method is,

(v) Maximize, $z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to, $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10,\ x_1, x_2, x_3, x_4 \geq 0$

Solution: Using a_1, a_2, a_3 as artificial variables, the modified function for big M method is.

$W = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2 - Ma_3$ .... (i)

$W = x_1 + 2x_2 + 3x_3 + 0x_4 + a_1 + 0a_2 + 0a_3 = 20$ .... (ii)

and $x_1 + 2x_2 + 3x_3 + 0x_4 + a_1 + 0a_2 + 0a_3 = 20$ .... (ii)

$2x_1 + x_2 + 5x_3 + 0x_4 + 0a_1 + a_2 + 0a_3 = 20$ .... (iii)

$x_1 + 2x_2 + x_3 + x_4 + 0a_1 + 0a_2 + a_3 = 10$ .... (iv)

Using (ii), (iii) & (iv) in (i), we have

Basic	W	x_1	x_2	x_3	x_4	a_1	a_2	a_3	b	$\frac{b_i}{a_{ij}}$
	1	$-(4M+1)$	$-(5M+2)$	$-(9M+3)$	$-(M-1)$	0	0	0	-45	
a_1	0	1	2	3	0	1	0	0	15	$\frac{15}{3}$
a_2	0	2	1	$\boxed{5}$	0	0	1	0	20	$\frac{20}{5} \rightarrow$ Exit
a_3	0	1	2	1	1	0	0	1	10	$\frac{10}{1}$

Basic, $a_1 = 15$, $a_2 = 20$, $a_3 = 10$ and non-basic $x_1 = x_2 = x_3 = x_4 = 0$, for this $W = -35M$. Also x_3-column is pivot column and R_3 is pivot row, so 5 is pivot element.

Applying, $R_1 \rightarrow R_1 + \frac{(9M+3)}{5} R_3$

$R_2 \rightarrow R_2 - \frac{3}{5} R_3$

$R_4 \rightarrow R_4 - \frac{1}{5} R_3$ the next simplex table is,

Basic	W	x_1	x_2	x_3	x_4	a_1	a_2	a_3	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{(2M-1)}{5}$	$-\frac{(16M+7)}{5}$	0	$-(M-1)$	0	$\frac{9M+3}{5}$	0	$-9M+2$	
a_1	0	$-\frac{1}{5}$	$\frac{7}{5}$	0	0	1	$-\frac{3}{5}$	0	3	$\frac{15}{7} \rightarrow$ Exit
x_3	0	2	1	5	0	0	1	0	20	$\frac{20}{1}$
a_3	0	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	$-\frac{1}{5}$	1	6	$\frac{30}{9}$

Basic, $a_1 = 3$, $x_3 = 4$, a_3 6 and non-basic $x_1 = x_2 = x_4 = a_2 = 0$, for this $W = -9M + 12$.

Also, x_2 column is pivot column and R_2 is pivot row, so $\frac{7}{5}$ is pivot element.

Applying, $R_1 \rightarrow R_1 + \frac{(16+7)}{7} R_2$, $R_3 \rightarrow R_3 - \frac{5}{7} R_2$, $R_4 \rightarrow R_4 - \frac{9}{7} R_2$

The next simplex table is,

Basic	W	entrant ↓ x_1	x_2	x_3	x_4	a_1	a_2	a_3	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{6M}{7}$	0	0	$-(M-1)$	$\frac{16M+7}{7}$	$\frac{3M}{7}$	0	$-\frac{15M+105}{7}$	
x_2	0	$-\frac{1}{5}$	$\frac{7}{5}$	0	0	1	$-\frac{3}{5}$	0	3	
x_3	0	$\frac{15}{7}$	0	5	0	$-\frac{5}{7}$	$\frac{10}{7}$	0	$\frac{125}{7}$	$\frac{125}{15}$
a_3	0	$\frac{6}{7}$	0	0	1	$\frac{9}{7}$	$\frac{4}{7}$	1	$\frac{15}{7}$	$\frac{15}{6} \rightarrow$ Exit

Basic, $x_2 = \frac{15}{7}$, $x_3 = \frac{25}{7}$, $a_3 = \frac{15}{7}$ and non-basic $x_1 = x_4 = a_1 = a_2 = 0$,

for this, $W = -\frac{15M+105}{7}$.

Also, x_1 column is pivot column and R_4 is pivot row, so $\frac{6}{7}$ is pivot element.

Applying, $R_1 \rightarrow r_1 + MR_4$, $R_2 \rightarrow \frac{7}{30} R_4$, $R_3 \rightarrow R_3 - \frac{5}{2} R_4$

The next simplex table is,

Basic	W	x_1	x_2	x_3	x_4	a_1	a_2	a_3	b
	1	0	0	0	1	$M+1$	$\frac{10M}{7}$	M	15
	0	0	$\frac{7}{5}$	0	$\frac{7}{30}$	$\frac{7}{10}$	$-\frac{7}{15}$	$\frac{7}{30}$	$\frac{175}{14}$
	0	0	0	5	$-\frac{5}{2}$	$-\frac{71}{70}$	0	$-\frac{5}{2}$	$\frac{15}{7}$
	0	$\frac{6}{7}$	0	0	1	$-\frac{9}{7}$	$\frac{4}{7}$	1	

Basic, $x_1 = \frac{5}{7}$, $x_2 = \frac{5}{2}$, $x_3 = \frac{5}{2}$ and non-basic $x_1 = a_1 = a_2 = a_3 = 0$, for this W = 15

Since $a_1 = a_2 = a_3 = 0$ at this step W and Z are same.

Hence Max Z = 15 at $x_1 = \frac{5}{2}$, $x_2 = \frac{5}{2}$, $x_3 = \frac{5}{2}$, $x_4 = 0$

4. Solve the following linear programming problems by simplex method, using two phase method.

(i) Minimize $z = x_1 + x_2$

Subject to, $2x_1 + x_2 \geq 4$ (0, 4) (2, 0)

(7, 0) (0, 1) **$x_1 + 7x_2 \geq 7,\ x_1, x_2 \geq 0$**

Solution: Using s_1 and s_2 as surplus variables and a_1 and a_2 as artificial variable, given Lρρ can be written as,

Maximize $W = -Z = -x_1 - x_2$ .... (i) [mini-max theorem]

and $2x_1 + x_2 - s_1 + 0s_2 + a_1 + 0a_2 = 4$ .... (ii)

$x_1 + 7x_2 + 0s_1 - s_2 + 0a_1 + a_2 = 7$ .... (iii)

Phase I : The auxiliary objective function of w is,

Maximize, $A = -a_1 - a_2$ and using (ii) & (iii)

$\therefore A = -(4 - 2x_1 - x_2 + s_1) - (7 - x_1 - 7x_2 + s_2)$

or, $A = 3x_1 + 8x_2 - s_1 - s_2 - 11$

$\therefore A - 3x_1 - 8x_2 + s_1 + s_2 = -11$

the simplex table of A with (ii) and (iii) is,

Basic	A	x_1	↓ entrant x_2	s_1	s_2	a_1	a_2	b	$\frac{b_i}{a_{ij}}$
	1	–3	–8	1	1	0	0	–11	
a_1	0	2	1	–1	0	1	0	4	$\frac{4}{1}$
a_2	0	1	$\boxed{7}$	0	–1	0	1	7	$\frac{7}{7}$ → Exit

Basic $a_1 = 4$, $a_2 = 7$ and non-basic $x_1 = x_2 = s_1 = s_2 = 0$, for this $A = -11$.

Also, x_2 column is pivot column and R_3 is pivot row so, 7 is pivot element.

Applying, $R_1 \to R_1 + \frac{8}{7}R_3$, $R_2 \to R_2 - \frac{1}{7}R_3$, the next simplex table, is

Basic	A	↓ entrant x_1	x_2	s_1	s_2	a_1	a_2	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{13}{7}$	0	1	$-\frac{1}{7}$	0	$\frac{8}{7}$	–3	
a_1	0	$\frac{13}{7}$	0	–1	$\frac{1}{7}$	1	$-\frac{1}{7}$	3	→ Exit
a_2	0	1	7	0	–1	0	1	7	

Basic, $a_1 = 3$, $x_2 = 1$, non-basic $x_1 = s_1 = s_2 = a_2 = 0$, for this $A = 0$.

Also, x_1–column is pivot column and R_2 is pivot row, so $\frac{13}{7}$ is pivot element.

Applying $R_1 \to R_1 + R_2$, $R_3 \to R_3 - \frac{13}{7}R_2$. The next table is.

Basic	A	x_1	x_2	s_1	s_2	a_1	a_2	b
	1	0	0	0	0	1	1	0
	0	$\frac{13}{7}$	0	–1	$\frac{1}{7}$	1	$-\frac{1}{7}$	3
	0	0	7	$\frac{7}{13}$	$-\frac{14}{13}$	$-\frac{7}{13}$	$\frac{14}{13}$	$\frac{70}{13}$

Since there are no negative terms in first row, so maximum value of $A = 0$ at $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$ and $s_1 = s_2 = a_1 = a_2 = 0$

So, the first phase is compelled.

Phase II. Rewriting above table for W.

Basic	W	x_1	x_2	s_1	s_2	b
	1	1	1	0	0	0
	0	$\frac{13}{7}$	0	–1	$\frac{1}{7}$	3
	0	0	7	$\frac{7}{13}$	$-\frac{14}{13}$	$\frac{70}{13}$

Since there are no negative numbers in first row so we need not to use simplex step, furthere and phase II is complete. Hence, 'W' has maximum value at $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{3}$

Also, max $W = -x_1 - x_2$

$= -\frac{21}{13} - \frac{10}{13} = -\frac{31}{13}$

So, min $z = -\text{Max}(w) = \frac{31}{13}$ at $x_1 = \frac{21}{13}$ and $x_2 = \frac{10}{13}$

(ii) Minimize $Z = 2x_1 + 9x_2 + x_3$

Subject to, $x_1 + 4x_2 + 2x_3 \geq 5$

$3x_1 + x_2 + 2x_3 \geq 4,\ x_1, x_2, x_3 \geq 0$

Solution: Using s_1 and s_2 as susplus variables and a_1 and a_2 as artificial variables, given Lρρ can be written as,

Maximize $W = -z = -2x_1 - 9x_2 - x_3$ ...... (i)

and $x_1 + 4x_2 + 2x_3 - s_1 + 0s_2 + a_1 + 0a_2 = 5$ .... (ii)

$3x_1 + x_2\ 2x_3 + 0s_1 - s_2 + 0a_1 + a_2 = 4$ .... (iii)

Phase I : The auxiliary objective function is,

max $A = -a_1 - a_2$, using (ii) and (iii)

$A = -(5 - x_1 -- 4x_2 - 2x_3 + s_1) - (4 - 3x_1 - x_2 - 2x_3 + s_2)$

or, $A = 4x_1 + 5x_2 + 4x_3 - s_1 - s_2 - 9$

$\therefore A - 4x_1 - 5x_2 - 4x_3 + s_1 + s_2 = -9$

The simplex table of A, with (ii) and (iii) is,

Basic	A	x_1	entrant x_2	x_3	s_1	s_2	a_1	a_2	b	$\frac{b}{a}$
	1	−4	−5	−4	1	1	0	0	−9	
a_1	0	1	$\boxed{4}$	2	−1	0	1	0	4	$\frac{5}{4}$ → Exit
a_2	0	3	1	2	0	−1	0	1	4	$\frac{4}{1}$

Basic, $a_1 = 5$, $a_2 = 4$ and non-basic $x_1 = x_2 = x_3 = s_1 = s_2 = 0$, for this $A = -9$.

Also, x_2 column is pivot column and R_2 is pivot row, so 4 is pivot element.

Applying $R_1 \to R_1 + \frac{5}{4} R_2$. $R_3 \to R_3 - \frac{1}{4} R_2$. the next table is,

Basic	A	entrant x_1	x_2	x_3	s_1	s_2	a_1	a_2	b	$\frac{b_i}{a_{ij}}$
	1	$-\frac{11}{4}$	0	$-\frac{3}{2}$	$-\frac{1}{4}$	1	$\frac{5}{4}$	0	$-\frac{11}{4}$	
	0	1	4	2	−1	0	1	0	5	
	0	$\frac{11}{4}$	0	$\frac{3}{2}$	$\frac{1}{4}$	−1	$-\frac{1}{4}$	1	$\frac{11}{4}$	→ Exit

Basic, $x_2 = \frac{5}{4}$, $a_2 = \frac{11}{4}$ and non-basic $x_1 = x_3 = s_1 = s_2 = a_1 = 0$ for this $A = -\frac{11}{4}$.

Also, x_1 column's pivot column and R_3 is pivot row, so $\frac{11}{4}$ is pivot element.

Applying $R_1 \to R_1 + R_3$, $R_2 \to R_2 - \frac{4}{11} R_3$, the next simplex table is,

Basic	A	x_1	x_2	x_3	s_1	s_2	a_1	a_2	b
	1	0	0	0	0	0	1	1	0
	0	0	4	$\frac{16}{11}$	$-\frac{12}{11}$	$\frac{4}{11}$	$\frac{12}{11}$	$-\frac{4}{11}$	4
	0	$\frac{11}{4}$	0	$\frac{3}{2}$	$\frac{1}{4}$	−1	$-\frac{1}{4}$	1	$\frac{11}{4}$

Since there are no negative numbers in first, row, so max A = 0 at $x_1 = 1$, $x_2 = 1$ and $x_3 = s_1 = s_2 = a_1 = a_2 = 0$.
Which completes first phase.
Phase II: Rewriting above table for W,

Basic	W	x_1	x_2	x_3	s_1	s_2	b
	1	2	9	1	0	0	0
	0	0	4	$\frac{16}{11}$	$-\frac{12}{11}$	$\frac{4}{11}$	4
	0	$\frac{11}{4}$	0	$\frac{3}{2}$	$\frac{1}{4}$	−1	$\frac{11}{4}$

Since there are no-negative numbers is first row, so it completes phase II. Hence at $x_1 = 1$, $x_2 = 1$, $x_3 = 0$

$$\text{Max } W = -2x_1 - 9x_2 - x_3$$
$$= -2.1 - 9.1 - 0$$
$$= -11$$

Hence, min $z = -\max(w) = 11$ at $x_1 = 1$, $x_2 = 1$, $x_3 = 0$

(iii) **Minimize, $Z = 5x_1 - 2x_2 + 3x_3$**

Subject to, $2x_1 + 2x_2 - x_3 \geq 2$

$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5,\ x_1, x_2, x_3 \geq 0$$

Solution: Using s_1 as surpluse, s_2 and s_3 as slack variables and a_1 as artificial variable, given Lpp can be written as,

Maximize, $W = -z = -5x_1 + 2x_2 - 3x_3$ .... (i)

and $2x_1 + 2x_2 - x_3 - s_1 + 0s_2 + 0_3 + a_1 = 2$ .... (ii)

$3x_1 - 4x_2 + 0x_3 + 0s_1 + s_2 + 0s_3 + 0a_1 = 3$ .... (iii)

$0x_1 + x_2 + 3x_3 + 0s_1 + 0s_3 + s_3 + 0a_1 = 5$ .... (iv)

Phase : I

The auxiliary object active function is,

Maximize $A = -a_1$

or, $A = -2(2 - 2x_1 - 2x_2 + x_3 + s_1)$

$\therefore A - 2x_1 - 2x_2 + x_3 + s_1 = -2$

The simplex table of A with (ii), (iii) & (iv) is,

Basic	A	entrant x_1	x_2	x_3	s_1	s_2	s_3	a_1	b	$\frac{b_i}{a_{ij}}$
	1	−2	−2	1	1	0	0	0	−2	
a_1	0	$\boxed{2}$	2	−1	−1	0	0	1	2	$\frac{2}{2}$ → Exit
s_2	0	3	−4	0	0	1	0	0	3	$\frac{2}{3}$
s_3	0	0	1	3	0	0	1	0	5	$\frac{5}{0}$

Basic, $S_2 = 3$, $s_3 = 5$, $a_1 = 2$ and non-basic $x_1 = x_2 = x_3 = s_1 = 0$, for this $A = -2$.

Also x_1 column is pivot column and R_2 row is pivot row, so '2' is pivot element.

Applying $R_1 \to R_1 + R_2$, $R_3 \to R_3 - \frac{3}{2} R_2$, $R_4 \to R_4$.

The next simplex table is,

Basic	A	x_1	x_2	x_3	s_1	s_2	s_3	a_1	b
	1	0	0	0	0	0	0	1	0
	0	2	2	–1	–1	0	0	1	2
	0	0	–7	$\frac{3}{2}$	$\frac{3}{2}$	1	0	$-\frac{3}{2}$	0
	0	0	1	3	0	0	1	0	5

Since, there are no. gegative negative numbers in first row, maximum value of $A = 0$ at $x_1 = 1, x_2 = 0, x_3 = 0$.
Also, $a_1 = 0$, which completes first phas,

Phase : II
Rewriting above table for w, from (i)

Basic	W	x_1	entrant ↓ x_2	x_3	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	5	–2	3	0	0	0	0	
	0	2	$\boxed{2}$	–1	–1	0	0	2	$\frac{2}{2}$ → Exit
s_2	0	0	–7	$\frac{3}{2}$	$\frac{3}{2}$	1	0	0	$\frac{0}{-7}$
s_3	0	0	1	3	0	0	1	5	$\frac{5}{0}$

x_2–column is pivot column and R_2 is pivot row, so '2' is pivot element.

Applying $R_1 \to R_1 + R_2$, $R_3 \to R_3 + \frac{7}{2} R_2$

$R_4 \to R_4 - \frac{1}{2} R_2$, the next simplex table is,

Basic	W	x_1	x_2	x_3	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	7	0	2	–1	0	0	2	
x_2	0	2	2	–1	–1	0	0	2	
s_2	0	7	0	–2	–2	1	0	7	
s_3	0	–1	0	$\frac{7}{2}$	$\frac{1}{2}$	0	1	4	→ Exit

Basic, $x_2 = 1$, $s_2 = 7$, $s_3 = 4$ and non-basic $x_1 = x_3 = s_1 = 0$, for this $w = 2$.

Also, s_1 column is pivot column and r_4 is pivot row, so, $\frac{1}{2}$ is pivot element.

Applying $R_1 \to R_1 + 2R_4$
$R_2 \to R_2 + 2R_4$

$R_3 \to R_3 + 4R_4$, the simplex table is,

Basic	W	x_1	x_2	x_3	s_1	s_2	s_3	b	$\frac{b_i}{a_{ij}}$
	1	5	0	9	0	0	2	10	
x_2	0	0	2	6	0	0	2	10	
s_2	0	4	0	12	0	1	4	23	
s_3	0	–1	0	$\frac{7}{2}$	$\frac{1}{2}$	0	1	4	

Since there are no negative number is first row, so max $W = 10$ at $x_1 = 0, x_2 = 5, x_3 = 0$.
Hence, min $W = -10$ at $x_1 = 0, x_2 = 5, x_3 = 0$

(iv) **Maximize,** $Z = 3x_1 - x_2$

$$2x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 2$$
$$x_2 \leq 4,\ x_1, x_2 \geq 0$$

Solution: Using s_1 as surplus variable s_2 and s_3 as black variables, and a_1 as artificial variable,

$z = 3x_1 - x_2$ .... (i)
and $2x_1 + x_2 - s_1 + 0s_2 + 0s_3 + a_1 = 2$ .... (ii)
$x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 + 0a_1 = 2$ .... (iii)
$0x_1 + x_2 + 0s_1 + 0s_2 + s_3 + 0a_1 = 4$ .... (iv)

The auxiliary objective function is,
$A = -a_1$
or, $A - (2 - 2x_1 - x_2 + s_1)$
$\therefore A - 2x_1 - x_2 + s_1 = -2$

The simplex table of A with (ii), (iii), & (iv) is,

Basic	A	entrant ↓ x_1	x_2	s_1	s_2	s_3	a_1	b	$\frac{b_i}{a_{ij}}$
	1	–2	–1	1	0	0	0	–2	
a_1	0	2	1	–1	0	0	1	2	$\frac{2}{2}$ → Exit
s_2	0	1	3	0	1	0	0	2	$\frac{2}{1}$
s_3	0	0	1	0	0	1	0	4	$\frac{4}{10}$

Basic $a_1 = 2$, $s_2 = 2$, $s_3 = 4$ and non-basic $x_1 = x_2 = s_1 = 0$, for this $A = -2$.

Also, x_1 column is pivot column and R_2 is pivot row, so 2 is pivot element.

Applying, $R_1 \to R_1 + R_2$

$R_3 \to R_3 - \frac{1}{2} R_2$, the next simplex table is,

Basic	A	x_1	x_2	s_1	s_2	s_3	a_1	b
	1	0	0	0	0	0	1	0
x_1	0	2	1	–1	0	0	1	2
s_2	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	1
s_3	0	0	1	0	0	1	0	4

Basic, $x_2 = 1$, $s_2 = 1$, $s_3 = 4$ and non-basic $x_2 = s_1 = a_1 = 0$, for this $A = 0$.

Since, there are no negative numbers in first row so, max $A = 0$ which completes first phase.

Phase II

Rewriting above table for z,

Basic	Z	x_1	x_2	s_1	s_2	s_3	b	$\frac{b_i}{a_{ii}}$
	1	–3	1	0	0	0	0	
	0	$\boxed{2}$	1	–1	0	0	2	→ exit
s_2	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	1	
s_3	0	0	1	0	0	1	4	

Basic, $s_2 = 1$, $s_3 = 4$ and non-basic $x_1 = x_2 = s_1 = 0$, for this $z = 0$.

Also, x_1 column is pivot column and R_2 is pivot row, so 2 is pivot element.

Applying $R_1 \to R_1 + \frac{3}{2} R_2$, $R_3 \to R_3$, $R_4 \to R_4$.

The next table is,

Basic	Z	x_1	x_2	s_1	s_2	s_3	b
	1	0	$\frac{5}{2}$	$-\frac{3}{2}$	0	0	3
x_1	0	2	1	–1	0	0	2
s_2	0	0	$\frac{5}{2}$	$\boxed{\frac{1}{2}}$	1	0	1
s_3	0	0	1	0	0	1	4

Basic, $x_1 = 1$, $s_1 = 1$, $s_3 = 4$ and non-basic $x_2 = s_1 = 0$, for this $Z = 3$.

Also, s_1 – column is pivot column and R_3 is pivot row, so $\frac{1}{2}$ is pivot element.

Applying. $R_1 \to R_1 + 3R_3$, $R_2 \to R_2 + 2R_3$, $R_4 \to R_4$

the next simplex table is,

Basic	Z	x_1	x_2	s_1	s_2	s_3	b
	1	0	4	0	$\frac{3}{2}$	0	6
	0	2	–4	0	–2	0	4
	0	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	1
	0	0	1	0	0	1	4

Since, there are no negative terms in first row, so the simplex process is completed. Hence max $z = 6$ at $x_1 = 2$ & $x_2 = 0$

Fourier Series

5

Formula

1. The Fourier series of a periodic function f(x) of period 2π defined in the internal $(\alpha, \alpha + 2\pi)$ is defined by

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos x + b_n \sin nx)$$

Where, $a_o = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\,dx,\ n = 1, 2, \ldots..$

and $b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\,dx,\ x = 1, 2, \ldots\ldots.$

For $\alpha = 0$, interval $(0, 2\pi)$

$a_o = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx,\ n = 1, 2, \ldots\ldots$

and $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx,\ n = 1, 2 \ldots\ldots\ldots$

For $\alpha = -\pi$, interval $(-\pi, \pi)$

$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx,\ n = 1, 2, \ldots\ldots\ldots..$

and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx,\ n = 1, 2, \ldots\ldots..$

Note : The interval $(\alpha, \alpha + 2\pi)$ may be closed interval $[\alpha, \alpha + 2\pi]$

2. The half range Fourier cosine series of a function f(x) in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx,\ n = 1, 2, \ldots\ldots..$

and $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

This formula is also known as the Fourier series of an even function f(x) : f(–x) = f(x) for all x.

3. The half range Fourier sine series of f(x) in $(0, \pi)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{Where, } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx, \; n = 1, 2, \ldots\ldots$$

This formula is also known as the Fourier series of an odd function f(x): $f(-x) = -f(x)$ for all x.

Important keys to make calculation easy.

1. $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_3 + \ldots\ldots\ldots$
   Where dashes in u denote successive differentiation w. r. t. x and subscripts in v denote the successive integration w. r. t. x. This formula is used when one of two u & v is algebraic.
2. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

   $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$
3. $\sin n\pi = 0, n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$
   $\cos n\pi = (-1)^n, \; n = 0, 1, 2, 3, \ldots\ldots$
   $\cos 2n\pi = 1, n = 0, 1, 2, 3, \ldots\ldots\ldots$
   $\cos (2n+1)\pi = -1 = \cos (2n-1)\pi, n = 0, 1, 2, \ldots\ldots$
4. $\int_{\alpha}^{\alpha+2\pi} \cos nx\, dx = 0, n \neq 0$
5. $\int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = 0, n \neq 0$
6. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx dx = 0, m \neq n$

   $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx dx = \pi$, for m = n
7. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx dx = 0$, for any m & n
8. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx dx = 0$ for $m \neq n$

   $= \pi$ for m = n
9. If f(x) is discontinuous at a point c in the interval $(\alpha, \alpha + 2\pi)$ then
   $f(c) = \frac{1}{2}[f(c-0) + f(c+0)]$
10. To deduce the numerical series from the calculated Fourier series put one of the value of x from the given interval looking the asked series/ that numerical series.

Formula of Fourier series in arbitary range (– L, L)

The Fourier series of a periodic function f(x) of period 2L in the interval (– L, L) is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi\frac{x}{L} + b_n \sin n\pi\frac{x}{L}\right)$$

$$\text{Where } a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos n\pi\frac{x}{L} dx, n = 1, 2, \ldots\ldots$$

$$\text{and } b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin n\pi\frac{x}{L} dx, n = 1, 2, \ldots\ldots\ldots$$

Notes : If the interval is (0, 2L) the limits only changes and rest keeps same.

2. Here L may be any non zero number like π, 1 etc.

The half range Fourier cosine series of f(x) in (0, L) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi\frac{x}{L}$$

$$\text{Where, } a_0 = \frac{2}{L}\int_0^{L} f(x)\, dx$$

$$\text{and } a_n = \frac{2}{L}\int_0^{L} f(x) \cos n\pi\frac{x}{L} dx, n = 1, 2, \ldots\ldots$$

and the half range Fourier sine series of f(x) in (0, L) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi\frac{x}{L} dx$$

$$\text{Where, } b_n = \frac{2}{L}\int_0^{L} f(x) \sin n\pi\frac{x}{L} dx, n = 1, 2, \ldots\ldots$$

Exercise - 17

1. **Obtain a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$ and deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \ldots\ldots$** [BE 2062]

Solution: let $f(x) = x - x^2$

The Fourier series of $f(x) = x - x^2$ in $[-\pi, \pi]$ is

$$f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi + b_n \sin n\pi) \;\ldots\ldots\ldots \text{(i)}$$

$$\text{Where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x dx - \int_{-\pi}^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}[0 - 2]\int_0^{\pi} x^2\, dx$$ [$\because$ using the properties of odd & even function resp.]

$$= -\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi$$

$$= -\frac{2}{\pi}\times\frac{\pi^3}{3} = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, 3, \ldots\ldots\ldots$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx - \int_{-\pi}^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\int_0^\pi x^2\cos nx\,dx\right]$$ [∵ x cosnx is an odd function & x^2 cosnx is an even function]

$$= \frac{-2}{\pi}\int_0^\pi x^2\cos nx\,dx$$

Let, $u = x^2$, $v = \cos nx$

Diff. u & integrating v w. r. t. x, we get

$u' = 2x$, $v_1 = \frac{\sin nx}{n}$

$u'' = 2$, $v_2 = -\frac{\cos nx}{n^2}$, $v_3 = -\frac{\sin nx}{n^3}$

Now, $a_n = -\frac{2}{n}\int_0^\pi x^2\cos nx\,dx$

$$= -\frac{2}{n}\int_0^\pi u\,v\,dx = -\frac{2}{n}\left[uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]_0^\pi$$

$$= -\frac{2}{\pi}\left[x^2.\frac{\sin nx}{n} + 2x.\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^\pi$$

$$= -\frac{2}{\pi}\left[2\pi.\frac{(-1)^n}{n^2}\right]$$

$$= -\frac{4(-1)^n}{n^2}$$

$$= \frac{4(-1)^{n+1}}{n^2}$$

and,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n = 1, 2, \ldots\ldots\ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin ndx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\,dx - \int_{-\pi}^{\pi} x^2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_0^\pi x\sin nx\,dx - 0\right]$$ [∵ x sin nx is even & x^2 sin nx is odd

function]

$$= \frac{2}{\pi}\left[x - \frac{\cos x}{n} + \frac{\sin nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n}\right]$$

$$= 2\frac{(-1)^{n+1}}{n}$$

Substituting a_0, a_n & b_n is (i), we get

$$x - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[4\frac{(-1)^{n+1}}{n^2}\cos nx - 2\frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$\therefore\ x - x^2 = \frac{-\pi^2}{3} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\left[2\frac{\cos nx}{n^2} - \frac{\sin nx}{n}\right]$$ is the req^d

Fourier series of $f(x) = x - x^2$

Next, put $x = 0$ in the above series

$$0 = \frac{-\pi^2}{3} + 2\sum_{n=1}^{\infty}(-1)^{n+1}2.\frac{1}{n^2}$$

$$\text{or, } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots$$

$$\therefore\ \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$$

2. **If $f(x) = x + x^2$ for $-\pi \le \pi$, find the Fourier series of f(x) and deduce het $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots$**

Solution: The Fourier series of $f(x) = x + x^2$ in $[-\pi, \pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\ \ldots\ldots\ (i)$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx + \int_{-\pi}^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + 2\int_0^\pi x^2\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi$$

$$= \frac{2}{\pi}.\frac{\pi^3}{3}$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \ldots\ldots\ldots$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx dx + \int_{-\pi}^{\pi} x^2 \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[0 + 2 \int_0^{\pi} x^2 \cos nx dx \right]$$

[$\because$ x cosnx is odd and x^2 cosnx is even function]

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[x^2 . \frac{\sin nx}{n} \Big|_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[0 - \frac{2}{n} \left\{ x . -\frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right\}_0^{\pi} \right]$$

$$\text{or, } a_n = \frac{4}{\pi} \left[\pi \frac{(-1)^n}{n^2} \right] = 4 \frac{(-1)^n}{n^2}, n = 1, 2, \dots\dots$$

$$\text{and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \; n = 1, 2, \dots\dots$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \sin nx dx + \int_{-\pi}^{\pi} x^2 \sin nx dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \sin nx dx + 0 \right]$$

[$\because$ x sin nx being even and x^2 sinnx being odd function]

$$= \frac{2}{\pi} \left[x . -\frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\pi \frac{(-1)^n}{n} \right]$$

$$= 2 \frac{(-1)^{n+1}}{n}$$

Substituting a_0, a_n & b_n is (i), we get,

$$x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[4 \frac{(-1)^n}{n^2} \cos nx + 2 \frac{(-1)^{n+1}}{n} \sin nx \right]$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2} 4 \cos nx + 2 \frac{(-1)^{n+1}}{n} \sin nx \right]$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[4 \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n} \right]$$

The range is $[-\pi, \pi]$, choosing $x = 0$ we get

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2}$$

$$\text{or, } -\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\text{or, } -\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots\dots\dots$$

$$\text{or, } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots\dots$$

$$\text{or, } \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots - \frac{2}{2^2} - \frac{2}{4^2} - \frac{2}{6^2} \dots\dots$$

$$= \sum_{n=n^2}^{\infty} \frac{1}{n^2} - \frac{2}{4} \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$= \sum_{n=n^2}^{\infty} \frac{1}{n^2} - \frac{2}{4} \sum \frac{1}{n^2}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\dots$$

3. **Expand the function f(x) = x sinx as a Fourier series in the interval $-\pi \le x \le \pi$.**

Solution: The Fourier series of $f(x) = x\sin x$ in $[-\pi, \pi]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \dots\dots \text{(i)}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin dx$$

$$= \frac{2}{\pi} [x . -\cos x + \sin x]_0^{\pi}$$

$$= 2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, n = 1, 2, \dots\dots\dots\dots$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx$$

[$\because$ x sinx cos nx being an even function]

$$= \frac{1}{\pi} \int_0^{\pi} x[\sin (x + nx) + \sin (x - 4x)] \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x\{\sin (n + 1) x - x \sin (n - 1) x\} \, dx$$

[Note, the term x sin $(n - 1)$ x carries the denominator $(n - 1)$ after integration which is problem for $n = 1$ case. So, we calculate $n = 1$ case separately and the rest cases together]

For $n \neq 1$

$$a_n = \frac{1}{\pi}\left[x.-\frac{\cos(n+1)x}{(n+1)} + \frac{\sin(n+1)x}{(n+1)^2} + x.\frac{\cos(n-1)x}{(n-1)} - \frac{\sin(n-1)x}{(n-1)^2}\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^{n+1}}{(n+1)} + \pi\frac{(-1)^{n-1}}{(n-1)}\right]$$

$$= \frac{(-1)^{n+2}}{(n+1)} - \frac{(-1)^{n-1}}{(n-1)}$$

$$= \frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{(n-1)}$$

$$= (-1)^n \frac{[n-1-n+1]}{(n^2-1)}$$

$$= \frac{2(-1)^n}{(n^2-1)}$$

For $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx = \frac{1}{\pi}\left[x.-\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2} + 0\right] = \frac{1}{2}$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$, $n = 1, 2, \ldots\ldots\ldots$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \sin nx\, dx = 0$ being $x \sin x \sin nx$ as an odd function.

Substituting a_0, a_n & b_n is ...... (i), we get

$$x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^n}{(n^2-1)}\cos nx$$

$$\text{or, } x \sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n^2-1)}\cos nx$$

This is the required Fourier series for $f(x) = x \sin nx$.

4. **Obtain a Fourier series for $f(x) = x^3$ in the interval $-\pi \le x \le \pi$.** [BE 065]

Solution: Let $f(x) = x^3 = \frac{a_0}{2} + \sum_{n=2}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\, dx = 0$ being x^3 an odd function.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \text{ con } nx\, dx$, $n = 1, 2, \ldots\ldots\ldots$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos nx\, dx = 0$ being $x^3 \cos nx$ an odd function.

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$, $n = 1, 2, \ldots\ldots\ldots$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\, dx$ being $x^3 \sin nx$ an even function.

$u = x^3$, $v = \sin nx$

Diff. u & integrating v w. r. t. x successively,

$u' = 3x^2$ $v_1 = -\frac{\cos nx}{n}$

$u'' = 6x$ $v_2 = -\frac{\sin nx}{n^2}$

$u''' = 6$ $v_3 = \frac{\cos nx}{n^3}$, $v_4 = \frac{\sin nx}{n^4}$

$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} uvx\, dx$

$$= \frac{2}{\pi}\left[uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^3.-\frac{\cos nx}{n} + 3x^2.\frac{\sin nx}{n^2} + 6x.\frac{\cos nx}{n^3} - 6\frac{\sin nx}{n^4}\right]$$

$$= \frac{2}{\pi}\left[-\pi^3.\frac{(-1)^n}{n} + 6\pi\frac{(-1)^n}{n^3}\right]$$

$$= 2\left[x^2\frac{(-1)^{n+1}}{n} + 6\frac{(-1)^n}{n^3}\right]$$

$$= 2\left[\frac{6(-1)^n}{n^3} - \pi^2\frac{(-1)^n}{n}\right]$$

Substituting a_0; a_n & b_n in (i)

$$x^3 = \sum_{n=1}^{\infty} 2\left[\frac{6(-1)^n}{n^3} - \pi^2\frac{(-1)^n}{n}\right]\sin nx$$

$$= 2\sum_{n=1}^{\infty}\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right](-1)^n \sin nx$$

This is the required Fourier series for $f(x) = x^3$

5. **Obtain the Fourier series to represent e^{-ax} from $-\pi$ to π and derive series for $\frac{\pi}{\sinh a\pi}$**

Solution: Let $f(x) = e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi} - e^{a\pi}}{-a}\right] = \frac{1}{\pi a}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+x^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}.-a(-1)^n + a.e^{a\pi}(-1)^n}{a^2+x^2}\right]$$

$$= \frac{a}{\pi}\left[\frac{e^{-a\pi}(-1)^{n+1} + (-1)^n e^{a\pi}}{a^2+x^2}\right]$$

$$= \frac{a}{\pi}\,\frac{(-1)^n}{a^2+n^2}\left[e^{a\pi} - e^{-a\pi}\right]$$

and, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ x = 1, 2, 3, \ldots\ldots\ldots$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx + n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}.-x(-1)^x + \frac{e^{a\pi}}{x^2+a^2}.x(-1)^x\right]$$

$$= \frac{1}{\pi}\,\frac{x}{(a^2+n^2)}\left[-e^{-a\pi}(-1)^x + e^{a\pi}(-1)^x\right]$$

$$= \frac{1}{\pi}\,\frac{x(-1)^x}{(a^2+n^2)}\left[e^{-a\pi} - e^{-a\pi}\right]$$

Substituting a_0, a_n & b_n in (i)

$$e^{-ax} = \frac{1}{2\pi a}\left[e^{a\pi} - e^{-a\pi}\right] + \sum_{n=1}^{\infty}\left[\frac{a}{\pi}\frac{(-1)^n}{(n^2+a^2)}(e^{a\pi} - e^{-a\pi})\cos nx + \frac{1}{\pi}\frac{(-1)^n}{(n^2+a^2)}(e^{a\pi} - e^{-a\pi})\sin nx\right]$$

$$= \frac{(e^{a\pi} - e^{-a\pi})}{\pi}\left[\frac{1}{2a} + \sum_{n=1}^{\infty}\left[\frac{a\cos nx + n\sin nx}{(n^2+a^2)}\right]\right](-1)^n$$

$$= \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{a\cos nx + \sin nx}{(n^2+a^2)}\right]\right]$$

Where $\sinh x = \frac{e^x - e^{-x}}{2}$, $x = a\pi$

This is the required Fourier series for $f(x) = e^{-ax}$

Next, putting $x = 0$ in the aove series, we get,

$$1 = \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n a}{(n^2+a^2)}\right]$$

or, $$\frac{2\sinh a\pi}{\pi} = \frac{1}{a} - 2a\left[\frac{-a}{(1^2+a^2)} + \frac{a}{2^2+a^2} + \frac{1}{3^2+a^2} + \ldots\ldots\right]$$

$$= \frac{1}{a} - 2a\left[\frac{1}{1^2+a^2} - \frac{1}{2^2+a^2} + \frac{1}{3^2+a^2} - \ldots\ldots\right]$$

$$\therefore \frac{\pi}{2\sinh a\pi} = \frac{1}{a} - 2a\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n^2+a^2)}$$

6. **Prove that in the interval $-\pi \le \pi$ the Fourier series for x cosx =** $-\frac{\sin x}{2} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\sin nx$.

Solution: Let $f(x) = x\cos x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ .... (i)

Where,

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\,dx = 0$ being x cosx odd function.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \ldots\ldots$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\cos nx\,dx = 0$ being x cosx cosnx odd function.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ x = 1, 2, \ldots\ldots$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,2\sin nx\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[x\sin(n+1)x + x\sin(n-1)x]\,dx$$

For $n \ne 1$, $$b_n = \frac{1}{\pi}\left[x.-\frac{\cos(n+1)x}{(n+1)} + \frac{\sin(n+1)x}{(n+1)^2} + x.-\frac{\cos(n-1)x}{(n-1)} + \frac{\sin(n-1)x}{(n-1)^2}\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^{n+1}}{(n+1)} - \pi\frac{(-1)^{n-1}}{(n-1)}\right]$$

$$= \frac{(-1)^{n+2}}{(n+1)} + \frac{(-1)^n}{(n-1)}$$

$$= (-1)^n \left[\frac{1}{(n+1)} - \frac{1}{(n-1)}\right]$$

$$= 2(-1)^n \frac{n}{(n^2-1)}$$

For n = 1, $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin 2x \, dx$

$$= \frac{1}{\pi}\left[x.-\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= -\frac{1}{2}$$

Substituting a_0, a_n, b_1 & b_n in (1)

$$x\cos x = -\frac{\sin x}{2} + 2\sum_{n=1}^{\infty}(-1)^n \frac{n}{(n^2-1)} \sin nx.$$

This is the required Fourier series for x cosx.

7. Obtain x Fourier series to represent the function f(x) = |x| for $-\pi \le x \le \pi$ and hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots$

[IOE 2060]

Solution: Let $f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx) \dots\dots\dots (1)$

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi}\int_0^{\pi} |x|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx$, x = 1, 2, ...

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx$, n = 1, 2, ...

$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx dx = 0$ being |x| sin nx odd function.

Substituting a_0, a_n & b_n in (1)

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right] \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos x - 0 - \frac{2}{3^2}\cos 3x - 0 - \frac{2}{5^2}\cos 5x - \dots\right]$$

$$\therefore |x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right]$$

This the required Fourier series for f(x) = |x|.

Next, putting x = 0 from [− π, π] in the above series

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\right]$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots$$

8. Obtain the Fourier series for the function f(x) given by

$$f(x) = 1 + \frac{2x}{\pi}, \; -\pi \le x \le 0$$

$$\quad\;\; 1 - \frac{2x}{\pi}, \; 0 \le x \le \pi$$

and hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \dots\dots (1)$

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_0^{\pi} f(x)dx\right]$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)dx + \int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx\right]$$

$$= \frac{1}{\pi}\left[x + \frac{x^2}{\pi}\right]_{-\pi}^{0} + \left[x - \frac{x^2}{\pi}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi - \frac{\pi^2}{\pi} + \pi - \frac{\pi^2}{\pi}\right] = 0$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx$, n = 1, 2, ...

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx + \int_0^{\pi} f(x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\cos nx dx + \int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\left(1+\frac{2x}{\pi}\right).\frac{\sin nx}{n} + \frac{2}{\pi}\frac{\cos nx}{n^2}\right|_{-\pi}^{0}$$

$$+ \left(1-\frac{2x}{\pi}\right).\frac{\sin nx}{n} - \frac{2}{\pi}\frac{\cos nx}{n^2}\Big|_0^{\pi}\Bigg]$$

$$= \frac{2}{\pi^2}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{4}{\pi^2}\left[\frac{1-(-1)^n}{n^2}\right]$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ x = 1, 2, \ldots\ldots$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\sin nx\,dx + \int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(1+\frac{2x}{\pi}\right).-\frac{\cos nx}{n} + \frac{2}{\pi}\frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$+\left(1-\frac{2x}{\pi}\right).\frac{\cos nx}{n} - \left[\frac{2}{\pi}\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n} + \frac{(-1)^n}{n} + \frac{1}{n}\right] = 0$$

Substituting a_0, a_n & b_n is (1)

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]$$

Putting x = 0 in the above series

$$1 = \frac{4}{\pi^2}\left[\frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + \ldots\ldots\right]$$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\right]$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots$$

9. Obtain the Fourier series for the function f(x) given by,

$$f(x) = -1 \text{ for } -\pi < x < -\frac{\pi}{2}$$

$$= 0 \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$= 1 \text{ for } \frac{\pi}{2} < x < \pi$$

Solution: Let $f(x) = \frac{a}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (1)

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_0 f(x)dx - \frac{\pi}{2} + \int_{\frac{\pi}{2}} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}}(-1)dx + \int_{\frac{\pi}{2}}^{\pi} 1dx\right]$$

$$= \frac{1}{\pi}\left[-x\right]_{-\pi}^{-\frac{\pi}{2}} + \left[-x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi}.\left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} f(x)\cos nx\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}}(-1)\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\sin nx}{n}\right]_{-\pi}^{-\frac{\pi}{2}} + \left[\frac{\sin nx}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin n\frac{\pi}{2}}{n} - \frac{\sin n\frac{\pi}{2}}{n}\right]$$

$$= 0$$

and, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} f(x)\sin nx\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\sin nx\,dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}}(-1)\sin nx\,dx + \int_{\frac{\pi}{2}}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\frac{\pi}{2}} - \left[\frac{\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{\pi}{2} - (-1)^n}{n}\right]$$

Substituting a_0, a_n and b_n is (1)

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\cos n\frac{\pi}{2} - (-1)^n\right]\frac{\sin nx}{x}$$

$$= \frac{2}{\pi}\left[\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right]$$

$$= \frac{2}{\pi}\left[\sin x - \sin 2x + \frac{\sin 3x}{3} + \ldots\ldots\right]$$

This is the required series.

(10) **Obtain the Fourier series of the function**

$$f(x) = -\frac{\pi}{4} \text{ for } -\pi < x < 0$$

$$= \frac{\pi}{4} \text{ for } 0 < x < \pi \text{ and } f(x + 2\pi) = f(x)$$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ldots\ldots (i)$

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4}dx + \int_{0}^{\pi}\frac{\pi}{4}dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{4}.x\Big|_{-\pi}^{0} + \frac{\pi}{4}.x\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{4} + \frac{\pi^2}{4}\right]$$

$$= 0$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \ldots\ldots\ldots\ldots\ldots$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nxdx + \int_{0}^{\pi} f(x)\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4}\cos nxdx + \int_{0}^{\pi}\frac{\pi}{4}\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{4}.\frac{\sin x}{n}\Big|_{-\pi}^{0} + \frac{\pi}{4}.\frac{\sin nx}{n}\Big|_{0}^{\pi}\right]$$

$$= 0$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ n = 1, 2, \ldots\ldots\ldots\ldots$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nxdx + \int_{0}^{\pi} f(x)\sin nxdx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4}\sin nxdx + \int_{0}^{\pi}\frac{\pi}{4}\sin nxdx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{4}.-\frac{\cos nx}{n}\Big|_{-\pi}^{0} + \frac{\pi}{4}.-\frac{\cos nx}{n}\Big|_{-\pi}^{0}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}\frac{1}{n} - \frac{\pi}{4}\frac{(-1)^n}{n} - \frac{\pi}{4}\frac{(-1)^n}{n} + \frac{\pi}{4}\frac{1}{n}\right]$$

$$= \frac{1}{\pi}\cdot\frac{2\pi}{4}\left[\frac{1-(-1)^n}{n}\right]$$

$$= \frac{1}{2}\left[\frac{1-(-1)^n}{n}\right]$$

Substituting a_0, a_n and b_n in (i)

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{-1(-1)^n}{n}\right]\sin nx$$

$$f(x) = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots$$

This is the required Fourier for f(x).

(11) **Obtain a Fourier series to represent the function $f(x) = |\sin x|$ & $|\cos x|$ for $-\pi \le x \le \pi$.**

Solution: Let $f(x) = |\sin x| = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ldots\ldots (i)$

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}|\sin x|\,dx = \frac{2}{\pi}\int_{0}^{\pi}\sin xdx = \frac{2}{\pi}\left[(-\cos x)\right]_{0}^{\pi}$$

$$= \frac{2}{\pi}[1 + 1]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, \ldots\ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}|\sin x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{\pi}(\sin(x+nx) + \sin(x-nx))]dx$$

For, $n \neq 1$,

$$a_n = \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)} - \frac{\cos(1-n)x}{(1-n)}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)} - \frac{(-1)^{1-n}}{(1-n)} + \frac{1}{(n+1)} + \frac{1}{(1-n)}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{1-n}}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n}}{n+1} - \frac{(-1)^{-n}}{(n-1)} + \frac{n-1-n-1}{(n^2-1)}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right] - \frac{2}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1-1-x-1)}{n^2-1}(-1)^n - \frac{2}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2(-1)^n}{n^2-1} - \frac{2}{n^2-1}\right]$$

$$= -\frac{2}{\pi}\left[\frac{(-1)^n+1}{(n^2-1)}\right]$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\, dx,\ x = 1, 2, \ldots\ldots$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\sin nx\, dx$$

$= 0$ being $|\sin x|\sin nx$ as odd function.

Substituting a_0, a_n & b_n is .... (i)

$$|\sin x| = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n+1}{(n^2+1)}\right]\cos nx$$

$$= \frac{2}{\pi} - \frac{2}{\pi}\left[0 + \frac{2}{3}\cos 2x + 0 + \frac{2}{15}\cos 4x + 0 + \frac{2}{35}\cos 6x + \ldots\ldots\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\ldots\right]$$

Next,

Let $f(x) = |\cos x| = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\, dx$$

Since, $|\cos x| = \cos x,\ 0 \le x \le \frac{\pi}{2}$

$= -\cos x,\ \frac{\pi}{2} \le x \le \pi.$

$$a_0 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} |\cos x|\, dx + \int_{\frac{\pi}{2}}^{\pi} |\cos x|\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos dx - \int_{\frac{\pi}{2}}^{\pi} \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\sin x\Big|_0^{\frac{\pi}{2}} - \sin x\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}[1+1]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\ n = 1, 2, \ldots\ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x|\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_{-\pi}^{\frac{\pi}{2}} |\cos x|\cos nx + \int_{\frac{\pi}{2}}^{\pi} |\cos x|\cos ndx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\cos nxdx - \int_{\frac{\pi}{2}}^{\pi} \cos x\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\{\cos(x+nx)+\cos(x-nx)\}\, dx - \int_{\frac{\pi}{2}}^{\pi}\{\cos(x+nx)+\cos(x-nx)\}\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\{\cos(n+1)x+\cos(n-1)x\}\, dx - \int_{\frac{\pi}{2}}^{\pi}\{\cos(n+1)x+\cos(n-1)x\}\, dx\right]$$

For $n \ne 1$,

$$a_n = \frac{1}{\pi}\left[\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)}\right]_0^{\frac{\pi}{2}} - \left[\frac{\sin(n+1)x}{(n+1)} - \frac{\sin(n-1)x}{(n-1)}\right]_{\frac{\pi}{2}}^{0}$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{(n+1)} + \frac{\sin(n-1)\frac{\pi}{2}}{(n-1)}\right]$$

For $n = 1$, $a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{\frac{\pi}{2}} \cos 2xdx - \int_{\frac{\pi}{2}}^{\pi} \cos 2xdx + \int_{-\pi}^{\frac{\pi}{2}} dx - \int_{\frac{\pi}{2}}^{\pi} dx\right]$

$$= \frac{1}{\pi}\left[\frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}} - \left[\frac{\sin 2x}{2}\right]_{\frac{\pi}{2}}^{\pi} + [x]_{0}^{\frac{\pi}{2}} - [x]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right]$$

$$= 0$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx, \; n = 1, 2, \ldots\ldots$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \sin nx \, dx$$

$= 0$ being $|\cos x| \sin nx$ as odd function.

Substituting a_0, a_1, a_n & b_n in (i)

$$|\cos x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{\sin(n-1)\frac{\pi}{2}}{(n-1)} + \frac{\sin(n-1)\frac{\pi}{2}}{(n-1)}\right]\cos nx$$

$$= \frac{2}{\pi} + \frac{2}{\pi}\left[\left(-\frac{1}{3}+1\right)\cos x - \left(\frac{1}{5}-\frac{1}{3}\right)\cos 4x - \ldots\ldots\right]$$

or, $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \ldots\ldots\right)$

This is required Fourier series for $|\cos x|$.

(12) Find the Fourier series for $f(x) = e^{x}, -\pi \le x \le \pi$.

Solution: Same as the solution of question 5 with $a = -1$

(13) If $f(x) = -x, -\pi < x < 0$
$= x, 0 < x < \pi$, then find Fourier series of $f(x)$ and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots = \frac{\pi^2}{8}.$$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x dx + \int_{0}^{\pi} x dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x^2}{2}\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \; n = 1, 2, \ldots$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx + \int_{0}^{\pi} f(x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\left[-x.\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \left[x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx, \; n = 1, 2, \ldots\ldots$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx dx + \int_{0}^{\pi} f(x)\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\left[-x.\frac{-\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \left[x.\frac{-\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n} - \pi\frac{(-1)^n}{n}\right]$$

$$= 0$$

Substituting a_0, a_n & b_n in (i)

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\frac{(-1)^n - 1}{n^2}\right)\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\cos nx \frac{(-1)^n - 1}{n^2}\right)$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2}{1^2}\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right]$$

This is the required Fourier series for $f(x)$.

Next, put $x = 0$ in the above series then,

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\right]$$

The function $f(x)$ is not continuas at $x = 0$.

So, $f(0) = \frac{1}{2}\,[f(0-0) + f(0+0)]$

$$= \frac{1}{2}[0+0]$$

$$= 0$$

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + ...\right]$$

$$\text{or, } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + ...$$

(14) Expand as a Fourier series, the function f(x) defined as,

$$f(x) = \pi + x \text{ for } -\pi < x < -\frac{\pi}{2}$$

$$= \frac{\pi}{2} \text{ for } -\frac{\pi}{2} < x < \pi$$

$$= \pi - x \text{ for } \frac{\pi}{2} < x < \pi$$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} f(x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} (\pi + x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2}dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)dx\right]$$

$$= \frac{1}{\pi}\left[\left|\pi x + \frac{x^2}{2}\right|_{-\pi}^{-\frac{\pi}{2}} + \left|\frac{\pi}{2}x\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{2} + \frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} + \frac{\pi^2}{4} + \frac{\pi^2}{4} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{3\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ n = 1, 2, 3, ...$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} f(x)\cos nxdx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos nxdx + \int_{\frac{\pi}{2}}^{\pi} f(x)\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} (\pi + x)\cos nxdx + \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos nxdx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\left|(\pi + x).\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_{-\pi}^{-\frac{\pi}{2}} + \left|\frac{\pi}{2}\frac{\sin nx}{n}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2x}\sin x\frac{\pi}{2} + \frac{\cos nx\frac{\pi}{2}}{n^2} - \frac{(-1)^n}{n^2} + \frac{\pi}{2n}\sin x\frac{\pi}{2} + \frac{\pi}{2n}\sin x\frac{\pi}{2}\right.$$

$$\left. - \frac{(-1)^n}{n^2} - \frac{\pi}{2n}\sin x\frac{\pi}{2} + \frac{\cos nx\frac{\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\frac{\pi}{2} - (-1)^n}{n^2}\right]$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\,dx, n = 1, 2, ...$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} f(x)\sin nxdx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\sin nxdx + \int_{\frac{\pi}{2}}^{\pi} f(x)\sin nxdx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} (\pi + x)\sin nxdx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2}\sin nxdx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nxdx\right]$$

$$= \frac{1}{\pi}\left[\left|(\pi + x).\frac{-\cos nx}{n} + \frac{\sin nx}{n^2}\right|_{-\pi}^{-\frac{\pi}{2}} + \left|\frac{\pi}{2}\frac{-\cos nx}{n}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left|(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\frac{\cos n\frac{\pi}{2}}{n^2} - \frac{\sin n\frac{\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\cos n\frac{\pi}{2}}{n} + \frac{\pi}{2} + \frac{\cos n\frac{\pi}{2}}{n} + \frac{\pi}{2}\frac{\cos n\frac{\pi}{2}}{n} + \frac{\sin n\frac{\pi}{2}}{n^2}\right]$$

$$= 0$$

Substituting a_0, a_n and b_n is .... (i)

$$f(x) = \frac{3\pi}{8} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos n\frac{\pi}{2} - (-1)^n}{n^2}\right]\cos nx$$

$$= \frac{3\pi}{8} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x - \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + ...\right]$$

This is the Fourier series of f(x).

(15) Find the Fourier sories of the function

$$f(x) = \begin{cases} \pi + x, -\pi < x < 0 \\ \pi - x, 0 < x < \pi \end{cases} \text{ of period } 2\pi. \text{ [IOE 2061]}$$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi + x)dx + \int_{0}^{\pi} (\pi - x)dx\right]$$

$$= \frac{1}{\pi}\left[\left|\pi x + \frac{x^2}{2}\right|_{-\pi}^{0} + \left|\pi x - \frac{x^2}{2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi^2 - \frac{\pi^3}{2} + \pi^2 - \frac{\pi^2}{2}\right]$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx, \; n = 1, 2, \ldots\ldots$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx + \int_{0}^{\pi} f(x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi - x)\cos nx dx + \int_{0}^{\pi} (\pi - x)\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\left|(\pi - x).\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{-\pi}^{0} + \left|(\pi - x).\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx, \; n = 1, 2, \ldots\ldots\ldots$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx dx + \int_{0}^{\pi} f(x)\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (\pi - x)\sin nx dx + \int_{0}^{\pi} (\pi - x)\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\left|(\pi + x).-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_{-\pi}^{0} + \left|(\pi - x).-\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n} + \frac{\pi}{n}\right]$$

$$= 0$$

Substituting a_0, a_n & b_n in (i)

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^2}\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{2}{1^2}\cos x + 0 + \frac{2}{3^2}\cos 3x + \ldots\right]$$

$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right]$. This is the required Fourier series of f(x).

Find the Fourier series of the function $f(x) = \frac{(\pi - x)}{2}$ in the internal $0 \le x \le 2\pi$ & deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$.

tion: Let $f(x) = \frac{(\pi - x)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where, $a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\int_{0}^{2\pi}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{0}^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi x^2 - \frac{4\pi^2}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,dx, \; n = 1, 2, \ldots\ldots$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi}(\pi - x)\cos nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x).\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{0}^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n^2} + \frac{1}{n^2}\right]$$

$$= 0$$

and $b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,dx, \; n = 1, 2, \ldots\ldots$

$$= \frac{1}{2\pi}\int_{0}^{2\pi}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x).\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{0}^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right]$$

$$= \frac{1}{n}$$

Substituting a_0, a_n & b_n in (i)

$$\frac{\pi-x}{2} = \sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\dots\dots$$

This is the required Fourier series of $\left(\frac{\pi-x}{2}\right)$.

Next, Putting $x = \frac{\pi}{2}$ from $[0, 2\pi]$ in the above series, we get

$$\frac{\pi}{4} = \frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\dots$$

If $f(x) = \frac{(\pi-x)^2}{4}$ in the range 0 to 2π, then show that

$$f(x) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}.\ \text{[IOE 2062]}$$

Solution: Let $f(x) = \frac{(\pi-x)^2}{4} = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where, $a_0 = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\,dx$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{3}.(-1)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[+\frac{\pi^3+\pi^3}{3}\right]$$

$$= \frac{1}{4\pi}\times\frac{2\pi^3}{3}$$

$$= \frac{\pi^2}{6}$$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ n = 1, 2, \dots\dots\dots$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

Let, $u = (\pi-x)^2$, $v = \cos nx$

Diff. u & integrating v w. r. t., x we get,

$u' = -2(\pi-x)$, $v_1 = \frac{\sin nx}{x}$

$u'' = 2$ $\quad v_2 = -\frac{\cos nx}{n^2}$, $v_3 = -\frac{\sin nx}{n^3}$

$u''' = 0$

$\therefore a_n = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$

$$= \frac{1}{4\pi}\int_0^{2\pi}uv\,dx$$

$$= \frac{1}{4\pi}\left[uv_1 - u'v_2 + u''v_3 - \dots\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2.\frac{\sin nx}{n} - 2(\pi-x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2\pi.\frac{1}{n^2}+2\pi.\frac{1}{n^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{n^2}+\frac{1}{n^2}\right]$$

$$= \frac{1}{n^2}$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx,\ x = 1, 2, \dots\dots$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

Let, $u = (\pi-x)^2 \sin nx\,dx$.

Diff. u and integrating v w. r. t. x, we get,

$u' = -2(\pi-x)$, $\quad v_1 = -\frac{\cos nx}{n}$, $v_2 = -\frac{\sin nx}{n^2}$, $v_3 = \frac{\cos nx}{n^3}$

$u'' = 2$, $u''' = 0$

$\therefore b_n = \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$

$$= \frac{1}{4\pi}\int_0^{2\pi}uv\,dx$$

$$= \frac{1}{4\pi}\left[uv_1 - u'v_2 + u''v_3 - \dots\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2.\frac{\cos nx}{n} - 2(\pi-x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\pi^2.\frac{1}{n}+\frac{2}{n^3}+\pi^2.\frac{1}{n}-\frac{2}{n^3}\right]$$

$$= 0$$

Substituting a_0, a_n & b_n in (i) we get,

$$\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

This is the required Fourier series for $\frac{(\pi-x)^2}{4}$.

(18) **Obtain a Fourier series of $f(x) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$ in the interval $(0, 2\pi)$ and decluce that $\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$**

Solution: Let $f(x) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ldots\ldots (i)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{12\pi}\int_0^{2\pi}(3x^2 - 6\pi x + 2\pi^2)\,dx$$

$$= \frac{1}{12\pi}\left[3\frac{x^3}{3} - 6\pi.\frac{x^2}{2} + 2\pi^2.x\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[x^3 - 3\pi x^2 + 2\pi^2 x\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[8\pi^3 - 12\pi^3 + 4\pi^3\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ n = 1, 2, \ldots\ldots$$

$$= \frac{1}{12\pi}\int_0^{2\pi}(3x^2 - 6\pi x + 2\pi^2)\cos nx\,dx$$

$$= \frac{1}{12\pi}\left[\int_0^{2\pi} 3x^2 \cos nx dx - 6\pi\int_0^{2\pi} x\cos nx dx + 2\pi^2\int_0^{2\pi}\cos nx\right]$$

$$= \frac{1}{12\pi}\left[3.\left\{\left|x^2.\frac{\sin nx}{n}\right|_0^{2\pi} - \int_0^{2\pi} 2x\frac{\sin nx}{n}dx\right\}\right.$$

$$\left. - 6\pi\ 3.\left\{\left|x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{2\pi} + \left|2x\frac{\sin nx}{n}\right|_0^{2\pi}\right\}\right]$$

$$= \frac{1}{12\pi}\left[3\left\{0 - \left(2x.-\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right)_0^{2\pi}\right\} - 6\pi.\frac{1}{n^2} + 6\pi.\frac{1}{n^2} + 0\right]$$

$$= \frac{1}{12\pi}\left[3\left(4\pi.\frac{1}{n^2} - 0\right)\right]$$

$$= \frac{1}{n^2}$$

$$\text{and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx,\ n = 1, 2, 3, \ldots\ldots$$

$$= \frac{1}{12\pi}\left[\int_0^{2\pi}(3x^2 - 6\pi x + 2\pi^2)\sin nx dx\right]$$

$$= \frac{1}{12\pi}\left[3\int_0^{2\pi} x^2 \sin nx dx - 6\pi\int_0^{2\pi} x\sin nx dx + 2\pi^2\int_0^{2\pi}\sin nx dx\right]$$

$$= \frac{1}{12\pi}\left[3.\left\{x^2.-\frac{\cos nx}{n} + 2x\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right\} - 6\pi\left\{x.-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right\} + 2\pi^2.-\frac{\cos nx}{n}\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[3\left\{4\pi^2.\left(-\frac{1}{n}\right) - 2.\frac{1}{n^2} - \frac{2}{n^3}\right\} - 6\pi\left\{2\pi.\left(-\frac{1}{n}\right)\right\} + 2\pi^2.-\frac{1}{n} + 2\pi^2.\frac{1}{n}\right]$$

$$= \frac{1}{12\pi}\left[-\frac{12\pi^2}{n} + \frac{12\pi^2}{n}\right]$$

$$= 0$$

Substituting a_0, a_n & b_n in (i) we get,

$$f(x) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

This is the required Fourier series for $f(x) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$

Next, Putting $x = 0$ in the above series, we get

$$f(0) = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

But $f(x)$ is not defined at $x = 0$ in the interval $(0, 2\pi)$.

$$\text{So, } f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right]$$

$$= \frac{1}{2}\left[\frac{2\pi^2}{12} + \frac{2\pi^2}{12}\right]$$

$$= \frac{\pi^2}{6}$$

$$\therefore \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$$

(19) **Expand $f(x) = x\sin x$, $0 < x < 2\pi$ in a Fourier series.**

Solution: Let $f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \ldots\ldots (i)$

$$\text{Where, } a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x dx$$

$$= \frac{1}{\pi}\left[x.-\cos x + \sin x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\right]$$

$$= -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ x = 1, 2, \ldots\ldots$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left[\sin(x+nx) + \sin(x+nx)\right]dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\,[\sin (n+1)x - \sin (n-1)x]\,dx$$

For $n \neq 1$,

$$a_n = \frac{1}{2\pi}\left[x.\left(\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{1}{(n+1)} + \frac{1}{(n-1)}\right)\right] = \left[\frac{-n+1+n+1}{n^2-1}\right] = \frac{2}{n^2-1}$$

For, $n = 1$,

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x.-\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi} = \frac{1}{2\pi} \times \frac{(-2\pi)}{2} = \frac{-1}{2}$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin nx\,dx,\ x = 1, 2, \ldots.$

$$= \frac{1}{\pi}\int_0^{2\pi} x\ \sin x \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(x+nx) + \cos(x-nx)]\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\ [\cos(n+1)x + \cos(n-1)x]\,dx$$

For $n \neq 1$

$$b_n = \frac{1}{2\pi}\left[x.\left(\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)}\right) - \left(\frac{-\cos(n+1)x}{(n+1)^2} - \frac{\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}\right]$$

$$= 0$$

For, $n = 1$,

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\ [\cos 2x + 1]\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x \cos 2x dx + \int_0^{2\pi} x dx\right]$$

$$= \frac{1}{2\pi}\left[x.\frac{\sin 2x}{2} + \frac{\cos 2x}{4} + \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{4} + \frac{4\pi^2}{2} - \frac{1}{4}\right]$$

$$= \frac{1}{2\pi}[2\pi^2]$$

$$= \pi$$

Substituting a_0, a_1, a_n, b_1 & b_n in (i) we get,

$$x \sin x = -1 + \left(-\frac{1}{2}\right)\cos x + \sum_{n=2}^{\infty}\left(\frac{2}{n^2-1}\right)\cos nx + \pi \sin x$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + 2\sum_{n=2}^{\infty}\frac{\cos nx}{(n^2-1)}$$

This is the required Fourier series for $f(x) = x \sin x$.

20. **Find the Fourier series to represent the function f(x) given by:**
$f(x) = x,\ 0 \leq x \leq \pi$
$= 2\pi - x,\ \pi \leq x \leq 2\pi$
and dedute that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ...... (i)

Where,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left|\frac{x^2}{2}\right|_0^{\pi} + \left|2\pi x - \frac{x^2}{2}\right|_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x) \cos nx dx + \int_{\pi}^{2\pi} f(x) \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \cos nx dx + \int_{\pi}^{2\pi}(2\pi - x) \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[x.+\left|\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi} + (2\pi - x).\left|\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

and $b_x = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin nx dx,\ n = 1, 2, \ldots\ldots$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x) \sin nx dx + \int_{\pi}^{2\pi} f(x) \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx dx + \int_{\pi}^{2\pi}(2\pi - x) \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[x\cdot\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2} + (2\pi - x)\cdot\left(-\frac{\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \pi\frac{(-1)^n}{n^2}\right] = 0$$

Substituting a_0, a_n & b_n in (1), we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos x + 0 - \frac{2}{3^2}\cos 3x + \ldots\ldots\right]$$

$$\text{or, } f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right]$$

This is the required Fourier series for f(x).

Next putting x = 0, we get,

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

But f(x) = 0, at x = 0

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

$$\text{i.e. } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\ldots$$

21. Find the Fourier expansion of the function defined in a single period given by,

$f(x) = 1$ for $0 \le x < \pi$

$= 2$ for $\pi < x < 2\pi$

and hence deduce that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} dx + 2\int_{\pi}^{2\pi} dx\right] = \frac{1}{\pi}\left[|x|_0^{\pi} + |2x|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}[\pi + 4\pi - 2\pi) = 3$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ n = 1, 2, 3, \ldots\ldots\ldots$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx + 2\int_{\pi}^{2\pi}\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin nx}{n}\right|_0^{\pi} + 2\left|\frac{\sin nx}{n}\right|_{\pi}^{2\pi}\right]$$

$$= 0$$

$$\text{and } b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx,\ n = 1, 2, 3, \ldots\ldots$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx + 2\int_{\pi}^{2\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\left|\frac{\cos nx}{n}\right|_0^{\pi} - 2\left|\frac{\cos nx}{n}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{-(-1)^n}{n} + \frac{1}{n} - \frac{2}{n} + \frac{2}{n}(-1)^n\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}(-1)^n - \frac{1}{n}\right] = \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n}\right]$$

Substituting a_0, a_n and b_n in (i), we get,

$$f(x) = \frac{3}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n}\right)\sin nx$$

$$= \frac{3}{2} + \frac{1}{\pi}\left[-\frac{2}{1}\sin x - 0 - \frac{2}{3}\sin 3x - \ldots\ldots\right]$$

$$\therefore f(x) = \frac{3}{2} - \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\ldots\right]$$

This is the required Fourier series expansion of f(x).

Next, putting $x = \frac{\pi}{2}$ in the above series, we get,

$$f\left(\frac{\pi}{2}\right) = \frac{3}{2} - \frac{2}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5}\ldots\ldots\right]$$

But f(x) is not continuous at $x = \frac{\pi}{2}$ in the given interval $(0, \pi)$.

$$\therefore f\left(\frac{\pi}{2}\right) = \frac{1}{2}\left[f\left(\frac{\pi}{2} + 0\right) + f\left(\frac{\pi}{2} - 0\right)\right] = \frac{1}{2}[1 + 1] = 1$$

$$\therefore 1 = \frac{3}{2} - \frac{2}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5}\ldots\ldots\right]$$

$$\text{or, } 1 - \frac{3}{2} = -\frac{2}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5}\ldots\ldots\right]$$

$$\text{or, } -\frac{1}{2} = -\frac{2}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5}\ldots\ldots\right]$$

$$\therefore \frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5}\ldots\ldots$$

Exercise - 18

1. **Expand e^{-x} as a Fourier series in the internal $(-L, L)$.**

(BE 2057)

Solution: Let $f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ..... (i)

Where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{-L}^{L} e^{-x}\, dx$

$= \frac{1}{L}\left[-e^{-x}\right]_{-L}^{L} = \frac{1}{L}\left[-e^{-L} + e^{L}\right]$

$= \frac{1}{L}\left[e^{L} - e^{-L}\right]$

$a_x = \frac{1}{L}\int_{-L}^{L} f(x)dx \cos x\pi \frac{x}{L}\, dx$, $x = 1, 2, \ldots\ldots\ldots$

$= \frac{1}{L}\int_{-L}^{L} e^{-x} \cos n\pi \frac{x}{L}\, dx$

$$= \frac{1}{L}\left[\frac{e^{-x}}{1+\frac{n^2\pi^2}{L^2}}\left((-1)\cos n\pi\frac{x}{L} + \frac{x\pi}{L}\sin n\pi\frac{x}{L}\right)\right]_{-L}^{L}$$

$$= \frac{1}{L}\left[\frac{e^{-x}}{1+\frac{n^2\pi^2}{L^2}}-(-1)^n + 0) - \frac{e^{L}}{1+\frac{n^2\pi^2}{L^2}}(-(-1)^n + 0\right]_{-L}^{L}$$

$$= L(e^{L} - e^{-L})\frac{(-1)^n}{n^2\pi^2 + L^2}$$

and $b_x = \frac{1}{L}\, f(x) \sin x\pi \frac{x}{L}\, dx$, $x = 1, 2, \ldots\ldots\ldots$

$= \frac{1}{L}\int_{-L}^{L} L^{-x} \sin x\pi \frac{x}{L}\, dx$

$$= \frac{1}{L}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{L}\right)^2}\left((-1)\sin n\pi\frac{x}{L} - \frac{n\pi}{L}\cos n\pi\frac{x}{L}\right)\right]_{-L}^{L}$$

$$= \frac{1}{L}\left[\frac{e^{-x}}{\frac{n^2\pi^2 + L^2}{L^2}}(-1)^n\frac{n\pi}{L}\right]$$

$$= \pi(e^{L} - e^{-L})\frac{(-1)^n n}{(x^2\pi^2 + L^2)}$$

Substituting a_0, a_n & b_n in (i), we get

$$e^{-x} = \frac{(e^{L} - e^{-L})}{2L} + \sum_{n=1}^{\infty}\left[L(e^{L} - e^{-L})\frac{(-1)^n}{n^2\pi^2 + L^2}\cos n\pi\frac{x}{L} + \pi(e^{L} - e^{-L})\frac{(-1)^n}{(n^2\pi^2 - L^2)}\sin n\pi\frac{x}{L}\right]$$

$$= (e^{L} - e^{-L})\left[\frac{1}{2L} + L\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi\frac{x}{L}}{n^2\pi^2 + L^2} + \pi\sum_{n=1}^{\infty}\frac{n(-1)^n\cos n\pi\frac{x}{L}}{n^2\pi^2 + L^2}\right]$$

$$= \left(\frac{e^{L} - e^{-L}}{2}\right)\left[\frac{1}{L} + 2L\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi\frac{x}{L}}{n^2\pi^2 + L^2} + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n n\cos n\pi\frac{x}{L}}{n^2\pi^2 + L^2}\right]$$

$$= \sinh L\left[\frac{1}{L} + 2L\left(-\frac{\cos n\pi\frac{x}{L}}{n^2 + L^2} + \frac{\cos 2\pi\frac{x}{L}}{n^2\pi^2 + L^2}\cdots\cdots\right) + 2\rho\left(\frac{-1.\sin n\pi\frac{x}{L}}{n^2 + L^2} + \frac{2\sin 2\pi\frac{x}{L}}{2^2\pi^2 + L^2}\cdots\cdots\right)\right]$$

This is the required Fourier series of e^{-x}.

2. **Find a Fourier series for $f(x) = 1 - x^2$ in the internal $-1 \le x \le 1$.**

[BE 2062/063/064]

Solution: The internal $[-1, 1]$ is like as $[-L, L]$, so $L = 1$.

Let $f(x) = 1 - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_x \sin x\pi x)$ .... (i)

Where,

$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = \int_{-1}^{1}(1 - x^2)dx$

$= \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 1 - \frac{1}{3} + 1 - \frac{1}{3} = \frac{4}{3}$

$a_x = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\, dx$, $n = 1, 2, \ldots\ldots\ldots$

$= \int_{-1}^{1}(1 - x^2)\cos n\pi x\, dx$

$= 2\int_{0}^{1}(1 - x^2)\cos n\pi\, dx$, being $(1 - x^2)\cos n\pi x$ even function.

Let, $u = 1 - x^2$, $V = \cos n\pi x$

Diff. u & integrating V w. r. t. u & V, we get,

$u' = -2x$, $V_1 = \frac{\sin n\pi x}{n\pi}$

$$u'' = -2,\ v_2 = -\frac{\cos n\pi x}{n^2\pi^2},\ v_3 = -\frac{\sin n\pi x}{n^3\pi^3}$$

$$\therefore a_n = 2\int_0^1 (1-x^2)\ \cos n\pi dx$$

$$= 2\int_0^1 uv\,dx$$

$$= 2\left[uv_1 - u'v_2 + u''v_3 \ldots\ldots\right]_0^1$$

$$= 2\left[(1-x^2)\frac{\sin n\pi x}{n\pi} - 2x\frac{\cos n\pi x}{n^2\pi^2} \div 2\frac{\sin n\pi x}{n^3\pi^3} - \ldots\right]$$

$$= 2\left[-2\frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{4}{\pi^2}\ \frac{(-1)^{n+1}}{n^2}$$

and $b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx,\ n = 1, 2, \ldots\ldots$

$$= \int_{-1}^{1}(1-x^2)\ \sin n\pi x\,dx$$

$$= 0$$

being $(1 - x^2)\sin n\pi x$ odd function.

Substituting a_0, a_n & b_x in (i), we get

$$1 - x^2 = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\ \cos n\pi x$$

This is the required Fourier series for $1 - x^2$.

3. **Find the Fourier series for the function $f(x) = x - x^2$ in the interval $-1 \le x \le 1$.**

Solution: Here $l = 1$

Let $f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_x \sin n\pi x)$ .... (i)

Where,

$$a_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{1}(x-x^2)dx = \int_{-1}^{1} x\,dx - \int_{-1}^{1} x^2\,dx$$

$$= 0 - 2\int_0^1 x^2\,dx = -2\left|\frac{x^3}{3}\right|_0^1 = -\frac{2}{3}$$

$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx,\ n = 1, 2, \ldots\ldots\ldots$

$$= \int_{-1}^{1}(x-x^2)\cos n\pi x\,dx = \int_{-1}^{1} x\cos n\pi x\,dx - \int_{-1}^{1} x^2\cos n\pi x\,dx$$

$= 0 - 2\int_0^1 x^2\ \cos n\pi x\,dx$, being $x\cos n\pi x$ odd & $x^2\cos n\pi x$ even function.

Let $u = x^2$, $v = \cos n\pi x$

Diff. u and integrating v w.r.t x, respect we get,

$$u' = 2x,\ v_1 = \frac{\sin n\pi x}{n\pi}$$

$$u'' = 2,\ v_2 = -\frac{\cos n\pi x}{n^2\pi^2},\ V_3 = -\frac{\sin n\pi x}{n^3\pi^3}$$

$$\therefore a_x = -2\int_0^1 u\ v\,dx = -2\left[uv_1 - u'v_2 + u''v_3 \ldots\ldots\right]$$

$$= 2\left[x^2.\frac{\sin n\pi x}{n\pi} + 2x\frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3} - \ldots\right]_0^1$$

$$= -2\times 2\frac{(-1)^n}{n^2\pi^2} = \frac{4}{n^2}\ \frac{(-1)^{n+1}}{n^2}$$

and $b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx,\ n = 1, 2, \ldots\ldots$

$$= \int_{-1}^{1}(x-x^2)\ \sin n\pi x\,dx = \int_{-1}^{1} x\sin n\pi x\,dx - \int_{-1}^{1} x^2\sin n\pi x\,dx$$

$= 2\int_0^1 x\ \sin n\pi x\,dx - 0$ being $x\sin n\pi x$ even function and x^2 $\sin n\pi x$ odd function.

$$\text{or, } b_n = 2\int_0^1 x\sin n\pi x\,dx$$

$$= 2\left[x.-\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$= -2\frac{(-1)^n}{n\pi} = \frac{2}{\pi}\ \frac{(-1)^{n+1}}{n}$$

Substituting a_0, a_x & b_x in (i), we get

$$x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nn\pi x + \frac{2}{\pi}\ \frac{(-1)^{n-1}}{n}$$

$$\therefore x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{4\cos n\pi x}{n^2\pi^2} + \frac{2\sin n\pi x}{n\pi}\right](-1)^{n+1}$$

This is the required Fourier series for $x - x^2$.

4. **Find the Fourier series for the function $f(a) = x^2 - 2$ in the interval $-2 \le x \le 2$.** (BE 2065)

Solution: Here $L = -2$

Let $f(x) = x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi\frac{x}{2} + b_n \sin n\pi\frac{x}{2}\right)$ ..... (i)

$$\text{Where, } a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{2}(x^2-2)\,dx$$

$= \frac{1}{2}.2\int_0^2 (x^2-2)\,dx$, being $x^2 - 2$ even function.

$$= \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)dx\ \cos n\pi\frac{x}{2}dx,\ n = 1, 2, 3\ldots\ldots$$

$$= \frac{1}{2}\int_{-2}^{2}(x^2-2)\cos n\pi\frac{x}{2}\,dx$$

$$= \frac{1}{2}\cdot 2\int_{0}^{2}(x^2-2)\cos n\pi\frac{x}{2}\,dx \quad \text{being } (x^2-2)\cos n\pi\frac{x}{2} \text{ even}$$

function.

$$= \int_{0}^{2}(x^2-2)\cos n\pi\frac{x}{2}\,dx$$

Let $u = x^2-2$, $v \cos n\pi\frac{x}{2}$

Diff. u & integrating v w. r. t. x^2, series P, we get

$$u' = 2x,\ v_1 = \frac{\sin\pi\frac{x}{2}}{\left(\frac{x\pi}{2}\right)^2}$$

$$u'' = 2,\ v_2 = -\frac{\cos\pi\frac{x}{2}}{\left(\frac{x\pi}{2}\right)^2},\ V_3 = -\frac{\sin n\pi\frac{x}{2}}{\left(\frac{x\pi}{2}\right)^3}$$

$$\therefore a_n = \int_0^2 uv\,dx = [uv_1 - u'v_2 + u''v_3 \ldots\ldots]_0^2$$

$$= \left[(x^2-2)\cdot\frac{\sin n\pi\frac{x}{2}}{n\frac{\pi}{2}} + 2x\frac{\cos n\pi\frac{x}{2}}{\left(n\frac{\pi}{2}\right)^2} - 2\frac{\sin n\pi\frac{x}{2}}{\left(n\frac{\pi}{2}\right)^3}\right]_0^2$$

$$= \frac{4(-1)^n}{\left(\frac{n\pi}{2}\right)^2} = \frac{16}{\pi^2}\frac{(-1)^n}{n^2}$$

and $b_n = \int_{-2}^{2} f(x)dx \sin n\pi\frac{x}{2}\,dx,\ n = 1, 2, \ldots\ldots$

$$= \frac{1}{2}\int_{-2}^{2}(x^2-2)\sin n\pi\frac{x}{2}\,dx = 0$$

being $(x^2-2)\sin n\pi\frac{x}{2}$ odd function.

Substituting a_0, a_x & b_x in (i), we get,

$$x^2-2 = -\frac{2}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

This is the required Fourier series for x^2-2.

5. **Find a Fourier series for f(x) in the interval (– 2, 2) if $f(x) = 0, -2 < x < 0$**
$= 1, 0 < x < 2$

Solution: Here L = 2

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos n\pi\frac{x}{2} + b_n\sin n\pi\frac{x}{2}\right)$ ..... (i)

Where $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$

$$= \frac{1}{2}\left[\int_{-2}^{0} f(x)dx + \int_{-2}^{2} f(x)dx\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 0.dx + \int_{-2}^{0} 1.dx\right]$$

$$= \frac{1}{2}\int_0^2 dx = \frac{1}{2}[x]_0^2 = 1$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos n\pi\frac{x}{2}\,dx,\ n = 1, 2, \ldots.$$

$$= \frac{1}{2}\left[\int_{-2}^{0} f(x)\cos n\pi\frac{x}{2} + \int_{0}^{2} f(x)\cos n\pi\frac{x}{2}dx\right]$$

$$= \frac{1}{2}\int_{-2}^{2}\cos n\pi\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\sin n\pi\frac{x}{2}\times\frac{2}{x\pi}\right]_0^2$$

$$= 0$$

and $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin n\pi\frac{x}{2}\,dx,\ n = 1, 2, \ldots..$

$$= \frac{1}{2}\left[\int_{-2}^{0} f(x)\sin n\pi\frac{x}{2}dx + \int_{0}^{2} f(x)\sin n\pi\frac{x}{2}dx\right]$$

$$= \frac{1}{2}\int_{-2}^{2}\sin n\pi\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[-\cos n\pi\frac{x}{2}\times\frac{2}{n\pi}\right]_0^2$$

$$= \frac{1}{2}\left[-(-1)^n\frac{2}{n\pi} + \frac{2}{n\pi}\right] = \frac{1}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

Substituting a_0, a_x and b_x in (i), we get

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{\pi}\left[\frac{1-(-1)^n}{n}\right]\sin n\pi\frac{x}{2}$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\frac{2}{1}\sin\pi\frac{x}{2} + 0 + \frac{2}{3}\sin 3\pi\frac{x}{2} + \ldots..\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin\pi\frac{x}{2}}{1} + \frac{\sin 3\pi\frac{x}{2}}{3} + \ldots..\right]$$

This is the required Fourier series for f(x).

6. Find the Fourier series expansion of the function

$f(x) = 0$ for $-2 < x < -1$

$= k$ for $-1 < x < 1$

$= 0$ for $1 < x < 2$ [BE 2060]

Solution: Here, L = 2

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\pi\frac{x}{2} + b_n \sin n\pi\frac{x}{2}\right)$ ...... (i)

Where, $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$

$= \frac{1}{2}\left[\int_{-2}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{2} f(x)dx\right]$

$= \frac{1}{2}\left[0 + \int_{-1}^{1} k\, dx + 0\right]$

$= \frac{k}{2}[x]_{-1}^{1} = \frac{k}{2}[1+1] = k.$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos n\pi\frac{x}{2}dx$, n = 1, 2, ...

$= \frac{k}{2}\int_{-1}^{1}\cos n\pi\frac{x}{2}dx$

$= \frac{k}{2}\left[\frac{\sin n\pi\frac{x}{2}}{n\frac{\pi}{2}}\right]_{-1}^{1}$

$= \frac{k}{2}\cdot\frac{2}{n\pi}\left[\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right]$

$= \frac{2k}{\pi}\cdot\frac{\sin n\frac{\pi}{2}}{n}$

and $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin n\pi\frac{x}{2}dx$, n = 1, 2, ...

$= \frac{k}{2}\int_{-1}^{1}\sin n\pi\frac{x}{2}dx = 0$ being $\sin n\pi\frac{x}{2}$ odd function.

Substituting a_0, a_x and b_x in (i) we get

$f(x) = \frac{k}{2} + \sum_{n=1}^{\infty}\frac{2k}{\pi}\frac{\sin n\frac{\pi}{2}}{n}\cos n\pi\frac{x}{2}$

$= \frac{k}{2} + \frac{2k}{\pi}\left[\frac{1}{1}\cos\pi\frac{x}{2} + 0 - \frac{1}{3}\cos\pi\frac{x}{2} + 0......\right]$

$\therefore f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\frac{\cos\pi\frac{x}{2}}{1} - \frac{\cos 3\pi\frac{x}{2}}{3} + ...\right]$

This is the required Fourier series for f(x).

Expand $f(x) = \frac{1}{4} - x$ if $0 < x < \frac{1}{2}$

$= x - \frac{3}{4}$ if $\frac{1}{2} < x < 1$ as the Fourier sine series.

(BE 2060)

Solution: Here, L = 1

The half-range Fourier sine series for f(x) in (0, 1) is,

$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ ...... (i)

Where $b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\, dx$, n = 1, 2, ...

$= 2\left[\int_0^{\frac{1}{2}} f(x)\sin n\pi x\, dx + \int_{\frac{1}{2}}^{1} f(x)\sin n\pi x\, dx\right]$

$= 2\left[\int_0^{\frac{1}{2}}\left(\frac{1}{4} - x\right)\sin n\pi x\, dx + \int_{\frac{1}{2}}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\, dx\right]$

$= 2\left[\left(\frac{1}{4} - x\right).-\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^{\frac{1}{2}} + \left[\left(x - \frac{3}{4}\right).-\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_{\frac{1}{2}}^{1}$

$= 2\left[\frac{1}{4}\cos n\frac{\pi}{2}.\frac{1}{n\pi} - \frac{\sin n\frac{\pi}{2}}{n^2\pi^2} + \frac{1}{4}.\frac{1\cdot(-1)^n}{n\pi\ n\pi} - \frac{1}{4}\frac{\cos n\frac{\pi}{2}}{n\pi} - \frac{\sin n\frac{\pi}{2}}{n^2\pi^2}\right]$

$= 2\left[\frac{1}{4\pi}\left(\frac{1-(-1)^n}{n}\right) - \frac{2\sin n\frac{\pi}{2}}{n^2\pi^2}\right]$

Substituting b_x in (i), we get,

$f(x) = 2\sum_{n=1}^{\infty}\left[\frac{1}{4\pi}\left(\frac{1-(-1)^n}{n}\right) - \frac{2\sin n\frac{\pi}{2}}{n^2\pi^2}\right]\sin n\pi x$

$= 2\left[\left(\frac{1}{4\pi}\times\frac{2}{1} - \frac{2}{1^2\pi^2}\right)\sin\pi x + 0 + \left(\frac{1}{4\pi}\times\frac{2}{3} + \frac{2}{3^2\pi^2}\right)\sin 3\pi x + ...\right]$

$= \left(\frac{1}{\pi} - \frac{4}{1^2\pi^2}\right)\sin\pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + ...$

This is the required Fourier series for f(x).

8. **If $f(x) = \pi x, 0 \le x \le 1$**

$= \pi(2 - x), 1 \le x \le 2$

then show that in the interval (0, 2) the Fourier seires is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right).$$

Solution: Here, L = 1 as 2L = 2, interval is like (0, 2L)

Let, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$ .... (i)

Where, $a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \pi\int_0^1 x\,dx + \pi\int_1^2 (2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \pi\left[\frac{1}{2} + 4 - \frac{4^2}{2} - 2 + \frac{1}{2}\right] = \pi$$

$a_n = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\,dx$, n = 1, 2, ...

$$= \int_0^1 f(x)\cos n\pi x\,dx + \int_1^2 f(x)\cos n\pi x\,dx$$

$$= \pi\int_0^1 x\cos n\pi x\,dx + \pi\int_1^2 (2-x)\cos n\pi x\,dx$$

$$= \pi\left[\left|x.\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right|_0^1 + \left|(2-x).\frac{\sin n\pi x}{n\pi} - \frac{\cos n\pi x}{n^2\pi^2}\right|_1^2\right]$$

$$= \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-1+(-1)^n}{n^2}\right]$$

and $b_x = \frac{1}{1}\int_0^2 f(x)\sin n\pi x\,dx$, x = 1, 2, ...

$$= \int_0^1 f(x)\sin n\pi x\,dx + \int_1^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[\left|x.-\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_0^1 + \left|(2-x).-\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_1^2\right]$$

$$= \pi\left[-1.\frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi}\right] = 0$$

Substituting a_0, a_n & b_n in (i), we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{-1+(-1)^n}{n^2}\right]\cos n\pi x$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\left[-\frac{2}{1^2}\cos\pi x + 0 - \frac{2}{3^2}\cos 3\pi x + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos n\pi}{1^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right]$$

This is the required Fourier series for f(x).

9. **Obtain a Fourier series to represent the function**
$f(x) = 0$ for $0 < x < L$
$= 1$ for $L < x < 2L$ [BE 2062]

Solution: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$ .... (i)

Where, $a_0 = \frac{1}{L}\left[\int_0^{2L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx + \int_L^{2L} f(x)\,dx\right]$..... (i)

$$= \frac{1}{L}\int_L^{2L} dx = \frac{1}{L}[x]_L^{2L} = 1$$

$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos n\pi\frac{x}{L}\,dx$, n = 1, 2, ...

$$= \frac{1}{L}\left[\int_0^L f(x)\cos n\pi\frac{x}{L}dx + \int_L^{2L} f(x)\cos n\pi\frac{x}{L}dx\right]$$

$$= \frac{1}{L}\int_L^{2L}\cos n\pi\frac{x}{L}dx$$

$$= \frac{1}{L}\left[\sin n\pi\frac{x}{L}.\frac{L}{n\pi}\right]_L^{2L} = 0$$

and $b_n = \frac{1}{L}\int_0^{2L} f(x)\sin n\pi\frac{x}{L}\,dx$, n = 1, 2, ...

$$= \frac{1}{L}\left[\int_0^L f(x)\sin n\pi\frac{x}{L}dx + \int_L^{2L} f(x)\sin n\pi\frac{x}{L}dx\right]$$

$$= \frac{1}{L}\int_L^{2L}\sin n\pi\frac{x}{L}dx$$

$$= \frac{1}{L}\left[-\cos n\pi\frac{x}{L}.\frac{L}{n\pi}\right]_L^{2L}$$

$$= \frac{1}{L}\left[\frac{-L}{n\pi} + (-1)^n\frac{L}{n\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1+(-1)^n}{n}\right]$$

Substituting a_0, a_n & b_n in (i), we get,

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{-1+(-1)^n}{n}\right]\sin n\pi\frac{x}{L}$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[-\frac{2}{1}\sin\pi\frac{x}{L} + 0 - \frac{2}{3}\sin 3\pi\frac{x}{L} + \ldots\right]$$

$$= \frac{1}{2} - \frac{2}{\pi}\left[\sin \pi\frac{x}{L} + \sin 3\pi\frac{x}{L} + \ldots\ldots\right]$$

This is the required Fourier series for f(x).

10. Find the half-range casine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$. [BE 2063]

Solution: Let $f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ .... (i)

Where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

& $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$, n = 1, 2, 3, ...

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \text{ csonx}\,dx$$

Let, $u = x^2$, $v = \cos nx$

Diff. u & integrating v w. r. t. x, resp, we get

$u' = 2x$, $v_1 = \frac{\sin nx}{x}$

$u'' = 2$, $v_2 = -\frac{\cos nx}{n^2}$, $v_3 = -\frac{\sin nx}{n^3}$.

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} uV\,dx = \frac{2}{\pi}\left[uV_1 - u'V_2 + u''V_3 - \ldots\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2.\frac{\sin nx}{n} + 2x.\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\frac{(-1)^n}{n^2}\right] = 4\frac{(-1)^n}{n^2}$$

substituting a_0 & a_n in (i), we get,

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$\text{i.e. } x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

This is the half-range Fourier cosine series for x^2.

11. Obtain half-range cosine and sine series for $f(x) = x$ in the interval $9 \le x \le \pi$. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots = \frac{\pi^2}{8}$.

Solution:. The half-range cosine series in $[0, \pi]$ is

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ .... (i)}$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$, n = 1, 2, ...

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$\therefore = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

Substituting a_0 & a_n in (i), we get,

$$x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{2}{1^2}\cos x + 0 - 2\frac{2}{3^2}\cos 3x + \ldots\right]$$

$$\therefore x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right]$$

Next, the half-range sine series in $[0, \pi]$ is

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \text{ .... (ii)}$$

Where $b_x = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$, x = 1, 2, ...

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x.-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi.\frac{(-1)^n}{n}\right] = 2\frac{(-1)^{n+1}}{n}$$

Substituting b_x in (ii), we get,

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Finally, putting x = 0 in (*) we get,

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \ldots..$$

12. Obtain the half-range sine series for e^x in $0 < x < 1$. [BE 2062]

Solution: Here, L = 1

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ .... (i)

Where $b_n = \frac{2}{1} \int_0^1 f(x) \sin n\pi x \, dx$, n = 1, 2, ...

$= 2 \int_0^1 e^x \sin n\pi x \, dx$

$= 2 \left[\frac{e^x}{1+n^2\pi^2} (1.\sin n\pi x - n\pi \cos n\pi x) \right]_0^1$

$= 2 \left[\frac{e^x}{1+n^2\pi^2} (0 - n\pi \cos n\pi) - \frac{1}{1+n^2\pi^2} (0 - n\pi) \right]$

$= 2 \left[\frac{1}{1+n^2\pi^2} (1 - e \cos n\pi) n\pi \right]$

$= \frac{2x\pi(1 - e\cos n\pi)}{1+n^2\pi^2}$

Substituting b_x in (i), we get,

$e^x = \sum_{n=1}^{\infty} \frac{2x\pi(1-e\cos n\pi)}{1+n^2\pi^2} \sin n\pi x$

This is the half-ragne Fourier cosine series for e^x.

13. **Obtain the half-range cosine series for x sinx in the interval (0, π) and deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - .... = \frac{1}{4}(\pi - 2)$.**

Solution: Let $f(x) = x \sin x = \sum_{n=1}^{\infty} a_n \cos nx$ .... (i)

Where, $a_0 = \frac{2}{\pi} \int_0^\pi f(x) \, dx$

$= \frac{2}{\pi} \int_0^\pi x \sin x \, dx$

$= \frac{2}{\pi} [x.-\cos x + \sin x]_0^\pi$

$= \frac{2}{\pi} (+\pi) = 2$

and $a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx$, n = 1, 2, ...

$= \frac{2}{\pi} \int_0^\pi x \sin x \cos nx \, dx$

$= \frac{1}{\pi} \int_0^\pi x[\sin(x+nx) + \sin(x-nx)] \, dx$

$= \frac{1}{\pi} \int_0^\pi x [\sin (n+1) x - \sin (n-1)x] \, dx$ ..... (ii)

For n > 1

$a_n = \frac{1}{\pi} \left[x.\left(-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)} \right) - \left(-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right) \right]_0^\pi$

$= \frac{1}{\pi} \left[\pi \left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} \right) \right]$

$= \frac{(-1)^{n+2}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}$

$= \frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{(n-1)} = (-1)^n \left[\frac{n-1-n-1}{n^2-1} \right]$

$= -\frac{2(-1)^n}{n^2-1} = 2\frac{(-1)^{n+1}}{n^2-1}$

For n = 1

$a_1 = \frac{1}{\pi} \int_0^\pi x \sin 2x \, dx$, from (ii)

$= \frac{1}{\pi} \left[x.-\frac{\cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^\pi$

$= \frac{1}{\pi} \left[\pi.-\frac{1}{2} + 0 \right] = -\frac{1}{2}$

Substituting a_0, a_n and a_1 in (i), we get,

$x \sin x = 1 - \frac{\cos x}{2} + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n+1} \cos nx}{(n^2-1)}$

$= 1 - \frac{\cos x}{2} + 2 \left(-\frac{\cos 2x}{3} + \frac{\cos 3x}{8} - \frac{\cos 4x}{15} + ... \right)$

$\therefore x \sin x = 1 - \frac{\cos x}{2} + 2 \left(-\frac{\cos 2x}{1.3} + \frac{\cos 3x}{2.4} - \frac{\cos 4x}{1.3.5} + ... \right)$

This is the half-range Fourier cosine series for x sinx.

Next,

Putting $x = \frac{\pi}{2}$ in the above series, we get,

$\frac{\pi}{2} = 1 - 0 - 2 \left(-\frac{1}{1.3} - 0 + \frac{1}{1.3.5} - 0 + ... \right)$

$= 1 + 2 \left(\frac{1}{3} - \frac{1}{3.5} + ... \right)$

or, $\frac{\pi}{2} - 1 = 2 \left(\frac{1}{3} - \frac{1}{3.5} + ... \right)$

$\therefore \frac{\pi - 2}{4} = \frac{1}{3} - \frac{1}{3.5} + ...$

14. If $f(x) = C$, then prove that Fourier sine series of the coustant function in the range $0 < x < \pi$ is,

$$\frac{4c}{\pi}\left\{\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\ldots\right\}$$ [BE 2063]

Solution: Let $f(x) = c = \sum_{n=1}^{\infty} b_n \sin nx$ …….. (i)

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx,\ n = 1, 2, \ldots\ldots$

$$= \frac{2}{\pi}\int_0^{\pi} c \sin nx\, dx$$

$$= \frac{2c}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{2c}{\pi}\left[\frac{(-1)^n-1}{n}\right]$$

Substituting b_n in (i), we get,

$$c = -\frac{2c}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n}\right]\sin nx$$

$$= -\frac{2c}{\pi}\left[-\frac{2}{1}\sin x+0-\frac{2}{3}\sin 3x+0-\frac{2}{5}\sin 5x+\ldots\ldots\right]$$

$$\therefore C = \frac{4c}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\ldots\right]$$

This is the Fourier sine series for $f(x) = C$.

15. Obtain a half-range sine series for the function $f(x) = x - x^2,\ 0 < x < 1$. (BE 2061)

Solution: Here, $L = 1$

Let, $f(x) = x - x^2 = \sum_{n=1}^{\infty} b_n \sin n\pi x$ ……. (i)

Where $b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\, dx,\ n = 1, 2, \ldots\ldots$

$$= 2\int_0^1 (x-x^2)\sin n\pi x\, dx$$

Let, $u = x - x^2$, $v = \sin n\pi x$

Diff. u and integrating v w. r. t., x respect, we get,

$u' = 1 - 2x,\ v_1 = -\frac{\cos n\pi x}{n\pi}$

$u'' = -2,\ v_2 = -\frac{\sin n\pi x}{n^2\pi}, v_3 = \frac{\cos n\pi x}{n^3\pi^3}$

$$\therefore b_n = 2\int_0^1 uv\, dx = 2\left[uv_1 - u'v_2 + u''v_3 \ldots\right]_0^1$$

$$= 2\left[(x-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right)+(1-2x)\frac{\sin n\pi x}{n^2\pi^2}-2\frac{\cos n\pi x}{n^3\pi^3}\right]_0^1$$

$$= 2\left[-2\frac{(-1)^n}{n^2\pi^2}+\frac{2}{n^3\pi^3}\right]$$

$$= \frac{4}{\pi^3}\left[\frac{1-(-1)^n}{n^3}\right]$$

Substituting b_n in (i), we get,

$$x - x^2 = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin n\pi x$$

$$= \frac{4}{\pi^3}\left[\frac{2}{1^3}\sin \pi x + 0 + \frac{2}{3^3}\sin 3x + \ldots\ldots\right]$$

$$\therefore x - x^2 = \frac{8}{\pi^3}\left[\frac{\sin \pi x}{1^3}+\frac{\sin 3x}{3^3}+\ldots\ldots\right]$$

This is the half-range Fourier sine series for $x - x^2$.

17. Obtain half-range cosine series for $f(x) = \sin x$, $0 < x < \pi$ and hence show that $\sum\frac{1}{4n^2-1} = \frac{1}{2}$. (BE 2058)

Solution: Let $f(x) = \sin x = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$ ……. (i)

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\, dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}[1+1] = \frac{4}{\pi}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx,\ n = 1, 2, \ldots\ldots\ldots$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(x+nx)+\sin(x-nx)]\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]\, dx$$

For $n > 1$

$$a_n = \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)}+\frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n+1}-\frac{(-1)^n}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1}-\frac{1}{n-1}\right)+\frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\frac{(n-1-n-1)}{n^2-1}-\frac{2}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n.\frac{-2}{n^2-1}-\frac{2}{n^2-1}\right]$$

$$= -\frac{2}{\pi}\left[\frac{(-1)^n+1}{n^2-1}\right]$$

For $n = 1$, $a_1 = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^\pi = \frac{1}{\pi}\left[\frac{-1+1}{2}\right] = 0$$

Substituting a_0, a_1 & a_n $n > 1$ in (i), we get,

$$\text{four} = \sin x = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{(-1)^n+1}{n^2-1}\right]\cos nx.$$

$$= \frac{2}{\pi} - \frac{2}{\pi}\left[\frac{2}{2^2-1}\cos 2x + 0 + \frac{2}{4^2-1}\cos 4x + 0 + \frac{2}{6^2-1}\cos 6x + \ldots\right]$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{4.1^2-1} + \frac{\cos 4x}{4.2^2-1} + \frac{\cos 6x}{4.3^2-1} + \ldots\right]$$

$$\therefore f(x) = \sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$$

This is the half range Fourier coine series for sine.

Next, putting $x = 0$, or π we get the result but $f(x) = \sin x$ is not continuous at 0 & π.

To find f(0), we have,

$$f(0) = \frac{1}{2}\left[\int (0-0) + \int (0+0)\right] = \frac{1}{2}\left[0+0\right] = 0$$

Now, putting $x = 0$ in the above series, we get,

$$f(0) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\text{or, } 0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\text{or, } -\frac{2}{\pi} = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\therefore \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

18. **Find the Fourier sine series for the function $f(x) = e^{ax}$ for $0 < x < \pi$.**

Solution: Let $f(x) = e^{ax} = \sum_{n=1}^{\infty} b_n \sin nx$ .... (i)

Where, $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$, $n = 1, 2, \ldots\ldots$

$$= \frac{2}{\pi}\int_0^\pi e^{ax}\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(0-n(-1)^n) - \frac{1}{a^2+n^2}(-n)\right]$$

$$= \frac{2}{\pi}\left[e^{a\pi}(-1)^{n+1}+1\right]\frac{n}{a^2+n^2}$$

Substituting b_n in (i), we get,

$$e^{ax} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[1-(-1)^n e^{a\pi}+1\right]\frac{n}{a^2+n^2}\sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left[1-(-1)^n e^{a\pi}+1\right]\frac{n\sin nx}{a^2+n^2}$$

This is the required Fourier series for e^{ax}.

19. **Prove that half range sine series for $f(x) = Lx - x^2$ in (0, L) is $\frac{8L^2}{\pi^3}\sum\frac{1}{(2n+1)^3}\sin\frac{(2n+1)}{L}\pi x$.**

Solution: Let $f(x) = Lx - x^2 = \sum_{n=1}^{\infty} b_n \sin \pi\frac{x}{L}$ ...... (i)

Where $b_n = \frac{2}{L}\int_0^L f(x)\sin n\pi\frac{x}{L}\,dx$, $n = 1, 2, \ldots\ldots$

$$= \frac{2}{L}\int_0^L (Lx - x^2)\sin n\pi\frac{x}{L}\,dx$$

Let $u = Lx - x^2$, $v = \sin n\pi\frac{x}{L}$

Diff. u and integrating v w. r. t. x, we get,

$$u' = L - 2x,\ v_1 = -\cos n\pi\frac{x}{L}\cdot\frac{L}{x\pi}$$

$$u'' = -2,\ v_2 = -\sin n\pi\frac{x}{L}\cdot\frac{L^2}{n^2\pi^2},\ v_3 = \cos n\pi\frac{x}{L}\cdot\frac{L^3}{n^3\pi^3}$$

$$\therefore b_n = \frac{2}{L}\int_0^L uv\,dx$$

$$= \frac{2}{L}\left[uv_1 - u'v_2 + u''v_3 \ldots\right]_0^L$$

$$= \frac{2}{L}\left[(Lx-x^2) = \cos n\pi\frac{x}{L}\cdot\frac{L}{n\pi} + (L-2x)\sin n\pi\frac{x}{L}\cdot\frac{L^2}{n^2\pi^2} - 2\cos n\pi\frac{x}{L}\cdot\frac{L^3}{n^3\pi^3}\right]_0^L$$

$$= \frac{2}{L}\left[-2.(-1)^n\frac{L^3}{n^3\pi^3} + 2\frac{L^3}{n^3\pi^3}\right]$$

$$= \frac{4L^2}{\pi^3}\left[\frac{1+(-1)^{n+1}}{n^3}\right]$$

Substituting b_n in (i), we get,

$$Lx - x^2 = \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1+(-1)^{n+1}}{n^3}\right]\sin n\pi\frac{x}{L}$$

$$= \frac{4L^2}{\pi^3}\left[\frac{2}{1^3}\sin\pi\frac{x}{L} + 0 + \frac{2}{3^3}\sin 3\pi\frac{x}{L} + \ldots\ldots\right]$$

$$= \frac{8L^2}{\pi^3}\left[\frac{\sin\pi\frac{x}{L}}{1^3} + \frac{\sin 3\pi\frac{x}{L}}{3^3} + \ldots..\right]$$

$$= \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi\frac{x}{L}}{(2n+1)^3}$$

This is the required Fourier sine-series for $Lx - x^2$.

20. **Obtain the half-range sine and cosine series for the function,**

$$f(x) = x, \ 0 \angle \frac{\pi}{2}$$

$$= \pi - x, \ \frac{\pi}{2} < x < \pi$$

Solution: The half-range Fourier sine series for f(x) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \ \ldots\ldots\ (i)$$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \ n = 1, 2, \ldots\ldots$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)\sin nx dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\sin nx dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin nx dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin nx dx\right]$$

$$= \frac{2}{\pi}\left[\left|x.-\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_0^{\frac{\pi}{2}} + \left|(\pi-x) - \frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos n\frac{\pi}{2}}{n} + \sin n\frac{\pi}{2}\frac{1}{n^2} + \frac{\pi}{2}\frac{\cos n\frac{\pi}{2}}{n} + \frac{\sin n\frac{\pi}{2}}{n^2}\right]$$

$$= \frac{4}{\pi}\frac{\sin n\frac{\pi}{2}}{n^2}$$

substituting b_n in (i), we get,

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\frac{\pi}{2}}{n^2}\sin nx$$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1^2} + 0 - \frac{\sin 3x}{3^2} + \ldots..\right]$$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \ldots..\right]$$

Next, the half-range cosine series of f(x) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ \ldots\ldots\ (i)$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\frac{\pi}{2}} + \left|\pi x - \frac{x^2}{2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{\pi}{2}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \ n = 1, 2, \ldots\ldots$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)\cos nx dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\cos nx dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nx dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx dx\right]$$

$$= \frac{2}{\pi}\left[\left|x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\frac{\pi}{2}} + \left|(\pi-x).\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\frac{\sin n\frac{\pi}{2}}{n} + \frac{\cos n\frac{\pi}{2}}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{\pi}{2}\frac{\sin n\frac{\pi}{2}}{n} + \frac{\cos n\frac{\pi}{2}}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos n\frac{\pi}{2} - 1 - (-1)^n}{n^2}\right]$$

Substituting a_0 & a_n in (i) we get

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2\cos n\frac{\pi}{2} - 1 - (-1)^n}{n^2}\right]\cos nx$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{(0-1+1)}{1^2}\cos x + \frac{(-2-1-1)}{2^2}\cos 2x + 0 + 0 + 0 + \frac{(-2-1-1)}{6^2}\cos\cos + ...\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + ...\right]$$

TU Examination 2068 (Baishakh)

Using the properties of determinant prove that :

1. $$\begin{vmatrix} x & 1 & y & 1 \\ 1 & y & 1 & x \\ 1 & x & 1 & y \\ y & 1 & x & 1 \end{vmatrix} = (x+y+2)(x-y)^2(x+y-2)$$

Sol^n. Let $\Delta = \begin{vmatrix} x & 1 & y & 1 \\ 1 & y & 1 & x \\ 1 & x & 1 & y \\ y & 1 & x & 1 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$ and taking $(x+y+z)$ common from R_1.

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & y & 1 & x \\ 1 & x & 1 & y \\ y & 1 & x & 1 \end{vmatrix}$$

Applying $C_3 \to C_3 - C_1$ and $C_4 \to C_4 - C_2$

$$= (x+y+2)\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & y & 0 & x-y \\ 1 & x & 0 & y-x \\ y & 1 & x-y & 0 \end{vmatrix}$$

Taking $(x-y)$ and $(x-y)$ common from C_3 and C_4 respectively. We get

$$= (x+y+2)(x-y)(x-y)\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & y & 0 & 1 \\ 1 & x & 0 & -1 \\ y & 1 & 1 & 0 \end{vmatrix}$$

Expanding along C_3

$$= (x+y+2)(x-y)^2(-1)\begin{vmatrix} 1 & 1 & 0 \\ 1 & y & 1 \\ 1 & x & -1 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$

$$= -(x+y+2)(x-y)^2\begin{vmatrix} 1 & 1 & 0 \\ 2 & x+y & 0 \\ 1 & x & -1 \end{vmatrix}$$

Expanding along C_3

$$= -(x+y+2)(x-y)^2(-1)\begin{vmatrix} 1 & 1 \\ 2 & x+y \end{vmatrix}$$

$$= (x+y+2)(x-y)^2(x+y-2)$$

$$\therefore \Delta = (x+y+2)(x-y)^2(x+y-2) \text{ Ans.}$$

2. **If A and b are two non-singular matrices of the same order, prove that $(AB)^{-1} = B^{-1}A^{-1}$**

Sol^n. Let A and B be two non-singular matrices of order n, then,

$|AB| = |A|\,|B| \neq 0$

$|AB| \neq 0$

Thus, AB is non-singular and inverse of AB is $(AB)^{-1}$

Now,

$(AB) . (B^{-1}A^{-1}) = A\,(BB^{-1})\,(A^{-1})$

$= A\,I\,A^{-1} = AA^{-1} = I$.

Similarly,

$(B^{-1}A^{-1}) . (AB) = B^{-1}\,(A^{-1}A)\,B = B^{-1}\,I\,B = B^{-1}B = I$

Hence, $B^{-1}A^{-1}$ is the inverse of AB.

i.e. $B^{-1}A^{-1} = (AB)^{-1}$

3. **Find the rank of the following matrix reduceing to normal form**

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{vmatrix}$$

Sol^n. See page 87

4. **Find the eigen-values and eigen vectors of the matrix**

$$\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Sol^n. The characteristic equation of the matrix is $|A - \lambda I| = 0$

$$\text{or,}\quad \begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

or, $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

Clearly, $\lambda = 2$ satisfies to this equation.

So, we can write this equation as $(\lambda - 2)\,(\lambda^2 - 4) = 0$

or, $(\lambda - 2)\,(\lambda - 2)\,(\lambda + 2) = 0$

$\therefore \lambda = 2, 2, -2$

Thus, the eigen-values of A are

$\lambda = 2, 2, -2$

If x, y, z be component of eigen-vector corresponding to eigen-values λ, then the matrix equation is

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 2$, we have,

$-2y + 2z = 0$ .... (i)

$x - y + z = 0$ .... (ii)

and $x + 3y - 3z = 0$ .... (iii)

Solving last two equations, we get

$$\frac{x}{3-3} = \frac{y}{1+3} = \frac{z}{3+1}$$

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{1}$$

Hence, the eigen-vector is (0, 1, 1)

Also, every non-zero multiple of this vector is an eigen-vector corresponding to $\lambda = 2$.

Similarly, putting $\lambda = -2$, we have

$4x - 2y + 2z = 0$ .... (iv)

$x + 3y + z = 0$ .... (v)

$x + 3y + z = 0$ .... (vi)

Solving first two equations, we get,

$$\frac{x}{-4} = \frac{y}{-1} = \frac{z}{7}$$

Hence, the eigen-vecotr is (– 4, – 1, 7)

Also, every non-zero multiple of the vector is an eigen-vector corresponding to $\lambda = -2$

Thus, the two eigen-vectors may be taken as

$a_1\,(0, 1, 1),\ a_1 \neq 0$ and $a_2\,(-4, -1, 7),\ a_2 \neq 0$

5. **Find the Laplace transform of the following functions:**

(a) $te^{-3t}\cos 2t$

Sol^n. We have $L\,[te^{-3t}\cos 2t]$

Now, $L\,[e^{-3t}\cos 2t] = \dfrac{s+3}{(s+3)^2+4} = \dfrac{s+3}{s^2+6s+13}$

By using the theorem of Laplace transform, we get

$$L\,[te^{-3t}\cos 2t] = (-1)^1 \frac{d}{ds}\left[\frac{s+3}{s^2+6s+13}\right]$$

$$= \frac{\left(s^2+6s+13-2s^2-6s-6s-18\right)}{\left(s^2+6s+13\right)^2}$$

$$= \frac{-(s^2-6s-5)}{(s^2+6s+13)^2} = \frac{s^2+6s+5}{(s^2+6s+13)^2} \text{ Ans.}$$

(b) $\dfrac{e^{at}-\cos 6t}{t}$

Solⁿ. We have $L\left[\dfrac{e^{at}-\cos 6t}{t}\right]$

Now, $L[e^{at}-\cos 6t] = L[e^{at}] - L[\cos 6t] = \dfrac{1}{s-a} - \dfrac{s}{s^2+36}$

By using the theorem of Laplace transform,
We get,

$$L\left[\frac{e^{at}-\cos 6t}{t}\right] = \int_s^\infty \left[\frac{1}{s-a}-\frac{s}{s^2+36}\right]ds$$

$$=\left[\log(s-a)-\frac{1}{2}\log(s^2+36)\right]_s^\infty = -\log(s-a)+\frac{1}{2}\log(s^2+36)$$

$$=\log\sqrt{s^2+36}-\log(s-a)=\log\frac{\sqrt{s^2+36}}{s-a} \text{ Ans.}$$

6. Find the inverse Laplace transform of the following functions

(a) $\dfrac{1}{(s-2)(s+2)^2}$

Solⁿ. We have $\dfrac{1}{(s-2)(s+2)^2} = L^{-1}\left[\dfrac{1}{s-2}\times\dfrac{1}{(s+2)^2}\right]$

Now, $L^{-1}\left[\dfrac{1}{s-2}\right] = e^{2t} = g(t)$ (say)

and $L^{-1}\left[\dfrac{1}{(s+2)^2}\right] = te^{-2t} = f(t)$ (say)

By using convolution theorem of inverse L. T., we get

$$L^{-1}\left[\frac{1}{s-2}\times\frac{1}{(s+2)^2}\right] = \int_0^t ue^{-2u}\,.\,e^{2t-2u}\,du$$

$$=\int_0^t u\,e^{2t-4u} = \left[\frac{Ue^{2t-4u}}{-4}-\frac{e^{2t-4u}}{16}\right]_0^t$$

$$=\frac{te^{-2t}}{-4}-\frac{e^{-2t}}{16}+\frac{e^{2t}}{16} = \frac{1}{16}\left[e^{2t}-e^{-2t}-4te^{-2t}\right]$$

$$\therefore L^{-1}\left[\frac{1}{s-2}\times\frac{1}{(s+2)^2}\right] = \frac{1}{16}\left[e^{2t}-e^{-2t}-4te^{-2t}\right] \text{ Ans.}$$

(b) $\dfrac{1}{s^2(s+2)}$

Solⁿ. We have, $L^{-1}\left[\dfrac{1}{s^2(s+2)}\right] = L^{-1}\left[\dfrac{1}{s^2}\times\dfrac{1}{s+2}\right]$

Now, $L^{-1}\left[\dfrac{1}{s^2}\right] = \dfrac{t}{1!} = t = g(t)$ (say)

or, $L^{-1}\left[\dfrac{1}{s+2}\right] = e^{-2t} = f(t)$ say

By using the convolusion theorem of inverses we get,

$$L^{-1}\left[\frac{1}{s^2}\times\frac{1}{s+2}\right] = \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t e^{-2u}(t-u)\,du = \int_0^t te^{-2tu}\,du - \int_0^t u\,e^{-2u}\,du$$

$$= \left[\frac{te^{-2tu}}{-2}\right]_0^t - \left[\frac{4e^{-2u}}{-2}\right]_0^t - \frac{1}{2}\int_0^t e^{2u}\,du$$

$$= \frac{te^{-2t}}{-2}+\frac{t}{2}+\frac{te^{-2t}}{2}+\frac{1}{4}\left[e^{-2u}\right]_0^t$$

$$= \frac{t}{2}+\frac{e^{-2t}}{4}-\frac{1}{4} = \frac{1}{4}[e^{-2t}+2t-1]$$

$$\therefore L^{-1}\left[\frac{1}{s^2}\times\frac{1}{s+2}\right] = \frac{1}{4}[e^{-2t}+2t-1] \text{ Ans.}$$

7. Solve using Laplace transform
$(D^2+4D+3)\,x = e^{-t}$, where $x(0)=x'(0)=1$

Solⁿ. Here, the differential equation is,
$(D^2+4D+3)\,x = e^{-t}$
Taking Laplace transform of both sides, we get
$[S^2X - sx(0) - x'(0)] + 4\,[SX - x'(0)] + 3X = L[e^{-t}]$
Using the given initial conditions $x(0)=x'(0)=1$, we get
$[S^2X - sx(0) - x'(0)] + 4\,(SX - x'(0)] + 3X = L[e^{-t}]$

$$(S^2+4s+3)\,X - s - 1 - 4 = \frac{1}{s+1}$$

$$\text{or, } (S^2+4s+3)\,X = S+5+\frac{1}{s+1} = \frac{S^2+6s+6}{s+1}$$

$$\text{or, } X = \frac{s^2+6s+6}{(s^2+4s+3)(s+1)}$$

$$\text{or, } X = \frac{s^2+6s+6}{(s+3)(s+1)(s+1)}$$

$$\text{or, } X = \frac{s^2+6s+6}{(s+3)(s+1)^2}$$

By partial fraction, we can write,

$$X = \frac{7}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{3}{4}\frac{1}{s+3}$$

By the inverse Laplace transform, we get,

$$x = \frac{7}{4} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4} L^{-1}\left[\frac{1}{s+3}\right]$$

$$\therefore x = \frac{7}{4} e^{-t} - \frac{3}{4} e^{-3t} + \frac{1}{2} t e^{-t} \text{ Ans.}$$

8. Obtain a Fourier series for $f(x) = x^3$ in the interval $-\pi \le x \le \pi$.

Sol^n. Let $f(x) = x^3 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ .... (i)

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3\,dx = 0$ being x^3 as odd function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \text{ con } nx\,dx,\ x = 1, 2, \ldots\ldots\ldots\ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos nx\,dx = 0 \text{ being } x^3 \cos nx \text{ as odd function.}$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx,\ x = 1, 2, \ldots\ldots\ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^3 \sin nx dx \text{ being } x^3 \sin nx \text{ as even function.}$$

$u = x^3$, $v = \sin nx$

Diff. u & integrating v w. r. t. x successively,

$u = 3x^2$ integrating $v_1 = -\frac{\cos nx}{x}$

$u'' = 6x$ integrating $v_2 = -\frac{\sin nx}{x^2}$

$u''' = 6$ integrating $v_3 = \frac{\cos nx}{x^3}$, $v_4 = \frac{\sin nx}{x^4}$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} uv\,dx$$

$$= \frac{2}{\pi}\left[uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^3.\frac{-\cos nx}{n} + 3x^2.\frac{\sin nx}{n^2} + 6x\frac{\cos nx}{n^3} - \frac{6\sin nx}{n^4}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi^3.\frac{(-1)^n}{n} + 6\pi\frac{(-1)^n}{n^3}\right] = 2\left[\pi^2\frac{(-1)^{n+1}}{n} + 6\frac{(-1)^n}{n^3}\right]$$

$$= 2\left[\frac{6(-1)^n}{n^3} - \pi^2\frac{(-1)^n}{n}\right]$$

Substituting a_0, a_n & b_n is (i)

$$x^3 = \sum_{n=1}^{\infty}\left[\frac{6(-1)^n}{n^3} - \pi^2\frac{(-1)^n}{n}\right]\sin nx.$$

This is the required fourier series for $f(x) = x^3$

9. Find the half range sine series for the function $f(x) = x - x^2$ in the internal $0 < x < 1$.

Sol^n. Let, $f(x) = x - x^2 = \sum_{n=1}^{\infty} b_n \sin n\pi x$ .... (i)

Where $b_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x\,dx,\ n = 1, 2, \ldots\ldots$

$$= 2\int_0^1 (x - x^2) \sin n\pi x\,dx$$

Let, $u = x - x^2$, $v = \sin n\pi x$

Diff. u and integrating v w. r. t., x respect, we get,

$u' = 1 - 2x$, $v_1 = -\frac{\cos n\pi x}{n\pi}$

$u'' = -2$, $v_2 = -\frac{\sin n\pi x}{n^2\pi^2}$, $v_3 = \frac{\cos n\pi x}{n^3\pi^3}$

$$\therefore b_n = 2\int_0^1 uv\,dx = 2\left[uv_1 - u'v_2 + u''v_3 \ldots\right]_0^1$$

$$= 2\left[(x - x^2).-\frac{\cos n\pi x}{n\pi} + (1-2x)\frac{\sin n\pi x}{n^2\pi^2} - 2\frac{\cos n\pi x}{n^3\pi^3}\right]_0^1$$

$$= 2\left[-2\frac{(-1)^n}{n^2\pi^2} + \frac{2}{n^3\pi^3}\right] = \frac{4}{\pi^3}\left[\frac{1-(-1)^n}{n^3}\right]$$

Substituting b_n in (i), we get,

$$x - x^2 = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^3}\right]\sin n\pi x$$

$$= \frac{4}{\pi^3}\left[\frac{2}{13}\sin \pi x + 0 + \frac{2}{3^3}\sin 3\pi x + \ldots\ldots\right]$$

$$\therefore x - x^2 = \frac{8}{\pi^3}\left[\frac{\sin \pi x}{1^3} + \frac{\sin 3x}{3^3} + \ldots\ldots\right]$$

This is the half-range Fourier sine series for $x - x^2$.

10. **Maximize $z = x_1 + 1.5x_2$ subject to constraints**
$2x_1 + 2x_2 \le 160$
$x_1 + 2x_2 \le 120$
$4x_1 + 2x_2 \le 280$
$x_1 \ge 0$ and $x_2 \ge 0$ graphically.

Sol^n. Here, the objective function is
$2x_1 + 2x_2 \le 160$
$x_1 + 2x_2 \le 120$
$4x_1 + 2x_2 \le 280$
$x_1 \ge 0$ and $x_2 \ge 0$

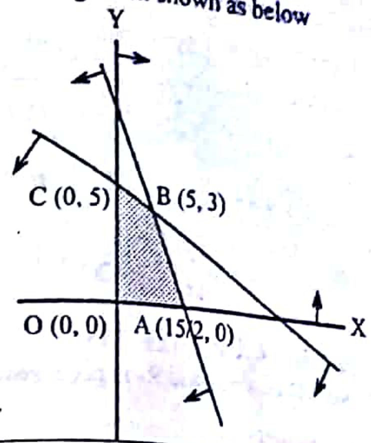
It is clear that $x_1 \ge 0$ and $x_2 \ge 0$ imply the feasible region lies in the first quadrant writing the inequilities in to equilities we get,

$2x_1 + 2x_2 = 160$ .... (i)
$x_1 + 2x_2 = 120$ .... (ii)
$4x_1 + 2x_2 = 280$ .... (iii)
$x_1 = 0$ and $x_2 = 0$ .... (iv)

These equation gives the boundary line.
In the equation (i) $2x_1 + 2x_2 = 160$
Put $x_1 = 0$, we get $x_2 = 80$
Put $x_2 = 0$, we get $x_1 = 80$
So, the points (0, 80) and (80, 0) lies in the line (i). Draw this line.
In the equation (ii) $x_1 + 2x_2 = 120$
Put $x_1 = 0$, we get $x_2 = 60$
Put $x_2 = 0$, we get $x_1 = 120$
So, the points (0, 60) and (120, 0) lie in the line (ii). Draw this line.
In the equation (iii) $4x_1 + 2x_2 = 280$
Put $x_1 = 0$, we get $x_2 = 140$
Put $x_2 = 0$, we get $x_1 = 70$
So the points (0, 140) and (70, 0) lie in the line (iii). Draw this line.
From the line (iv), $x_1 = 0$ and $x_2 = 0$ represents the line of x_2 – axis and x_1 – axis respectively.
Draw these lines.
The coordinates of the points O, B, E and (0, 0) to (0) and (60, 0) respectively.

From (i) and (iii) $\quad 2x_1 + 2x_2 = 160$
Solving these, $\quad 4x_1 + 2x_2 = 280$
Solving these, we get $x_1 = 60$, and $x_2 = 20$
From (i) and (ii) $\quad 2x_1 + 2x_2 = 160$
$\quad x_1 + 2x_2 = 120$
Solving these, we get $x_1 = 40$ and $x_2 = 40$
So the corner points C and D are
(40, 40) and (60, 20) respectively.

Put $x_1 = 0$ and $x_2 = 0$, they are satisfied
So, the feasible region in shown as below



Vertices	Objective function $z = x_1 + 1.5x_2$
1. O (0, 0)	1. $z = 0 + 1.5 \times 0 = 0$
2. A (70, 0)	2. $z = 70 + 1.5 \times 0 = 70$
3. C (40, 40)	3. $z = 40 + 1.5 \times 40 = 100$
4. D (60, 20)	4. $z = 60 + 1.5 \times 20 = 90$

From the above table, we see that,
Maximum of $z = x_1 + 1.5x_2$ occur at the vertex (40, 40)
$\therefore$ Maximum value = 100 at (40, 40)

11. **Solve the following linear programming problem by simplex method maximize $z = 15x_1 + 10x_2$**
Subject to $2x_1 + 2x_2 \le 10$
$x_1 + 3x_2 \le 10$ and $x_1, x_2 \ge 0$

Sol^n. Using slack variables s_1 and s_2, given LPP can be written in the normal form as,
$z - 15x_1 - 10x_2 + 0.s_1 + 0.s_2 = 0$
$2x_1 + x_2 + s_1 + 0.s_2 = 10$
$x_1 + 3x_2 + 0.s_1 + s_2 = 10$
the first simplex table is,

Basic	z	$\downarrow x_1$	x_2	s_1	s_2	b	$\frac{b_i}{b_{ij}}$
	1	–15	–10	0	0	0	
s_1	0	$\boxed{2}$	1	1	0	10	$\frac{10}{2} \rightarrow$ exit
s_2	0	1	3	0	1	10	$\frac{10}{1}$

Basic $s_1 = 10$, $s_2 = 10$
Non-basic, $x_1 = x_2 = 0$ for this $z = 0$
Also, x_1-column is pivot column and R_2 is pivot row.
So, '2' is pivot element.

Applying $R_1 \to R_1 + \frac{15}{2} R_2,\ R_3 \to R_3 - \frac{1}{2} R_2$

Basic	z	x_1	x_2	s_1	s_2	b	$\frac{b_i}{b_{ij}}$
	1	0	$-\frac{5}{2}$	$\frac{15}{2}$	0	75	
s_1	0	2	1	1	0	10	$\frac{10}{1}$
s_2	0	0	$\boxed{\frac{5}{2}}$	$-\frac{1}{2}$	1	5	$\frac{5}{2} \to$ exit

Basic $x_1 = 5, s_2 = 5$
Non-basic, $x_2 = s_1 = 0$, for this $z = 75$
x_2-column is pivot column and R_3 is pivot row
So, $\frac{5}{2}$ is pivot element.

Applying $R_1 \to R_1 + R_3,\ R_2 \to R_2 - \frac{5}{2} R_3$

The next simplex table is

Basic	z	x_1	x_2	s_1	s_2	b
	1	0	0	7	1	80
x_1	0	2	0	$\frac{6}{5}$	$-\frac{2}{5}$	8
x_2	0	0	$\frac{5}{2}$	$-\frac{1}{2}$	1	5

Since, there are no negative terms in first row, So simplex process is completed.

$\therefore x_1 = \frac{8}{2} = 4 \qquad x_2 = \frac{5}{\frac{5}{2}} = 2$

$\therefore$ From table max. $z = 80$ at $x_1 = 4$ and $x_2 = 2$

12. Show that the vector field $\bar{F} = (x^2 - yz)\,\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$ is irrotational. Find the scalar function ϕ (x, y, z) such that $\bar{F} = \nabla\phi$.

Sol^n. Here $\bar{F} = (x^2 - yz)\,\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$

$$\therefore \nabla \times \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (x^2 - yz)\,\bar{i} + (y^2 - zx)\,\bar{j} + (z^2 - xy)\,\bar{k}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right] - \bar{j}\left[\frac{\partial}{\partial z}(x^2 - yz) - \frac{\partial}{\partial x}(z^2 - xy)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right]$$

$$= \bar{i}\ [-x + x] - \bar{j}\,[-y + y] + \bar{k}\ [-z + z] = 0$$

$\therefore \nabla \times \bar{F} = 0$

Hence, vector function $\bar{F}$ is irrotational

Also, we have $\bar{F} = \nabla\phi$

$$\bar{F} \cdot d\bar{r} = \nabla\phi \cdot d\bar{r}$$

or, $(x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz = d\phi$

or, $x^2dx + y^2dx + z^2dz - [yz\,dx + zx\,dy + xy\,dz] = d\phi$

or, $d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + d\left(\frac{z^3}{3}\right) - d[xyz] = d\phi$

On integration, we get

$$\frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz = \phi$$

$$\therefore \phi = \frac{1}{3}[x^3 + y^3 + z^3] - xyz + c$$

13. If S be the part of surface $z = 9 - x^2 - y^2$ with $z \geq 0$ and $\bar{F} = 3x\,\bar{i} + 3y\,\bar{j} + z\,\bar{k}$, find the flux of F through S.

Sol^n. Here $\bar{F} = 3x\,\bar{i} + 3y\,\bar{j} + z\,\bar{k}$ and S is the surface $z = 9 - x^2 - y^2$ with $z \geq 0$, Normal to the surface is,

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(z - 9 + x^2 + y^2) = 2x\,\bar{i} + 2y\,\bar{j} + \bar{k}$$

Unit normal to the surface is,

$$\bar{n} = \frac{2x\bar{i} + 2y\bar{j} + \bar{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$\therefore \bar{F} \cdot \bar{n} = (3x\,\bar{i} + 3y\,\bar{j} + 3z\,\bar{k}) \cdot \left(\frac{2x\bar{i} + 2y\bar{j} + \bar{k}}{\sqrt{4x^2 + 4y^2 + 1}}\right)$$

$$= \frac{6x^2+6y^2 j+z}{\sqrt{4x^2+4y^2+1}}$$

Let R be the projection of surface on the xy-plane then

$$\vec{n}\,.\,\vec{k} = \frac{2x\vec{i}+2y\vec{j}+\vec{k}}{\sqrt{4x^2+4y^2+1}}\,.\,\vec{k} = \frac{1}{\sqrt{4x^2+4y^2+1}}$$

$$\therefore \iint_s \vec{A}\,.\,\vec{n}\;ds = \iint_R \vec{F}\,.\,\vec{n}\,\frac{dxdy}{|\vec{n}\,.\,\vec{k}|}$$

$$= \iint_R \frac{6x^2+6y^2+z}{\sqrt{4x^2+4y^2+1}}\cdot\frac{dxdy}{\frac{1}{\sqrt{4x^2+4y^2+1}}}$$

$$= \iint_R (6x^2+6y^2+z)\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (6x^2+6y^2+9-x^2-y^2)\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5x^2+5y^2+9)\,dy\,dx = 4\int_0^3\int_0^{\sqrt{9-x^2}} [5(x^2+y^2)+9]\,dy\,dx$$

To change it polar form put $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$, $dx\,dy = r\,dr\,d\theta$

and r various from $r = 0$ to $r = 3$ and $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$\text{So, } \iint_s \vec{A}\,.\,\vec{n}\;ds = 4\int_0^{\frac{\pi}{2}}\int_0^3 (5r^2+9)\,r\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^3 (5r^3+9r)\,dr\,d\theta$$

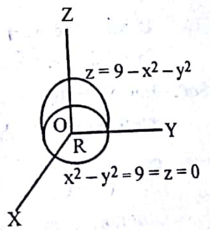
$$= 4\int_0^{\frac{\pi}{2}}\left[\frac{5r^4}{4}+\frac{9r^2}{2}\right]_0^3 d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left(\frac{4\times 81}{4}+\frac{81}{2}-0\right)d\theta$$

$$= 4\left(\frac{40s}{4}+\frac{81}{2}\right)[\theta]_0^{\frac{\pi}{2}}$$

$$= 4\times\left(\frac{405+162}{4}\right)\times\frac{\pi}{2} = \frac{567\pi}{2}$$

$$\therefore \iint_S \vec{A}\,.\,\vec{n}\;ds = \frac{567\pi}{2}\ \text{Ans.}$$



14. State and prove that Green's theorem in the plane.

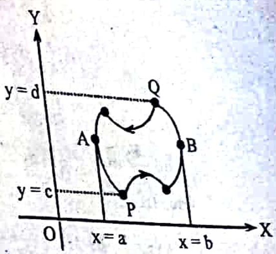
Sol^n. State,ment of Green's Theorems in the plane

If $F_1(x, y)$ and $F_2(x, y)$ are two functions having continuous first order partial derivatives in the region R on (xy-Plane) bounded by the closed curve C then,

$$\int_C (F_1\,dx + F_2\,dy) = \iint_R \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy$$

Proof

Let the curve C (APBQA) is bounded by the limit of y from $y = f_1(x)$ to $f_2(x)$ and x from $x = a$ to b



$$\therefore \iint_R \frac{\partial F_1}{\partial y}\,dxdy = \int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)}\frac{\partial F_1}{\partial y}dy\right]dx$$

$$= \int_a^b\int_{f_1(x)}^{f_2(x)}\{d\,F_1(x,y)\}\,dx = \int_a^b [F_1(x,y)]_{f_1(x)}^{f_2(x)}\,dx$$

$$= \int_a^b [F_1(x, f_2(x)] - F_1(x, f_1(x)]\,dx$$

$$= \int_a^b F_1(x, y)\,dx - \int_a^b F_1(x, y)\,dx$$

$$= -\int_b^a F_1(x, y)\,dx - \int_a^b F_1(x, y)\,dx$$

$$= -\int_{BQA} F_1(x, y)\,dx - \int_{APB} F_1(x, y)\,dx$$

$$= -\int_{APBQA} F_1(x, y)\,dx = -\int_C F_1(x)\,dx$$

$$\therefore \iint_R \frac{\partial F_1}{\partial y}\,dxdy = -\int_C F_1(x)\,dx \;..........\;(i)$$

Also, if the curve c (QAPBQ) is bounded by the limit of x from $x = g_1(y)$ to $g_2(y)$ and y from $y = c$ to d.

$$\therefore \iint_R \frac{\partial F_2}{\partial x}\,dxdy = \int_{y=c}^{d}\left[\int_{x=g_1(y)}^{g_2(y)}\frac{\partial F_2}{\partial x}dx\right]dy$$

$$= \int_c^d \int_{g_1(y)}^{g_2(y)} \{d\, F_2(x,y)\}\, dy = \int_c^d [F_2(x,y)]_{g_1(y)}^{g_2(y)}\, dy$$

$$= \int_c^d [F_2(g_2(y), y) - F_2(g_1(y))]\, dy$$

$$= \int_c^d F_2(x,y)\, dy - \int_c^d F_2(x,y)\, dy$$

$$= \int_c^d F_2(x,y)\, dy + \int_d^c F_2(x,y)\, dy$$

$$= \int_{PBQ} F_2(x,y)\, dy + \int_{QAP} F_2(x,y) dy$$

$$= \int_{PBQAP} F_2(x,y)\, dy = \int_C F_2\, dy$$

$\therefore \iint_R \dfrac{\partial F_2}{\partial x} dxdy = \int_C F_2\, dy$ .... (ii)

From (i) and (ii)

$\int_C (F_1\, dx + F_2\, dy) = \iint_R \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) dxdy$ Proved.

15. Evaluate by Stoke's theorem $\int_C (e^x dx + 2ydy - dz)$ where C is the curve $x^2 + y^2 = 4$, $z = 2$.

Sol^n. Here, $\int_C (e^x dx + 2ydy - dz)$

So that $F_1 = e^x$, $F_2 = 2y$, $F_3 = -1$

$$\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$$
$$= e^x.\, \vec{i} + 2y\, \vec{j} - \vec{k}$$

By Stoke's theorem, we have,

$\int_C \vec{F}.\, d\vec{r} = \iint_S (\nabla \times \vec{F}).\, \vec{n}\, ds$

$$(\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\,(0-0) = 0$$

Thus, by Stoke's theorem $\int_C \vec{F}.\, d\vec{r} = \iint_S (\nabla \times \vec{F}).\, \vec{n}\, ds = 0$

$\therefore \int_C \vec{F}.\, d\vec{r} = 0$ Ans.

OR

Verify Gauss divergence theorem for the vector function $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$, taken over the unit cube bounded by the plane: $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

Sol^n. Given $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ and V is the unit cube, $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$

We have the Gauss divergence theorem is

$\iint_S \vec{F}.\vec{n}\, ds = \iiint_V \ldots$

Now, $\nabla \times \vec{F} = \left(\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right).(x^2\vec{i} + z\vec{j} + yz\vec{k}) = 2x + y$

So, $\iiint_V (\nabla \times \vec{F}) = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2x+y)\,(x^2\vec{i} + z\vec{j} + yz\vec{k})dzdydx$

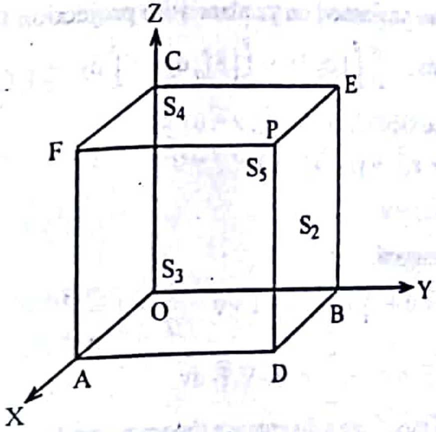
$$= \int_0^1\int_0^1 (2x+y)\, [z]_0^1\, dy\, dx = \int_0^1\int_0^1 (2x+y)\, dydx$$

$$= \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \left(2x + \frac{1}{2}\right) dx = \left[x^2 + \frac{1}{2}x\right]_0^1 = 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

Let, ADBECF is a cube which consists six surface so that,

$$\iint_S \vec{F}.\vec{n}\, ds = \iint_{S_1} \vec{F}.\vec{n}\, ds + \iint_{S_2} \vec{F}.\vec{n}\, ds + \iint_{S_3} \vec{F}.\vec{n}\, ds + \iint_{S_4} \vec{F}.\vec{n}\, ds + \iint_{S_5} \vec{F}.\vec{n}\, ds + \iint_{S_6} \vec{F}.\vec{n}\, ds$$



For the surface OABD $\vec{n} = \vec{k}$ and $z = 0$

$\vec{F}.\vec{n} = (x^2\vec{i} + z\vec{j} + yz\vec{k}).(-\vec{k}) = yz = y.0 = 0$

$\therefore \iint_{S_1} \vec{F}.\vec{n}\, ds = \iint_{S_2} zdzdx = \int_0^1\int_0^1 zdzdx$

$$= \int_0^1 \left[\frac{z^2}{2}\right]_0^1 dx = \int_0^1 \frac{1}{2} dx = \left[\frac{1}{2}x\right]_0^1 = \frac{1}{2}$$

For the surface EPFC, $\vec{n} = \vec{k}$, $z = 1$

$\vec{F}.\vec{n} = (x^2\vec{i} + z\vec{j} + yz\vec{k}).\, \vec{k} = yz = y.1 = y$

But, the surface projection on XY plane with projection R_2

$$\therefore \iint_{S_4} \vec{F}.\vec{n}\, ds = \int_0^1\int_0^1 y\, dydx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \int_0^1 \frac{1}{2} dx = \left[\frac{1}{2}x\right]_0^1 = \frac{1}{2}$$

For the surface AOCF = $\vec{n} = -\vec{j}$, $y = 0$

$\vec{F}.\vec{n} = (x^2\vec{i} + z\vec{j} + yz\vec{k}).(-\vec{j}) = -z$

Let the surface projection on XZ plane with projection R_3

$$\therefore \iint_{S_6} \vec{F}.\vec{n}\, ds = \int_0^1\int_0^1 -z\, dz\, dx$$

$$= -\int_0^1 \left[\frac{z^2}{2}\right]_0^1 dx = -\int_0^1 \frac{1}{2} dx = -\left[\frac{1}{2}x\right]_0^1 = -\frac{1}{2}$$

For the surface ADPF, $\vec{n} = \vec{i}$, $x = 1$

$\vec{F}.\vec{n} = (x^2\vec{i} + z\vec{j} + yz\vec{k}).\ \vec{i} = x^2 = 1$

But, the surface projection on yz plane with projection R_3

$$\therefore \iint_{S_3} \vec{F}.\vec{n}\, ds = \int_0^1\int_0^1 1\, dydz = \int_0^1 [z]_0^1 dy = \int_0^1 dy = [y]_0^1 = 1$$

For the surface OBFC, $\vec{n} = -\vec{i}$, $x = 0$

$\vec{F}.\vec{n} = (x^2\vec{i} + z\vec{j} + yz\vec{k}).(-\vec{i}) = x^2 = 0$

$$\therefore \iint_{S_5} \vec{F}.\vec{n}\, ds = 0$$

The surface integral,

$$\iint_{S_5} \vec{F}.\vec{n}\, ds = 0 + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + 1 + 0 = \frac{1}{2} + 1 = \frac{3}{2}$$

Hence, $\iint_{S_5} \vec{F}.\vec{n}\, ds = \frac{3}{2} = \iiint_V \nabla.\vec{F}\, dv$

Thus, Gauss divergence theorem verifies.

TU Examination 2068 (Shrawan)

1. **Prove that the determinant $\begin{vmatrix} -a^2 & ba & ca \\ ab & -b^2 & cb \\ ac & bc & -c^2 \end{vmatrix}$ is a perfect square and find it's value. (5)**

Solⁿ. Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2 & R_3 resp. We get

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Again taking a, b, c common from C_1, C_2 & C_3 resp.

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ we get

$$= a^2b^2c^2 \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Expanding along C_1

$$= a^2b^2c^3\, 2 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 2a^2b^2c^2 (1 + 1)$$

$$= 4a^2b^2c^2$$

$$= \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} 2a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2$$

Which shows that $\begin{vmatrix} -a^2 & ba & ca \\ ab & -b^2 & cb \\ ac & bc & -c^2 \end{vmatrix}$ is a perfect square and it's value is $4a^2b^2c^2$ Ans.

2. **Show that every square matrix can be uniquely expressed as the sum of Hermitian and Skew-Hermitian matrices. (5)**

Solⁿ. Lat A be a given square matrix, then

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = B + C \text{ (say)} \quad \text{.............. (i)}$$

Where $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2}(A - A^*)$

So, $\overline{B} = \frac{1}{2}\left(\overline{A + A^*}\right) = \frac{1}{2}\left(\overline{A} + \overline{A^*}\right)$

or, $\overline{B}^T = \frac{1}{2}\left(\overline{A} + \overline{A^*}\right)^T = \frac{1}{2}\left[\left(\overline{A}\right)^T + \left(\overline{A^*}\right)^T\right]$

$= \frac{1}{2}[A^* + (A^*)^*] = \frac{1}{2}[A^* + A] = B$

$\therefore \overline{B}^T = B$

Hence, $B = \frac{1}{2}(A + A^*)$ is a Hermitian Matrix.

Also,

$\overline{C} = \frac{1}{2}\left(\overline{A - A^*}\right) = \frac{1}{2}\left(\overline{A} - \overline{A^*}\right)$

or, $\overline{C}^T = \frac{1}{2}\left(\overline{A} - \overline{A^*}\right)^T = \frac{1}{2}\left[\left(\overline{A}\right)^T - \left(\overline{A^*}\right)^T\right]$

$= \frac{1}{2}[A^* - (A^*)^*] = \frac{1}{2}[A^* - A] = -\frac{1}{2}[A - A^*] = -C$

$\therefore \overline{C}^T = -C$

Hence $C = \frac{1}{2}(A - A^*)$ is a skew Hermitian Matrix.

Thus the square matrix A is the Sum of Hermitian and Skew Hermitian Matrix.

Uniqueness

If possible, Let $A = P + Q$ .... (ii)

Where P is Hermitaian and Q is Skew Hermitaian Matrices. So that

$P^* = P$ and $Q^* = -Q$

Then $A^* = (P + Q)^* = P^* + Q^* = P - Q$ .... (iii)

From (ii) and (iii) we get

$A + A^* = 2P$

$P = \frac{1}{2}(A + A^*) = B$

Also, $A - A^* = 2Q$

$Q = \frac{1}{2}(A - A^*) = C$

$P + Q = B + C$

This implies that the expression (i) is unique. Hence Proved.

Find the rank of the matrix $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ **reducing to normal form. (5)**

Sol. Let, $A = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$

Applying

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 - 4R_1$ $\sim \begin{bmatrix} 1 & 0 & -5 & 6 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \end{bmatrix}$

Applying,

$C_3 \to C_3 - 5C_1$, $C_4 \to C_4 - 6C_1$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \end{bmatrix}$

Applying,

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 16 & -16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying,

$C_3 \to C_3 + 8C_2$, $C_4 \to C_4 - 8C_2$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $R_2 \to \frac{R_2}{-2}$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Here the rank of $A = 2$ Ans.

4. **State Cayley-Hamilton theorem. Verify the theorem for the square matrix** $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. **(5)**

Sol^n. Statement of Cayley-Hamilton theorem

Every square matrix satisfies its own characteristic equation.

If the characteristic equation of the saure matrix A is

$|A - \lambda I| = 0$

i.e. $(-1)^n \lambda^n + k_1 \lambda^{n-1} + \ldots + k_n = 0$ of order n then,

$(-1)^n A^n + k_1 A^{n-1} + \ldots\ldots + k_n = 0$

Next part :

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The characteristic equation of the matrix is,

$|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)\{(2-\lambda)(1-\lambda) - 6\} - 3\{4(1-\lambda) - 3\} + 7\{8 - 1(2-\lambda)\} = 0$

or, $(1-\lambda)(2 - 2\lambda - \lambda + \lambda^2 - 6) - 3(4 - 4\lambda - 3) + 7(8 - 2 + \lambda) = 0$

or, $(1-\lambda)(\lambda^2 - 3\lambda - 4) - 3(1 - 4\lambda) + 7(6 + \lambda) = 0$

or, $\lambda^2 - 3\lambda - 4 - \lambda^3 + 3\lambda^2 + 4\lambda - 3 + 12\lambda + 42 + 7\lambda = 0$

or, $-\lambda^3 + 4\lambda^2 + 20\lambda + 35 = 0$

or, $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

Now, we have to verify that,

$A^3 - 4A^2 - 20A - 35\,I = 0$ .... (i)

For this,

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$\therefore A^3 - 4A^2 - 20A - 3I$

Find the rank of the matrix $\begin{bmatrix} \ldots \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ **reducing to normal form. (5)**

Sol^n. Let, $A = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$

Applying

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 5R_1$
$R_4 \to R_4 - 4R_1$ $\sim \begin{bmatrix} 1 & 0 & -5 & 6 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \end{bmatrix}$

Applying,

$C_3 \to C_3 - 5C_1$
$C_4 \to C_4 - 6C_1$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \end{bmatrix}$

Applying,

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 16 & -16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying,

$C_3 \to C_3 + 8C_2$
$C_4 \to C_4 - 8C_2$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Applying $R_2 \to \dfrac{R_2}{-2} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Here the rank of A = 2 Ans.

4. **State Cayley-Hamilton theorem. Verify the theorem for the square matrix** $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. **(5)**

Sol^n. Statement of Cayley-Hamilton theorem
Every square matrix satisfies its own characteristic equation.
If the characteristic equation of the saure matrix A is

$|A - \lambda I| = 0$

i.e. $(-1)^n \lambda^n + k_1 \lambda^{n-1} + \ldots\ldots + k_n = 0$ of order n then,

$(-1)^n A^n + k_1 A^{n-1} + \ldots\ldots\ldots + k_n = 0$

Next part :

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The characteristic equation of the matrix is,

$|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)\{(2-\lambda)(1-\lambda) - 6\} - 3\{4(1-\lambda) - 3\} + 7\{8 - 1(2-\lambda)\} = 0$

or, $(1-\lambda)(2 - 2\lambda - \lambda + \lambda^2 - 6) - 3(4 - 4\lambda - 3) + 7(8 - 2 + \lambda) = 0$

or, $(1-\lambda)(\lambda^2 - 3\lambda - 4) - 3(1 - 4\lambda) + 7(6 + \lambda) = 0$

or, $\lambda^2 - 3\lambda - 4 - \lambda^3 + 3\lambda^2 + 4\lambda - 3 + 12\lambda + 42 + 7\lambda = 0$

or, $-\lambda^3 + 4\lambda^2 + 20\lambda + 35 = 0$

or, $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

Now, we have to verify that,

$A^3 - 4A^2 - 20A - 35I = 0$ .... (i)

For this,

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}.$$

$\therefore A^3 - 4A^2 - 20A - 3I$

$$= \begin{bmatrix} 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135-80-20-35 & 152-92-60-0 & 232-92-140-0 \\ 140-60-80-0 & 163-88-40-35 & 208-148-60-0 \\ 60-40-20-0 & 76-36-40-0 & 111-56-20-35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore A^3 - 4A^2 - 20A - 3I = 0$

Hence, the cayley - Hamilton theorem is verified.

5. **Evaluate :** $\int_C \vec{F}.d\vec{r}$ **where** $\vec{F} = (\sin y)\vec{i} + x(1 + \cos y)\vec{j}$ **and the curve is the circular path given by** $x^2 + y^2 = a^2, z = 0$. **(5)**

Sol^n. The parametric equation of the circular path $x^2 + y^2 = a^2, z = 0$ is,

$x = a\cos t$, and $y = a \sin t$

$\therefore dx = -a \sin t\, dt$ and $dy = a \cos t\, dt$

as t varies from $t = 0 + 0$ to $t = 2\pi$

Hence, $\int_c \vec{F}.d\vec{r} = \int_C \sin y\, dx + x\, dy + x \cos y\, dy$

$= \int_0^{2\pi} \sin(a \sin t)(-a \sin t)\, dt + \int_0^{2\pi} a^2 \cos^2 t\, dt + \int_0^{2\pi} a \cos t \cos(a \sin t) a \cos t\, dt$

$= a \int_0^{2\pi} \{-\sin(a \sin t) . \sin t + a \cos^2 t \cos(a \sin t)\}\, dt + \frac{a^2}{2} \int_0^{2\pi} (1 + \cos 2t) dt$

$= a \int_0^{2\pi} d\{\sin(a \sin t) \cos t\} + \frac{a^2}{2} \int_0^{2\pi} (1 + \cos 2t)\, dt$

$= a\left[\{\sin(a \sin t) \cos t\}\right]_0^{2\pi} + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$

$= a(0 - 0) + \frac{a^2}{2}(2\pi + 0 - 0)$

$= \frac{a^2}{2} . 2\pi = \pi a^2$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135-80-20-35 & 152-92-60-0 & 232-92-140-0 \\ 140-60-80-0 & 163-88-40-35 & 208-148-60-0 \\ 60-40-20-0 & 76-36-40-0 & 111-56-20-35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore A^3 - 4A^2 - 20A - 3I = O$

Hence, the cayley - Hamilton theorem is verified.

5. **Evaluate : $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = (\sin y)\bar{i} + x(1+\cos y)\bar{j}$ and the curve is the circular path given by $x^2 + y^2 = a^2, z = 0$. (5)**

Sol^n. The parametric equation of the circular path $x^2 + y^2 = a^2, z = 0$ is,

$x = a\cos t$, and $y = a\sin t$

$\therefore dx = -a\sin t\, dt$ and $dy = a\cos t\, dt$

as t varies from $t = 0 + 0$ to $t = 2\pi$

Hence, $\int_c \bar{F}.d\bar{r} = \int_C \sin y\, dx + x\, dy + x\cos y\, dy$

$= \int_0^{2\pi} \sin(a\sin t)(-a\sin t)\,dt + \int_0^{2\pi} a^2\cos^2 t\,dt + \int_0^{2\pi} a\cos t\,\cos(a\sin t)\, a\cos t\,dt$

$= a\int_0^{2\pi}\{-\sin(a\sin t)\,.\,\sin t + a\cos^2 t\cos(a\sin t)\}\,dt + \dfrac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)dt$

$= a\int_0^{2\pi} d\{\sin(a\sin t)\cos t\} + \dfrac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$

$= a\left[\{\sin(a\sin t)\cos t\}\right]_0^{2\pi} + \dfrac{a^2}{2}\left[t + \dfrac{\sin 2t}{2}\right]_0^{2\pi}$

$= a(0-0) + \dfrac{a^2}{2}(2\pi + 0 - 0)$

$= \dfrac{a^2}{2}\,.\,2\pi = \pi a^2$

$\therefore \int_C \bar{F}.d\bar{r} = \pi a^2$ Ans.

6. **Evaluate : $\iint_S \bar{F}\hat{n}ds$ where $\bar{F} = (2x-z)\bar{i} + x^2y\bar{j} - xz^2\bar{k}$ and s is the surface of the plnes x = 0, x = 1, y = 0, z = 0, z = 1. (5)**

Sol^n. Here, $\bar{F} = \{(2x-z)\bar{i} + x^2y\bar{j} - xz^2\bar{k}\}$

and S in the surface of the planes

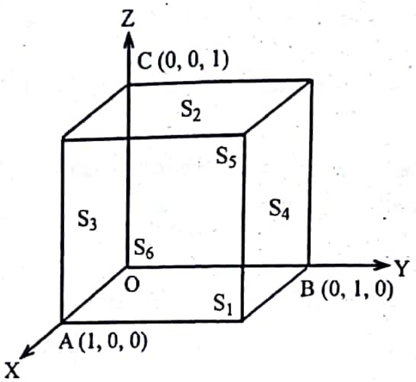
$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

We have,

$\iint_S \bar{F}\hat{n}ds = \iint_{S_1} \bar{F}\hat{n}ds + \iint_{S_2} \bar{F}\hat{n}ds + \iint_{S_3} \bar{F}\hat{n}ds + \iint_{S_4} \bar{F}\hat{n}ds + \iint_{S_5} \bar{F}\hat{n}ds + \iint_{S_6} \bar{F}\hat{n}ds$

For the surface S_1; $z = 0$

$\hat{n} = -\bar{k}$



$\bar{F}.\hat{n} = \{2x-z)\bar{i} + x^2y\bar{j} - xz^2\bar{k}\}.(\bar{k})$

$= xz^2 = x \times 0^2$

$= 0$

$\therefore \iint_{S_1} \bar{F}\hat{n}ds = 0$

For the surface S_2;

$Z = 1, \hat{n} = \bar{k}$

$\bar{F}.\hat{n} = \{(2x-z)\bar{i} + x^2y\bar{j} - xz^2\bar{k}\}.\bar{k}$

$= -xz^2 = -x \times 1 = -x$

$\therefore \iint_{S_2} \bar{F}\hat{n}ds = \int_0^1\int_0^1 (-x)\,dx\,dy$

$= \int_0^1\left[-\dfrac{x^2}{2}\right]_0^1 dy = -\dfrac{1}{2}\int_0^1 dy$

$\therefore \int_C \vec{F}.d\vec{r} = \pi a^2$ Ans.

6. **Evaluate : $\iint_S \vec{F}\hat{n}ds$ where $\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$ and s is the surface of the plnes x = 0, x = 1, y = 0, z = 0, z = 1.** (5)

Sol^n. Here, $\vec{F} = \{(2x-z)\,\vec{i} + x^2y\,\vec{j} - xz^2\,\vec{k}\}$

and S in the surface of the planes

$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

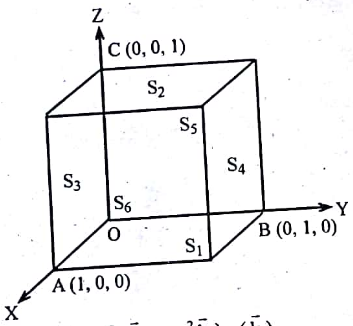
We have,

$\iint_S \vec{F}\hat{n}ds = \iint_{S_1} \vec{F}\hat{n}ds + \iint_{S_2} \vec{F}\hat{n}ds + \iint_{S_3} \vec{F}\hat{n}ds + \iint_{S_4} \vec{F}\hat{n}ds$

$+ \iint_{S_5} \vec{F}\hat{n}ds + \iint_{S_6} \vec{F}\hat{n}ds$

For the surface S_1; $z = 0$

$\hat{n} = -\vec{k}$



$\vec{F}.\hat{n} = \{(2x-z)\,\vec{i} + x^2y\,\vec{j} - xz^2\vec{k}\}.(\vec{k})$

$= xz^2 = x \times 0^2$

$= 0$

$\therefore \iint_{S_1} \vec{F}\hat{n}ds = 0$

For the surface S_2;

$Z = 1,\ \hat{n} = \vec{k}$

$\vec{F}.\hat{n} = \{(2x-z)\,\vec{i} + x^2y\,\vec{j} - xz^2\vec{k}\}.\vec{k}$

$= -xz^2 = -x \times 1 = -x$

$\therefore \iint_{S_2} \vec{F}\hat{n}ds = \int_0^1\int_0^1 (-x)\, dx\, dy$

$= \int_0^1 \left[-\frac{x^2}{2}\right]_0^1 dy = -\frac{1}{2}\int_0^1 dy$

$= -\frac{1}{2}[y]_0^1 = -\frac{1}{2}$

For the surface s_3; $y = 0,\ \hat{n} = -\vec{j}$

$\vec{F}.\hat{n} = \{(2x-z)\,\vec{i} + x^2y\,\vec{j} - xz^2\vec{k}\}.(-\vec{j})$

$= -x^2y = -x^2 \times 0 = 0$

$\therefore \iint_{S_3} \vec{F}\hat{n}ds = 0$

For the surface s_4; $y = 1,\ \hat{n} = \vec{j}$

$\vec{F}.\hat{n} = \{(2x-z)\,\vec{i} + x^2y\,\vec{j} - xz^2\vec{k}\}.(\vec{j})$

$= x^2y = x^2 \times 1 = x^2$

$\therefore \iint_{S_4} \vec{F}\hat{n}ds = \int_0^1\int_0^1 x^2\, dz\, dx$

$= \int_0^1 x^2 [z]_0^1 dx$

$= \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

For the surface S_5; $x = 0,\ \hat{n} = -\vec{i}$

$\vec{F}.\hat{n} = \{(2x-z)\,\vec{i} + x^2y\,\vec{j} - xz^2\vec{k}\}.(-\vec{i})$

$= (2x-z)(-1) = (0-z)(-1) = z$

$\therefore \iint_{S_5} \vec{F}\hat{n}ds = \int_0^1\int_0^1 z\, dy\, dz$

$= \int_0^1 z[y]_0^1 dz$

$= \int_0^1 z\, dz = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}$

For the surface S_6; $x = 1,\ \hat{n} = \vec{i}$

$\vec{F}.\hat{n} = \{(2x-z)\,\vec{i} + x^2y\,\vec{j} - xz^2\vec{k}\}.(\vec{i})$

$= (2x-z) = (2-z)$

$\therefore \iint_{S_6} \vec{F}\hat{n}ds = \int_0^1\int_0^1 (2-z)\, dy\, dz$

$= \int_0^1 (2-z)\,[y]_0^1\, dz$

$= \int_0^1 (2-z)\, dz = \left[2z - \frac{z^2}{2}\right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$

Thus,

$\iint_S \vec{F}\hat{n}ds = \iint_{S_1} \vec{F}\hat{n}ds + \iint_{S_2} \vec{F}\hat{n}ds + \iint_{S_3} \vec{F}\hat{n}ds$

$+ \iint_{S_4} \bar{F}.\hat{n}ds + \iint_{S_5} \bar{F}.\hat{n}ds + \iint_{S_6} \bar{F}.\hat{n}ds$

$$= 0 - \frac{1}{2} + 0 + \frac{1}{3} + \frac{1}{2} + \frac{3}{2}$$

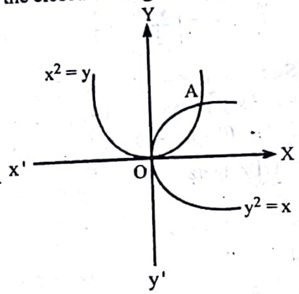
$$= \frac{1}{3} + \frac{3}{2}$$

$$= \frac{2+9}{6} = \frac{11}{6}$$

$\therefore \iint_{S_3} \bar{F}.\hat{n}ds = \frac{11}{6}$ Ans.

7. **Verify Green's theorem for $\int_c [(2xy - x^2)\, dx + (x + y^2)dy]$ where C is the cosed curve of the region bounded by $y = x^2$ and $x = y^2$.** **(5)**

Sol^n. Here, the line integral is $\int_C [(2xy - x^2)\, dx + (x + y^2)dy]$
Where C is the closed curve given by $y = x^2$ and $x = y^2$



So, $\int_C [(2xy - x^2)dx + (x + y^2)\, dy]$
$= \int_{OA} [(2xy - x^2)dx + (x + y^2)dy] + \int_{AO} [(2xy - x^2)\, dx + (x + y^2)\, dy]$

Along, OA, $y = x^2$
So that $dy = 2xdx$ as x varies from $x = 0$ to $x = 1$

$\int_{OA} [(2xy - x^2)\, dx + (x + y^2)dy] = \int_0^1 [(2x^3 - x^2)\, dx + (x + x^4)\, 2x\, dx]$

$$= \int_0^1 (2x^3 - x^2 + 2x^2 + 2x^5)\, dx$$

$$= \int_0^1 (2x^3 + x^2 + 2x^5)\, dx$$

$$= \left[\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\right]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3}$$

$$= \frac{1}{2} + \frac{2}{3} = \frac{3+4}{6} = \frac{7}{4}$$

Along AO, $y^2 = x$ so that $2ydy = dx$
as y varies from $y = 1$ to $y = 0$

$\int_{OA} (2xy - x^2)\, dx + (x + y^2)dy = \int_1^0 [(2y^3 - y^4)\, 2ydy + (y^2 + y^2)\, dy]$

$$= \int_1^0 (4y^4 - 2y^5 + 2y^2)\, dy = \left[\frac{4y^5}{5} - \frac{y^6}{3} + \frac{2y^3}{3}\right]_1^0$$

$$= \left[0 - \frac{4}{5} + \frac{1}{3} - \frac{2}{3}\right] = -\frac{4}{5} - \frac{1}{3} = -\frac{12+5}{15} = -\frac{17}{15}$$

$$\therefore \int_C [(2xy - x^2)\, dx + (x + y^2)dy] = \frac{7}{6} - \frac{17}{15} = \frac{35-34}{30} = \frac{1}{30} \quad \text{...... (i)}$$

Also, $F_1 = 2xy - x^2$ and $F_2 = x + y^2$ so that,

$$\frac{\partial F_1}{\partial y} = 2x, \quad \frac{\partial F_2}{\partial x} = 1$$

We have,

$$\iint_F \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - 2x)\, dx\, dy$$

$$= \int_0^1 (1 - 2x)\, [y]_{x^2}^{\sqrt{x}}\, dx$$

$$= \int_0^1 (1 - 2x)(\sqrt{x} - x^2)\, dx$$

$$= \int_0^1 (x^{\frac{1}{2}} - x^2 - 2x^{\frac{3}{2}} + 2x^3)\, dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} - \frac{4x^{\frac{5}{2}}}{5} + \frac{x^4}{2}\right]_0^1$$

$$= \left[\frac{2}{3} - \frac{1}{3} - \frac{4}{5} + \frac{1}{2}\right] = \frac{20 - 10 - 24 + 15}{30} = \frac{1}{30}$$

$$\therefore \iint_F \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \frac{1}{30} \quad \text{........ (ii)}$$

From (i) and (ii) Green's theorem is verified.

8. **Verify Stoke's theorem for $\bar{F} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ where S is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$ and C is it's boundary.** **(5)**

Sol^n. By Stoke's theorem
We have to verify that,
$\int_C \bar{F}.d\bar{r} = \iint_S (\nabla \times \bar{F}).\hat{n}\, ds$

Here, $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \bar{k}$

$\therefore \iint_S (\nabla \times \bar{F}) . \hat{n}\, ds = \iint_S \bar{k} . \hat{n}\, ds = \iint_R dx\, dy$

Since $\bar{k} . \hat{n}\ ds = dxdy$ and R is the projection of the surface S on xy-plane, then it becomes a circle,

$x^2 + y^2 = a^2$

$\therefore \iint_S (\nabla \times \bar{F}) . \hat{n}\, ds = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$

$= 2 \int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} dy\, dx = 2 \int_{-a}^{a} [y]_0^{\sqrt{a^2-x^2}} dx$

$= 4 \int_0^a \sqrt{a^2-x^2}\, dx = 4 \left[x\frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a = \pi a^2$

Also the surface S on xy. Plane is a,

Circle, $x^2 + y^2 = a^2$, $z = 0$

It's parametric equation is,

$x = r\cos\theta$ and $y = r\sin\theta$

$dx = -a \sin\theta d\theta$ and $dy = a \cos\theta d\theta$ as θ varies from $\theta = 0$ to $\theta = 2\pi$

Therefore,

$\bar{F} . d\bar{r} = \{(2x-y)\,\bar{i} - yz^2\,\bar{j} - y^2z\,\bar{k}\} . (\bar{i}\ dx + \bar{j}\ dy + \bar{k}\ dz)$

$= (2x-y)\, dx - yz^2dy - y^2zdz$

$= [(2a\cos\theta - a\sin\theta)(-a\sin\theta) - 0]\, d\theta$

$= -(2a\cos\theta - a\sin\theta)\, a \sin\theta\, d\theta$

So,

$\int_C \bar{F} . d\bar{r} = -\int_0^{2\pi} (2a\cos\theta - a\sin\theta)\, a\sin\theta\, d\theta$

$= -a^2 \int_0^{2\pi} (2\cos\theta\sin\theta - \sin^2\theta)\, d\theta$

$= -a^2 \int_0^{2\pi} \left(\sin 2\theta - \frac{1-\cos 2\theta}{2}\right) d\theta$

$= -a^2 \left[-\frac{\cos 2\theta}{2} - \frac{1}{2}\theta + \frac{\sin 2\theta}{4} \right]_0^{2\pi}$

$= -a^2 \left[-\frac{1}{2} - \frac{1}{2}2\pi + 0 + \frac{1}{2} \right] = \pi a^2$

Hence, $\int_C \bar{F} . d\bar{r} = \iint_S (\nabla \times \bar{F}) . \hat{n}\, ds$

Thus Stoke's theorem is verified.

Evaluate : $\iiint_V \bar{F} dv$, **where** $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ **and V is region bounded by** $x = 0, z = x^2, y = 0, y = 6$ **and** $z = 4$. **(5)**

Sol^n. See page 192

9. **Find the Laplace transforms of : (5)**

(i) $e^{-3t} \cos 4t$

Sol^n. We have $L\,[e^{-3t} \cos 4t)$

By using the first shifting theorem of Laplace transform, we get,

$L\,[e^{-3t} \cos 4t] = \frac{S+3}{(S+3)^2+4^2} = \frac{S+3}{S^2+6S+9+16} = \frac{S+3}{S^2+6S+25}$ Ans.

(ii) $\frac{e^{2t} - \cos at}{t}$

Sol^n. We have, $L\left[\frac{e^{2t} - \cos at}{t}\right]$

Now, $L\,[e^{2t} - \cos at] = L\,[e^{2t}] - L\,[\cos at]$

$= \frac{1}{S-2} - \frac{S}{S^2+a^2}$

By using the theorem of Laplace transform we get,

$L\left[\frac{e^{2t} - \cos at}{t}\right] = \int_S^\infty \left[\frac{1}{S-2} - \frac{S}{S^2+a^2}\right] ds$

$= \left[\log(S-2) - \frac{1}{2}\log\left(S^2+a^2\right)\right]_S^\infty$

$= -\log(S-2) + \frac{1}{2}\log(S^2+a^2)$

$= \log\frac{\sqrt{S^2+a^2}}{S-2}$ Ans.

10. **State and prove the convolution theorem of Inverse Laplace transform. (1 + 4)**

Sol^n. **Statement of the Convolution theorem of inverse Laplace transform.**

It $L^{-1}\,[F(s)] = f(t)$ and $L^{-1}\,[G(s)] = g(t)$

Then, $L^{-1}\,[F(s)\,G(s)]$

$= \int_0^t f(u)\ g(t-u)\, du$

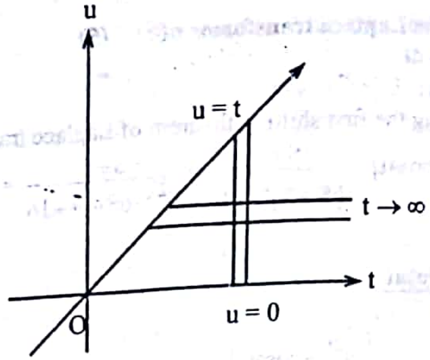
Proof

Let $\phi\,(t) = \int_0^t f(u)\ g(t-u)\, du$

Let, $[\phi(t)] = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\,g(t-u)\,du\right\}dt$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,d.t.$$

Here, this is first integrated. With respect to u. Which extends u = 0 to u = t and then with respect to t which extends t = 0 to t →∞.



By changing order of integration, we first integrate with respect to t which extends

t = u to t→∞ and then with respect to u which extends u = 0 to u →∞.

$$\therefore L[\phi(t)] = \int_0^\infty \int_u^\infty e^{-st} f(u)\,g(t-u)\,dt\,du$$

$$= \int_0^\infty f(u)\left\{\int_u^\infty e^{-st} g(t-u)\,dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_u^\infty e^{-s(t-u)} g(t-u)\,dt\right\}du$$

Put $t - u = v$

$dt = dv$

When $t \to u$. Then $v \to 0$

When $t \to \infty$ then $v \to \infty$

$$L[\phi(t)] = \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv} g(v)\,dv\right\}$$

$$= \int_0^\infty e^{-su} f(u)\,G(s)\,du = \int_0^\infty e^{-su} f(u)\,du \,.\, G(s)$$

$$= F(s)\,.\,G(s)$$

or, $L[\phi(t)] = F(s)\,.\,G(s)$

$$L\left[\int_0^t f(u)\,g(t-u)\,du\right] = F(s)\,.\,G(s)$$

Taking inverse Laplace transform on both sides.

$$\int_0^t f(u)\,g(t-u)\,du = L^{-1}[F(s), G(s)]$$

Hence, $L^{-1}[F(s)\,.\,G(s)] = \int_0^t f(u)\,g(t-u)\,du$ Proved.

11. **Solve the differential equation using Laplace transform : $y'' + y' - 2y = x$ given that $y(0) = 1$, $y'(0) = 0$.**

Solⁿ. Given differential equation is,

$y'' - y' - 2y = x$

Taking Laplace transform or both sides, we get,

$L[y''] + L[y'] - 2L[y] = L[x]$

or, $s^2 Y(s) - S\,y(0) - y'(0) + S\,Y(s) - y(0) - 2Y(s) = \frac{1}{s^2}$

Using the given conditions y(i0) = 1 and y'(0) = 0 we get,

or, $S^2 Y(s) - S \times 1 - 0 + S\,Y(s) - 1 - 2Y(s) = \frac{1}{s^2}$

or, $[s^2 + s - 2]\,Y(s) - s - 1 = \frac{1}{s^2}$

or, $[s^2 + s - 2]\,Y(s) = S + 1 + \frac{1}{s^2} = \frac{s^3+s^2+1}{s^2}$

$$\therefore Y(s) = \frac{s^3+s^2+1}{s^2(s^2+s-2)} = \frac{s^3+s^2+1}{s^2(s+2)(s-1)}$$

By the partial fraction, we can write,

$$Y(s) = -\frac{1}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{(s+2)} + \frac{1}{s-1}$$

Taking inverse Laplace transform we get,

$$L^{-1}[Y(s)] = -\frac{1}{4}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{1}{s-1}\right]$$

$$\therefore y(x) = -\frac{1}{4} - \frac{1}{2}x + \frac{1}{4}e^{-2x} + e^{x} \text{ Ans.}$$

12. **Find the Fourier series to represent $f(x) = e^{ax}$ in $(0, 2\pi)$.**

Solⁿ. Let $f(x) = e^{ar} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$

$$= \frac{1}{\pi}\frac{(e^{2\pi a}-1)}{a} = \frac{e^{2\pi a}-1}{\pi a}$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,dx$

$$= \frac{1}{\pi\left(n^2+a^2\right)}\left[e^{ax}\left(a\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi\left(n^2+a^2\right)}\left(ae^{2\pi a}-a\right) = \left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{a}{n^2+a^2}\right)$$

$$\therefore a_1 = \left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{a}{1^2+a^2}\right),\ a_2 = \left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{a}{4+a^2}\right)$$

$$a_3 = \left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{a}{9+a^2}\right) \text{ etc.}$$

and $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin x\,dx$

$$= \frac{1}{\pi\left(n^2+a^2\right)}\left[e^{ax}\left(a\sin nx + n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi\left(n^2+a^2\right)}\left[e^{2\pi a}.(-n)+n\right] = -\left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{n}{n^2+a^2}\right)$$

$$\therefore b_1 = -\left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{1}{1^2+a^2}\right)$$

$$b_2 = -\left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{2}{4+a^2}\right),\ b_3 = -\left(\frac{e^{2\pi a}-1}{\pi}\right)\cdot\left(\frac{3}{9+a^2}\right)$$

Substituting the values of a_0, a_n, b_n in (1), we get,

$$e^{ax} = \left(\frac{e^{2\pi a}-1}{\pi}\right)\left[\left\{\frac{1}{2a}+\left(\frac{a}{1+a^2}\right)\cos x+\left(\frac{a}{4+a^2}\right)\cos 2x+\left(\frac{a}{9+a^2}\right)\cos 3x+\ldots\right\}\right.$$

$$\left.-\left\{\left(\frac{1}{1+a^2}\right)\sin x+\left(\frac{2}{4+a^2}\right)\sin 2x+\left(\frac{3}{9+a^2}\right)\sin 3x+\ldots\right\}\right] \text{ Ans.}$$

13. Obtain half range cosine series for $f(x) = \sin x$, $0 < x < \pi$ and hence show that $\Sigma\frac{1}{4n^2-1} = \frac{1}{2}$.

Sol^n. See page 389

14. Maximize $P = 2x_1 + 3x_2$, subject to $x_1 - x_2 \leq 2$, $x_1 + x_2 \geq 4$ and x_1, $x_2 > 0$. Use the graphical method. (5)

Sol^n. See page 292

15. Solve the given LPP using duality : Minimize $p = 8x_1 + 9x_2$ subject to : $x_1 + 3x_2 \geq 4$, $2x_1 + x_2 \geq 5$ with $x_1, x_2 \geq 0$.

Sol^n. Here, the objective function is,

Min $P = 8x_1 + 9x_2$

Subject to

$x_1 + 3x_2 \geq 4$

$2x_1 + x_2 \geq 5$ with $x_1, x_2 \geq 0$

By the theorem of duality of LPP, its dual problem is defined by

Maximize : $P^* = 4y_1 + 5y_2$

Subject to $y_1 + 2y_2 \leq 8$

$3y_1 + y_2 \leq 9$ with $y_1, y_2 \geq 0$

By introducting the slack k variables, this can be wirtten as,

$P^* - 4y_1 - 5y_2 + 0.x_1 + 0.x_2 = 0$

$y_1 + 2y_2 + x_1 + 0.x_2 = 8$

$3y_1 + y_2 + 0.x_1 + x_2 = 9$

and $y_1, y_2, x_1, x_2 \geq 0$

Simplex table I :

P^*	y_1	y_2	x_1	x_2	b
1	−4	−5	0	0	0
0	1	2	1	0	8
0	3	1	0	1	9

Since the least negative number in the first row in – 5. So select third column. Dividing the remaning third column entires into the corresponding entries in the last column.

We have,

$\frac{8}{2} = 4$ (Smallest quotient), $\frac{9}{1} = 9$

Taking 3 as the pivot element of the second row.

Operating, $R_1 \rightarrow R_1 + \frac{5}{2}R_2$, $R_3 \rightarrow R_3 - \frac{1}{2}R_2$

Simplex table II :

P^*	y_1	y_2	x_1	x_2	b
1	$-\frac{3}{2}$	0	$\frac{5}{2}$	0	20
0	1	2	1	0	8
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	5

Since three in least negative term in the first row is $-\frac{3}{2}$, so select second column.

Dividing the remaining second column entries into the corresponding entries in last column.

We have,

$\frac{8}{1} = 8,\ \frac{5}{\frac{5}{2}} = 2$ (smallest quotient)

Taking $\frac{5}{2}$ as the pivot element of the third row.

Operating, $R_1 \to R_1 + \frac{3}{5} R_3,\ R_2 \to R_2 - \frac{2}{5} R_3$

Simplex table III :

P^*	y_1	y_2	x_1	x_2	b
1	0	0	$\frac{11}{5}$	$\frac{3}{5}$	23
0	0	2	$\frac{6}{5}$	$-\frac{2}{5}$	6
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	5

Since there is no negative term, in the first row. our procedure in end. Thus,

$\frac{5}{2}y_1 = 5 \quad \therefore y_1 = 2$

$2y_2 = 6 \quad \therefore y_2 = 3$

$x_1 = \frac{11}{5} \quad \therefore x_1 = \frac{11}{5}$

$x_2 = \frac{3}{5} \quad \therefore x_2 = \frac{3}{5}$

Max $P^* = 23$ at $y_1 = 2,\ y_2 = 3$

Min $P = 23$ at $x_1 = \frac{11}{5},\ x_2 = \frac{3}{5}$

TU Examination 2069 (Ashad)

1. **Find the value of the determinant:**

$$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$$

Solⁿ. *See exercise- 1, Q. No. 20 in this book.*

2. **Prove that every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrices.**

Solⁿ. Let A be a given square matrix, then

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = B + C$ .... (i) (say)

Where, $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$

So, $B^T = \left[\frac{1}{2}\left(A + A^T\right)\right]^T = \frac{1}{2}[A^T + (A^T)^T] = \frac{1}{2}[A^T + A]$

or, $B^T = B$

Hence, $B = \frac{1}{2}[A^T + A]$ is a symmetric matrix

Also, $C^T = \left[\frac{1}{2}\left(A - A^T\right)\right]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}[A^T - A]$

$= -\frac{1}{2}[A - A^T] = -C$

or, $C^T = -C$

Hence, $C = \frac{1}{2}(A - A^T)$ is a skew symmetric matrix.

Thus the equation (i) express A as the sum of a symmetric and skew symmetric matrices.

Uniqueness : If possible, let $A = P + Q$ .... (ii)

Where P is symmetric and Q is skew symmetric matrices, so that $P^T = P$ and $Q^T = -Q$

Then, $A^T = (P + Q)^T = P^T + Q^T$ .... (iii)

$= P - Q$

From (ii) and (iii) we get,

$A + A^T = 2P$

$P = \frac{1}{2}[A + A^T] = B$

Also, $A - A^T = 2Q$

$Q = \frac{(A - A^T)}{2} = C$

$\therefore P + Q = B + C$

This implies the expression (i) is unique. Hence Proved.

3. **Find the rank of matrix :** $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$ **reducing to echelon form.**

Sol^n. *See exercise-5, Q. No. 17 in this book.*

4. **Verify Cayley Hamilton theorem for the matrix:** $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Sol^n. *See exercise-6, Q. No. 8 (iii) in this book.*

5. **Find the Laplace transforms of : (a) $te^{-t} \sin t$ (b) $\frac{e^{at} - \cos 6t}{t}$**

Sol^n. (a) *See exercise- 11, Q. No. 2(xvi) in this book.*
(b) *See exercise- 11, Q. No. 3(iv) in this book.*

6. **If $L[f(t)] = F(s)$, then prove that $L[f'(t)] = SF(s) - f(0)$.**

Sol^n. Given that $L[f(t)] = F(s)$

or, $\int_0^\infty e^{-st} f(t)\, dt = F(s)$

By using the definition of Laplace transform we get

$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$

Integrating by parts, we get

$\left[e^{-st} f(t)\right]_0^\infty (-s) e^{-st} f(t)\, dt$

$= \lim_{t \to \infty} e^{-st} f(t) - f(0) + S \int_0^\infty e^{-st} f(t)\, dt$

$= \lim_{t \to \infty} \frac{f(t)}{e^{st}} - f(0) + SL[f(t)]$

Now assuming f(t) to be such that

$\lim_{t \to \infty} e^{-st} f(t) = 0$, we get

$L[f'(t)] = -f(0) + SL[f(t)]$

$L[f'(t)] = SL[f(t)] - f(0)$ .... (i)

$= S\,F(s) - f(0)$

$\therefore L[f'(t)] = SF(s) - f(0)$

Use Laplace ... $x(0); x'(0) = 1$.

Sol^n. *See exercise- 14, Q. No. 9 in this book.*

Obtain the Fourier series for $f(x) = x^3$ in the interval $-\pi \le x \le \pi$.

Sol^n. *See exercise- 17, Q. No.4 in this book.*

Obtain half-range sine series for e^x in (0, 1).

Sol^n. *See exercise- 18, Q. No. 12 in this book.*

10. **Maximize $z = 2x_1 + 3x_2$ subject to constaints $x_1 - x_2 \le 2$, $x_1 + x_2 \ge 4$ and $x_1, x_2 \ge 0$ graphically.**

Sol^n. *See exercise- 15, Q. No. 2(v) in this book.*

11. **Solve the linear programming problems by simplex method constructing the duality.**
Minimize $Z = 3x_1 + 2x_2$
Subject to $2x_1 + 4x_2 \ge 10$
$4x_1 + 2x_2 \ge 10$
$x_2 \ge 4$ and $x_1, x_2 \ge 0$

Sol^n. Minimize $Z = 3x_1 + 2x_2$
Subject to $2x_1 + 4x_2 \ge 10$
$4x_1 + 2x_2 \ge 10$
$x_2 \ge 4$

Then the dual of the above problem is
Maximize $z^* = 10y_1 + 10y_2 + 4y_3$
Subject to $2y_1 + 4y_2 + 0.y_3 - 3$
$4y_1 + 2y_2 + y_3 = 2$

By introducting the variables, x_1, x_2 together with objective function these can be written as,

$z^* - 10y_1 - 10y_2 - 4y_3 + 0\,.\,x_1 + 0\,.\,x_2 = 0$
$2y_1 + 4y_2 + 0\,.\,y_3 + x_1 + 0\,.\,x_2 = 3$
$4y_1 + 2y_2 + y_3 + 0\,.\,x_1 + x_2 = 2$

In simplex table from,

Simplex table I :

z	y_1	y_2	y_3	x_1	x_2	b
1	– 10	– 10	– 4	0	0	0
0	2	$\boxed{4}$	0	1	0	3
0	4	2	1	0	1	2

There are negative numbers in the first row and the last negative number (– 10) is one the second and third column. Taking third column (one of them). Dividing the remaining third column entires into the corresponding entries in the last column.

We have,

$\frac{3}{4} = 0.75$ (Smallest quotient), $\frac{2}{2} = 01$

Taking 4 is pivot element of the second row.

Operating,

$R_1 \to R_1 + \frac{5}{2}R_2$

$R_3 \to R_3 - \frac{1}{2}R_2$ we obtain

Simplex table II:

z	y_1	y_2	y_3	x_1	x_2	b
1	-5	0	-4	$\frac{5}{2}$	0	$\frac{15}{2}$
0	2	4	0	1	0	3
0	2	0	1	$-\frac{1}{2}$	1	$\frac{1}{2}$

We see that, there is last negative term (-5) in the first and the second column. Dividing the remaining second column entries to the corresponding entries in the last column.

We have,

$\frac{3}{4} = 1.5$, $\frac{\frac{1}{2}}{2} = 0.25$ (Smallest quotient)

Taking 2 is the pivot element of the third row.

Operating $R_1 \to R_1 + \frac{5}{2}R_3$ and $R_2 \to R_2 - R_3$, we obtain,

Simplex table II:

z	y_1	y_2	y_3	x_1	x_2	b
1	0	0	$-\frac{3}{2}$	$\frac{5}{4}$	$\frac{5}{2}$	$\frac{35}{4}$
0	0	4	-1	$\frac{3}{2}$	-1	$\frac{5}{2}$
0	2	0	1	$-\frac{1}{2}$	1	$\frac{1}{2}$

We see that there is negative terms $\left(-\frac{3}{2}\right)$ in the first row and fourth column. Dividing the remaning fourth column entries into the corresponding entries in the last column.

we have,

$\frac{\frac{1}{2}}{1} = 0.5$ (Smallest quotient)

Taking 1 is pivot element of the operating $R_1 \to R_1 + \frac{3}{2}R_3$

R_2 ...

Simplex table III:

z	y_1	y_2	y_3	x_1	x_2	b
1	3	0	0	$\frac{1}{2}$	4	$\frac{19}{2}$
0	2	4	0	1	0	3
0	2	0	1	$-\frac{1}{2}$	1	$\frac{1}{2}$

We see that there are no negative entries in the first row. So our procedure is end.

Setting $x_1 = \frac{1}{2}$, $x_2 = 4$ we obtain the feasible solutions.

$y_1 = 0$ $\quad\therefore y_1 = 0$

$4y_2 = 3$ $\quad\therefore y_2 = \frac{3}{4}$

$1 \cdot y_3 = \frac{1}{2}$ $\quad\therefore y_3 = \frac{1}{2}$

$z^* = \frac{19}{2}$

Max. $z^* = \frac{19}{2}$

At $y_1 = 0$, $y_2 = \frac{3}{4}$, $y_3 = \frac{1}{2}$

Thus min. $z = \frac{19}{2}$ at $x_1 = \frac{1}{2}$, $x_2 = 4$

12. Prove that $\bar{F} = (2xz^3 + 6y)\,\bar{i} + (6x - 2yz)\,\bar{j} + (3x^2z^2 - y^2)\,\bar{k}$ is conservative vector field and find its scalar potential function.

Sol^n. Here, curl $\bar{F} = \nabla \times \bar{F}$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times \{(2xz^3 + 6y)\bar{i} + (6x - 2yz)\bar{j} + (3x^2z^2 - y^2)\bar{k}\}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^3 + 6y & 6x - 2yz & 3x^2z^2 - y^2 \end{vmatrix} = 0$$

Hence, $\bar{F}$ is conservative field

Also, we have $\bar{F} = \nabla\phi$

$\bar{F} \cdot d\bar{r} = \nabla\phi \cdot d\bar{r}$

or, $(2xz^3 + 6y)\,dx + (6x - 2yz)\,dy + (3x^2z^2 - y^2)\,dz = d\phi$

or, $(2xz^3dx + 3x^2z^2dz) + 6(xdy + ydx) - (2yzdy + y^2dz) = d\phi$

or, $d(x^2z^3) + 6d(xy) - d(y^2z) = d\phi$

On integration

$\phi = x^2z^3 + 6xy - y^2z + c$

13. **Evaluate :** $\iint_S \vec{F}\cdot n\,ds$ **Where,** $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ **and S is the finite plane $x + y + z = 1$ between the coordinates planes.**

Sol^n. *See exercise- 8, Q. No. 4 in this book.*

14. **Using Green's theorem, find the area of the hypocycloid** $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1.$

Sol^n. *See exercise-9, Q. No. 9(iii) in this book.*

15. **Evaluate** $\iint_S \vec{F}\cdot\hat{n}\,ds$ **where** $\vec{F} = 2x\vec{i} + 3y\vec{j} + 4z\vec{k}$ **& S is the surface of sphere $x^2 + y^2 + z^2 = 1$ by Gauss' divergence theorem.**

Sol^n. *See exercise- 10, Q. No. 11 in this book.*

OR

Verify Stoke's theorem for $\vec{F} = 2y\vec{i} + 3x\vec{j} - z^2\vec{k}$ **where S is upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is boundary.**

Sol^n. *See exercise 9, Q. No. 13 in this book.*

TU Examination 2069 (Chaitra)

1. Find the value of the determinant $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ac \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$

Sol^n. *See exercise 1, Q. No. 8 in this book*

2. **Show that the matrix $B^\theta AB$ is Hermitian or skew Hermitian according as A is Hermitian and skew Hermitian.**

Sol^n. *See exercise 4, Q. No. 17 in this book*

3. **Find the rank of the matrix** $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ **reducing this into the triangular form.**

Sol^n. Let $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

$A \sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 2 & 4 & 6 & -1 \\ 3 & 10 & 9 & 7 \\ 4 & 16 & 12 & 15 \end{bmatrix}$ Applying $c_1 \leftrightarrow c_2$

$A \sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 0 & -8 & 0 & 17 \\ 0 & -8 & 0 & 17 \\ 0 & -8 & 0 & -17 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$
$R_2 \to R_3 - 3R_1$
$R_4 \to R_4 - 4R_1$

$\sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 0 & -8 & 0 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Applying $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 6 & 11 & 8 \\ 0 & -8 & -17 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{Applying } c_3 \to c_3 + c_4$$

This is a triangular matrix and it has two non-zero rows hence rank of A = 2.

4. **Obtain the characteristic equation of the matrix A =** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ **and verify that it is satisfied by A.**

Sol^n. *See exercise 6, Q. No. 8(iv) in this book.*

5. **Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x-y)\,\vec{i} + (x+y)\,\vec{j}$ along the closed curve C bounded by $y^2 = x$ and $x^2 = y$.**

Sol^n. *See exercise 7, Q. No. 1(vii) in this book.*

6. **Find the value of the normal surface integral $\iint_S \vec{F}.\hat{n}\, ds$ for $\vec{F} = x\vec{i} - y\vec{j} + (z^2-1)\vec{k}$ where S is the surface bounded by the cylinder $x^2 + y^2 = 4$ between the planes Z = 0 and Z = 1.**

Sol^n. Here the surface is the cylinder formed by the planes z = 0 to z = 1 and $x^2 + y^2 = 4$ so that the projection of the surface s_1 on zx - plane is rectangular bounded by x = 2, x = – 2, z = 0, z = 1

The surface in $\phi = x^2 + y^2 - 4 = 0$

$$\text{Grad } \phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)(x^2 + y^2 - 4) = 2x\vec{i} + 2y\vec{j}$$

Unit normal vector to the surface in

$$\hat{n} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{1}{2}(x\vec{i} + y\vec{j})$$

$$\text{Also, } \vec{F}.\hat{n} = \left\{x\vec{i} - y\vec{j} + (z^2-1)\vec{k}\right\} \cdot \frac{1}{2}(x\vec{i} + y\vec{j}) = \frac{1}{2}(x^2 + y^2)$$

$$\text{and } \hat{n}.\vec{j} = \frac{1}{2}(x\vec{i} + y\vec{j}).\vec{j} = \frac{y}{2}$$

$$\therefore \iint_{S_1} \vec{F}.\hat{n}\, dx = \iint_R \vec{F}.\hat{n} \frac{dx\,dz}{|\hat{n}.\vec{j}|} = \iint_R \frac{(x^2 - y^2)}{y}\, dx\, dz$$

$$= \int_2^{-2}\int_0^1 \frac{2x^2 - 4}{\sqrt{4 - x^2}}\, dx\, dz = \int_2^{-2} \frac{2x^2 - 4}{\sqrt{4 - x^2}}\, dx$$

Put x = 2 sinθ

$\therefore dx = 2\cos\theta\, d\theta$

$$\therefore \iint_{S_1} \vec{F}.\hat{n}\, ds = -4\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (1 - 2\sin^2\theta)\, d\theta$$

$$= 4\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos 2\theta\, d\theta = 0$$

For two surface s_2 . z = 0, $\hat{n} = -\vec{k}.\vec{F}.\hat{n} = 1$

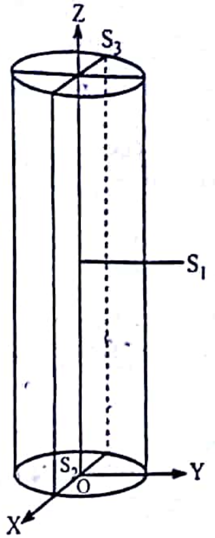
$$\iint_{S_2} \vec{F}.\hat{n}\, ds = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx\, dy$$

$$= 2\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 4\pi$$

For surface s_3; z = 1

$\hat{n} = \vec{k}.\vec{F}.\hat{n} = 0$

$$\iint_{S_2} \vec{F}.\hat{n}\, ds = 0 + 4\pi + 0 = 4\pi$$



7. **Using Green's theorem, find the area of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.**

Sol^n. We have Green's theorem in the plane,

$$\int_c (F_1 dx + F_2 dy) = \iint_E \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

Let $F_1(x, y) = -y$, $F_2(x, y) = x$

$$\therefore \int_c x dy - y dx = \iint_E (1+1)\, dxdy = \iint_E 2dxdy = 1A$$

Therefore, the area enclosed by a place curve is,

$$A = \frac{1}{2}\int_c (xdy - ydx)$$

Also, the parametric equation of hypocycloid

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $x = a\cos^3\theta$, $y = a\sin^3\theta$ and θ varies from θ = 0 to θ = 2π

$$\text{Area (A)} = \frac{1}{2}\int_c (xdy - ydx)$$

$$= \frac{1}{2}\int_0^{2\pi} a\cos^3\theta\, d(a\sin^3\theta) - a\sin^3\theta\, d(a\cos^3\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} 3a^2 \sin^2\theta\cos^2\theta d\theta$$

$$= \frac{3a^2}{8}\int_0^{2\pi} \sin^2 2\theta d\theta = \frac{3a^2}{16}\int_0^{2\pi} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{3a^2}{16}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi} = \frac{3\pi a^2}{8}$$

8. **Verify Stoke's theorem for $\vec{F} = 2y\,\vec{i} + 3x\,\vec{j} - z^2\vec{k}$ where S is upper half of the sphere $x^2 + y^2 + z^2 = 9$ and C is boundary.**

Sol^n. *See exercise 9, Q. No. 13 in this book.*

OR

Evaluate $\iiint_v \vec{F}\, dv$ where v is the region bounded by the surface $x = 0$, $y = 0$, $z = x^2$, $y = 6$, $z = 4$ & $\vec{F} = 2xz\vec{i} - x\vec{j} + y^2\vec{k}$.

Sol^n. *See exercise 10, Q. No. 3 in this book*

9. **Find the Laplace transforms of the following functions**

(a) $te^{-4t}\sin 3t$ (b) $\dfrac{\cos at - \cos bt}{t}$

Sol^n. (a) We know,

$$L[\cos at - \cos bt] = L[\cos at] - L[\cos bt]$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = F(s) \text{ say}$$

By using the theorem of Laplace transform, we get

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty F(s)\, ds$$

$$= \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds = \frac{1}{2}\left[\log\left(s^2+a^2\right) - \log\left(s^2+b^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

(b) We know, $L[\sin 3t] = \dfrac{3}{s^2+9}$

By using the theorem of Lapalce transform we get,

$$L[e^{-4t}\sin 3t] = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+16+9} = \frac{3}{s^2+8s+25}$$

Again by using the theorem of Lapalce transform we get,

$$L[t\,e^{-4t}\sin 3t] = (-1)\frac{d}{ds}\left[\frac{3}{s^2+8s+25}\right]$$

$$= -2\frac{d\left(s^2+8s+25\right)}{d\left(s^2+8s+25\right)} \cdot \frac{d\left(s^2+8s+25\right)}{ds}$$

$$= (-3)(-1)(s^2+8s+25)^{-2}(2s+8)$$

$$= \frac{3(25+8)}{\left(s^2+8s+25\right)^2} = \frac{6(-s+4)}{\left(s^2+8s+25\right)^2}$$

10. **State and prove the second shifting theorem of the Laplace transform.**

Sol^n. Given that $L[f(t)] = F(s)$

$$\int_0^\infty e^{-st} f(t)\, dt = F(s)$$

By using the definition of Lapace transform we get,

$$L[f(t-a)\,u(t-a)] = \int_0^\infty e^{-st}[f(t-a)\,u(t-a)]\, dt$$

$$= \int_0^a e^{-st} f(t-a)\,u(t-a)\, dt + \int_a^\infty e^{-st} f(t-a)\,u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,.\,1\,.\, dt$$

Put $t - a = u$ so that, $dt = du$

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du = e^{-as}\int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\, dt = e^{-as} L[f(t)]$$

$\therefore L[f(t-a)\,u(t-a)] = e^{-as} F(s)$

11. **Solve the following differential equation using Laplace transform.**

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x \text{ given } y(0) = 1,\ y'(0) = 0$$

Sol^n. ***See T.U. exam 2068 (shrawan), Q. No. 14, page no. 423 in this book.***

12. Obtain the Fourier series for $f(x) = x2$ in the interval $-\pi < x < \pi$ and hence show that $\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} = \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}$

Sol^n. Let $x^2 = \dfrac{1}{2} a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ....(i)

Where $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx$

$a_0 = \frac{2\pi^2}{3}$

$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{4\pi}{\pi}\frac{\cos nx}{x^2} = \frac{4\cos n\pi}{x^2}$

$= \frac{4}{x^2}$ for n is even $= -\frac{4}{x^2}$ for n is odd

$a_n = \frac{4(-1)^n}{n^2}$, $a_1 = -4$, $a_2 = 4\cdot\frac{1}{2^2}$, $a_3 = -4\cdot\frac{1}{3^2}$ etc.

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0$ ($\because x^2 \sin nx$ is odd function)

Substituting the values of a_0, a_n, b_n in .... (i) we get

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

or, $x^2 = \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^3} + ...\right]$

Putting x = 0 on both sides we get,

$0 = \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + ...\right)$

or, $\frac{-\pi^2}{3} = -4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right)$

or, $\frac{\pi^2}{6} = 2\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... - \frac{2}{2^2} - \frac{2}{4^2} - \frac{2}{6^2}\right)$

$= I\frac{1}{n^2} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right) = 2I\frac{1}{n^2} - \sum\frac{1}{n^2}$

$\therefore \frac{\pi^2}{6} = \sum\frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$

13. Express $f(x) = x$ as a half-range sine series in $0 < x < 2$

Sol^n. The function may be extended as an odd function,

$f(x) = x$ in $-2 < x < 2$

$\therefore a_n = 0$ for $x = 0, 1, 2, 3, ...$

Hence, the Fouria series for f(x) over the full period (−2, 2) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

Where $b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx = \int_0^2 x\sin\frac{n\pi x}{2}dx$

$= \left[\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2 = -\frac{4}{n\pi}\cos n\pi$

$= -\frac{4}{n\pi}$ for n is even and $\frac{4}{n\pi}$ for n is odd.

Thus, $b_1 = \frac{4}{\pi}$, $b_2 = \frac{-4}{2\pi}$, $b_3 = \frac{4}{3\pi}$, $b_4 = \frac{-4}{4\pi}$ etc.

Hence thf fourier series for f(x) over the half range (0, 2) is,

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} + ...\right)$$

14. Maximize $Z = 4x_1 + 5x_2$ subject to constraints $2x_1 + 5x_2 \le 25$, $6x_1 + 5x_2 \le 45$, $x_1 \ge 0$ and $x_2 \ge 0$ graphically

Sol^n. Here, the objective functions $z = 4x_1 + 5x_2$

Subject to $2x_1 + 5x_2 \le 25$, $6x_1 + 5x_2 \le 45$ and $x_1 \ge 0$, $x_2 \le 0$

It is clear that $x_1 \ge 0$ and $x_2 \ge 0$ imply that the feasible region lies in the first quadrant..

Writing the inequalities into equalities, we get

$2x_1 + 5x_2 = 25$ .... (i), $6x_1 + 5x_2 = 45$ .... (ii)

$x_1 = 0$, $x_2 = 0$ .... (iii)

These equation give the boundary lines,

In the equation (i) $2x_1 + 5x_2 = 25$

Put $x_1 = 0$ we get, $x_2 = 5$

Put $x_2 = 0$ we get $x_1 = \frac{25}{2}$

So the point (0, 5) and $\left(\frac{25}{2}, 0\right)$ lie on the line (1) draw this line.

In the equation (ii) $6x_1 + 5x_2 = 45$

Put $x_1 = 0$, we get $x_2 = 8$

Put $x_2 = 0$, we get $x_1 = \frac{15}{2}$

So the points (0, 9) and $\left(\frac{15}{2}, 0\right)$

lie in the line (ii) draw this line.

From the line (iii) $x_1 = 0$ and $x_2 = 0$

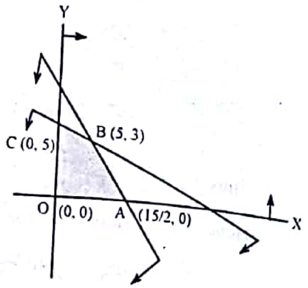
Represents the line of x_2 axis and x_1 − axis respectively draw these lines.

To test the plane region determined by the constraints, we consider the point (0,0). Put $x_1 = 0$, $x_2 = 0$ in the first two constraints to get the plane determined by the given constratints the feasible region in shown as below.

The feasible region in bounded by the points O(0, 0), A $\left(\frac{15}{2}, 0\right)$, c(0, 5) the corner point is obtained by solving the equations.

$2x_1 + 5x_2 = 25, \quad 6x_1 + 5x_2 = 45$

We get $x_1 = 5$, $x_2 = 4$



Vertices	Objective function $z = 4x_1 + 5x_2$
(a) A $\left(\frac{15}{2}, 0\right)$	(a) $z = 4 \times \frac{15}{2} + 5 \times 0 = 30$
(b) B (5, 3)	(b) $z = 4 \times 5 + 5 \times 3 = 35$
(c) c (0, 5)	(c) $z = 4 \times 0 + 5 \times 5 = 25$
(d) 0 (0, 0)	(d) $z = 4 \times 0 + 5 \times 0 = 0$

From the above table, we see that maximum of $z = 4x_1 + 5x_2$ occur at the vertex (5, 3) and its minimum at vertex 0(0, 0)

Max. profit = Rs. 35 at (5, 3)

Min. profit = Rs. 0 at (0, 0)

15. Solve the following linear programming problem using the simplex method.

Maximize $P = 50x_1 + 80x_2$, Subject to $x_1 + 2x_2 \leq 32$

$3x_1 + 4x_2 \leq 84$, $x_1, x_2 \geq 0$

Sol^n. Here maximize $p = 50x_1 + 80x_2$

Subject to $x_1 + 2x_2 \leq 32$, $3x_1 + 4x_2 \leq 84$

By introducting soack variables s_1 and s_2 together with objective function, these can be written as,

Manimize $p - 50x_1 - 80x_2 + 0.\ S_1 + 0.s_2 = 0$-

Subject to $x_1 + 2x_2 + s_1 + 0.s_2 = 32$

$3x_1 + 4x_2 + 0.\ s_1 + s_2 = 84$

In simplex tableau form,

Simplex table I

P	x_1	x_2	s_1	s_2		b
1	– 50	– 80	0	0		0
0	1	$\boxed{2}$	1	0	:	32
0	3	4	0	1	:	84

We see that x_1, x_2 are non basic variables and the variables s_1, s_2 are basic.

First step

Operation 1 : Selection of column of the pivot element.

There are negative number in the first row and the least we negative number (– 80) is on the third column. Select the third column.

Operation 2 : Selection of row of the pivot element dividing the remaining third column entries into the corresponding entries in the last column, we get

$\frac{32}{2} = 16\left(\begin{matrix}\text{Smallest}\\ \text{quotient}\end{matrix}\right), \frac{84}{4} = 21$

Taking 2 is pivot element in the third row.

Operation 3 : Elimination by row operation.

Operation $R_1 \rightarrow R_1 + \frac{80}{2} R_2$ and $R_3 \rightarrow R_3 - \frac{4}{2} R_2$ we obtain

Simplex table II

P	x_1	x_2	s_1	s_2		b
1	– 10	0	40	0	:	1280
0	1	2	1	0	:	32
– 2	$\boxed{1}$	0	– 2	1	:	20

Second step

Operation 1 : Selection of column of the pivot element.

There is negative number in the first row and the least negative number (– 10) is on second column. Select the second column.

Operation 2 : Selection of row of the pivot element.

Dividing the remain by second column entries in to corresponding entries in the last column we get,

$\frac{18}{1} = 18$ (smallest quotient), $\frac{32}{1} = 32$

Taking 1 is pivot element of the third row.

Operations 3 : Elimination by row operation

Operating $R_1 \rightarrow R_1 + 10R_3$, $R_2 \rightarrow R_2 - R_3$

Simplex table III.

P	x_1	x_2	s_1	s_2		b
1	0	0	20	10	:	1480
0	0	2	3	– 1	:	12
0	1	0	– 2	1	:	20

We see that, there are no negative entries in the first row, so our procedure is end.

Setting $s_1 = 0$, $s_2 = 0$

We obtain the feasible solution.

$2x_2 = 12 \Rightarrow x_2 = 6$, $\quad 1x_1 = 20 \Rightarrow x_1 = 20$

Thus max. $p = 1480$ at $x_1 = 6$, $x_2 = 20$

TU Examination 2070 (Ashad)

1. Prove that : $\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ a & a & b & a \\ b & b & b & a \end{vmatrix} = -(b-a)^4$

Soln. *See exercise 1, Q. No. 22 in this book.*

2. Prove that every matrix A can uniquely be expressed as a sum of a symmetric and a skew symmetric matrix.

Soln. Let A be a given square matrix, then

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = B + C$ (say) .... (i)

Where $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$

So,

$$B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}\left[A^T + (A^T)^T\right] = \frac{1}{2}(A^T + A)$$

or, $B^T = B$

Hence, $B = \frac{1}{2}(A^T + A$ is a symmetric matrix .

Also,

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}[A - A^T] = -C$$

or, $C^T = -C$

Hence, $C = \frac{1}{2}(A - A^T)$ is a skew-symenctric matrix.

Thus the equation (i) express as the sum of a symmetric and skew symmetric matrices.

Uniqueness : If possible let $A = P + Q$ .... (ii)

Where P is symmetric and Q is a symmetric matrices

So that $P^T = P$ and $Q^T = -Q$

$A^T = (P + Q)^T = P^T + Q^T = P - Q$ .... (iii)

From (ii) and (iii), we get,

$A + A^T = 2P$

or, $P = \frac{1}{2}(A + A^T) = B$

Also, $A - A^T = 2Q$, i.e. $Q = \frac{1}{2}(A - A^T) = C$

$\therefore P + Q = B + C$ hence proved.

3. Test the consistency of the system $x + y + z = 6,\ x + 2y + 3z = 10,\ x = 2y + 4z = 1$ and solve.

Soln. *See exercise 5, Q. No. 20(ix) in this book.*

4. **Verify Cayley-Hamilton theorem for matrix A and find the inverse of** $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Soln. *See exercise 6, Q. No. 8(i) in this book*

5. **State and prove Green's theorem in the plane.**

Soln. *See T.U. exam 2068 (Baishak), Q. No. 14, page no. 407 in this book.*

OR

Verify stoke's theorem for $\bar{F} = (x^2 - y^2)\,\bar{i} - 2xy\,\bar{j}$ **taken round the rectangle in the xy-plane bounded by** $x = 0, y = 0; x = a$ **and** $y = b$.

Soln. *See exercise 9, Q. No. 10 in this book.*

6. **Find the work done in moving particle once round the circle** $x^2 + y^2 = 9, z = 0$ **under the force field** $\bar{F}$ **given by** $\bar{F} = (2x - y + z)\,\bar{i} + (x + y - z^2)\,\bar{j} + (3x - 2y + 4z)\,\bar{k}$.

Soln. Parametric equation of the circle $x^2 + y^2 = 9, z = 0$ is $x = 3\cos t$ and $y = 3\sin t$

$\therefore dx = -3\sin t\,dt$ and $dy = 3\cos t\,dt$ as t varies from $t = 0$ to $t = 2\pi$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (2x - y)\,dx + (x + y)\,dy$$

$$= \int_0^{2\pi} (6\cos t - 3\sin t)(-3\sin t\,dt) + \int_0^{2\pi} (3\cos t + 3\sin t)(3\cos t)\,dt$$

$$= 9\int_0^{2\pi} (1 - \sin t\cos t)dt = 9\int_0^{2\pi}\left[1 - \frac{1}{2}\sin 2t\right]dt$$

$$= 9\left[t + \frac{\cos 2t}{4}\right]_0^{2\pi} = 9\ 9\left[2\pi + \frac{1}{4}(1-1)\right] = 18\pi$$

7. **Evaluate** $\iint_S \bar{F}\cdot\bar{n}\ ds$ **where** $\bar{F} = xy\bar{i} - x^2\bar{j} + (x+z)\bar{k}$, **s is the portion of the plane** $2x + 2y + z = 6$ **included in the first octant.**

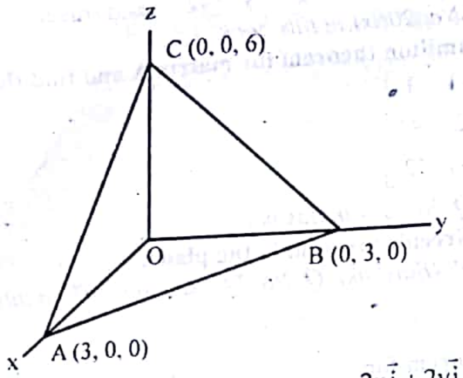
Soln. Here, $\bar{F} = xy\bar{i} - x^2\bar{j} + (x+z)\bar{k}$ and S is the portion of the plane

$2x + 2y + z = 6$

The normal to the plane

$\phi = 2x + 2y + z - 6$ is

$$\Delta\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(2x + 2y + z - 6) = 2\vec{i} + 2\vec{j} + \vec{k}$$



Unit normal to the surface is $\hat{n} = \dfrac{2x\vec{i} + 2y\vec{j} + \vec{k}}{\sqrt{4+4+1}} = \dfrac{2x\vec{i} + 2y\vec{j} + \vec{k}}{9}$

and, $\vec{F}\cdot\hat{n} = \left(xy\vec{i} - x^2\vec{j} + (x+z)\vec{k}\right)\cdot\left(\dfrac{2\vec{i} + 2\vec{j} + \vec{k}}{3}\right) = \dfrac{2xy - 2x^2 + x + z}{3}$

Let R be the projection of the plane in the xy-plane then

$2x + 2y = 6,\ z = 0$

$x + y = 3$

So, y varies from $y = 0$ to $y = 3 - x$ and x varies

From $x = 0$ to $x = 4$

$$\hat{n}\cdot\vec{k} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}\cdot\vec{k} = \frac{1}{3}$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \frac{2xy - 2x^2 + x + z}{3}\,\frac{dxdy}{\frac{1}{3}}$$

$$= \iint_R \left(2xy - 2x^2 + x + 2\right) dx\, dy$$

$$= \int_0^3\int_0^{3-x} \left(2xy - 2x^2 + x + 6 - 2x - 2y\right) dy\, dx$$

$$= \int_0^3 \left[xy^2 - 2x^2y - xy - y^2 + 6y\right]_0^{3-x} dx$$

$$= \int_0^3 \left(6x + 3x^3 - 12x^2 + 9\right) dx$$

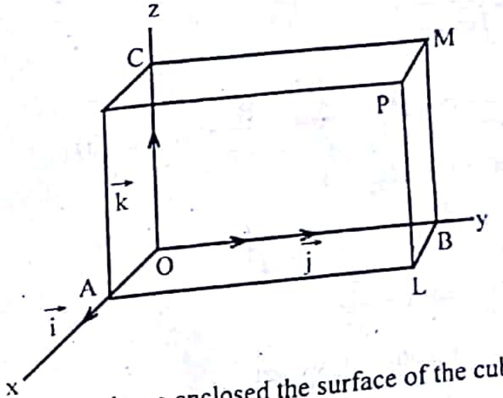
$$= 3\int_0^3 \left(x^3 - 4x^2 + 2x + 3\right) dx$$

$$= 3\left[\frac{x^4}{x} - \frac{4x^3}{3} + x^3 + 3x\right]_0 = 3\left[\frac{81}{4} - 18\right] = 3\times\frac{9}{4} = \frac{27}{4}$$

8. **Show that $\iint_S \left[\left(x^3 - yz\right)\vec{i} - 2x^2y\vec{j} + 2\vec{k}\right]\cdot\hat{n}\, ds = \frac{a^5}{3}$ where s is the surface of the cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$**

Solⁿ. We have to verify that Gaus's divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \left(\nabla\cdot\vec{F}\right) dv$$



Where v is the volume enclosed the surface of the cube.

and $\vec{F} = \left(x^3 - yz\right)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{F} = 3x^2 - 2x^2 = x^2$$

$$\iint_S \vec{F}\cdot\hat{n} = \int_0^a\int_0^a\int_0^a x^2 dx\, dy\, dz = \int_0^a\int_0^a ax^2\, dx\, dy$$

$$= \int_0^a a^2x^2\, dx = \frac{a^5}{3}$$

9. **Find the Lapalce transform of (i) $f(t) = \dfrac{1-\cos t}{t}$ (ii) $f(t) = te^{-t}\sin t$**

Solⁿ. *See exercise 11, Q. No. 2(xvi) and Q. No. 3(iii) in this book*

10. **Find the inverse Lapalce transform of**

(i) $\dfrac{(s+2)^3}{s^4}$ (ii) $\dfrac{1}{s^2\left(s^2+a^2\right)}$

Solⁿ. (i) $\dfrac{s^3 + 6s^2 + 12s + 8}{s^4}$

$$\frac{(s+2)^3}{s^4} = \frac{1}{s} + \frac{6}{s^2} + \frac{12}{s^3} + \frac{8}{s^4}$$

Taking inverse Laplace transform on both sides.

$$L^{-1}\left[\frac{(s+2)^3}{s^4}\right] = L^{-1}\left[\frac{1}{s}\right] + 6L^{-1}\left[\frac{1}{s^2}\right] + 12L^{-1}\left[\frac{1}{s^3}\right] + 8L^{-1}\left[\frac{1}{s^4}\right]$$

$$= 1 + 6t + 12 + 8 . \frac{t^3}{3!} \qquad \therefore L^{-1}\left[\frac{(s+2)^3}{s^4}\right] = 1 + 6t + 6t^2 + \frac{4}{3}t^3$$

(ii) $\frac{1}{s^2(s^2+a^2)}$

$$L^{-1}\left[\frac{1}{s^2}(s^2+a^2)\right] = \frac{A}{s^2} + \frac{B}{s^2+a^2} \quad \text{............ (i)}$$

$\therefore A(s^2 + a^2) + Bs^2 = 1$

Put $s^2 = -a^2$ $\quad \therefore B = \frac{-1}{a^2}$

Put s = 0 then $A = \frac{1}{a^2}$

Then from equation (i) we get,

$$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}L^{-1}\frac{1}{s^2} - \frac{1}{a^2}L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{t}{a^2} - \frac{1}{a^3}\sin at$$

11. Using Laplace transform to solve :
$y'' + 4y = \sin t$; $y(0) = 0 = y'(0)$

Sol^n. Given differential equation $y'' + 4y = \sin t$, $y(0) = 0$, $y'(0) = 0$

$L[y'' + 4y] = L[\sin t]$

or, $L[y''] + 4L[y] = \frac{1}{1+s^2}$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{1+s^2}$$

Using the initial condition, $y(0) = 0$, $y'(0) = 0$

$$s^2Y(s) - 0 - 0 + 4Y(s) = \frac{1}{1+s^2} \qquad \text{or, } (s^2+4)\,Y(s) = \frac{1}{(1+s^2)}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)}$$

Taking inverse Laplace transform

$$L[Y(s)] = L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right] = L^{-1}\left\{\frac{1}{3}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2-4)}\right]\right\}$$

$$= \frac{1}{3}\left\{L^{-1}\left[\frac{1}{s^2+1}\right] - L^{-1}\left[\frac{1}{s^2+4}\right]\right\} \qquad \therefore y(t) = \frac{1}{3}\sin t - \frac{1}{2}\sin 2t$$

12. Find a fourier series to represent $f(x) = x - x^2$ from $x = -\Pi$ to $x = \Pi$

Sol^n. *See exercise 17, Q. No. (i) in this book*

13. Find a fourier series to represent $f(x) = 2x - x^2$ in the range (0, 2)

Sol^n. We have Fourier series for the function $f(x) = 2x - x^2$ in the interval (0, 2) is,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right) \quad \text{........... (i)}$$

$$\text{Where, } a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos n\pi x\,dx = \int_0^2 (2x - x^2)\cos n\pi x\,dx$$

$$= \left[(2x - x^2)\frac{\sin n\pi x}{n\pi}\right]_0^2 - \int_0^2 (2-2x)\frac{\sin n\pi x}{n\pi}\,dx$$

$$= 0 - \frac{2}{n\pi}\left[(1-x)\frac{(-\cos n\pi x)}{n\pi}\right]_0^2 + \frac{2}{n\pi}\int_0^2 (-1)\frac{(-\cos n\pi x)}{n\pi}\,dx$$

$$= \frac{2}{n^2\pi^2}[-1-1] = \frac{-4}{n^2\pi^2}$$

Similarly, $b_n = \int_0^2 (2x - x^2)\sin n\pi x\,dx = 0$

Putting the value of a_n and b_n in equation (i), we get

$$(2x - x^2) = \frac{2}{3} - \frac{4}{n^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$$

OR

Express $f(x) = x$ as a half range sine series in $0 < x < \Pi$

Sol^n. *See exercise 11, Q. no. 11 in this book*

14. Use simplex method to, Maximize $p = 15x_1 + 10x_2$
Subject to $2x_1 + x_2 \leq 10$

$x_1 + 3x_2 \le 10,\ x_1, x_2 \ge 0$

Sol^n. Here, Maximize $z = 15x_1 + 10x_2$

Subject to $2x_1 + x_2 \le 10$

$x_1 + 3x_2 \le 10$

By introducing slack variables s_1, s_2

We have, Maximize $z - 15x_1 - 10x_2 + 0.s_1 + 0.s_2 = 0$

Subject to $2x_1 + x_2 + s_1 + 0.s_2 = 10$

$x_1 + 3x_2 + 0.s_1 + s_2 = 0$

In simplex table as,

z	x_1	x_2	s_1	s_2	:	b
1	-15	-10	0	0	:	0
0	$\boxed{2}$	1	1	0	:	10
0	1	3	0	1	:	10

Here leatest negative number -15 on the second column

So, Dividing the remaining second column entries,

$\frac{10}{2} = 5$ (small), $\frac{10}{1} = 10$

z	x_1	x_2	s_1	s_2	:	b
1	0	$-\frac{5}{2}$	$\frac{15}{2}$	0	:	75
0	2	1	1	0	:	10
0	0	$\boxed{\frac{5}{2}}$	$-\frac{1}{2}$	1	:	5

Again, Least negative number $\left(-\frac{2}{5}\right)$

Dividing $\frac{5}{12} = 2$ (smallest number), $\frac{10}{1} = 10$

Taking $\frac{5}{2}$ is pivot element

z	x_1	x_2	s_1	s_2	:	b
1	0	0	7	1	:	80
0	2	0	$\frac{6}{5}$	$-\frac{2}{5}$	:	8
0	0	$\frac{5}{2}$	$-\frac{1}{2}$	1	:	5

We see that there are no negative entries in the first row,

Setting $s_1 = 0$, $s_2 = 0$

So, $2x_1 = 8$ $\quad \therefore x_1 = 4$

$\frac{5}{2} x_2 = 5$ $\quad \therefore x_2 = 5$

Thus, max. $z = 80$, at $x_1 = 4$, $x_2 = 2$