Probability for Data Manipulation

Contents

In this lecture, we will look at some basic elements of probability theory which will help us manipulate data in a more accurate and principled way.

4.1. Uncertainty & Importance of Probability Theory

We have seen that dealing with data means dealing with "some variables assuming some values".

However, **what can we say about what values such variables will assume**?

In some cases, it is possible to predict the values of these variables with perfect accuracy, given a set of initial conditions. For instance, think about a system of equations describing the *speed of objects according to Newtonian laws*.

In other cases, modeling the relationship between variables in a deterministic way is not possible. We often say that this is due to **uncertainty**.

As reported here, **uncertainty in a system can be due to different factors**:

- **Inherent stochasticity in the system being modeled.** Some events such as *drawing a card from a deck of cards*, *rolling a die*, or the movements of subatomic particles can be seen as truly random events. The outcomes of such events cannot be predicted with perfect accuracy, so they are stochastic (random).
- **Incomplete observability.** Sometimes, even in a deterministic system, we cannot observe anything. For instance, *in the Monty Hall problem, the participant to the game is asked to choose between three doors*. One of the doors contains a prize, while the others lead to a goat. Even if the event is deterministic, the game contestant cannot observe everything, so from their point of view, the outcome is uncertain.
- **Incomplete modeling.** Sometimes a system has to discard some of the information needed to make a decision. For instance, imagine a robotic system aiming to pick objects from a table with a single RGB camera. *The system cannot reconstruct the 3D position of the objects, but the RGB camera an allow to obtain an estimate of the 2D coordinates of each objects with respect to the robot's point of view*. All the information which cannot be observed is uncertain, despite the problem is a deterministic one.

4.1.1. Examples

- **Tossing a coin or rolling a die**: these kinds of experiments are generally impossible to model in a deterministic way. This can be due to our limited ability to model the event (i.e., rolling a die might be deterministic, but deriving a set of equations to determine the outcome given the initial motion of the hand is intractable).

- **Determining if a patient has a given pathology**: different pathologies might have similar symptoms. Hence, observing some of them does not allow to determine with perfect accuracy if the patient has that pathology. In this case, uncertainty might arise from incomplete observability.

4.1.2. Importance of Probability Theory

Probability theory provides a consistent framework to work with uncertain events.

It allows to quantify and manipulate uncertainty with a set of axioms, as well as to derive new uncertain statements.

Probability theory is hence **an important tool to work with data**. We will start to revise the main concepts behind probability theory by talking about random variables.

4.1.3. Random Experiments

In practice, when the acquisition of data is affected by uncertainty, we will use the term **random experiment**. We will informally define a random experiment as:

> An experiment which can be repeated any number of times, leading to different outcomes.

We will use the following terminology:

- **sample space** $\Omega = \{\omega_1, \ldots, \omega_k\}$: the set of all possible outcomes of the experiment;
- **simple event** ω_i: a possible outcome of a random experiment;

- **event** $A \subseteq \Omega$: a subset of the sample space including certain events. We usually denote $\overline{A} = \Omega \setminus A$ as the complementary event to A, i.e., the event that A does not happen.

Given the definitions above, Ω is often called the **sure event** or **certain event**, because it contains all possible outcomes. The null set $\emptyset$ is called the **impossible event**.

4.1.3.1. Example

Let us consider the random experiment of rolling a die. Our outcomes will be numbers which we read on the top face of the die when it lands. We will have:

- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$;
- Simple event: an example of a simple event would be $\omega_1 = 1$;
- Event: the event "we obtain 1" could be denoted by $A = \{1\}$. The event "we obtain an even number" can be denoted by $A = \{2, 4, 6\}$. We would have $\overline{A} = \{1, 3, 5\}$, the event that "we obtain an odd number".

4.2. Random Variables

We have so far talked about "statistical variables". When dealing with uncertain events, we need to use the concept of 'random variables'. Informally (from wikipedia):

> A random variable is a variable whose values depend on outcomes of a random phenomenon.

A random variable is characterized by a set of possible values often called *sample space*, *probability space*, or *alphabet* (this last term comes from information theory, where we often deal with sources emitting symbols from an alphabet, in which case the values of X will be the symbols).

The definition of a random variable is very similar to that of a statistical variable.

Formally, if Ω is the *sample space*, we will define a random variable as a function:

$$X : \Omega \rightarrow E$$

Where **E is a measurable space** and often $E = \mathbb{R}$. This definition is similar to the one of statistical variable we have given before.

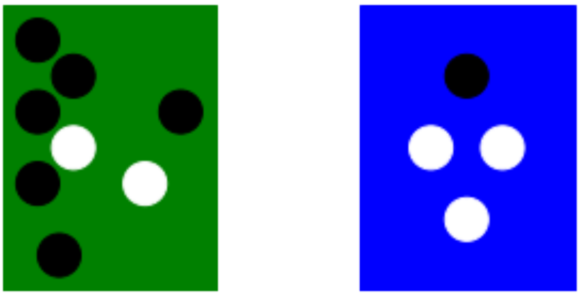
A random variable is generally denoted by a *capital letter*, such as X.

Similar to statistical variables, random variables can be *discrete* or *continuous*, *scalar* or *multi-dimensional*.

The following table lists some examples.

	Discrete	**Continuous**
Scalar	Tossing a coin	Height of a person
Multidimensional	Pair of dice	Coordinates of a car

4.2.1. Urns and Marble Example



We will now introduce an example that we will use in the rest of this lecture to reason on some basic properties of probabilities. Let's assume we have two urns urns: one green and one blue.

Each box contains marbles of two colors: black and white.

We consider the experiment of randomly drawing a marble from one of the two urns. This happens in two stages:

- We first randomly pick one of the two urns;
- Then we randomly pick one of the marbles in the urn;
- After observing the type of the marble, we replace it in the same urn.

The outcome of the experiment can be characterized by *two random variables*:

- U represents the color of the urn and can take values g (green) and b (blue).
- M represents the color of the marble and can take values b (black) and w (white).
- If we pick a white marble from the blue urn, then the outcome of the experiment can be characterized by the values $M = w$, $U = b$;

4.3. Working Definition of Data

We will define "data" as follows:

> The values assumed by a random variable

4.3.1. Example

- For instance, if the outcome of tossing a coing is $head$, then $X = head$ *is data*;
- It should be clear that the 'data' is *the pair* <random variable, value> and not just the value. Indeed $head$ alone would not be very useful (we don't know which phenomenon it is related to), whereas $X = head$ can be useful, as we know that X is the random variable describing the outcome of tossing a coin;
- In this example, the data $X = head$ is representing a fact: 'I tossed a coin and the outcome was $head$'. This is also called 'an event';

4.4. Probability

Since random variables are related to stochastic phenomena, we cannot say much about the outcome of a single phenomenon.

However, we expect to be able to characterize the class of experiments related to a random variable, to infer rules on what values the random variable is likely to take.

For instance, in the case of coin tossing, we can observe that, if I toss a coin for a large number of times, the number of heads will be roughly similar to the number of tails.

This kind of observations is useful, as it can give us a prior on what values we are likely to encounter and what are not.

To formally express such rules, we can define the concept of probability on a random variable.

Specifically, it is possible to assign a probability value to a given outcome. This is generally represented with a capital P:

- For instance, $P(U=b)$ represents the probability of picking a blue urn in the previous example;
- A probability $P(U=b)$ is a number comprised between 0 and 1 which quantifies how likely we believe the event to be;
    - 0 means impossible;
    - 1 means certain;

When it is clear from the context which variable we are referring to, the probability can also be expressed simply as:

$$P(b) = P(U=b)$$

4.4.1. Laplace Probability

When all outcomes in a random experiment are considered equally probable, this is called a **Laplace experiment**. In this case, we can calculate the probability of a given event A as the ratio between the favorable outcomes and the possible outcomes:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

For instance, if we are tossing a die:

- The probability of obtaining any of the faces will be: $\frac{1}{6}$;
- The probability of obtaining an even number will be: $\frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$.

4.4.2. Estimating probabilities from observations

What about the cases in which we cannot make any assumption on the equal probability of events? In those cases we would like to estimate probabilities from observations. There are two main approaches to do so: frequentist and Bayesian.

4.4.2.1. Frequentist approach

Probability theory was initially developed to analyze the frequency of events. For instance, it can be used to study events like drawing a certain hand of cards in a poker game. These events are repeatable and can be dealt with using frequencies. In this sense, when we say that an event has probability p of occurring, it means that if we repeat the experiment infinitely many times, then a proportion p of the repetitions would result in that outcome.

According to the frequentist approach, we can estimate probabilities by repeating an experiment for a large number of times and then computing:

- The number of trials: how many times we performed the experiment;
- The number of favorable outcomes: how many times the outcome of the experiment was favorable.

The probability is hence obtained by dividing the number of favorable outcomes by the number of trials.

For instance, let's suppose we want to estimate the probability of obtaining a 'head' by tossing a coin. Let's suppose we toss the coin 1000 times and obtain 499 heads and 501 tails. We can compute the probability of obtaining head as follows:

- Number of trials: 1000;
- Number of favorable outcomes: 499.

The probability of obtaining head will be 499/1000=0.49

This is the approach we have seen so far in the course when dealing with relative frequencies.

4.4.2.1.1. Examples

- The probability of obtaining 'head' when tossing a coin is 0.5. We know that because, if we toss a coin for a large number of times, about half of the times, we will obtain 'head';
- The probability of picking a red ball from a box with 40 red balls and 60 blue balls is 0.4. We know this because, if we repeat the experiment for a large number of times, we will observe that proportion.

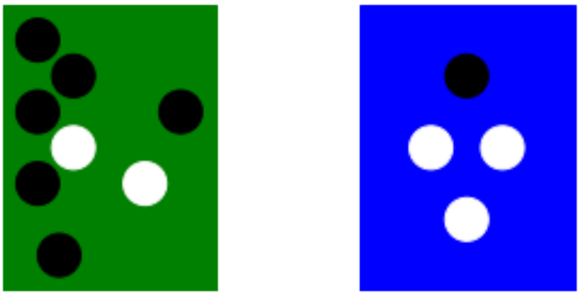
4.4.2.2. Bayesian Approach

The Bayesian approach to probability offers a different perspective on probability theory. Unlike the frequentist approach, which focuses on analyzing the frequency of events based on observations, the **Bayesian approach allows us to incorporate prior knowledge and update our beliefs as new information becomes available**.

In Bayesian probability, we view **probability as a measure of uncertainty or belief**. When we assign a probability to an event, it reflects our **subjective degree of belief in the event's likelihood**. This approach is particularly useful when dealing with unique or one-time events where frequency-based analysis may not be applicable (e.g., “**what is the probability that the sun will extinguish in 5 billion years?**”).

We will discuss better the Bayesian approach when we’ll discuss Bayes theorem.

4.4.3. Example of Probability



Let’s consider the previous example of drawing marbles from the two urns.

Suppose we repeat this experiment for many times and observe that:

- We pick the green urn 40% of the times;
- We pick the blue urn 60% of the times;
- Once we selected a urn, we are equally likely to select any of the marble contained in it, but we know that colors are not distributed evenly in the urns (see figure).

Using a frequentist approach, we can define the probabilities:

- $P(U=b) = \frac{6}{10}$
- $P(U=r) = \frac{4}{10}$

This is done by using the formula:

- $P(X=x) = \frac{\# \; of \; times \; X=x}{\# \; trials}$

4.5. Joint probability

We can define **univariate** (= with respect to only one variable) probabilities $P(U)$ and $P(M)$ as we have seen in the previous examples.

However, in some cases, it is useful to define probabilities on more than one variable at the time. For instance, we could be interested in studying the probability of picking a given fruit from a given box. In this case, we would be interested in the **joint probability** $P(B, F)$.

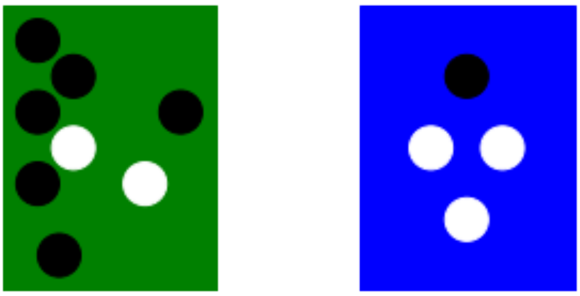
In general, we can have joint probabilities with arbitrary numbers of variable. For instance, $P\left(X_1, X_2, X_3, \ldots, X_n\right)$.

Joint probabilities are symmetric, i.e., $P(X, Y) = P(Y, X)$.

We should note that, when dealing with multiple unidimensional variables, we can always define a new multi-variate variable comprising all of them:

- $X = [X_1, X_2]$;
- $P(X) = P\left(X_1, X_2\right)$.

4.5.1. Example



We can see the concept of joint probability in the context of the examples of the two urns.

We have seen how to define the univariate probability $P(U)$ over the whole probability space of U.

However, we could be interested in the probability of both variables jointly: $P(U, M)$, i.e., the joint probability of U and M.

To 'measure' the joint probability, we could repeat the experiment for many times and observe the outcomes.

We can then build a table which keeps track of how many times we observed a given combination:

	Green Urn	**Blue Urn**	**All**
White	10	15	25
Black	30	45	75
All	40	60	100

A table like this is called a "**contincency table**".

From the table above, we can easily derive the joint probability of a given pair of values using the **frequentist approach**. For instance:

$$P(U=b, M=b) = \frac{\#\, times\ (b,a)\ occurs}{\#\, trials} = \frac{15}{100}$$

Similarly, we can derive the other values:

- $F(U=b, M=b) = \frac{45}{100}$
- $F(U=g, M=w) = \frac{10}{100}$
- $F(U=g, M=b) = \frac{30}{100}$

Note that we can also obtain the univariate probabilities by using the values in the "sum" row and column. For instance:

$$P(U=b) = \frac{\#\, times\ b\ occurred}{\#\, trials} = \frac{15+45}{100} = \frac{60}{100}$$

Similarly:

- $P(U = g) = \frac{40}{100}$
- $P(M = w) = \frac{25}{100}$
- $P(M = b) = \frac{30}{100}$

These univariate probabilities computed starting from joint probabilities are usually called "marginal probabilities" (we are using the sums in the margin of the table).

We can obtain a joint probability table by dividing the table by the total number of trials (100):

	Green Urn	Blue Urn	Sum
White	10/100	15/100	25/100
Black	30/100	45/100	75/100
Sum	40/100	60/100	100/100

4.6. Sum Rule (Marginal Probability)

In the previous example, we have seen how we can compute **marginal (univariate) probabilities** from the contingency table. This is possible because the contingency table contains information on how the different possible outcomes distribute over the sample space.

In general, we can compute marginal probabilities form joint probabilities (i.e., we don't need to have the non-normalized frequency counts of the contingency table). Let us consider the general contingency table:

	Y=y_1	**Y=y_2**	**...**	**Y=y_l**	**Total**
X=x_1	n_{11}	n_{12}	...	n_{1l}	n_{1+}
X=x_2	n_{21}	n_{22}	...	n_{2l}	n_{2+}
...	...	...	...	...	...
X=x_k	n_{k1}	n_{k2}	...	n_{kl}	n_{k+}
Total	n_{+1}	n_{+2}	...	n_{+l}	n

We can compute the joint probability $P(X = x_i, Y = y_j)$ with a frequentist approach using the formula:

$$P\left(X = x_i,\ Y = y_j\right) = \frac{n_{ij}}{n}$$

Note that these are the joint frequencies f_{ij} we mentioned in the past.

Also, we note that we can define the marginal probabilities of X and Y as follows:

- $P\left(X = x_i\right) = \frac{n_{i+}}{n}$.
- $P\left(Y = y_j\right) = \frac{n_{+j}}{n}$.

Note that n_{i+} can be seen as the sum of all occurrences in which $X = x_i$ (i.e., we are summing all values in row i):

$$n_{i+} = \sum_j n_{ij}$$

We can write the marginal probability of X as follows:

$$P\left(X = x_i\right) = \frac{n_{i+}}{n} = \frac{\sum_j n_{ij}}{n} = \sum_j \frac{n_{ij}}{n} = \sum_j P(X = x_i, Y = y_j)$$

This result is known as the **sum rule of probability**, which allows to estimate marginal probabilities from joint probabilities. This can be seen in more general terms as:

$$P(X=x)=\sum_y P(X=x,Y=y)$$

The act of computing $P(X)$ from $P(X,Y)$ is also known as marginalization.

4.7. Conditional Probability

In many cases, we are interested in the probability of some event, given that some other event happened.

This is called **conditional probability** and is denoted as $P\left(X=x|Y=y\right)$ and read as "P of X=y given that Y=y". In this context, $Y=y$ is the condition, and we are interested in studying the probability of X only in the cases in which the condition is verified.

For instance, in the case of the two urns, we could be interested in $P(M=w|U=b)$, i.e., what is the probability of picking a white marble, given that we know that we are drawing from the blue urn?

Let's consider our example contingency table of two variables again:

	Y=y_1	**Y=y_2**	**...**	**Y=y_l**	**Total**
X=x_1	n_{11}	n_{12}	...	n_{1l}	n_{1+}
X=x_2	n_{21}	n_{22}	...	n_{2l}	n_{2+}
...	...	...	...	...	...
X=x_k	n_{k1}	n_{k2}	...	n_{kl}	n_{k+}
Total	n_{+1}	n_{+2}	...	n_{+l}	n

We can compute the conditional probability $P(X=x_i|Y=y_j)$ using the frequentist approach:

$$P\left(X=x_i|Y=y_j\right) = \frac{\#\ cases\ in\ which\ X=x_i\ and\ Y=y_i}{\#\ cases\ in\ which\ Y=y_j} = \frac{n_{ij}}{n_{+j}}$$

If we multiply the expression above by $1=\frac{n}{n}$, we obtain:

$$P\left(X=x_i|Y=y_j\right) = \frac{n_{ij}}{n}\frac{n}{n_{+j}} = \frac{\frac{n_{ij}}{n}}{\frac{n_{+j}}{n}} = \frac{P\left(X=x_i, Y=y_j\right)}{P(Y=y_j)}$$

This leads us to the general definition of conditional probability:

$$P\left(X=x|Y=y\right) = \frac{P(X=x,\ Y=y)}{P(Y=y)}$$

The conditional probability is defined only when $P(Y=y)>0$, that is, we cannot define a probability conditioned on an event that never happens. It should be noted that, in general $P\left(X|Y\right) \neq P(X)$.

4.8. Product Rule (Factorization)

We can see the definition of conditional probability:

$$P\left(X=x|Y=y\right) = \frac{P(X=x,\ Y=y)}{P(Y=y)}$$

As follows:

$$P(X=x,Y=y) = P\left(X=x|Y=y\right)P(Y=y)$$

which is often referred to as **the product rule**.

The product rule allows to compute joint probabilities starting from conditional probabilities and marginal probabilities. This is useful because measuring joint probabilities generally involves creating large tables, whereas conditional and marginal probabilities might be easier to derive.

This operation of expressing a joint probability in terms of two factors is known as *factorization*.

4.8.1. How to compute a conditional probability?

We just said that factorization can be useful for computing joint probabilities starting from conditional probabilities. However, two questions arise: “how can we compute a conditional probability?” and “Is it easier than computing a joint probability?”.

Since conditional probabilities are obtained by *restricting* the probability space to *a subset* of the events, we can compute conditional probabilities by considering the observations which satisfy the condition.

For example, let’s say we want to compute the conditional probability:

$$P(M|U=b)$$

That is to say, the probability of taking a marble of a given color, given that we know that we are considering the blue urn. Let’s consider again our contingency table:

	Green Urn	**Blue Urn**
White	10	15
Black	30	45

To compute this probability, we can just *consider all the observations that satisfy the condition* $U=b$, which is equivalent to taking the second column of the full contingency table and compute the probabilities in a frequentist way:

	Blue Urn
White	15
Black	45

$$P(M=w|U=b) = \frac{15}{60}$$

$$P(M=b|U=b) = \frac{45}{60}$$

Note that, in general, when the number of variables is large, this approach allows to save a lot of space and time as it is not necessary to even build the first contingency table, but only the second, restricted one is required (for instance, one may choose not to record all observations in which the user has drawn from the red box).

4.9. The Chain Rule of Conditional Probabilities

When dealing with multiple variables, the product rule can be applied in an iterative fashion, thus obtaining the 'chain rule' of conditional probabilities.

For instance:

$$P(X,Y,Z) = P\left(X|Y,Z\right)P(Y,Z)$$

Since:

$$P(Y,Z) = P\left(Y|Z\right)P(Z)$$

We obtain:

$$P(X,Y,Z) = P\left(X|Y,Z\right)P\left(Y|Z\right)P(Z)$$

Since joint probabilities are symmetric, we could equally obtain:

$$P(X,Y,Z) = P\left(Z|Y,X\right)P\left(Y|X\right)P(X)$$

This rule can be formalized as follows:

$$P\left(X_1,\ldots,\ X_n\right) = P\left(X_1\right)\prod_{i=2}^{n}P(X_i|X_1,\ldots,\ X_{i-1})$$

4.10. Bayes' Theorem

Given two variables A and B, from the product rule, we obtain:

- $P(A,B) = P\left(A|B\right)P(B)$
- $P(B,A) = P\left(B|A\right)P(A)$

Since joint probabilities are symmetric, we have:

$$P(A,B) = P(B,A) \rightarrow P\left(A|B\right)P(B) = P\left(B|A\right)P(A)$$

Which implies:

$$P\left(A|B\right) = \frac{P\left(B|A\right)P(A)}{P(B)}$$

This last expression is known as Bayes' Theorem (or Bayes' rule).

Technically speaking, the Bayes' rule can be used to "turn" probabilities of the kind $P(A|B)$ into probabilities of the kind $P(B|A)$.

More formally, the Bayes' rule can be used to **update our expectation that some**

event will happen (event A) when we observe some evidence (event B). This links to the Bayesian interpretation of probability, according to which probability can be seen as **a *reasonable expectation* representing *the quantification of a degree of belief*** (i.e., how much we believe some event will happen).

The different terms in the Bayes' rule have specific names:

- $P(A)$ is called 'the prior' – this is our expectation that A happens when we do not have any other data to rely on
- $P(B|A)$ is called 'the likelihood' – this quantifies how likely it is to observe event B happening if we assume that event A has happened
- $P(B)$ is called 'the evidence' – this models the probability of observing event B
- $P(A|B)$ is called 'the posterior' – this is our updated probability, i.e., how likely it is A to happen once we have observed B happening

We'll see an example in the section below.

4.10.1. Bayesian Probability Example

Let's imagine we are trying to understand if a friend has COVID or not. If we do not know anything about our friend's symptoms (i.e., we don't know if they have any symptoms or not), then we would expect our friend to have COVID with a "prior" probability $P(C)$. If we know that currently one people over two has COVID, we expect:

$$P(C) = \frac{1}{2}$$

Now, if our friend tells us he has fever, things change a bit. We are now interested in modeling the probability: $P\left(C|F\right)$. We can try to model it using Bayes' rule:

$$P\left(C|F\right) = \frac{P\left(F|C\right)P(C)}{P(F)}$$

Note that it is not straightforward to estimate $P(C|F)$ in a frequentist way. Ideally, we should take all people with a fever on earth and check how many of them has COVID. This is not feasible as people with just fever may never do a COVID test. On the contrary, measuring $P(F|C)$ is easier: we take all people which we know have COVID (these may not be all people with COVID, but probably a large enough sample) and see how many of them have a fever. Let's suppose that one people with COVID out of three has a fever. Then we can expect:

$$P\left(F|C\right) = \frac{1}{3}$$

Now we need to estimate the evidence $P(F)$. This can be done by considering how frequent it is for people to have a fever. Let's say we use historical data and find out that one person out of five has fever. We finally have:

$$P\left(C|F\right) = \frac{\frac{1}{3}\frac{1}{2}}{\frac{1}{5}} = \frac{5}{6}$$

We can interpret this use of the Bayes' theorem as follows:

- Before knowing anything about symptoms, we could only guess that our friend had COVID with a prior probability of $P(C) = \frac{1}{2}$
- When we get to know that our friend has a fever, our expectation changes. We know that people with COVID often get a fever, but we also know that not all people with COVID get a fever, so we are not going to say that we are 100% certain that our friend has fever. Instead, we use Bayes' rule to obtain an updated estimate $P\left(C|F\right) = \frac{5}{6}$ based on our knowledge of the likelihood $P(F|C)$ (how likely it is for people with symptom to have COVID) and the evidence $P(F)$ (how common is this symptom – if it is too common, then it is not so informative)

4.11. Independence and Conditional Independence

Two variables X and Y are independent if the outcome of one of the two does not influence the outcome of the other. Formally speaking, two variables are **said to be independent** if and only if:

$$P(X,Y) = P(X)P(Y)$$

We have seen why definition this makes sense in the context of the Pearson's χ^2 statistic. The same considerations apply here.

It should be noted that the expression above **is generally not true** as we cannot always assume that two variables are independent.

Independence can be denoted as:

$$X \perp Y$$

Moreover, if two variables X and Y are independent, then:

$$P\left(X|Y\right) = \frac{P(X,Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$$

This makes sense because it means that "the fact that Y happens does not influence the fact that X happens", which is what we would expect of two independent variables X and Y.

4.11.1. Examples

Intuitively, two variables are independent if the values of one of them do not affect the values of the other one:

- Weight and height of a person are **not independent**. Indeed, taller people are usually heavier.
- Height and richness are **independent**, as the richness does not depend on the height of a person.

4.11.2. Conditional independence

Two random variables X and Y are said to be **conditionally independent** given a random variable Z if:

$$P(X, Y|Z) = P(X|Z)P(Y|Z)$$

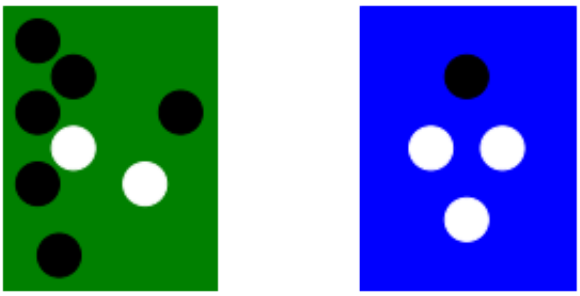
Conditional independence can be denoted as:

$$X \perp Y \mid Z$$

4.11.2.1. Example

Height and vocabulary are not independent: taller people are usually older, and hence they have a more sophisticated vocabulary (they know more words). However, if we condition on age, they become independent. Indeed, among people of the same age, height should not influence vocabulary. Hence, height and vocabulary are **conditionally independent** with respect to age.

4.12. Example of Probability Manipulation



Let's get back to our example of the two boxes and let's suppose that, by repeating several trials, we discovered the following probabilities (in a frequentist way):

- $P(U = g) = \frac{4}{10}$

- $P(U = b) = \frac{6}{10}$

We also focused on a given urn and performed different trials, observing the following proportions:

- $P\left(M = w|U = g\right) = \frac{1}{4}$
- $P\left(M = b|U = g\right) = \frac{3}{4}$
- $P\left(M = w|U = b\right) = \frac{3}{4}$
- $P\left(M = b|U = b\right) = \frac{1}{4}$

We shall note that these probabilities are normalized such that:

- $P(U = g) + P(U = b) = 1$
- $P\left(M = w|U = g\right) + P\left(M = b|U = g\right) = 1$
- $P\left(M = w|U = b\right) + P\left(M = b|U = b\right) = 1$

We can now use the rules we have seen before to answer questions such as:

- What is the overall probability of choosing a white marble?
- What is the probability of picking a green urn, given that we have drawn a black marble?

What is the overall probability of choosing an white marble?

To answer this question, we need to find $P(M = w)$. We note that, by the sum rule:

$$P(M = w) = P(M = w, U = b) + P(M = w, B = g)$$

We also observe that the joint probabilities can be recovered using the product rule:

$$P(M = w, U = b) = P\left(M = w|U = b\right)P(U = b)$$

$$P(M = w, U = g) = P\left(M = w|U = g\right)P(U = g)$$

Hence, our probability can be found using the formula:

$$P(M=w) = P\left(M=w|U=b\right)P(U=b) + P\left(M=w|U=g\right)P(U=g) =$$

$$= \frac{3}{4} \cdot \frac{6}{10} + \frac{1}{4} \cdot \frac{4}{10} = \frac{18+4}{40} = \frac{22}{40} = \frac{11}{20}$$

From the definition of probability, we have:

$$P(M=w) + P(M=b) = 1$$

Hence:

$$P(M=b) = 1 - P(M=w) = 1 - \frac{11}{20} = \frac{9}{20}$$

What is the probability of picking a green urn, given that we have drawn an black marble?

To answer this question, we need to find the conditional probability $P(U=g|M=b)$. To find this probability, we need to 'invert' our conditional probabilities using the Bayes' rule:

$$P\left(U=g|M=b\right) = \frac{P\left(M=b|U=g\right)P(U=g)}{P(M=b)} = \frac{\frac{3}{4} \cdot \frac{4}{10}}{\frac{9}{20}} = \frac{12}{40} \cdot \frac{20}{9} =$$

4.12.1. Excercise

Suppose that we have three colored boxes r (red), b (blue), and g (green). Box r contains 3 apples, 4 oranges, and 3 limes, box b contains 1 apple, 1 orange, and 0 limes, and box g contains 3 apples, 3 oranges, and 4 limes. If a box is chosen at random with probabilities P(r)=0.2, P(b)=0.2, P(g)=0.6, and a piece of fruit is extracted from the box (with equal probability of selecting any of the items in the box), then what is the probability of selecting an apple? If we observe that the selected fruit is in fact an orange, what is the probability that it came from the green box?

4.13. References

- Parts of chapter 1 of [1];
- Most of chapter 3 of [2];
- Parts of chapters 5-7 of [3].

[1] Bishop, Christopher M. *Pattern recognition and machine learning*. springer, 2006. https://www.microsoft.com/en-us/research/uploads/prod/2006/01/Bishop-Pattern-Recognition-and-Machine-Learning-2006.pdf

[2] Goodfellow, Ian, Yoshua Bengio, and Aaron Courville. *Deep learning*. MIT press, 2016. https://www.deeplearningbook.org/

[3] Heumann, Christian, and Michael Schomaker Shalabh. Introduction to statistics and data analysis. Springer International Publishing Switzerland, 2016.

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