Fórmula de Leibniz para Determinantes

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1) Deduza o determinante de uma matriz 4x4 usando a fórmula:

$$\det(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} (-1)^{\operatorname{sgn}(\sigma)} ai\sigma(i) \right)$$

Permutações

$$\begin{aligned} S_4 = \{&(1,2,3,4),(1,2,4,3),(1,3,2,4),(1,3,4,2),(1,4,2,3),(1,4,3,2),\\ &(2,1,3,4),(2,1,4,3),(2,3,1,4),(2,3,4,1),(2,4,1,3),(2,4,3,1)\\ &(3,1,2,4),(3,1,4,2),(3,2,1,4),(3,2,4,1),(3,4,1,2),(3,4,2,1),\\ &(4,1,2,3),(4,1,3,2),(4,2,1,3),(4,2,3,1),(4,3,1,2),(4,3,2,1)\} \end{aligned}$$

$$\begin{aligned} \operatorname{sgn}(1,2,3,4) &= 0 \\ \operatorname{sgn}(1,2,4,3) &= 1 \\ \operatorname{sgn}(1,3,2,4) &= 1 \\ \operatorname{sgn}(1,3,4,2) &= 2 \\ \operatorname{sgn}(1,4,2,3) &= 2 \\ \operatorname{sgn}(1,4,3,2) &= 3 \\ \operatorname{sgn}(2,1,3,4) &= 1 \\ \operatorname{sgn}(2,1,4,3) &= 2 \\ \operatorname{sgn}(2,3,1,4) &= 2 \\ \operatorname{sgn}(2,3,4,1) &= 3 \\ \operatorname{sgn}(2,4,1,3) &= 3 \\ \operatorname{sgn}(2,4,3,1) &= 4 \end{aligned}$$

$$\begin{aligned} \operatorname{sgn}(3,1,2,4) &= 2 \\ \operatorname{sgn}(3,1,4,2) &= 3 \\ \operatorname{sgn}(3,2,1,4) &= 3 \\ \operatorname{sgn}(3,2,4,1) &= 4 \\ \operatorname{sgn}(3,4,1,2) &= 4 \\ \operatorname{sgn}(3,4,2,1) &= 5 \\ \operatorname{sgn}(4,1,2,3) &= 3 \\ \operatorname{sgn}(4,1,3,2) &= 4 \\ \operatorname{sgn}(4,2,1,3) &= 4 \\ \operatorname{sgn}(4,2,3,1) &= 5 \\ \operatorname{sgn}(4,3,1,2) &= 5 \\ \operatorname{sgn}(4,3,2,1) &= 6 \end{aligned}$$

$$n = 4$$

$$\det(A) = \sum_{\sigma \in S_4} \left(\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(\sigma)} ai\sigma(i) \right) = \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(1,2,3,4)} ai_{(1,2,3,4)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(1,2,4,3)} ai_{(1,2,4,3)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(1,3,2,4)} ai_{(1,3,2,4)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(1,3,4,2)} ai_{(1,3,4,2)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(1,4,2,3)} ai_{(1,4,2,3)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(1,4,3,2)} ai_{(1,4,3,2)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(2,1,3,4)} ai_{(2,1,3,4)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(2,1,4,3)} ai_{(2,1,4,3)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(2,3,1,4)} ai_{(2,3,1,4)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(2,3,4,1)} ai_{(2,3,4,1)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(2,4,1,3)} ai_{(2,4,1,3)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(2,4,3,1)} ai_{(2,4,3,1)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(3,1,2,4)} ai_{(3,1,2,4)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(3,1,4,2)} ai_{(3,1,4,2)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(3,2,1,4)} ai_{(3,2,1,4)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(3,2,4,1)} ai_{(3,2,4,1)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(3,4,1,2)} ai_{(3,4,1,2)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(3,4,2,1)} ai_{(3,4,2,1)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(4,1,2,3)} ai_{(4,1,2,3)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(4,1,3,2)} ai_{(4,1,3,2)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(4,2,1,3)} ai_{(4,2,1,3)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(4,2,3,1)} ai_{(4,2,3,1)(i)} +$$

$$\prod_{i=1}^{4} (-1)^{\operatorname{sgn}(4,3,1,2)} ai_{(4,3,1,2)(i)} + \prod_{i=1}^{4} (-1)^{\operatorname{sgn}(4,3,2,1)} ai_{(4,3,2,1)(i)}$$

$$\det(A) = \prod_{i=1}^{4}(-1)^0 ai_{(1,2,3,4)(i)} + \prod_{i=1}^{4}(-1)^1 ai_{(1,2,4,3)(i)} + \prod_{i=1}^{4}(-1)^1 ai_{(1,3,2,4)(i)} +$$

$$\prod_{i=1}^{4}(-1)^2 ai_{(1,3,4,2)(i)} + \prod_{i=1}^{4}(-1)^2 ai_{(1,4,2,3)(i)} + \prod_{i=1}^{4}(-1)^3 ai_{(1,4,3,2)(i)} +$$

$$\prod_{i=1}^{4}(-1)^1 ai_{(2,1,3,4)(i)} + \prod_{i=1}^{4}(-1)^2 ai_{(2,1,4,3)(i)} + \prod_{i=1}^{4}(-1)^2 ai_{(2,3,1,4)(i)} +$$

$$\prod_{i=1}^{4}(-1)^3 ai_{(2,3,4,1)(i)} + \prod_{i=1}^{4}(-1)^3 ai_{(2,4,1,3)(i)} + \prod_{i=1}^{4}(-1)^4 ai_{(2,4,3,1)(i)} +$$

$$\prod_{i=1}^{4}(-1)^2 ai_{(3,1,2,4)(i)} + \prod_{i=1}^{4}(-1)^3 ai_{(3,1,4,2)(i)} + \prod_{i=1}^{4}(-1)^3 ai_{(3,2,1,4)(i)} +$$

$$\prod_{i=1}^{4}(-1)^4 ai_{(3,2,4,1)(i)} + \prod_{i=1}^{4}(-1)^4 ai_{(3,4,1,2)(i)} + \prod_{i=1}^{4}(-1)^5 ai_{(3,4,2,1)(i)} +$$

$$\prod_{i=1}^{4}(-1)^3 ai_{(4,1,2,3)(i)} + \prod_{i=1}^{4}(-1)^4 ai_{(4,1,3,2)(i)} + \prod_{i=1}^{4}(-1)^4 ai_{(4,2,1,3)(i)} +$$

$$\prod_{i=1}^{4}(-1)^5 ai_{(4,2,3,1)(i)} + \prod_{i=1}^{4}(-1)^5 ai_{(4,3,1,2)(i)} + \prod_{i=1}^{4}(-1)^6 ai_{(4,3,2,1)(i)} +$$

Posições

$'1,2,3,4'(1)=1$
’1,2,3,4’(2)= 2
’1,2,3,4’(3)= 3
’1,2,3,4’(4)= 4
’1,2,4,3’(1)= 1
’1,2,4,3’(2)= 2
’1,2,4,3’(3)= 4
’1,2,4,3’(4)= 3
’1,3,2,4’(1)= 1
’1,3,2,4’(2)= 3
’1,3,2,4’(3)= 2
’1,3,2,4’(4)= 4
’1,3,4,2’(1)= 1
’1,3,4,2’(2)= 3
’1,3,4,2’(3)= 4
’1,3,4,2’(4)= 2
’1,4,2,3’(1)= 1
’1,4,2,3’(2)= 4
’1,4,2,3’(3)= 2
’1,4,2,3’(4)= 3
’2,3,1,4’(1)= 2
’2,3,1,4’(2)= 3
’2,3,1,4’(3)= 1
’2,3,1,4’(4)= 4
’2,3,4,1’(1)= 2
’2,3,4,1’(2)= 3
’2,3,4,1’(3)= 4
’2,3,4,1’(4)= 1
’2,4,1,3’(1)= 2
’2,4,1,3’(2)= 4
’2,4,1,3’(3)= 1
’2,4,1,3’(4)= 3
’2,4,3,1’(1)= 2
’2,4,3,1’(2)= 4
’2,4,3,1’(3)= 3
’2,4,3,1’(4)= 1
’3,1,2,4’(1)= 3
’3,1,2,4’(2)= 1
’3,1,2,4’(3)= 2
’3,1,2,4’(4)= 4
’3,1,4,2’(1)= 3
’3,1,4,2’(2)= 1
’3,1,4,2’(3)= 4
’3,1,4,2’(4)= 2
’3,2,1,4’(1)= 3
’3,2,1,4’(2)= 2
’3,2,1,4’(3)= 1
’3,2,1,4’(4)= 4
’3,2,4,1’(1)= 3
’3,2,4,1’(2)= 2
’3,2,4,1’(3)= 4
’3,2,4,1’(4)= 1
’3,4,1,2’(1)= 3
’3,4,1,2’(2)= 4
’3,4,1,2’(3)= 1
’3,4,1,2’(4)= 2
’3,4,2,1’(1)= 3
’3,4,2,1’(2)= 4
’3,4,2,1’(3)= 2
’3,4,2,1’(4)= 1
’4,1,2,3’(1)= 4
’4,1,2,3’(2)= 1
’4,1,2,3’(3)= 2
’4,1,2,3’(4)= 3
’4,1,3,2’(1)= 4
’4,1,3,2’(2)= 1
’4,1,3,2’(3)= 3
’4,1,3,2’(4)= 2
’4,2,1,3’(1)= 4
’4,2,1,3’(2)= 2
’4,2,1,3’(3)= 1
’4,2,1,3’(4)= 3
’4,2,3,1’(1)= 4
’4,2,3,1’(2)= 2
’4,2,3,1’(3)= 3
’4,2,3,1’(4)= 1
’4,3,1,2’(1)= 4
’4,3,1,2’(2)= 3
’4,3,1,2’(3)= 1
’4,3,1,2’(4)= 2
’4,3,2,1’(1)= 4
’4,3,2,1’(2)= 3
’4,3,2,1’(3)= 2
’4,3,2,1’(4)= 1

$$
\begin{aligned}
\det(A) = {} & a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{43}a_{34} - a_{11}a_{32}a_{23}a_{44} + a_{11}a_{32}a_{43}a_{24} \\
& + a_{11}a_{42}a_{23}a_{34} - a_{11}a_{42}a_{33}a_{24} - a_{21}a_{12}a_{33}a_{44} + a_{21}a_{12}a_{43}a_{34} \\
& + a_{21}a_{32}a_{13}a_{44} - a_{21}a_{32}a_{43}a_{14} - a_{21}a_{42}a_{13}a_{34} + a_{21}a_{42}a_{33}a_{14} \\
& + a_{31}a_{12}a_{23}a_{44} - a_{31}a_{12}a_{43}a_{24} - a_{31}a_{22}a_{13}a_{44} + a_{31}a_{22}a_{43}a_{14} \\
& + a_{31}a_{42}a_{13}a_{24} - a_{31}a_{42}a_{23}a_{14} - a_{41}a_{12}a_{23}a_{34} + a_{41}a_{12}a_{33}a_{24} \\
& + a_{41}a_{22}a_{13}a_{34} - a_{41}a_{22}a_{33}a_{14} - a_{41}a_{32}a_{13}a_{24} + a_{41}a_{32}a_{23}a_{14}
\end{aligned}
$$

Resultado Final

$$
\boxed{
\begin{aligned}
\det(A) = {} & a_{11}a_{22}a_{33}a_{44} + a_{11}a_{32}a_{43}a_{24} + a_{11}a_{42}a_{23}a_{34} + a_{21}a_{12}a_{43}a_{34} \\
& + a_{21}a_{32}a_{13}a_{44} + a_{21}a_{42}a_{33}a_{14} + a_{31}a_{12}a_{23}a_{44} + a_{31}a_{22}a_{43}a_{14} \\
& + a_{31}a_{42}a_{13}a_{24} + a_{41}a_{12}a_{33}a_{24} + a_{41}a_{22}a_{13}a_{34} + a_{41}a_{32}a_{23}a_{14} \\
& - a_{11}a_{22}a_{43}a_{34} - a_{11}a_{32}a_{23}a_{44} - a_{11}a_{42}a_{33}a_{24} - a_{21}a_{12}a_{33}a_{44} \\
& - a_{21}a_{32}a_{43}a_{14} - a_{21}a_{42}a_{13}a_{34} - a_{31}a_{12}a_{43}a_{24} - a_{31}a_{22}a_{13}a_{44} \\
& - a_{31}a_{42}a_{23}a_{14} - a_{41}a_{12}a_{23}a_{34} - a_{41}a_{22}a_{33}a_{14} - a_{41}a_{32}a_{13}a_{24}
\end{aligned}
}
$$

2) Calcule o determinante usando o que foi deduzido, de duas matrizes definidas pelo autor (det = 0 / det $\neq$ 0):

- det = 0

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
\det(A) &= 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 \\
&\quad + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 + 1\cdot1\cdot1\cdot1 \\
&\quad - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 \\
&\quad - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 - 1\cdot1\cdot1\cdot1 \\
\det(A) &= 1+1+1+1+1+1+1+1+1+1+1+1-1-1-1-1-1-1 \\
&\qquad -1-1-1-1-1-1 \\
\det(A) &= 12-12 \\
\det(A) &= 0
\end{aligned}$$

- det $\neq$ 0

$$B = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned}
\det(B) &= 2\cdot2\cdot2\cdot2 + 2\cdot0\cdot0\cdot1 + 2\cdot1\cdot0\cdot0 + 0\cdot0\cdot0\cdot2 + 0\cdot0\cdot1\cdot2 + 0\cdot1\cdot2\cdot0 \\
&\quad + 1\cdot0\cdot0\cdot2 + 1\cdot2\cdot0\cdot0 + 1\cdot1\cdot1\cdot1 + 0\cdot0\cdot2\cdot1 + 0\cdot2\cdot1\cdot0 + 0\cdot0\cdot0\cdot1 \\
&\quad - 0\cdot2\cdot0\cdot0 - 0\cdot0\cdot0\cdot2 - 2\cdot1\cdot2\cdot1 - 0\cdot0\cdot2\cdot2 - 1\cdot0\cdot0\cdot0 - 0\cdot1\cdot1\cdot0 \\
&\quad - 1\cdot0\cdot0\cdot0 - 1\cdot2\cdot1\cdot2 - 1\cdot1\cdot0\cdot0 - 0\cdot0\cdot0\cdot0 - 0\cdot2\cdot2\cdot0 - 0\cdot0\cdot1\cdot1 \\
\det(B) &= 16+0+0+0+0+0+0+0+1+0+0+0-0-0-4-0-0-0 \\
&\qquad -0-4-0-0-0-0 \\
\det(B) &= 17-8 \\
\det(B) &= 9
\end{aligned}$$