Section 3.2: Properties of Determinants

Theorem 1. *(Theorem 3: Row operations)Let A be a square matrix*

- *(Replacement) If a multiple of one row of A is added to another row to produce matrix B, then*
$$\det(B) = \det(A)$$
.
- *(Interchange) If two rows of A are interchanged to produce B, then*
$$\det(B) = -\det(A)$$
.
- *(Scaling) If one row of A is multiplied by a constant k to make B, then*
$$\det(B) = k \cdot \det(A)$$
.

Example 2. Let $A = \begin{bmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{bmatrix}$. Find the determinant of A.

Observe, using only the row operation of replacement we are able to row reduce matrix A to an upper triangular matrix.

$$\begin{bmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

By Theorem 1 above, since row replacement does not change the determinant we have $\det(A) = \det \begin{bmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix} = 1(1)(3) = 3$, by the Determinant of Triangular Matrix theorem.

Example 3. Let $A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix}$. Find the determinant of A.

(**Approach:**) First we will try to use row operations to row reduce matrix A to an upper triangular matrix. Note, the matrix does not need to in RREF so we do not need our pivots to be 1. We only need a triangular matrix, then we can apply the Row Operations theorem and the Determinant of a Triangular matrix theorem. We will try to use mostly row replacements since this does not change the determinant of the matrix.

$$A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note, in our row reduction we used row replacements in the first 2 steps and interchanged rows R_3 and R_4 in the last step. Now applying the Row Operations Theorem, and keeping track of which row operations we did in each step, we have

$$\begin{aligned} \det(A) = \det \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{bmatrix} &= \det \begin{bmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{bmatrix} \\ &= -1 \cdot \det \begin{bmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= -1 \cdot (-3) = 3. \end{aligned}$$

Example 4. Let $A = \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$. Find the determinant of A.

$$\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the last matrix was obtained by row replacements and has determinant 0, we have $\det(A) = 0$.

Theorem 5. *(Theorem 4) A square matrix A is invertible if and only if* $\det(A) \neq 0$*.*

Theorem 6. *(Theorem 5) If A is an $n \times n$ matrix, then* $\det(A^T) = \det(A)$*.*

Theorem 7. *(Multiplicative Property of Determinants) If A and B are $n \times n$ matrices, then*

$$\det(AB) = \det(A) \cdot \det(B).$$

WARNING!!!! For matrices A and B, in general $\det(A + B) \neq \det(A) + \det(B)$.

Practice Problems

1. Suppose A is invertible, find a formula for $\det(A^{-1})$. (*Hint:* Recall if A is invertible then $AA^{-1} = I_n$.)

2. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det(A)$.

3. Let $M = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$. Use row operations to show that $\det(M) = (b-a)(c-a)(c-b)$.

Textbook Practice Problems: Section 3.2 (page 177) # 5-10, 15, 17, 19, 24-26, 27, 28.

END OF SECTION 3.2