By signing below, you attest that you have neither given nor received help of any kind on this exam. Signature: ____________________ Printed Name: ____________________

Instructions: Show work to get full credit (the correct answer may NOT be enough). Do all your work on the paper provided. Write clearly! Double check your answers!

You will **not** receive full credit for using methods other than those discussed in class.

Final Exam

MATH 214 – Linear Algebra

Directions: Read all directions carefully and show all work. There are a total of 12 questions and the exam is worth 200 points. All the best and have a great summer!!!

1. For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer. [16]

   (a) T or F? The set of vectors of the form $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ is a subspace of $\mathbb{R}^3$.

   (b) T or F? A $n \times n$ matrix A is not invertible if $\dim(\mathrm{Nul}(A)) = 0$.

   (c) T or F? The vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$

   (d) T or F? The matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.

   (e) T or F? The eigenvalue of a matrix is always a real number.

   (f) T or F? Let A be an $n \times n$ matrix. If the rows of A span $\mathbb{R}^n$, then the columns of A must be linearly independent.

   (g) T or F? Every homogenous linear system has at least one solution.

   (h) T or F? The zero vector of a vector space V is a subspace of V.

2. In each part below, you're given information about a 4×4 matrix A. Based on the given information in each part, write "**invertible**", "**not invertible**", or "**impossible to tell**". No explanations or justification is necessary. [14]

   (a) A has 4 distinct eigenvalues

   (b) A has 2 identical rows

   (c) No column of A is a scalar multiple of another column

   (d) $\det(A) = \pi$

   (e) A is diagonalizable

   (f) $\mathrm{rank}(A) = 4$

   (g) A^T is invertible

3. Find all values of h such that the linear system [20]

$$\begin{array}{rcrcrcr} & & 3x_2 & + & 6hx_3 & = & 0 \\ x_1 & + & 2x_2 & + & 6x_3 & = & 2 \\ 3hx_1 & & & + & 6x_3 & = & 3 \end{array}$$

has

(a) no solutions.

(b) a unique solution.

(c) infinitely many solutions.

4. Let [15]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}.$$

(a) Find the inverse of A.

(b) Use part (a) to write $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ as a linear combination of the columns of A.

5. Use cofactor expansion across the 2^{nd} row to compute the determinant of A [20]

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 4 & 0 & 0 & 7 \\ 0 & 1 & 2 & 8 & 2 \end{bmatrix}.$$

6. Given the matrix $A = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. [15]

(a) Find a basis for Col(A). What is rank(A)?

(b) Find a basis for Row(A). What is the dim(Row(A))?

(c) Find a basis for Nul(A). What is dim(Nul(A))?

7. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the transformation defined by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+2c+d \\ b+c+2d \\ 3c \end{bmatrix}$. [20]

   (a) Is T a linear transformation? Be sure to justify your answer.

   (b) Determine whether T is one-to-one and whether T is onto? Be sure to justify your answer.

   (c) Find a basis for the kernel of T.

8. Consider the matrix [15]

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) Find the eigenvalues of A.

   (b) For each eigenvalue, find a basis for its eigenspace.

9. Let A be an unknown 3×3 matrix that has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$ and corresponding eigenspaces [10]

$$E_{\lambda_1} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad E_{\lambda_2} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

   Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$. (You do not need to compute S^{-1} here).

10. Let $W = \left\{ \begin{bmatrix} x \\ y \\ w \\ x+y \end{bmatrix} : x, y, w \text{ in } \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^4$. [15]

   (a) Find basis for W. What is the dimension of W?

   (b) Find a basis for $W^{\perp}$. What is the dimension of $W^{\perp}$?

11. Let $\vec{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. [20]

(a) Show that S is an orthogonal set.

(b) Is S an orthonormal set? Explain.

(c) Is S a basis for $\mathbb{R}^3$? Explain.

(d) Use inner products to write $\vec{b} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S.

12. Let $\vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. [20]

(a) Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$.

(b) Write $\vec{y}$ as the sum of two orthogonal vectors, one in $L = \text{Span}\{\vec{u}\}$ and one orthogonal to $\vec{u}$.

(c) Find the distance from $\vec{y}$ to L.