Exam 3 Study Guide
Math 214

Exam 3 will be **Monday April 13, 2020**. You will have the entire 60-minutes class period to complete the exam and 15 minutes to upload your Exam to Moodle. The exam will cover sections 2.3, 3.1, 3.2, 4.1, 4.2, 4.3, 4.5 of the textbook.

A. Outline of Topics

- Determining if a matrix is invertible without using row reduction, i.e., the Invertible Matrix Theorem (2.3)
- Determining if a linear transformation is invertible and finding the inverse transformation (2.3)
- Computing the determinant of an $n \times n$ matrix using cofactor expansion and row reduction (3.1)
- Finding the determinant of a matrix A, given a matrix B that is row equivalent to A and the row operations needed to get from A to B (3.2)
- Using properties of determinants (e.g., $\det(AB) = \det A \det B$, etc.) (3.2)
- Using determinants to check if a matrix is invertible (3.2)
- Determining if a given subset of a vector space is a subspace (4.1)
- Determine the kernel of given transformation (4.2)
- Finding bases for Nul A and Col A (4.2, 4.3)
- Determining a basis for a given subspace (4.3)
- Determining the dimension of a vector space (4.5)

B. Definitions you should know:

- Inverse of a matrix (2.2)
- Subspace (4.1)
- Null space of a matrix (4.2)
- Column space of a matrix (4.2)
- Kernel of a transformation (4.2)
- Basis of a vector space (4.3)
- Dimension of a vector space (and subspace) (4.5)

C. Practice problems: These are meant to illustrate for you the varying types of problems which may appear on the exam. They are **NOT** meant to indicate in any way what specific topics will be addressed on the exam, nor the length of the exam.

1. Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 & 1 \\ 2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$.

   (a) Using cofactor expansion, compute the determinant of A.

   (b) Using row operations, compute the determinant of A.

2. Determine if the following subsets are subspaces of $\mathbb{R}^4$. If it is, find a basis. If it's not, explain why.

   (a) $H = \left\{ \begin{bmatrix} a+2c \\ b \\ a+b+2c \\ -a-2c \end{bmatrix} : a, b, c \text{ are in } \mathbb{R} \right\}$

   (b) Find a basis for the set of all vectors of the form $\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix}$.

3. Determine if $\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in $\text{Nul}(A)$, where $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$.

4. Define a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ defined by $T((x, y, z, w)) = (y, z)$, where (x, y, z, w) is an element of $\mathbb{R}^4$.

   (a) Is $(1, 0, 0, -7)$ in the kernel of the transformation? How about $(0, 1, -7, 0)$?

   (b) Find the $\ker(T)$.

5. For the following transformations find the kernel

   (a) $T : M_{3\times 3} \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} a \\ d+f \\ g+h \end{bmatrix}$

   (b) $T : \mathbb{R}^4 \to \mathbb{R}^6$ by $T(x) = Ax$ where $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 3 & 0 & 0 \\ 5 & 4 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 6 \\ -2 & 0 & 3 & 1 \end{bmatrix}$.

   (c) $T : M_{3\times 3} \to M_{2\times 2}$ by $T\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} a & c \\ g & i \end{bmatrix}$.

6. For each vector space V, determine if the given set is linearly independent in V

   (a) $V = \mathbb{R}^4$; $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

   (b) $V = \mathbb{P}_3$; $\{x, 3x+4, x^2\}$

   (c) $V = M_{2\times 2}$; $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

7. Find a spanning set for $\text{Nul}(A)$ and $\text{Col}(A)$, where $A = \begin{bmatrix} 1 & -5 & 0 & 2 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

8. A matrix A is row equivalent to, but may not be equal to, the matrix $B = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   What is the dimension of Col A? Find a basis for Nul(A). What is dim(Nul(A))? With the given information, is it possible to determine bases for Col A?

9. Let $B = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$. Find a basis for each of Nul(B) and Col(B).

10. Let $\vec{v_1} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} -3 \\ 6 \\ -7 \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$. Find a basis for $H = \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

11. For the following questions provide a brief explanation to support your answer.
    (a) If A, B and C are $n \times n$ matrices and $AB = AC$, does $B = C$? Prove it or find a counterexample.
    (b) Let $T : M_{2\times 2} \to M_{2\times 2}$ be the transformation defined by $T(A) = 3A + 3A^T$. Show that T is a linear transformation.
    (c) Let V be a vector space. Show that $H = \{0\}$, i.e. the set containing the zero vector of V, is a subspace of V.