By signing below, you attest that you have neither given nor received help of any kind on this exam.

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Instructions: Show work to get full credit (the correct answer may NOT be enough). Do all your work on the paper provided. Write clearly! Double check your answers!

You will **not** receive full credit for using methods other than those discussed in class.

EXAM II

MATH 214 – LINEAR ALGEBRA

Problem Number	Available Points	Your Points
1	16	16
2	12	12
3	10	8
4	4	3
5	20	20
6	15	15
7	15	15
8	8	8
Total	100	97

have neither given nor ...

1. Determine if the following sets of vectors are linearly independent or linearly dependent. Justify your answer.

(a) $\left\{ \begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix} \right\}$ [4]

$v_1 \quad v_2 \quad v_3$

Since we have 3 vectors in $\mathbb{R}^2$, we know by a theorem that the set of vectors is linearly dependent because $p = 3 > 2 = n$.

(b) $\left\{ \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}6\\3\\3\end{bmatrix} \right\}$ [4]

Since $3\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}6\\3\\3\end{bmatrix}$, we know by a theorem that the set of vectors is linearly dependent since one is a multiple of the other.

(c) $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$ $A = \begin{bmatrix}1&0&0\\1&1&1\\1&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ [4]

$-R_1 + R_2 \to R_2$
$-R_1 + R_3 \to R_3$

$-R_3 + R_2 \to R_2$

Since $Ax = \vec{0}$ has only the trivial solution (the RREF of A has a pivot in every column), then we know the corresponding vector equation has only the trivial solution, and therefore the set of vectors is linearly independent.

(d) $\left\{ \begin{bmatrix}2\\5\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\10\\8\end{bmatrix} \right\}$ [4]

Since this set of vectors contains the zero vector, we know by a theorem that the set is linearly dependent.

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2. Let $A = \begin{bmatrix} 1 & 4 & 8 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ and let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation give by $T(x) = Ax$.

(a) Find the image of $u = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$ under the transformation T. [4]

$$\begin{bmatrix} 1 & 4 & 8 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 4\cdot 0 + 8\cdot(-3) + 1\cdot 2 \\ 0\cdot 1 + 2\cdot 0 + 1\cdot(-3) + 3\cdot 2 \\ 0\cdot 1 + 0\cdot 0 + 0\cdot(-3) + 5\cdot 2 \end{bmatrix} = \begin{bmatrix} 1-24+2 \\ -3+6 \\ 10 \end{bmatrix} = \begin{bmatrix} -21 \\ 3 \\ 10 \end{bmatrix}$$

(b) Is T one-to-one? Be sure to justify your answer. [4]

T is not one-to-one. because it is not linearly independent since there is not a pivot in every column.

(c) Is T onto? Be sure to justify your answer. [

T is onto since the columns of A span $\mathbb{R}^3$ because there is a pivot in every row.

3. Let $A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which has reduced echelon form of $\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$ $\begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & \underbrace{0}_{x_2} & 0 & \underbrace{0}_{x_4} & 0 & \underbrace{0}_{x_6} \end{bmatrix}$.

(a) Do the columns of A span $\mathbb{R}^4$? Explain your answer. [2]

We know by a theorem that the columns of A do not span $\mathbb{R}^4$ because there is not a pivot in every row. (TFAE). ✓

2pt.

(b) Write the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric vector form. [5]

$x_1 - 4x_2 + 5x_6 = 0 \Rightarrow x_1 = 4x_2 - 5x_6$

$x_3 - x_6 = 0 \Rightarrow x_3 = x_6$

$x_5 - 4x_6 = 0 \Rightarrow x_5 = -4x_6$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ -4x_6 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

x_4 is free

3pt.

(c) Given that $\begin{bmatrix} 7 \\ -1 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ is a solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ -10 \\ 0 \end{bmatrix}$ write the entire solution set of $A\mathbf{x} = \mathbf{b}$. [3]

$\vec{w} = \vec{p} + v_h$

$$\vec{w} = \begin{bmatrix} 7 \\ -1 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

✓ 3pt.

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4. Carefully write out the definition of a **linear transformation**. [4]

A transformation is linear if:

$T(u+cv) = T(u) + c\,T(v)$ ✓ 3pt.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-2y \\ y \\ -2x+y \end{bmatrix}.$$

(a) Show T is a linear transformation. [15]

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$T(u+cv)$

$= T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$

$= T\left(\begin{bmatrix} u_1 + cv_1 \\ u_2 + cv_2 \end{bmatrix}\right)$

$= \begin{bmatrix} u_1 + cv_1 - 2u_2 - 2cv_2 \\ u_2 + cv_2 \\ -2u_1 - 2cv_1 + u_2 + cv_2 \end{bmatrix}$ ✓

$T(u) + c\,T(v)$

$= T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + c\,T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$

$= \begin{bmatrix} u_1 - 2u_2 \\ u_2 \\ -2u_1 + u_2 \end{bmatrix} + c\begin{bmatrix} v_1 - 2v_2 \\ v_2 \\ -2v_1 + v_2 \end{bmatrix}$

$= \begin{bmatrix} u_1 + cv_1 - 2u_2 - 2cv_2 \\ u_2 + cv_2 \\ -2u_1 - 2cv_1 + u_2 + cv_2 \end{bmatrix}$

✓

(b) Find the standard matrix for T. [5]

$A = [\,T(e_1)\ \ T(e_2)\,]_{3\times 2}$

$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}$ ✓

$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

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6. Given the following matrices, perform the indicated operation if possible. If it is not possible to perform the operation, explain why it could not be performed.

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 0 & 8 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 4 & 0 \end{bmatrix},\ C = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix},\ D = \begin{bmatrix} 7 & -1 & 4 \\ 0 & 9 & -2 \end{bmatrix}.$$

(a) $-2A + D$ [5]

$$-2\begin{bmatrix} 2 & 4 & 0 \\ 3 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 7 & -1 & 4 \\ 0 & 9 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & 0 \\ -6 & 0 & -16 \end{bmatrix} + \begin{bmatrix} 7 & -1 & 4 \\ 0 & 9 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -9 & 4 \\ -6 & 9 & -18 \end{bmatrix}$$

(b) CB [5]

$$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 4 & 0 \end{bmatrix}_{3\times3} = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

CB is not defined by the definition of matrix multiplication because $C\vec{b_1}$, $C\vec{b_2}$, and $C\vec{b_3}$ are not computable, since there is not enough columns in C for the size of the vector.

(c) AB [5]

$$\begin{bmatrix} 2 & 4 & 0 \\ 3 & 0 & 8 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 4 & 0 \end{bmatrix}_{3\times3} = \begin{bmatrix} 2+0+0 & 4+4+0 & 6+4+0 \\ 3+0+24 & 6+0+32 & 9+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 & 10 \\ 27 & 38 & 9 \end{bmatrix}_{2\times3}$$

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7. Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$. Find A^{-1}. [15]

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

$3R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$3R_2 + R_3 \to R_3$

$R_3 + R_2 \to R_2$
$R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$\frac{1}{2}R_3 \to R_3$

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$ ✓

15pt.

8. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible. [8]

The columns of an $n \times n$ matrix A are linearly independent when A is invertible because there is a pivot in every column, meaning the only solution ✓ to the vector equation $c_1v_1 + \ldots + c_pv_p = 0$ is the trivial one.

8pt