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Exam 4

- 1) Since $m = \# \text{ of rows} = 4$, the number of pivot positions cannot exceed the number of rows or the number of columns, hence the largest possible rank of C is 4. By the rank theorem, $\text{rank } A + \dim \text{Nul } A = n$, so if $\text{rank } A$ is at its largest possible value of 4, then $\dim \text{Nul } A$ must be 1, and the smallest possible dimension of $\text{Nul } A$ is 1 also.

$$2) a) \begin{bmatrix} 1 & -4 & 0 & 5 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 1/2 x_3 \\ x_3 \\ 0 \end{bmatrix} \Rightarrow \text{Nul } A = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \dim \text{Nul } A = 1$$

$$b) \text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \right\} \text{ and } \dim \text{Col } A = 3$$

$$c) \text{Row } A = \text{Span} \{ (1, 0, -2, 0), (0, 1, -1/2, 0), (0, 0, 0, 1) \}$$
$$\dim \text{Row } A = 3$$

$$d) \dim \text{Col } A = \dim \text{Row } A = \text{rank } A = 3 = \# \text{ of pivot columns}$$
$$\dim \text{Nul } A = 1 = \# \text{ of non-pivot columns}$$
$$\text{rank } A + \dim \text{Nul } A = n$$
$$3 + 1 = 4 \quad \checkmark$$

$$\begin{aligned}
 3) \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2} - 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2(-\sqrt{2} - 1) + 1(1) \\ 1(-\sqrt{2} - 1) + 1(4) \end{bmatrix} \\
 &= \begin{bmatrix} -2\sqrt{2} - 2 + 1 \\ -\sqrt{2} - 1 + 4 \end{bmatrix} \\
 &= \begin{bmatrix} -2\sqrt{2} - 1 \\ -\sqrt{2} + 3 \end{bmatrix} \\
 &= \lambda \begin{bmatrix} -\sqrt{2} - 1 \\ 1 \end{bmatrix} \Rightarrow \lambda = -\sqrt{2} + 3
 \end{aligned}$$

$$\begin{aligned}
 4) \det(A - \lambda I) &= \det \begin{bmatrix} 4-\lambda & 0 & 1 & 0 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & 2-\lambda \end{bmatrix} \\
 &= (4-\lambda) \det \begin{bmatrix} 4-\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \\
 &= (4-\lambda) \left[-\lambda \det \begin{bmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} \right] \\
 &= (4-\lambda) [-\lambda (4-\lambda)(2-\lambda)] = 0
 \end{aligned}$$

$\lambda = 4$ (multiplicity 2), 0 (multiplicity 1), 2 (multiplicity 1)

A is not invertible by the IMT since it has an eigenvalue of 0 .

$$5) \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A is not diagonalizable since the sums of the dimensions of the eigenspaces do not add up to $n=3$.

$$6) A^k = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3^k & 0 \\ 0 & -2^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot -3^k & -2 \cdot -2^k \\ 2 \cdot -3^k & -1 \cdot -2^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1(3 \cdot -3^k) + (-2)(-2 \cdot -2^k) & 2(3 \cdot -3^k) + 3(-2 \cdot -2^k) \\ -1(2 \cdot -3^k) + (-2)(-1 \cdot -2^k) & 2(2 \cdot -3^k) + 3(-1 \cdot -2^k) \end{bmatrix}$$

$$= \begin{bmatrix} -3^{k+1} & -2^{k+2} \\ -2(-3^k) - (-2^{k+1}) & 6(-3^k) + 3(-2^{k+1}) \end{bmatrix}$$

$$4(-3^k) - 3(-2^k)$$

$$7) a) \begin{bmatrix} 2 & 3 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2(1) + 3(2) - 6(2) = 2 + 6 - 12 = -4$$

$$b) \|v\| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

$$\frac{1}{\|v\|} v = \begin{bmatrix} 2/7 \\ 3/7 \\ -6/7 \end{bmatrix}$$

$$8) a) \begin{bmatrix} 8 \\ -5 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = (8)(-2) + (-5)(-3) = -16 + 15 = -1 \neq 0$$

Since $u \cdot v \neq 0$, then the vectors are not orthogonal and do not satisfy the Pythagorean Theorem.

$$b) \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 12(2) + 3(-3) + (-5)(3) = 24 - 9 - 15 = 0$$

Since $u \cdot v = 0$, then the vectors are orthogonal and satisfy the Pythagorean Theorem.