Inner Product, Length, and Orthogonality

We will first start with a definition.

Definition: Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. The **inner product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is the number $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. That is

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example 1. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$. Then the inner product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = 2 + 0 + 3 + 12 = 17$$

Theorem: Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(c) $(c\vec{u}) \cdot \vec{w} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

(d) $\vec{u} \cdot \vec{u} \geq 0$

(e) $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$.

Definition: The **length** (or **norm**) of a vector $\vec{v}$ in $\mathbb{R}^n$, denoted $||\vec{v}||$, is

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \ldots v_n^2}$$

Example 2. Let $\vec{v} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$. Compute $||\vec{v}||$.

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{2^2 + (-5)^2 + (-1)^2} = \sqrt{4 + 25 + 1} = \sqrt{30}.$$

Example 3. Let $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$. Compute $||\vec{v}||$.

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{3^2 + 2^2 + (-5)^2 + 0^2} = \sqrt{9 + 4 + 25} = \sqrt{38}.$$

Remark:

1. $||\vec{v}||^2 = \left(\sqrt{\vec{v} \cdot \vec{v}}\right)^2 = \vec{v} \cdot \vec{v}$
2. If c is a scalar, then $||c\vec{v}|| = |c|||\vec{v}||$
3. If $\vec{v}$ is in $\mathbb{R}^2$ and $\mathbb{R}^3$, this length is the length of the line segment between the origin and the point $\vec{v}$.

Definition: A **unit vector** is a vector of length 1. The process of dividing a nonzero vector $\vec{v}$ by its length-that is multiplying by $1/||\vec{v}||$- we form a unit vector is called **normalizing** $\vec{v}$. We say that the unit vector is in the same direction as $\vec{v}$.

Example 4. From the previous Example 2, for $\vec{v} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ we have $||\vec{v}|| = \sqrt{30}$. Find a unit vector $\vec{u}$ in the same direction as $\vec{v}$.

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v} = \frac{1}{\sqrt{30}}\vec{v} = \begin{bmatrix} \frac{2}{\sqrt{30}} \\ \\ \frac{-5}{\sqrt{30}} \\ \\ \frac{-1}{\sqrt{30}} \end{bmatrix}.$$

Definition: For $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, the **distance between $\vec{u}$ and $\vec{v}$**, written as $\text{dist}(\vec{u},\vec{v})$, is the length of the vector $\vec{u}-\vec{v}$, that is

$$\text{dist}(\vec{u},\vec{v}) = ||\vec{u}-\vec{v}||.$$

Example 5. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$, find $\text{dist}(\vec{u},\vec{v})$.

$$\text{dist}(\vec{u},\vec{v}) = ||\vec{u}-\vec{v}|| = || \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} || = \sqrt{(-1)^2+2^2+2^2+1^1} = \sqrt{1+4+4+1} = \sqrt{10}$$

Remark: In $\mathbb{R}^2$ and $\mathbb{R}^3$, this is the usual distance between points $\vec{u}$ and $\vec{v}$.

Definition: Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are **orthogonal** if $\vec{u}\cdot\vec{v} = \vec{0}$.

Example 6. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ -7 \\ 8 \\ -3 \end{bmatrix}$.

$$\vec{u}\cdot\vec{v} = 1(2)+2(0)+3(1)+4(3) = 2+3+12 = 17 \neq 0$$

$$\vec{u}\cdot\vec{w} = 1(2)+2(-7)+3(8)+4(-3) = 2-14+24-12 = 26-26 = 0$$

So $\vec{u}$ and $\vec{v}$ are NOT orthogonal, but $\vec{u}$ and $\vec{w}$ ARE orthogonal.

Remark:

1. In $\mathbb{R}^2$ and $\mathbb{R}^3$, $\vec{u}$ and $\vec{v}$ being orthogonal means that the lines through the origin and $\vec{u}$ and $\vec{v}$, respectively are perpendicular.
2. In any $\mathbb{R}^n$, $\vec{0}$ is orthogonal to every vector since $\vec{0}\cdot\vec{v} = \vec{0}$ for all $\vec{v}$.

Theorem 2 (The Pythagorean Theorem): Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if

$$||\vec{u}+\vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2.$$

Definition: Let W be a subspace of $\mathbb{R}^n$. If $\vec{x}$ in $\mathbb{R}^n$ is orthogonal to every vector in W, then $\vec{x}$ is **orthogonal** to W. The set of all vectors orthogonal to W is the **orthogonal complement** of W, denoted $W^\perp$, and read "W perp".

Remark:

1. $\vec{x}$ is in $W^\perp$ if and only if $\vec{x}$ is orthogonal to every vector in a spanning set of W.
2. $W^\perp$ is a subspace of $\mathbb{R}^n$.

Example 7. Let $W = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ in $\mathbb{R}^2$. Geometrically, this is the line through the origin passing through the point $(1,1)$.

$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ is in $W^\perp$ if and only if $\vec{x}$ is orthogonal to $\begin{bmatrix}1\\1\end{bmatrix}$, that is if and only if

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = 1(x_1) + 1(x_2) = 0 \implies x_1 = -x_2.$$

So $\vec{x} = \begin{bmatrix}-x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}-1\\1\end{bmatrix}$. Therefore we have $W^\perp = \text{Span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Example 8. Let $W = \text{Span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right\}$ in $\mathbb{R}^4$. Find $W^\perp$.

$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ is in $W^\perp$ if and only if $\vec{x}$ is orthogonal to $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$.

That is, if and only if, $\vec{x} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \vec{0}$ and $\vec{x} \cdot \begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \vec{0}$.

That is, if and only if, $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 + x_3 = 0$. We therefore set up an augmented matrix for this linear system and row reduce it,

$$\begin{bmatrix}1 & 1 & 1 & 1 & 0\\1 & 0 & 1 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 1 & 0\end{bmatrix}$$

$$\vec{x} = \begin{bmatrix}-x_3\\-x_4\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + x_4\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} \implies W^\perp = \text{Span}\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}.$$

Theorem 3: Let A be an $m \times n$ matrix. Then

$$(\text{Row}(A))^\perp = \text{Nul}(A) \text{ and } (\text{Col}(A))^\perp = \text{Nul}(A^T).$$

Textbook Practice Problems: Section 6.1 (page 338-339) # 1, 2, 3, 4, 5, 6, 7, 9, 10, 16, 17, 19, 20, 26.

END OF SECTION 6.1