Section 5.3: Diagonalization

Definition: An $n \times n$ A is **diagonalizable** if it is similar to a diagonal matrix.

- That is $A = S^{-1}DS$ for some invertible matrix S and diagonal matrix D.

This factorization $A = S^{-1}DS$ is a **diagonalization** of A.

Theorem: (**The Diagonalization Theorem**) Let A be an $n \times n$ matrix. Then

(1) A is diagonalizable if and only if A has n linearly independent eigenvectors

(2) In this case, $A = SDS^{-1}$, where the columns of S are n linearly independent eigenvectors of A and the diagonal matrix D are the corresponding eigenvalues of A.

Our goal now, is given a matrix A determine if A is diagonalizable. If A is diagonalizable, find a diagonalization of A. (We will use the diagonalization theorem to do this).

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

What happens when A does not have distinct eigenvalues. That is to say, what happens when A has an eigenvalue with multiplicity more than 1. Can A be diagonalizable

Theorem: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(1) The dimension of the eigenspace of λ_k is less than or equal to the multiplicity of λ_k.

(2) A is diagonalizable if and only if the sum of the dimensions of the eigenspaces is n. This happens if and only if the characteristic polynomial factors completely into linear factors and the dimension of each eigenspace is equal to the multiplicity of the eigenvalues.

(3) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace of λ_k, then the set of vectors in $\mathcal{B}_1, \ldots, \mathcal{B}_p$ form an eigenvector basis for $\mathbb{R}^n$.

Example 1. Let $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$. Diagonalize A, if possible.

- **Step 1:** Find all eigenvalues.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{bmatrix}\right) = (3-\lambda)(-\lambda) - (-2) = \lambda^2 - 3\lambda + 2$$
$$= (\lambda - 2)(\lambda - 1)$$

  Therefore, the eigenvalues are $\lambda = 1, 2$, each with multiplicity 1.

- **Step 2:** Find a basis for each eigenspace.

  For $\lambda = 1$:

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \implies \mathcal{B}_1 = \left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}$$

  For $\lambda = 2$:

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \implies \mathcal{B}_2 = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

- **Step 3:** Conclude if A is diagonalizable. If so, find S and D.

  Since the dimension of each eigenspace is the same as the multiplicity of the corresponding eigenvalue, we have A is diagonalizable. Following the Diagonalization Theorem we have

$$S = \begin{bmatrix} -1/2 & -1 \\ 1 & 1 \end{bmatrix} \qquad \text{and } D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- **Step 4:** Check that $A = S^{-1}DS$ which is equivalent to $SA = DS$.

  You can do this last step on your own.

Example 2. Diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$, if possible. The eigenvalues are $\lambda = 0$ and $\lambda = -1$ with multiplicity 2.

- **Step 1:** Find all eigenvalues.

  Already given in the problem; $\lambda = 0$ and $\lambda = -1$ with multiplicity 2.

- **Step 2:** Find a basis for each eigenspace.

  For $\lambda = -1$:

$$\begin{bmatrix} 2 & 1 & 1 \\ -2 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathcal{B}_1 = \left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

  For $\lambda = 0$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathcal{B}_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- **Step 3:** Conclude if A is diagonalizable. If so, find S and D.

  Since the dimension of the eigenspace corresponding to $\lambda = -1$ is 2 (which is the multiplicity of $\lambda = 2$) and the dimension of the eigenspace corresponding to $\lambda = 0$ is 1 (which is the multiplicity of $\lambda = 0$) we know A is diagonalizable.

$$S = \begin{bmatrix} -1/2 & -1/2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Step 4:** Check that $A = S^{-1}DS$ which is equivalent to $SA = DS$.

  You can do this last step on your own.

Example 3. Diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, if possible.

- **Step 1:** Find all eigenvalues.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & & 1-\lambda \end{bmatrix}\right) = (2-\lambda)\det\begin{bmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$
$$= (2-\lambda)(2-\lambda)(1-\lambda)$$

Therefore, the eigenvalues are $\lambda = 2$ with multiplicity 2 and $\lambda = 1$ with multiplicity 1.

- **Step 2:** Find a basis for each eigenspace.

For $\lambda = 1$:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathcal{B}_1 = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- **Step 3:** Conclude if A is diagonalizable. If so, find S and D.

Since the dimension of the eigenspace corresponding to $\lambda = 2$ is 2 (which is the multiplicity of $\lambda = 2$) and the dimension of the eigenspace corresponding to $\lambda = 1$ is 1 (which is the multiplicity of $\lambda = 1$) we know A is diagonalizable.

$$S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \text{and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 4. Diagonalize $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$, if possible.

A is *not* diagonalizable. The eigenvalues are $\lambda = 2$ with multiplicity 2 and $\lambda = 3$ with multiplicity 2. Observe, for $\lambda = 3$ the eigenspace has dimension 1 which is not equal to the multiplicity of $\lambda = 3$.

For $\lambda = 3$: we have $A - 3I$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \mathcal{B}_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Textbook Practice Problems: Section 5.3 (page 288-289) # 1, 3, 5, 11, 18, 20, 25, 26.

END OF SECTION 5.3