Section 4.2: Null Space, Column Space, and Linear Transformations

Definition: The **null space** of a $m \times n$ matrix A, written $\text{Nul}(A)$, is the set of all solutions of the homogeneous equation $Ax = 0$.

$$\text{Nul}(A) = \{x : x \in \mathbb{R}^n \text{ and } Ax = 0\}.$$

That is the set of all vectors $x \in \mathbb{R}^n$, such that Ax equals 0.

Example 1. Let $A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}_{3\times 5}$. Are the vectors $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ in $\text{Nul}(A)$?

Solution: For $\vec{u}$ and $\vec{v}$ to be in $\text{Nul}(A)$, we need to check if $A\vec{u} = 0$ and $A\vec{v} = 0$.

$$A\vec{u} = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 + -2 + -3 + 6 + 0 \\ -2 + -4 + -3 + 9 + 0 \\ 1 + 2 + -3 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$A\vec{v} = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 + -5 + -1 + 0 + 2 \\ 8 + -10 + -1 + 06 \\ -4 + 5 + -1 + 0 + -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -6 \end{bmatrix}.$$

Therefore, $\vec{u} \in \text{Nul}(A)$ and $\vec{v} \notin \text{Nul}(A)$.

Theorem 2: For matrix $A_{m\times n}$, the $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Proof. First note, $\text{Nul}(A)$, is a subset of $\mathbb{R}^n$ since for $x \in \text{Nul}(A)$ we have x is a $n \times 1$ vector. Furthermore, $\vec{0} \in \text{Nul}(A)$, since $A0 = 0$. This is always true since the trivial solution is always a solution to the homogeneous solution. Now let $\vec{u}, \vec{v} \in \text{Nul}(A)$ and c be a scalar. Then we know $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$ and we have,

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0} \qquad \text{and} \qquad A(c\vec{u}) = c(A\vec{u}) = c \cdot \vec{0} = \vec{0}.$$

Therefore, we have $\text{Nul}(A)$ satisfies the 3 conditions of a subspace and thus $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$. $\square$

Example 2. Let $H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : 2a + 3b - c = 0 \text{ and } a + b - d = 0 \right\}$. Is H a subspace of $\mathbb{R}^4$?

Solution: Observe, $\begin{array}{ccccccccc} 2a & + & 3b & - & c & - & d & = & 0 \\ a & + & b & & & - & d & = & 0 \end{array} \implies \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, we have a vector in H is a solution to the matrix equation $Ax = 0$, thus $H = \text{Nul}(A)$. By Theorem 2 above , H a subspace of $\mathbb{R}^4$.

Remark: We can find an explicit description of $\text{Nul}(A)$ (i.e. writing it as the span of a set of vectors) by finding the RREF of A and expressing our solution set in parametric vector form.

Example 3. Let us consider $A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}$. Find a spanning set for $\text{Nul}(A)$.

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 & 0 \\ 2 & -2 & -1 & 3 & 3 & 0 \\ -1 & 1 & -1 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} x_2 - x_4 - 2x_5 \\ x_2 \\ x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, we have $\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Definition 2: The **column space** of a $m \times n$ matrix A, written $\text{Col}(A)$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \vec{a_1} & \dots & \vec{a_n} \end{bmatrix}$, then

$$\text{Col}(A) = \text{Span}\left\{\vec{a_1}, \dots, \vec{a_n}\right\}.$$

Theorem 3: For matrix $A_{m\times n}$, the $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Proof. Since by definition $\text{Col}(A)$ is the span of vectors in $\mathbb{R}^m$, from Section 4.1 we have $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$. $\square$

Example 4. Let $A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ in $\text{Col}(A)$.

Solution: For $\vec{u}$ and $\vec{v}$ to be in $\text{Col}(A)$, $\vec{u}$ and $\vec{v}$ need to be in the span of the the column vectors of A. This is the same thing as determining if the augmented matrices $[A|u]$ and $[A|v]$ are consistent.

$$[A|u] = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 & 1 \\ 2 & -2 & -1 & 3 & 3 & 2 \\ -1 & 1 & -1 & 0 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \vec{u} \notin \text{Col}(A)$$

$$[A|v] = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 & 2 \\ 2 & -2 & -1 & 3 & 3 & 5 \\ -1 & 1 & -1 & 0 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v} \in \text{Col}(A).$$

Therefore $\vec{u}$ is not in the column space of A and $\vec{v}$ is in in the column space of A.

Example 5. Let $H = \left\{ \begin{bmatrix} 3a+2b \\ 2b-4c \\ 3c \end{bmatrix} |, b, c \in \mathbb{R} \right\}$. Is $H = \text{Col}(A)$ for some matrix A.

Solution: Observe, $\begin{bmatrix} 3a+2b \\ 2b-4c \\ 3c \end{bmatrix} = a\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right\}$.

So $H = \text{Col}(A)$ where $A = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}$.

Definition 3: A **linear transformation** T from a vector space V to a vector space W (written $T : V \to W$) is a rule that assigns to each vector $\vec{v} \in V$ a unique vector $T(\vec{v}) \in W$, such that

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in V and
- $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in V and scalars c.

Definition 4: The **kernel** (or **null space**) of such T is the set of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$.

Definition 5: The **range** of T is the set of all $\vec{w} \in W$ of the form $T(\vec{v})$ for some $\vec{v}$ in V.

Example 6. Let $V = \mathbb{P}_2$ and $W = \mathbb{R}^2$. Let $T : V \to W$ by $T(\mathbf{p}) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

- Let $\mathbf{p}(t) = 3 + 5t + 7t^2$. Compute $T(\mathbf{p})$.

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} 3 + 5(0) + 7(0)^2 \\ 3 + 5(1) + 7(1)^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

- Show T is linear. Let $\mathbf{p} = a_0 + a_1x + a_2x^2$ and $\mathbf{q} = b_0 + b_1x + b_2x^2$ be in $\mathbb{P}_2$. Then

$$\begin{aligned} T\left((\mathbf{p} + \mathbf{q})\right) &= \begin{bmatrix} (\mathbf{p} + \mathbf{q})(0) \\ (\mathbf{p} + \mathbf{q})(1) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(1) + \mathbf{q}(1) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} \\ &= T(\mathbf{p}) + T(\mathbf{q}) \\ T(c\mathbf{p}) &= \left[c\mathbf{p}(0), c\mathbf{p}(1)\right] \\ &= c\left[\mathbf{p}(0), \mathbf{p}(1)\right] \\ &= cT(\mathbf{p}) \end{aligned}$$

- Determine the $\ker(T)$.

  Let $\mathbf{p}(x) = a_0 + a_1x + a_2x^2 \in \mathbb{P}_2$. For $\mathbf{p} \in \ker(T)$ we have $\mathbf{p}(0) = 0$ and $\mathbf{p}(1) = 01$.

$$\mathbf{p}(0) = 0 \implies a_0 = 0 \quad \text{and} \quad \mathbf{p}(1) = 0 \implies a_1 + a_2 = 0 \implies a_1 = -a_2$$

  So we have $\mathbf{p} = -a_2x + a_2x^2 = a_2x(x-1)$. So $\ker(T) = \{\mathbf{p} \in \mathbb{P}_2 | \mathbf{p}(x) = a_2x(x-1) \text{ for } a_2 \in \mathbb{R}\}$.

- Determine the range(T).

  Solution: The range$(T) = \mathbb{R}^2$. Let $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. We wish to find a polynomial in $\mathbb{P}_2$ such that $T(p(x)) = \begin{bmatrix} a \\ b \end{bmatrix}$. Observe, $\mathbf{p}(x) = a + (b-a)x$ has

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} a \\ a + (b-a) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Textbook Practice Problems: Section 4.2 (page 208-209) # 3, 5, 7-16, 17, 19, 25, 26.

END OF SECTION 4.2