Section 4.1: Vector Spaces and Subspaces

Definition 1. A vector space is a nonempty set V of objects called vectors along with addition and scalar multiplication such that the following properties hold for all vectors u, v and W in V and all scalars c and d:

1. (Closed under addition) $u + v$ is in V
2. (Closed under scalar multiplication) cv is in V
3. (Commutativity of addition) $u + v = v + u$
4. (Associativity of addition) $(u + v) + w = u + (v + w)$
5. (Additive identity) $u + 0 = u = 0 + u$
6. (Additive Inverse) $u + (-u) = 0 = (-u) + u$
7. (First distributive law) $c(u + v) = cu + cv$
8. (Second distributive law) $(c + d)u = cu + du$
9. (Relation to ordinary multiplication) $c(du) = (cd)u$
10. (Multiplicative identity) $1u = u$.

Examples of Vector Spaces

1. $\mathbb{R}^n := \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : \ x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}$. Algebraic properties of $\mathbb{R}^n$ in our textbook on page 27, show that the 10 properties of a vector space hold for $\mathbb{R}^n$.

2. $\mathbb{P}_n := \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_0, \ldots a_n \in \mathbb{R}\}$. This vector space is the set of all polynomials of degree at most n where the coefficients are all real numbers. Let us consider consider the vector space $\mathbb{P}_4$. Below are examples of vectors in $\mathbb{P}_4$ and vectors not in $\mathbb{P}_4$:

   (a) $p(x) = 2x^3 - x + 8$, is in $\mathbb{P}_4$ since $p(x)$ has degree at most 4 and all coefficients are real numbers.

   (b) $q(x) = \dfrac{-1}{8}x^4 - x^2$, is in $\mathbb{P}_4$ since $q(x)$ has degree at most 4 and all coefficients are real numbers.

   (c) $h(x) = 2x^{1/2} - x + 1$, is not in $\mathbb{P}_4$ since $h(x)$ is not a polynomial because of the $2x^{1/2}$ term.

3. $M_{2\times 2} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$. This is the vector space of all 2×2 matrices with real number entries.

   (a) $A = \begin{bmatrix} 2 & 3 \\ -8 & 4 \end{bmatrix}$, is in $M_{2\times 2}(\mathbb{R})$ since A is a 2×2 matrix with real number entries.

   (b) $B = \begin{bmatrix} 2 & 3 & 1 \\ -8 & 4 & 0 \end{bmatrix}$, is not in $M_{2\times 2}(\mathbb{R})$ since B is not a 2×2 matrix.

4. $C[a, b] = \{$ all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ $\}$. This vector space is the set of all continuous functions $f(x) : [a, b] \to \mathbb{R}$. Let us consider consider the vector space $C[-1, 1]$. Below are examples of vectors in $C[-1, 1]$ and vectors not in $C[-1, 1]$:

   (a) $f(x) = x + 1$, is in $C[-1, 1]$ since $f(x)$ is continuous on the closed interval $[-1, 1]$.

   (b) $g(x) = \frac{1}{x}$, is not in $C[-1, 1]$ since $g(x)$ is not continuous on the entire closed interval $[-1, 1]$. In particular $g(0)$ is undefined.

Definition 2. A subspace of a vector space V is a nonempty subset H that satisfy the following properties:

(a) The zero vector of V is in H

(b) If u, v are in H, then $u + v$ is in H

(c) If c is a scalar and u is in H, then cu is in H

Examples of Subspaces

1. $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

   To show H is a subspace, we must show it satisfies the 3 condition of a subspace:

   - H contains the zero vector of $\mathbb{R}^3$, since if we let $s = 0$ and $t = 0$, we have
   $$\begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H.$$

   - Now we must show for $x, y \in H$, tthat $x + y \in H$. Let $x, y \in H$. Then we have $x = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} q \\ r \\ 0 \end{bmatrix}$ for $s, t, q, r \in \mathbb{R}$. Then we have
   $$x + y = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} q \\ r \\ 0 \end{bmatrix} = \begin{bmatrix} s+q \\ t+r \\ 0 \end{bmatrix}.$$
   Since $s + q$ and $t + r$ are also real numbers we have $x + y \in H$.

   - Lastly, we show $cx \in H$ for $c \in \mathbb{R}$ and $x \in H$. For $x = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \in H$ and $c \in \mathbb{R}$ we have
   $$cx = c \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} cs \\ ct \\ 0 \end{bmatrix}.$$
   Since cs, ct are real numbers we have $cx \in H$.

2. $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since vectors in $\mathbb{R}^3$ have 3 entries while vectors in $\mathbb{R}^2$ have 2 entries.

3. Let $W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} : s,t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

   - W contains the zero vector of $\mathbb{R}^3$, since if we let $s = 0$ and $t = 0$, we have
   $$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = \begin{bmatrix} 0+3(0) \\ 0-0 \\ 2(0)-0 \\ 4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in w.$$

   - Now we must show for $x, y \in w$, that $x + y \in w$. Let $x, y \in W$. Then we have $x = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$ and $y = \begin{bmatrix} q+3r \\ q-r \\ 2q-r \\ 4r \end{bmatrix}$ for $s, t, q, r \in \mathbb{R}$. Then we have
   $$x + y = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} + \begin{bmatrix} q+3r \\ q-r \\ 2q-r \\ 4r \end{bmatrix} = \begin{bmatrix} s+3t+q+3r \\ s-t+q-r \\ 2s-t+2q-r \\ 4t+4r \end{bmatrix} = \begin{bmatrix} (s+q)+3(t+r) \\ (s+q)-(t+r) \\ 2(s+q)-(t+r) \\ 4(t+r) \end{bmatrix}.$$
   Since $x + y$ can be written in the same form as vectors in W, we have $x + y \in W$.

   - Lastly, we show $cx \in W$ for $c \in \mathbb{R}$ and $x \in W$. For $x = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} \in W$ and $c \in \mathbb{R}$ we have
   $$cx = c\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = \begin{bmatrix} (cs)+3(ct) \\ cs-ct \\ 2(cs)-(ct) \\ 4(ct) \end{bmatrix}.$$
   Since cs, ct are real numbers we have $cx \in W$.

Theorem 3. *If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are in a vector space V, then span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a subspace of V.*

This theorem tells use if we express a set W as a span (the set of all linear combinations) of vectors from your vector space, then W is a subspace.

Example 4. Let $W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} : s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

In the previous example we showed W was a subspace of $\mathbb{R}^4$ by showing the 3 conditions of a subspace hold. Another way to show W is a subspace is observing for $x in W$ we have

$$x = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

This gives us that $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$, and applying Theorem 3 above we have W is a subspace of $\mathbb{R}^4$.

Textbook Practice Problems: Section 4.1 (page 197-199) # 3, 5, 9-11, 13, 15-18, 21, 22, 24, 25.

END OF SECTION 4.1