Orthogonal Sets

Definition: A set of vectors $\{\vec{u}_1, \ldots, \vec{u}_p\}$ in $\mathbb{R}^n$ is an **orthogonal set** if each pair of distinct vectors is orthogonal.

Example 1. The standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is an orthogonal set.

Example 2. Let $\vec{u} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 11 \\ -1 \\ 7 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 11 - 4 - 7 = 0 \\ \vec{u} \cdot \vec{w} &= 3 - 8 + 5 = 0 \\ \vec{v} \cdot \vec{w} &= 33 + 2 - 35 = 0 \end{aligned}$$

So the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthogonal set.

Theorem 4: If $\mathcal{S} = \{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then $\mathcal{S}$ is linearly independent and hence a basis for the subspace of $\mathbb{R}^n$ spanned by $\mathcal{S}$.

Example 3. Looking back at our previous example, since $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthogonal set, by Theorem 4, it is also linearly independent and hence a basis for the subspace of $\mathbb{R}^3$ spanned by $\{\vec{u}, \vec{v}, \vec{w}\}$. But three linearly independent vectors in $\mathbb{R}^3$ forms a basis for $\mathbb{R}^3$, and so $\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$.

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W which is an orthogonal set.

Remark: For a subspace W, we can write every vector in W uniquely as a linear combination of the vectors in a basis for W. If the basis is an orthogonal basis, we can compute the weights easily.

Theorem 5: Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for the subspace W of $\mathbb{R}^n$. For each $\vec{w}$ in W, the weights in $\vec{w} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$ are given by

$$c_I = \frac{\vec{w} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}$$

Example 4. Let $\mathcal{B}$ be our orthogonal basis for $\mathbb{R}^3$ from before, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 11 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} \right\}$. Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the weights of $\vec{x}$ with respect to $\mathcal{B}$.

We wish to find c_1, c_2, c_3 such that $\vec{x} = c_1 \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 11 \\ -1 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$, i.e. $[\vec{x}]_{\mathcal{B}}$

Using Theorem 5 we have

$$c_1 = \frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = \frac{1+8-3}{1+16+1} = \frac{6}{18} = 1/3$$
$$c_2 = \frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} = \frac{11-2+21}{121+1+49} = 30/171$$
$$c_3 = \frac{\vec{x}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3} = \frac{3-4-15}{9+4+25} = \frac{-16}{38} = -8/19$$

So we have $\vec{x} = \frac{1}{3}\vec{u}_1 + \frac{30}{171}\vec{u}_2 - \frac{8}{19}\vec{u}_3$.

Orthogonal Projection: Let $\vec{u}$ be a nonzero vector in $\mathbb{R}^n$. Let $\vec{y}$ be any vector in $\mathbb{R}^n$. Consider the problem of decomposing the vector $\vec{y}$ as the sum of two vectors, one a scalar multiple of $\vec{u}$ and the other orthogonal to $\vec{u}$. That is we want:

$$\vec{y} = \hat{y} + \vec{z},$$

where $\hat{y}$ is a multiple of $\vec{u}$ and $\vec{z}$ is orthogonal to $\vec{u}$. Since $\hat{\vec{y}}$ is a scalar multiple of $\vec{u}$ we have

$$\hat{\vec{y}} = \alpha\vec{u},$$

For some scalar α. This gives us that

$$\vec{y} = \hat{y} + \vec{z} = \alpha\vec{u} + \vec{z} \implies \vec{z} = \vec{y} - \alpha\vec{u}$$

For $\vec{z}$ to be orthogonal to $\vec{u}$, this means that $\vec{z}\cdot\vec{u} = 0$, or

$$\vec{z}\cdot\vec{u} = (\vec{y}-\alpha\vec{u})\vec{u} = \vec{y}\cdot\vec{u} - \alpha\vec{u}\cdot\vec{u} = 0 \implies \alpha = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}} \text{ and } \hat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\cdot\vec{u}$$

Definition: The vector $\hat{y}$ is called the **orthogonal projection of $\vec{y}$ onto $\vec{u}$**. The vector $\vec{z}$ is called the **component of $\vec{y}$ orthogonal to $\vec{u}$**.

$$\hat{y} = \text{proj}_L\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

Remark: If we replace $\vec{u}$ with $c\vec{u}$, for some nonzero scalar c, then we get the same decomposition. So, the projection is determined not by $\vec{u}$, but by the subspace spanned by $\vec{u}$, that is $L = \text{Span}\{\vec{u}\}$ (or the line through $\vec{0}$ and $\vec{u}$).

Sometimes, $\hat{y}$ is denoted by $\mathbf{proj}_L\vec{y}$ and is called the **orthogonal projection of $\vec{y}$ onto L**.

$$\hat{y} = \mathbf{proj}_L\vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \quad\text{and}\quad \vec{z} = \vec{y} - \hat{\vec{y}}$$

Example 5. Let $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Find the orthogonal projection of $\vec{y}$ onto $\vec{u}$. Then write $\vec{y}$ as the sum of two orthogonal vectors, one in the $L = \text{Span}\{\vec{u}\}$ and one orthogonal to $\vec{u}$.

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{14+6}{49+1} \vec{u} = 20/50 \vec{u} = 2/5 \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$
$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

Observe, we have $\hat{y}$ and $\vec{z}$ are in fact orthogonal:

$$\hat{y} \cdot \vec{z} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} \cdot \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix} = \begin{bmatrix} -56/25 + 56/25 \end{bmatrix} = 0$$

Since the line segment between $\vec{y}$ and $\hat{y}$ is perpendicular to L, by construction of $\hat{y}$, the point identified with $\hat{y}$ is the "closet point of L to $\vec{y}$".

Example 6. Find the distance from $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ to $L = \text{Span}\{\vec{u}\}$.

$$||\vec{y} - \hat{y}||^2 = ||\vec{z}||^2 = \vec{z} \cdot \vec{z} = \frac{16}{25} + \frac{784}{25} = \frac{800}{35} = 32$$
$$||\vec{y} - \hat{y}|| = \sqrt{32}$$

Definition: A set $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors.

Example 7. The standard basis for $\mathbb{R}^n$ is an orthonormal basis.

Example 8. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 11 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} \right\}$ is NOT an orthonormal basis.

Example 9. Let $\mathcal{B} = \left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis.

END OF SECTION 6.2