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Exam 4

1) Since $m = \#$ of rows $= 4$, the number of pivot positions cannot exceed the number of rows or the number of columns, hence the largest possible rank of C is 4.
By the rank theorem, $\operatorname{rank} A + \dim \operatorname{Nul} A = n$, so if $\operatorname{rank} A$ is at its largest possible value of 4, then $\dim \operatorname{Nul} A$ must be 1, and the smallest possible dimension of $\operatorname{Nul} A$ is 1 also.

2) a) $\begin{bmatrix} 1 & -4 & 0 & 5 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ \frac{1}{2}x_3 \\ x_3 \\ 0 \end{bmatrix} \Rightarrow \operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$ and $\dim \operatorname{Nul} A = 1$

b) $\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \right\}$ and $\dim \operatorname{Col} A = 3$

c) $\operatorname{Row} A = \operatorname{Span}\{(1,0,-2,0), (0,1,-\frac{1}{2},0), (0,0,0,1)\}$
$\dim \operatorname{Row} A = 3$

d) $\dim \operatorname{Col} A = \dim \operatorname{Row} A = \operatorname{rank} A = 3 = \#$ of pivot columns
$\dim \operatorname{Nul} A = 1 = \#$ of non-pivot columns
$\operatorname{rank} A + \dim \operatorname{Nul} A = n$
$3 + 1 = 4$ ✓

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3) $$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -\sqrt{2}-1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-\sqrt{2}-1) + 1(1) \\ 1(-\sqrt{2}-1) + 1(4) \end{bmatrix}$$

$$= \begin{bmatrix} -2\sqrt{2} - 2 + 1 \\ -\sqrt{2} - 1 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2\sqrt{2} - 1 \\ -\sqrt{2} + 3 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} -\sqrt{2} - 1 \\ 1 \end{bmatrix} \Rightarrow \lambda = -\sqrt{2} + 3$$

2+2= ith

4) $$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 & 1 & 0 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (4-\lambda) \det \begin{bmatrix} 4-\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$= (4-\lambda)\left[-\lambda \det \begin{bmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix}\right]$$

$$= (4-\lambda)\left[-\lambda(4-\lambda)(2-\lambda)\right] = 0$$

$\lambda = 4$ (multiplicity 2), 0 (multiplicity 1), 2 (multiplicity 1)

A is not invertible by the IMT since it has an eigenvalue of 0.

5) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} -3 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

A is <u>not</u> diagonalizable since the sums of the dimensions of the eigenspaces do not add up to $n = 3$.

6) $A^K = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3^K & 0 \\ 0 & -2^K \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 \cdot -3^K & -2 \cdot -2^K \\ 2 \cdot -3^K & -1 \cdot -2^K \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

$-3 \cdot -3^k$

$= \begin{bmatrix} -1(3 \cdot -3^k) + (-2)(-2 \cdot -2^k) & 2(3 \cdot -3^K) + 3(-2 \cdot -2^k) \\ -1(2 \cdot -3^k) + (-2)(-1 \cdot -2^k) & 2(2 \cdot -3^K) + 3(-1 \cdot -2^k) \end{bmatrix}$

$= \begin{bmatrix} (-3)^{k+1} (-2)^{k+2} & 6(-3^K) + 3(-2^{k+1}) \\ -2(-3)^{k} - (-2^{k+1}) & 4(-3^K) - 3(-2^K) \end{bmatrix}$

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7) a) $\begin{bmatrix} 2 & 3 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2(1) + 3(2) - 6(2) =$
$= 2 + 6 - 12 = -4$

b) $||v|| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

$\frac{1}{||v||} v = \begin{bmatrix} 2/7 \\ 3/7 \\ -6/7 \end{bmatrix}$

8) a) $\begin{bmatrix} 8 \\ -5 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = (8)(-2) + (-5)(-3) = 6 + 15 = 21 \neq 0$

Since $u \cdot v \neq 0$, then the vectors are not orthogonal and do not satisfy the Pythagorean Theorem.

b) $\begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 12(2) + 3(-3) + (-5)(3)$
$24 - 9 - 15 = 0$

Since $u \cdot v = 0$, then the vectors are orthogonal and satisfy the Pythagorean Theorem

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