Section 5.1: Eigenvectors and Eigenvalues

Definition: An **eigenvector** of A is a nonzero vector $\vec{x}$ with $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is an **eigenvalue** of A if there is a nontrivial solution to $A\vec{x} = \lambda\vec{x}$, such an $\vec{x}$ is an eigenvector corresponding to λ.

Example 1. Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

For $\vec{u}$ and $\vec{v}$ to be eigenvectors we need $A\vec{u} = \lambda_1\vec{u}$ and $A\vec{v} = \lambda_2\vec{v}$ for some scalars λ_1 and λ_2.

$$A\vec{u} = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \implies \lambda_1 = 7$$

$$A\vec{v} = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \neq \lambda_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \implies \vec{v} \text{ is not an eigenvector.}$$

Example 2. Let $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$ and $\lambda = 3$. Is 3 an eigenvalue for A?

For 3 to be an eigenvalue, we need $A\vec{x} = 3\vec{x}$ for some nonzero vector $\vec{x}$. That is we need $A\vec{x} - 3\vec{x} = \vec{0}$ to have a nontrivial solution.

$$\begin{aligned} A\vec{x} - 3\vec{x} &= \vec{0} \\ (A - 3I)\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \vec{x} &= \vec{0} \\ \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \vec{x} &= \vec{0} \end{aligned}$$

We want to know if $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \vec{x} = \vec{0}$ has a nontrivial solution. From here we set up the following augmented matrix $\begin{bmatrix} 2 & 2 & 0 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since we have a free variable, we know the matrix equation $A\vec{x} = 3\vec{x}$ has a nontrivial solution, hence 3 is an eigenvalue for A.

FACTS:

- λ is an eigenvalue of A if and only if $(A-\lambda I)\vec{x}=\vec{0}$ has non trivial solutions.

- **Def.** The set of solutions of $(A-\lambda I)\vec{x}=\vec{0}$, called the **eigenspace corresponding to** λ, is the null space of the matrix $(A-\lambda I)$, i.e. $\mathrm{Nul}((A-\lambda I))$.

- The eigenspace contains the zero vector, which is excluded from being an eigenvector by definition. Every *nonzero vector* in the eigenspace is an eigenvector corresponding to λ.

Example 3. Let $A=\begin{bmatrix}4 & 0 & -1\\ 3 & 0 & 3\\ 2 & -2 & 5\end{bmatrix}$. Find a basis for the eigenspace corresponding to $\lambda=3$.

Solution: A basis for the eigenspace corresponding to $\lambda=3$ is the basis for the null space of the matrix $A-3I$. A basis for the eigenspace corresponding to $\lambda=3$ is a basis for $\mathrm{Nul}(A-3I)$.

$$A-3I=\begin{bmatrix}4 & 0 & -1\\ 3 & 0 & 3\\ 2 & -2 & 5\end{bmatrix}-\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & -1\\ 3 & -3 & 3\\ 2 & -2 & 2\end{bmatrix}$$

The eigenspace corresponding to $\lambda=3$ is the null space of the matrix $\begin{bmatrix}1 & 0 & -1\\ 3 & -3 & 3\\ 2 & -2 & 2\end{bmatrix}$.

To find a basis for the null space we row reduce the following matrix

$$\begin{bmatrix}1 & 0 & -1 & 0\\ 3 & -3 & 3 & 0\\ 2 & -2 & 2 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\implies \text{A solution has the form } \vec{x}=x_3\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}.$$

So, a basis for the eigenspace corresponding to $\lambda=3$, is $\mathcal{B}=\left\{\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}\right\}$.

Theorem The eigenvalues of a triangular matrix are the entries on the diagonal.

Example 4. Let $A=\begin{bmatrix}3 & 1 & 2\\ 0 & -4 & 2\\ 0 & 0 & 0\end{bmatrix}$. Then the eigenvalues of A are $\lambda=3,-4,0$. Notice for A, we have that A is not invertible as the $\det(A)=3(-4)(0)=0$.

Theorem: Invertible Matrix Theorem (Cont.) Let A be an $n \times n$ matrix. Then A is invertible if and only if

(s) The number 0 is *not* an eigenvalue of A

(t) The determinant of A is *not* zero.

Theorem: If $\vec{v}_1, \dots \vec{v}_p$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_p$ of an $n \times n$ matrix A, then $\{\vec{v}_1, \dots \vec{v}_p\}$ are linearly independent.

Example 5. Let A be a 3×3 matrix and suppose $\vec{u}, \vec{v}, \vec{w}$ are nonzero vectors in $\mathbb{R}^3$ such that $A\vec{u} = 2\vec{u}$, $A\vec{v} = 3\vec{v}$, and $A\vec{w} = -4\vec{w}$. Is the set $\mathcal{B} = \{\vec{u}, \vec{v}, \vec{w}\}$ a basis for $\mathbb{R}^3$.

Solution: Observe since $\vec{u}, \vec{v}, \vec{w}$ are nonzero vectors and there exist scalars $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = -4$ such that $A\vec{u} = 2\vec{u}$, $A\vec{v} = 3\vec{v}$, and $A\vec{w} = -4\vec{w}$, we have $\vec{u}, \vec{v}, \vec{w}$ are eigenvectors of A. Furthermore, by the above theroem since $\vec{u}, \vec{v}, \vec{w}$ correspond to distinct eigenvectors, thhen $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent. Lastyle, by the Basis Theorem in chapter 4 since we know $\dim(\mathbb{R}^3) = 3$ and we have three linearly independent vectors in $\mathbb{R}^3$, we have the set $\{\vec{u}, \vec{v}, \vec{w}\}$ forms a basis for $\mathbb{R}^3$.

Fact: An $n \times n$ matrix A can have at most n distinct eigenvalues.

Textbook Practice Problems: Section 5.1 (page 273-274) # 1, 3, 5, 8, 10, 13, 15, 17, 18, 21, 22.

END OF SECTION 5.1