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Final Exam

1) a) FALSE

b) FALSE

c) TRUE

d) TRUE

e) FALSE

f) FALSE

g) TRUE

h) TRUE

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 4 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \underset{x_2}{0} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 2 \\ 6 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

2) a) impossible to tell – 0 could be a distinct eigenvalue

b) not invertible

c) invertible

d) invertible

e) impossible to tell

f) invertible

g) invertible

3) $$\left[\begin{array}{ccc|c} 0 & 3 & 6h & 0 \\ 1 & 2 & 6 & 2 \\ 3h & 0 & 6 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 3 & 6h & 0 \\ 3h & 0 & 6 & 3 \end{array}\right] \xrightarrow{-3hR_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2h & 0 \\ 0 & -6h & -18h+6 & -6h+3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2h & 0 \\ 0 & 0 & 12h^2-18h+6 & -6h+3 \end{array}\right]$$

no solutions

$12h^2 - 18h + 6 = 0$ $\quad -6h + 3 \neq 0$

$(3h-3)(4h-2) = 0$ $\quad h \neq \frac{1}{2}$

$h = \frac{1}{2}, 1$

a unique solution

$12h^2 - 8h + 6 \neq 0$

$h = \frac{1}{2}, 1$

infinitely many solutions

$h = \frac{1}{2}$

4) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \sim$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

$c_1 \begin{bmatrix} -24 \\ 20 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 18 \\ -15 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

5) $-(-1)\det\begin{vmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 7 \\ 0 & 2 & 8 & 2 \end{vmatrix} + 9\det\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 1 & 2 & 8 \end{vmatrix}$

$-(-1)\det\begin{vmatrix} 1 & 3 & 4 \\ -1 & 0 & 0 \\ 0 & 2 & 8 \end{vmatrix} + 9\left[-1\det\begin{vmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 1 & 2 & 8 \end{vmatrix} - 4\det\begin{vmatrix} 1 & 3 & 4 \\ -1 & 0 & 0 \\ 0 & 2 & 8 \end{vmatrix}\right]$

$-1\cdot\det\begin{vmatrix} 3 & 4 \\ 2 & 8 \end{vmatrix} + 9\left[-1\left(2\det\begin{vmatrix} 0 & 0 \\ 2 & 8 \end{vmatrix} + 1\det\begin{vmatrix} 3 & 4 \\ 0 & 0 \end{vmatrix}\right)\right]$

$-1[3\cdot 8 - 4\cdot 2] + 9(-1)(0+0)$

$-1[24-8] + 0$

-16

$\det A = -16$

6) $\text{Col}A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\text{rank}A = 3$

$\text{Row}A = \{(1,4,0,2,0), (0,0,1,-1,0), (0,0,0,0,1)\}$

$\dim \text{Row}A = 3$

$\text{Nul}A = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim \text{Nul}A = 2$

$T(u+cv) = Tu + cTv$

7) $T(u+cv)$

$$= \begin{bmatrix} u_1 + 2u_3 + u_4 + cv_1 + 2cv_3 + cv_4 \\ u_2 + u_3 + 2u_4 + cv_2 + cv_3 + 2cv_4 \\ 3u_3 + 3cv_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + 2u_3 + u_4 \\ u_2 + u_3 + 2u_4 \\ 3u_3 \end{bmatrix} + c \begin{bmatrix} v_1 + 2v_3 + v_4 \\ v_2 + v_3 + 2v_4 \\ 3v_3 \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$= T(u) + cT(v)$

$A = \begin{bmatrix} \underline{1} & 0 & 2 & 1 \\ 0 & \underline{1} & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

T is not one-to-one since the columns of A are not linearly independent (i.e. not a pivot in every column.

T is onto since the columns of A span $\mathbb{R}^3$ (i.e. a pivot in every row).

7c) $T(v)=\begin{bmatrix} a+2c+d \\ b+c+2d \\ 3c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a+2c+d=0 \Rightarrow a+d=0 \Rightarrow d=-a,\ a=-d$
$b+c+2d=0 \Rightarrow b+2d=0 \Rightarrow b=-2d,\ b=$
$3c=0 \Rightarrow c=0$

$\mathrm{Ker}(T)=\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a+2c+d=0 \text{ and } b+c+2d=0 \text{ and } 3c=0 \right\}$

8) $\begin{bmatrix} -\lambda & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 2 & 0 & 0 & 1-\lambda \end{bmatrix}$ $-(-\lambda)\det \begin{vmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = \lambda\left[-(-\lambda)\det \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix}\right]$

$= \lambda^2[(-\lambda)(1-\lambda)-(2)(1)]$
$= \lambda^2[\lambda^2-\lambda-2]$
$= \lambda^2[(\lambda+1)(\lambda-2)]$
eigenvalues $\Rightarrow \lambda = 0, 0, -1, 2$

$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \beta = \left\{ \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

9) $S = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10) $\begin{bmatrix} x \\ y \\ w \\ x+y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + w\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$W = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

$\dim W = 3$

$x \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0, \quad x \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0, \quad x \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$x_1 + x_4 = 0$
$x_2 + x_4 = 0$
$x_3 = 0$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -x_4 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix} \Rightarrow W^{\perp} = \text{Span}\left\{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

11) $u_1 \cdot u_2 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 3(2) + (-3)(2) + 0(-1) = 0$

$u_1 \cdot u_3 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 3(1) + (-3)(1) + (0)(4) = 0$

$c_1 = \frac{w \cdot u_1}{u_1 \cdot u_1}$

$u_2 \cdot u_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2(1) + 2(1) + (-1)(4) = 0$

S is not an orthonormal set since the vectors in the set are not unit vectors.

S is a basis for $\mathbb{R}^3$ since it is an orthogonal set of vectors which are nonzero in $\mathbb{R}^3$, so S is linearly independent and hence a basis for the subspace of $\mathbb{R}^3$ spanned by S.

$$c_1 = \frac{5(3) + (-3)(-3) + 1(0)}{3^2 + (-3)^2 + 0^2} = \frac{24}{18} = \frac{12}{9} = \frac{4}{3}$$

$$c_2 = \frac{5(2) + (-3)(2) + 1(-1)}{2^2 + 2^2 + (-1)^2} = \frac{3}{5}$$

$$c_3 = \frac{5(1) + -3(1) + 1(4)}{1^2 + 1^2 + 4^2} = \frac{6}{18} = \frac{3}{9} = \frac{1}{3}$$

$$b = \frac{4}{3} u_1 + \frac{3}{5} u_2 + \frac{1}{3} u_3$$

$\frac{13}{\frac{8}{39}}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

12) $\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{6+3}{4+9} u = \frac{9}{13} u = \frac{9}{13} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 27/13 \\ 9/13 \end{bmatrix}$

$\frac{26}{13} - \frac{27}{13}$ $\quad \frac{39}{13} - \frac{9}{13}$

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 27/13 \\ 9/13 \end{bmatrix} = \begin{bmatrix} -1/13 \\ 30/13 \end{bmatrix}$

$z \cdot z = \left(-\frac{1}{13}\right)^2 + \left(\frac{30}{13}\right)^2 = \left(\frac{1}{26}\right) + \left(\frac{900}{26}\right) = \frac{901}{26}$

$\|\vec{y} - \hat{y}\| = \sqrt{\frac{901}{26}}$