Section 2.3: Characterization of Invertible Matrices

Theorem 1. *(The Invertible Matrix Theorem:) Let A be a square matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false:*

(a) A is an invertible matrix

(b) A is row equivalent to the $n \times n$ identity matrix I_n

(c) A has n pivot positions

(d) The equation $A\vec{x} = \vec{0}$ has only the trivial solution

(e) The columns of A form a linearly independent set

(f) The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one

(g) The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b} \in \mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$

(i) The linear transformation $T(\vec{x}) = A\vec{x}$ is onto

(j) There is an $n \times n$ matrix C such that $CA = I_n$

(k) There is an $n \times n$ matrix D such that $AD = I_n$

(ℓ) A^T is an invertible matrix.

Example 2. Determine if the following matrices are invertible

(a) $A = \begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$

Solution: A is not invertible since it has a row of zeroes and therefore A can not have a pivot in every row. Hence, the columns of A do not span $\mathbb{R}^n$ and condition (h) above fails.

(b) $B = \begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$

Solution: Observe, using row operations we have

$$\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since B row reduces to a matrix with 4 pivots condition (c) above is satisfied and therefore B is invertible.

(c) $C = \begin{bmatrix} 3 & 4 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: Observe, C is already in REF and has 3 pivots therefore condition (c) above is satisfied and therefore C is invertible.

(d) $D = \begin{bmatrix} 5 & 3 & 1 & 6 & -2 \\ 6 & 0 & 2 & 0 & -8 \\ 6 & 1 & 3 & 2 & 9 \\ 9 & -2 & 4 & -4 & -5 \\ 0 & 5 & 2 & 10 & 4 \end{bmatrix}$

Solution: Observe, column 4 is a scalar multiple column 1. Thus, the columns of D do not form a linearly independent set and hence condition (e) fails. Therefore, D is not invertible.

Definition 3. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\vec{x})) = \vec{x} \text{ and } T(S(\vec{x}))$$

Theorem 4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given by $T^{-1}(\vec{x}) = A^{-1}\vec{x}$ is the inverse of T.*

Example 5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}$. Determine if T is invertible. If so, find the inverse transformation.

Since $\det(A) = 6 - 3 = 3$ we know A is invertible. The inverse of A is

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = 1/3 \begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}.$$

Therefore, from Theorem 4 above we have $T^{-1}(\vec{x}) = A^{-1}\vec{x}$, that is

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Example 6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T((x_1, x_2, x_3) = (4x_1, x_1, x_2 - 3x_3)$. Determine if T is invertible and find T^{-1}.

First observe the standard matrix A for the linear transformation T is

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}.$$

Since we have,

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the standard matrix A is not invertible, then the linear transformation T is not invertible and thus T^{-1} does not exists.

Textbook Practice Problems: Section 2.3 (page 117) # 1-8, 22, 23, 24, 33, 34.

END OF SECTION 2.3