<u>**Introduction to Determinants**</u>

Remark 1. Recall

- For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the $\det(A) = ad - bc$. A 2×2 matrix is invertible if and only if $\det(A) \neq 0$.

We extend this useful fact to larger $n \times n$ matrices. Let A be a $n \times n$ matrix. Recall A_{ij} s the entry in the row i and column j of A.

Let A_{ij} denote the submatrix formed by deleting row i and column j.

Example 2. Let $A = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 2 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$. Compute A_{32}, A_{23}, A_{33}.

$$A_{32} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix} \qquad A_{33} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

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Remark 3. Note A_{ij} will be an $(n-1) \times (n-1)$ matrix.

Definition 4. For $n \geq 2$, the **determinant** of a $n \times n$ matrix $A = [a_{ij}]$, denoted either by $\det(A)$ or $|A|$ is

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(A_{1n}) = \sum_{j=1}^{n}(-1)^{1+j}a_{ij}\det(A_{1j})$$

Using the definition above and notation, we have

Example 5. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 4 & 5 & -1 \end{bmatrix}$. Find the determinant of A.

$$\begin{aligned}
\det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) \\
&= 3 \cdot \det \begin{bmatrix} 0 & 2 \\ 5 & -1 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix} \\
&= 3(0-10) - 2(-1-8) + 1(5-0) \\
&= 3(-10) - 2(-9) + 1(5) \\
&= -30 + 18 + 5 \\
&= -7.
\end{aligned}$$

Definition 6. Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

Remark 7. In this notation,

$$\begin{aligned} \det(A) &= a_{i11} \det(A_{11}) - a_{12} \det(A_{12}) + \cdots + (-1)^{1+n} a_{1n} \det(A_{1n}) \\ &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}, \end{aligned}$$

Which is called the cofactor expansion across the first row.

Theorem 8. *(Cofactor Expansion:) The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. That is*

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \text{ for any row } i$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \text{ for any column } j$$

Example 9. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ 4 & 5 & -1 \end{bmatrix}$. We already know $\det(A) = -7$ using cofactor expansion across the first row.

Now let's compute the determinant again, this time using cofactor expansion along the second column and using the notation above in Definition 6.

$$\begin{aligned} \det(A) &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= 2 \cdot C_{12} + 0 \cdot C_{22} + 5 \cdot C_{32} \\ &= 2 \cdot (-1)^{1+2} \det(A_{12}) + 0 \cdot (-1)^{2+2} \det(A_{22}) + 5 \cdot 5^{3+2} \det(A_{32}) \\ &= -2 \det \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} + 0 \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} - 5 \det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \\ &= -2(-1-8) + 0 - 5(6-1) \\ &= -2(-9) - 5(5) \\ &= 18 - 25 \\ &= -7. \end{aligned}$$

And we get the same answer, as we should, since the determinant of a matrix is unique!

Example 10. Compute the determinant by cofactor expansion. At each step choose a row or column that involves the least amount of computations (this means contains the most zeroes).

$$A = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$$

First we expand across the second row

$$\det(A) = (-1)^{2+3} \cdot 2 \cdot \det \begin{bmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix} = -2 \cdot \det \begin{bmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Now we expand down the second column

$$\begin{aligned} \det(A) &= -2 \cdot \left((-1)^{2+2} \cdot 3 \cdot \det \begin{bmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \right) \\ &= -2 \cdot \left(3 \cdot \det \begin{bmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \right) \\ &= -6 \cdot \det \begin{bmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \end{aligned}$$

Now we expand down the first column

$$\begin{aligned} \det(A) &= -6 \cdot \det \begin{bmatrix} 4 & 3 & -5 \\ 5 & 0 & -3 \\ 0 & -1 & 2 \end{bmatrix} \\ &= -6 \left((-1)^{1+1} \cdot 4 \cdot \det \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} + (-1)^{2+1} \cdot 5 \cdot \det \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} + (-1)^{3+1} \cdot 0 \cdot \det \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix} \right) \\ &= -6 \left(4 \left(4 - 3 \right) - 5 \left(6 - 5 \right) + 0 \right) \\ &= -6 \left(4(1) - 5(1) \right)) \\ &= -6(4 - 5) \\ &= -6(-1) \\ &= 6. \end{aligned}$$

Theorem 11. *(Determinant of Triangular Matrix) If A is a triangular matrix, then* $\det(A)$ *is the product of the entries on the main diagonal of A.*

Example 12. Let $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{bmatrix}$. Find the determinant of A.

Using the theorem above, since A is a triangular matrix (in particular a lower triangular matrix since the non-zero terms appear below the main diagonal) we have

$$\det(A) = 4 \cdot (-1) \cdot (3) \cdot (-3) = 36$$

Textbook Practice Problems: Section 3.1 (page 169-170) # 1,3,5, 9-14, 39, 30.

END OF SECTION 3.1