Section 4.3: Linearly Independent Sets; Bases

Recall a set of vectors $\{v_1, \ldots, v_p\}$ are linearly independent if the vector equation

$$c_1v_1 + c_2v_{2+} \cdots + c_pv_p = 0$$

has only the trivial solution.

Example 1. Let $V = \mathbb{P}_3$ and consider the vectors $p_1(t) = 1$, $p_2(t) = t$, and $p_3(t) = 4 - t$. Determine if the set of vectors $\{p_1, p_2, p_3\}$ are linearly independent.

Solution: The set is not linearly independent. Observe, since

$$p_3(t) = 4 - t = 4(1) + (-1)(t) = 4p_1(t) + (-1)p_2(t).$$

This means that $p_3(t)$ is a linear combination of the polynomials $p_1(t) = 1$ and $p_2(t) = t$.

Example 2. Let $V = C[0, 1]$ and consider the vectors $\sin(x)$ and $\cos(x)$. Determine if the set of vectors $\{\sin(x), \cos(x)\}$ are linearly independent.

Solution: Yes, the set is linearly independent. Since $\sin(x) \neq c\cos(x)$ for any constant $c \in \mathbb{R}$.

Example 3. Let $V = C[0, 1]$. Determine if the set of vectors $\{\sin(x)\cos(x), \sin(2x)\}$ are linearly independent.

Solution: The set is linearly dependent. Since from calculus/trig we know

$$\sin(2x) = 2\sin(x)\cos(x),$$

so the function $\sin(2x)$ is a scalar multiple of $\sin(x)\cos(x)$.

Definition 4. Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{b_1, \ldots, b_p\}$ in V is a basis for H if

1. $\mathcal{B}$ is a linearly independent set, and
2. The subspace spanned by $\mathcal{B}$ coincides with H, that is

$$H = \text{Span}\{b_1, \ldots, b_p\}.$$

Example 5. Let A be an invertible $n \times n$ matrix. The columns of A form a basis for $\mathbb{R}^n$.

- (Linearly Independent) Since A is invertible by the Invertible Matrix Theorem the columns of A form a linearly independent set (IMT part e)
- (Span$\{a_1, \ldots, a_n\} = \mathbb{R}^n$) Additionally, by the IMT the columns of A span $\mathbb{R}^n$ (IMT part h)

Example 6. Recall the e_i's for $\mathbb{R}^n$. The set $\{e_1, \ldots, e_n\}$ for a basis for $\mathbb{R}^n$. In particular, this set is called the **standard basis**.

Example 7. Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

- Observe $A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 6 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 15 & 9 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix}$
- Since A has a pivot in every column, we know the vectors v_1, v_2, v_3 are linearly independent. Also, since we have a pivot in every row Span$\{v_1, v_2, v_3\} = \mathbb{R}^3$. Therefore, $\{v_1, v_2, v_3\}$ forms a basis for $\mathbb{R}^3$.

Theorem 8. *(Spanning Set Theorem) Let $S = \{v_1, v_2, \ldots, v_p\}$ be a set in V and let $H =$ Span$\{v_1, v_2, \ldots, v_p\}$.*

- *If one of the vectors in S-say, v_k- is a linear combination of the remaining vectors in S, then the set formed by removing v_k still spans H.*
- *If $H \neq \{0\}$, some subset of S is a basis for H.*

Example 9. Let $B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for Col$(B)$.

Solution: Since B is already in RREF, we see B has three pivot columns. Observe, each non pivot column is a linear combination of the pivot columns.

$$\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + -1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

By the Spanning set theorem we can remove these vectors from a spanning set for Col(B) and get

$$\text{Col}(B) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since these three vectors are linearly independent (pivot in every column) we have these three vectors form a basis for Col(B).

Theorem 10. *The pivot columns of a matrix A form a basis for Col(B).*

Example 11. Suppose $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for Nul$(A)$ and Col(A).

$$\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, a basis for Nul(A) is

$$\text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since the set is also linearly independent it forms a basis for Nul(A). A basis for Col(A) are the pivot columns of A, so we have

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}.$$

Textbook Practice Problems: Section 4.3 (page 215-217) # 3, 5, 9, 10, 13, 14, 15, 16, 21, 22.

END OF SECTION 4.3