Section 4.6: Rank

Definition: The **row space** of a $m \times n$ matrix A, written $\text{Row}(A)$, is the set of all linear combinations of the row vectors of A. The row space of A an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

Theorem 1: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as a basis for the row space of B.

Example 1. Let us consider $A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}$. Find a basis for the row space of A, $\text{Row}(A)$, and the column space of A, $\text{Col}(A)$.

Solution: To find a basis for $\text{Row}(A)$, using Theorem 1, we put A in REF and take the nonzero rows of the REF matrix. So we have,

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we have a basis for $\text{Row}(A) = \text{Span}\,\{(1,-1,0,1,2),(0,0,1,-1,1)\}$. The pivot columns of A form a basis for $\text{Col}(A)$. Since columns 1 and 3 are pivot columns, we have a basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \right\}$.

Definition: The **rank** of a $m \times n$ matrix A, is the dimension of the column space of A.

Theorem 2 (Rank Theorem): The dimension of the column space of A and the row space of A of a $m \times n$ matrix A are equal. The common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank}(A) + \dim(\text{Nul}(A)) = n.$$

Example 2. If A is a 7×9 matrix with a 2-dimensional null space, what is the rank of A?

Solution: Since A is a 7×9 matrix and the $\dim(\text{Nul}(A)) = 2$, this means that

$$\begin{aligned} \text{rank}(A) + \dim(\text{Nul}(A)) &= n \\ \text{rank}(A) + 2 &= 9 \\ \text{rank}(A) &= 7. \end{aligned}$$

Example 3. Can a 6×9 matrix have a 2-dimensional null space?

Solution: Suppose A is a 6×9 matrix and that $\dim(\text{Nul}(A)) = 2$, this means that $\text{rank}(A) + 2 = 9$, hence the $\text{rank}(A) = 7$. But since A is a 6×9 matrix, the rank of A, which is the number of pivot column of A can not exceed 6. Therefore, a 6×9 matrix can not have a null space of dimension 2.

Theorem 3 (Invertible Matrix Theorem (Continued): Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis for $\mathbb{R}^n$

n. $\text{Col}(A) = \mathbb{R}^n$

o. $\dim(\text{Col}(A)) = n$

p. $\text{rank}(A) = n$

q. $\text{Nul}(A) = \{0\}$

r. $\dim(\text{Nul}(A)) = 0$

Example 4. For each of the following statements, please *circle* T (True) or F (False).

(a) If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A. **Solution:** False, you need to take the pivot columns of A to form a basis for the column space of A.

(b) Row operations preserve the linear independence relations among rows of A. **Solution:** True, since the nonzero rows of an echelon form of A form a basis for $\text{Row}(A)$.

(c) The dimension of the null space of A is the number of columns of A that are not pivot columns. **Solution:** True, since the number of columns that are not pivot columns in the number of free variables.

(d) The row space of A^T is the same as the column space of A. **Solution:** True, since in A^T to row become columns (and vice versa) the row space of A^T is the set of all linear combinations of the columns of A, hence the column space of A.

Textbook Practice Problems: Section 4.6 (page 238-239) # 1, 2, 3, 4, 5, 7, 10, 13, 17.

END OF SECTION 4.6