Section 5.2: Characteritic Equation

Last time: Given a $n \times n$ matrix A we can...

- Determine if a vector $\vec{x}$ is an eigenvector of A. [See if you can find λ such that $A\vec{x} = \lambda\vec{x}$.]
- Determine if λ is an eigenvalue of A. [See if $A - \lambda I$ has a free variable.]
- Find the eigenspace corresponding to λ. [Find the Null space of $(A - \lambda I)$.]

Today, we wish to answer the following question: Given A, find all of its eigenvalues.

$$\begin{aligned} \text{Recall } \lambda \text{ is an eigenvalue of } A \quad &\iff \quad A - \lambda I \text{ as a free variable} \\ &\iff \quad A - \lambda I \text{ is not invertible} \\ &\iff \quad \det(A - \lambda I) = 0. \end{aligned}$$

So we can find all eigenvalues of A if we are able to determine when $\det(A - \lambda I) = 0$.

Example 1. Let $A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$. Find all the eigenvalues of A.

$$A - \lambda I = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) = \det\left(\begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix} \right) &= (-4-\lambda)(1-\lambda) - (-1)(6) \\ &= \lambda^2 + 3\lambda - 4 + 6 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda+1)(\lambda+2) \end{aligned}$$

So $\det(A - \lambda I) = 0$ if and only if $(\lambda+1)(\lambda+2) = 0$, that is when $\lambda = -1$ and $\lambda = -2$.

Definition: The equation $\det(A - \lambda I) = 0$ is the **characteristic equation** of A. The **characteristic polynomial** is just the polynomial you get when you find $\det(A - \lambda I)$.

Remark: A scalar λ is an eigenvalue of A if and only if λ is a solution to the characteristic equation $\det(A - \lambda I) = 0$.

Example 2. Find all eigenvalues of $A = \begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 8-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 8-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}\right) = (8-\lambda)(3-\lambda)-6 = 24-11\lambda+\lambda^2-6 = \lambda^2-11\lambda+18 = (\lambda-9)(\lambda-2)$$

$$\begin{aligned} \det(A-\lambda I) = 0 &\implies (\lambda-9)(\lambda-2) = 0 \\ &\implies \lambda = 9,\ 2 \ \text{ are the eigenvalues.} \end{aligned}$$

Example 3. Let $A = \begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$. Find the eigenvalues of A.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 9-\lambda & -2 \\ 2 & 5-\lambda \end{bmatrix}\right) = (9-\lambda)(5-\lambda)+4 = \lambda^2-14\lambda+49 = (\lambda-7)^2$$

So the eigenvalue of A is $\lambda = 7$ with multiplicity 2.

Facts/Comments: Let A be an $n \times n$ matrix. Then

- A has at most n eigenvalues
- A has exactly n eigenvalues, counting multiplicity.
- If n is odd, then A must have at least one real eigenvalue.
- If A is even, then A may not any real eigenvalues.

Our next example illustrates that we can have 1 real eigenvalue, and 2 complex eigenvalues.

Example 4. Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -7 & 0 & 7 \end{bmatrix}$. Find all the real eigenvalues of A.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 & 1 \\ 0 & 5-\lambda & 0 \\ -7 & 0 & 7-\lambda \end{bmatrix}\right) = (5-\lambda)\det\left(\begin{bmatrix} 3-\lambda & 1 \\ -7 & 7-\lambda \end{bmatrix}\right)$$
$$= (5-\lambda)\left[(3-\lambda)(7-\lambda)+7\right]$$
$$= (5-\lambda)\left(\lambda^2-10\lambda+28\right)$$

Since the discriminant $b^2-4ac = (-10)^2-4(1)(28) = 100-112 = -12 < 0$, we know $\lambda^2-10\lambda+28$ has two complex roots. So our real eigenvalue is $\lambda = 5$ and, using the quadratic formula, our complex eigenvalues are

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{10 \pm \sqrt{-12}}{2} = \frac{10 \pm 2\sqrt{3}i}{2} = 5 \pm \sqrt{3}i \quad \Longrightarrow \lambda = 5+\sqrt{3}i \quad \text{and } \lambda = 5-\sqrt{3}i.$$

Definition: Let A and B be $n \times n$ matrices. Then A and B are **similar** if there is an invertible matrix S such that $A = SBS^{-1}$.

Theorem: If A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicity).

WARNINGS:

- Matrices can have the same eigenvalues without being similar.
  (a) Observe, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ have the same eigenvalues $\lambda = 0$ with multiplicity 2, but are not similar.

- Similarity is NOT the same as row equivalent. In general, row operations will change the eigenvalues.
  (a) Observe, for $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we have $A \sim B$ by scaling row 2, but A has eigenvalues $\lambda = 1, 2$ and B has eigenvalues $\lambda = 1$ with multiplicity 2.

Textbook Practice Problems: Section 5.2 (page 281-282) # 2, 4, 9, 11, 12, 15, 17, 18.

END OF SECTION 5.2