Section 4.5: The Dimension of a Vector Space

Definition 1. If V is spanned by a finite set, then V is said to be a **finite-dimensional**, and the **dimension** of V, written $\dim(V)$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Example 2. Let's discuss the dimension of vector spaces we have already studied.

1. $\dim(\mathbb{R}^n) = n$

   Why: The standard basis for $\mathbb{R}^n$ is the set $\{e_1, e_2, \dots e_n\}$. Since this set has n vectors we have $\dim(\mathbb{R}^n) = n$.

2. $\dim(\mathbb{P}_n) = n + 1$

   Why: The standard basis for $\mathbb{P}_n$ is the set $\{1, x, x^2, \dots, x^n\}$. Since this set has $n + 1$ vectors in it we have $\dim(\mathbb{P}_n) = n + 1$.

3. $\dim(M_{2\times 2}(\mathbb{R})) = 4$

   Why: The standard basis for $M_{n\times n}(\mathbb{R})$ is the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Since this set has $n + 1$ vectors in it we have $\dim(M_{n\times n}(\mathbb{R})) = 4$.

   In general $\dim(M_{m\times n}(\mathbb{R})) = mn$.

4. $\mathbb{P}$ is infinite dimensional

   Why: The standard basis for $\mathbb{P}$ is the set $\{1, x, x^2, x^3, x^4, \dots\}$. Since this set is an infinite set we have $\mathbb{P}$ is an infinite dimensional v vector space.

5. Let A be a $m \times n$ matrix. The dimension of the subspaces $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$ are:

$$\begin{aligned} \dim(\mathrm{Nul}(A)) = &\text{ the number of free variables (non pivot columns)} \\ \dim(\mathrm{Col}(A)) = &\text{ the number of pivot columns} \end{aligned}$$

Theorem 3. *If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.*

Remark: This theorem tells us that if $\dim(V) = n$, then every single basis for V *must* have n vectors.

Theorem 4. *If a vector space V has a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.*

Example 5. Determine which sets are a basis for $\mathbb{R}^3$.
Solution: Since $\dim(\mathbb{R}^3) = 3$, we know every basis for $\mathbb{R}^3$ must contain exactly 3 vectors.

- $\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \\ 3 \end{bmatrix} \right\}$; not a basis since it contains 4 vectors and thus linearly dependent.
- $\left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \right\}$; not a basis since it contains 2 vectors.
- $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$; although this set contains 3 vectors it is not a basis since it is not a linearly independent set.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$; is a basis since it contains 3 vectors AND is a linearly independent set.

Theorem 6. *Let H be a subspace of a finite dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is a finite-dimensional and*

$$\dim(H) \leq \dim(V).$$

Remark 7. This theorem tells us the dimension of a subspace H of a vector space V is at most the dimension of V.

Theorem 8. *Let V be a p-dimensional vector space, for $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.*

Remark 9. The $\dim(\mathrm{Nul}(A))$ is the number of free variables in the equation $Ax = 0$ and $\dim(\mathrm{Col}(A))$ is the number of pivot columns in A.

Example 10. Suppose $A = \begin{bmatrix} -2 & 4 & -2 \\ 2 & -6 & -3 \\ -3 & 8 & 2 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. Find the dimension of subspaces $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$.

Solution: Since A has two pivot columns (columns 1 and 2) we have $\dim(\mathrm{Col}(A)) = 2$. Also, since A has 1 free variable we have $\dim(\mathrm{Nul}(A)) = 1$.

Example 11. Find a basis for the subspace $H = \left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ of $\mathbb{R}^4$.

Solution: Observe, vectors in H have the form

$$\begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

Therefore we have $H = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$. Next, we determine if this set is linearly independent. Since

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and we have a pivot in every column, we have the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$ is linearly independent. Therefore this set forms a basis for H, and thus $\dim(H) = 3$.

Example 12. Find the dimension of the subspace spanned by the given vectors.

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$

  Solution: So we have our subspace is the

$$H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\} = \operatorname{Col}(A),$$

  where $A = \begin{bmatrix} 1 & -7 & 9 & 3 \\ 0 & -3 & 4 & 1 \\ 2 & 1 & -2 & 1 \end{bmatrix}$. We row reduce A to find our pivot columns. This gives us

$$\begin{bmatrix} 1 & -7 & 9 & 3 \\ 0 & -3 & 4 & 1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 9 & 3 \\ 0 & 1 & -4/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

  Therefore, our pivot columns are the first 2 columns so we have a basis for H is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix} \right\}$ and thus $\dim(H) = 2$.

Textbook Practice Problems: Section 4.5 (page 231-232) # 1-6, 11,12, 13-18.

END OF SECTION 4.5