Função Gama e Beta

Define-se função gama por

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} \, dt \quad , \; n > 0$$

As formas de recorrência mais usadas são

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(n+1) = n!$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\left\{ \Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \quad , \; n > 0 \right.$$

$$\Gamma(n+1) = \int_0^\infty x^{(n+1)-1} e^{-x}\, dx$$

$$= \int_0^\infty x^n e^{-x}\, dx = \lim_{b\to\infty} \int_0^b x^n e^{-x}\, dx$$

Integ. por partes $\Rightarrow$

$$= \lim_{b\to\infty} \left(x^n \left(-e^{-x}\right)\Big|_0^b - \int_0^b -e^{-x}\, n\, x^{n-1}\, dx \right)$$

$$= \lim_{b\to\infty} x^n \left(-e^{-x}\right)\Big|_0^b + n \lim_{b\to\infty} \int_0^b e^{-x} x^{n-1}\, dx$$

Verifiquem!!!

$$= 0 + n \int_0^\infty e^{-x} x^{n-1}\, dx = n\,\Gamma(n) \quad \#$$

$\forall\, n > 0$.

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt, \quad n > 0$$

$$\Gamma(n+1) = n!$$

Prove que $0! = 1$

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt$$

$$= \int_0^\infty e^{-t}\, dt = \lim_{b \to \infty} \int_0^b e^{-t}\, dt$$

$$= \lim_{t \to \infty} \left(-e^{-t}\right)\Big|_0^b = \lim_{t \to \infty} \cancel{-e^{-b}}^{\,0} - \left(-e^{0}\right) = 1$$

mas $\Gamma(1) = \Gamma(0+1) = 0!$

$\therefore$ $\boxed{0! = 1}$

Prove que $\Gamma(1/2) = \sqrt{\pi}$ $\overset{\text{ou}}{\Rightarrow}$ $\sqrt{\Gamma(1/2)^2} = \pi$

$$\Gamma(1/2) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,dx = \int_0^\infty x^{-1/2} e^{-x}\,dx$$

fazendo $x = u^2$

$dx = 2u\,du$

$$= \int_0^\infty (u^2)^{-1/2} e^{-u^2} (2u\,du)$$

$$= \int_0^\infty u^{-1} e^{-u^2} \cdot 2u\,du = 2\int_0^\infty e^{-u^2}\,du$$

usando o fato de que

$$\Gamma(1/2) = \sqrt{(\Gamma(1/2))^2} \Rightarrow$$

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt, \quad n>0$$

$$\Gamma(n+1) = n!$$

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2}\,du \Rightarrow \Gamma(1/2)^2 = \left(2\int_0^\infty e^{-u^2}\,du\right)^2 = 4\left(\int_0^\infty e^{-u^2}\,du\right)\left(\int_0^\infty e^{-u^2}\,du\right)$$

$$\Gamma(1/2)^2 = 4\int_0^\infty e^{-u^2}\,du * \int_0^\infty e^{-v^2}\,dv = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,dv\,du$$

usando coordenadas esféricas temos

$$\iint_R f(u,v)\,dv\,du = \iint_R f(r\cos\theta, r\,\mathrm{sen}\,\theta)\,r\,dr\,d\theta$$

$$u = r\cos\theta \Rightarrow r = \frac{u}{\cos\theta}$$

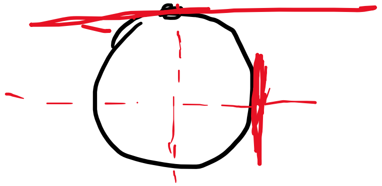
$$v = r\,\mathrm{sen}\,\theta \Rightarrow r = \frac{v}{\mathrm{sen}\,\theta}$$

$$\frac{u}{\cos\theta} = \frac{v}{\mathrm{sen}\,\theta} \Rightarrow \frac{\mathrm{sen}\,\theta}{\cos\theta} = \frac{v}{u} \Rightarrow \mathrm{tg}\,\theta = \frac{v}{u}$$

$$\operatorname{tg}\theta = \frac{v}{u} \quad \begin{cases} v = 0 \Rightarrow \operatorname{tg}\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \\ v = \infty \Rightarrow \operatorname{tg}\theta \cong \infty \Rightarrow \theta = 0 \end{cases}$$

$$u = r\cos\theta$$

$$r = \frac{u}{\cos\theta} \quad \begin{cases} u = 0 \Rightarrow r = 0 \\ u = \infty \Rightarrow r = \infty \end{cases}$$



$$w = -r^2$$
$$dw = -2r\,dr$$

$$\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\,dv\,du = \int_0^{\pi/2} \underbrace{\int_0^\infty e^{-r^2}\, r\,dr}\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\int_0^\infty e^{-w}\,dw\,d\theta = \frac{1}{2}\int_0^{\pi/2} -e^{-w}\Big|_0^\infty dw = -\frac{1}{2}\int_0^{\pi/2} -dw = \frac{1}{2}\, w\Big|_0^{\pi/2} = \frac{\pi}{4}$$

logo

$$\Gamma(\pi/2)^2 = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,dv\,du = 4\cdot\frac{\pi}{4} = \pi \Rightarrow \Gamma(\pi/2) = \sqrt{\pi}$$

Calcule $\Gamma(3)$

$$\Gamma(n)=\int_0^\infty t^{n-1}e^{-t}\,dt\ ,\ n>0$$

$$\Gamma(n+1)=n!$$

$$\Gamma(3)=\Gamma(2+1)=2!=2$$

ou

$$\Gamma(3)=\int_0^\infty t^2e^{-t}\,dt=\lim_{b\to\infty}\int_0^b t^2e^{-t}\,dt=$$

$$=\lim_{b\to\infty}-(2+2t+t^2)e^{-t}\Big|_0^b=-\lim_{b\to\infty}$$

$$=-\lim_{b\to\infty}(2+2b+b^2)e^{-b}+\lim_{b\to\infty}(2+2.0+0^2)$$

$$=\left(-2\lim_{b\to\infty}\cancelto{0}{e^{-b}}-2\lim_{b\to\infty}\cancelto{0}{\frac{b}{e^b}}-\lim_{b\to\infty}\cancelto{0}{\frac{b^2}{e^b}}\right)+2$$

$$\Rightarrow\ \boxed{\Gamma(3)=2}$$

$$\lim_{b\to\infty}\frac{b}{e^b}=\frac{\infty}{\infty}$$

$$=\lim_{b\to\infty}\frac{1}{e^b}=0$$

Exercícios: Calcule

1) $\int_0^{\infty} x^3 e^{-x}\, dx$

2) $\int_0^{\infty} x^6 e^{-2x}\, dx$

3) $\int_0^{\infty} e^{-x^2}\, dx = \int_0^{\infty} e^{-u} \cdot \frac{du}{2\sqrt{u}} = \frac{1}{2}\int_0^{\infty} u^{-1/2} e^{-u}\, du = \frac{1}{2}\int_0^{\infty} u^{\frac{1}{2}-1} e^{-u}\, du$

fazendo

$u = x^2 \Rightarrow x = \sqrt{u}$

$du = 2x\,dx \Rightarrow dx = \frac{du}{2x} \Rightarrow dx = \frac{du}{2\sqrt{u}}$

$= \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$ #

$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t}\, dt, \quad n > 0$

$\Gamma(n+1) = n!$

Exercícios (Entregar !!!)

Calcule:

1) $\Gamma(-3)$

2) $\Gamma(-5/2)$

3) $\displaystyle\int_0^{\infty} t^2 e^{-x} \, dx$

4) $\displaystyle\int_0^{\infty} x^3 e^{-x} \, dx$

Função Beta

Define-se a função Beta por

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dx, \quad m \in \mathbb{R},\ n \in \mathbb{R},$$ com as seguintes propriedades

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}; \qquad B(m,n) = B(n,m); \qquad B(n,1-n) = \Gamma(n)\,\Gamma(1-n)$$

Exemplo; calcule

$$\int_0^1 x^3(1-x)^4\,dx = \int_0^1 x^{4-1}(1-x)^{5-1}\,dx = B(4,5)$$

$$B(4,5) = \frac{\Gamma(4)\,\Gamma(5)}{\Gamma(4+5)} = \frac{3!\,4!}{8!} = \frac{1}{280}$$

Funções Ortogonais

Conceitos vetoriais de produto interno (escalar) e ortogonalidade podem ser estendidos a funções. Relembrando tais conceitos:

Sejam $\vec{u}$ e $\vec{v}$ vetores no espaço tridimensional. O produto interno $(\vec{u},\vec{v})$ de dois vetores denotado por $\vec{u}\cdot\vec{v}$ representa as seguintes propriedades:

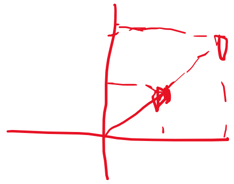
$(\vec{u},\vec{v}) = (\vec{v},\vec{u})$

$(K\vec{u},\vec{v}) = K(\vec{u},\vec{v})$, K = escalar

$(\vec{u},\vec{u}) = \vec{0} \Rightarrow \vec{u} = \vec{0}$, $(\vec{u},\vec{u}) > 0$ se $\vec{u} \neq 0$

$(\vec{u}+\vec{v},\vec{w}) = (\vec{u},\vec{w}) + (\vec{v},\vec{w})$

Vetores // e ⊥

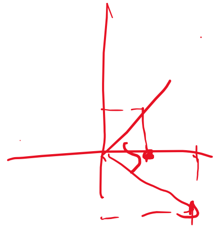


$\vec{u} \perp \vec{v} \Leftrightarrow (\vec{u}\cdot\vec{v}) = 0$

$\vec{u} \,//\, \vec{v} \Leftrightarrow \vec{u} = K\vec{v}$, K = escalar

Ex: $\vec{u} = (1,1)$ $\vec{v} = (2,2)$, $\vec{w} = (2,-2)$

$\vec{u} = \frac{1}{2}\vec{v} \Rightarrow \vec{u} \,//\, \vec{v}$



$(\vec{u}\cdot\vec{v}) = (1\cdot2 + 1\cdot2) = 4$

$(\vec{u}\cdot\vec{w}) = (1\cdot2 + 1(-2)) = 0 \Rightarrow \vec{u} \perp \vec{w}$

A norma, ou comprimento de um vetor pode ser expressa em termos do produto interno, ou seja

$(u,u) = \|u\|^2 \Rightarrow \|u\| = \sqrt{(u,u)}$, se $\vec{u} = (u_1, u_2) \Rightarrow \|u\| = \sqrt{u_1^2 + u_2^2}$

Vetor unitário é definido por $\hat{v} = \dfrac{(u,u)}{\|u\|} = \dfrac{u_1^2 + u_2^2}{\sqrt{u_1^2 + u_2^2}} = 1$ Ex: $\vec{v} = (\cos t, \operatorname{sen} t)$

Definição: O produto interno de duas funções f_1 e f_2 em um intervalo $[a,b]$ é o número

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx$$

Se $(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx = 0$

$\Rightarrow$ f_1 e f_2 são ortogonais.

OBS: a palavra ortogonal, aqui não tem significado FÍSICO

Exemplo: As funções $f_1(x) = x^2$ e $f_2(x) = x^3$ são ortogonais no intervalo $[-1,1]$ pois

$$(f_1, f_2) = \int_{-1}^{1} f_1(x)\, f_2(x)\, dx = \int_{-1}^{1} x^2 \cdot x^3\, dx = \int_{-1}^{1} x^5\, dx = \left.\frac{x^6}{6}\right|_{-1}^{1} = \frac{1^6}{6} - \frac{(-1)^6}{6}$$

$$= \frac{1}{6} - \frac{1}{6} = 0$$

Simples demais $\Rightarrow$ Vamos complicar!!

Conjuntos ortogonais

Um conjunto de funções com valores reais $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ é ortogonal em um intervalo $[a,b]$ se

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0 \qquad m \neq n.$$

Exemplo:

Mostre que o conjunto $\{1, \cos x, \cos 2x, \cos 3x, \dots\}$ é ortogonal no intervalo $[-\pi, \pi]$

Solução: Fazendo $\phi_0(x) = 1$ e $\phi_n(x) = \cos nx$, $n = 1, 2, \dots$

Devemos mostrar que

$$\int_{-\pi}^{\pi} \phi_0(x)\,\phi_n(x)\,dx = 0, \quad n \neq 0$$

$$\int_{-\pi}^{\pi} 1 \cdot \cos nx\,dx = \int_{-\pi}^{\pi} \cos nx\,dx$$

$$= \frac{1}{n} \operatorname{sen} nx \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n} \left(\operatorname{sen} n\pi - \operatorname{sen}(-n\pi) \right)$$

$$= \frac{1}{n} \cdot 0 = 0, \quad n \neq 0$$

$$\int_{-\pi}^{\pi} \phi_m(x)\,\phi_n(x)\,dx = 0 \quad m \neq n$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx =$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left[\cos(m+n)x + \cos(m-n)x \right] dx$$

$$= \frac{1}{2} \left[\frac{\operatorname{sen}((m+n)x)}{m+n} + \frac{\operatorname{sen}((m-n)x)}{m-n} \right]_{-\pi}^{\pi}$$

$$= 0, \quad m \neq n.$$

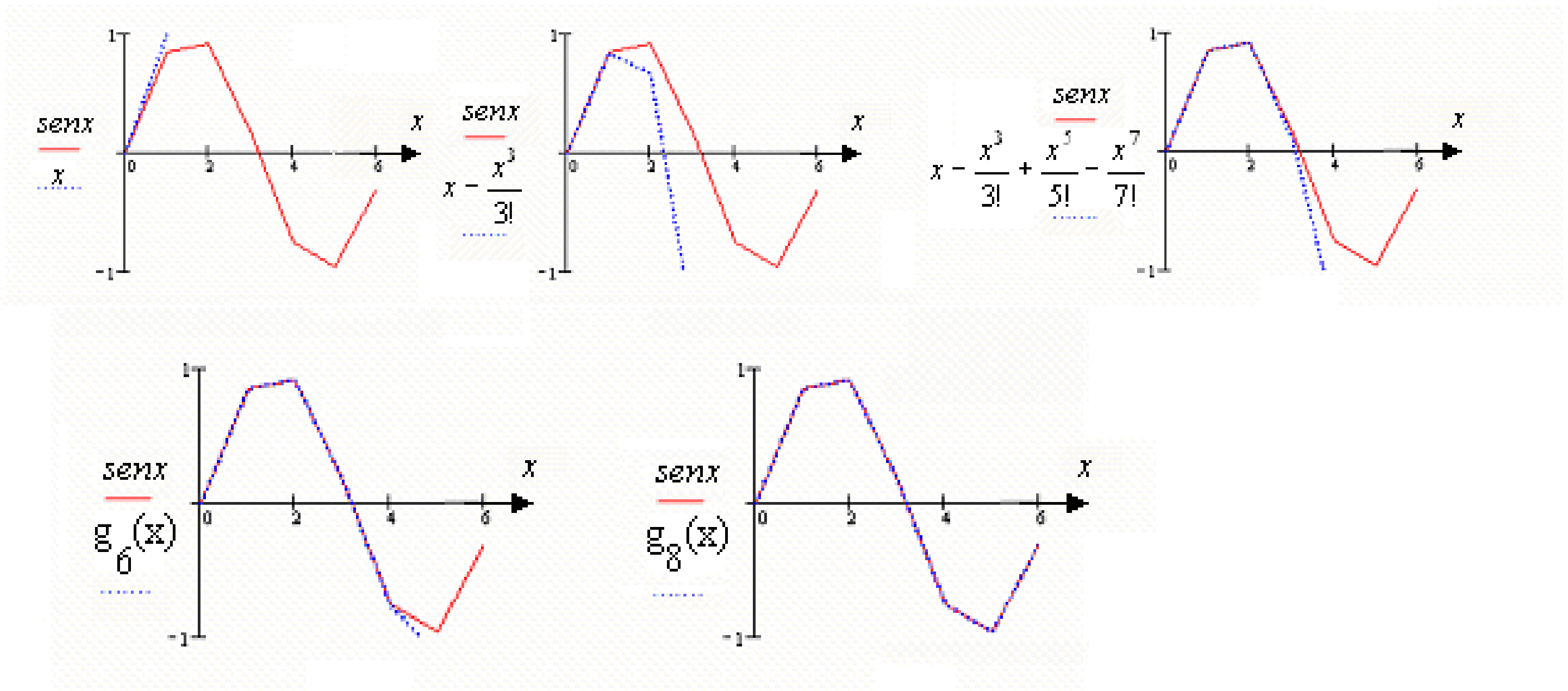
$\operatorname{sen} x = \operatorname{sen}(-x) \to$ função ímpar

Identidade trigonométrica

$\cos u \cos v = \frac{1}{2}\left(\cos(u+v) + \cos(u-v) \right)$

Aproximações de funções

$$\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$



Apresenta-se na Tabela 1 alguns exemplos de funções e sua representação em série de potências.

Tabela 1: funções e séries de potências

$\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^{n}=1+x+x^{2}+\cdots+x^{n}+\cdots$
$e^{x}=\sum_{n=0}^{\infty}\dfrac{x^{n}}{n!}=1+\dfrac{x}{1!}+\dfrac{x^{2}}{2!}+\cdots+\dfrac{x^{n}}{n!}+\cdots$
$sen x=\sum_{n=0}^{\infty}(-1)^{n}\dfrac{x^{2n+1}}{(2n+1)!}=x-\dfrac{x^{3}}{3!}+\dfrac{x^{5}}{5!}-\dfrac{x^{7}}{7!}+\cdots$
$\cos x=\sum_{n=0}^{\infty}(-1)^{n}\dfrac{x^{2n}}{(2n)!}=1-\dfrac{x^{2}}{2!}+\dfrac{x^{4}}{4!}-\dfrac{x^{6}}{6!}+\cdots$