XPBD: position-based simulation of compliant constrained dynamics

IG3DA/IMA904 : Advanced 3D Computer Graphics

Sammy Rasamimanana

November 25, 2022

Table of Contents

Introduction

A basic strategy for physic simulation

Consider a particle with

1. mass m
2. position p
3. speed v

Newton's Second Law of Motion

$$m\dot{v} = F_{ext}$$

$$\dot{p} = v$$

Force based strategy for physic simulation

Consider a particle with

1. mass m
2. position p
3. speed v

Explicit Euler: discretization of Newton's Second Law of Motion

Let dt the time step and t_k the time at the k-th step of the animation. Thus

$$m(v_{k+1} - v_k) = F_{ext} dt$$

$$p_{k+1} - p_k = v_k dt$$

Explicit Euler : a basic strategy for physic simulation

Consider a particle with mass m, position p, speed v

Explicit Euler: discretization of Newton's Second Law of Motion

Let dt the time step and t_k the time at the k-th step of the animation. Thus

$$v_{k+1} = v_k + \frac{F_{ext}}{m} dt$$

$$p_{k+1} = p_k + v_k dt$$

Explicit Euler : a basic strategy for physic simulation

Consider a particle with mass m, position p, speed v

Explicit Euler: discretization of Newton's Second Law of Motion

Let dt the time step and t_k the time at the k-th step of the animation. Thus

$$v_{k+1} = v_k + \frac{F_{ext}}{m} dt$$

$$p_{k+1} = p_k + v_k dt$$

But it diverges and has a low accuracy !

Semi-implicit Euler : a basic strategy for physic simulation

Let's make a little change

Semi-implicit Euler/Symplectic integrator

Let dt the time step and t_k the time at the k-th step of the animation. Thus

$$v_{k+1} = v_k + \frac{F_{ext}}{m}dt$$

$$p_{k+1} = p_k + v_{k+1}dt$$

Algorithm

Algorithm 1 Simulation loop

for all particles **do**
  $v \longleftarrow v + f_{ext} \cdot m^{-1} \cdot dt$
  $p \longleftarrow p + v \cdot dt$
end for

Position Based Dynamics (PBD)

Still some unstability

Semi-implicit Euler/Symplectic integrator

Let dt the time step and t_k the time at the k-th step of the animation. Thus

$$v_{k+1} = v_k + \frac{F_{ext}}{m} dt$$

$$p_{k+1} = p_k + v_{k+1} dt$$

$\Longrightarrow$ More stable than explicit Euler, but still unstable depending on time step and types of force ...

$\Longrightarrow$ A dynamic object : its center of mass (usually)

Another approach: a constraints based strategy

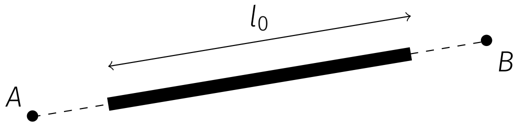
A dynamic object : a set of N vertices (= particles) and M constraints

A Constraint $j \in [\![1, M]\!]$ consists of

- a cardinality n_j and a set of indices $i_1, \cdots, i_{n_j} \subset [\![1, N]\!]$
- a function $C_j : \mathbb{R}^{3n_j} \longrightarrow \mathbb{R}$
- a stiffness parameter $k_j \in [0, 1]$
- a type of either *equality* ($C_j = 0$) or *inequality* ($C_j \geqslant 0$)

Constraint examples

Two particles A and B.



Distance constraint : $C(p_A, p_B) = |p_A - p_B| - l_0$

It is an *equality* constraint : satisfied when $C(p_A, p_B) = 0$.

Main idea : projecting the points

Move the points such that they satisfy the constraint

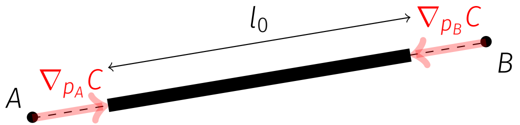
Main idea : move along gradient

Two particles A and B with Distance constraint : $C(p_A, p_B) = |p_A - p_B| - l_0$

Denote $p = \begin{pmatrix} p_A^T \\ p_B^T \end{pmatrix} = \begin{pmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \end{pmatrix}$. How to find Δp such that $C(p + \Delta p) = 0$?

Gradient $\nabla_p C$ indicates the direction of maximal increase for C

The correction term Δp will be along the gradient $\nabla_p C$



PBD Algorithm

Algorithm 2 Simulation loop

for all particles **do**
    $p_{old} \longleftarrow p$
    $v \longleftarrow v + f_{ext} \cdot m^{-1} \cdot dt$
    $p \longleftarrow p + v \cdot dt$
end for
while $i < \texttt{nbIterations}$ **do**
    for all constraints C **do**
        $\Delta p \longleftarrow \texttt{solveConstraint(C)}$
        $p \longleftarrow p + \Delta p$
    end for
    $i \longleftarrow i + 1$
end while
$v \longleftarrow (p - p_{old})/dt$

PBD : pros and cons

- ➕ direct manipulation of positions of vertices and parts of objects
- ➕ gives control over explicit integration and removes typical instability problems (it is unconditionnally stable)
- ➕ can handle non-linear constraints
- ➖ low accuracy, no energy preserving
- ➖ stiffness of the model depends on the number of sub-iterations and the timpe step when moving along the gradient

Extended Position Based Dynamics (XPBD)

Compliant constraint formulation

Idea 1: treat a constraint as an energy potential

$$U(p) = \frac{1}{2} C(p)^T \alpha^{-1} C(p)$$

α corresponds to the inverse stiffness

Newton's second law : $M\ddot{p} = -\nabla_p U^T = -\nabla_p C^T(p) \alpha^{-1} C(p)$

Compliant constraint formulation

Newton's second law : $M\ddot{p} = -\nabla_p U^T = -\nabla_p C^T(p)\alpha^{-1}C(p)$

Idea 2: introduce the Lagrange multiplier

$$\lambda = -\tilde{\alpha}^{-1}C(p)$$

The equation becomes the system

$$\begin{aligned} M\ddot{p} - \nabla_p C(p)^T\lambda &= 0 \\ C(p) + \tilde{\alpha}\lambda &= 0 \end{aligned}$$

$\Longrightarrow$ Discretization of the system and iterative resolution (Gauss-Seidel or Jacobi) to find λ then p at each time.

XPBD

- ➕ direct manipulation of positions of vertices and parts of objects
- ➕ gives control over explicit integration and removes typical instability problems (it is unconditionnally stable)
- ➕ can handle non-linear constraints
- ➖ no energy preserving
- ➕ stiffness of the model qualitatively independent of time step and number of iterations

XPBD Algorithm

Algorithm 3 Simulation loop

```
for all particles do
    p_old ⟵ p
    v ⟵ v + f_ext · m^{-1} · dt
    p ⟵ p + v · dt
end for
Initialize multipliers λ ⟵ 0
while i < nbIterations do
    for all constraints C do
        Compute Δλ
        Compute Δp
        λ ⟵ λ + Δλ
        p ⟵ p + Δp
    end for
    i ⟵ i + 1
end while
v ⟵ (p − p_old)/dt
```

Conclusion

Conclusion

Goals of the project

- ✓ Implementation of distance constraint and fixed constraint (often used)
- ✓ Tests on a spring model and on a cloth
- Implementation of bending constraint and collision/penetration constraints
- Tests cloth and deformable ball interaction with ground
- Implementation of Rayleigh dissipation potential (for damping) and generic constraint

Thank You

Questions?