$$
\begin{aligned}
Y_{i,j}(Q) &= (E_{i,n+j} - E_{j,n+i})(\sum (e_p \wedge e_{n+p})) \\
&= 2 \cdot e_j \wedge e_i \\
&\neq 0
\end{aligned}
$$

so that whenever $a < i, j \leq n - b$,

$$
\begin{aligned}
&Y_{i,j}(w^{(a,b)}) \\
&= Y_{i,j}(e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2}) \\
&= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Y_{i,j}((\sum (e_p \wedge e_{n+p}))^{(k-a-b)/2}) \\
&= (k-a-b) \cdot (e_1 \wedge \cdots \wedge e_a \wedge e_j \wedge e_i \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b-2)/2}) \\
&\neq 0.
\end{aligned}
$$

It is always possible to find a pair (i, j) satisfying the conditions $a < i, j \leq n - b$ since we are assuming $a + b < k < n$; this concludes the proof of part (i).

The proof of part (ii) requires only one further step: we have to check the vectors $w^{(a,b)}$ with $a + b = k = n$ to see if any of them might be highest weight vectors for $\mathfrak{so}_{2n}\mathbb{C}$. In fact (as the statement of the theorem implies), two of them are: It is not hard to check that, in fact, $w^{(n,0)}$ and $w^{(n-1,1)}$ are killed by every positive root space $\mathfrak{g}_{L_i+L_j}$. To see that no other vector $w^{(a,n-a)}$ is, look at the action of $Y_{a+1,a+2} \in \mathfrak{g}_{L_{a+1}+L_{a+2}}$: we have

$$
\begin{aligned}
&Y_{a+1,a+2}(w^{(a,n-a)}) \\
&= (E_{a+1,n+a+2} - E_{a+2,n+a+1})(e_1 \wedge \cdots \wedge e_a \wedge e_{n+a+1} \wedge \cdots \wedge e_{2n}) \\
&= e_1 \wedge \cdots \wedge e_a \wedge e_{a+1} \wedge e_{n+a+1} \wedge e_{n+a+3} \wedge \cdots \wedge e_{2n} \\
&\quad - e_1 \wedge \cdots \wedge e_a \wedge e_{a+2} \wedge e_{n+a+2} \wedge \cdots \wedge e_{2n} \\
&\neq 0.
\end{aligned}
$$

□

Remarks. (i) This theorem will be a consequence of the Weyl character formula, which will tell us a priori that the dimension of the irreducible representation of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weight $L_1 + \cdots + L_k$ has dimension $\binom{2n}{k}$ if $k < n$, and half that if $k = n$.

(ii) Note also that by the above, $\wedge^n V$ is the direct sum of the two irreducible representations $\Gamma_{2\alpha}$ and $\Gamma_{2\beta}$ with highest weights $2\alpha = L_1 + \cdots + L_n$ and $2\beta = L_1 + \cdots + L_{n-1} - L_n$. Indeed, the inclusion $\Gamma_{2\alpha} \oplus \Gamma_{2\beta} \subset \wedge^n V$ can be seen just from the weight diagram: $\wedge^n V$ possesses a highest weight vector with highest weight $L_1 + \cdots + L_n$, and so contains a copy of $\Gamma_{2\alpha}$; but this representation does not possess the weight 2β, and so $\wedge^n V$ must contain $\Gamma_{2\beta}$ as well. (Alternatively, we observed in the preceding lecture that in choosing an ordering of the roots we could have chosen our linear functional $l = c_1 H_1 + \cdots + c_n H_n$ with $c_1 > c_2 > \cdots > -c_n > 0$ without altering the positive roots or the Weyl chamber; in this case the weight λ of $\bigwedge^n V$ with $l(\lambda)$ maximal would be 2β, showing that $\Gamma_{2\beta} \subset \bigwedge^n V$.)

(iii) If we want to avoid weight diagrams altogether, we can still see that $\bigwedge^n V$ must be reducible, because the action of $\mathfrak{so}_{2n}\mathbb{C}$ preserves two bilinear forms: first, we have the bilinear form induced on $\bigwedge^n V$ by the form Q on V; and second we have the wedge product

$$\varphi\colon \textstyle\bigwedge^n V \times \bigwedge^n V \to \bigwedge^{2n} V = \mathbb{C},$$

the last map taking $e_1 \wedge \cdots \wedge e_{2n}$ to 1. It follows that $\bigwedge^n V$ is reducible; indeed, if we want to see the direct sum decomposition asserted in the statement of the theorem we can look at the composition

$$\tau\colon \textstyle\bigwedge^n V \to \bigwedge^n V^* \to \bigwedge^n V,$$

where the first map is the isomorphism given by Q and the second is the isomorphism given by φ. The square of this map is the identity, and decomposing $\bigwedge^n V$ into $+1$ and -1 eigenspaces for this map gives two subrepresentations.

Exercise 19.3*. Part (i) of Theorem 19.2 can also be proved by showing that for any nonzero vector $w \in \bigwedge^k V$, the linear span of the vectors $X(w)$, for $X \in \mathfrak{so}_m\mathbb{C}$, is all of $\bigwedge^k V$. For these purposes take, instead of the basis we have been using, an orthonormal basis $v_1, \ldots, v_m$ for $V = \mathbb{C}^m$, $m = 2n$, so $Q(v_i, v_j) = \delta_{i,j}$. The vectors $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$, $I = \{i_1 < \cdots < i_k\}$, form a basis for $\bigwedge^k V$, and $\mathfrak{so}_m\mathbb{C}$ has a basis consisting of endomorphisms $V_{p,q}$, $p < q$, which takes v_q to v_p, v_p to $-v_q$, and takes the other v_i to zero. Compute the images $V_{p,q}(v_I)$, and prove the claim, first, when $w = v_I$ for some I, and then by induction on the number of nonzero coefficients in the expression $w = \sum a_I v_I$. For (ii) a similar argument shows that $\bigwedge^n V$ is an irreducible representation of the group $\mathrm{O}_n\mathbb{C}$, and the ideas of §5.1 (cf. §19.5) can be used to see how it decomposes over the subgroup $\mathrm{SO}_n\mathbb{C}$ of index two.

We return now to our analysis of the representations of $\mathfrak{so}_{2n}\mathbb{C}$. By the theorem, the exterior powers $V, \bigwedge^2 V, \ldots, \bigwedge^{n-2} V$ provide us with the irreducible representations with highest weight the fundamental weight along the first $n-2$ edges of the Weyl chamber (of course, the exterior power $\bigwedge^{n-1} V$ is irreducible as well, but as we have observed, $L_1 + \cdots + L_{n-1}$ is not on an edge of the Weyl chamber, and so $\bigwedge^{n-1} V$ is not as useful for our purposes). For the remaining two edges, we have found irreducible representations with highest weights located there, namely the two direct sum factors of $\bigwedge^n V$; but the highest weights of these two representations are not primitive ones; they are divisible by 2. Thus, given the theorem above, we see that we have constructed exactly one-half the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$, namely, those whose highest weight lies in the sublattice $\mathbb{Z}\{L_1, \ldots, L_n\} \subset \Lambda_W$. Explicitly, any weight γ in the closed Weyl chamber can be expressed (uniquely) in the form

$$\gamma = a_1L_1 + \cdots + a_{n-2}(L_1 + \cdots + L_{n-2})$$
$$+ a_{n-1}(L_1 + \cdots + L_{n-1} - L_n)/2 + a_n(L_1 + \cdots + L_n)/2$$

with $a_i \in \mathbb{N}$. If $a_{n-1} + a_n$ is even, with $a_{n-1} \geq a_n$ we see that the representation

$$\mathrm{Sym}^{a_1}V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2}V) \otimes \mathrm{Sym}^{a_n}(\wedge^{n-1}V) \otimes \mathrm{Sym}^{(a_{n-1}-a_n)/2}(\Gamma_{2\beta})$$

will contain an irreducible representation Γ_γ with highest weight γ; whereas if $a_n \geq a_{n-1}$, we will find Γ_γ inside

$$\mathrm{Sym}^{a_1}V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2}V) \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1}V) \otimes \mathrm{Sym}^{(a_n-a_{n-1})/2}(\Gamma_{2\alpha}).$$

There remains the problem of constructing irreducible representations Γ_γ whose highest weight γ involves an odd number of α's and β's. To do this, we clearly have to exhibit irreducible representations Γ_α and Γ_β with highest weights α and β. These exist, and are called the *spin representations* of $\mathfrak{so}_{2n}\mathbb{C}$; we will study them in detail in the following lecture. We see from the above that once we exhibit the two representations Γ_α and Γ_β, we will have constructed all the representations of $\mathfrak{so}_{2n}\mathbb{C}$. The representation Γ_γ with highest weight γ written above will be found in the tensor product

$$\mathrm{Sym}^{a_1}V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-2}}(\wedge^{n-2}V) \otimes \mathrm{Sym}^{a_{n-1}}(\Gamma_\beta) \otimes \mathrm{Sym}^{a_n}(\Gamma_\alpha).$$

For the time being, we will assume the existence of the spin representations of $\mathfrak{so}_{2n}\mathbb{C}$; there is a good deal we can say about these representations just on the basis of their weight diagrams.

Exercise 19.4*. Find the weights (with multiplicities) of the representations $\wedge^k V$, and also of $\Gamma_{2\alpha}$, $\Gamma_{2\beta}$, Γ_α, and Γ_β.

Exercise 19.5. Using the above, show that Γ_α and Γ_β are dual to one another when n is odd, and that they are self-dual when n is even.

Exercise 19.6. Give the complete decomposition into irreducible representations of $\mathrm{Sym}^2\Gamma_\alpha$ and $\wedge^2\Gamma_\alpha$. Show that

$$\Gamma_\alpha \otimes \Gamma_\alpha = \Gamma_{2\alpha} \oplus \wedge^{n-2}V \oplus \wedge^{n-4}V \oplus \wedge^{n-6}V \otimes \cdots.$$

Exercise 19.7. Show that

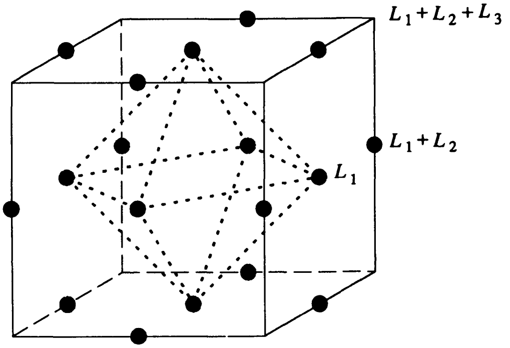
$$\Gamma_\alpha \otimes \Gamma_\beta = \wedge^{n-1}V \oplus \wedge^{n-3}V \oplus \wedge^{n-5}V \oplus \cdots.$$

Exercise 19.8. Verify directly the above statements in the case of $\mathfrak{so}_6\mathbb{C}$, using the isomorphism with $\mathfrak{sl}_4\mathbb{C}$.

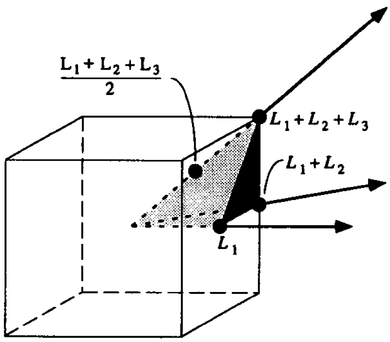
Exercise 19.9. Show that the automorphism of $\mathbb{C}^{2n}$ that interchanges e_n and e_{2n}, leaving the other e_i fixed, determines an automorphism of $\mathfrak{so}_{2n}\mathbb{C}$ that preserves the $n-2$ roots $L_1 - L_2, \ldots, L_{n-2} - L_{n-1}$ and interchanges $L_{n-1} - L_n$ and $L_{n-1} + L_n$. This automorphism takes the representation V to itself, but interchanges Γ_α and Γ_β.

§19.3. Representations of $\mathfrak{so}_7\mathbb{C}$

While we might reasonably be apprehensive about the prospect of a family of Lie algebras even more strangely behaved than the even orthogonal algebras, there is some good news: even though the roots systems of the odd Lie algebras appear more complicated than those of the even, the representation theory of the odd algebras is somewhat tamer. We will describe these representations, starting with the example of $\mathfrak{so}_7\mathbb{C}$; we begin, as always, with a picture of the root diagram:

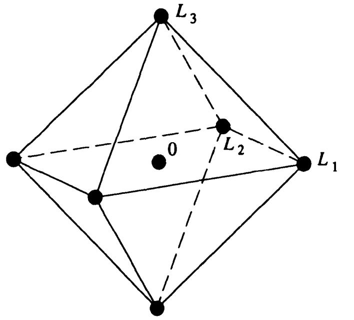


As we said, this looks like the root diagram for $\mathfrak{sp}_6\mathbb{C}$, except that the roots $\pm 2L_i$ have been shortened to $\pm L_i$. Unlike the case of $\mathfrak{so}_5\mathbb{C}$, however, where the long and short roots could be confused and the root diagram was correspondingly congruent to that of $\mathfrak{sp}_4\mathbb{C}$, in the present circumstance the root diagram is not similar to any other; the Lie algebra $\mathfrak{so}_7\mathbb{C}$, in fact, is *not* isomorphic to any of the others we have studied. Next, the Weyl chamber:



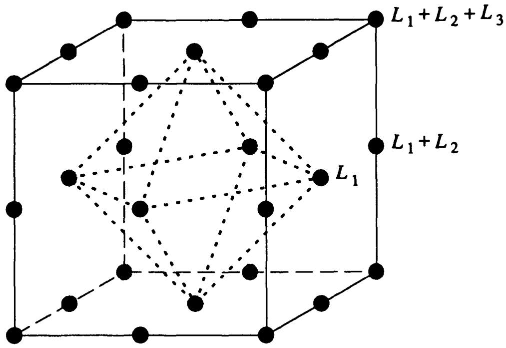
Again, the Weyl chamber itself looks just like that of $\mathfrak{sp}_6\mathbb{C}$; the difference in this picture is in the weight lattice, which contains the additional vector $(L_1 + L_2 + L_3)/2$.

As usual, we start our study of the representations of $\mathfrak{so}_7\mathbb{C}$ with the standard representation, whose weights are $\pm L_i$ and 0:



Note that the highest weight L_1 of this representation lies along the front edge of the Weyl chamber. Next, the weights of the exterior square $\wedge^2 V$ are $\pm L_i \pm L_j$, $\pm L_i$, and 0 (taken three times); this, of course, is just the adjoint representation. Note that the highest weight $L_1 + L_2$ of this representation is the same as that of the exterior square of the standard representation for $\mathfrak{so}_6\mathbb{C}$, but because of the smaller Weyl chamber this weight does indeed lie on an edge of the chamber.

Next, consider the third exterior power $\wedge^3 V$ of the standard. This has weights $\pm L_1 \pm L_2 \pm L_3$, $\pm L_i \pm L_j$, $\pm L_i$ (with multiplicity 2) and 0 (with multiplicity 3), i.e., at the midpoints of all the vertices, edges, and faces of the cube:



It is not obvious, from the weight diagram alone, that this is an irreducible representation; it could be that $\wedge^3 V$ contains a copy of the standard representation V and that the irreducible representation $\Gamma_{L_1+L_2+L_3}$ thus has multiplicity 1 on the weights $\pm L_i$ and multiplicity 2 (or 1) at 0. We can rule out this possibility by direct calculation: for example, if this were the case, then $\wedge^3 V$ would contain a highest weight vector with weight L_1. The weight space with eigenvalue L_1 in $\bigwedge^3 V$ is spanned by the tensors $e_1 \wedge e_2 \wedge e_5$ and $e_1 \wedge e_3 \wedge e_6$, however, and if we apply to these the generators $X_{1,2} = E_{1,2} - E_{5,4}$, $X_{2,3} = E_{2,3} - E_{6,5}$, and $U_3 = E_{3,7} - E_{7,6}$ of the root spaces corresponding to the positive roots $L_1 - L_2$, $L_2 - L_3$, and L_3, we see that

$$X_{2,3}(e_1 \wedge e_3 \wedge e_6) = e_1 \wedge e_2 \wedge e_6,$$

$$U_3(e_1 \wedge e_3 \wedge e_6) = e_1 \wedge e_3 \wedge e_7 \neq 0;$$

$$X_{2,3}(e_1 \wedge e_2 \wedge e_5) = e_1 \wedge e_2 \wedge e_6,$$

$$U_3(e_1 \wedge e_2 \wedge e_5) = 0.$$

There is thus no linear combination of $e_1 \wedge e_2 \wedge e_5$ and $e_1 \wedge e_3 \wedge e_6$ killed by both U_3 and $X_{2,3}$, showing that $\bigwedge^3 V$ has no highest weight vector of weight L_1.

Exercise 19.10. Verify that $\bigwedge^3 V$ does not contain the trivial representation.

We have thus found irreducible representations of $\mathfrak{so}_7\mathbb{C}$ with highest weight vectors along the three edges of the Weyl chamber, and as in the case of $\mathfrak{so}_6\mathbb{C}$ we have thereby established the existence of the irreducible representations of $\mathfrak{so}_7\mathbb{C}$ with highest weight in the sublattice $\mathbb{Z}\{L_1, L_2, L_3\}$. To complete the description, we need to know that the representation Γ_α with highest weight $\alpha = (L_1 + L_2 + L_3)/2$ exists, and what it looks like, and this time there is no isomorphism to provide this; we will have to wait until the following lecture. In the meantime, we can still have fun playing around both with the representations we do know exist, and also with those whose existence is simply asserted.

Exercise 19.11. Find the decomposition into irreducible representations of the tensor product $V \otimes \bigwedge^2 V$; in particular find the multiplicities of the irreducible representation $\Gamma_{2L_1+L_2}$ with highest weight $2L_1 + L_2$.

Exercise 19.12. Show that the symmetric square of the representation Γ_α decomposes into a copy of $\bigwedge^3 V$ and a trivial one-dimensional representation.

Exercise 1913. Find the decomposition into irreducible representations of $\bigwedge^2 \Gamma_\alpha$.

§19.4. Representations of the Odd Orthogonal Algebras

We will now describe as much as we can of the general pattern for representations of the odd orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$. As in the case of the even orthogonal Lie algebras, the proof of the existence part of the basic theorem (14.18) (that is, the construction of the irreducible representation with given highest weight) will not be complete until the following lecture, but we can work around this pretty well.

To begin with, recall that the weight lattice of $\mathfrak{so}_{2n+1}\mathbb{C}$ is, like that of $\mathfrak{so}_{2n}\mathbb{C}$, generated by $L_1, \ldots, L_n$ together with the further vector $(L_1 + \cdots + L_n)/2$. The Weyl chamber, on the other hand, is the cone

$$\mathscr{W} = \{\sum a_i L_i \colon a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.$$

The Weyl chamber is as we have pointed out the same as for $\mathfrak{sp}_{2n}\mathbb{C}$, that is, it is a simplicial cone with faces corresponding to the n planes $a_1 = a_2, \ldots, a_{n-1} = a_n$ and $a_n = 0$. The edges of the Weyl chamber are thus the rays generated by the vectors $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}$ and $L_1 + \cdots + L_n$ (note that $L_1 + \cdots + L_{n-1}$ *is* on an edge of the Weyl chamber). Again, the intersection of the weight lattice with the closed Weyl cone is a free semigroup, in this case generated by the fundamental weights $\omega_1 = L_1$, $\omega_2 = L_1 + L_2$, $\ldots$, $\omega_{n-1} = L_1 + \cdots + L_{n-1}$ and the weight $\omega_n = \alpha = (L_1 + \cdots + L_n)/2$. Moreover, as we saw in the cases of $\mathfrak{so}_5\mathbb{C}$ and $\mathfrak{so}_7\mathbb{C}$, the exterior powers of the standard representation do serve to generate all the irreducible representations whose highest weights are in the sublattice $\mathbb{Z}\{L_1, \ldots, L_n\}$: in general we have the following theorem.

Theorem 19.14. *For $k = 1, \ldots, n$, the exterior power $\bigwedge^k V$ of the standard representation V of $\mathfrak{so}_{2n+1}\mathbb{C}$ is the irreducible representation with highest weight $L_1 + \cdots + L_k$.*

Proof. We will leave this as an exercise; the proof is essentially the same as in the case of $\mathfrak{so}_{2n}\mathbb{C}$, with enough of a difference to make it interesting. □

We have thus constructed one-half of the irreducible representations of $\mathfrak{so}_{2n+1}\mathbb{C}$: any weight γ in the closed Weyl chamber can be written

$$\gamma = a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1}) + a_n(L_1 + \cdots + L_n)/2$$

with $a_i \in \mathbb{N}$; and if a_n is even, the representation

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\textstyle\bigwedge^{n-1} V) \otimes \mathrm{Sym}^{a_n/2}(\bigwedge^n V)$$

will contain an irreducible representation Γ_γ with highest weight γ. We are still missing, however, any representation whose weights involve odd multiples of α; to construct these, we clearly have to exhibit an irreducible representation Γ_α with highest weight α. This exists and is called (as in the case of the even orthogonal Lie algebras) the *spin representation* of $\mathfrak{so}_{2n+1}\mathbb{C}$. We see from the above that once we exhibit the spin representation Γ_α, we will have constructed all the representations of $\mathfrak{so}_{2n+1}\mathbb{C}$; for any γ as above the tensor

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\textstyle\bigwedge^{n-1} V) \otimes \mathrm{Sym}^{a_n}(\Gamma_\alpha)$$

will contain a copy of Γ_γ.

As in the case of the spin representation Γ_α of the even orthogonal Lie algebras, we can say some things about Γ_α even in advance of its explicit construction; for example, we can do the following exercises.

Exercise 19.15. Find the weights (with multiplicities) of the representations $\bigwedge^k V$, and also of Γ_α.

Exercise 19.16. Give the complete decomposition into irreducible representations of $\mathrm{Sym}^2\Gamma_\alpha$ and $\bigwedge^2\Gamma_\alpha$. Show that

$$\Gamma_\alpha \otimes \Gamma_\alpha = \textstyle\bigwedge^n V \oplus \bigwedge^{n-1} V \oplus \bigwedge^{n-2} V \oplus \cdots \oplus \bigwedge^1 V \oplus \bigwedge^0 V.$$

Exercise 19.17. Verify directly the above statements in the case of $\mathfrak{so}_5\mathbb{C}$, using the isomorphism with $\mathfrak{sp}_4\mathbb{C}$.

§19.5. Weyl's Construction for Orthogonal Groups

The same procedure we saw in the symplectic case can be used to construct representations of the orthogonal groups, this time generalizing what we saw directly for $\bigwedge^k V$ in §§19.2 and 19.4. For the symmetric form Q on $V = \mathbb{C}^m$, the same formula (17.9) determines contractions from $V^{\otimes d}$ to $V^{\otimes(d-2)}$. Denote the intersection of the kernels of all these contractions by $V^{[d]}$. For any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ of d, let

$$\mathbb{S}_{[\lambda]}V = V^{[d]} \cap \mathbb{S}_\lambda V. \tag{19.18}$$

As before, this is a representation of the orthogonal group $\mathrm{O}_m\mathbb{C}$ of Q.

Theorem 19.19. *The space $\mathbb{S}_{[\lambda]}V$ is an irreducible representation of $\mathrm{O}_m\mathbb{C}$; $\mathbb{S}_{[\lambda]}V$ nonzero if and only if the sum of the lengths of the first two columns of the Young diagram of λ is at most m.*

The tensor power $V^{\otimes d}$ decomposes exactly as in Lemma 17.15, with everything the same but replacing the symbol $\langle d\rangle$ by $[d]$. In particular,

$$\mathbb{S}_{[\lambda]}V = V^{[d]} \cdot c_\lambda = \mathrm{Im}(c_\lambda\colon V^{[d]} \to V^{[d]}).$$

Exercise 19.20. Verify that $\mathbb{S}_{[\lambda]}V$ is zero when the sum of the lengths of the first two columns is greater than m by showing that $\bigwedge^a V \otimes \bigwedge^b V \otimes V^{(d-a-b)}$ is contained in $\sum_I \Psi_I(V^{\otimes(d-2)})$ when $a + b > m$. Show that $\mathbb{S}_{[\lambda]}V$ is not zero when the sum of the lengths of the first two columns is at most m.

Exercise 19.21*. (i) Show that the kernel of the contraction from $\mathrm{Sym}^d V$ to $\mathrm{Sym}^{d-2}V$ is the irreducible representation $\mathbb{S}_{[d]}V$ of $\mathfrak{so}_m\mathbb{C}$ with highest weight dL_1.

(ii) Show that

$$\mathrm{Sym}^d V = \mathbb{S}_{[d]}V \oplus \mathbb{S}_{[d-2]}V \oplus \cdots \oplus \mathbb{S}_{[d-2p]}V,$$

where p is the largest integer $\leq d/2$.

The proof of the theorem proceeds exactly as in §17.3. The fundamental fact from invariant theory is the same statement as (17.19), with, of course, the operators $\vartheta_I = \Psi_I \circ \Phi_I$ defined using the given symmetric form, and the group $\mathrm{Sp}_{2n}\mathbb{C}$ replaced by $\mathrm{O}_m\mathbb{C}$ (and the same reference to Appendix F.2 for the proof). The theorem then follows from Lemma 6.22 in exactly the same way as for the symplectic group.

To find the irreducible representations over $\mathrm{SO}_m\mathbb{C}$ one can proceed as in §5.1. Weyl calls two partitions (each with the sum of the first two column lengths at most m) *associated* if the sum of the lengths of their first columns is m and the other columns of their Young diagrams have the same lengths. Representations of associated partitions restrict to isomorphic representations of $\mathrm{SO}_m\mathbb{C}$. Note that at least one of each pair of associated partitions will have a Young diagram with at most $\frac{1}{2}m$ rows. If $m = 2n + 1$ is odd, no λ is associated to itself, but if $m = 2n$ is even, any λ with a Young diagram with n nonzero rows will be associated to itself, and its restriction will be the sum of two conjugate representations of $\mathrm{SO}_m\mathbb{C}$ of the same dimension. The final result is:

Theorem 19.22. (i) *If $m = 2n + 1$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, then $\mathbb{S}_{[\lambda]}V$ is the irreducible representation of $\mathfrak{so}_m\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

(ii) *If $m = 2n$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0)$, then $\mathbb{S}_{[\lambda]}V$ is the irreducible representation of $\mathfrak{so}_m\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

(iii) *If $m = 2n$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n > 0)$, then $\mathbb{S}_{[\lambda]}V$ is the sum of two irreducible representations of $\mathfrak{so}_m\mathbb{C}$ with highest weights $\lambda_1 L_1 + \cdots + \lambda_n L_n$ and $\lambda_1 L_1 + \cdots + \lambda_{n-1} L_{n-1} - \lambda_n L_n$.*

Exercise 19.23. When m is odd, show that $\mathrm{O}_m\mathbb{C} = \mathrm{SO}_m\mathbb{C} \times \{\pm I\}$. Show that if λ and μ are associated, then $\mu = \lambda \otimes \varepsilon$, where ε is the sign of the determinant.

We postpone to Lecture 25 all discussion of multiplicities of weight spaces, or decomposing tensor products or restrictions to subgroups.

As we saw in Lecture 15 for $\mathrm{GL}_n\mathbb{C}$ and in Lecture 17 for $\mathrm{Sp}_{2n}\mathbb{C}$, it is possible to make a commutative algebra $\mathbb{S}^{[\cdot]} = \mathbb{S}^{[\cdot]}(V)$ out of the sum of all the irreducible representations of $\mathrm{SO}_m\mathbb{C}$, where $V = \mathbb{C}^m$ is the standard representation. First suppose $m = 2n + 1$ is odd. Define the ring $\mathbb{S}^{\cdot}(V, n)$ as in §15.5, which is a sum of all the representations $\mathbb{S}_\lambda(V)$ of $\mathrm{GL}(V)$ where λ runs over all partitions with at most n parts. As in the symplectic case, there is a canonical decomposition

$$\mathbb{S}_\lambda(V) = \mathbb{S}_{[\lambda]}(V) \oplus J_{[\lambda]}(V),$$

and the direct sum $J^{[\cdot]} = \bigoplus_\lambda J_{[\lambda]}(V)$ is an ideal in $\mathbb{S}^{\cdot}(V, n)$. The quotient ring

$$\mathbb{S}^{[\cdot]}(V) = A^{\cdot}(V, n)/J^{[\cdot]} = \bigoplus_\lambda \mathbb{S}_{[\lambda]}(V)$$

is a commutative graded ring which contains each irreducible representation of $\mathrm{SO}_{2n+1}\mathbb{C}$ once.

If $m = 2n$ is even, the above quotient will contain each representation $\mathbb{S}_{[\lambda]}(V)$ twice if λ has n rows. To cut it down so there is only one of each, one can add to $J^{[\cdot]}$ relations of the form $x - \tau(x)$, for $x \in \bigwedge^n V$, where $\tau\colon \bigwedge^n V \to \bigwedge^n V$ is the isomorphism described in the remark (iii) after the proof of Theorem 19.2. For a detailed discussion, with explicit generators for the ideas, see [L-T].

LECTURE 20

Spin Representations of $\mathfrak{so}_m\mathbb{C}$

In this lecture we complete the picture of the representations of the orthogonal Lie algebras by constructing the spin representations $S^{\pm}$ of $\mathfrak{so}_m\mathbb{C}$; this also yields a description of the spin groups $\mathrm{Spin}_m\mathbb{C}$. Since the representation-theoretic analysis of the spaces $S^{\pm}$ was carried out in the preceding lecture, we are concerned here primarily with the algebra involved in their construction. Thus, §20.1 and §20.2, while elementary, involve some fairly serious algebra. Section 20.3, where we briefly sketch the notion of triality, may seem mysterious to the reader (this is at least in part because it is so to the authors); if so, it may be skipped. Finally, we should say that the subject of the spin representations of $\mathfrak{so}_m\mathbb{C}$ is a very rich one, and one that accommodates many different points of view; the reader who is interested is encouraged to try some of the other approaches that may be found in the literature.

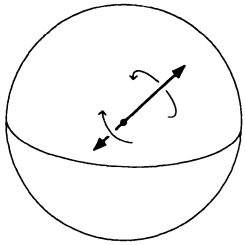
§20.1. Clifford Algebras and Spin Representations of $\mathfrak{so}_m\mathbb{C}$

We begin this section by trying to motivate the definition of Clifford algebras. We may begin by asking, why were we able to find all the representations of $\mathrm{SL}_n\mathbb{C}$ or $\mathrm{Sp}_{2n}\mathbb{C}$ inside tensor powers of the standard representation, but only half the representations of $\mathrm{SO}_m\mathbb{C}$ arise this way? One difference that points in this direction lies in the topology of these groups: $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$ are simply connected, while $\mathrm{SO}_m\mathbb{C}$ has fundamental group $\mathbb{Z}/2$ for $m > 2$ (for proofs see §23.1). Therefore $\mathrm{SO}_m\mathbb{C}$ has a double covering, the *spin group* $\mathrm{Spin}_m\mathbb{C}$. (For $m \leq 6$, these coverings could also be extracted from our identifications of the adjoint group $PSO_m\mathbb{C}$ with the adjoint group of other simply connected groups; e.g. the double cover of $SO_3\mathbb{C}$ is $SL_2\mathbb{C}$.) We will see that the missing representations are those representations of $Spin_m\mathbb{C}$ that do not come from representations of $SO_m\mathbb{C}$.

This double covering may be most readily visible, and probably familiar, for the case of the real subgroup $SO_3\mathbb{R}$ of rotations: a rotation is specified by an axis to rotate about, given by a unit vector u, and an angle of rotation about u; the two choices $\pm u$ of unit vector give a two-sheeted covering. In other words, if D^3 is the unit ball in $\mathbb{R}^3$, there is a double covering

$$S^3 = D^3/\partial D^3 \to SO_3\mathbb{R},$$

which sends a vector v in D^3 to rotation by the angle $2\pi\|v\|$ about the unit vector $v/\|v\|$ (the origin and the unit sphere ∂D^3 are sent to the identity transformation).



This covering is even easier to see for the entire orthogonal group $O_3\mathbb{R}$, which is generated by reflections R_v in unit vectors v (with $\pm v$ determining the same reflection): we can describe the double cover of $O_3\mathbb{R}$ as the group generated by unit vectors v, with relations

$$v_1 \cdot \ldots \cdot v_n = w_1 \cdot \ldots \cdot w_m$$

whenever the compositions of the corresponding reflections are equal, i.e., whenever

$$R_{v_1} \circ \cdots \circ R_{v_n} = R_{w_1} \circ \cdots \circ R_{w_m};$$

and also relations

$$(-v)\cdot(-w) = v \cdot w$$

for all pairs of unit vectors v and w. (Note that if we restricted ourselves to products of even numbers of the generators $v \in \partial D^3$ we would get back the double cover of the special orthogonal group $SO_3\mathbb{C}$.)

How should we generalize this? The answer is not obvious. For one thing, for various reasons we will not try to construct directly a group that covers the orthogonal group in general. Instead, given a vector space V (real or complex) and a quadratic form Q on V, we will first construct an algebra $\mathrm{Cliff}(V, Q)$, called the *Clifford algebra*. The algebra $\mathrm{Cliff}(V, Q)$ will then turn out to contain in its multiplicative group a subgroup which is a double cover of the orthogonal group $O(V, Q)$ of automorphisms of V preserving Q.

By analogy with the construction of the double cover of $SO_3\mathbb{R}$, the Clifford algebra $\mathrm{Cliff}(V, Q)$ associated to the pair (V, Q) is an associative algebra containing and generated by V. (When we want to describe the spin group inside $\mathrm{Cliff}(V, Q)$ we will restrict ourselves to products of even numbers of elements of V having a fixed norm $Q(v, v)$; if odd products are allowed as well, we get a group called "Pin" which is a double covering of the whole orthogonal group.) To motivate the definition, we would like $\mathrm{Cliff}(V, Q)$ to be the algebra generated by V subject to relations analogous to those above for the double cover of the orthogonal group. In particular, for any vector v with $Q(v, v) = 1$, since the reflection R_v in the hyperplane perpendicular to v is an involution, we want

$$v \cdot v = 1$$

in $\mathrm{Cliff}(V, Q)$. By polarization, this is the same as imposing the relation

$$v \cdot w + w \cdot v = 2Q(v, w)$$

for all v and w in V. In particular, $w \cdot v = -v \cdot w$ if v and w are perpendicular. In fact, the Clifford algebra[1] will be defined below to be the associative algebra generated by V and subject to the equation $v \cdot v = Q(v, v)$.

Looking ahead, we will see later in this section that each complex Clifford algebra contains an orthogonal Lie algebra as a subalgebra. The key theorem is then that $\mathrm{Cliff}(V, Q)$ *is isomorphic either to a matrix algebra or to a sum of two matrix algebras*. This in turn determines either one or two representations of the orthogonal Lie algebras, which turn out to be the representations which were needed to complete the story in the last lecture. Just as in the special linear and symplectic cases, the corresponding Lie groups are not really needed to construct the representations; they can be written down directly from the Lie algebra. In this section we do this, using the Clifford algebras to construct these representations of $\mathfrak{so}_m\mathbb{C}$ directly, and verify that they give the missing spin representations. In the second section of this lecture we will show how the spin groups sit as subgroups in their multiplicative groups.

Clifford Algebras

Given a symmetric bilinear form Q on a vector space V, the *Clifford algebra* $C = C(Q) = \mathrm{Cliff}(V, Q)$ is an associative algebra with unit 1, which contains and is generated by V, with $v \cdot v = Q(v, v) \cdot 1$ for all $v \in V$. Equivalently, we have the equation

$$v \cdot w + w \cdot v = 2Q(v, w), \tag{20.1}$$

[1] The mathematical world seems to be about evenly divided about the choice of signs here, and one must translate from Q to $-Q$ to go from one side to the other.

for all v and w in V. The Clifford algebra can be defined to be the universal algebra with this property: if E is any associative algebra with unit, and a linear mapping $j\colon V \to E$ is given such that $j(v)^2 = Q(v, v)\cdot 1$ for all $v \in V$, or equivalently

$$j(v)\cdot j(w) + j(w)\cdot j(v) = 2Q(v, w)\cdot 1 \tag{20.2}$$

for all $v, w \in V$, then there should be a unique homomorphism of algebras from $C(Q)$ to E extending j. The Clifford algebra can be constructed quickly by taking the tensor algebra

$$T^{\bullet}(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots,$$

and setting $C(Q) = T^{\bullet}(V)/I(Q)$, where $I(Q)$ is the two-sided ideal generated by all elements of the form $v \otimes v - Q(v, v)\cdot 1$. It is automatic that this $C(Q)$ satisfies the required universal property.

The facts that the dimension of C is 2^m, where $m = \dim(V)$, and that the canonical mapping from V to C is an embedding, are part of the following lemma:

Lemma 20.3. *If $e_1, \ldots, e_m$ form a basis for V, then the products $e_I = e_{i_1}\cdot e_{i_2}\cdot\ldots\cdot e_{i_k}$, for $I = \{i_1 < i_2 < \cdots < i_k\}$, and with $e_\phi = 1$, form a basis for $C(Q) = \mathrm{Cliff}(V, Q)$.*

Proof. From the equations $e_i\cdot e_j + e_j\cdot e_i = 2Q(e_i, e_j)$ it follows immediately that the elements e_I generate $C(Q)$. Their independence is not hard to verify directly; it also follows by seeing that the images in the matrix algebras under the mappings constructed below are independent. For another proof, note that when $Q \equiv 0$, the Clifford algebra is just the exterior algebra $\bigwedge^{\bullet}V$. In general, the Clifford algebra can be filtered by subspaces F_k, consisting of those elements which can be written as sums of at most k products of elements in V; one checks that the associated graded space F_k/F_{k+1} is $\bigwedge^k V$. For a third proof, one can verify that the Clifford algebra of the direct sum of two orthogonal spaces is the skew commutative tensor product of the Clifford algebras of the two spaces (cf. Exercise B.9), which reduces one to the trivial case where $\dim V = 1$. $\square$

Since the ideal $I(Q) \subset T(V)$ is generated by elements of even degree, the Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$ grading:

$$C = C^{\mathrm{even}} \oplus C^{\mathrm{odd}} = C^+ \oplus C^-,$$

with $C^+ \cdot C^+ \subset C^+, C^+ \cdot C^- \subset C^-, C^- \cdot C^+ \subset C^-, C^- \cdot C^- \subset C^+$; C^+ is spanned by products of an even number of elements in V and C^- is spanned by products of an odd number. In particular, C^{even} is a subalgebra of dimension 2^{m-1}.

Since $C(Q)$ is an associative algebra, it determines a Lie algebra, with bracket $[a, b] = a\cdot b - b\cdot a$. From now on we assume Q is nondegenerate. The new representations of $\mathfrak{so}_m\mathbb{C}$ will be found in two steps:

(i) embedding the Lie algebra $\mathfrak{so}(Q) = \mathfrak{so}_m\mathbb{C}$ inside the Lie algebra of the even part of the Clifford algebra $C(Q)$;
(ii) identifying the Clifford algebras with one or two copies of matrix algebras.

To carry out the first step we make explicit the isomorphism of $\bigwedge^2 V$ with $\mathfrak{so}(Q)$ that we have discussed before. Recall that

$$\mathfrak{so}(Q) = \{X \in \operatorname{End}(V)\colon Q(Xv, w) + Q(v, Xw) = 0 \text{ for all } v, w \text{ in } V\}.$$

The isomorphism is given by

$$\textstyle\bigwedge^2 V \xrightarrow{\cong} \mathfrak{so}(Q) \subset \operatorname{End}(V), \qquad a \wedge b \mapsto \varphi_{a\wedge b},$$

for a and b in V, where $\varphi_{a\wedge b}$ is defined by

$$\varphi_{a\wedge b}(v) = 2(Q(b, v)a - Q(a, v)b). \tag{20.4}$$

It is a simple verification that $\varphi_{a\wedge b}$ is in $\mathfrak{so}(Q)$. One sees that the natural bases correspond up to scalars, e.g., $e_i \wedge e_{n+j}$ maps to $2(E_{i,j} - E_{n+j,n+i})$, so the map is an isomorphism. (The choice of scalar factor is unimportant here; it was chosen to simplify later formulas.) One calculates what the bracket on $\bigwedge^2 V$ must be to make this an isomorphism of Lie algebras:

$$\begin{aligned}
[\varphi_{a\wedge b}, \varphi_{c\wedge d}](v) &= \varphi_{a\wedge b} \circ \varphi_{c\wedge d}(v) - \varphi_{c\wedge d} \circ \varphi_{a\wedge b}(v)\\
&= 2\varphi_{a\wedge b}(Q(d, v)c - Q(c, v)d) - 2\varphi_{c\wedge d}(Q(b, v)a - Q(a, v)b)\\
&= 4Q(d, v)(Q(b, c)a - Q(a, c)b)\\
&\quad - 4Q(c, v)(Q(b, d)a - Q(a, d)b)\\
&\quad - 4Q(b, v)(Q(d, a)c - Q(c, a)d)\\
&\quad + 4Q(a, v)(Q(d, b)c - Q(c, b)d)\\
&= 2Q(b, c)\varphi_{a\wedge d}(v) - 2Q(b, d)\varphi_{a\wedge c}(v)\\
&\quad - 2Q(a, d)\varphi_{c\wedge b}(v) + 2Q(a, c)\varphi_{d\wedge b}(v).
\end{aligned}$$

This gives an explicit formula for the bracket on $\bigwedge^2 V$:

$$\begin{aligned}
[a \wedge b, c \wedge d] = 2Q(b, c)a \wedge d - 2Q(b, d)a \wedge c&\\
- 2Q(a, d)c \wedge b + 2Q(a, c)d \wedge b.&
\end{aligned} \tag{20.5}$$

On the other hand, the bracket in the Clifford algebra satisfies

$$\begin{aligned}
[a\cdot b, c\cdot d] &= a\cdot b\cdot c\cdot d - c\cdot d\cdot a\cdot b\\
&= (2Q(b, c)a\cdot d - a\cdot c\cdot b\cdot d) - (2Q(a, d)c\cdot b - c\cdot a\cdot d\cdot b)\\
&= 2Q(b, c)a\cdot d - (2Q(b, d)a\cdot c - a\cdot c\cdot d\cdot b)\\
&\quad - 2Q(a, d)c\cdot b + (2Q(a, c)\cdot d\cdot b - a\cdot c\cdot d\cdot b)\\
&= 2Q(b, c)a\cdot d - 2Q(b, d)a\cdot c - 2Q(a, d)c\cdot b + 2Q(a, c)\cdot d\cdot b.
\end{aligned}$$

It follows that the map $\psi: \bigwedge^2 V \to \mathrm{Cliff}(V, Q)$ defined by

$$\psi(a \wedge b) = \tfrac{1}{2}(a \cdot b - b \cdot a) = a \cdot b - Q(a, b) \tag{20.6}$$

is a map[2] of Lie algebras, and by looking at basis elements again one sees that it is an embedding. This proves:

Lemma 20.7. *The mapping* $\psi \circ \varphi^{-1}: \mathfrak{so}(Q) \to C(Q)^{\text{even}}$ *embeds* $\mathfrak{so}(Q)$ *as a Lie subalgebra of* $C(Q)^{\text{even}}$.

Exercise 20.8. Show that the image of ψ is

$$F_2 \cap C(Q)^{\text{even}} \cap \mathrm{Ker}(\text{trace}),$$

where F_2 is the subspace of $C(Q)$ spanned by products of at most two elements of V, and the trace of an element of $C(Q)$ is the trace of left multiplication by that element on $C(Q)$.

We consider first the *even* case: write $V = W \oplus W'$, where W and W' are n-dimensional isotropic spaces for Q. (Recall that a space is isotropic when Q restricts to the zero form on it.) With our choice of standard Q on $V = \mathbb{C}^{2n}$, W can be taken to be the space spanned by the first n basis vectors, W' by the last n.

Lemma 20.9. *The decomposition* $V = W \oplus W'$ *determines an isomorphism of algebras*

$$C(Q) \cong \mathrm{End}(\textstyle\bigwedge^{\bullet} W),$$

where $\bigwedge^{\bullet} W = \bigwedge^0 W \oplus \cdots \oplus \bigwedge^n W$.

PROOF. Mapping $C(Q)$ to the algebra $E = \mathrm{End}(\bigwedge^{\bullet} W)$ is the same as defining a linear mapping from V to E, satisfying (20.2). We must construct maps $l: W \to E$ and $l': W' \to E$ such that

$$l(w)^2 = 0, \qquad l'(w')^2 = 0, \tag{20.10}$$

and

$$l(w) \circ l'(w') + l'(w') \circ l(w) = 2Q(w, w')I$$

for any $w \in W$, $w' \in W'$. For each $w \in W$, let $L_w \in E$ be left multiplication by w on the exterior algebra $\bigwedge^{\bullet} W$:

$$L_w(\xi) = w \wedge \xi, \quad \xi \in \textstyle\bigwedge^{\bullet} W.$$

For $\vartheta \in W^*$, let $D_\vartheta \in E$ be the derivation of $\bigwedge^{\bullet} W$ such that $D_\vartheta(1) = 0$, $D_\vartheta(w) = \vartheta(w) \in \bigwedge^0 W = \mathbb{C}$ for $w \in W = \bigwedge^1 W$, and

[2] Note that the bilinear form ψ given by (20.6) is alternating since $\psi(a \wedge a) = 0$, so it defines a linear map on $\bigwedge^2 V$.

$$D_\vartheta(\zeta \wedge \xi) = D_\vartheta(\zeta) \wedge \xi + (-1)^{\deg(\zeta)}\zeta \wedge D_\vartheta(\zeta).$$

Explicitly, $D_\vartheta(w_1 \wedge \cdots \wedge w_r) = \sum(-1)^{i-1}\vartheta(w_i)(w_1 \wedge \cdots \wedge \hat{w}_i \wedge \cdots \wedge w_r)$. Now set

$$l(w) = L_w, \qquad l'(w') = D_\vartheta, \tag{20.11}$$

where $\vartheta \in W^*$ is defined by the identity $\vartheta(w) = 2Q(w, w')$ for all $w \in W$. The required equations (20.10) are straightforward verifications: one checks directly on elements in $W = \bigwedge^1 W$, and then that, if they hold on ζ and ξ, they hold on $\zeta \wedge \xi$. Finally, one may see that the resulting map is an isomorphism by looking at what happens to a basis. □

Exercise 20.12. The left $C(Q)$-module $\bigwedge^\bullet W$ is isomorphic to a left ideal in $C(Q)$. Show that if f is a generator for $\bigwedge^n W'$, then $C(Q)\cdot f = \bigwedge^\bullet W \cdot f$, and the map $\zeta \mapsto \zeta \cdot f$ gives an isomorphism

$$\textstyle\bigwedge^\bullet W \to \bigwedge^\bullet W \cdot f = C(Q)\cdot f$$

of left $C(Q)$-modules.

Now we have a decomposition $\bigwedge^\bullet W = \bigwedge^{\text{even}} W \oplus \bigwedge^{\text{odd}} W$ into the sum of even and odd exterior powers, and $C(W)^{\text{even}}$ respects this splitting. We deduce from Lemma 20.9 an isomorphism

$$C(Q)^{\text{even}} \cong \operatorname{End}(\textstyle\bigwedge^{\text{even}} W) \oplus \operatorname{End}(\bigwedge^{\text{odd}} W). \tag{20.13}$$

Combining with Lemma 20.7, we now have an embedding of Lie algebras:

$$\mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\textstyle\bigwedge^{\text{even}} W) \oplus \mathfrak{gl}(\bigwedge^{\text{odd}} W), \tag{20.14}$$

and hence we have two representations of $\mathfrak{so}(Q) = \mathfrak{so}_{2n}\mathbb{C}$, which we denote by

$$S^+ = \textstyle\bigwedge^{\text{even}} W \quad \text{and} \quad S^- = \bigwedge^{\text{odd}} W.$$

Proposition 20.15. *The representations $S^\pm$ are the irreducible representations of $\mathfrak{so}_{2n}\mathbb{C}$ with highest weights $\alpha = \frac{1}{2}(L_1 + \cdots + L_n)$ and $\beta = \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$. More precisely,*

$$S^+ = \Gamma_\alpha \quad \textit{and} \quad S^- = \Gamma_\beta \quad \textit{if } n \textit{ is even};$$

$$S^+ = \Gamma_\beta \quad \textit{and} \quad S^- = \Gamma_\alpha \quad \textit{if } n \textit{ is odd}.$$

Proof. We show that the natural basis vectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $\bigwedge^\bullet W$ are weight vectors. Tracing through the isomorphisms established above, we see that $H_i = E_{i,i} - E_{n+i,n+i}$ in $\mathfrak{h} \subset \mathfrak{so}_{2n}\mathbb{C}$ corresponds to $\frac{1}{2}(e_i \wedge e_{n+i})$ in $\bigwedge^2 V$, which corresponds to $\frac{1}{2}(e_i \cdot e_{n+i} - 1)$ in $C(Q)$, which maps to

$$\tfrac{1}{2}(L_{e_i} \circ D_{2e_i^*} - I) = L_{e_i} \circ D_{e_i^*} - \tfrac{1}{2}I \in \operatorname{End}(\textstyle\bigwedge^\bullet W).$$

A simple calculation shows that

$$L_{e_i} \circ D_{e_i^*}(e_I) = \begin{cases} e_I & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

Therefore, e_I spans a weight space with weight $\frac{1}{2}(\sum_{i \in I} L_i - \sum_{j \notin I} L_j)$. All such weights with given $|I|$ mod 2 are congruent by the Weyl group, so each of $S^+ = \bigwedge^{\text{even}} W^+$ and $S^- = \bigwedge^{\text{odd}} W$ must be an irreducible representation. The highest weights are easy to read off. For example, the highest weight for $\bigwedge^{\text{even}} W$ is $\frac{1}{2}\sum L_i = \alpha$ if n is even, while if n is odd, its highest weight is β. □

These two representations S^+ and S^- are usually called the *half-spin representations* of $\mathfrak{so}_{2n}\mathbb{C}$, while their sum $S = S^+ \oplus S^- = \bigwedge^\bullet W$ is called the *spin representation*. Frequently, especially when we speak of the even and odd cases together, we call them all simply "spin representations." Elements of S are called *spinors*. For other proofs of the proposition see Exercises 20.34 and 20.35.

For the *odd* case, write $V = W \oplus W' \oplus U$, where W and W' are n-dimensional isotropic subspaces, and U is a one-dimensional space perpendicular to them. For our standard Q on $\mathbb{C}^{2n+1}$, these are spanned by the first n, the second n, and the last basis vector.

Lemma 20.16. *The decomposition $V = W \oplus W' \oplus U$ determines an isomorphism of algebras*

$$C(Q) \cong \operatorname{End}(\textstyle\bigwedge^\bullet W) \oplus \operatorname{End}(\bigwedge^\bullet W').$$

PROOF. Proceeding as in the even case, to map V to $E = \operatorname{End}(\bigwedge^\bullet W)$, map $w \in W$ to L_w, $w' \in W'$ to D_ϑ, where $\vartheta(w) = 2Q(w, w')$ as before. Let u_0 be the element in U such that $Q(u_0, u_0) = 1$, and send u_0 to the endomorphism that is the identity on $\bigwedge^{\text{even}} W$, and minus the identity on $\bigwedge^{\text{odd}} W$. Since this involution skew commutes with all L_w and D_ϑ, the resulting map from $V = W \oplus W' \oplus U$ to E determines an algebra homomorphism from $C(Q)$ to E. The map to $\operatorname{End}(\bigwedge^\bullet W')$ is defined similarly, reversing the roles of W and W'. Again one checks that the map is an isomorphism by looking at bases. □

Exercise 20.17*. Find a generator for a left ideal of $C(Q)$ that is isomorphic to $\bigwedge^\bullet W$.

The subalgebra $C(Q)^{\text{even}}$ of $C(Q)$ is mapped isomorphically onto either of the factors by the isomorphism of the lemma, so we have an isomorphism in the odd case:

$$C(Q)^{\text{even}} \cong \operatorname{End}(\textstyle\bigwedge^\bullet W). \tag{20.18}$$

As before, this gives a representation $S = \bigwedge^\bullet W$ of Lie algebras:

$$\mathfrak{so}_{2n+1}\mathbb{C} = \mathfrak{so}(Q) \subset C(Q)^{\text{even}} \cong \mathfrak{gl}(\textstyle\bigwedge^\bullet W) = \mathfrak{gl}(S). \tag{20.19}$$

Proposition 20.20. *The representation $S = \wedge^{\bullet} W$ is the irreducible representation of $\mathfrak{so}_{2n+1}\mathbb{C}$ with highest weight*

$$\alpha = \tfrac{1}{2}(L_1 + \cdots + L_n).$$

PROOF. Exactly as in the even case, each e_I is an eigenvector with with weight $\frac{1}{2}(\sum_{i \in I} L_i - \sum_{j \notin I} L_j)$. This time all such weights are congruent by the Weyl group, so this must be an irreducible representation, and the highest weight is clearly $\frac{1}{2}(L_1 + \cdots + L_n)$. □

As we saw in Lecture 19, the construction of this *spin representation* S finishes the proof of the existence theorem for representations of $\mathfrak{so}_m\mathbb{C}$, and hence for all of the classical complex semisimple Lie algebras.

Exercise 20.21*. Use the above identification of the Clifford algebras with matrix algebras (or direct calculation) to compute their centers. In particular, show that the intersection of the center of C with the even subalgebra C^{even} is always the one-dimensional space of scalars. Show similarly that if x is in C^{odd} and $x \cdot v = -v \cdot x$ for all v in V, then $x = 0$.

Exercise 20.22*. For $X \in \mathfrak{so}(Q)$ and $v \in V$, we have $X \cdot v \in V$ by the standard action of $\mathfrak{so}(Q)$ on V. On the other hand, we have identified $\mathfrak{so}(Q)$ and V as subspaces of the Clifford algebra C, so we can compute the commutator $[X, v]$. Show that these agree:

$$X \cdot v = [X, v] \in V \subset C.$$

Problem 20.23*. Let $C(p, q)$ be the real Clifford algebra corresponding to the quadratic form with p positive and q negative eigenvalues. Lemmas 20.9 and 20.16 actually construct isomorphisms of $C(n, n)$ with a real matrix algebra, and of $C(n + 1, n)$ with a product of two real matrix algebras. Compute $C(p, q)$ for other p and q. All are products of one or two matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

§20.2. The Spin Groups $\mathrm{Spin}_m\mathbb{C}$ and $\mathrm{Spin}_m\mathbb{R}$

The Clifford algebra $C = C(Q)$ is generated by the subspace $V = \mathbb{C}^m$, and C has an anti-involution $x \mapsto x^*$, determined by

$$(v_1 \cdot \ldots \cdot v_r)^* = (-1)^r v_r \cdot \ldots \cdot v_1$$

for any $v_1, \ldots, v_r$ in V. This operation $*$, sometimes called the *conjugation*, is the composite of:

the *main antiautomorphism* or reversing map $\tau\colon C \to C$ determined by

$$\tau(v_1 \cdot \ldots \cdot v_r) = v_r \cdot \ldots \cdot v_1 \tag{20.24}$$

for $v_1, \ldots, v_r$ in V, and

the *main involution* α which is the identity on C^{even} and minus the identity on C^{odd}, i.e.,

$$\alpha(v_1 \cdot \ldots \cdot v_r) = (-1)^r v_1 \cdot \ldots \cdot v_r. \tag{20.25}$$

Note that $(x \cdot y)^* = y^* \cdot x^*$, which comes from the identities $\tau(x \cdot y) = \tau(y) \cdot \tau(x)$ and $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$.

Exercise 20.26. Use the universal property for C to verify that these are well defined: show that α is a homomorphism from C to C and τ is a well-defined homomorphism from C to the opposite algebra of C (the algebra with the same vector space structure, but with reversed multiplication: $x \tilde{\cdot} y = y \cdot x$).

Instead of defining the spin group as the set of products of certain elements of V, it will be convenient to start with a more abstract definition. Set

$$\operatorname{Spin}(Q) = \{x \in C(Q)^{\text{even}} \colon x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}. \tag{20.27}$$

We see from this definition that $\operatorname{Spin}(Q)$ forms a closed subgroup of the group of units in the (even) Clifford algebra. Any x in $\operatorname{Spin}(Q)$ determines an endomorphism $\rho(x)$ of V by

$$\rho(x)(v) = x \cdot v \cdot x^*, \quad v \in V.$$

Proposition 20.28. *For $x \in \operatorname{Spin}(Q)$, $\rho(x)$ is in $\operatorname{SO}(Q)$. The mapping*

$$\rho \colon \operatorname{Spin}(Q) \to \operatorname{SO}(Q)$$

is a homomorphism, making $\operatorname{Spin}(Q)$ a connected two-sheeted covering of $\operatorname{SO}(Q)$. The kernel of ρ is $\{1, -1\}$.

PROOF. We will prove something more. Define a larger subgroup, this time of the multiplicative group of $C(Q)$, by

$$\operatorname{Pin}(Q) = \{x \in C(Q) \colon x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}, \tag{20.29}$$

and define a homomorphism

$$\rho \colon \operatorname{Pin}(Q) \to \operatorname{O}(Q), \qquad \rho(x)(v) = \alpha(x) \cdot v \cdot x^*, \tag{20.30}$$

where $\alpha \colon C(Q) \to C(Q)$ is the main involution.

To see that $\rho(x)$ preserves the quadratic form Q, we use the fact that for w in V, $Q(w, w) = w \cdot w = -w \cdot w^*$, and calculate:

$$\begin{aligned} Q(\rho(x)(v), \rho(x)(v)) &= -\alpha(x) \cdot v \cdot x^* \cdot (\alpha(x) \cdot v \cdot x^*)^* \\ &= -\alpha(x) \cdot v \cdot x^* \cdot x \cdot v^* \cdot \alpha(x)^* \end{aligned}$$

$$\begin{aligned} &= -\alpha(x)\cdot v\cdot v^*\cdot \alpha(x^*) \\ &= Q(v, v)\alpha(x)\cdot\alpha(x^*) \\ &= Q(v, v)\alpha(x\cdot x^*) = Q(v, v). \end{aligned}$$

We claim next that ρ is surjective. This follows from the standard fact (see Exercise 20.32) that the orthogonal group $\mathrm{O}(Q)$ is generated by reflections. Indeed, if R_w is the reflection in the hyperplane perpendicular to a vector w, normalized so that $Q(w, w) = -1$, it is easy to see that w is in $\mathrm{Pin}(Q)$ and $\rho(w) = R_w$; in fact,

$$w\cdot w^* = w\cdot(-w) = -Q(w, w) = 1,$$

and so

$$\rho(w)(w) = \alpha(w)\cdot w\cdot w^* = -w\cdot 1 = -w;$$

and if $Q(w, v) = 0$,

$$\rho(w)(v) = \alpha(w)\cdot v\cdot w^* = -w\cdot v\cdot w^* = v\cdot w\cdot w^* = v.$$

The next claim is that the kernel of ρ on the larger group $\mathrm{Pin}(Q)$ is ± 1. Suppose x is in the kernel, and write $x = x_0 + x_1$ with $x_0 \in C^{\mathrm{even}}$ and $x_1 \in C^{\mathrm{odd}}$. Then $x_0\cdot v = v\cdot x_0$ for all $v \in V$, so x_0 is in the center of C. And $x_1\cdot v = -v\cdot x_1$ for all $v \in V$. By Exercise 20.21, x_0 is in $\mathbb{C}\cdot 1$, and $x_1 = 0$. So $x = x_0$ is in $\mathbb{C}$ and $x^2 = 1$; so $x = \pm 1$.

It follows that if $R \in \mathrm{O}(Q)$ is written as a product of reflections $R_{w_1}\circ\ldots\circ R_{w_r}$, then the two elements in $\rho^{-1}(R)$ are $\pm w_1\cdot\ldots\cdot w_r$. In particular, we get another description of the spin groups:

$$\begin{aligned} \mathrm{Spin}(Q) &= \mathrm{Pin}(Q)\cap C(Q)^{\mathrm{even}} = \rho^{-1}(\mathrm{SO}(Q)) \\ &= \{\pm w_1\cdot\ldots\cdot w_{2k} \colon w_i \in V, Q(w_i, w_i) = -1\}. \end{aligned} \tag{20.31}$$

Since $-1 = v\cdot v$ for any v with $Q(v, v) = -1$, we see that the spin group consists of even products of such elements.

To complete the proof, we must check that $\mathrm{Spin}(Q)$ is connected or, equivalently, that the two elements in the kernel of ρ can be connected by a path. We leave this now as an exercise, since much more will be seen shortly. □

Exercise 20.32*. Let Q be a nondegenerate symmetric bilinear form on a real or complex vector space V.

(a) Show that if v and w are vectors in V with $Q(v, v) = Q(w, w) \neq 0$, then there is either a reflection or a product of two reflections that takes v into w.

(b) Deduce that every element of the orthogonal group of Q can be written as the product of at most $2\cdot\dim(V)$ reflections.

Exercise 20.33*. Since $\mathrm{Spin}(Q)$ is a subgroup of the multiplicative group of $C(Q)$, its Lie algebra is a subalgebra of $C(Q)$ with its usual bracket. Verify that this subalgebra is the subalgebra $\mathfrak{so}(Q)$ that was constructed in §20.1.

Exercise 20.34. The fact that $\bigwedge^\bullet W$ (and $\bigwedge^\bullet W'$ in the odd case) is an irreducible module over $C(Q)$ is equivalent to the fact that it is an irreducible representation of the group $\mathrm{Pin}(Q)$ since the linear span of $\mathrm{Pin}(Q)$ is dense in $C(Q)$.

(a) Apply the analysis of §5.1 to the subgroup

$$\mathrm{Spin}(Q) \subset \mathrm{Pin}(Q)$$

of index two. In the odd case, $\bigwedge^\bullet W$ and $\bigwedge^\bullet W'$ are conjugate representations, so their restrictions to $\mathrm{Spin}(Q)$ are isomorphic and irreducible: this is the spin representation. In the even case, $\bigwedge^\bullet W$ is self-conjugate, and its restriction to $\mathrm{Spin}(Q)$ is a sum of two conjugate irreducible representations, which are the two half-spin representations.

(b) Of the representations of $\mathrm{Spin}(Q)$ (i.e., the representations of $\mathfrak{so}_m\mathbb{C}$), which induce irreducible representations of $\mathrm{Pin}(Q)$ and which are restrictions of irreducible representations of $\mathrm{Pin}(Q)$?

Exercise 20.35. Deduce the irreducibility of the spin and half-spin representations from the fact that their restrictions to the 2-groups of Exercise 3.9 are irreducible representations of these finite groups.

Exercise 20.36*. Show that the center of $\mathrm{Spin}_m(\mathbb{C})$ is $\rho^{-1}(1) = \{\pm 1\}$ if m is odd. If m is even show that the center is

$$\rho^{-1}(\pm 1) = \{\pm 1, \pm\omega\},$$

where, in terms of our standard basis,

$$\omega = \frac{ie_1\cdot e_{n+1} - ie_{n+1}\cdot e_1}{2}\cdot\ldots\cdot\frac{ie_n\cdot e_{2n} - ie_{2n}\cdot e_n}{2}.$$

Exercise 20.37*. Show that the spin representation $\mathrm{Spin}(Q) \to \mathrm{GL}(S)$ maps into the special linear group $\mathrm{SL}(S)$. Show that for $m = 2n$ and n even, the half-spin representations also map into the special linear groups $\mathrm{SL}(S^+)$ and $\mathrm{SL}(S^-)$.

Exercise 20.38*. Construct a nondegenerate bilinear pairing β on the spinor space $S = \bigwedge^\bullet W$ by choosing an isomorphism of $\bigwedge^n W$ with $\mathbb{C}$ and letting $\beta(s, t)$ be the image of $\tau(s) \wedge t \in \bigwedge^\bullet W$ by the projection to $\bigwedge^n W = \mathbb{C}$, where τ is the main antiautomorphism).

(a) When $m = 2n$, show that β can also be defined by the identity $\beta(s, t)f = \tau(s\cdot f)\cdot t\cdot f$ for an appropriate generator f of $\bigwedge^n W'$. Deduce that the action of $\mathrm{Spin}(Q)$ on S respects the bilinear form β.

(b) Show that β is symmetric if n is congruent to 0 or 3 modulo 4, and skew-symmetric otherwise. So the spin representation is a homomorphism

$$\mathrm{Spin}_{2n+1}\mathbb{C} \to \mathrm{SO}_{2^n}\mathbb{C} \quad \text{if } n \equiv 0, 3\ (4),$$

$$\mathrm{Spin}_{2n+1}\mathbb{C} \to \mathrm{Sp}_{2^n}\mathbb{C} \quad \text{if } n \equiv 1, 2\ (4).$$

(c) If $m = 2n$, the restrictions of β to S^+ and S^- are zero if n is odd. For n even, deduce that the half-spin representations are homomorphisms

$$\mathrm{Spin}_{2n}\mathbb{C} \to \mathrm{SO}_{2^{n-1}}\mathbb{C} \quad \text{if } n \equiv 0\ (4),$$
$$\mathrm{Spin}_{2n}\mathbb{C} \to \mathrm{Sp}_{2^{n-1}}\mathbb{C} \quad \text{if } n \equiv 2\ (4).$$

Note in particular that $\mathrm{Spin}_8\mathbb{C}$ has two maps to $\mathrm{SO}_8\mathbb{C}$ in addition to the original covering. "Triality," which we discuss in the next section, describes the relation among these three homomorphisms.

Exercise 20.39. Show that the spin and half-spin representations give the isomorphisms we have seen before:

$$\mathrm{Spin}_2\mathbb{C} \cong \mathrm{GL}(S^+) = \mathrm{GL}_1\mathbb{C} = \mathbb{C}^*,$$
$$\mathrm{Spin}_3\mathbb{C} \cong \mathrm{SL}(S) = \mathrm{SL}_2\mathbb{C},$$
$$\mathrm{Spin}_4\mathbb{C} \cong \mathrm{SL}(S^+) \times \mathrm{SL}(S^-) = \mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C},$$
$$\mathrm{Spin}_5\mathbb{C} \cong \mathrm{Sp}(S) = \mathrm{Sp}_4\mathbb{C},$$
$$\mathrm{Spin}_6\mathbb{C} \cong \mathrm{SL}(S^+) = \mathrm{SL}_4\mathbb{C}.$$

Exercise 20.40. Let C_m denote the Clifford algebra of the vector space $\mathbb{C}^m$ with our standard quadratic form Q_m.

(a) The embedding of $\mathbb{C}^{2n} = W \oplus W'$ in $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$ as indicated induces an embedding of C_{2n} in C_{2n+1}, and corresponding embedding of $\mathrm{Spin}_{2n}\mathbb{C}$ in $\mathrm{Spin}_{2n+1}\mathbb{C}$ and of $\mathrm{SO}_{2n}\mathbb{C}$ in $\mathrm{SO}_{2n+1}\mathbb{C}$. Show that the spin representation S of $\mathrm{Spin}_{2n+1}\mathbb{C}$ restricts to the spin representation $S^+ \oplus S^-$ of $\mathrm{Spin}_{2n}\mathbb{C}$.

(b) Similarly there is an embedding of $\mathrm{Spin}_{2n+1}\mathbb{C}$ in $\mathrm{Spin}_{2n+2}\mathbb{C}$ coming from an embedding of $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$ in $\mathbb{C}^{2n+2} = W \oplus W' \oplus U_1 \oplus U_2$; here $U_1 \oplus U_2 = \mathbb{C} \oplus \mathbb{C}$ with the quadratic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $U = \mathbb{C}$ is embedded in $U_1 \oplus U_2$ by sending 1 to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Show that each of the half-spin representations of $\mathrm{Spin}_{2n+2}\mathbb{C}$ restricts to the spin representation of $\mathrm{Spin}_{2n+1}\mathbb{C}$.

Very little of the above discussion needs to be changed to construct the real spin groups $\mathrm{Spin}_m(\mathbb{R})$, which are double coverings of the real orthogonal groups $\mathrm{SO}_m(\mathbb{R})$. One uses the real Clifford algebra $\mathrm{Cliff}(\mathbb{R}^m, Q)$ associated to the real quadratic form $Q = -Q_m$, where Q_m is the standard positive definite quadratic form on $\mathbb{R}^m$. If v_i are an orthonormal basis, the products in this Clifford algebra are given by

$$v_i \cdot v_j = -v_j \cdot v_i \quad \text{if } i \neq j, \qquad \text{and} \quad v_i \cdot v_i = -1.$$

The same definitions can be given as in the complex case, giving rise to coverings $\mathrm{Pin}_m(\mathbb{R})$ of $\mathrm{O}_m(\mathbb{R})$ and $\mathrm{Spin}_m(\mathbb{R})$ of $\mathrm{SO}_m(\mathbb{R})$.

Exercise 20.41. Show that $\mathrm{Spin}_m\mathbb{R}$ is connected by showing that if v and w are any two perpendicular elements in V with $Q(v, v) = Q(w, w) = -1$, the path

$$t \mapsto (\cos(t)v + \sin(t)w)\cdot(\cos(t)v - \sin(t)w), \quad 0 \le t \le \pi/2$$

connects -1 to 1.

Exercise 20.42. Show that $i \mapsto v_2 \cdot v_3$, $j \mapsto v_3 \cdot v_1$, $k \mapsto v_1 \cdot v_2$ determines an isomorphism of the quaternions $\mathbb{H}$ onto the even part of $\mathrm{Cliff}(\mathbb{R}^3, -Q_3)$, such that conjugation $^-$ in $\mathbb{H}$ corresponds to the conjugation $*$ in the Clifford algebra. Show that this maps $\mathrm{Sp}(2) = \{q \in \mathbb{H} \mid q\bar{q} = 1\}$ isomorphically onto $\mathrm{Spin}_3\mathbb{R}$, and that this isomorphism is compatible with the map to $\mathrm{SO}_3\mathbb{R}$ defined in Exercise 7.15.

More generally, if Q is a quadratic form on $\mathbb{R}^m$ with p positive and q negative eigenvalues, we get a group $\mathrm{Spin}^+(p, q)$ in the Clifford algebra $C(p, q) = \mathrm{Cliff}(\mathbb{R}^m, Q)$, with double coverings

$$\mathrm{Spin}^+(p, q) \to \mathrm{SO}^+(p, q).$$

Exercise 20.43*. Show that $\mathrm{Spin}^+(p, q)$ is connected if p and q are positive, except for the case $p = q = 1$, when it has two components. Show that if in the definition of spin groups one relaxes the condition $x \cdot x^* = 1$ to the condition $x \cdot x^* = \pm 1$, one gets coverings $\mathrm{Spin}(p, q)$ of $\mathrm{SO}(p, q)$.

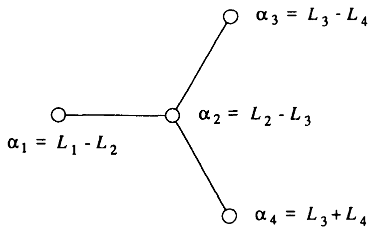
§20.3. $\mathrm{Spin}_8\mathbb{C}$ and Triality

When m is even, there is always an outer automorphism of $\mathrm{Spin}_m(\mathbb{C})$ that interchanges the two spin representations S^+ and S^-, while preserving the basic representation $V = \mathbb{C}^m$ (cf. Exercise 19.9). In case $m = 8$, all three of these representations V, S^+, and S^- are eight dimensional. One basic expression of *triality* is the fact that there are automorphisms of $\mathrm{Spin}_8\mathbb{C}$ or $\mathfrak{so}_8\mathbb{C}$ that permute these three representations arbitrarily. (In fact, the group of outer automorphisms modulo inner automorphisms is the symmetric group on three elements.) We give a brief discussion of this phenomenon in this section, in the form of an extended exercise.

To see where these automorphisms might come from, consider the four simple roots:

$$\alpha_1 = L_1 - L_2, \quad \alpha_2 = L_2 - L_3, \quad \alpha_3 = L_3 - L_4, \quad \alpha_4 = L_3 + L_4.$$

Note that α_1, α_3, and α_4 are mutually perpendicular, and that each makes an angle of 120° with α_2:



Exercise 20.44*. For each of the six permutations of $\{\alpha_1, \alpha_3, \alpha_4\}$ find the orthogonal automorphism of the root space which fixes α_2 and realizes the permutation of α_1, α_3, and α_4.

Each automorphism of this exercise corresponds to an automorphism of the Cartan subalgebra $\mathfrak{h}$. In the next lecture we will see that such automorphisms can be extended (nonuniquely) to automorphisms of the Lie algebra $\mathfrak{so}_8(\mathbb{C})$. (For explicit formulas see [Ca2].)

There is also a purely geometric notion of triality. Recall that an even-dimensional quadric Q can contain linear spaces Λ of at most half the dimension of Q, and that there are two families of linear spaces of this maximal dimension (cf. [G-H], [Ha]). In case Q is six-dimensional, each of these families can themselves be realized as six-dimensional quadrics, which we may denote by Q^+ and Q^- (see below). Moreover, there are correspondences that assign to a point of any one of these quadrics a 3-plane in each of the others:

$$\begin{array}{ccccc} & \text{Point in } Q & \longrightarrow & \text{3-plane in } Q^+ & \\ \swarrow & & & & \nwarrow \\ \text{3-plane in } Q^- & & & & \text{Point in } Q^- \\ \nwarrow & & & & \swarrow \\ & \text{Point in } Q^+ & \longrightarrow & \text{3-plane in } Q & \end{array} \qquad (20.45)$$

Given $P \in Q$, $\{\Lambda \in Q^+ : \Lambda \text{ contains } P\}$ is a 3-plane in Q^+, and $\{\Lambda \in Q^- : \Lambda \text{ contains } P\}$ is a 3-plane in Q^-.

Given $\Lambda \in Q^+$, Λ itself is a 3-plane in Q, and $\{\Gamma \in Q^- : \Gamma \cap \Lambda \text{ is a 2-plane}\}$ is a 3-plane in Q^-.

Given $\Lambda \in Q^-$, Λ itself is a 3-plane in Q, and $\{\Gamma \in Q^+ : \Gamma \cap \Lambda \text{ is a 2-plane}\}$ is a 3-plane in Q^+.

To relate these two notions of triality, take Q to be our standard quadric in $\mathbb{P}^7 = \mathbb{P}(V)$, with $V = W \oplus W'$ with our usual quadratic space, and let $S^+ = \bigwedge^{\text{even}} W$ and $S^- = \bigwedge^{\text{odd}} W$ be the two spin representations. In Exercise 20.38 we constructed quadratic forms on S^+ and S^-, by choosing an isomorphism of $\bigwedge^4 W$ with $\mathbb{C}$. This gives us two quadrics Q^+ and Q^- in $\mathbb{P}(S^+)$ and $\mathbb{P}(S^-)$.

To identify Q^+ and Q^- with the families of 3-planes in Q, recall the action of V on $S = \bigwedge^{\bullet} W = S^+ \oplus S^-$ which gave rise to the isomorphism of the Clifford algebra with $\mathrm{End}(S)$ (cf. Lemma 20.9). This in fact maps S^+ to S^-

and S^- to S^+; so we have bilinear maps

$$V \times S^+ \dashrightarrow S^- \quad \text{and} \quad V \times S^- \dashrightarrow S^+. \tag{20.46}$$

Exercise 20.47. Show that for each point in Q^+, represented by a vector $s \in S^+$, $\{v \in V: v \cdot s = 0\}$ is an isotropic 4-plane in V, and hence determines a projective 3-plane in Q. Similarly, each point in Q^- determines a 3-plane in Q. Show that every 3-plane in Q arises uniquely in one of these ways.

Let $\langle\ ,\ \rangle_V$ denote the symmetric form corresponding to the quadratic form in V, and similarly for S^+ and S^-. Define a product

$$S^+ \times S^- \dashrightarrow V, \qquad s \times t \mapsto s \cdot t, \tag{20.48}$$

by requiring that $\langle v, s \cdot t\rangle_V = \langle v \cdot s, t\rangle_{S^-}$ for all $v \in V$.

Exercise 20.49. Use this product, together with those in (20.46), to show that the other four arrows in the hexagon (20.45) for geometric triality can be described as in the preceding exercise.

This leads to an algebraic version of triality, which we sketch following [Ch2]. The above products determine a commutative but nonassociative product on the direct sum $A = V \oplus S^+ \oplus S^-$. The operation

$$(v, s, t) \mapsto \langle v \cdot s, t\rangle_{S^-}$$

determines a cubic form on A, which by polarization determines a symmetric trilinear form Φ on A.

Exercise 20.50*. One can construct an automorphism J of A of order three that sends V to S^+, S^+ to S^-, and S^- to V, preserving their quadratic forms, and compatible with the cubic form. The definition of J depends on the choice of an element $v_1 \in V$ and $s_1 \in S^+$ with $\langle v_1, v_1\rangle_V = \langle s_1, s_1\rangle_{S^+} = 1$; set $t_1 = v_1 \cdot s_1$, so that $\langle t_1, t_1\rangle_{S^-} = 1$ as well. The map J is defined to be the composite $\mu \circ \nu$ of two involutions μ and ν, which are determined by the following:

(i) μ interchanges S^+ and S^-, and maps V to itself, with $\mu(s) = v_1 \cdot s$ for $s \in S^+$; $\mu(v) = 2\langle v, v_1\rangle_V v_1 - v$ for $v \in V$.
(ii) ν interchanges V and S^-, maps S^+ to itself, with $\nu(v) = v \cdot s_1$ for $v \in V$; $\nu(s) = 2\langle s, s_1\rangle_{S^+} s_1 - s$ for $s \in S^+$.

Show that this J satisfies the asserted properties.

Exercise 20.51*. In this algebraic form, triality can be expressed by the assertion that there is an automorphism j of $\mathrm{Spin}_8\mathbb{C}$ of order 3 compatible with J, i.e., such that for all $x \in \mathrm{Spin}_8\mathbb{C}$, the following diagrams commute:

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "\rho(x)"] & S^+ \arrow[r, "j"] \arrow[d, "\rho^+(j(x))"] & S^- \arrow[r, "j"] \arrow[d, "\rho^-(j^2(x))"] & V \arrow[d, "\rho(x)"] \\
V \arrow[r, "j"'] & S^+ \arrow[r, "j"'] & S^- \arrow[r, "j"'] & V
\end{tikzcd}$$

If $j' \colon \mathfrak{so}_8\mathbb{C} \to \mathfrak{so}_8\mathbb{C}$ is the map induced by j, the fact that j is compatible with the trilinear form Φ (cf. Exercise 20.49) translates to the "local triality" equation

$$\Phi(Xv, s, t) + \Phi(v, Ys, t) + \Phi(v, s, Zt) = 0$$

for $X \in \mathfrak{so}_8\mathbb{C}$, $Y = j'(X)$, $Z = j'(Y)$.

PART IV
LIE THEORY

The purpose of this final part of the book is threefold.

First of all, we want to complete the program stated in the introduction to Part II. We have completed the first two steps of this program, showing in Part II how the analysis of representations of Lie groups could be reduced to the study of representations of complex Lie algebras, of which the most important are the semisimple; and carrying out in Part III such an analysis for the classical Lie algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$. To finish the story, we want now to translate our answers back into the terms of the original problem. In particular, we want to deal with representations of Lie groups as well as Lie algebras, and real groups and algebras as well as complex. The passage back to groups is described in Lecture 21, and the analysis of the real case in Lecture 26.

Another goal of this Part is to establish a framework for some of the results of the preceding lectures—to describe the general theory of semisimple Lie algebras and Lie groups. The key point here is the introduction of the Dynkin diagram and its use in classifying all semisimple Lie algebras over $\mathbb{C}$. From one point of view, the impact of the classification theorem is not great: it just tells us that we have in fact already analyzed all but five of the simple Lie algebras in existence. Beyond that, however, it provides a picture and a language for the description of the general Lie algebra. This both yields a description of the five remaining simple Lie algebras and allows us to give uniform descriptions of associated objects: for example, the compact homogeneous spaces associated to simple Lie groups, or the characters of their representations. The classification theory of semisimple Lie algebras is given in Lecture 21; the description in these terms of their representations and characters is given in Lecture 23. The five exceptional simple Lie algebras, whose existence is revealed from the Dynkin diagrams, are studied in Lecture

22; we give a fairly detailed account of one of them ($\mathfrak{g}_2$), with only brief descriptions of the others.

Third, all this general theory makes it possible to answer the main outstanding problem left over from Part III: a description of the multiplicities of the weights in the irreducible representations of the simple Lie algebras. We give in Lectures 24 and 25 a number of formulas for these multiplicities.

This, it should be said, represents in some ways a shift in style. In the previous lectures we would typically analyze special cases first and deduce general patterns from these cases; here, for example, the Weyl character formula is stated and proved in general, then specialized to the various individual cases (this is the approach more often taken in the literature on the subject). In some ways, this is a fourth goal of Part IV: to provide a bridge between the naive exploration of Lie theory undertaken in Parts II and III, and the more general theory readers will find elsewhere when they pursue the subject further.

Finally, we should repeat here the disclaimer made in the Preface. This part of the book, to the extent that it is successful, will introduce the reader to the rich and varied world of Lie theory; but it certainly undertakes no serious exploration of that world. We do not, for example, touch on such basic constructions as the universal enveloping algebra, Verma modules, Tits buildings; and we do not even hint at the fascinating subject of (infinite-dimensional) unitary representations. The reader is encouraged to sample these and other topics, as well as those included here, according to background and interest.

LECTURE 21

The Classification of Complex Simple Lie Algebras

In the first section of this lecture we introduce the Dynkin diagram associated to a semisimple Lie algebra $\mathfrak{g}$. This is an amazingly efficient way of conveying the structure of $\mathfrak{g}$: it is a simple diagram that not only determines $\mathfrak{g}$ up to isomorphism in theory, but in practice exhibits many of the properties of $\mathfrak{g}$. The main use of Dynkin diagrams in this lecture, however, will be to provide a framework for the basic classification theorem, which says that with exactly five exceptions the Lie algebras discussed so far in these lectures are all the simple Lie algebras. To do this, in §21.2 we show how to list all diagrams that arise from semisimple Lie algebras. In §21.3 we show how to recover such a Lie algebra from the data of its diagram, completing the proof of the classification theorem. All three sections are completely elementary, though §21.3 gets a little complicated; it may be useful to read it in conjunction with §22.1, where the process described is carried out in detail for the exceptional algebra $\mathfrak{g}_2$. (Note that neither §21.3 or §22.1 is a prerequisite for §22.3, where another description of $\mathfrak{g}_2$ will be given.)

§21.1: Dynkin diagrams associated to semisimple Lie algebras
§21.2: Classifying Dynkin diagrams
§21.2: Recovering a Lie algebra from its Dynkin diagram

§21.1. Dynkin Diagrams Associated to Semisimple Lie Algebras

For the following, we will let $\mathfrak{g}$ be a semisimple Lie algebra; as usual, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ will be fixed throughout. As we have seen, the roots R of $\mathfrak{g}$ span a real subspace of $\mathfrak{h}^*$ on which the Killing form is positive definite. We denote this Euclidean space here by $\mathbb{E}$, and the Killing form on $\mathbb{E}$ simply by

$(\ , \ )$ instead of $B(\ , \ )$. The geometry of how R sits in $\mathbb{E}$ is very rigid, as indicated by the pictures we have seen for the classical Lie algebras. In this section we will classify the possible configurations, up to rotation and multiplication by a positive scalar in $\mathbb{E}$. In the next section we will see that this geometry completely determines the Lie algebra.

The following four properties of the root system are all that are needed:

(1) *R is a finite set spanning $\mathbb{E}$.*
(2) *$\alpha \in R \Rightarrow -\alpha \in R$, but $k \cdot \alpha$ is not in R if k is any real number other than ± 1.*
(3) *For $\alpha \in R$, the reflection W_α in the hyperplane $\alpha^\perp$ maps R to itself.*
(4) *For $\alpha, \beta \in R$, the real number*

$$n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

is an integer.

Except perhaps for the second part of (2), these properties have been seen in Lecture 14. For example, (4) is Corollary 14.29. Note that $n_{\beta\alpha} = \beta(H_\alpha)$, and

$$W_\alpha(\beta) = \beta - n_{\beta\alpha}\alpha. \tag{21.1}$$

For (2), consider the representation $\mathfrak{i} = \bigoplus_k \mathfrak{g}_{k\alpha}$ of the Lie algebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$. Note that all the nonzero factors but $\mathfrak{h} = \mathfrak{g}_0$ are one dimensional. We may assume α is the smallest nonzero root that appears in the string. Now, decompose $\mathfrak{i}$ as an $\mathfrak{s}_\alpha$-module:

$$\mathfrak{i} = \mathfrak{s}_\alpha \oplus \mathfrak{i}'.$$

By the hypothesis that α is the smallest nonzero root that appears in the string, $\mathfrak{i}'$ is a representation of $\mathfrak{s}_\alpha$ having no eigenspace with eigenvalue 1 or 2 for H_α. It follows that $\mathfrak{i}'$ must be trivial, i.e., $\mathfrak{g}_{k\alpha} = (0)$ for $k \neq 0$ or ± 1.

Any set R of elements in a Euclidean space $\mathbb{E}$ satisfying conditions (1) to (4) may be called an (*abstract*) *root system.*

Property (4) puts very strong restrictions on the geometry of the roots. If ϑ is the angle between α and β, we have

$$n_{\beta\alpha} = 2\cos(\vartheta)\frac{\|\beta\|}{\|\alpha\|}. \tag{21.2}$$

In particular,

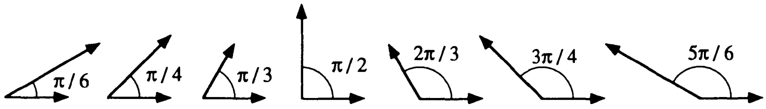
$$n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2(\vartheta) \tag{21.3}$$

is an integer between 0 and 4. The case when this integer is 4 occurs when $\cos(\vartheta) = \pm 1$, i.e. $\beta = \pm\alpha$. Omitting this trivial case, the only possibilities are therefore those given in the following table. Here we have ordered the two roots so that $\|\beta\| \geq \|\alpha\|$, or $|n_{\beta\alpha}| \geq |n_{\alpha\beta}|$.

Table 21.4

$\cos(\vartheta)$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
ϑ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$n_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$n_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\dfrac{\lVert\beta\rVert}{\lVert\alpha\rVert}$	$\sqrt{3}$	$\sqrt{2}$	1	*	1	$\sqrt{2}$	$\sqrt{3}$

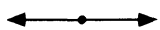
In other words, the relation of any two roots α and β is one of



The dimension $n = \dim_{\mathbb{R}}\mathbb{E} = \dim_{\mathbb{C}}\mathfrak{h}$ is called the *rank* (of the Lie algebra, or the root system). It is easy to find all those of smallest ranks. As we write them down, we will label them by the labels (A_n), (B_n), ... that have become standard.

Rank 1. The only possibility is

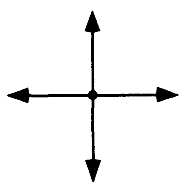
(A_1)



which is the root system of $\mathfrak{sl}_2\mathbb{C}$.

Rank 2. Note first that by Property (3), the angle between two roots must be the same for any pair of adjacent roots in a two-dimensional root system. As we will see, any of the four angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$ can occur; once this angle is specified the relative lengths of the roots are determined by Property (4), except in the case of right angles. Thus, up to scalars there are exactly four root systems of dimension two. First we have the case $\vartheta = \pi/2$,

$(\mathrm{A}_1 \times \mathrm{A}_1)$



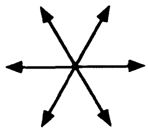
which is the root system of $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_4\mathbb{C}$.

(In general, the orthogonal direct sum of two root systems is a root system;

a root system that is not such a sum is called *irreducible*. Our task will be to classify all irreducible root systems.)

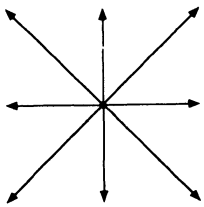
The other root systems of rank 2 are

(A_2)



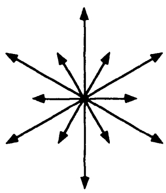
the root system of $\mathfrak{sl}_3\mathbb{C}$;

(B_2)



the root system of $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$; and

(G_2)



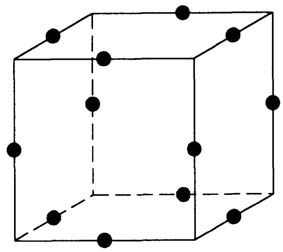
Although we have not yet seen a Lie algebra with this root system, we will see that there is one.

Exercise 21.5. Show that these are all the root systems of rank 2.

Exercise 21.6. Show that a semisimple Lie algebra is simple if and only if its root system is irreducible.

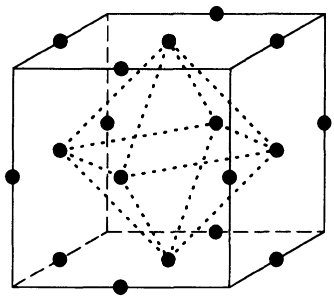
Rank 3. Besides the direct sums of (A_1) with one of those of rank 2, we have the irreducible root systems we have seen; we draw only dots at the ends of the vectors, the origins being in the centers of the reference cubes:

(A_3)



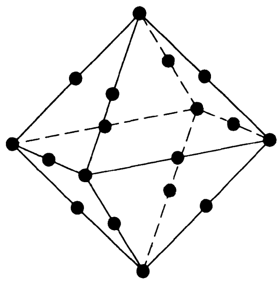
which is the root system of $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$;

(B_3)



the root system of $\mathfrak{so}_7\mathbb{C}$;

(C_3)



the root system of $\mathfrak{sp}_6\mathbb{C}$.

Exercise 21.7. Show that there are no other root systems of rank 3.

We can further reduce the data of a root system by introducing a subset of the roots, called the simple roots. First, choose as in Lecture 14 a direction

$l\colon \mathbb{E} \to \mathbb{R}$, so that $R = R^+ \cup R^-$ is a disjoint union of positive and negative roots. Call a positive root *simple* if it is not the sum of two other positive roots. For the classical Lie algebras, keeping the notations and conventions of Lectures 15–20, the simple roots are

(A_n)	$\mathfrak{sl}_{n+1}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n - L_{n+1}$,
(B_n)	$\mathfrak{so}_{2n+1}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n$,
(C_n)	$\mathfrak{sp}_{2n}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, 2L_n$,
(D_n)	$\mathfrak{so}_{2n}\mathbb{C}$	$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_{n-1} + L_n$.

Exercise 21.8. Verify this list, and find two simple roots for (G_2).

We next deduce a few consequences of properties (1)–(4), which indicate how strong these axioms are. They will be used in the present classification of abstract systems, as well as in the following section.

(5) *If α, β are roots with $\beta \neq \pm\alpha$, then the α-string through β, i.e., the roots of the form*

$$\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + q\alpha$$

has at most four in a string, i.e. $p + q \leq 3$; in addition, $p - q = n_{\beta\alpha}$.

Indeed, since $W_\alpha(\beta + q\alpha) = \beta - p\alpha$, and

$$W_\alpha(\beta + q\alpha) = (\beta - n_{\beta\alpha}\alpha) - q\alpha,$$

we must have $p = n_{\beta\alpha} + q$, which is the second equality. For the first, we may take $p = 0$, and then $q = -n_{\beta\alpha}$, which we have seen is an integer no larger than three. As a consequence of (5) we have

(6) *Suppose α, β are roots with $\beta \neq \pm\alpha$. Then*

$(\beta, \alpha) > 0 \Rightarrow \alpha - \beta$ *is a root*;
$(\beta, \alpha) < 0 \Rightarrow \alpha + \beta$ *is a root.*

If $(\beta, \alpha) = 0$, then $\alpha - \beta$ and $\alpha + \beta$ are simultaneously roots or nonroots.

(7) *If α and β are distinct simple roots, then $\alpha - \beta$ and $\beta - \alpha$ are not roots.*

This follows from the definition of simple, since from the equation $\alpha = \beta + (\alpha - \beta)$, $\alpha - \beta$ cannot be in R^+, and similarly $-(\alpha - \beta) = \beta - \alpha$ cannot be in R^+. From (6) and (7) we deduce that $(\alpha, \beta) \leq 0$, i.e.,

(8) *The angle between two distinct simple roots cannot be acute.*

(9) *The simple roots are linearly independent.*

This follows from (8) by

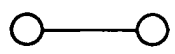
Exercise 21.9*. If a set of vectors lies on one side of a hyperplane, with all mutual angles at least 90°, show that they must be linearly independent.

(10) *There are precisely n simple roots. Each positive root can be written uniquely as a non-negative integral linear combination of simple roots.*

Since R spans $\mathbb{E}$, the first statement follows from (9), as does the uniqueness of the second statement. The fact that any positive root can be written as a positive sum of simple roots follows readily from the definition, for if α were a positive root with minimal $l(\alpha)$ that could not be so written, then α is not simple, so $\alpha = \beta + \gamma$, with β and γ positive roots with $l(\beta)$, $l(\gamma) < l(\alpha)$.

Note that as an immediate corollary of (10) it follows that *no root is a linear combination of the simple roots α_i with coefficients of mixed sign.* For example, (7) is just a special case of this.

The *Dynkin diagram* of the root system is drawn by drawing one node ○ for each simple root and joining two nodes by a number of lines depending on the angle ϑ between them:

no lines		if $\vartheta = \pi/2$
one line		if $\vartheta = 2\pi/3$
two lines		if $\vartheta = 3\pi/4$
three lines		if $\vartheta = 5\pi/6$.

When there is one line, the roots have the same length; if two or three lines, an arrow is drawn pointing from the *longer* to the *shorter* root.

Exercise 21.10. Show that a root system is irreducible if and only if its Dynkin diagram is connected.

We will see later that the Dynkin diagram of a root system is independent of the choice of direction, i.e., of the decomposition of R into R^+ and R^-.

§21.2. Classifying Dynkin Diagrams

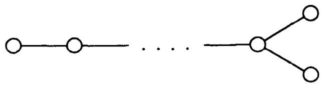
The wonderful thing about Dynkin diagrams is that from this very simple picture one can reconstruct the entire Lie algebra from which it came. We will see this in the following section; for now, we ask the complementary question of which diagrams arise from Lie algebras. Our goal is the following classification theorem, which is a result in pure Euclidean geometry. (The subscripts on the labels (A_n), ... are the number of nodes.)

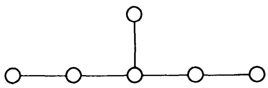
Theorem 21.11. *The Dynkin diagrams of irreducible root systems are precisely*:

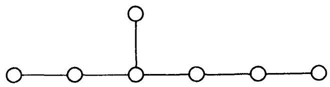
(A_n)  $(n \geq 1)$

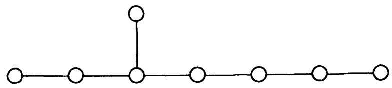
(B_n)  $(n \geq 2)$

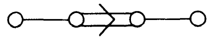
(C_n)  $(n \geq 3)$

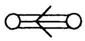
(D_n)  $(n \geq 4)$

(E_6) 

(E_7) 

(E_8) 

(F_4) 

(G_2) 

The first four are those belonging to the classical series we have been studying:

$$\begin{array}{ll} (\mathrm{A}_n) & \mathfrak{sl}_{n+1}\mathbb{C} \\ (\mathrm{B}_n) & \mathfrak{so}_{2n+1}\mathbb{C} \\ (\mathrm{C}_n) & \mathfrak{sp}_{2n}\mathbb{C} \\ (\mathrm{D}_n) & \mathfrak{so}_{2n}\mathbb{C} \end{array}$$

The restrictions on n in these series are to avoid repeats, as well as degenerate cases. Indeed, the diagrams can be used to recall all the coincidences we have seen:

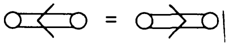
When $n = 1$, all four of the diagrams become one node. The case (D_1) is degenerate, since $\mathfrak{so}_2\mathbb{C}$ is not semisimple, while the coincidences $(\mathrm{C}_1) = (\mathrm{B}_1) = (\mathrm{A}_1)$ correspond to the isomorphisms

$$\mathfrak{sp}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \qquad \circ.$$

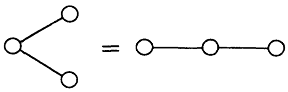
For $n = 2$, $(D_2) = (A_1) \times (A_1)$ consists of two disjoint nodes, corresponding to the isomorphism

$\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$  .

The coincidence $(C_2) = (B_2)$ corresponds to the isomorphism

$\mathfrak{sp}_4\mathbb{C} \cong \mathfrak{so}_5\mathbb{C}$  .

For $n = 3$, the fact that $(D_3) = (A_3)$ reflects the isomorphism

$\mathfrak{so}_6\mathbb{C} \cong \mathfrak{sl}_4\mathbb{C}$  .

Proof of the Theorem. Our desert-island reader would find this a pleasant pastime. For example, if there are two simple roots with angle $5\pi/6$, the plane of these roots must contain the G_2 configuration of 12 roots. It is not hard to see that one cannot add another root that is not perpendicular to this plane, without some of the 12 angles and lengths being wrong. This shows that (G_2) is the only connected diagram containing a triple line. At the risk of spoiling your fun, we give the general proof of a slightly stronger result.

In fact, the angles alone determine the possible diagrams. Such diagrams, without the arrows to indicate relative lengths, are often called *Coxeter diagrams* (or Coxeter graphs). Define a diagram of n nodes, with each pair connected by 0, 1, 2, or 3 lines, to be *admissible* if there are n independent unit vectors $e_1, \dots, e_n$ in a Euclidean space $\mathbb{E}$ with the angle between e_i and e_j being $\pi/2$, $2\pi/3$, $3\pi/4$, or $5\pi/6$, according as the number of lines between corresponding nodes is 0, 1, 2, or 3. The claim is that the diagrams of the above Dynkin diagrams, ignoring the arrows, are the only connected admissible diagrams. Note that

$$(e_i, e_j) = 0,\ -1/2,\ -\sqrt{2}/2, \text{ or } -\sqrt{3}/2, \tag{21.12}$$

according as the number of lines between them is 0, 1, 2, or 3; equivalently,

$$4(e_i, e_j)^2 = \text{number of lines between } e_i \text{ and } e_j. \tag{21.13}$$

The steps of the proof are as follows:

(i) *Any subdiagram of an admissible diagram, obtained by removing some nodes and all lines to them, will also be admissible.*

(ii) *There are at most $n - 1$ pairs of nodes that are connected by lines. The diagram has no cycles (loops).*

Indeed, if e_i and e_j are connected, $2(e_i, e_j) \le -1$, and

$$0 < \left(\sum e_i, \sum e_i\right) = n + 2 \sum_{i<j} (e_i, e_j),$$

which proves the first statement of (ii). The second follows from the first and (i).

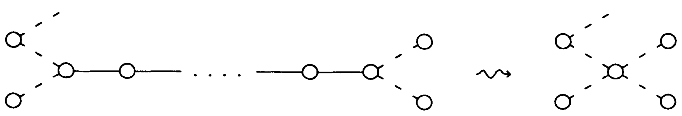
(iii) *No node has more than three lines to it.*

By (i), we may assume that e_1 is connected to each of the other nodes; by (ii), no other nodes are connected to each other. We must show that $\sum_{j=2}^{n} 4(e_1, e_j)^2 < 4$. Since $e_2, \ldots, e_n$ are perpendicular unit vectors, and e_1 is not in their span,

$$1 = (e_1, e_1)^2 > \sum_{j=2}^{n} (e_1, e_j)^2,$$

as required.

(iv) *In an admissible diagram, any string of nodes connected to each other by one line, with none but the ends of the string connected to any other nodes, can be collapsed to one node, and resulting diagram remains admissible*:

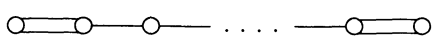


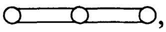
If $e_1, \ldots, e_r$ are the unit vectors corresponding to the string of nodes, then $e' = e_1 + \cdots + e_r$ is a unit vector, since

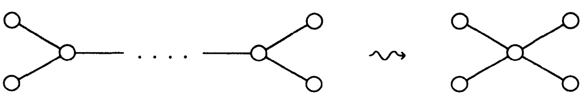
$$\begin{aligned}(e', e') &= r + 2((e_1, e_2) + (e_2, e_3) + \cdots + (e_{r-1}, e_r)) \\ &= r - (r - 1).\end{aligned}$$

Moreover, e' satisfies the same conditions with respect to the other vectors since (e', e_j) is either (e_1, e_j) or (e_r, e_j).

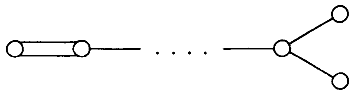
Now we can rule out the other admissible connected diagrams not on our list. First, from (iii) we see that the diagram (G_2) has the only triple edge. Next, there cannot be two double lines, or we could find a subdiagram of the form:



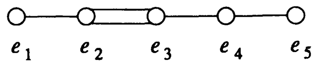
and then collapse the middle to get , contradicting (iii). Similarly there can be at most one triple node, i.e., a node with single lines to three other nodes, by



By the same reasoning, there cannot be a triple node together with a double line:



To finish the case with double lines, we must simply verify that



is not admissible. Consider general vectors $v = a_1e_1 + a_2e_2$, and $w = a_3e_3 + a_4e_4 + a_5e_5$. We have

$$\|v\|^2 = a_1^2 + a_2^2 - a_1a_2, \qquad \|w\|^2 = a_3^2 + a_4^2 + a_5^2 - a_3a_4 - a_4a_5,$$

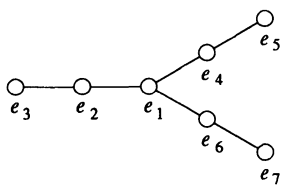
and $(v, w) = -a_2a_3/\sqrt{2}$. We want to choose v and w to contradict the Cauchy-Schwarz inequality $(v, w)^2 < \|v\|^2\|w\|^2$. For this we want $|a_2|/\|v\|$ and $|a_3|/\|w\|$ to be as large as possible.

Exercise 21.14. Show that these maxima are achieved by taking $a_2 = 2a_1$ and $a_3 = 3a_5$, $a_4 = 2a_5$.

In fact, $v = e_1 + 2e_2$, $w = 3e_3 + 2e_4 + e_5$ do give the contradictory

$$(v, w)^2 = 18, \quad \|v\|^2 = 3, \quad \text{and} \quad \|w\|^2 = 6.$$

Finally, we must show that the strings coming out from a triple node cannot be longer that those specified in types (D_n), (E_6), (E_7), or (E_8). First, we rule out



Consider the three perpendicular unit vectors:

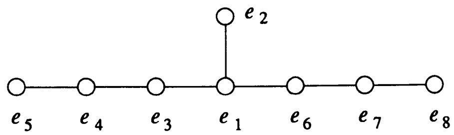
$$u = (2e_2 + e_3)/\sqrt{3}, \qquad v = (2e_4 + e_5)/\sqrt{3}, \qquad w = (2e_6 + e_7)/\sqrt{3}.$$

Then as in (iii), since e_1 is not in the span of them,

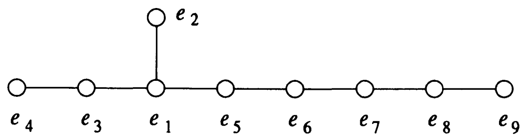
$$1 = \|e_1\|^2 > (e_1, u)^2 + (e_1, v)^2 + (e_1, w)^2 = 1/3 + 1/3 + 1/3 = 1,$$

a contradiction.

Exercise 21.15*. Similarly, rule out



and



(The last few arguments can be amalgamated, by showing that if the legs from a triple node have lengths p, q, and r, then $1/p + 1/q + 1/r$ must be greater than 1.)

This finishes the proof of the theorem. □

§21.3. Recovering a Lie Algebra from Its Dynkin Diagram

In this section we will complete the classification theorem for simple Lie algebras by showing how one may recover a simple Lie algebra from the data of its Dynkin diagram. This will proceed in two stages: first, we will see how to reconstruct a root system from its Dynkin diagram (which a priori only tells us the configuration of the simple roots). Secondly, we will show how to describe the entire Lie algebra in terms of its root system. (In the next lecture we will do all this explicitly, by hand, and independently of the general discussion here, for the simplest exceptional case (G_2); as we have noted, the reader may find it useful to work through §22.1 before or while reading the general story described here.)

To begin with, to recover the root system from the Dynkin diagram, let α_1, $\ldots$, α_n be the simple roots corresponding to the nodes of a connected Dynkin diagram. We must show which non-negative integral linear combinations $\sum m_i \alpha_i$ are roots. Call $\sum m_i$ the *level* of $\sum m_i \alpha_i$. Those of level one are the simple roots. For level two, we see from Property (2) that no $2\alpha_i$ is a root, and by

Property (6) that $\alpha_i + \alpha_j$ is a root precisely when $(\alpha_i, \alpha_j) < 0$, i.e., when the corresponding nodes are joined by a line.

Suppose we know all positive roots of level at most m, and let $\beta = \sum m_i \alpha_i$ be any positive root of level m. We next determine for each simple root $\alpha = \alpha_j$, whether $\beta + \alpha$ is also a root. Look at the α-string through β:

$$\beta - p\alpha, \ldots, \beta, \ldots, \beta + q\alpha.$$

We know p by induction (no root is a linear combination of the simple roots α_i with coefficients of mixed sign, so $p \le m_j$ and $\beta - p\alpha$ is a positive root). By Property (5), $q = p - n_{\beta\alpha}$. So $\beta + \alpha$ is a root exactly when

$$p > n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = \sum_{i=1}^{n} m_i n_{\alpha_i \alpha}.$$

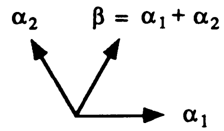
In effect, the additional roots we will find in this way are those obtained by reflecting a known positive root in the hyperplane perpendicular to a simple root α_i (and filling in the string if necessary).

To finish the proof, we must show that we get all the positive roots in this way. This will follow once from the fact that any positive root of level $m + 1$ can be written in at least one way as a sum of a positive root of level m and a simple root. If $\gamma = \sum r_i \alpha_i$ has level $m + 1$, from

$$0 < (\gamma, \gamma) = \sum r_i(\gamma, \alpha_i),$$

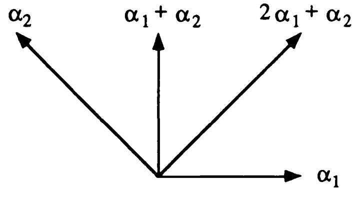
some (γ, α_i) must be positive, with $r_i > 0$. By property (6), $\gamma - \alpha_i$ is a root, as required.

By way of example, consider the rank 2 root systems. In the case of $\mathfrak{sl}_3\mathbb{C}$, we start with a pair of simple roots α_1, α_2 with $n_{\alpha_1, \alpha_2} = -1$, i.e., at an angle of $2\pi/3$; as always, we know that $\beta = \alpha_1 + \alpha_2$ is a root as well.



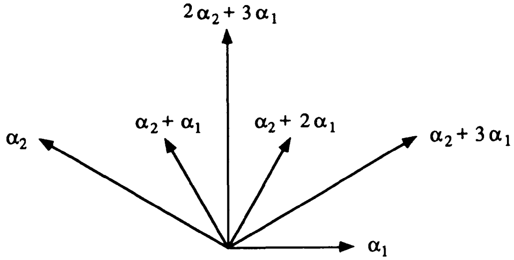
On the other hand, since $\beta - 2\alpha_1 = \alpha_2 - \alpha_1$ is not a root, $\beta + \alpha_1$ cannot be either, and likewise $\beta + \alpha_2$ is not; so we have all the positive roots.

In the case of $\mathfrak{sp}_4\mathbb{C}$, we have two simple roots α_1 and α_2 at an angle of $3\pi/4$; in terms of an orthonormal basis L_1 and L_2 these may be taken to be L_1 and $L_2 - L_1$, respectively.



We then see that in addition to $\beta = \alpha_1 + \alpha_2$, the sum $\beta + \alpha_1 = 2\alpha_1 + \alpha_2$ is a root—it is just the reflection of α_2 in the plane perpendicular to α_1—but $\beta + \alpha_2 = \alpha_1 + 2\alpha_2$ and $3\alpha_1 + \alpha_2$ are not because $\alpha_1 - \alpha_2$ and $\alpha_2 - \alpha_1$ are not respectively (alternatively, we could note that they would form inadmissible angles with α_1 and α_2 respectively).

Finally, in the case of (G_2), we have two simple roots α_1, α_2 at an angle of $5\pi/6$, which in terms of an orthonormal basis for $\mathbb{E}$ may be taken to be L_1 and $(-3L_1 + \sqrt{3}L_2)/2$ respectively.



Reflecting α_2 in the plane perpendicular to α_1 yields a string of roots $\alpha_2 + \alpha_1$, $\alpha_2 + 2\alpha_1$ and $\alpha_2 + 3\alpha_2$. Moreover, reflecting the last of these in the plane perpendicular to α_2 yields one more root, $2\alpha_2 + 3\alpha_3$. Finally, these are all the positive roots, giving us the root system for the diagram (G_2).

We state here the results of applying this process to the exceptional diagrams (F_4), (E_6), (E_7), and (E_8) (in addition to (G_2)). In each case, $L_1, \ldots, L_n$ is an orthogonal basis for $\mathbb{E}$, the simple roots α_i can be taken to be as follows, and the corresponding root systems are given:

(G_2) $$\alpha_1 = L_1, \qquad \alpha_2 = -\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2;$$

$$R^+ = \left\{L_1, \sqrt{3}L_2, \pm L_1 + \frac{\sqrt{3}}{2}L_2, \pm\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2\right\}.$$

(G_2) thus has 6 positive roots.

(F_4) $$\alpha_1 = L_2 - L_3, \qquad \alpha_2 = L_3 - L_4, \qquad \alpha_3 = L_4,$$

$$\alpha_4 = \frac{L_1 - L_2 - L_3 - L_4}{2};$$

$$R^+ = \{L_i\} \cup \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \left\{\frac{L_1 \pm L_2 \pm L_3 \pm L_4}{2}\right\}.$$

In particular, (F_4) has 24 positive roots.

(E_6) $$\alpha_1 = \frac{L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6}{2}, \qquad \alpha_2 = L_1 + L_2,$$

$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2,$$
$$\alpha_5 = L_4 - L_3, \qquad \alpha_6 = L_5 - L_4;$$
$$R^+ = \{L_i + L_j\}_{i<j\le 5} \cup \{L_i - L_j\}_{j<i\le 5}$$
$$\cup \left\{\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 \pm L_5 + \sqrt{3}L_6}{2}\right\}_{\text{number of minus signs even}}.$$

(E_6) has 36 positive roots.

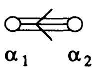
(E_7) $$\alpha_1 = \frac{L_1 - L_2 - \cdots - L_6 + \sqrt{2}L_7}{2}, \qquad \alpha_2 = L_1 + L_2,$$
$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2, \qquad \alpha_5 = L_4 - L_3,$$
$$\alpha_6 = L_5 - L_4, \qquad \alpha_7 = L_6 - L_5;$$
$$R^+ = \{L_i + L_j\}_{i<j\le 6} \cup \{L_i - L_j\}_{j<i\le 6} \cup \{\sqrt{2}L_7\}$$
$$\cup \left\{\frac{\pm L_1 \pm L_2 \pm \cdots \pm L_6 + \sqrt{2}L_7}{2}\right\}_{\text{number of minus signs odd}}.$$

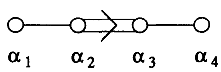
Thus, (E_7) has 63 positive roots.

(E_8) $$\alpha_1 = \frac{L_1 - L_2 - \cdots - L_7 + L_8}{2}, \qquad \alpha_2 = L_1 + L_2,$$
$$\alpha_3 = L_2 - L_1, \qquad \alpha_4 = L_3 - L_2, \qquad \alpha_5 = L_4 - L_3,$$
$$\alpha_6 = L_5 - L_4, \qquad \alpha_7 = L_6 - L_5, \qquad \alpha_8 = L_7 - L_6.$$
$$R^+ = \{L_i + L_j\}_{i<j\le 8} \cup \{L_i - L_j\}_{j<i\le 8}$$
$$\cup \left\{\frac{\pm L_1 \pm L_2 \pm \cdots \pm L_7 + L_8}{2}\right\}_{\text{number of minus signs even}}.$$

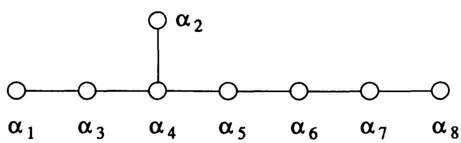
(E_8) has 120 positive roots.

For (G_2) and (F_4) the simple roots are listed in order reading from left to right in their Dynkin diagrams





as in the classical series (A_n)–(D_n). For (E_8), the numbering is



while those for (E_7) and (E_6) are obtained by removing the last one or two nodes. Note that, given the root system of (E_8), we can find the root system of (E_7) or (E_6) by taking the subspace spanned by the first seven or six simple roots.

Exercise 21.16*. (a) Verify the above lists of roots.

(b) In each case, calculate the corresponding fundamental weights.

Exercise 21.17*. Show that no two of the root systems of (A_n)–(E_8) are isomorphic, and deduce that the Dynkin diagram of a root system is independent of choice of positive roots.

A more satisfying reason for the last fact is the observation that any two choices of positive roots differ by an element of the Weyl group—the group generated by reflections W_α in the simple roots. This can be seen directly for each of the diagrams (A_n)–(E_8); for a general proof that two choices differ by an element of the Weyl group, see Proposition D.29.

We should mention here another way of conveying the data of a Dynkin diagram. This is simply the $n \times n$ matrix of integers $(n_{i,j} = n_{\alpha_i \alpha_j})$, where we take $n_{i,i} = 2$; it is called the *Cartan matrix* of the Dynkin diagram (or of the Lie algebra). Thus, for example, the Cartan matrix of (A_n) is

$$\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 \end{pmatrix}.$$

These matrices pop up remarkably often, in a variety of seemingly unrelated areas of mathematics. They will not play a major role in the present text, but the reader has probably encountered them already in one form or another, and will probably do so again.

Exercise 21.18*. Compute the Cartan matrix, and its determinant, for each Dynkin diagram.

The next task is to see how the root system determines the Lie algebra. We concentrate on the uniqueness, since there are other ways to see the existence; indeed, for all but the five exceptions we have already seen the Lie algebras. We will describe several approaches to this problem, starting with a straightforward and computational method and finishing with a slick but abstract approach.

Assume as before that $\mathfrak{g}$ is a simple Lie algebra, with a chosen Cartan subalgebra $\mathfrak{h}$ and decomposition of the roots R into positive and negative roots; let $\alpha_1, \ldots, \alpha_n$ be the simple roots. The Dynkin diagram information is the knowledge of (α_i, α_j) for all $i \neq j$. Let $H_i = H_{\alpha_i}$ be the corresponding basis of $\mathfrak{h}$, defined by the rule we have seen in Lecture 14: if $\{T_i\}$ is the basis corresponding via the Killing form to $\{\alpha_i\}$, set $H_i = 2T_i/(\alpha_i, \alpha_i)$.

Choose any nonzero element X_i in the root space $\mathfrak{g}_{\alpha_i}$, for $1 \leq i \leq n$. This determines elements Y_i in $\mathfrak{g}_{-\alpha_i}$ such that $[X_i, Y_i] = H_i$. We claim first that these $3n$ elements $\{H_i, X_i, Y_i\}$ generate $\mathfrak{g}$ as a Lie algebra. This follows from

Claim 21.19. *If α, β, and $\alpha + \beta$ are roots, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

PROOF. Again look at the α-string through $\mathfrak{g}_\beta$, i.e., $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$. This is an irreducible representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$, since all the terms are one dimensional (this follows from the fact that no $\beta + k\alpha$ can be zero, given that $\beta \neq \pm\alpha$). But now if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, $\bigoplus_{k \leq 0} \mathfrak{g}_{\beta+k\alpha}$ would be a nontrivial subrepresentation. □

For each positive root β, we have seen that can write β as a sum of simple roots $\beta = \alpha_{i_1} + \cdots + \alpha_{i_r}$ such that each of the sums $\alpha_{i_1} + \cdots + \alpha_{i_s}$ is a root, $1 \leq s \leq r$. If we choose such a presentation for each β, and set

$$X_\beta = [X_{i_r}, [X_{i_{r-1}}, \ldots, [X_{i_2}, X_{i_1}] \ldots]]$$

and

$$Y_\beta = [Y_{i_r}, [Y_{i_{r-1}}, \ldots, [Y_{i_2}, Y_{i_1}] \ldots]],$$

then the collection

$$\{H_i, 1 \leq i \leq n; X_\beta, Y_\beta, \beta \in R^+\} \tag{21.20}$$

forms a basis for $\mathfrak{g}$. Note that if β is not simple, there is no reason to expect $[X_\beta, Y_\beta]$ to be the distinguished element H_β in $\mathfrak{h}$.

We want to show that the multiplication table for these basis elements is completely determined by the Dynkin diagram. The main difficulty is that the ordering of the simple roots in the above expression for β may not be unique. For example, suppose

$$\beta = (\alpha_1 + \alpha_2) + \alpha_3 = (\alpha_2 + \alpha_3) + \alpha_1,$$

with $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ roots. We must compare $[X_3, [X_2, X_1]]$ with $[X_1, [X_3, X_2]]$. In fact, they must be negatives of each other. For, by Jacobi, we have

$$[X_1, [X_3, X_2]] = -[X_3, [X_2, X_1]] - [X_2, [X_1, X_3]] = -[X_3, [X_2, X_1]],$$

noting that $[X_1, X_3] = 0$ since $\alpha_1 + \alpha_3$ cannot be a root, e.g., by step (ii) of the preceding section.

For any sequence $I = (i_1, \ldots, i_r)$, $1 \leq i_j \leq n$, set

$$\alpha_I = \alpha_{i_1} + \cdots + \alpha_{i_r},$$
$$X_I = [X_{i_r}, [X_{i_{r-1}}, \ldots, [X_{i_2}, X_{i_1}] \ldots]],$$
$$Y_I = [Y_{i_r}, [Y_{i_{r-1}}, \ldots, [Y_{i_2}, Y_{i_1}] \ldots]].$$

Call I *admissible* if each partial sum $\alpha_{i_1} + \cdots + \alpha_{i_s}$ is a root, $1 \leq s \leq r$; note that I is admissible exactly when X_I is not zero.

Lemma 21.21. *If I and J are two admissible sequences for which $\alpha_I = \alpha_J$, then there is a nonzero rational number q determined by I, J, and the Dynkin diagram, such that $X_J = q \cdot X_I$.*

PROOF. Let $k = i_r$ be the last entry in I. If $j_r = k$ as well, the result follows by induction on r. We reduce the general case to this case, by maneuvering to replace j_r by k. We have first

$$X_J = q_1 \cdot [X_k, [Y_k, X_J]],$$

with q_1 a nonzero rational number depending only on J, k, and the Dynkin diagram, since $\alpha_J - \alpha_k = \alpha_I - \alpha_k$ is a root; the point is that we know how $\mathfrak{s}_{\alpha_k} \cong \mathfrak{sl}_2$ acts on the α_k-string through α_J as soon as we know the length of the string, and this is Dynkin diagram information. Next, let s be the largest integer such that $j_s = k$. Then

$$[Y_k, X_J] = [X_{j_r}, \ldots [X_{j_{s+1}}, [Y_k, [X_k, X_K]]] \ldots],$$

where $K = (j_1, \ldots, j_{s-1})$, since $[Y_k, [X_i, Z]] = [X_i, [Y_k, Z]]$ when $i \neq k$. Finally,

$$[Y_k, [X_k, X_K]] = q_2 \cdot X_K,$$

with q_2 a nonzero rational number depending only on K, k, and the Dynkin diagram, since $\alpha_K + \alpha_k$ is a root. Combining these three equations, we get

$$X_J = q_1 q_2 \cdot [X_k, [X_{j_r}, \ldots [X_{j_{s+1}}, X_K] \ldots]],$$

which suffices since the sequence for the term on the right ends in the same integer k as I. □

Proposition 21.22. *The bracket of any two basis elements in* (21.20) *is a rational multiple of another basis element, that multiple determined from the Dynkin diagram.*

PROOF. This is clear for brackets of an H_i with any basis element. Lemma 21.21 handles brackets of the form $[X_I, X_J]$, and those involving only Y's are similar. For brackets $[Y_I, X_J]$, it suffices inductively to compute $[Y_k, X_J]$ as a rational multiple of some X_K, with K shorter than J (or of H_k if J has one term); but this was worked out in the proof of the lemma. □

Exercise 21.23*. (i) Show that in (G_2) each positive root can be written in only one way as a sum of simple roots, up to the order of the first two roots.

(ii) Work out the multiplication table from the Dynkin diagram. (iii) Verify that the result is indeed a Lie algebra, which is (visibly) simple.

This exercise will be worked out in detail to start the next lecture. Of course, there is nothing but lack of time to keep us from verifying that the other four exceptional Dynkin diagrams do lead, by the same prescription, to honest Lie algebras, but doing it by hand gets pretty laborious, and we will describe some of the other methods available.

The fact that the multiplication table can be defined with rational coefficients becomes important when one wants to reduce them modulo prime numbers, which we will not discuss here. The fact that they can be taken to be real, on the other hand, will come up later, when we discuss real forms of complex Lie algebras and groups.

There is a more general and elegant way to proceed, given by Serre [Se3]. Write n_{ij} in place of $n_{\alpha_i\alpha_j}$. Form the free Lie algebra on generators

$$H_1, \ldots, H_n, X_1, \ldots, X_n, Y_1, \ldots, Y_n,$$

i.e., form the free (tensor) algebra with this basis, and divide modulo by the relations $[A, B] + [B, A] = 0$ and the Jacobi relation. Then take this free Lie algebra, and divide by the relations

$$[H_i, H_j] = 0 \text{ (all } i, j); \qquad [X_i, Y_i] = H_i \text{ (all } i); \qquad [X_i, Y_j] = 0 \ (i \neq j);$$

$$[H_i, X_j] = n_{ji}X_j \text{ (all } i, j); \qquad [H_i, Y_j] = -n_{ji}Y_j \text{ (all } i, j);$$

and, for all $i \neq j$,

$$\begin{aligned}
[X_i, X_j] &= 0, & [Y_i, Y_j] &= 0 & &\text{if } n_{ij} = 0;\\
[X_i, [X_i, X_j]] &= 0, & [Y_i, [Y_i, Y_j]] &= 0 & &\text{if } n_{ij} = -1;\\
[X_i, [X_i, [X_i, X_j]]] &= 0, & [Y_i, [Y_i, [Y_i, Y_j]]] &= 0 & &\text{if } n_{ij} = -2;\\
[X_i, [X_i, [X_i, [X_i, X_j]]]] &= 0, & [Y_i, [Y_i, [Y_i, [Y_i, Y_j]]]] &= 0 & &\text{if } n_{ij} = -3.
\end{aligned}$$

Exercise 21.24. Verify that if one starts with a semisimple Lie algebra with a given Dynkin diagram, the above equations must hold.

Serre shows ([Se3, Chap. VI App.], cf. [Hul §18]) that the resulting Lie algebra is a finite-dimensional semisimple Lie algebra, with Cartan subalgebra generated by $H_1, \ldots, H_n$ and given root system. In particular, this includes a proof of the existence of all the simple Lie algebras.

Here is a third approach to uniqueness. Suppose $\mathfrak{g}$ and $\mathfrak{g}'$, with given Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, and choice of positive roots, have isomorphic root systems. There is an isomorphism $\mathfrak{h} \to \mathfrak{h}'$, taking corresponding H_i to H_i'. Choose arbitrarily nonzero vectors X_i and X_i' in the root spaces of $\mathfrak{g}$ and $\mathfrak{g}'$ corresponding to the simple roots.

Claim 21.25. *There is a unique isomorphism from* $\mathfrak{g}$ *to* $\mathfrak{g}'$ *extending the isomorphism of* $\mathfrak{h}$ *with* $\mathfrak{h}'$, *and mapping* X_i *to* X_i' *for all* i.

PROOF. The uniqueness of the isomorphism is easy: the resulting map is determined on the Y_i by $\mathfrak{sl}_2$ considerations, and the H_i, X_i, and Y_i generate $\mathfrak{g}$. For the existence of the isomorphism consider the subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g} \oplus \mathfrak{g}'$ generated by $\tilde{H}_i = H_i \oplus H_i'$, $\tilde{X}_i = X_i \oplus X_i'$, and $\tilde{Y}_i = Y_i \oplus Y_i'$. It suffices to prove that the two projections from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphisms. The kernel of the second projection is $\mathfrak{k} \oplus 0$, where $\mathfrak{k}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{k}$ is either 0, as required, or $\mathfrak{k} = \mathfrak{g}$. In the latter case, we must have $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$.

To see that this is impossible, consider a maximal positive root β, take nonzero vectors X_β, X_β' in the corresponding root spaces, and set $\tilde{X}_\beta = X_\beta \oplus X_\beta'$, a highest weight vector in $\tilde{\mathfrak{g}}$. Let W be the subspace of $\tilde{\mathfrak{g}}$ obtained by successively applying all $\tilde{Y}_i$'s. Then W is a proper subspace of $\tilde{\mathfrak{g}}$, since its weight space W_β corresponding to β is one dimensional. By the argument we have seen several times, $\tilde{\mathfrak{g}}$ preserves W. Now if $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$, W would be an ideal in $\mathfrak{g} \oplus \mathfrak{g}'$, and this would force $X_\beta \oplus 0$ to belong to W, making W_β two dimensional again. □

To finish this story, we should show that the simple Lie algebras corresponding to two different Dynkin diagrams cannot be isomorphic, i.e., that the two choices made in going from a semisimple Lie algebra to Dynkin diagram do not change the answer. The general facts are:

(1) Any two Cartan subalgebras of a semisimple Lie algebra are conjugate, i.e., there is an inner automorphism by an element in the corresponding adjoint group, which takes one into the other.
(2) Any two decompositions of a root system into positive and negative roots differ by an element of the Weyl group.

These are standard facts which are proved in Appendix D. Both statements are subsumed in the fact that any two *Borel subalgebras* of a semisimple Lie algebra are conjugate, a Borel subalgebra being the subspace spanned by the Cartan subalgebra and the root spaces $\mathfrak{g}_\alpha$ for positive α. For those readers who crave logical completeness but do not want to go through so much general theory, we observe that most possible coincidences can be ruled out by such simple considerations as computing dimensions, and others can be ruled out by simple ad hoc methods, cf. Exercise 21.17.

Finally, we must also prove the "existence theorem": that there is a simple Lie algebra for each Dynkin diagram. Serre's theorem quoted above gives a unified proof of existence. But we have seen and studied the Lie algebras for the classical cases (A_n)–(D_n), and it is more in keeping with the spirit of these lectures to at least try to see the five exceptions explicitly. This is the subject of the next lecture.

LECTURE 22

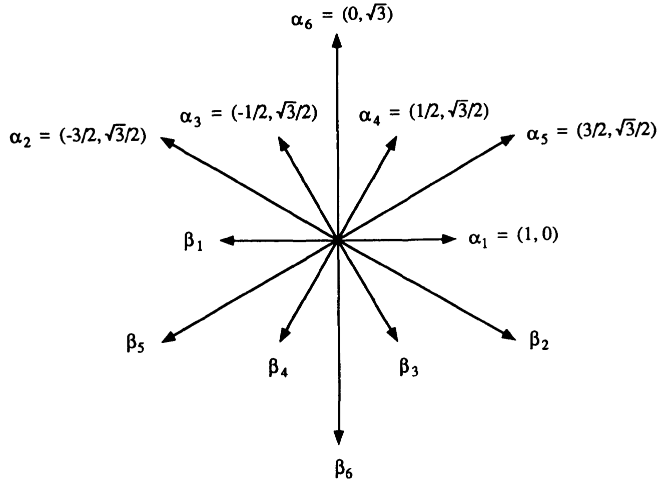
$\mathfrak{g}_2$ and Other Exceptional Lie Algebras

This lecture is mainly about $\mathfrak{g}_2$, with just enough discussion of the algebraic constructions of the other exceptional Lie algebras to give the reader a sense of their complexity. $\mathfrak{g}_2$, being only 14-dimensional, is different: we can reasonably carry out in practice the process described in §21.3 to arrive at an explicit description of the algebra by specifying a basis and all pairwise products; we do this in §22.1 and verify in §22.2 that the result really is a Lie algebra. In §22.3 we analyze the representations of $\mathfrak{g}_2$, and arrive in particular at another description of $\mathfrak{g}_2$: it is the algebra of endomorphisms of a seven-dimensional vector space preserving a general trilinear form. (Note that §22.3 may be read independently of either §22.1, §21.2, or §21.3.) Finally, in the fourth section we will sketch some of the more abstract (i.e., coordinate free) approaches to the construction of the five exceptional Lie algebras. While the first two sections are completely elementary, the constructions given in §22.4 involve some fairly serious algebra.

§22.1. Construction of $\mathfrak{g}_2$ from Its Dynkin Diagram

In this section we will carry out explicitly the process described in the preceding section for the Dynkin diagram (G_2), constructing in this way a Lie algebra $\mathfrak{g}_2$ with diagram (G_2) (and in particular proving its existence).

The first step is to find the root system from the Dynkin diagram. In the case of $\mathfrak{g}_2$ this is immediate; we may draw the root system $R \subset \mathfrak{h}^*$ associated to the diagram G_2 as follows:



Here the positive roots are denoted α_i, with α_1 and α_2 the simple roots. The coordinate system here has no particular significance (in particular, recall that the configuration of roots α_i and β_i is determined only up to a real scalar), but is convenient for calculating inner products. Note that the Weyl group is the dihedral group generated by rotation through an angle of $\pi/3$ and reflection in the horizontal; the Weyl chamber associated to the choice of ordering of the roots given is the cone between the roots α_6 and α_4.

As indicated in the preceding section, we start by letting X_1 be any eigenvector for the action of $\mathfrak{h}$ with eigenvalue α_1, and X_2 any eigenvector for the action of $\mathfrak{h}$ with eigenvalue α_2. We similarly let Y_1 and Y_2 be eigenvectors with eigenvalues β_1 and β_2 and set

$$H_1 = [X_1, Y_1] \quad \text{and} \quad H_2 = [X_2, Y_2].$$

We can choose Y_1 and Y_2 so that the elements $H_i \in \mathfrak{h}$ satisfy $\alpha_1(H_1) = \alpha_2(H_2) = 2$, i.e.,

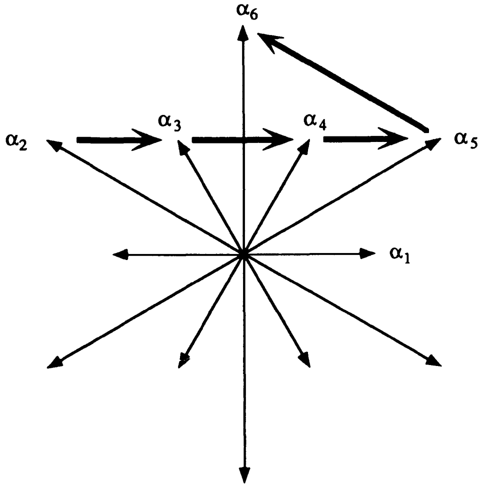
$$[H_1, X_1] = 2 \cdot X_1 \quad \text{and} \quad [H_2, X_2] = 2 \cdot X_2.$$

It follows that

$$[H_1, Y_1] = -2 \cdot Y_1 \quad \text{and} \quad [H_2, Y_2] = -2 \cdot Y_2,$$

i.e., H_i, X_i, and Y_i span a subalgebra $\mathfrak{s}_{\alpha_i} \cong \mathfrak{sl}_2\mathbb{C}$, with H_i, X_i, and Y_i a normalized basis for this copy of $\mathfrak{sl}_2\mathbb{C}$.

Now, it is clear from the diagram above that there is a unique way of writing each positive root α_i as a sum of simple roots $\alpha_{i_1} + \cdots + \alpha_{i_k}$ so that the partial sums $\alpha_{i_1} + \cdots + \alpha_{i_l}$ are roots for each $l \leq k$ (modulo exchanging the first two terms): we go through the root system by the path



i.e., we write

$$\alpha_3 = \alpha_1 + \alpha_2,$$

$$\alpha_4 = \alpha_1 + \alpha_3 = \alpha_1 + \alpha_1 + \alpha_2,$$

$$\alpha_5 = \alpha_1 + \alpha_4 = \alpha_1 + \alpha_1 + \alpha_1 + \alpha_2,$$

$$\alpha_6 = \alpha_2 + \alpha_5 = \alpha_2 + \alpha_1 + \alpha_1 + \alpha_1 + \alpha_2.$$

According to the general recipe, this means we now set

$$X_3 = [X_1, X_2], \qquad X_4 = [X_1, X_3],$$
$$X_5 = [X_1, X_4], \qquad X_6 = [X_2, X_5],$$

and define $Y_3, \ldots, Y_6$ similarly. The elements $H_1, H_2, X_1, \ldots, X_6, Y_1, \ldots, Y_6$ then form a basis for the 14-dimensional $\mathfrak{g}_2$, with H_1 and H_2 a basis for $\mathfrak{h}$, X_i a generator of the eigenspace $\mathfrak{g}_{\alpha_i}$, and Y_i a generator of $\mathfrak{g}_{\beta_i}$ for $i = 1, \ldots, 6$.

The task at hand now is to write down the multiplication table for $\mathfrak{g}_2$ in terms of this basis. Of course, some products are already known: we know, for example, that H_i, X_i, and Y_i form a normalized basis for $\mathfrak{sl}_2\mathbb{C}$ for $i = 1, 2$, and we have the relations defining $X_3, \ldots, X_6$ and $Y_1, \ldots, Y_6$ above. In addition, since we know that the product $[X_i, X_j]$ lies in the root space $\mathfrak{g}_{\alpha_i+\alpha_j}$ for each i and j, we see immediately that $[X_i, X_j] = 0$ whenever $\alpha_i + \alpha_j$ is not a root. We deduce that

$$\begin{aligned}[X_1, X_5] &= [X_1, X_6] = [X_2, X_3] = [X_2, X_4] = [X_2, X_6] = [X_3, X_5] \\ &= [X_3, X_6] = [X_4, X_5] = [X_4, X_6] = [X_5, X_6] = 0,\end{aligned}$$

and likewise

$$\begin{aligned}[Y_1, Y_5] &= [Y_1, Y_6] = [Y_2, Y_3] = [Y_2, Y_4] = [Y_2, Y_6] = [Y_3, Y_5] \\ &= [Y_3, Y_6] = [Y_4, Y_5] = [Y_4, Y_6] = [Y_5, Y_6] = 0.\end{aligned}$$

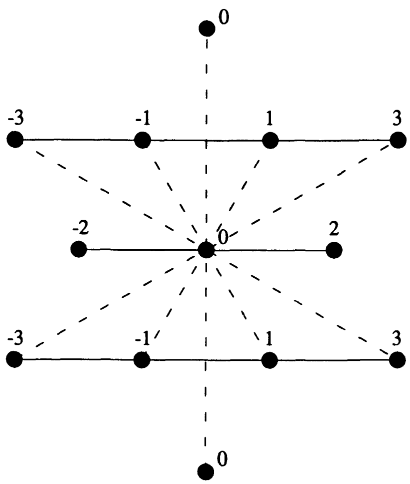
Similarly, we know that $[X_i, Y_j] = 0$ whenever $\alpha_i + \beta_j = \alpha_i - \alpha_j$ is not a root; this tells us as well that

$$[X_1, Y_2] = [X_1, Y_6] = [X_2, Y_1] = [X_2, Y_4] = [X_2, Y_5] = [X_3, Y_5]$$
$$= [X_4, Y_2] = [X_5, Y_2] = [X_5, Y_3] = [X_6, Y_1] = 0.$$

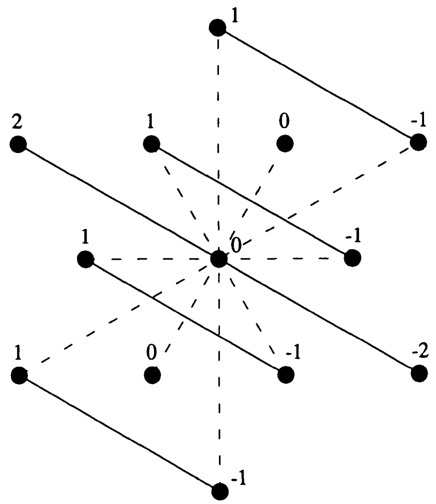
The multiplication table thus far looks like

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	*	*	*	*	*	*	*	*	*	*
H_2		*	*	$2X_2$	$-2Y_2$	*	*	*	*	*	*	*	*
X_1			H_1	X_3	0	X_4	*	X_5	*	0	*	0	0
Y_1				0	Y_3	*	Y_4	*	Y_5	*	0	0	0
X_2					H_2	0	*	0	0	X_6	0	0	*
Y_2						*	0	0	0	0	Y_6	*	0
X_3							*	*	*	0	0	0	*
Y_3								*	*	0	0	*	0
X_4									*	0	*	0	*
Y_4										*	0	*	0
X_5											*	0	*
Y_5												*	0
X_6													*

The next thing to do is to describe the action of H_1 and H_2 on the various vectors X_i and Y_i. This can be done using the inner product on $\mathfrak{h}$, but it is perhaps simpler to go back to the basic idea of restriction to the subalgebras $\mathfrak{s}_{\alpha_1}$ and $\mathfrak{s}_{\alpha_2}$. For example, if we want to determine the action of H_1 on the various X_i, consider how the algebra $\mathfrak{g} = \mathfrak{h} \bigoplus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{\beta_i})$ decomposes as a representation of $\mathfrak{s}_{\alpha_1}$:



We get two trivial representations (the spans of X_6 and Y_6, as already noted); one copy of the adjoint representation $\mathrm{Sym}^2 V$ (the subalgebra $\mathfrak{s}_{\alpha_1}$ itself) spanned by X_1, Y_1, and H_1; and two copies of the irreducible four-dimensional representation $\mathrm{Sym}^3 V$ spanned by X_2, X_3, X_4, and X_5 and Y_5, Y_4, Y_3, and Y_2. In particular, it follows that X_2, X_3, X_4, and X_5 are eigenvectors for the action of H_1 with eigenvalues of -3, -1, 1, and 3, respectively; and likewise Y_5, Y_4, Y_3, and Y_2 are eigenvectors with eigenvalues -3, -1, 1, and 3. In similar fashion, we consider the decomposition of $\mathfrak{g}$ under the action of $\mathfrak{s}_{\alpha_2} = \mathbb{C}\{H_2, X_2, Y_2\}$: diagrammatically, this looks like



Here we have two trivial representations, spanned by X_4 and Y_4, one adjoint ($\mathfrak{s}_{\alpha_2}$ itself), and four copies of the standard two-dimensional representation V, spanned by X_6 and X_5, X_3 and X_1, Y_1 and Y_3, and Y_5 and Y_6. It follows that X_6, X_3, Y_1, and Y_5 are eigenvectors for the action of H_2 with eigenvalue 1, and likewise X_5, X_1, Y_3, and Y_6 are eigenvectors with eigenvalue -1.

Including this information, we can fill in the top two rows of the multiplication table:

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$

Decomposing $\mathfrak{g}_2$ according to the action of $\mathfrak{s}_{\alpha_1}$ and $\mathfrak{s}_{\alpha_2}$ gives us information about the action of X_1, X_2, Y_1, and Y_2 on the other basis vectors as well. For example, we saw a moment ago that X_5 and X_6 together span a subrepresentation of $\mathfrak{g}_2$ under the action of $\mathfrak{s}_{\alpha_2}$, with $\operatorname{ad}(X_2)$ carrying X_5 to X_6. It follows from this that $\operatorname{ad}(Y_2)$ must carry X_6 back to X_5: we have

$$\begin{aligned}\operatorname{ad}(Y_2)(X_6) &= \operatorname{ad}(Y_2)\operatorname{ad}(X_2)(X_5)\\ &= \operatorname{ad}(X_2)\operatorname{ad}(Y_2)(X_5) - \operatorname{ad}([X_2, Y_2])(X_5)\\ &= 0 - \operatorname{ad}(H_2)(X_5) = X_5.\end{aligned}$$

Similarly, since $\operatorname{ad}(X_2)$ carries X_1 into $-X_3$, which together with X_1 spans a copy of the standard two-dimensional representation of $\mathfrak{s}_{\alpha_2} \cong \mathfrak{sl}_2\mathbb{C}$, it follows that $\operatorname{ad}(Y_2)$ will carry $-X_3$ back to X_1. Likewise from the fact that $\operatorname{ad}(Y_2)$ carries Y_1 to $-Y_3$ we see that $\operatorname{ad}(Y_2)(Y_3) = -Y_1$, and since $\operatorname{ad}(Y_2)$: $Y_5 \mapsto Y_6$, $\operatorname{ad}(X_2)$: $Y_6 \mapsto Y_5$.

We can in the same way use the action of $\mathfrak{s}_{\alpha_1}$ to determine the values of $\operatorname{ad}(X_1)$ and $\operatorname{ad}(Y_2)$ on various basis vectors, though because the representation of $\mathfrak{s}_{\alpha_1}$ on $\mathfrak{g}_2$ has larger-dimensional components this is slightly more complicated. To begin with, consider the representation of $\mathfrak{s}_{\alpha_1}$ on the subspace spanned by X_2, X_3, X_4, and X_5. We know that $\operatorname{ad}(X_1)$ carries X_2 to X_3, and since X_2 is an eigenvector for the action of the commutator $[X_1, Y_1] = H_1$ with eigenvalue -3, it follows that $\operatorname{ad}(Y_1)$ must carry X_3 to $3X_2$: we have

$$\begin{aligned}\operatorname{ad}(Y_1)(X_3) &= \operatorname{ad}(Y_1)\operatorname{ad}(X_1)(X_2)\\ &= \operatorname{ad}(X_1)\operatorname{ad}(Y_1)(X_2) - \operatorname{ad}([X_1, Y_1])(X_2)\\ &= 0 - \operatorname{ad}(H_1)(X_2) = 3X_2.\end{aligned}$$

Using this, we can next determine the action of Y_1 on X_4:

$$\begin{aligned}\operatorname{ad}(Y_1)(X_4) &= \operatorname{ad}(Y_1)\operatorname{ad}(X_1)(X_3)\\ &= \operatorname{ad}(X_1)\operatorname{ad}(Y_1)(X_3) - \operatorname{ad}(H_1)(X_3)\\ &= \operatorname{ad}(X_1)(3X_2) + X_3 = 4X_3,\end{aligned}$$

and we calculate likewise that $\operatorname{ad}(Y_1)(X_5) = 3X_4$. Analogously, knowing that $\operatorname{ad}(Y_1)$ carries Y_2 to Y_3 to Y_4 to Y_5 yields the information that $\operatorname{ad}(X_1)$ must carry Y_3, Y_4, and Y_5 to $3Y_2$, $4Y_3$ and, $3Y_4$, respectively. Including all this information in the chart, the next four rows of our multiplication table are

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
X_1			H_1	X_3	0	X_4	$3Y_2$	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1				0	Y_3	$3X_2$	Y_4	$4X_3$	Y_5	$3X_4$	0	0	0
X_2					H_2	0	$-Y_1$	0	0	X_6	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	X_5	0

We next have to find the commutators of the basis elements X_i and Y_j for $i, j \geq 3$. We cannot do this by looking at the action of the subalgebras generated by X_i and Y_i, since for $i \geq 3$ we do not know the commutator $[X_i, Y_i]$. Rather, the way to do this is outlined in the general proof in the preceding section: we just use the expression of the X_i and Y_j as brackets of the generators X_1, X_2, Y_1, and Y_2 to reduce the problem to brackets with these generators, which we now know. Thus, for example, the first unknown entry in the table at present is the bracket $[X_3, Y_3]$. We calculate this by writing X_3 as $[X_1, X_2]$, so that

$$\begin{aligned}
\operatorname{ad}(X_3)(Y_3) &= \operatorname{ad}([X_1, X_2])(Y_3) \\
&= \operatorname{ad}(X_1)\operatorname{ad}(X_2)(Y_3) - \operatorname{ad}(X_2)\operatorname{ad}(X_1)(Y_3) \\
&= \operatorname{ad}(X_1)(-Y_1) - \operatorname{ad}(X_2)(3Y_2) \\
&= -H_1 - 3H_2.
\end{aligned}$$

Likewise, to evaluate $[X_3, X_4]$ we have

$$\begin{aligned}
\operatorname{ad}(X_3)(X_4) &= \operatorname{ad}([X_1, X_2])(X_4) \\
&= \operatorname{ad}(X_1)\operatorname{ad}(X_2)(X_4) - \operatorname{ad}(X_2)\operatorname{ad}(X_1)(X_4) \\
&= -\operatorname{ad}(X_2)(X_5) = -X_6.
\end{aligned}$$

In this way, we can evaluate all brackets with X_3; knowing these, we can reduce any bracket with X_4 to one involving X_1 and X_3 by writing $X_4 = [X_1, X_3]$, and so on. Continuing in this way, we may complete our multiplication table:

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	X_4	$3Y_2$	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1				0	Y_3	$3X_2$	Y_4	$4X_3$	Y_5	$3X_4$	0	0	0
X_2					H_2	0	$-Y_1$	0	0	X_6	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	X_5	0
X_3							$-H_1$ $-3H_2$	$-X_6$	$4Y_1$	0	0	0	$3Y_4$
Y_3								$4X_1$	$-Y_6$	0	0	$3X_4$	0
X_4									$8H_1$ $+12H_2$	0	$-12Y_1$	0	$12Y_3$
Y_4										$-12X_1$	0	$12X_3$	0
X_5											$-36H_1$ $-36H_2$	0	$36Y_2$
Y_5												$36X_2$	0
X_6													$36H_1$ $+72H_2$

Of course, in retrospect we see that the basis we have chosen is far from the most symmetric one possible: for example, if we divided X_4 and Y_4 by 2 and X_5, X_6, Y_5, and Y_6 by 6, and changed the signs of X_5 and Y_3, the form of the table would be

Table 22.1

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y_1				0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X_2					H_2	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	$-X_5$	0
X_3							H_1+3H_2	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y_3								$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X_4									$2H_1+3H_2$	0	$-Y_1$	0	$-Y_3$
Y_4										X_1	0	X_3	0
X_5											H_1+H_2	0	$-Y_2$
Y_5												X_2	0
X_6													H_1+2H_2

There was another good reason for these changes: now each of the brackets $[X_i, Y_i]$ will be the distinguished element of $\mathfrak{h}$ corresponding to the root α_i. If we denote this element by H_i, then we read off from the table that

$$H_3 = H_1 + 3H_2, \qquad H_4 = 2H_1 + 3H_2, \qquad H_5 = H_1 + H_2, \qquad H_6 = H_1 + 2H_2, \tag{22.2}$$

and

$$H_i = [X_i, Y_i], \qquad [H_i, X_i] = 2X_i, \qquad [H_i, Y_i] = -2Y_i, \tag{22.3}$$

for $i = 1, 2, 3, 4, 5, 6$.

§22.2. Verifying That $\mathfrak{g}_2$ Is a Lie Algebra

The calculation of the preceding section gives a complete description of what the Lie algebra $\mathfrak{g}_2$ must look like, but there is still some work to be done: unless we know that there is a Lie algebra with diagram (G_2), we do not know that the above multiplication table defines a Lie algebra, let alone a simple one. In fact, the simplicity is not much of a problem (cf. Exercise 14.34), but to know that it is a Lie algebra requires knowing that the Jacobi identity is valid. One could simply check this from the table for all $\binom{14}{3}$ triples of elements from the basis, a rather uninviting task.

There is another way, which gives more structure to the preceding calculations, and which will give a clue for possible constructions of other Lie algebras. The root diagram for (G_2) is made up of two hexagons, one with long arrows, the other with short. This suggests that we should find a copy of the corresponding Lie algebra $\mathfrak{sl}_3\mathbb{C}$ inside $\mathfrak{g}_2$. The subspace spanned by $\mathfrak{h}$ and the root spaces corresponding to the six longer roots is clearly closed under brackets, so is the obvious candidate. The long roots are α_5, α_2, and $\alpha_6 = \alpha_5 + \alpha_2$, and their inverses. So we define $\mathfrak{g}_0$ to be the subspace spanned by the corresponding vectors:

$$\mathfrak{g}_0 = \mathbb{C}\{H_5, H_2, X_5, Y_5, X_2, Y_2, X_6, Y_6\}.$$

The multiplication table for $\mathfrak{g}_0$ is read off from Table 22.1:

	H_2	X_5	Y_5	X_2	Y_2	X_6	Y_6
H_5	0	$2X_5$	$-2Y_5$	$-X_2$	Y_2	X_6	$-Y_6$
H_2		$-X_5$	Y_5	$2X_2$	$-2Y_2$	X_6	$-Y_6$
X_5			H_5	X_6	0	0	$-Y_2$
Y_5				0	$-Y_6$	X_2	0
X_2					H_2	0	Y_5
Y_2						$-X_5$	0
X_6							$H_5 + H_2$

This is exactly the multiplication table for $\mathfrak{sl}_3\mathbb{C}$, with its standard basis (in the same order):

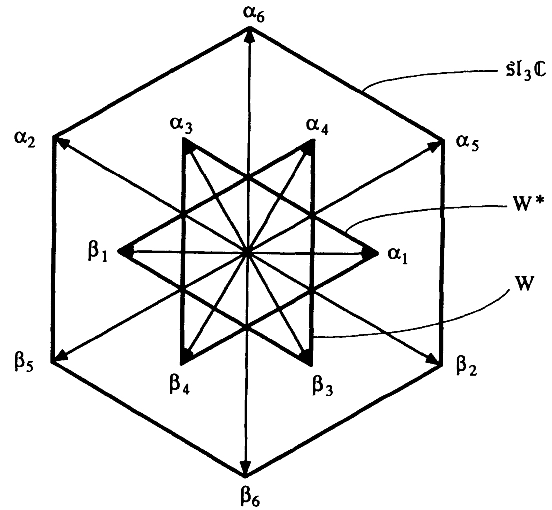
$$\mathfrak{sl}_3\mathbb{C} = \mathbb{C}\{E_{1,1} - E_{2,2}, E_{2,2} - E_{3,3}, E_{1,2}, E_{2,1}, E_{2,3}, E_{3,2}, E_{1,3}, E_{3,1}\}.$$

So we have determined an isomorphism

$$\mathfrak{g}_0 \cong \mathfrak{sl}_3\mathbb{C}.$$

(Note right away that this verifies the Jacobi identity for triples taken from $\mathfrak{g}_0$.)

The rest of the Lie algebra must be a representation of the subalgebra $\mathfrak{g}_0 \cong \mathfrak{sl}_3\mathbb{C}$, and we know what this must be: the smaller hexagon is the union of the two triangles which are the weight diagrams for the standard representation of $\mathfrak{sl}_3$ and its dual, which we denote here by W and W^*; W is the sum of the root spaces for α_4, β_1, and β_3, while W^* is the sum of those for β_4, α_1, and α_3.



Again, a look at the table shows that the vectors X_4, Y_1, and Y_3 form a basis for $W = \mathbb{C}^3$ that corresponds to the standard basis e_1, e_2, and e_3, and similarly Y_4, X_1, and X_3 form a basis for $W^* = (\mathbb{C}^3)^*$ that corresponds to the dual basis e_1^*, e_2^*, and e_3^*: we have

$$W = \mathbb{C}\{X_4, Y_1, Y_3\}; \qquad W^* = \mathbb{C}\{Y_4, X_1, X_3\};$$

$$\mathfrak{g}_2 = \mathfrak{g}_0 \oplus W \oplus W^*.$$

With these isomorphisms, the brackets

$$\mathfrak{g}_0 \times W \to W \quad \text{and} \quad \mathfrak{g}_0 \times W^* \to W^*$$

correspond to the standard operations of $\mathfrak{sl}_3\mathbb{C}$ on $\mathbb{C}^3$ and $(\mathbb{C}^3)^*$.

Next we look at brackets of elements in W. Note that $[W, W]$ is contained in W^*, either by weights or by looking at the table. The table is

	Y_1	Y_3
X_4	$-2X_3$	$2X_1$
Y_1	0	$-2Y_4$

or

	e_2	e_3
e_1	$-2e_3^*$	$2e_2^*$
e_2	0	$-2e_1^*$

Identifying $W = \mathbb{C}^3$, $W^* = (\mathbb{C}^3)^*$ as above, we see that the bracket $W \times W \to W^*$ becomes the map

$$W \times W \to W^* = \Lambda^2 W, \qquad v \times w \mapsto -2 \cdot v \wedge w.$$

Similarly for W^*, we have $[W^*, W^*] \subset W$, and the bracket is identified with the map

$$W^* \times W^* \to W = \Lambda^2 W^*, \qquad \varphi \times \psi \mapsto 2 \cdot \varphi \wedge \psi.$$

Finally we must look at brackets of elements of W with those of W^*, which land in $\mathfrak{g}_0$. Here the table is

	Y_4	X_1	X_3
X_4	$2H_5 + H_2$	$3X_5$	$3X_6$
Y_1	$3Y_5$	$H_2 - H_5$	$3X_2$
Y_3	$3Y_6$	$3Y_2$	$-H_5 - 2H_2$

In terms of the standard bases, $[e_i, e_j^*] = 3E_{i,j} - \delta_{ij}I$. Intrinsically, this mapping

$$[\ ,\ ]\colon W \times W^* \to \mathfrak{sl}_3\mathbb{C} \subset \mathfrak{gl}(W)$$

can be described by the formula

$$[v, \varphi](w) = 3\varphi(w)v - \varphi(v)w \tag{22.4}$$

for $v, w \in W$ and $\varphi \in W^*$.

Exercise 22.5*. Show that $[v, \varphi]$ is the element of $\mathfrak{sl}_3\mathbb{C}$ characterized by the formula

$$B([v, \varphi], Z) = 18\varphi(Z \cdot v) \quad \text{for all } Z \in \mathfrak{sl}_3\mathbb{C},$$

where B is the Killing form on $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}$. In other words, if we write $v * \varphi$ for the element in $\mathfrak{g}_0 = \mathfrak{sl}_3$ satisfying the identity

$$B(v * \varphi, Z) = \varphi(Z \cdot v) \quad \text{for all } Z \in \mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}, \tag{22.6}$$

then the bracket $[v, \varphi]$ can be written in the form

$$[v, \varphi] = 18 \cdot v * \varphi. \tag{22.7}$$

It is now a relatively painless task to verify the Jacobi identity, since, rather than having to check it for triples from a basis, it suffices to check it on triples of arbitrary elements of the three spaces $\mathfrak{g}_0$, W, and W^* using the above linear algebra descriptions for the brackets. We will write out this exercise, since the same reasoning will be used later. For example, for three or two elements from $\mathfrak{g}_0$, this amounts to the fact that $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}$ is a Lie algebra and W and W^* are representations.

For one element Z in $\mathfrak{g}_0$, and two elements v and w in W, the Jacobi identity for these three elements is equivalent to the identity

$$Z \cdot (v \wedge w) = (Z \cdot v) \wedge w + v \wedge (Z \cdot w),$$

which we know for the action of a Lie algebra on an exterior product; and similarly for one element in $\mathfrak{g}_0$ and two in W^*.

The Jacobi identity for $Z \in \mathfrak{g}_0$, $v \in W$, and $\varphi \in W^*$ amounts to

$$[Z, v * \varphi] = (Z \cdot v) * \varphi + v * (Z \cdot \varphi).$$

Applying $B(Y, —)$ to both sides, and using the identity $B(Y, [Z, X]) = B([Y, Z], X)$, this becomes

$$\varphi([Y, Z] \cdot v) = \varphi(Y \cdot (Z \cdot v)) + (Z \cdot \varphi)(Y \cdot v).$$

Since $\varphi([Y, Z] \cdot v) = \varphi(Y \cdot (Z \cdot v)) - \varphi(Z \cdot (Y \cdot v))$, this reduces to

$$(Z \cdot \varphi)(w) = -\varphi(Z \cdot w),$$

for $w = Y \cdot v$, which comes from the fact that W and W^* are dual representations.

For triples u, v, w in W, the Jacobi identity is similarly reduced to the identity

$$(u \wedge v)(Z \cdot w) + (v \wedge w)(Z \cdot u) + (w \wedge u)(Z \cdot v) = 0$$

for all $z \in \mathfrak{g}_0$, which amounts to

$$\begin{aligned} &u \wedge v \wedge (Z \cdot w) + u \wedge (Z \cdot v) \wedge w + (Z \cdot u) \wedge v \wedge w \\ &= Z \cdot (u \wedge v \wedge w) = 0 \quad \text{in } \textstyle\bigwedge^3 W = \mathbb{C}; \end{aligned}$$

and similarly for triples from W^*.

For $v, w \in W$, and $\varphi \in W^*$, noting that

$$[[v, w], \varphi] = -2 \cdot [v \wedge w, \varphi] = -4 \cdot (v \wedge w) \wedge \varphi = -4 \cdot (\varphi(v)w - \varphi(w)v),$$

the Jacobi identity for these elements reads

$$-4\cdot(\varphi(v)w - \varphi(w)v) = -[w, \varphi](v) + [v, \varphi](w). \tag{22.8}$$

The right-hand side is

$$-[w, \varphi](v) + [v, \varphi](w) = -(3\varphi(v)w - \varphi(w)v) + (3\varphi(w)v - \varphi(v)w),$$

which proves this case. (This last line was the only place where we needed to use the definition (22.4) in place of the fancier (22.7).)

The last case is for one element v in W and two elements φ and ψ in W^*. This time identity to be proved comes down to

$$-4\cdot(\psi(v)\varphi - \varphi(v)\psi) = [v, \varphi]\cdot\psi - [v, \psi]\cdot\varphi.$$

Applying both sides to an element w in W, this becomes

$$-4\cdot(\psi(v)\varphi(w) - \varphi(v)\psi(w)) = \varphi([v, \psi]\cdot w) - \psi([v, \varphi]\cdot w).$$

If we apply ψ to the previous case (22.8) we have

$$-4\cdot(\varphi(v)\psi(w) - \varphi(w)\psi(v)) = -\psi([w, \varphi]\cdot v) + \psi([v, \varphi]\cdot w).$$

And these are the same, using the symmetry of the Killing form:

$$18\cdot\varphi([v, \psi]\cdot w) = B([v, \psi], [w, \varphi]) = B([w, \varphi], [v, \psi]) = 18\psi([w, \varphi]\cdot v).$$

This completes the proof that the algebra with multiplication table (22.1) is a Lie algebra. With the hindsight derived from working all this out, of course, we see that there is a quicker way to construct $\mathfrak{g}_2$, without any multiplication table: simply start with $\mathfrak{sl}_3\mathbb{C} \oplus W \oplus W^*$, and define products according to the above rules.

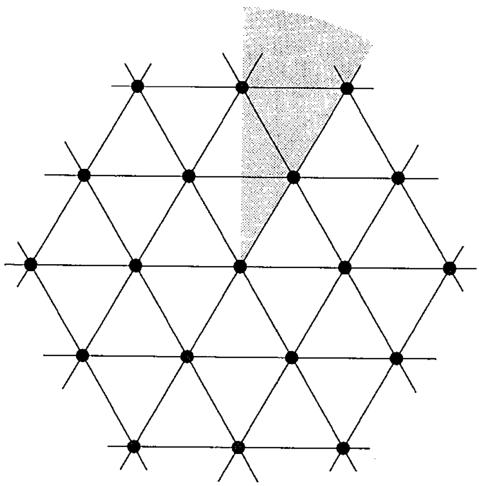
§22.3. Representations of $\mathfrak{g}_2$

We would now like to use the standard procedure, outlined in Lecture 14 (and carried out for the classical Lie algebras in Lectures 15–20) to say something about the representations of $\mathfrak{g}_2$. One nice aspect of this is that, working simply from the root system of $\mathfrak{g}_2$ and analyzing its representations, we will arrive at what is perhaps the simplest description of the algebra: we will see that $\mathfrak{g}_2$ is the algebra of endomorphisms of a seven-dimensional vector space preserving a general trilinear form.

The first step is to find the weight lattice for $\mathfrak{g}_2$. This is the lattice $\Lambda_W \subset \mathfrak{h}^*$ dual to the lattice $\Gamma_W \subset \mathfrak{h}$ generated by the six distinguished elements H_i. By (22.2), Γ_W is generated by H_1 and H_2. Since the values of the eigenvalues α_1 and α_2 on H_1 and H_2 are given by

$$\alpha_1(H_1) = 2, \qquad \alpha_1(H_2) = -1,$$
$$\alpha_2(H_1) = -3, \qquad \alpha_2(H_2) = 2,$$

it follows that the weight lattice is generated by the eigenvalues α_1 and α_2 (and in particular the weight lattice Λ_W is equal to the root lattice Λ_R). The picture is thus

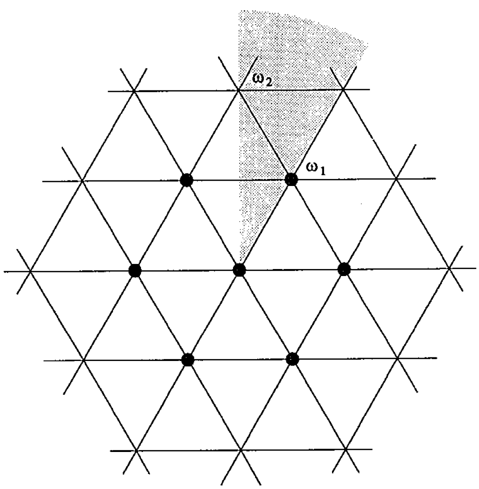


As in the case of the classical Lie algebras, the intersection of the (closed) Weyl chamber $\mathscr{W}$ with the weight lattice is a free semigroup on the two fundamental weights

$$\omega_1 = 2\alpha_1 + \alpha_2 \quad \text{and} \quad \omega_2 = 3\alpha_1 + 2\alpha_2.$$

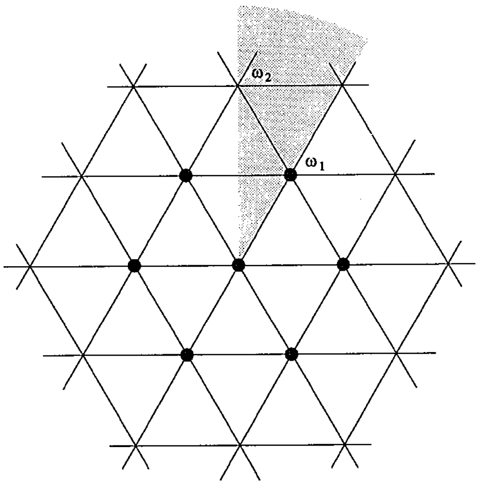
Any irreducible representation of $\mathfrak{g}_2$ will thus have a highest weight vector λ which is a non-negative linear combination of these two. As usual, we write $\Gamma_{a,b}$ for the irreducible representation with highest weight $a\omega_1 + b\omega_2$.

Let us consider first the representation $\Gamma_{1,0}$ with highest weight ω_1. Translating ω_1 around by the action of the Weyl group, we see that the weight diagram of $\Gamma_{1,0}$ looks like



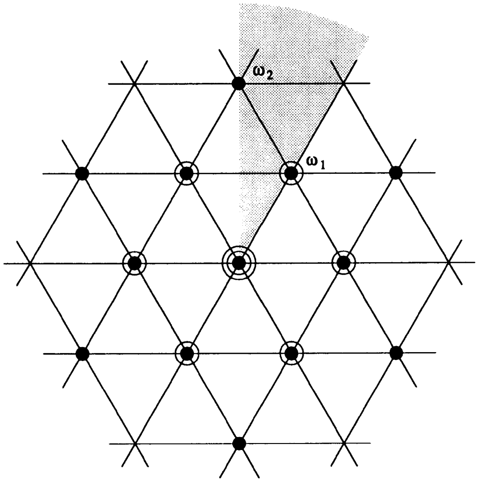
Since there is only one way of getting from the weight ω_1 to the weight 0 by subtraction of simple positive roots, the multiplicity of the weight 0 in $\Gamma_{1,0}$ must be 1. $\Gamma_{1,0}$ is thus a seven-dimensional representation. It is the smallest of the representations of $\mathfrak{g}_2$, and moreover has the property (as we will verify below) that every irreducible representation of $\mathfrak{g}_2$ appears in its tensor algebra; we will therefore call it the *standard* representation of $\mathfrak{g}_2$ and denote it V.

The next smallest representation of $\mathfrak{g}_2$ is the representation $\Gamma_{0,1}$ with highest weight ω_2; this is just the adjoint representation, with weight diagram



Note that the multiplicity of 0 as a weight of $\Gamma_{0,1}$ is 2, and the dimension of $\Gamma_{0,1}$ is 14.

Consider next the exterior square $\wedge^2 V$ of the standard representation $V = \Gamma_{1,0}$ of $\mathfrak{g}_2$. Its weight diagram looks like

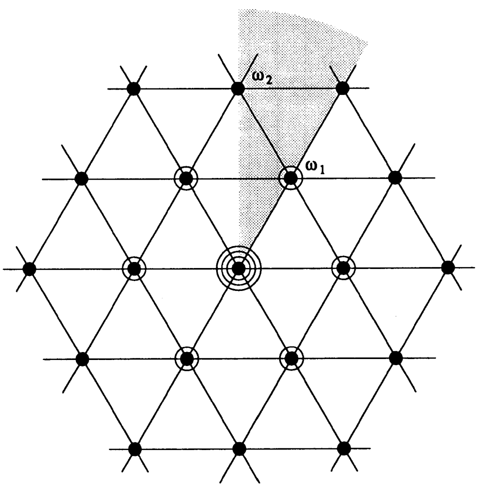


from which we may deduce that

$$\wedge^2 V \cong \Gamma_{0,1} \oplus V.$$

In particular, since the adjoint representation $\Gamma_{0,1}$ of $\mathfrak{g}_2$ is contained in $\wedge^2 V$, and the irreducible representation $\Gamma_{a,b}$ with highest weight $a\omega_1 + b\omega_2$ is contained in the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b \Gamma_{0,1}$, we see that *every irreducible representation of* $\mathfrak{g}_2$ *appears in some tensor power* $V^{\otimes m}$ *of the standard representation*, as stated above.

Next, look at the symmetric square $\mathrm{Sym}^2 V$ of the standard representation. It has weight diagram

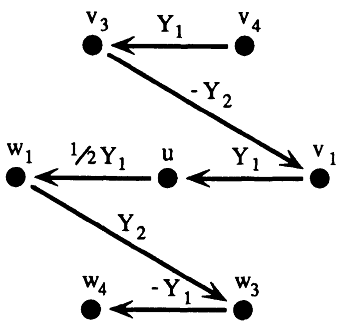


Clearly, this contains a copy of the irreducible representation $\Gamma_{2,0}$ of $\mathfrak{g}_2$ with highest weight $2\omega_1$. Depending on the multiplicities of this representation, it may also contain a copy of V itself, of the trivial representation, or both; or it may be irreducible. To see which is in fact the case, we need to know more about the action of $\mathfrak{g}_2$ on the standard representation V. We will do this in two ways, first by direct calculation, and second using the decomposition of $\mathfrak{g}_2$ into $\mathfrak{sl}_3 \oplus W \oplus W^*$. Although the second approach is shorter, the first illustrates how one can calculate for the exceptional Lie algebras very much as we have been doing in the classical cases.

To describe V explicitly, start with a highest weight vector for V, i.e., any nonzero element v_4 of the eigenspace $V_4 \subset V$ for the action of $\mathfrak{h}$ with eigenvalue α_4. The image of v_4 under the root vector Y_1 will then be a nonzero element of the eigenspace V_3 with eigenvalue α_3 (this follows from the fact that the direct sum $V_3 \oplus V_4$, as a representation of the subalgebra $\mathfrak{s}_{\alpha_1} \subset \mathfrak{g}$, is a copy of the standard representation of $\mathfrak{s}_{\alpha_1} \cong \mathfrak{sl}_2\mathbb{C}$). Similarly, the image of v_3 under Y_2 is a generator v_1 of the eigenspace V_1 with eigenvalue α_1, the image of v_1 under Y_1 is a generator of the eigenspace V_0 with eigenvalue 0, and so on. We may thus choose as a basis for V the vectors

$$v_4, \qquad v_3 = Y_1(v), \qquad v_1 = -Y_2(v_3), \qquad u = Y_1(v_1),$$
$$w_1 = \tfrac{1}{2}Y_1(u), \qquad w_3 = Y_2(w_1), \qquad \text{and} \qquad w_4 = -Y_1(w_3),$$

where v_i (resp. w_i) is an eigenvector with eigenvalue α_i (resp. β_i). (The signs and coefficient $\frac{1}{2}$ in the definition of w_1 are there for reasons of symmetry—see Exercise 22.10.) Diagrammatically, the action of $\mathfrak{g}_2$ may be represented by the arrows



Exercise 22.9. (i) Verify that the vectors v_i, w_i, and u, as defined above, are indeed generators of the corresponding eigenspaces. (ii) Find, in terms of this basis for V, the images of v_4 under the elements Y_3, Y_4, Y_5, and Y_6.

Exercise 22.10. Show that the elements X_i and $Y_i \in \mathfrak{g}_2$ all carry basis vectors v_j and w_j into other basis vectors, up to sign (or to zero, of course), and carry u to twice basis vectors, that is, $X_i u = 2v_i$ and $Y_i u = 2w_i$ for $i = 1, 3, 4$.

Now, the representation $\mathrm{Sym}^2 V$ has, as basis, the pairwise products of the basis vectors for V; and the subrepresentation $\Gamma_{2,0}$ is just the subspace generated by the images of the highest weight vector v_4^2 under (repeated applications of) the generators Y_1, Y_2 of the negative root spaces of $\mathfrak{g}_2$. Thus, for example, the eigenspace in $\mathrm{Sym}^2 V$ with eigenvalue α_4 is the span of the products $u \cdot v_4$ and $v_3 \cdot v_1$; the part of this lying in $\Gamma_{2,0}$ will be the span of the two vectors $Y_2 Y_1 Y_1(v_4^2)$ and $Y_1 Y_2 Y_1(v_4^2)$. We calculate:

$$\begin{aligned} Y_2 Y_1 Y_1(v_4^2) &= Y_2 Y_1(2v_3 \cdot v_4) = Y_2(2v_3^2) \\ &= -4v_1 \cdot v_3 \end{aligned}$$

and

$$\begin{aligned} Y_1 Y_2 Y_1(v_4^2) &= Y_1 Y_2(2v_3 \cdot v_4) = -Y_1(2v_1 \cdot v_4) \\ &= -2v_1 \cdot v_3 - 2u \cdot v_4. \end{aligned}$$

We see, in other words, that $\Gamma_{2,0}$ assumes the weight α_4 with multiplicity 2, so that in particular $\mathrm{Sym}^2 V$ does not contain a copy of V.

Similarly, to see whether or not $\mathrm{Sym}^2 V$ contains a copy of the trivial representation, we have to calculate the multiplicity of the weight 0 in $\Gamma_{2,0}$. Since any path in the weight lattice from the eigenvalue $2\alpha_4$ to 0 obtained by subtracting α_1 and α_2 must pass through α_4, we can do this by evaluating the products of Y_1 and Y_2 on the generators $v_1 \cdot v_3$ and $u \cdot v_4$ of the eigenspace with eigenvalue α_4: we have

$$\begin{aligned} Y_1 Y_1 Y_2(v_1 v_3) &= -Y_1 Y_1(v_1^2) = -Y_1(2u \cdot v_1) \\ &= -4w_1 \cdot v_1 - 2u^2; \\ Y_1 Y_1 Y_2(u \cdot v_4) &= 0; \\ Y_1 Y_2 Y_1(v_1 v_3) &= Y_1 Y_2(u \cdot v_3) = -Y_1(u \cdot v_1) \\ &= -2w_1 \cdot v_1 - u^2; \\ Y_1 Y_2 Y_1(u \cdot v_4) &= Y_1 Y_2(u \cdot v_3 + 2w_1 \cdot v_4) \\ &= Y_1(-u \cdot v_1 + 2w_3 \cdot v_4) \\ &= -2w_1 \cdot v_1 - u^2 - 2w_4 \cdot v_4 + 2w_3 v_3; \\ Y_2 Y_1 Y_1(v_1 v_3) &= Y_2 Y_1(u \cdot v_3) = Y_2(2w_1 \cdot v_3) \\ &= -2w_1 \cdot v_1 + 2w_3 \cdot v_3; \end{aligned}$$

and

$$\begin{aligned} Y_2 Y_1 Y_1(u \cdot v_4) &= Y_2 Y_1(u \cdot v_3 + 2w_1 \cdot v_4) = Y_2(4w_1 \cdot v_3) \\ &= -4w_1 \cdot v_1 + 4w_3 \cdot v_3. \end{aligned}$$

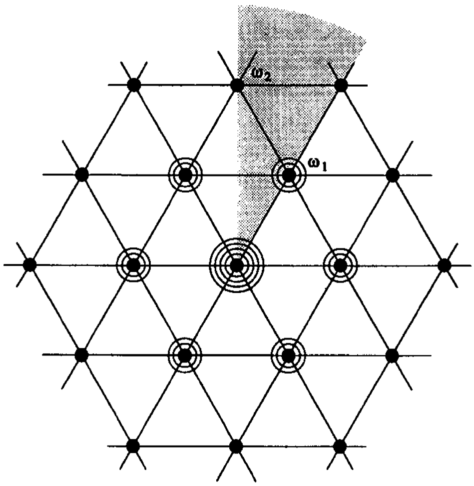
We see from this that the 0-eigenspace of $\Gamma_{2,0}$ is three dimensional; we thus have the decomposition

$$\mathrm{Sym}^2 V \cong \Gamma_{2,0} \oplus \mathbb{C}.$$

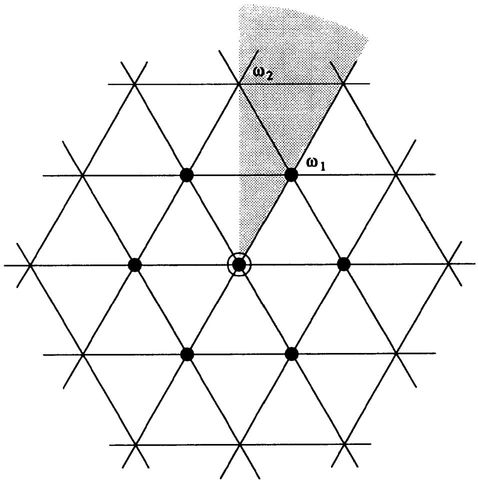
In particular, we deduce that *the action of* $\mathfrak{g}_2$ *on the standard representation* $V = \mathbb{C}^7$ *preserves a quadratic form*; and correspondingly that the subalgebra $\mathfrak{g}_2 \subset \mathfrak{sl}(V) = \mathfrak{sl}_7\mathbb{C}$ is actually contained in the algebra $\mathfrak{so}_7\mathbb{C}$. We will see this again in the following section, where we will give alternative descriptions of the exceptional Lie algebras, and again in §23.3 where we describe compact homogeneous spaces for Lie groups.

Exercise 22.11. Analyze in general the symmetric powers $\mathrm{Sym}^k V$ of the standard representation V of $\mathfrak{g}_2$.

Finally, consider the exterior cube $\wedge^3 V$ of the standard representation. The weight diagram is



and after we remove one copy of the representation $\Gamma_{2,0}$ with highest weight $2\omega_1$ (this is the sum of the three highest weights α_4, α_3, and α_1 of V), we are left with



This, by what we have seen, can only be the direct sum of the standard representation V with the trivial representation $\mathbb{C}$. In sum, then, we conclude that

$$\wedge^3 V \cong \Gamma_{2,0} \oplus V \oplus \mathbb{C}.$$

Note in particular that, as a corollary, *the action of* $\mathfrak{g}_2$ *on the standard representation preserves a skew-symmetric trilinear form* ω *on* V. It is not hard to write down this form: it is a linear combination of the five vectors $w_3 \wedge u \wedge v_3, v_4 \wedge u \wedge w_4, w_1 \wedge u \wedge v_1, v_1 \wedge v_3 \wedge w_4$, and $w_1 \wedge w_3 \wedge v_4$; and the fact that it is preserved by X_1 and X_2 is enough to determine the coefficients: we have

$$\omega = w_3 \wedge u \wedge v_3 + v_4 \wedge u \wedge w_4 + w_1 \wedge u \wedge v_1$$
$$+ 2v_1 \wedge v_3 \wedge w_4 + 2w_1 \wedge w_3 \wedge v_4.$$

The fact that the action of $\mathfrak{g}_2$ on V preserves the skew-symmetric cubic form ω takes on additional significance when we make a naive dimension count. The space $\wedge^3 V$ of all such alternating forms has dimension 35, while the algebra $\mathfrak{gl}(V)$ of endomorphisms of V has dimension 49; the difference is exactly the dimension of the algebra $\mathfrak{g}_2$. In fact, we can check directly that the linear map

$$\varphi\colon \mathfrak{gl}(V) \to \wedge^3 V$$

sending $A \in \mathrm{End}(V)$ to $A(\omega)$ is surjective. We deduce that ω is a general cubic alternating form [i.e., an open dense subset of $\wedge^3 V$ corresponds to forms equivalent to ω under $\mathrm{Aut}(V)$], and hence that

Proposition 22.12. *The algebra* $\mathfrak{g}_2$ *is exactly the algebra of endomorphisms of a seven-dimensional vector space* V *preserving a general skew-symmetric cubic form* ω *on* V.

Exercise 22.13*. Verify that the map φ above is surjective by direct calculation of the action of $\mathfrak{gl}(V)$ on $\omega \in \wedge^3 V$.

Exercise 22.14. As an alternative to the preceding exercise, analyze skew-symmetric trilinear forms on $\mathbb{C}^n$ to show that for $n \leq 7$ there are only finitely many such forms, up to the action of $\mathrm{GL}_n\mathbb{C}$. Verify that the form ω above is general in $\wedge^3\mathbb{C}^7$. (In fact, there are only finitely many cubic alternating forms on $\mathbb{C}^8$ as well, though this is fairly complicated; for $n \geq 9$ a simple dimension count shows that there is a continuously varying family of such forms.)

Note that the cubic form ω preserved by the action of $\mathfrak{g}_2$ gives us explicitly the inclusion

$$V \hookrightarrow \wedge^2 V$$

deduced earlier from their weight diagrams: this is just the map $V^* \to \wedge^2 V$ given by contraction/wedge product with ω, composed with the isomorphism of V with V^*.

Exercise 22.15*. Find the algebra of endomorphisms of a six-dimensional vector space preserving a general skew-symmetric trilinear form.

We will see the form ω again when we describe $\mathfrak{g}_2$ in the following section.

These calculations using the table amount to using all the information that can be extracted from the subalgebras $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$ of $\mathfrak{g}_2$. Using the copy of $\mathfrak{sl}_3\mathbb{C}$ that we found in the second section can make some of this more transparent. Make the identification

$$\mathfrak{g}_2 = \mathfrak{g}_0 \oplus W \oplus W^* = \mathfrak{sl}_3\mathbb{C} \oplus W \oplus W^*.$$

As a representation of $\mathfrak{sl}_3\mathbb{C}$, the seven-dimensional representation V must be the sum of W, W^*, and the trivial representation $\mathbb{C}$. If we make this identification,

$$V = W \oplus W^* \oplus \mathbb{C},$$

it is not hard to work out how the rest of $\mathfrak{g}_2$ acts. This is given in the following table:

		W	W^*	$\mathbb{C}$
		w	ψ	z
$\mathfrak{g}_0$	X	$X \cdot w$	$X \cdot \psi$	0
W	v	$-v \wedge w$	$\psi(v)$	$2z \cdot v$
W^*	φ	$\varphi(w)$	$\varphi \wedge \psi$	$2z \cdot \varphi$

With this identification, we have $u = 1$ in $\mathbb{C}$, and

$$
\begin{aligned}
v_4 &= e_1, \qquad w_1 = e_2, \qquad w_3 = e_3 \quad \text{in } W = \mathbb{C}^3;\\
w_4 &= e_1^*, \qquad v_1 = e_2^*, \qquad v_3 = e_3^* \quad \text{in } W^* = (\mathbb{C}^3)^*.
\end{aligned}
$$

Conversely, it is not hard to verify that the above table defines a representation of $\mathfrak{g}_2$, by checking the various cases of the identity $[\xi, \eta]\cdot y = \xi\cdot(\eta\cdot y) - \eta\cdot(\xi\cdot y)$ for ξ, η in $\mathfrak{g}_2$ and y in V. Note that the cubic form ω becomes

$$\omega = \sum_{i=1}^{3} e_i \wedge u \wedge e_i^* + 2(e_1 \wedge e_2 \wedge e_3 + e_1^* \wedge e_2^* \wedge e_3^*).$$

This description of V can be used to verify the calculations made earlier, and also to study its symmetric and exterior powers. For example, $\mathrm{Sym}^2 V$ decomposes over $\mathfrak{sl}_3\mathbb{C}$ into

$$
\begin{aligned}
&\mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 W^* \oplus \mathrm{Sym}^2\mathbb{C} \oplus W \otimes \mathbb{C} \oplus W^* \otimes \mathbb{C} \oplus W \otimes W^*\\
&\quad = \mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 W^* \oplus \mathbb{C} \oplus W \oplus W^* \oplus \mathfrak{sl}_3\mathbb{C} \oplus \mathbb{C}.
\end{aligned}
$$

To get the weights around the outside ring, the irreducible representation $\Gamma_{2,0}$ must include $\mathrm{Sym}^2 W$, $\mathrm{Sym}^2 W^*$, and $\mathfrak{sl}_3\mathbb{C}$. Checking that $W \subset \mathfrak{g}_2$ maps $\mathrm{Sym}^2 W^*$ nontrivially to W^* shows that it must also include W and W^*. To finish it suffices to compute the part killed by $\mathfrak{g}_2$, which must lie in the sum of the two components which are trivial for $\mathfrak{sl}_3\mathbb{C}$; checking that this is one dimensional, one recovers the decomposition

$$\mathrm{Sym}^2 V = \Gamma_{2,0} \oplus \mathbb{C}.$$

Exercise 22.16. Use this method to decompose $\bigwedge^3 V$ and $\mathrm{Sym}^3 V$.

§22.4. Algebraic Constructions of the Exceptional Lie Algebras

In this section we will sketch a few of the abstract approaches to the construction of the five exceptional Lie algebras. The constructions are not as easy as you might wish: although the exceptional Lie groups and their Lie algebras have a remarkable way of showing up unexpectedly in many areas of mathematics and physics, they do not have such simple descriptions as the classical series. Indeed, they were not discovered until the classification theorem forced mathematicians to look for them.

To begin with, the method we used to construct $\mathfrak{g}_2$ in the second section of this lecture can be generalized to construct other Lie algebras. This is the construction of Freudenthal, which we do first. It can be used to construct the Lie algebra $\mathfrak{e}_8$ for the diagram (E_8). From $\mathfrak{e}_8$ it is possible to construct $\mathfrak{e}_7$ and $\mathfrak{e}_6$ and $\mathfrak{f}_4$. Then we will present (or at least sketch) several other approaches to their construction. Since it is a rather technical subject, probably not really suited for a first course, we will touch on several approaches rather than give a detailed discussion of one.

The construction of $\mathfrak{g}_2$ as a sum $\mathfrak{g}_0 \oplus W \oplus W^*$ that we found in the second section works more generally, with very little change. Suppose $\mathfrak{g}_0$ is a semisimple Lie algebra, and W is a representation of $\mathfrak{g}_0$; let W^* be the dual representation, and set

$$\mathfrak{g} = \mathfrak{g}_0 \oplus W \oplus W^*.$$

We also need maps

$$\wedge : \Lambda^2 W \to W^* \quad \text{and} \quad \wedge : \Lambda^2 W^* \to W$$

of representations of $\mathfrak{g}_0$. We assume these are given by trilinear maps of $\mathfrak{g}_0$-representations $T: \Lambda^3 W \to \mathbb{C}$ and $T': \Lambda^3 W^* \to \mathbb{C}$, which means that

$$(u \wedge v)(w) = T(u, v, w) \quad \text{and} \quad \vartheta(\varphi \wedge \psi) = T'(\varphi, \psi, \vartheta).$$

We can then define a bracket on $\mathfrak{g}$ by the same rules as in the second section. To describe it, we let $X, Y, Z, \ldots$ denote arbitrary elements of $\mathfrak{g}_0$, $u, v, w, \ldots$ elements of W, and $\varphi, \psi, \vartheta, \ldots$ elements of W^*. The bracket in $\mathfrak{g}$ is determined by setting:

(i) $[X, Y] = [X, Y]$ (the given bracket in $\mathfrak{g}_0$),
(ii) $[X, v] = X \cdot v$ (the action of $\mathfrak{g}_0$ on W),
(iii) $[X, \varphi] = X \cdot \varphi$ (the canonical action of $\mathfrak{g}_0$ on W^*),
(iv) $[v, w] = a \cdot (v \wedge w)$ (for a scalar a to be determined),
(v) $[\varphi, \psi] = b \cdot (\varphi \wedge \psi)$ (for a scalar b to be determined)
(vi) $[v, \varphi] = c \cdot (v * \varphi)$ (for a scalar c to be determined).

As before, $v * \varphi$ is the element of $\mathfrak{g}_0$ such that

$$B(v * \varphi, Z) = \varphi(Z \cdot v) \quad \text{for all } Z \in \mathfrak{g}_0,$$

where B is the Killing form on $\mathfrak{g}_0$. The rules (i)–(vi) determine a bilinear product $[\ ,\ ]$ on all of $\mathfrak{g}$, and the fact that it is skew follows from the facts that $[X, X] = 0$, $[v, v] = 0$, and $[\varphi, \varphi] = 0$.

The argument that we gave showing that $\mathfrak{g}_2$ satisfies the Jacobi identity works in this general case without essential change, except for the last two cases, where explicit calculation is needed. For $v, w \in W$, and $\varphi \in W^*$, the Jacobi identity is equivalent to the identity

$$ab((v \wedge w) \wedge \varphi) = c((v * \varphi) \cdot w - (w * \varphi) \cdot v). \tag{22.17}$$

For $v \in W$, $\varphi, \psi \in W^*$, the Jacobi identity amounts to

$$ab((\varphi \wedge \psi) \wedge v) = c((v * \psi) \cdot \varphi - (v * \varphi) \cdot \psi). \tag{22.18}$$

We will see in Exercise 22.20 that (22.17) and (22.18) are equivalent. Again, the simplicity of the resulting Lie algebra is easy to see, provided all the weight spaces are one dimensional, using Exercise 14.34, so we have:

Proposition 22.19 (Freudenthal). *Given a representation W of a semisimple Lie algebra $\mathfrak{g}_0$ and trilinear forms T and T' inducing maps $\bigwedge^2 W \to W^*$ and $\bigwedge^2 W^* \to W$, such that (22.17) and (22.18) are satisfied, the above products make*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus W \oplus W^*$$

into a Lie algebra. If the weight spaces of W are all one dimensional, and the weights of W, W^*, and the roots of $\mathfrak{g}_0$ are all distinct, and $abc \neq 0$, then $\mathfrak{g}$ is semisimple, with the same Cartan subalgebra as $\mathfrak{g}_0$.

Exercise 22.20*. (a) Show that the trilinear map T determines a map $\wedge$: $\bigwedge^2 W \to W^*$ of representations if and only if it satisfies the identity

$$T(X \cdot u, v, w) + T(u, X \cdot v, w) + T(u, v, X \cdot w) = 0 \qquad \forall X \in \mathfrak{g}_0,$$

and similarly for T'.

(b) Show that each of (22.17) and (22.18) is equivalent to the identity

$$ab \cdot (v \wedge w)(\varphi \wedge \psi) = c \cdot (B(w * \psi, v * \varphi) - B(w * \varphi, v * \psi)).$$

The Lie algebra $\mathfrak{e}_8$ for (E_8) can be constructed by this method. This time $\mathfrak{g}_0$ is taken to be the Lie algebra $\mathfrak{sl}_9\mathbb{C}$; if $V = \mathbb{C}^9$ is the standard representation of $\mathfrak{sl}_9\mathbb{C}$, let $W = \bigwedge^3 V$, so $W^* = \bigwedge^3 V^*$; the trilinear map is the usual wedge product

$$\textstyle\bigwedge^3 V \otimes \bigwedge^3 V \otimes \bigwedge^3 V \to \bigwedge^9 V = \mathbb{C},$$

and similarly for $\bigwedge^3 V^*$. We leave the verifications to the reader:

Exercise 22.21*. (i) Verify the conditions on the roots of $\mathfrak{sl}_9$ and the weights of $\bigwedge^3 V$ and $\bigwedge^3 V^*$. (ii) Use the fact that $B(X, Y) = 18 \cdot \mathrm{Tr}(XY)$ for $\mathfrak{sl}_9$ to show that (22.17) holds precisely if $c = -18ab$. (iii) Show that the Dynkin diagram of the resulting Lie algebra is (E_8).

Note that the dimension of $\mathfrak{sl}_9\mathbb{C}$ is 80, and that of W and W^* is 84, so the sum has dimension 248, as predicted by the root system of (E_8).

Once the Lie algebra $\mathfrak{e}_8$ is constructed, $\mathfrak{e}_7$ and $\mathfrak{e}_6$ can be found as subalgebras, as follows. Note that removing one or two nodes from the long arm of the Dynkin diagram of (E_8) leads to the Dynkin diagrams (E_7) and (E_6).

In general, if $\mathfrak{g}$ is a simple Lie algebra, with Dynkin diagram D, consider a subdiagram D° of D obtained by removing some subset of nodes, together with all the lines meeting these nodes.[1] Then we can construct a semisimple subalgebra $\mathfrak{g}^\circ$ of $\mathfrak{g}$ with D° as its Dynkin diagram. In fact, $\mathfrak{g}^\circ$ is the subalgebra generated by all the root spaces $\mathfrak{g}_{\pm\alpha}$, where α is a root in D°.

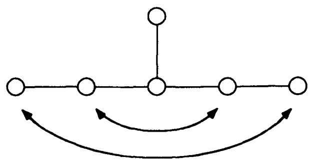
[1] If there are double or triple lines between two nodes, both nodes should be removed or kept together.

Exercise 22.22. (a) Prove this by verifying that the positive roots of $\mathfrak{g}^\circ$ are the positive roots β of $\mathfrak{g}$ that are sums of the roots in D°, and the Cartan subalgebra $\mathfrak{h}^\circ$ is spanned by the corresponding vectors $H_\beta \in \mathfrak{h}$.

(b) Carry this out for $\mathfrak{e}_7$ and $\mathfrak{e}_6$; in particular, show again that $\mathfrak{e}_7$ has 63 positive roots, so dimension $7 + 2(63) = 133$, and $\mathfrak{e}_6$ has 36 positive roots, so dimension $6 + 2(36) = 78$.

Exercise 22.23. For each of the simple Lie algebras, find the subalgebras obtained by removing one node from an end of its Dynkin diagram.

The last exceptional Lie algebra $\mathfrak{f}_4$ can be constructed by taking an invariant subalgebra of $\mathfrak{e}_6$ by an involution. This involution corresponds to the evident symmetry in the Dynkin diagram:



In general, an automorphism of a Dynkin diagram arises from an automorphism of the corresponding semisimple Lie algebra, as follows from the fact that the multiplication table is determined by the Dynkin diagram, cf. Proposition 21.22 and Claim 21.25.

Exercise 22.24*. (a) Show that the invariant subalgebra for the indicated involution of $\mathfrak{e}_6$ is a simple Lie algebra $\mathfrak{f}_4$ with Dynkin diagram (F_4).

(b) Find the invariant subalgebra for the involutions of (A_n) and (D_n), and for an automorphism of order three of (D_4).

Exercise 22.25*. For each automorphism of the Dynkin diagrams (A_n) and (D_n), find an explicit automorphism of $\mathfrak{sl}_{n+1}\mathbb{C}$ and $\mathfrak{so}_{2n}\mathbb{C}$ that induces it.

The exceptional Lie algebras can also be realized as the Lie algebras of derivations of certain nonassociative algebras. This also gives realizations of corresponding Lie groups as groups of automorphism of these algebras (see Exercise 8.28). Some examples of this for associative algebras should be familiar. The group of automorphisms of the algebra $\mathbb{H}$ of (real) quaternions is $\mathrm{O}(3)$, so the Lie algebra of derivations is $\mathfrak{so}_3\mathbb{R}$. The Lie algebra of derivations of the complexification $\mathbb{H}_\mathbb{C}$ is $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$.

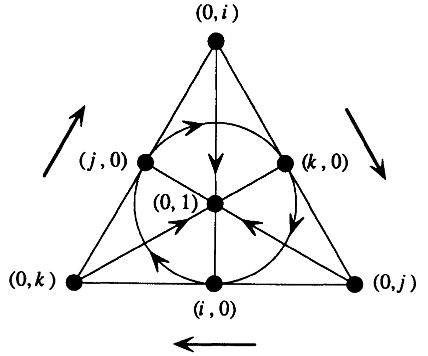
The exceptional group G_2 can be realized as the group of automorphisms of the complexification of the eight-dimensional *Cayley algebra*, or algebra of *octonions*. Recall that the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ can be constructed as the set of pairs (a, b) of complex numbers. In a similar way the Cayley algebra, which we denote by $\mathbb{O}$, can be constructed as the set of pairs (a, b), with a and b quaternions. The addition is componentwise, with multiplication

$$(a, b) \circ (c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where $^-$ denotes conjugation in $\mathbb{H}$. This algebra $\mathbb{O}$ also has a conjugation, which takes (a, b) to $(\bar{a}, -b)$. It has a basis $1 = (1, 0)$, together with seven elements $e_1, \ldots, e_7$:

$$(i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k).$$

These satisfy $e_p \circ e_p = -1$ and $e_p \circ e_q = -e_q \circ e_p$ for $p \neq q$, and the conjugate $\overline{e_p}$ of e_p is $-e_p$. The multiplication table can be encoded in the diagram:



Here, if e_p, e_q, and e_r appear on a line in the order shown by the arrow, then

$$e_p \circ e_q = e_r, \qquad e_q \circ e_r = e_p, \qquad e_q \circ e_p = e_r.$$

Note in particular that any two of these basic elements generate a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

Exercise 22.26. Show that the subalgebra of $\mathbb{O}$ generated by any two elements is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Deduce that, although $\mathbb{O}$ is noncommutative and nonassociative, it is "alternative," i.e., it satisfies the identities $(x \circ x) \circ y = x \circ (x \circ y)$ and $y \circ (x \circ x) = (y \circ x) \circ x$.

A trace and norm can be defined on $\mathbb{O}$ by

$$\operatorname{Tr}(x) = \tfrac{1}{2}(x + \bar{x}), \qquad N(x) = x \circ \bar{x};$$

these satisfy the relation $x^2 - 2\operatorname{Tr}(x) + N(x) = 0$. Let $\beta(x, y) = \frac{1}{2}(x \circ \bar{y} + y \circ \bar{x})$ be the bilinear form associated to N; note that the above basis is an orthonormal basis for this inner product.

Let G be the group of algebra automorphisms of the real algebra $\mathbb{O}$. The next exercise sketches a proof that the complexification of G is a Lie group of type (G_2).

Exercise 22.27*. The center of $\mathbb{O}$ is $\mathbb{R}\cdot 1$, which is preserved by G. Let Y be orthogonal space to $\mathbb{R}\cdot 1$ with respect to the quadratic form N. Then G is imbedded in the group $SO(Y)$ of orthogonal transformations of Y.

(a) Define a "cross product" $\times$ on Y by the formula $v \times w = v\cdot w + \beta(v, w)\cdot 1$. Show that G can be identified with the group of orthogonal transformations of Y that preserve the cross product.

(b) Show that $G = \mathrm{Aut}(\mathbb{O})$ acts transitively on the 6-sphere

$$S^6 = \{\sum r_i e_i \colon \sum r_i^2 = 1\},$$

and the subgroup K that fixes $i = e_1$ is mapped onto the 5-sphere in $e_1^\perp$ by the map $g \mapsto g\cdot j$. Conclude from this that G is 14-dimensional and simply connected.

(c) Show that $\{D \in \mathrm{Der}(\mathbb{O})\colon D(i) = 0\}$ is isomorphic to $\mathfrak{su}_3$.

(d) Verify that the Lie algebra of derivations of the complex octonians is the simple Lie algebra of type (G_2).

Exercise 22.28*. The octonions can also be constructed from the Clifford algebra of an eight-dimensional vector space with a nondegenerate quadratic form. With V, S^+, and S^- as in §20.3, with $v_1 \in V$, $s_1 \in S^+$, $t_1 = v_1\cdot s_1 \in S^-$ chosen so the values of the quadratic forms are 1 on each of them as in Exercise 20.50, define a product $V \times V \to V$, $(v, w) \mapsto v \circ w$ by the formula

$$v \circ w = (v\cdot t_1)\cdot(w\cdot s_1).$$

Note that $v\cdot t_1 \in S^+$, $w\cdot s_1 \in S^-$, so their product $(v\cdot t_1)\cdot(w\cdot s_1)$ is back in V.

(a) Show that V with this product is isomorphic to the complex octonians $\mathbb{O}$, with unit v_1, with the map $v \mapsto -\rho(v_1)(v)$ corresponding to conjugation in $\mathbb{O}$.

Conversely, starting with the complex octonians $\mathbb{O}$, one can reconstruct the algebra of §20.3: define $A = \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$, define an automorphism J of order 3 of A by $J(x, y, z) = (z, x, y)$, and define a product $\cdot$ from each succession of two factors to the third by the formulas $x\cdot y = \bar{x} \circ \bar{y}$, $y\cdot z = \bar{y} \circ \bar{z}$, $z\cdot x = \bar{z} \circ \bar{x}$.

(b) Show that A is isomorphic to the algebra described in §20.3.

(c) Identifying $\mathfrak{so}_8\mathbb{C}$ with the space of skew linear transformations of $\mathbb{O}$, show that for each A in $\mathfrak{so}_8\mathbb{C}$ there are unique B and C in $\mathfrak{so}_8\mathbb{C}$ such that

$$A(x \circ y) = B(x) \circ y + x \circ C(y)$$

for all complex octonions x and y. Equivalently, if one defines a trilinear form $(\ ,\ ,\ )$ on the octonions by $(x, y, z) = \mathrm{Tr}((x \circ y) \circ z) = \mathrm{Tr}(x \circ (y \circ z))$,

$$(Ax, y, z) + (x, By, z) + (x, y, Cz) = 0$$

for all x, y, z. Show that this trilinear form agrees with that defined in Exercise 20.49, and the mapping $A \mapsto B$ determines the triality automorphism j' of $\mathfrak{so}_8\mathbb{C}$ of order three described in Exercise 20.51.

Exercise 22.29. Define three homomorphisms from the real Clifford algebra $C_7 = C(0, 7)$ to $\mathrm{End}_{\mathbb{R}}(\mathbb{O})$ by sending $v \in \mathbb{R}^7 = \sum \mathbb{R}e_i$ to the maps L_v, R_v, and T_v defined by $L_v(x) = v \circ x$, $R_v(x) = x \circ v$, and $T_v(x) = v \circ (x \circ v) = (v \circ x) \circ v$.

(a) Show that these do determine maps of the Clifford algebra, and that the induced maps

$$\mathrm{Spin}_8 \mathbb{R} \hookrightarrow C_8^{\mathrm{even}} = C_7 \to \mathrm{End}_{\mathbb{R}}(\mathbb{O})$$

are the two spin representations and the standard representation, respectively.

(b) Verify that $T_v(x \circ y) = L_v(x) \cdot L_v(y)$ for all v, x, y, and use this to verify the triality formula in (c) of the preceding exercise.

The algebra $\mathfrak{f}_4$ can be realized as the derivation algebra of the complexification of a 27-dimensional *Jordan algebra* $\mathbb{J}$. This can be constructed as the set of matrices of the form

$$\begin{pmatrix} a & \alpha & \beta \\ \bar{\alpha} & b & \gamma \\ \bar{\beta} & \bar{\gamma} & c \end{pmatrix},$$

with a, b, c scalars, and α, β, γ in $\mathbb{O}$. The product $\circ$ in $\mathbb{J}$ is given by

$$x \circ y = \tfrac{1}{2}(xy + yx),$$

where the products on the right-hand side are defined by usual matrix multiplication. This algebra is commutative but not associative, and satisfies the identity $((x \circ x) \circ y) \circ x = (x \circ x) \circ (y \circ x)$. In fact, (F_4) is the group of automorphisms of this 27-dimensional space that preserve the scalar product $(x, y) = \mathrm{Tr}(x \circ y)$ and the scalar triple product $(x, y, z) = \mathrm{Tr}((x \circ y) \circ z)$. The kernel of the trace map is an irreducible 26-dimensional representation of $\mathfrak{f}_4$. For details see [Ch-S], [To], [Pos].

In addition, there is a cubic form "det" on $\mathbb{J}$ such that the linear automorphisms of $\mathbb{J}$ that preserve this form is a group of type (E_6). This again shows $\mathfrak{f}_4$ as a subalgebra of $\mathfrak{e}_6$.

The other exceptional Lie algebras can also be constructed as derivations of appropriate algebras. We refer for this to [Ti2], [Dr], [Fr2], [Jac2], and the references found in these sources. Other constructions were given by Witt, cf. [Wa]. The simple Lie algebras are also constructed explicitly in [S-K, §1]. See also [Ch-S], [Fr1], and [Sc].

What little we will have to say about the representations of the four exceptional Lie algebras besides $\mathfrak{g}_2$ can wait until we have the Weyl character formula.

LECTURE 23
Complex Lie Groups; Characters

This lecture serves two functions. First and foremost, we make the transition back from Lie algebras to Lie groups: in §23.1 we classify the groups having a given semisimple Lie algebra, and say which representations of the Lie algebra, as described in the preceding lectures, lift to which groups. Secondly, we introduce in §23.2 the notion of *character* in the context of Lie theory; this gives us another way of describing the representations of the classical groups, and also provides a necessary framework for the results of the following two lectures. Then in §23.3 we sketch the beautiful interrelationships among Dynkin diagrams, compact homogeneous spaces and the irreducible representations of a Lie group. The first two sections are elementary modulo a little topology needed to calculate the fundamental groups of the classical groups in §23.1. The third section, by contrast, may appear impossible: it involves, at various points, projective algebraic geometry, holomorphic line bundles, and their cohomology. In fact, a good deal of §23.3 can be understood without these notions; the reader is encouraged to read as much of the section as seems intelligible. A final section §23.4 gives a very brief introduction to the related Bruhat decomposition, which is included because of its ubiquity in the literature.

§23.1. Representations of Complex Simple Lie Groups

In Lecture 21 we classified all simple Lie algebras over $\mathbb{C}$. This in turn yields a classification of simple complex Lie groups: as we saw in Lecture 7, for any Lie algebra $\mathfrak{g}$ there is a unique simply connected group G, and all other (connected) complex Lie groups with Lie algebra $\mathfrak{g}$ are quotients of G by discrete subgroups of the center $Z(G)$. In this section, we will first describe the groups associated to the classical Lie algebras, and then proceed to describe which of the representations of the classical algebras we have described in Part III lift to which of the groups. We start with

Proposition 23.1. *For all $n \geq 1$, the Lie groups $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$ are connected and simply connected. For $n \geq 1$, $\mathrm{SO}_n\mathbb{C}$ is connected, with $\pi_1(\mathrm{SO}_2\mathbb{C}) = \mathbb{Z}$, and $\pi_1(\mathrm{SO}_n\mathbb{C}) = \mathbb{Z}/2$ for $n \geq 3$.*

Proof. The main tool needed from topology is the long exact homotopy sequence of a fibration. If the Lie group G acts transitively on a manifold M, and H is the isotropy group of a point P_0 of M, then $G/H = M$, and the map $G \to M$ by $g \mapsto g \cdot P_0$ is a fibration with fiber H. The resulting long exact sequence is, assuming the spaces are connected,

$$\cdots \to \pi_2(M) \to \pi_1(H) \to \pi_1(G) \to \pi_1(M) \to \{1\}. \tag{23.2}$$

(The base points, which are omitted in this notation, can be taken to be the identity elements of H and G, and the point P_0 in M.) In practice we will know M and H are connected, from which it follows that G is also connected. From this exact sequence, if M and H are also simply connected, the same follows for G.

To apply the long exact homotopy sequence in our present circumstance we argue by induction, noting first that $\mathrm{SL}_1\mathbb{C} = \mathrm{SO}_1\mathbb{C} = \{1\}$. Now consider the action of $G = \mathrm{SL}_n\mathbb{C}$ on the manifold $M = \mathbb{C}^n\backslash\{0\}$. The subgroup H fixing the vector $P_0 = (1, 0, \ldots, 0)$ consists of matrices whose first column is $(1, 0, \ldots, 0)$ and whose lower right $(n-1)$ by $(n-1)$ matrix is in $\mathrm{SL}_{n-1}\mathbb{C}$; it follows that as topological spaces $H \cong \mathrm{SL}_{n-1}\mathbb{C} \times \mathbb{C}^{n-1}$. Since M is simply connected for $n \geq 2$ (having the sphere S^{2n-1} as a deformation retract), and H has $\mathrm{SL}_{n-1}\mathbb{C}$ as a deformation retract, the claim for $\mathrm{SL}_n\mathbb{C}$ follows from (23.2) by induction on n.

The group $\mathrm{SO}_2\mathbb{C}$ is isomorphic to the multiplicative group $\mathbb{C}^*$, which has the circle as a deformation retract, so $\pi_1(\mathrm{SO}_2\mathbb{C}) = \mathbb{Z}$. The group $G = \mathrm{SO}_n\mathbb{C}$ acts transitively on $M = \{v \in \mathbb{C}^n \colon Q(v, v) = 1\}$, where Q is the symmetric bilinear form preserved by G. (The transitivity of the action is more or less equivalent to knowing that all nondegenerate symmetric bilinear forms are equivalent.) For explicit calculations take the standard Q for which the standard basis $\{e_i\}$ of $\mathbb{C}^n$ is an orthonormal basis. This time the subgroup H fixing e_1 is $\mathrm{SO}_{n-1}\mathbb{C}$. From the following exercise, it follows that M has the sphere S^{n-1} as a deformation retract. By (23.2) the map

$$\pi_1(\mathrm{SO}_{n-1}\mathbb{C}) \to \pi_1(\mathrm{SO}_n\mathbb{C})$$

is an isomorphism for $n \geq 4$. So it suffices to look at $\mathrm{SO}_3\mathbb{C}$. This could be done by looking at the maps in the sme exact sequence, but we saw in Lecture 10 that $\mathrm{SO}_3\mathbb{C}$ has a two-sheeted covering by $\mathrm{SL}_2\mathbb{C}$, which is simply connected by the preceding paragraph, so $\pi_1(\mathrm{SO}_3\mathbb{C}) = \mathbb{Z}/2$, as required.

The group $G = \mathrm{Sp}_{2n}\mathbb{C}$ acts transitively on

$$M = \{(v, w) \in \mathbb{C}^{2n} \times \mathbb{C}^{2n} : Q(v, w) = 1\},$$

where Q is the skew form preserved by G, and the isotropy group is $\mathrm{Sp}_{2n-2}\mathbb{C}$. Since $\mathrm{Sp}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}$, the first case is known. By the following exercise, since M is defined in $\mathbb{C}^{4n}$ by a nondegenerate quadratic form, M has S^{4n-1} as a deformation retract, so we conclude again by induction. □

Exercise 23.3*. Show that $\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum z_i^2 = 1\}$ is homeomorphic to the tangent bundle to the $(n-1)$-sphere, i.e., to

$$T_{S^{n-1}} = \{(u, v) \in S^{n-1} \times \mathbb{R}^n : u \cdot v = 0\}.$$

Using the exact sequence $\{1\} \to \mathrm{SL}_n\mathbb{C} \to \mathrm{GL}_n\mathbb{C} \to \mathbb{C}^* \to \{1\}$ we deduce from the proposition and (23.2) that

$$\pi_1(\mathrm{GL}_n\mathbb{C}) = \mathbb{Z}. \tag{23.4}$$

Exercise 23.5. Show that for all the above groups G, the second homotopy groups $\pi_2(G)$ are trivial.

We digress a moment here to mention a famous fact. Each of the above groups G has an associated compact subgroup: $\mathrm{SU}(n) \subset \mathrm{SL}_n\mathbb{C}$, $\mathrm{Sp}(n) \subset \mathrm{Sp}_{2n}\mathbb{C}$, and $\mathrm{SO}(n) \subset \mathrm{SO}_n\mathbb{C}$. In fact, each of these subgroups is connected, and these inclusions induce isomorphisms of their fundamental groups.

Exercise 23.6. Prove these assertions by finding compatible actions of the subgroups on appropriate manifolds. Alternatively, observe that in each case the compact subgroup in question is just the subgroup of G preserving a Hermitian form on $\mathbb{C}^n$ or $\mathbb{C}^{2n}$, and use Gram–Schmidt to give a retraction of G onto the subgroup.

Now, by Proposition 23.1 the simply-connected complex Lie groups corresponding to the Lie algebras $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$ are

$$\tilde{G} = \mathrm{SL}_n\mathbb{C}, \quad \mathrm{Sp}_{2n}\mathbb{C}, \quad \text{and } \mathrm{Spin}_m\mathbb{C}.$$

We also know the center $Z(\tilde{G})$ of each of these groups. From Lecture 7 we also know the other connected groups with these Lie algebras:

- The complex Lie groups with Lie algebra $\mathfrak{sl}_n\mathbb{C}$ are $\mathrm{SL}_n\mathbb{C}$ and quotients of $\mathrm{SL}_n\mathbb{C}$ by subgroups of the form $\{e^{2\pi l i/m} \cdot I\}_l$ for m dividing n (in particular, if n is prime the only such groups are $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{PSL}_n\mathbb{C}$).
- The complex Lie groups with Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ are $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathrm{PSp}_{2n}\mathbb{C}$.
- The complex Lie groups with Lie algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ are $\mathrm{Spin}_{2n+1}\mathbb{C}$ and $\mathrm{SO}_{2n+1}\mathbb{C}$.
  and
- The complex Lie groups with Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$ are $\mathrm{Spin}_{2n}\mathbb{C}$, $\mathrm{SO}_{2n}\mathbb{C}$ and $\mathrm{PSO}_{2n}\mathbb{C}$; in addition, if n is even, there are two other groups covered doubly by $\mathrm{Spin}_{2n}\mathbb{C}$ and covering doubly $\mathrm{PSO}_{2n}\mathbb{C}$ [cf. Exercise 20.36].

These are called the *classical groups*. In the cases where we have observed coincidences of Lie algebras, we have the following isomorphisms of groups:

$$\mathrm{Spin}_3\mathbb{C} \cong \mathrm{SL}_2\mathbb{C} \quad \text{and} \quad \mathrm{SO}_3\mathbb{C} \cong \mathrm{PSL}_2\mathbb{C};$$

$$\mathrm{Spin}_4\mathbb{C} \cong \mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C} \quad \text{and} \quad \mathrm{PSO}_4\mathbb{C} \cong \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C};$$

$$\mathrm{Spin}_5\mathbb{C} \cong \mathrm{Sp}_4\mathbb{C} \quad \text{and} \quad \mathrm{SO}_5\mathbb{C} \cong \mathrm{PSp}_4\mathbb{C};$$

and

$$\mathrm{Spin}_6\mathbb{C} \cong \mathrm{SL}_4\mathbb{C} \quad \text{and} \quad \mathrm{PSO}_6\mathbb{C} \cong \mathrm{PSL}_4\mathbb{C}.$$

Note that in the first case $n = 4$ where there is an intermediate subgroup between $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{PSL}_n\mathbb{C}$, the subgroup in question is interesting: it turns out to be $\mathrm{SO}_6\mathbb{C}$. In general, however, these intermediate groups seldom arise.

Consider now representations of these classical groups. According to the basic result of Lecture 7, representations of a complex Lie algebra $\mathfrak{g}$ will correspond exactly to representations of the associated simply connected Lie group $\tilde{G}$: specifically, for any representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$$

of $\mathfrak{g}$, setting

$$\tilde{\rho}(\exp(X)) = \exp(\rho(X))$$

determines a well-defined homomorphism

$$\tilde{\rho}\colon \tilde{G} \to \mathrm{GL}(V).$$

For any other group with algebra $\mathfrak{g}$, given as the quotient $\tilde{G}/C$ of $\tilde{G}$ by a subgroup $C \subset Z(\tilde{G})$, the representations of G are simply the representations of $\tilde{G}$ trivial on C. It is therefore enough to see which of the representations of the classical Lie algebras described in Part III are trivial on which subgroups $C \subset Z(\tilde{G})$.

This turns out to be very straightforward. To begin with, we observe that the center of each group G with Lie algebra $\mathfrak{g}$ lies in the image of the chosen Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ under the exponential map. It will therefore be enough to know when $\exp(\rho(X)) = I$ for $X \in \mathfrak{h}$; and since the representations ρ of $\mathfrak{g}$ are particularly simple on $\mathfrak{h}$ this presents no difficulty.

What we do have to do first is to describe the restriction of the exponential map to $\mathfrak{h}$, so that we can say which elements of $\mathfrak{h}$ exponentiate to elements of $Z(\tilde{G})$. For the groups that are given as matrix groups, this will all be perfectly obvious, but for the spin groups we will need to do a little calculation. We will also want to describe the *Cartan subgroup* H of each of the classical groups G, which is the connected subgroup whose Lie algebra is the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For $G = \mathrm{SL}_n\mathbb{C}$, H is just the diagonal matrices in G, i.e.,

$$H = \{\mathrm{diag}(z_1, \ldots, z_n)\colon z_1 \cdot \ldots \cdot z_n = 1\}.$$

Similarly in $\mathrm{Sp}_{2n}\mathbb{C}$ or $\mathrm{SO}_{2n}\mathbb{C}$, $H = \{\mathrm{diag}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})\}$, whereas in $\mathrm{SO}_{2n+1}\mathbb{C}$, $H = \{\mathrm{diag}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1)\}$. In each of these cases the exponential mapping from $\mathfrak{h}$ to H is just the usual exponentiation of diagonal matrices.

To calculate the exponential mapping for $\mathrm{Spin}_m\mathbb{C}$, we need to describe the elements in $\mathrm{Spin}_m\mathbb{C}$ that lie over the diagonal matrices in $\mathrm{SO}_m\mathbb{C}$. This is not a difficult task. Calculating as in §20.2, we find that for any nonzero complex number z and any $1 \le j \le n$, and with $m = 2n + 1$ or $m = 2n$, the elements

$$w_j(z) = \frac{1}{2}(ze_j \cdot e_{n+j} + z^{-1}e_{n+j} \cdot e_j) = z^{-1} + \left(\frac{z - z^{-1}}{2}\right)e_j \cdot e_{n+j} \qquad (23.7)$$

in the Clifford algebra are in fact elements of $\mathrm{Spin}_m\mathbb{C}$. Moreover, if $\rho\colon \mathrm{Spin}_m\mathbb{C} \to \mathrm{SO}_m\mathbb{C}$ is the covering, the image $\rho(w_j(z))$ is the diagonal matrix whose jth entry is z^2, $(n + j)$th entry is z^{-2}, and other diagonal entries are 1. These elements $w_j(z)$ also commute with each other, so for any nonzero complex numbers $z_1, \ldots, z_n$ we can define

$$w(z_1, \ldots, z_n) = w_1(z_1) \cdot w_2(z_2) \cdot \ldots \cdot w_n(z_n). \qquad (23.8)$$

Then $\rho(w(z_1, \ldots, z_n)) = \mathrm{diag}(z_1^2, \ldots, z_n^2, z_1^{-2}, \ldots, z_n^{-2})$ if $m = 2n$, while if $m = 2n + 1$, we get the same diagonal matrix but with a 1 at the end.

Let $H_i = E_{i,i} - E_{n+i,n+i}$, the usual basis for $\mathfrak{h} \subset \mathfrak{so}_m\mathbb{C}$.

Lemma 23.9. *For any complex numbers* $a_1, \ldots, a_n$,

$$\exp(a_1 H_1 + \cdots + a_n H_n) = w(e^{a_1/2}, \ldots, e^{a_n/2})$$

in $\mathrm{Spin}_m\mathbb{C}$.

PROOF. Since the map $\exp\colon \mathfrak{h} \to \mathrm{Spin}_m\mathbb{C}$ is determined by the facts that it is continuous, it takes 0 to 1, and its composite with ρ is the exponential for $\mathrm{SO}_m\mathbb{C}$, this follows from the preceding formulas. $\square$

Exercise 23.10*. Show that $\exp(\sum a_j H_j) = 1$ if and only if each a_j is in $2\pi i\mathbb{Z}$ and $\sum a_j \in 4\pi i\mathbb{Z}$.

We see also that $\exp(\mathfrak{h})$ contains the center of $\mathrm{Spin}_m\mathbb{C}$. Indeed, $-1 = w(-1, 1, \ldots, 1)$, and if m is even, the other central elements are $\pm\omega$, with $\omega = w(i, \ldots, i)$, as we calculated in Exercise 20.36. (This, of course, also contains the fact that there is a path between 1 and -1, proving again that $\mathrm{Spin}_m\mathbb{C}$ is connected.)

Exercise 23.11*. Verify for all the classical groups G that: (i) $H = \exp(\mathfrak{h})$ is a closed subgroup of G that contains the center of G; (ii) the map of fundamental groups $\pi_1(H, e) \to \pi_1(G, e)$ is surjective; (iii) for any connected covering $\pi\colon G' \to G$, $\pi^{-1}(H)$ is connected and is the Cartan subgroup of G'.

Now let $G = \tilde{G}/C$ be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$. Choose an ordering of the roots, and let Γ_λ be the irreducible representation of $\mathfrak{g}$ with highest weight λ. The basic fact that we need is

Lemma 23.12. *The representation Γ_λ is a representation of $G = \tilde{G}/C$ if and only if*

$$\lambda(X) \in 2\pi i\mathbb{Z} \quad \textit{whenever } \exp(X) \in C.$$

PROOF. The representation Γ_λ is a representation of G when $g \cdot v = v$ for all $g \in C$, where v is a highest weight vector in Γ_λ. Since $\exp(\mathfrak{h})$ contains C, this says $\exp(X) \cdot v = v$ for all $X \in \mathfrak{h}$ such that $\exp(X) \in C$. Now by the naturality of the exponential map, and since $X \cdot v = \lambda(X)v$ for $X \in \mathfrak{h}$, we have $\exp(X) \cdot v = e^{\lambda(X)}v$. Hence the condition is that $e^{\lambda(X)}v = v$, or that $e^{\lambda(X)} = 1$ if $\exp(X) \in C$, which is the displayed criterion. □

Let us work this out explicitly for each of the classical groups. It may help to introduce a notation for the irreducible representations which, among other virtues, allows some common terminology in the various cases. Note that for each of $\mathfrak{sl}_{n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, and $\mathfrak{so}_{2n+1}$ the root space $\mathfrak{h}^*$ is spanned by weights we have called $L_1, \ldots, L_n$, so a weight can be written uniquely in form $\lambda_1 L_1 + \cdots + \lambda_n L_n$. We may sometimes write λ in place of the weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$. In the rest of this lecture at least, we write Γ_λ for the irreducible representation with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$. Note that by our choice of Weyl chambers the highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ that arise satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \quad \text{for } \mathfrak{sl}_{n+1}, \mathfrak{sp}_{2n}, \text{ and } \mathfrak{so}_{2n+1},$$

where the λ_i are all integers in the first two cases, and for $\mathfrak{so}_{2n+1}$ they are either all integers or all half-integers; and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \geq 0 \quad \text{for } \mathfrak{so}_{2n},$$

with the λ_i all integers or all half-integers.

Proposition 23.13. *For each subgroup C of the center of $\tilde{G}$, the representation Γ_λ is a representation of $\tilde{G}/C$ precisely under the following conditions:*

(i) $\tilde{G} = \mathrm{SL}_{n+1}\mathbb{C}$, *$C$ has order m dividing $n + 1$: $\sum \lambda_j \equiv 0 \bmod(m)$.*
(ii) $\tilde{G} = \mathrm{Sp}_{2n}\mathbb{C}$, $C = \{\pm 1\}$: *$\sum \lambda_j$ is even.*
(iii) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$ *or* $\mathrm{Spin}_{2n+1}\mathbb{C}$, $C = \{\pm 1\}$: *all λ_i are integers.*
(iv) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$, $C = \{\pm 1, \pm\omega\}$: *all λ_i are integers, $\sum \lambda_j$ is even.*
(v) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$, *$n$ even,* $C = \{1, \omega\}$: *$\sum \lambda_j$ is an even integer; and for* $C = \{1, -\omega\}$: *$\sum \lambda_j - n/2$ is an odd integer.*

In particular, representations of $\mathrm{PSL}_{n+1}\mathbb{C}$ are given by partitions λ with $\sum \lambda_j \equiv 0 \bmod(n + 1)$, and those for $\mathrm{PSp}_{2n}\mathbb{C}$ have $\sum \lambda_j$ even. Case (iii) verifies what we saw in Lecture 19 about representations of $\mathrm{SO}_m\mathbb{C}$. Representations of $\mathrm{PSO}_m\mathbb{C}$ correspond to integral partitions λ with $\sum \lambda_j$ even.

PROOF. With the preceding lemma and the explicit description of everything in sight, the calculations are routine. In case (i), for example, a generator for

C is of the form $\exp(X)$, with

$$X = (2\pi i/m)\left(\sum_{j=1}^{n} E_{j,j} - nE_{n+1,n+1}\right),$$

and so $\lambda(X) = (2\pi i/m)(\sum \lambda_j)$ will be a multiple of $2\pi i$ exactly when $\sum \lambda_j$ is divisible by m. For $\mathrm{Sp}_{2n}\mathbb{C}$, $\exp(X) = -1$ when $X = \pi i(\sum H_j)$, so $\lambda(X) = \pi i \sum \lambda_j$, and (ii) follows. The calculations are similar for $\mathrm{Spin}_m\mathbb{C}$, noting that $\exp(2\pi i(H_1) = -1$ and $\exp(\pi i(\sum H_j)) = \omega$. □

By way of an example, recall that any irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is of the form $\mathrm{Sym}^k V$, where V is the standard two-dimensional representation. Any such representation, of course, lifts to the group $\mathrm{SL}_2\mathbb{C}$; but it lifts to $\mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}_3\mathbb{C}$ if and only if k is even (in particular, the "standard" representation of $\mathrm{SO}_3\mathbb{C}$ on $\mathbb{C}^3$ is the symmetric square $\mathrm{Sym}^2 V$). For another example, we have seen that any irreducible representation of $\mathfrak{sp}_4\mathbb{C}$ may be found in a tensor product $\mathrm{Sym}^k V \otimes \mathrm{Sym}^l W$, where V is the standard four-dimensional representation of $\mathfrak{sp}_4\mathbb{C}$ and $W \subset \wedge^2 V$ the complement of the trivial one-dimensional representation. All such representations lift to $\mathrm{Sp}_4\mathbb{C}$, but they lift to $\mathrm{PSp}_4\mathbb{C} \cong \mathrm{SO}_5\mathbb{C}$ if and only if k is even—equivalently, if they are contained in a representation of the form $\mathrm{Sym}^l W \otimes \mathrm{Sym}^k(\wedge^2 W)$, where W is the "standard" representation of $\mathrm{SO}_5\mathbb{C}$.

Exercise 23.14. Show that each of these semisimple complex Lie groups G has a finite-dimensional faithful representation.

The result of the proposition can be put in a more formal setting, which brings out a feature that our alert reader has surely noticed: the center of the simply-connected form of $\mathfrak{g}$ is isomorphic to the quotient group Λ_W/Λ_R of the weight lattice modulo the root lattice. We note first that this abelian group Λ_W/Λ_R is *finite*. We have seen this for the classical Lie algebras. In general, we have

Lemma 23.15. *The group Λ_W/Λ_R is finite, of order equal to the determinant of the Cartan matrix.*

PROOF. The simple roots α form a basis for the root lattice Λ_R. The corresponding elements H_α form a basis for

$$\Gamma_R = \mathbb{Z}\{H_\gamma : \gamma \in R\},$$

a lattice in $\mathfrak{h}$; this is proved in Appendix D.4. Since Λ_W is defined to be the lattice of elements of $\mathfrak{h}^*$ that take integral values on Γ_R, the determinant

$$\det(\alpha(H_\beta)) = \det(n_{\alpha\beta})$$

is the index $[\Lambda_W : \Lambda_R]$. □

In particular, for the exceptional groups, Λ_W/Λ_R is trivial for (G_2), (F_4), and (E_8), and cyclic of order two for (E_7) and order three for (E_6).

In fact, the center of the simply-connected group is naturally isomorphic to the *dual* of Λ_W/Λ_R. To express this, consider the natural dual of this last group. The lattice Γ_R defined in the preceding proof is a sublattice of the lattice

$$\Gamma_W = \{X \in \mathfrak{h}: \alpha(X) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Note that Λ_W was defined to be the lattice of elements of $\mathfrak{h}^*$ that take integral values on Γ_R. It follows formally from the definitions and the fact that Λ_W/Λ_R is finite that we have a perfect pairing

$$\Gamma_W/\Gamma_R \times \Lambda_W/\Lambda_R \to \mathbb{Q}/\mathbb{Z}, \qquad (X, \alpha) \mapsto \alpha(X).$$

The claim is that there is a natural isomorphism from Γ_W/Γ_R to the center of $\tilde{G}$, which is given by the exponential. More precisely, let $e_G: \mathfrak{h} \to H \subset G$ be the homomorphism defined by

$$e_G(X) = \exp(2\pi i X).$$

We claim that when $G = \tilde{G}$ is the simply-connected group, $\operatorname{Ker}(e_{\tilde{G}}) = \Gamma_R$ and $e_{\tilde{G}}(\Gamma_W)$ is the center of $\tilde{G}$, from which it follows that $e_{\tilde{G}}$ induces an isomorphism

$$\Gamma_W/\Gamma_R \cong Z(\tilde{G}).$$

More generally, for any $G = \tilde{G}/C$, define a lattice $\Gamma(G)$ between Γ_R and Γ_W by

$$\Gamma(G) = \operatorname{Ker}(e_G).$$

Then e_G determines an isomorphism

$$\Gamma_W/\Gamma(G) \cong Z(G).$$

We may thus state our result as

Theorem 23.16. *There is a one-to-one correspondence between connected Lie groups G with the Lie algebra $\mathfrak{g}$ and lattices $\Lambda \subset \mathfrak{h}^*$ such that*

$$\Lambda_R \subset \Lambda \subset \Lambda_W.$$

The correspondence is given by associating to a group G the lattice dual to the kernel of the exponential map $\exp: \mathfrak{g} \to G$; *in particular, the largest lattice Λ_W corresponds to the simply-connected group, the smallest Λ_R to the adjoint group with no center. In terms of this correspondence, the irreducible representation V_λ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ will lift to a representation of the group G corresponding to $\Lambda \subset \mathfrak{h}^*$ if and only if $\lambda \in \Lambda$.*

Note also that

$$H = \mathfrak{h}/\Gamma(G) \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*,$$

with $n = \dim_{\mathbb{C}} \mathfrak{h}$ copies of $\mathbb{C}^*$.

Exercise 23.17*. Show that these claims follow formally from what we have seen: that the image of the exponential map contains the center, and that for any weight α there is a representation V of $\mathfrak{g}$ whose weight space V_α is not zero. Show also that e_G determines an isomorphism $\Gamma(G)/\Gamma_R \cong \pi_1(G)$. In diagram form,

$$\begin{array}{ccc} \Gamma_W & & G_0 \\ \cup & \text{Center}(G) & \uparrow \\ \Gamma(G) & & G \\ \cup & \pi_1(G) & \uparrow \\ \Gamma_R & & \tilde{G} \end{array}$$

Exercise 23.18. Find the kernels of each of the spin and half-spin representations $\mathrm{Spin}_m\mathbb{C} \to \mathrm{GL}(S)$ and $\mathrm{Spin}_m\mathbb{C} \to \mathrm{GL}(S^\pm)$.

Exercise 23.19*. Classify the irreducible representations of the full orthogonal group $\mathrm{O}_m\mathbb{C}$.

Note that by our analysis of the Lie algebra $\mathfrak{g}_2$ there is a unique group G_2 with this Lie algebra, which is simultaneously the simply-connected and adjoint forms; the representations of this group are exactly those of the algebra $\mathfrak{g}_2$. The same is true for the Lie algebras of type (F_4) and (E_8), while (E_7) and (E_6) each have two associated groups, an adjoint one with fundamental group $\mathbb{Z}/2$ and $\mathbb{Z}/3$, and a simply-connected form with center $\mathbb{Z}/2$ and $\mathbb{Z}/3$ respectively.

It may be worth pointing out that each complex simple Lie group G can be realized as a closed subgroup defined by polynomial equations in some general linear group, i.e., that G is an *affine algebraic group*. Every irreducible representation $G \to \mathrm{GL}(V)$ is also defined by polynomials in appropriate coordinates. This explains why the whole subject can be developed from the point of view of algebraic groups, as in [Bor1] and [Hu2].

The Weyl group $\mathfrak{W}$, which we defined as a subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$, can be interpreted in terms of any connected Lie group G with Lie algebra $\mathfrak{g}$. Let H be the Cartan subgroup corresponding to $\mathfrak{h}$, and let $N(H)$ be the normalizer:

$$N(H) = \{g \in G\colon gHg^{-1} = H\}.$$

We have homomorphisms:

$$N(H) \to \mathrm{Aut}(H) \to \mathrm{Aut}(\mathfrak{h}) \to \mathrm{Aut}(\mathfrak{h}^*),$$

the first defined by conjugation, the second by differentiation at the identity, and the third using the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Killing form. Fact 14.11 can be sharpened to the claim that this map determines an *isomorphism*

$$N(H)/N \xrightarrow{\cong} \mathfrak{W}. \tag{23.20}$$

When G is the adjoint form of the Lie algebra, this isomorphism is proved in Appendix D. The general case follows, using:

Exercise 23.21. Show that if $\pi: G' \to G$ is a connected covering, with Cartan subgroups $H' = \pi^{-1}(H)$, then the induced map $N(H')/H' \to N(H)/H$ is an isomorphism.

Exercise 23.22. For each of the classical groups, and each simple root α, find an element in $N(H)$ that maps to the reflection W_α in $\mathfrak{W}$.

§23.2. Representation Rings and Characters

Just as with finite groups, we can form the representation ring R of a semisimple Lie algebra or Lie group: take the free abelian group on the isomorphism classes $[V]$ of finite-dimensional representations V, and divide by the relations $[V] = [V'] + [V'']$ whenever $V \cong V' \oplus V''$. By the complete reducibility of representations, it follows as before that R is a free abelian group on the classes $[V]$ of irreducible representations. Again, the tensor product of representations makes R into a ring: $[V]\cdot[W] = [V \otimes W]$. Many of our questions about decomposing representations and tensor products of representations can be nicely encoded by describing R more fully. We do this first for the Lie algebras.

For a semisimple Lie algebra $\mathfrak{g}$, let $\Lambda = \Lambda_W$ be the weight lattice, and let $\mathbb{Z}[\Lambda]$ be the integral group ring on the abelian group Λ. We write $e(\lambda)$ for the basis element of $\mathbb{Z}[\Lambda]$ corresponding to the weight λ; for now at least these are just formal symbols, having nothing to do with exponentials (but see (23.40)). Elements of $\mathbb{Z}[\Lambda]$ are expressions of the form $\sum n_\lambda e(\lambda)$, i.e., they assign an integer n_λ to each weight λ, with all but a finite number being zero. So $\mathbb{Z}[\Lambda]$ is a natural carrier for the information about multiplicities of representations. Define a *character homomorphism*

$$\mathrm{Char}: R(\mathfrak{g}) \to \mathbb{Z}[\Lambda] \tag{23.23}$$

by the formula $\mathrm{Char}[V] = \sum \dim(V_\lambda)e(\lambda)$, where V_λ is the weight space of V for the weight λ and $\dim(V_\lambda)$ its multiplicity. This is clearly an additive homomorphism.

The first assertion about this character map is that it is *injective*. This comes down to the fact that a representation is determined by the multiplicities of its weight spaces, which is something we saw in Lecture 14.

The product in the group ring $\mathbb{Z}[\Lambda]$ is determined by $e(\alpha)\cdot e(\beta) = e(\alpha + \beta)$. We claim next that Char is a *ring homomorphism*. This comes from the familiar fact that

$$(V \otimes W)_\lambda = \bigoplus_{\mu+\nu=\lambda} V_\mu \otimes W_\nu.$$

The Weyl group $\mathfrak{W}$ acts on $\mathbb{Z}[\Lambda]$, and a third simple claim is that the image of Char is contained in the ring of invariants $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$. This comes down to the fact that, for an irreducible (and hence for any) representation V, the weight spaces obtained by reflecting in walls of the Weyl chambers all have the same dimension.

Let $\omega_1, \ldots, \omega_n$ be a set of fundamental weights; as we have seen, these are the first weights along edges of a Weyl chamber, and they are free generators for the lattice Λ. Let $\Gamma_1, \ldots, \Gamma_n$ be the classes in $R(\mathfrak{g})$ of the irreducible representations with highest weights $\omega_1, \ldots, \omega_n$.

Theorem 23.24. (a) *The representation ring* $R(\mathfrak{g})$ *is a polynomial ring on the variables* $\Gamma_1, \ldots, \Gamma_n$.

(b) *The homomorphism* $R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}$ *is an isomorphism.*

In particular, this says that $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ is a polynomial ring on the variables $\mathrm{Char}(\Gamma_1), \ldots, \mathrm{Char}(\Gamma_n)$. In fact, the theorem is equivalent to this assertion, since if we take variables $U_1, \ldots, U_n$ and map the polynomial ring on the U_i to $R(\mathfrak{g})$ by sending U_i to Γ_i, we have

$$\mathbb{Z}[U_1, \ldots, U_n] \to R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}.$$

If the composite is an isomorphism, the second being injective, both must be isomorphisms, which is what the theorem says.

In spite of its fancy appearance, we will see that the theorem follows quite easily from what we know about the action of the Weyl group $\mathfrak{W}$ on the weights.

For any $P \in \mathbb{Z}[\Lambda]$ let us say that α is a *highest weight* for P if the coefficient of $e(\alpha)$ in P is nonzero, and, with a chosen ordering of weights as before, α is the largest such weight. We first observe that if P is invariant under $\mathfrak{W}$, then the highest weight for P is in $\mathscr{W} \cap \Lambda$, where $\mathscr{W}$ is our chosen (closed) Weyl chamber. In general, weights in $\mathscr{W} \cap \Lambda$ are often referred to as *dominant weights*.

Now suppose $\{P_\lambda\}$ is any collection of elements in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, one for each dominant weight λ, such that P_λ has highest weight λ and the coefficient of $e(\lambda)$ is 1. We claim that the P_λ form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ over $\mathbb{Z}$. This is easy to see and is the same argument used in the theory of symmetric polynomials in any algebra text: given P with highest weight λ, if the coefficient of $e(\lambda)$ is m, then $P - mP_\lambda$ is invariant whose highest weight is lower, and one continues inductively until one reaches weight zero, i.e., the constants.

Let $P_i = \mathrm{Char}(\Gamma_i)$, which has highest weight ω_i, and suppose the coefficient of $e(\omega_i)$ is 1. Since any weight $\lambda \in \mathscr{W} \cap \Lambda$ can be uniquely expressed in the form $\lambda = \sum m_i \omega_i$, for some non-negative integers m_i, and the highest weight of $\prod (P_i)^{m_i}$ is $\sum m_i \omega_i$, it follows that the monomials $\prod (P_i)^{m_i}$ in $P_1, \ldots, P_n$ form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$. This says precisely that $\mathbb{Z}[P_1, \ldots, P_n] = \mathbb{Z}[\Lambda]^{\mathfrak{W}}$, and completes the proof. □

Let us work this out concretely for each of our cases $\mathfrak{sl}_{n+1}\mathbb{C}$, $\mathfrak{sp}_n\mathbb{C}$, $\mathfrak{so}_{2n+1}\mathbb{C}$, and $\mathfrak{so}_{2n}\mathbb{C}$. Each lattice Λ contains weights we have called $L_1, \ldots, L_n$; in the first case we also have L_{n+1} with $L_1 + \cdots + L_{n+1} = 0$. We set

$$x_i = e(L_i), \qquad x_i^{-1} = e(-L_i) \in \mathbb{Z}[\Lambda]. \tag{23.25}$$

Note that in case $L_1, \ldots, L_n$ is a basis for Λ, then

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}] = \mathbb{Z}[x_1, \ldots, x_n, (x_1 \cdot \ldots \cdot x_n)^{-1}]$$

as a subring of the field $\mathbb{Q}(x_1, \ldots, x_n)$.

(A_n) For $\mathfrak{sl}_{n+1}\mathbb{C}$, fundamental weights are

$$L_1, \quad L_1 + L_2, \quad L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_n,$$

corresponding to the irreducible representations V, $\bigwedge^2 V$, $\ldots$, $\bigwedge^n V$, with $V = \mathbb{C}^{n+1}$ the standard representation. The character of $\bigwedge^k V$ is $\sum e(\alpha)$, the sum over all α that are sums of k different L_i for $1 \le i \le n+1$. So $\mathrm{Char}(\bigwedge^k V) = A_k$, where A_k is the kth elementary symmetric function of $x_1, \ldots, x_{n+1}$. The Weyl group is the symmetric group $\mathfrak{S}_{n+1}$, acting by permutation on the indices, so the theorem in this case says that

$$R(\mathfrak{sl}_{n+1}) = \mathbb{Z}[\Lambda]^{\mathfrak{S}_{n+1}} = \mathbb{Z}[A_1, \ldots, A_n]. \tag{23.26}$$

Note that $\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \ldots, x_n, x_{n+1}]/(x_1 \cdot \ldots \cdot x_{n+1} - 1)$, so $\mathbb{Z}[\Lambda]$ has an additive basis consisting of all monomials x^α, with α an n-tuple of non-negative integers, but with not all α_i positive.

(C_n) For $\mathfrak{sp}_{2n}\mathbb{C}$, the lattice Λ and fundamental weights have the same description as in the preceding case. The corresponding irreducible representations are the kernels $V^{(k)}$ of the contraction maps $\bigwedge^k V \to \bigwedge^{k-2} V$, with now $V = \mathbb{C}^{2n}$ the standard representation, $k = 1, \ldots, n$. The character of $\bigwedge^k V$ is $\sum e(\alpha)$, the sum over all α that are sums of k different $\pm L_i$ for $1 \le i \le n$. The character $\mathrm{Char}(\bigwedge^k V)$ is thus the elementary symmetric polynomial C_k in the variables $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$. The theorem then says that

$$\begin{aligned} R(\mathfrak{sp}_{2n}\mathbb{C}) &= \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[C_1, C_2 - 1, C_3 - C_1, \ldots, C_n - C_{n-2}] \\ &= \mathbb{Z}[C_1, C_2, C_3, \ldots, C_n]. \end{aligned} \tag{23.27}$$

(B_n) For $\mathfrak{so}_{2n+1}\mathbb{C}$, Λ is spanned by the L_i together with $\frac{1}{2}(L_1 + \cdots + L_n)$. The fundamental representations are V, $\bigwedge^2 V$, $\ldots$, $\bigwedge^{n-1} V$, and the spin representation S. The character of $\bigwedge^k V$ is the kth elementary symmetric function of the $2n+1$ elements $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$, and 1; denote this by B_k. The character of S, which we denote by B, is the sum $\sum x_1^{\pm 1/2} \cdot \ldots \cdot x_n^{\pm 1/2}$, where

$$x_i^{+1/2} = e(L_i/2), \qquad x_i^{-1/2} = e(-L_i/2). \tag{23.28}$$

So B is the nth elementary symmetric polynomial in the variables $x_i^{+1/2} + x_i^{-1/2}$. Therefore,

$$R(\mathfrak{so}_{2n+1}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[B_1, \ldots, B_{n-1}, B]. \tag{23.29}$$

(D_n) For $\mathfrak{so}_{2n}\mathbb{C}$, Λ and $\mathbb{Z}[\Lambda]$ are the same as in the preceding case. The fundamental representations are V, $\bigwedge^2 V$, $\ldots$, $\bigwedge^{n-2} V$, and the half-spin representations S^+ and S^-. The character of $\bigwedge^k V$, denoted D_k, is the kth elementary symmetric function of the $2n$ elements $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$. The character $D^{\pm}$ of $S^{\pm}$ is the sum $\sum x_1^{\pm 1/2} \cdot \ldots \cdot x_n^{\pm 1/2}$, where the number of plus signs is even or odd according to the sign. We have

$$R(\mathfrak{so}_{2n}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[D_1, \ldots, D_{n-2}, D^+, D^-]. \tag{23.30}$$

Exercise 23.31*. (a) Prove the following relation in $R(\mathfrak{so}_{2n+1}\mathbb{C})$:

$$B^2 = B_n + \cdots + B_1 + 1,$$

corresponding to the isomorphism

$$S \otimes S \cong \wedge^n V \oplus \cdots \oplus \wedge^1 V \oplus \wedge^0 V.$$

This describes $R(\mathfrak{so}_{2n+1}\mathbb{C})$ as a quadratic extension of the ring $\mathbb{Z}[B_1, \ldots, B_n]$.

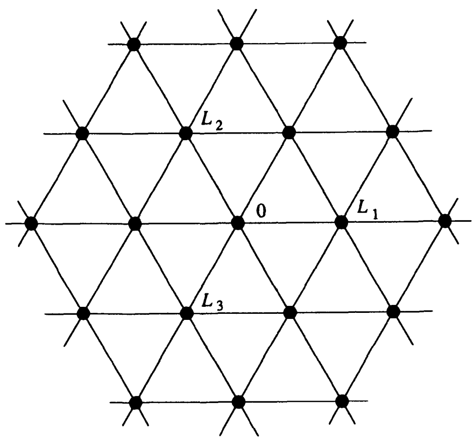
(b) Let D_n^+ (respectively, D_n^-) be the character of the representation whose highest weight is twice that of D^+ (resp., D^-), so that, for example, the sum of the representations D_n^+ and D_n^- is $\wedge^n V$. Prove the relations in $R(\mathfrak{so}_{2n}\mathbb{C})$:

$$D^+ \cdot D^+ = D_n^+ + D_{n-2} + D_{n-4} + \cdots,$$

$$D^- \cdot D^- = D_n^- + D_{n-2} + D_{n-4} + \cdots,$$

$$D^+ \cdot D^- = D_{n-1} + D_{n-3} + D_{n-5} + \cdots.$$

We can likewise describe the representation ring for $\mathfrak{g}_2$. Here, we may take as generators for the weight lattice the weights L_1 and L_2 as pictured in the diagram



and correspondingly write $\mathbb{Z}[\Lambda]$ as $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}]$, where $x_i = e(L_i)$. It will be a little more symmetric to introduce $L_3 = -L_1 - L_2$ as pictured and $x_3 = x_1^{-1} \cdot x_2^{-1} = e(L_3)$, and write

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, x_2, x_3]/(x_1 x_2 x_3 - 1).$$

In these terms the Weyl group is the group $\mathfrak{W}$ generated by the symmetric group $\mathfrak{S}_3$ permuting the variables x_i and the involution sending each x_i to x_i^{-1}. The standard representation has weights $\pm L_i$ and 0, and so has character

$$A = A(x_1, x_2, x_3) = 1 + x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_3 + x_3^{-1}.$$

Similarly, the adjoint representation has weights $\pm L_i$, $\pm(L_i - L_j)$, and 0 (taken twice); its character is

$$B = A(x_1, x_2, x_3) + A(x_1/x_2, x_2/x_3, x_3/x_1).$$

The theorem thus implies in this case the equality

$$R(\mathfrak{g}_2) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[A, B]. \tag{23.32}$$

Exercise 23.33. Verify directly the statement that any element of $\mathbb{Z}[x_1, x_2, x_3]/(x_1x_2x_3 - 1)$ invariant under the group $\mathfrak{W}$ as described is in fact a polynomial in A and B.

Similarly we can define the representation ring $R(G)$ of a semisimple group G. When G is the simply-connected form of its Lie algebra $\mathfrak{g}$, $R(G) = R(\mathfrak{g})$, so $R(\mathrm{SL}_n\mathbb{C})$, $R(\mathrm{Sp}_{2n}\mathbb{C})$, $R(\mathrm{Spin}_{2n+1}\mathbb{C})$, and $R(\mathrm{Spin}_{2n}\mathbb{C})$ are given by (23.26), (23.27), (23.29), and (23.30). In general, $R(G)$ is a subring of $R(\mathfrak{g})$; we can read off which subring by looking at Proposition 23.13. We have, in fact,

$$R(\mathrm{SO}_{2n+1}\mathbb{C}) = \mathbb{Z}[B_1, \ldots, B_n]; \tag{23.34}$$

$$R(\mathrm{SO}_{2n}\mathbb{C}) = \mathbb{Z}[D_1, \ldots, D_{n-1}, D_n^+, D_n^-], \tag{23.35}$$

with D_n^+ and D_n^- as in Exercise 23.31. But this time there is one relation:

$$(D_n^+ + D_{n-2} + D_{n-4} + \cdots + 1)(D_n^- + D_{n-2} + D_{n-4} + \cdots + 1)$$
$$= (D_{n-1} + D_{n-3} + \cdots +)^2.$$

Exercise 23.36*.

(a) Prove (23.34).
(b) Show that the relation in (23.35) comes from Exercise 23.31(b). Show that $R(\mathrm{SO}_{2n}\mathbb{C})$ is the polynomial ring in the $n + 1$ generators shown, modulo the ideal generated by the one polynomial indicated.
(c) Describe the representation rings for the other groups with these simple Lie algebras.
(d) Prove the isomorphism

$$R(\mathrm{GL}_n\mathbb{C}) = \mathbb{Z}[E_1, \ldots, E_n, E_n^{-1}],$$

where the E_k are the elementary symmetric functions of $x_1, \ldots, x_n$.

Exercise 23.37*. (a) Show that the image of $R(\mathrm{O}_m\mathbb{C})$ in $R(\mathrm{SO}_m\mathbb{C})$ is the polynomial ring $\mathbb{Z}[B_1, \ldots, B_n]$ if $m = 2n + 1$, and $\mathbb{Z}[D_1, \ldots, D_n]$ if $m = 2n$.

(b) Show that

$$R(\mathrm{O}_{2n+1}\mathbb{C}) = R(\mathrm{SO}_{2n+1}\mathbb{C}) \otimes R(\mathbb{Z}/2)$$
$$= \mathbb{Z}[B_1, \ldots, B_n, B_{2n+1}]/((B_{2n+1})^2 - 1)$$

and

$$R(\mathrm{O}_{2n}\mathbb{C}) = \mathbb{Z}[D_1, \ldots, D_n, D_{2n}]/I,$$

where I is the ideal generated by $(D_{2n})^2 - 1$ and $D_n D_{2n} - D_n$.

Exercise 23.38*. The mapping that takes a representation V to its dual V^* induces an involution of the representation ring: $[V]^* = [V^*]$. The ring $\mathbb{Z}[\Lambda]$ has an involution determined by $(e(\lambda))^* = e(-\lambda)$. Show that the character homomorphism commutes with these involutions. Show that for $\mathfrak{sl}_{n+1}$, $(A_k)^* = A_{n+1-k}$; for $\mathfrak{so}_{2n+1}\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_{2n}\mathbb{C}$ for n even, the involution is the identity; while for $\mathfrak{so}_{2n}\mathbb{C}$ with n odd, $(D_k)^* = D_k$, $(D^+)^* = D^-$, $(D^-)^* = D^+$. Deduce that all representations of all symplectic and orthogonal groups are self-dual. Note that when $*$ is the identity, all representations are self-dual. In the other cases, compute the duals of irreducible representations with given highest weight.

The following exercise deals with a special property of the representation rings of semisimple Lie groups and algebras.

Exercise 23.39*. The representation rings $R = R(\mathfrak{g})$ and $R(G)$ have another important structure: they are *λ-rings*. There are operators

$$\lambda^i \colon R(G) \to R(G), \quad i = 0, 1, 2, \ldots,$$

determined by $\lambda^i([V]) = [\Lambda^i V]$ for any representation V.

(a) Show that this determines well-defined maps, satisfying $\lambda^0 = 1$, $\lambda^1 = \mathrm{Id}$, and

$$\lambda^i(x + y) = \sum_{i+j=k} \lambda^i(x) \cdot \lambda^j(y)$$

for any x and y in R. In fact, R is what is called a *special* λ-ring: there are formulas for $\lambda^i(x \cdot y)$ and $\lambda^i(\lambda^j(x))$, valid as if x and y could be written as sums of one-dimensional representations (see, e.g., [A-T]).

(b) Show that λ^i extends to $\mathbb{Z}[\Lambda]$, and use this to verify that $R(G)$ is a special λ-ring.

Define *Adams operators* $\psi^k \colon R \to R$ by $\psi^k(x) = P_k(\lambda^1 x, \ldots, \lambda^n x)$, where P_k is the expression for the kth power sum (cf. Exercise A.32) in terms of the elementary symmetric functions, $n \geq k$. Equivalently,

$$\psi^k(x) - \psi^{k-1}(x)\lambda^1(x) + \cdots + (-1)^k k \lambda^k(x) = 0.$$

(c) Show that, regarding R as the ring of functions on the group G, $(\psi^k x)(g) = x(g^k)$. Equivalently, $\psi^k(e(\lambda)) = e(k\lambda)$.

(d) Show that each ψ^k is a ring homomorphism, and $\psi^k \circ \psi^l = \psi^{k+l}$.
(e) Show that for a representation V,

$$\mathrm{Char}(\mathrm{Sym}^2 V) = \tfrac{1}{2}\,\mathrm{Char}(V)^2 + \tfrac{1}{2}\psi^2(\mathrm{Char}(V)),$$
$$\mathrm{Char}(\Lambda^2 V) = \tfrac{1}{2}\,\mathrm{Char}(V)^2 - \tfrac{1}{2}\psi^2(\mathrm{Char}(V)).$$

Show that $\mathrm{Char}(\mathrm{Sym}^d V)$ and $\mathrm{Char}(\Lambda^d V)$ can be written as polynomials in $\psi^k(\mathrm{Char}(V))$, $1 \le k \le d$.

Formal Characters and Actual Characters

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any representation V of $\mathfrak{g}$, the image of $[V] \in R(\mathfrak{g})$ in $\mathbb{Z}[\Lambda]$ is called the *formal character* of V. As it turns out, this formal character can be identified with the honest character of the corresponding representation of the group G, restricted to the Cartan subgroup H:

(23.40) *If* $\mathrm{Char}(V) = \sum m_\alpha e(\alpha)$ *is the formal character, and* $\exp(X)$ *is an element of* H, *then the trace of* $\exp(X)$ *on* V *is* $\sum m_\alpha e^{\alpha(X)}$.

This is simply because $\exp(X)$ acts on the weight space V_μ by multiplication by $e^{\mu(X)}$, as we have seen. In particular, a representation is determined by the character of its restriction to a Cartan subgroup.

Another common notation for this is to set $e(X) = \exp(2\pi i X)$, and $e(z) = \exp(2\pi i z)$. Then the trace of $e(X)$ is $\sum m_\alpha e(\alpha(X))$.

Exercise 23.41. As a function on H, the character of a representation is invariant under the Weyl group $\mathfrak{W} = N(H)/H$. Describe $R(G)$ as a ring of $\mathfrak{W}$-invariant functions on H.

This is also compatible with our descriptions of elements of $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ as Laurent polynomials in variables x_i or $x_i^{1/2}$. For $\mathrm{SL}_{n+1}\mathbb{C}$, for example, if the character $\mathrm{Char}(W)$ of a representation W is $P(x_1, \ldots, x_{n+1})$, the trace of the matrix $\mathrm{diag}(z_1, \ldots, z_{n+1})$ on V is $P(z_1, \ldots, z_{n+1})$. Similarly for the other groups, using the diagonal matrices described in the first section of this lecture. For the spin groups, the element $w(z_1, \ldots, z_n)$ defined in (23.8) has trace given by substituting z_i for $x_i^{1/2}$, and z_i^{-1} for $x_i^{-1/2}$ in the corresponding Laurent polynomial.

Exercise 23.42*. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two semisimple Lie algebras, show that

$$R(\mathfrak{g}_1 \times \mathfrak{g}_2) = R(\mathfrak{g}_1) \otimes R(\mathfrak{g}_2).$$

Exercise 23.43*. (a) For the natural inclusion $\mathfrak{sl}_n\mathbb{C} \subset \mathfrak{sl}_{n+1}\mathbb{C}$, restriction of representations gives a homomorphism $R(\mathfrak{sl}_{n+1}\mathbb{C}) \to R(\mathfrak{sl}_n\mathbb{C})$, which can be described by saying what happens to the polynomial generators. Since $\bigwedge^k(\mathbb{C}^n \oplus \mathbb{C}) = \bigwedge^k(\mathbb{C}^n) \oplus \bigwedge^{k-1}(\mathbb{C}^n)$, this is

$$A_k \mapsto A_k + A_{k-1}.$$

Give the analogous descriptions for the following inclusions:

$$\mathfrak{sp}_{2n-2}\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}, \qquad \mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}, \qquad \mathfrak{so}_{2n-1}\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C};$$

$$\mathfrak{sl}_n\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}, \qquad \mathfrak{sl}_n\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}, \qquad \mathfrak{sl}_n\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C};$$

$$\mathfrak{sp}_{2n}\mathbb{C} \subset \mathfrak{sl}_{2n}\mathbb{C}, \qquad \mathfrak{so}_{2n+1}\mathbb{C} \subset \mathfrak{sl}_{2n+1}\mathbb{C}, \qquad \mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{sl}_{2n}\mathbb{C}.$$

(b) The inclusion $\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_m\mathbb{C} \subset \mathfrak{sl}_{n+m}\mathbb{C}$ determines a restriction homomorphism $R(\mathfrak{sl}_{n+m}\mathbb{C}) \to R(\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_m\mathbb{C}) = R(\mathfrak{sl}_n\mathbb{C}) \otimes R(\mathfrak{sl}_m\mathbb{C})$, which takes polynomial generators A_k to $A_k \otimes 1 + A_{k-1} \otimes A_1 + \cdots + 1 \otimes A_k$. Compute analogously for

$$\mathfrak{sp}_{2n}\mathbb{C} \times \mathfrak{sp}_{2m}\mathbb{C} \subset \mathfrak{sp}_{2n+2m}\mathbb{C}, \qquad \mathfrak{so}_n\mathbb{C} \times \mathfrak{so}_m\mathbb{C} \subset \mathfrak{so}_{n+m}\mathbb{C}.$$

Which of these inclusions correspond to removing nodes from the Dynkin diagrams?

Exercise 23.44. Compute the isomorphisms of representation rings corresponding to the isomorphisms $\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C}$, $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$, and $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$.

§23.3. Homogeneous Spaces

In this section we will introduce and describe the compact homogeneous spaces associated to the classical groups. As we will see, these are classified neatly in terms of Dynkin diagrams, and are, in turn, closely related to the representation theory of the groups acting on them. Unfortunately, we are unable to give here more than the barest outline of this beautiful subject; but we will at least try to say what the principal objects are, and what connections among them exist. In particular, we give at the end of the section a diagram (23.58) depicting these objects and correspondences to which the reader can refer while reading this section.

We begin by introducing the notion of Borel subalgebras and Borel subgroups. Recall first that a choice of Cartan subalgebra $\mathfrak{h}$ in a semisimple Lie algebra $\mathfrak{g}$ determines, as we have seen, a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. To each choice of ordering of the root system $R = R^+ \cup R^-$, we can associate a subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

called a *Borel subalgebra*. Note that $\mathfrak{b}$ is solvable, since $\mathscr{D}\mathfrak{b} \subset \bigoplus \mathfrak{g}_\alpha$, $\mathscr{D}^2\mathfrak{b} \subset \bigoplus \mathfrak{g}_{\alpha+\beta}$, etc. In fact, $\mathfrak{b}$ is a maximal solvable subalgebra (Exercise 14.35).

If G is a Lie group with semisimple Lie algebra $\mathfrak{g}$, the connected subgroup B of G with Lie algebra $\mathfrak{b}$ is called a *Borel subgroup.*

Claim 23.45. *B is a closed subgroup of G, and the quotient G/B is compact.*

Proof. Consider the adjoint representation of G on $\mathfrak{g}$. The action of the Borel subalgebra $\mathfrak{b}$ obviously preserves the subspace $\mathfrak{b} \subset \mathfrak{g}$, and, in fact, $\mathfrak{b}$ is just the inverse image of the subalgebra of $\mathfrak{gl}(\mathfrak{g})$ preserving this subspace: if $X = \sum X_\alpha$ is any element of $\mathfrak{g}$ with $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\alpha \neq 0$ for some $\alpha \in R^-$, we could find an element H of $\mathfrak{h} \subset \mathfrak{b}$ with $\mathrm{ad}(X)(H) \notin \mathfrak{b}$—any H not in the annihilator of $\alpha \in \mathfrak{h}^*$ would do. B is thus (the connected component of the identity in) the inverse image in G of the subgroup of $\mathrm{GL}(\mathfrak{g})$ carrying $\mathfrak{b}$ into itself. It follows that B is closed; and the quotient G/B is contained in a Grassmannian and hence compact. (Alternatively, we could consider the action of G on the projective space $\mathbb{P}(\bigwedge^m \mathfrak{g})$, where m is the number of positive roots, and observe that B is the stabilizer of the point corresponding to the exterior product of the positive root spaces.)

In fact, in the case of the classical groups, it is easy to describe the Borel subgroups and the corresponding quotients.

For $G = \mathrm{SL}_{n+1}\mathbb{C}$, B is the group of all upper-triangular matrices in G, i.e., those automorphisms preserving the standard flag. It follows that G/B is the usual (complete) flag manifold, i.e., the variety of all flags

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}\}$$

of subspaces with $\dim(V_r) = r$.

For $G = \mathrm{SO}_{2n+1}\mathbb{C}$ the orthogonal group of automorphisms of $\mathbb{C}^{2n+1}$ preserving a quadratic form Q, B is the subgroup of automorphisms which preserve a fixed flag $V_1 \subset \cdots \subset V_n$ of isotropic subspaces with $\dim(V_r) = r$. All such flags being conjugate, G/B is the variety of all such flags, i.e.,

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{2n+1} \colon Q(V_n, V_n) \equiv 0\}.$$

Note that B automatically preserves the flag of orthogonal subspaces, so that we could also characterize G/B as the space of complete flags equal to their orthogonal complements, i.e.,

$$G/B = \{V_1 \subset \cdots \subset V_{2n} \subset \mathbb{C}^{2n+1} \colon Q(V_i, V_{2n+1-i}) = 0\}.$$

The same holds for $\mathrm{Sp}_{2n}\mathbb{C}$: the Borel subgroups $B \subset \mathrm{Sp}_{2n}\mathbb{C}$ are just the subgroups preserving a half-flag of isotropic subspaces, or equivalently a full flag of pairwise complementary subspaces; and the quotient G/B is correspondingly the variety of all such flags.

For $G = \mathrm{SO}_{2n}\mathbb{C}$, B fixes an isotropic flag $V_1 \subset \cdots \subset V_{n-1}$, and

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^{2n} \colon Q(V_{n-1}, V_{n-1}) = 0\}.$$

Exercise 23.46. With our choice of basis $\{e_i\}$, let V_r be the subspace spanned by the first r basic vectors. If B is defined to be the subgroup that preserves V_r for $1 \leq r \leq n$, verify that the Lie algebra of B is spanned by the Cartan subalgebra and the positive root spaces described in Lectures 17 and 19.

We now want to consider more general quotients of a semisimple complex group G. To begin with, we say that a (closed, complex analytic, and connected[1]) subgroup P of G is *parabolic* if the quotient G/P can be realized as the orbit of the action of G on $\mathbb{P}(V)$ for some representation V of G. In particular, G/P is a projective algebraic variety. It follows from the proof of Claim 23.45 that any Borel subgroup B of G is parabolic. The following two claims characterize parabolic subgroups as those containing a Borel subgroup, i.e., the Borel subgroups are exactly the *minimal* parabolic subgroups.

Claim 23.47. *If B is a Borel subgroup and P a parabolic subgroup of G, then there is an $x \in G$ with*

$$B \subset xPx^{-1}.$$

Claim 23.48. *If a subgroup P of G contains a Borel subgroup B, then P is parabolic.*

The first claim is deduced from a version of *Borel's fixed point theorem*: if B is a connected solvable group, V a representation of B and $X \subset \mathbb{P}V$ a projective variety carried into itself under the action of B on $\mathbb{P}V$, then B must have a fixed point on X. This is straightforward: we observe (by Lie's theorem (9.11)) that the action of the solvable group B on V must preserve a flag of subspaces

$$0 \subset V_1 \subset \cdots \subset V_n = V$$

with $\dim(V_i) = i$. We can thus find a subspace $V_i \subset V$ fixed by B such that X intersects $\mathbb{P}V_i$ in a finite collection of points, which must then be fixed points for the action of B on X. As for Claim 23.48 we will soon see directly how G/P is a projective variety whenever P is a subgroup containing B.

We can now completely classify the parabolic subgroups of a simple group, up to conjugacy. By the above, we may assume that P contains a Borel subgroup B. Correspondingly, its Lie algebra $\mathfrak{p}$ is a subspace of $\mathfrak{g}$ containing $\mathfrak{b}$ and invariant under the action of B on $\mathfrak{g}$; i.e., it is a direct sum

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}_\alpha$$

for some subset T of R that contains all positive roots. Now, in order for $\mathfrak{p}$ to be a subalgebra of $\mathfrak{g}$, the subset T must be closed under addition (that is, if two roots are in T, then either their sum is in T or is not a root). Since, in addition, T contains all the positive roots, we may observe that if α, β, and γ are positive roots with $\alpha = \beta + \gamma$, then we must have

$$-\alpha \in T \Rightarrow -\beta \in T \text{ and } -\gamma \in T.$$

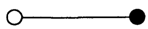
[1] It is a general fact that P must be connected if G/P is a projective variety.

Clearly, any such subset T must be generated by R^+ together with the negatives of a subset Σ of the set of simple roots. Thus, if for each subset Σ of the set of simple roots we let $T(\Sigma)$ consist of all roots which can be written as sums of negatives of the roots in Σ, together with all positive roots, and form the subalgebra

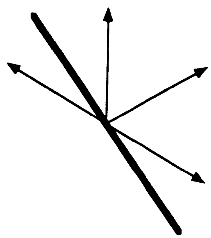
$$\mathfrak{p}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in T(\Sigma)} \mathfrak{g}_\alpha, \tag{23.49}$$

then $\mathfrak{p}(\Sigma)$ is a parabolic subalgebra, the corresponding Lie group $P(\Sigma)$ is a parabolic subgroup containing B, and we obtain in this way all the parabolic subgroups of G. We can express this as the observation that, up to conjugacy, *parabolic subgroups of the simple group G are in one-to-one correspondence with subsets of the nodes of the Dynkin diagram, i.e., with subsets of the set of simple roots.*

Examples. In the case of $\mathfrak{sl}_3\mathbb{C}$, there is a symmetry in the Dynkin diagram, so that there is only one parabolic subgroup other than the Borel, corresponding to the diagram



This, in turn, gives the subset of the root system



R$^+$

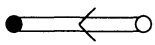
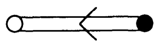
corresponding to the subgroup

$$P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

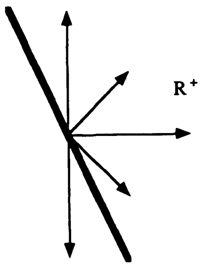
and the homogeneous space

$$G/P = \mathbb{P}^2.$$

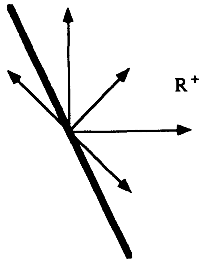
In the case of $\mathfrak{sp}_4\mathbb{C}$, there are two subdiagrams of the Dynkin diagram:

 and 

these correspond to the subsets of the root system



and

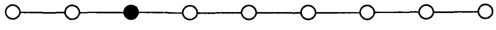


(Here we are using a black dot to indicate an omitted simple root, a white dot to indicate an included one.) The corresponding subgroups of $Sp_4\mathbb{C}$ are those preserving the vector e_1, and preserving the subspace spanned by e_1 and e_2, respectively. The quotients G/B are thus the variety of one-dimensional isotropic subspaces (i.e., the variety $\mathbb{P}^3$ of all the one-dimensional spaces) and the variety of two-dimensional isotropic subspaces.

Exercise 23.50. Interpret the diagrams above as giving rise to parabolic subgroups of the group $SO_5\mathbb{C}$ of automorphisms of $\mathbb{C}^5$ preserving a symmetric bilinear form. Show that the corresponding homogeneous spaces are the variety of isotropic planes and lines in $\mathbb{C}^5$, respectively. In particular, deduce the classical algebraic geometry facts that:

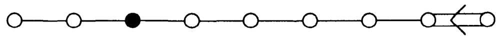
(i) The variety of isotropic 2-planes for a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^4$ is isomorphic to a quadric hypersurface in $\mathbb{P}^4$.
(ii) The variety of isotropic 2-planes for a nondegenerate symmetric bilinear form on $\mathbb{C}^5$ (equivalently, lines on a smooth quadric hypersurface in $\mathbb{P}^4$) is isomorphic to $\mathbb{P}^3$.

In general, it is not hard to see that any parabolic subgroup P in a classical group G may be described as the subgroup that preserves a partial flag in the standard representation. In particular, a maximal parabolic subgroup, corresponding to omitting one node of the Dynkin diagram, may be described as the subgroup of G preserving a single subspace. Thus, for $G = SL_m\mathbb{C}$, the kth node of the Dynkin diagram

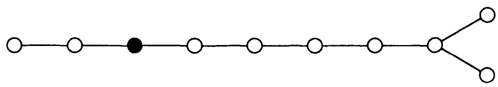


corresponds to the Grassmannian $G(k, m)$ of k-dimensional subspaces of $\mathbb{C}^m$. (Note that the symmetry of the diagram reflects the isomorphism of the Grassmannians $G(k, m)$ and $G(m - k, m)$.)

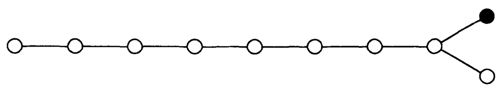
For $Sp_{2n}\mathbb{C}$, the kth node of the Dynkin diagram



corresponds to the *Lagrangian Grassmannian* of isotropic k-planes, for $k = 1, 2, \ldots, n$. Similarly, for $G = \mathrm{SO}_{2n+1}\mathbb{C}$, the kth node of the Dynkin diagram corresponds to the *orthogonal Grassmannian* of isotropic k-planes in $\mathbb{C}^{2n+1}$. Finally, for $\mathrm{SO}_{2n}\mathbb{C}$, for $k = 1, 2, \ldots, n-2$ the kth node of the Dynkin diagram



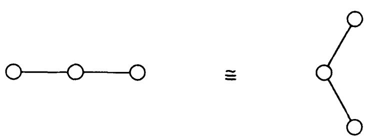
yields the orthogonal Grassmannian of isotropic k-planes in $\mathbb{C}^{2n}$, but there is one anomaly: either of the last two nodes



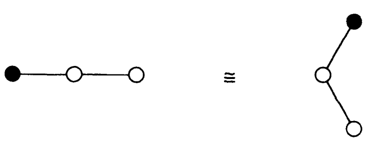
gives one of the two connected components of the Grassmannian of isotropic n-planes.

Exercise 23.51*. Compute $\mathfrak{p}(\Sigma)$ directly for each of the classical groups, and verify the above statements. Why is the orthogonal Grassmannian of isotropic $(n-1)$-planes in $\mathbb{C}^{2n}$ not included on the list?

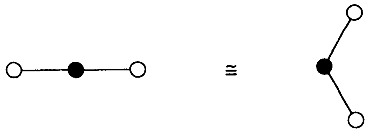
As we saw already in Exercise 23.50, the low-dimensional coincidences between Dynkin diagrams can be used to recover some facts we have seen before. For example, the coincidence $(\mathrm{D}_2) = (\mathrm{A}_1) \times (\mathrm{A}_1)$ identifies the two family of lines on a quadratic surface in $\mathbb{P}^3$ with two copies of $\mathbb{P}^1$. The coincidence $(\mathrm{A}_3) = (\mathrm{D}_3)$



gives rise to two identifications of marked diagrams: we have



corresponding to the isomorphism between the Grassmann varieties $\mathbb{P}^3 = G(1, 4)$, $\check{\mathbb{P}}^3 = G(3, 4)$ and the two components of the family of 2-planes on a quadric hypersurface Q in $\mathbb{P}^5$; and



corresponding to the isomorphism of the Grassmannian $G(2, 4)$ with the quadric hypersurface Q itself. Finally, an observation that is not quite so elementary, but which we saw in §20.3: the identification of the diagrams

says that *either connected component of the variety of 3-planes on a smooth quadric hypersurface Q in $\mathbb{P}^7$ is isomorphic to the quadric Q itself.*

There is another way to realize the compact homogeneous spaces associated to a simple group G. Let $V = \Gamma_\lambda$ be an irreducible representation of G with highest weight λ, and consider the action of G on the projective space $\mathbb{P}V$. Let $p \in \mathbb{P}V$ be the point corresponding to the eigenspace with eigenvalue λ. We have then

Claim 23.52. *The orbit $G \cdot p$ is the unique closed orbit of the action of G on $\mathbb{P}V$.*

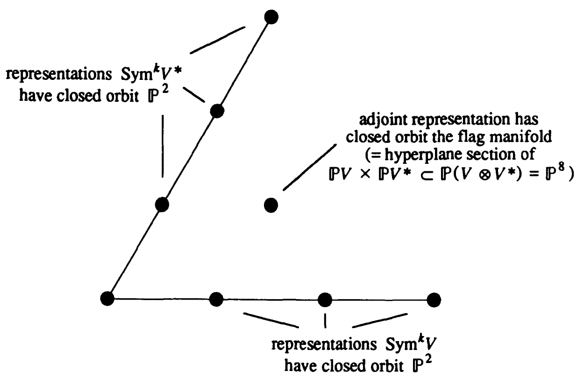
Proof. The point p is fixed under the Borel subgroup B, so that the stabilizer of p is a parabolic subgroup P_λ; the orbit G/P_λ is thus compact and hence closed. Conversely, by the Borel fixed point theorem, any closed orbit of G contains a fixed point for the action of B; but p is the unique point in $\mathbb{P}V$ fixed by B. □

In fact, it is not hard to say which parabolic subgroup P_λ is, in terms of the classification above: *it is the parabolic subgroup corresponding to the subset of simple roots that are perpendicular to the weight λ.* Now, sets Σ of simple roots correspond to faces of the Weyl chamber, namely, the face that is the intersection of all hyperplanes perpendicular to all roots in Σ.

We thus have a correspondence between faces of the Weyl chamber and parabolic subgroups P, such that if $V = \Gamma_\lambda$ is the irreducible representation with highest weight λ, then the unique closed orbit of the action of G on $\mathbb{P}V$

is of the form G/P, where P is the parabolic subgroup corresponding to the open face of $\mathscr{W}$ containing λ. In particular, weights in the interior of the Weyl chamber correspond to $P_\lambda = B$, and so determine the full flag manifold G/B, whereas weights on the edges give rise to the quotients of G by maximal parabolics. Note that we do obtain in this way all compact homogeneous spaces for G.

For example, we have the representations of $\mathrm{SL}_3\mathbb{C}$: as we have seen, the representations $\mathrm{Sym}^k V$ and $\mathrm{Sym}^k V^*$, with highest weights on the boundaries of the Weyl chamber, have closed orbits $\{v^k\}_{v \in V}$ and $\{l^k\}_{l \in V^*}$, isomorphic to $\mathbb{P}V$ and $\mathbb{P}V^*$. By contrast, the adjoint representation—the complement of the trivial representation in $\mathrm{Hom}(V, V) = V \otimes V^*$—has as closed orbit the variety of traceless rank 1 homomorphisms, which is isomorphic to the flag manifold via the map sending a homomorphism φ to the pair $(\mathrm{Im}\ \varphi, \mathrm{Ker}\ \varphi)$. The picture is

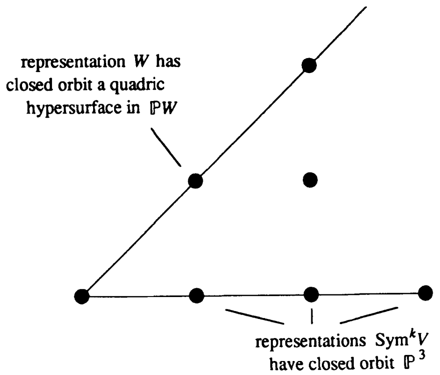


In general, if V is the standard representation of $\mathrm{SL}_n\mathbb{C}$, in the representations of $\mathrm{SL}_n\mathbb{C}$ of the form $W = \mathrm{Sym}^k V$ we saw that the vectors of the form $\{v^k\}_{v \in V}$ formed a closed orbit in $\mathbb{P}W$, called the Veronese embedding of $\mathbb{P}^{n-1}$. Likewise, in representations of the form $W = \bigwedge^k V$ the decomposable vectors $\{v_1 \wedge v_2 \wedge \cdots \wedge v_k\}$ formed a closed orbit in $\mathbb{P}W$; this is the Plücker embedding of the Grassmannian.

Similarly, we may identify the closed orbits in representations of $\mathrm{Sp}_4\mathbb{C}$. Recall here that the basic representations of $\mathrm{Sp}_4\mathbb{C}$ are the standard representation $V \cong \mathbb{C}^4$ and the complement W of the trivial representation in the exterior square $\bigwedge^2 V$; all other representations are contained in a tensor product of symmetric powers of these. Now, $\mathrm{Sp}_4\mathbb{C}$ acts transitively on $\mathbb{P}V$; the closed orbit is all of $\mathbb{P}^3$. In general, in $\mathbb{P}(\mathrm{Sym}^k V)$ the closed orbit is just the set of vectors $\{v^k\}_{v \in V} \cong \mathbb{P}^3$. By contrast, the closed orbit in $\mathbb{P}W$ is just the intersection of the hyperplane $\mathbb{P}W \subset \mathbb{P}(\bigwedge^2 V)$ with the locus of decomposable vectors $\{v \wedge w\}_{v, w \in V}$; this is the variety

$$X = \{v \wedge w \colon Q(v, w) = 0\}$$

of isotropic 2-planes $\Lambda \subset V$ for the skew form Q.



For the group $\mathrm{Spin}_{2n+1}\mathbb{C}$, the closed orbit of the spin representation S is the orthogonal Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n+1}$. The corresponding subvariety

$$G/P \hookrightarrow \mathbb{P}(S)$$

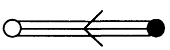
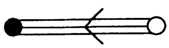
is a variety of dimension $(n + 1)n/2$ in $\mathbb{P}^N$, $N = 2^n - 1$, called the *spinor variety*, or the variety of *pure spinors*. Similarly for $\mathrm{Spin}_{2n}\mathbb{C}$, the two spin representations S^+ and S^- give embeddings of the two components of the orthogonal Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$, one in $\mathbb{P}(S^+)$, one in $\mathbb{P}(S^-)$. These spinor varieties have dimension $n(n - 1)/2$ in projective spaces of dimension $2^{n-1} - 1$.

Exercise 23.53. Show that the spinor variety for $\mathrm{Spin}_{2n-1}\mathbb{C}$ is isomorphic to each of the spinor varieties for $\mathrm{Spin}_{2n}\mathbb{C}$. In fact they are projectively equivalent as subvarieties of projective space $\mathbb{P}^N$, $N = 2^{n-1} - 1$.

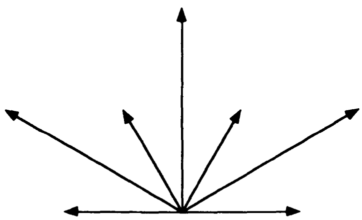
It follows that, for $m \leq 8$, the spinor varieties for $\mathrm{Spin}_m\mathbb{C}$ are isomorphic to homogeneous spaces we have described by other means. The first new one is the 10-dimensional variety in $\mathbb{P}^{15}$, which comes from $\mathrm{Spin}_9\mathbb{C}$ or $\mathrm{Spin}_{10}\mathbb{C}$.

It is worth going back to interpret some of the "geometric plethysm" of earlier lectures (e.g., Exercises 11.36 and 13.24) in this light.

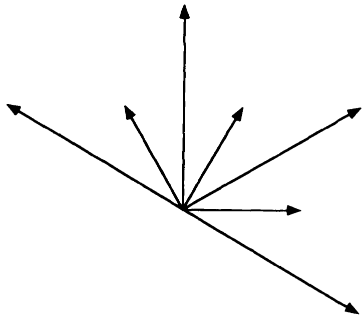
Finally, we can describe (at least one of) the compact homogeneous spaces for the group G_2 in this way. To begin with, G_2 has two maximal parabolic subgroups, corresponding to the diagrams

 and 

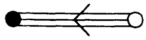
These are the groups whose Lie algebras are the parabolic subalgebras spanned by the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ together with the root spaces corresponding to the roots in the diagrams



and



In particular, each of these parabolic subgroups will have dimension 9, so that both the corresponding homogeneous spaces will be five-dimensional varieties. We can use this to identify one of these spaces: if V is the standard seven-dimensional representation of G_2, the closed orbit in $\mathbb{P}V \cong \mathbb{P}^6$ will be a hypersurface, which (since it is homogeneous) can only be a quadric hypersurface. Thus, the homogeneous space for G_2 corresponding to the diagram



is a quadric hypersurface in $\mathbb{P}^6$. In particular, we see again that the action of G_2 on V preserves a nondegenerate bilinear form, i.e., we have an inclusion

$$G_2 \hookrightarrow SO_7\mathbb{C}.$$

The other homogeneous space Y of $\mathfrak{g}_2$ is less readily described. One way to describe it is to use the fact that the adjoint representation W of $\mathfrak{g}_2$ is contained in the exterior square $\wedge^2 V$ of the standard. Since the Grassmannian $\mathbb{G}(1, 7) \subset \mathbb{P}(\wedge^2 V)$ of lines in $\mathbb{P}V$ is closed and invariant in $\mathbb{P}(\wedge^2 V)$, it follows that Y is contained in the intersection of $\mathbb{G}$ with the subspace $\mathbb{P}W \subset \mathbb{P}(\bigwedge^2 V)$. In other words, in terms of the skew-symmetric trilinear form ω on V preserved by the action of G_2, we can say that Y is contained in the locus

$$\Sigma = \{\Lambda \subset V : \omega(\Lambda, \Lambda, \cdot) \equiv 0\} \subset G(2, V).$$

Problem 23.54. Is $Y = \Sigma$?

Exercise 23.55. Show that the representation of E_6 whose highest weight is the first fundamental weight ω_1 determines a 16-dimensional homogeneous space in $\mathbb{P}^{26}$.

These homogeneous spaces have an amazing way of showing up as extremal examples of subvarieties of projective spaces, starting with a discovery of Severi that the Veronese surface in $\mathbb{P}^5$ is the only surface in $\mathbb{P}^5$ (nonsingular and not contained in a hyperplane) whose chords do not fill up $\mathbb{P}^5$. For recent work along these lines, see [L-VdV], with its appendix by Zak on interesting projective varieties that arise from representation theory.

Although we have described homogeneous spaces only for semisimple Lie groups, this is no real loss of generality: any irreducible representation V of a Lie group G comes from a representation of its semisimple quotient, up to multiplying by a character (see Proposition 9.17), and this character does not change the orbits in $\mathbb{P}(V)$.

It is possible to take this whole correspondence one step further and use it to give a construction of the irreducible representations of G; this is the modern approach to constructing the irreducible representations, due primarily to Borel, Weil, Bott, and, in a more general setting, Schmid. We do not have the means to do this in detail in the present circumstances, but we will sketch the construction.

The idea is very straightforward. We have just seen that for every irreducible representation V of G there is a unique closed orbit $X = G/P$ of the action of G on $\mathbb{P}V$. We obtain in this way from V a projective variety X together with a line bundle L on X invariant under the action of G (the restriction of the universal bundle from $\mathbb{P}V$). In fact, we may recover V from this data simply as the vector space of holomorphic sections of the line bundle L on X. What ties this all together is the fact that this gives us a one-to-one correspondence between irreducible representations of G and ample (positive) line bundles on compact homogeneous spaces G/P. More generally, using the projection maps $G/B \to G/P$, we may pull back all these line bundles to line bundles on G/B. This then extends to give an isomorphism between the weight lattice of $\mathfrak{g}$ and the group of line bundles on G/B, with the wonderful property that for dominant weights λ, the space of holomorphic sections of the associated line bundle L_λ is the irreducible representation of G with highest weight λ.

The point of all this, apart from its intrinsic beauty, is that we can go backward: starting with just the group G, we can construct the homogeneous space G/B, and then realize all the irreducible representations of G as cohomology groups of line bundles on G/B. To carry this out, start with a weight $\lambda \in \mathfrak{h}^*$ for $\mathfrak{g}$. We have seen that λ exponentiates to a homomorphism $H \to \mathbb{C}^*$, i.e., it gives a one-dimensional representation $\mathbb{C}_\lambda$ of H. We want to induce this representation from H to G. If $H \subset B \subset G$ is a Borel subgroup, the representation extends trivially to B, since B is a semidirect product of H and the nilpotent subgroup N whose Lie algebra is the direct sum of those $\mathfrak{g}_\alpha$ for positive roots α. Then we can form

$$\begin{aligned} L_\lambda &= G \times_B \mathbb{C}_\lambda \\ &= (G \times \mathbb{C}_\lambda)/\{(g, v) \sim (gx, x^{-1}v), x \in B\}, \end{aligned}$$

which, with its natural projection to G/B, is a holomorphic line bundle on the projective variety G/B. The cohomology groups of such a line bundle are finite dimensional, and since G acts on L_λ, these cohomology groups are representations of G.

We have Bott's theorem for the vanishing of the cohomology of this line bundle:

Claim 23.56. $H^i(G/B, L_\lambda) = 0$ *for* $i \neq i(\lambda)$,

where $i(\lambda)$ is an integer depending on which Weyl chamber λ belongs to. If λ is a dominant weight (i.e., belongs to the closure of the positive Weyl chamber for the choice of positive roots used in defining B), then $i(-\lambda) = 0$. In this case the sections $H^0(G/B, L_{-\lambda})$ are a finite-dimensional vector space, on which G acts.

Claim 23.57. *For λ a dominant weight, the space of sections $H^0(G/B, L_{-\lambda})$ is the irreducible representation with highest weight λ.*

In this context the Riemann–Roch theorem can be applied to give a formula for the dimension of the irreducible representation. In fact, the dimension part of Weyl's character formula can be proved this way. More refined analysis, using the Woods Hole fixed point theorem, can be used to get the full character formula (cf. [A-B]). For a very readable introduction to this, see [Bot].

We conclude this discussion by giving a diagram showing the relationships among the various objects associated to an irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. The objects and maps in diagram (23.58) are explained next.

First of all, as we have indicated, the term "Grassmannians" means the ordinary Grassmannians in the case of the groups $\mathrm{SL}_n\mathbb{C}$, and the Lagrangian Grassmannians and the orthogonal Grassmannians of isotropic subspaces in the cases of $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathrm{SO}_m\mathbb{C}$, respectively. Likewise, "flag manifolds" refers to the spaces parametrizing nested sequences of such subspaces. In the cases

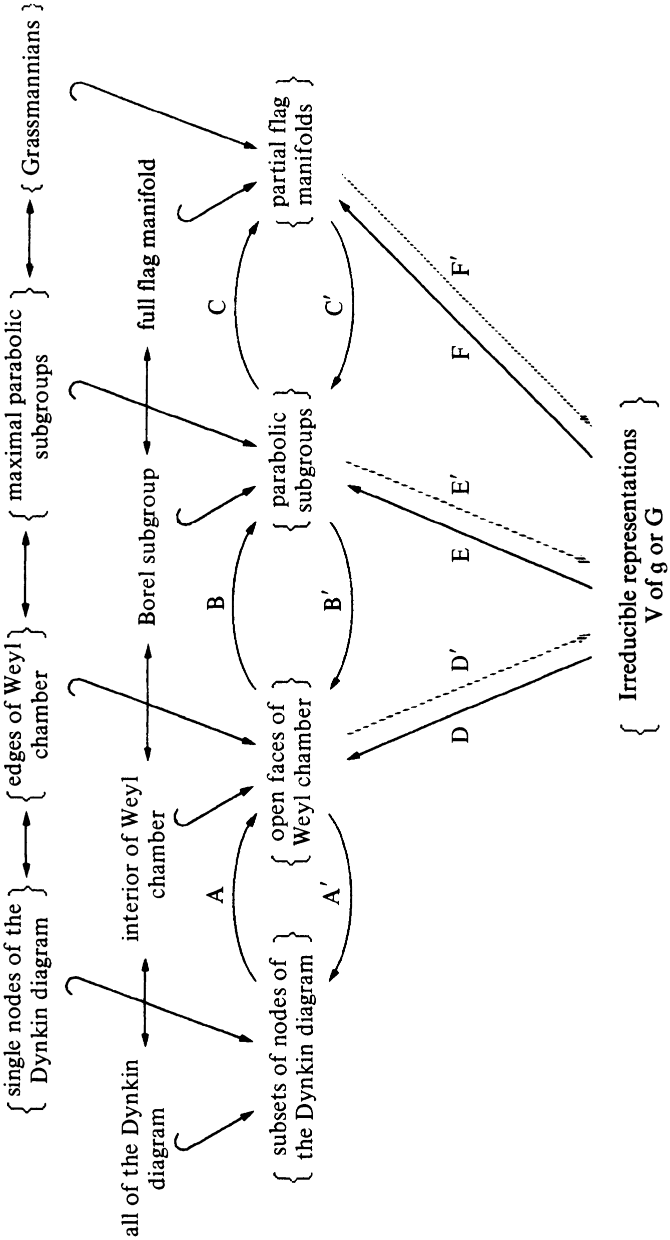


Diagram 23.58

of the exceptional Lie algebras, the term "Grassmannian" should just be ignored; except for the quotient of G_2 by one of its two maximal parabolic subgroups, the homogeneous spaces for the exceptional groups are not varieties with which we are likely to be a priori familiar.

With this said, we may describe the maps A, B, etc., as follows:

A, A': the map A associates to a subset of the nodes of the Dynkin diagram (equivalently, a subset S of the set of simple roots) the face of the Weyl chamber described by

$$\mathscr{W}_S = \left\{\lambda\colon \begin{matrix}(\lambda, \alpha) > 0, \forall\alpha \in S;\\ (\lambda, \alpha) = 0, \forall\alpha \notin S\end{matrix}\right\},$$

where (,) is the Killing form; the inverse is clear.

B, B': the map B associates to a face $\mathscr{W}_S$ of the Weyl chamber the subalgebra $\mathfrak{g}_S$ spanned by the Cartan subalgebra $\mathfrak{h}$, the positive root spaces $\mathfrak{g}_\alpha$, $\alpha \in R^+$, and the root spaces $\mathfrak{g}_{-\alpha}$ corresponding to those positive roots α perpendicular to $\mathscr{W}_S$. Equivalently, in terms of the corresponding subset S of the simple roots, $\mathfrak{g}_S$ will be generated by the Borel subalgebra, together with the root spaces $\mathfrak{g}_{-\alpha}$ for $\alpha \notin S$. Again, since every parabolic subalgebra is conjugate to one of this form, the inverse map is clear.

C, C': The map C simply associates to a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ the quotient G/P of G by the corresponding parabolic subgroup $P \subset G$. In the other direction, given the homogeneous space $X = G/P$, with the action of G, the group P is just the stabilizer of a point in X. Note that the connected component of the identity in the automorphism group of G/P may be strictly larger: for example, $\mathbb{P}^{2n-1}$ is a compact homogeneous space for $\mathrm{Sp}_{2n}\mathbb{C}$, and we have seen that a quadric hypersurface in $\mathbb{P}^6$ is a homogeneous space for G_2.

D, D': The map D associates to the irreducible representation V of $\mathfrak{g}$ with highest weight λ the open face of the Weyl chamber containing λ. In the other direction, given an open face $\mathscr{W}_S$ of $\mathscr{W}$, choose a lattice point $\lambda \in \mathscr{W}_S \cap \Lambda_W$ and take $V = \Gamma_\lambda$.

E: We send the representation V to the subalgebra or subgroup fixing the highest weight vector $v \in V$.

F, F': We associate to the representation V the (unique) closed orbit of the corresponding action of the group G on the projective space $\mathbb{P}V$. Going in the other direction, we have to choose an ample line bundle L on the space G/P, and then take its vector space of holomorphic sections.

§23.4. Bruhat Decompositions

We end this lecture with a brief introduction to the *Bruhat decomposition* of a semisimple complex Lie group G, and the related *Bruhat cells* in the flag manifold G/B. These ideas are not used in this course, but they appear so often elsewhere that it may be useful to describe them in the language we have developed in this lecture. We will give the general statements, but verify them only for the classical groups. General proofs can be found in [Bor1] or [Hu2].

As we have seen, a choice of positive roots determines a Borel subgroup B and Cartan subgroup H, with normalizer $N(H)$, so $N(H)/H$ is identified with the Weyl group $\mathfrak{W}$. For each $W \in \mathfrak{W}$ fix a representative n_W in $N(H)$. The double coset $B \cdot n_W \cdot B$ is clearly independent of choice of n_W, and will be denoted $B \cdot W \cdot B$.

Theorem 23.59 (Bruhat Decomposition). *The group G is a disjoint union of the $|\mathfrak{W}|$ double cosets $B \cdot W \cdot B$, as W varies over the Weyl group.*

Let us first see this explicitly for $G = \mathrm{SL}_m\mathbb{C}$. Here $N(H)$ consists of all monomial matrices in $\mathrm{SL}_m\mathbb{C}$, i.e., matrices with exactly one nonzero entry in each row and each column, and $\mathfrak{W} = \mathfrak{S}_m$; a monomial matrix with nonzero entry in the $\sigma(j)$th row of the jth column maps to the permutation σ. To see that the double cosets cover G, given $g \in G$, use elementary row operations by left multiplication by elements in B to get an element $b \cdot g^{-1}$, with $b \in B$ chosen so that the total number of zeros appearing at the left in the rows in $b \cdot g^{-1}$ is as large as possible. If two rows of $b \cdot g^{-1}$ had the same number of zeros at the left, one could increase the total by an elementary row operation. Since all the rows of $b \cdot g^{-1}$ start with different numbers of zeros, this matrix can be put in upper-triangular form by left multiplication by a monomial matrix; therefore, there is a permutation σ so that $b' = n_\sigma \cdot b \cdot g^{-1}$ is upper triangular, i.e., $g = (b')^{-1} \cdot n_\sigma \cdot b$ is in $B \cdot \sigma \cdot B$. To see that the double cosets are disjoint, suppose $n_{\sigma'} = b' \cdot n_\sigma \cdot b$ for some b and b' in B. From the equation $b = (n_\sigma)^{-1} \cdot (b')^{-1} \cdot n_{\sigma'}$ one sees that b must have nonzero entries in each place where $(n_\sigma)^{-1} \cdot n_{\sigma'}$ does, from which it follows that $\sigma' = \sigma$.

In fact, this can be strengthened as follows. Let U (resp. U^-) be the subgroup of G whose Lie algebra is the sum of all root spaces $\mathfrak{g}_\alpha$ for all positive (resp. negative) roots α. For $G = \mathrm{SL}_m\mathbb{C}$, U (resp. U^-) consists of upper- (resp. lower-) triangular matrices with 1's on the diagonal. For W in the Weyl group, define subgroups

$$U(W) = U \cap n_W \cdot U^- \cdot n_W^{-1}, \qquad U(W)' = U \cap n_W \cdot U \cdot n_W^{-1}$$

of U, which are again independent of the choice of representative n_W for W.

Corollary 23.60. *Every element in $B \cdot W \cdot B$ can be written $u \cdot n_W \cdot b$ for unique elements u in $U(W)$ and b in B.*

To see the existence of such an expression, note first that the Lie algebra of $U(W)$ is the sum of all root spaces $\mathfrak{g}_\alpha$ for which α is positive and $W^{-1}(\alpha)$ is negative; and the Lie algebra of $U(W)'$ is the sum of all root spaces $\mathfrak{g}_\alpha$ for which α and $W^{-1}(\alpha)$ are positive. One sees from this that $U(W) \cdot U(W)' \cdot H$ is the entire Borel group B. Since $H \cdot n_W = n_W \cdot H$ and $U(W)' \cdot n_W = n_W \cdot U$, and H and U are subgroups of B,

$$\begin{aligned} B\cdot n_W\cdot B &= U(W)\cdot U(W)'\cdot H\cdot n_W\cdot B \\ &= U(W)\cdot U(W)'\cdot n_W\cdot B \\ &= U(W)\cdot n_W\cdot B. \end{aligned}$$

To see the uniqueness, suppose that $n_W = u\cdot n_W\cdot b$ for some u in $U(W)$ and b in B. Then $n_W^{-1}\cdot u\cdot n_W$ is in $U^- \cap B = \{1\}$, so $u = 1$, as required.

Note in particular that the dimension of $U(W)$ is the cardinality of $R^+ \cap W(R^-)$, where R^+ and R^- are the positive and negative roots; this is also the minimum number $l(W)$ of reflections in simple roots whose product is W, cf. Exercise D.30. It is a general fact, which we will see for the classical groups, that $U(W)$ is isomorphic to an affine space $\mathbb{C}^{l(W)}$.

It follows from the Bruhat decomposition that G/B is a disjoint union of the cosets $X_W = B\cdot n_W\cdot B/B$, again with W varying over the Weyl group. These X_W are called *Bruhat cells.* From the corollary we see that X_W is isomorphic to the affine space $U(W) \cong \mathbb{C}^{l(W)}$.

For $G = \mathrm{SL}_m\mathbb{C}$ and σ in $\mathfrak{W} = \mathfrak{S}_m$, the group $U(\sigma)$ consists of matrices with 1's on the diagonal, and zero entry in the i, j place whenever either $i > j$ or $\sigma^{-1}(i) < \sigma^{-1}(j)$, which is an affine space of dimension $l(\sigma) = \#\{(i, j)\colon i > j \text{ and } \sigma(i) < \sigma(j)\}$.

Exercise 23.61. Identifying $\mathrm{SL}_m\mathbb{C}/B$ with the space of all flags, show that X_σ consists of those flags $0 \subset V_1 \subset V_2 \subset \cdots$ such that the dimensions of intersections with the standard flag are governed by σ, in the following sense: for each $1 \le k \le m$, the set of k numbers d such that $V_k \cap \mathbb{C}^{d-1} \subsetneqq V_k \cap \mathbb{C}^d$ is precisely the set $\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}$.

We will verify the Bruhat decomposition for $\mathrm{Sp}_{2n}\mathbb{C}$ by regarding it as a subgroup of $\mathrm{SL}_{2n}\mathbb{C}$ and using what we have just seen for $\mathrm{SL}_{2n}\mathbb{C}$, following [Ste2]. Our description of $\mathrm{Sp}_{2n}\mathbb{C}$ in Lecture 16 amounts to saying that it is the fixed point set of the automorphism φ of $\mathrm{SL}_{2n}\mathbb{C}$ given by $\varphi(A) = M^{-1}\cdot {}^t\!A^{-1}\cdot M$, with $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Borel subgroup of $\mathrm{Sp}_{2n}\mathbb{C}$ will be the intersection of the Borel subgroup B of $\mathrm{SL}_{2n}\mathbb{C}$ with $\mathrm{Sp}_{2n}\mathbb{C}$, provided we change the order of the basis of $\mathbb{C}^{2n}$ to $e_1, \ldots, e_n, e_{2n}, \ldots, e_{n+1}$, so that B consists of matrices whose upper left block is upper triangular, whose lower left block is zero, and whose lower right block is lower triangular. The automorphism φ maps this B to itself, and also preserves the diagonal subgroup H and its normalizer $N(H)$, and the groups U and U^-. The Weyl group of $\mathrm{Sp}_{2n}\mathbb{C}$ can be identified with the permutations in $\mathfrak{S}_{2n}$ such that $\sigma(n+i) = \sigma(i) \pm n$ for all $1 \le i \le n$, and it is exactly for these σ for which one can choose a monomial representative n_σ in $\mathrm{Sp}_{2n}\mathbb{C}$. Now if g is any element in $\mathrm{Sp}_{2n}\mathbb{C}$, write $g = u\cdot n_\sigma\cdot b$ according to the above corollary. Then

$$g = \varphi(g) = \varphi(u)\cdot\varphi(n_\sigma)\cdot\varphi(b),$$

and by uniqueness of the decomposition we must have $\varphi(u) = u$, $\varphi(n_\sigma) = n_\sigma \cdot h$, $h \in H$, and $\varphi(b) = h^{-1} \cdot b$. It follows that σ belongs to the Weyl group of $\mathrm{Sp}_{2n}\mathbb{C}$. This gives the Bruhat decomposition, and, moreover, a unique decomposition of $g \in \mathrm{Sp}_{2n}\mathbb{C}$ into $u \cdot n_\sigma \cdot b$, with u in $U(\sigma) \cap \mathrm{Sp}_{2n}\mathbb{C}$. Since this latter is an affine space, this shows that the corresponding Bruhat cell in the symplectic flag manifold is an affine space.

Exactly the same idea works for the orthogonal groups $\mathrm{SO}_m\mathbb{C}$, by realizing them as fixed points of automorphisms of $\mathrm{SL}_m\mathbb{C}$ of the form $A \mapsto M^{-1} \cdot {}^tA^{-1} \cdot M$, with M the matrix giving the quadratic form.

Note finally that if W' is the element in the Weyl group that takes each Weyl chamber to its negative, then $B \cdot W' \cdot B$ is a dense open subset of G, a fact which is evident for the classical groups by the above discussion. The corresponding Bruhat cell $X_{W'}$ is the image of U^- in G/B, which is also a dense open set. It follows that a function or section of a line bundle on G/B is determined by its values on U^-. For treatises developing representation theory via functions on U^-, see [N-S] or [Žel].

The following exercise uses these ideas to sketch a proof of Claim 23.57 that the sections of the bundle $L_{-\lambda}$ on G/B form the irreducible representation with highest weight λ:

Exercise 23.62*. (a) Show that sections s of $L_{-\lambda}$ are all of the form $s(gB) = (g, f(g))$, where f is a holomorphic function on G satisfying

$$f(g \cdot x) = \lambda(x) f(g) \quad \text{for all } x \in B.$$

(b) Let $n' \in N(H)$ be a representative of the element W' in the Weyl group which takes each element to its negative. Show that f is determined by its value at n'.

(c) Show that any highest weight for f must be λ, and conclude that $H^0(G/B, L_{-\lambda})$ is the irreducible representation Γ_λ with highest weight λ.

The holomorphic functions f of this exercise are functions on the space G/U. In other words, all irreducible representations of G can be found in spaces of functions on G/U. This is one common approach to the study of representations, especially by the Soviet school, cf. [N-S], [Žel].

Functions on G/U form a commutative ring, which indicates how to make the sum of all the irreducible representations into a commutative ring. In fact, for the classical groups, these rings are the algebras $\mathbb{S}^{\bullet}$, $\mathbb{S}^{\langle\bullet\rangle}$, and $\mathbb{S}^{[\bullet]}$ constructed in Lectures 15, 17, and 19, cf. [L-T]. They are also coordinate rings for natural embeddings of flag manifolds in products of projective spaces.

LECTURE 24

Weyl Character Formula

This lecture is pretty straightforward: we simply state the Weyl character formula in §24.1, then show how it may be worked out in specific examples in §24.2. In particular, we derive in the case of the classical algebras formulas for the character of a given irreducible representation as a polynomial in the characters of certain basic ones (either the alternating or the symmetric powers of the standard representation for $\mathfrak{sl}_n\mathbb{C}$ and their analogues for $\mathfrak{sp}_{2n}\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$). The proofs of the formula are deferred to the following two lectures. The techniques involved here are elementary, though the determinantal formulas are fairly complex, involving all the algebra of Appendix A.

§24.1. The Weyl Character Formula

We have already seen the Weyl character formula in the case of $\mathfrak{sl}_n\mathbb{C}$, and it is one reason why we were able to calculate so many more representations in that case. We saw in Lectures 6 and 15 that for the representation $\Gamma_\lambda = \mathbb{S}_\lambda\mathbb{C}^n$ of $\mathrm{SL}_n\mathbb{C}$ with highest weight $\lambda = \sum \lambda_i L_i$, the trace of the action of a diagonal matrix $A \in \mathrm{SL}_n\mathbb{C}$ with entries $x_1, \ldots, x_n$ is the symmetric function called the Schur polynomial $S_\lambda(x_1, \ldots, x_n)$. This included a formula for the multiplicities, which are the coefficients of the monomials in these variables.

In order to extend this formula to the other Lie algebras, let us try to rewrite this Schur polynomial in a way that may generalize. The Schur polynomial is defined to be a quotient of two alternating polynomials:

$$S_\lambda(x_1, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|}.$$

These determinants can be expanded as usual as a sum over the symmetric group $\mathfrak{S}_n$, which is the Weyl group $\mathfrak{W}$. Writing $x_i = e(L_i)$ in $\mathbb{Z}[\Lambda]$ as in the preceding lecture, and writing $(-1)^W$ for $\operatorname{sgn}(W) = \det(W)$ for W in the Weyl group, the numerator may be expanded in the form

$$\begin{aligned}\sum_{W\in\mathfrak{W}} (-1)^W x_{W(1)}^{\lambda_1+n-1}\cdot\ldots\cdot x_{W(n)}^{\lambda_n} &= \sum_{W\in\mathfrak{W}} (-1)^W e(W(\Sigma(\lambda_i+n-i)L_i))\\ &= \sum_{W\in\mathfrak{W}} (-1)^W e(W(\lambda+\rho)),\end{aligned}$$

where we write λ for $\Sigma\lambda_i L_i$ and we set $\rho = \Sigma(n-i)L_i$. Our formula therefore takes the form

$$\operatorname{Char}(\Gamma_\lambda) = \frac{\sum(-1)^W e(W(\lambda+\rho))}{\Sigma(-1)^W e(W(\rho))}.$$

The denominator is the discriminant

$$\Delta(x_1,\ldots,x_n) = \prod_{i<j}(x_i - x_j) = \prod_{i<j}(e(L_i) - e(L_j)).$$

This can be written in terms of the positive roots $L_i - L_j$, $i<j$, as

$$\Delta(x_1,\ldots,x_n) = \prod_{i<j}(e(\tfrac{1}{2}(L_i - L_j)) - e(-\tfrac{1}{2}(L_i - L_j))).$$

Note also that

$$\begin{aligned}\rho &= \Sigma(n-i)L_i = L_1 + (L_1+L_2) + \cdots + (L_1 + \cdots + L_{n-1})\\ &= \frac{1}{2}\sum_{i<j}(L_i - L_j),\end{aligned}$$

which is the *sum of the fundamental weights*, and *half the sum of the positive roots.*

These are the formulas that generalize to the other semisimple Lie algebras: For any weight μ, define $A_\mu \in \mathbb{Z}[\Lambda]$ by

$$A_\mu = \sum_{W\in\mathfrak{W}} (-1)^W e(W(\mu)). \tag{24.1}$$

Note that A_μ is not invariant by the Weyl group, but is *alternating*: $W(A_\mu) = (-1)^W A_\mu$ for $W\in\mathfrak{W}$. The ratio of two alternating polynomials will be invariant.

Theorem 24.2 (Weyl Character Formula). *Let ρ be half the sum of the positive roots. Then ρ is a weight, and $A_\rho \neq 0$. The character of the irreducible representation Γ_λ with highest weight λ is*

$$\operatorname{Char}(\Gamma_\lambda) = \frac{A_{\lambda+\rho}}{A_\rho}. \tag{WCF}$$

The assertions about ρ are part of the following lemma and exercise, which will also be useful in the applications:

Lemma 24.3. *The denominator A_ρ of Weyl's formula is*

$$\begin{aligned} A_\rho &= \prod_{\alpha \in R^+} (e(\alpha/2) - e(-\alpha/2)) \\ &= e(\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha)) \\ &= e(-\rho) \prod_{\alpha \in R^+} (e(\alpha) - 1). \end{aligned}$$

Proof. Since $e(\rho) = e(\sum \alpha/2) = \prod e(\alpha/2)$, the equality of the three displayed expressions is evident; denote these expressions temporarily by A. The key point is to see that A is alternating. For this, it suffices to see that A changes sign when a reflection in a hyperplane perpendicular to one of the simple roots is applied to it, since these reflections generate the Weyl group. This follows immediately from the first expression for A and (a) in Exercise 24.4 below.

Now, by the second displayed expression, the highest weight term that appears in A is $e(\rho)$, which is the same as that appearing in A_ρ. Calculating $1/A$ formally as in (24.5) below, we see that A_ρ/A is a formal sum $\sum m_\mu e(\mu)$ that is invariant by the Weyl group, and, using part (c) of the following exercise, it has weight 0. As in Theorem 23.24 it follows that A_ρ/A is constant; and, since A and A_ρ have the same leading term $e(\rho)$, we must have $A_\rho = A$. □

Exercise 24.4*. (a) If $W = W_{\alpha_i}$ is the reflection in the hyperplane perpendicular to a simple root α_i, show that $W(\alpha_i) = -\alpha_i$, and W permutes the other positive roots.

(b) With W as in (a), show that $W(\rho) = \rho - \alpha_i$. Deduce that ρ is the element in $\mathfrak{h}^*$ such that $\rho(H_{\alpha_i}) = 2(\rho, \alpha_i)/(\alpha_i, \alpha_i) = 1$ for each simple root α_i. Equivalently, ρ is the sum of the fundamental weights. In particular, ρ is a weight.

(c) For any $W \neq 1$ in the Weyl group, show that $\rho - W(\rho)$ is a sum of distinct positive roots. Deduce that $W(\rho)$ is not in the closure of the positive Weyl chamber.

Proofs of the character formula will be given in §25.2 and again in §26.2. For now we should at least verify that it is plausible, i.e., that $A_{\lambda+\rho}/A_\rho$ is in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ and that the highest weight that occurs is λ. Note that since the numerator and denominator are alternating, the ratio is invariant. The fact that A_ρ is not zero follows from the second expression in the preceding lemma. To see that the ratio is actually in $\mathbb{Z}[\Lambda]$, however, we must verify that it has only a finite number of nonzero coefficients. Write

$$\frac{1}{A_\rho} = e(-\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha))^{-1} = e(-\rho) \prod_{\alpha} \sum_{n=0}^{\infty} e(-n\alpha). \tag{24.5}$$

When this is multiplied by $A_{\lambda+\rho} = \sum (-1)^W e(W(\lambda + \rho))$, we get a formal sum where the highest weight that occurs is the weight λ. This means in particular that there are only a finite number of nonzero terms corresponding to weights in the fundamental (positive) Weyl chamber $\mathscr{W}$. But since the ratio is invariant by the Weyl group, the same is true for all Weyl chambers, so $A_{\lambda+\rho}/A_\rho$ is in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, and has highest weight λ. It follows in particular that the $A_{\lambda+\rho}/A_\rho$, as λ varies over $\mathscr{W} \cap \Lambda$, form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$.

Before considering the proof or any other special cases, we apply (WCF) to give a formula for the dimension of Γ_λ:

Corollary 24.6. *The dimension of the irreducible representation Γ_λ is*

$$\dim \Gamma_\lambda = \prod_{\alpha \in R^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \prod_{\alpha \in R^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where $\langle \alpha, \beta \rangle = \alpha(H_\beta) = 2(\alpha, \beta)/(\beta, \beta)$ and $(\ ,\ )$ is the Killing form.

PROOF. The dimension of Γ_λ is obtained by adding the coefficients of all $e(\alpha)$ in $\mathrm{Char}(\Gamma_\lambda)$, i.e., computing the image of $\mathrm{Char}(\Gamma_\lambda)$ by the homomorphism from $\mathbb{Z}[\Lambda]$ to $\mathbb{C}$ which sends each $e(\alpha)$ to 1. However, as in the case of the Schur polynomial, the denominator vanishes if we try to do this directly. To get around this, we factor this homomorphism through the ring of power series:

$$\mathbb{Z}[\Lambda] \xrightarrow{\Psi} \mathbb{C}[[t]] \to \mathbb{C},$$

where the second homomorphism sets the variable t equal to zero, i.e., picks off the constant term of the power series, and the first homomorphism Ψ takes $e(\alpha)$ to $e^{(\rho,\alpha)t}$. More generally, for any weight μ define a homomorphism

$$\Psi_\mu\colon \mathbb{Z}[\Lambda] \to \mathbb{C}[[t]], \qquad e(\alpha) \mapsto e^{(\mu,\alpha)t}.$$

We claim that $\Psi_\mu(A_\lambda) = \Psi_\lambda(A_\mu)$ for all λ and μ. This is a simple consequence of the invariance of the metric $(\ ,\ )$ under the Weyl group:

$$\begin{aligned}\Psi_\mu(A_\lambda) &= \sum (-1)^W e^{(\mu, W(\lambda))t}\\ &= \sum (-1)^W e^{(W^{-1}(\mu), \lambda)t}\\ &= \sum (-1)^W e^{(W(\mu), \lambda)t}\\ &= \Psi_\lambda(A_\mu).\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi(A_\lambda) &= \Psi_\rho(A_\lambda) = \Psi_\lambda(A_\rho)\\ &= \prod_{\alpha \in R^+} (e^{(\lambda,\alpha)t/2} - e^{-(\lambda,\alpha)t/2})\end{aligned}$$

$$= \left(\prod_{\alpha \in R^+} (\lambda, \alpha) \right) t^{\#(R^+)} + \text{terms of higher degree in } t.$$

Hence,

$$\begin{aligned} \Psi(A_{\lambda+\rho}/A_\rho) &= \Psi(A_{\lambda+\rho})/\Psi(A_\rho) \\ &= \frac{\prod(\lambda + \rho, \alpha)}{\prod(\rho, \alpha)} + \text{terms of positive degree in } t, \end{aligned}$$

which finishes the proof. □

Exercise 24.7. In the case of $\mathfrak{sl}_n\mathbb{C}$, verify that the above corollary gives the dimension we found in Lecture 6.

Exercise 24.8. Verify directly that the right-hand side of the formula for the dimension is positive.

Since $\chi_\lambda = A_{\lambda+\rho}/A_\rho$ is the character of a virtual representation which takes on a positive value at the identity, as in the case of finite groups, to prove that it is the character of an irreducible representation, it suffices to show that $\int_G \chi_\lambda \overline{\chi}_\lambda = 1$ for an appropriate compact group G. This was the original approach of Weyl, which we will describe in the last lecture. Since the highest weight appearing is λ, we will know then that this irreducible representation must be Γ_λ.

Exercise 24.9. Use Corollary 24.6 to show that if λ is a dominant weight (i.e., in the closure of the positive Weyl chamber), and ω is a fundamental weight, then the dimension of $\Gamma_{\lambda+\omega}$ is greater than the dimension of Γ_λ. Conclude that the nontrivial representations of smallest dimension must be among the n representations Γ_ω with ω a fundamental weight.

§24.2. Applications to Classical Lie Algebras and Groups

In the case of the general linear group $GL_n\mathbb{C}$, the character[1] of the representation Γ_λ is the Schur polynomial

$$S_\lambda(x_1, \ldots, x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|},$$

[1] We use the representation of $GL_n\mathbb{C}$ instead of its restriction to $SL_n\mathbb{C}$, since the latter would require the product of the variables x_i to be 1.

which has several expressions in terms of simpler symmetric functions. Note that the character of the dth symmetric power of the standard representation is the dth complete symmetric polynomial H_d in n variables (Appendix A.1):

$$H_d = \mathrm{Char}(\mathrm{Sym}^d(\mathbb{C}^n)).$$

The first "Giambelli" or determinantal formula (A.5) of Appendix A gives the character of the representation with highest weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ as an $r \times r$ determinant:

$$\mathrm{Char}(\Gamma_\lambda) = |H_{\lambda_i+j-i}| = \begin{vmatrix} H_{\lambda_1} & H_{\lambda_1+1} \cdots H_{\lambda_1+k-1} \\ H_{\lambda_2-1} H_{\lambda_2} \cdots & \\ \vdots & \\ H_{\lambda_k-k+1} \cdots & H_{\lambda_k} \end{vmatrix}. \tag{24.10}$$

Equivalently, this expresses a general element $\Gamma_\lambda \in R(G)$ of the representation ring as a polynomial in the representations $\mathrm{Sym}^d(\mathbb{C}^n)$. A second determinantal formula, from (A.6), expresses Γ_λ in terms of the basic representations $\bigwedge^d(\mathbb{C}^n)$, whose characters are the elementary symmetric polynomials

$$E_d = \mathrm{Char}(\textstyle\bigwedge^d(\mathbb{C}^n)).$$

This formula is, with μ the conjugate partition to λ,

$$\mathrm{Char}(\Gamma_\lambda) = |E_{\mu_i+j-i}| = \begin{vmatrix} E_{\mu_1} & E_{\mu_1+1} \cdots E_{\mu_1+l-1} \\ E_{\mu_2-1} E_{\mu_2} \cdots & \\ \vdots & \\ E_{\mu_l-l+1} \cdots & E_{\mu_l} \end{vmatrix} \tag{24.11}$$

In this section we work out the character formula for the other classical Lie algebras, including analogues of these determinantal formulas. The analogues of the first determinantal formula (24.10) were given by Weyl, but the analogues of (24.11) were found only recently ([D'H], [Ko-Te]). We also pay, at least by way of exercises, the debts to (WCF) that we owe from earlier lectures.

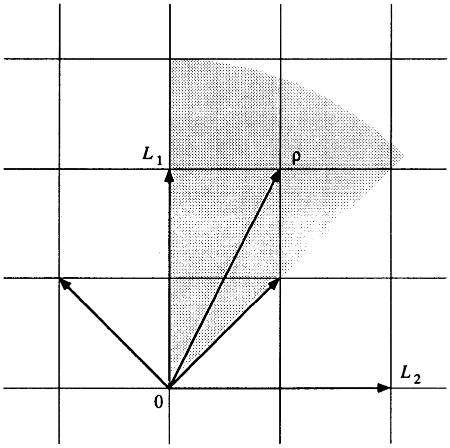
The Symplectic Case

The weights for $\mathfrak{sp}_{2n}\mathbb{C}$ are integral linear combinations of $L_1, \ldots, L_n$. We often write $\mu = (\mu_1, \ldots, \mu_n)$ for the weight $\mu_1 L_1 + \cdots + \mu_n L_n$.

The positive roots are $\{L_i - L_j\}_{i<j}$ and $\{L_i + L_j\}_{i \leq j}$, from which we find

$$\rho = \sum (n + 1 - i)L_i = L_1 + (L_1 + L_2) + \cdots + (L_1 + \cdots + L_n), \tag{24.12}$$

i.e., $\rho = (n, n - 1, \ldots, 1)$.



As we saw in Lecture 16, an element in the Weyl group can be written uniquely as a product $\varepsilon\sigma$, where σ is a permutation of $\{L_1, \ldots, L_n\}$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, with $\varepsilon_i = \pm 1$. Hence

$$A_\mu = \sum_\sigma (-1)^\sigma \sum_\varepsilon (-1)^\varepsilon e\left(\sum_{i=1}^{n} \varepsilon_i \mu_i L_{\sigma(i)}\right); \tag{24.13}$$

here the sign $(-1)^\varepsilon$ is the product of the ε_i. Now with $x_i = e(L_i)$, this can be written

$$A_\mu = \sum_\sigma (-1)^\sigma \prod_{i=1}^{n} \left(x_{\sigma(i)}^{\mu_i} - x_{\sigma(i)}^{-\mu_i}\right)$$

or

$$A_\mu = |x_j^{\mu_i} - x_j^{-\mu_i}|, \tag{24.14}$$

where $|a_{i,j}|$ denotes the determinant of the $n \times n$ matrix $(a_{i,j})$. In particular,

$$A_\rho = |x_j^{n-i+1} - x_j^{-(n-i+1)}|. \tag{24.15}$$

From (24.14) or Exercise A.52 we have

$$A_\rho = \Delta(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1}) \cdot (x_1 - x_1^{-1}) \cdot \ldots \cdot (x_n - x_n^{-1}), \tag{24.16}$$

where Δ is the discriminant.

Exercise 24.17. Show that

$$A_\rho = \prod_{i<j} (x_i - x_j)(x_i x_j - 1) \cdot \prod_i (x_i^2 - 1)/(x_1 \cdot \ldots \cdot x_n)^n.$$

The character of the irreducible representation Γ_λ with highest weight $\lambda = \sum \lambda_i L_i$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, is therefore:

$$\operatorname{Char}(\Gamma_\lambda) = \frac{|x_j^{\lambda_i+n-i+1} - x_j^{-(\lambda_i+n-i+1)}|}{|x_j^{n-i+1} - x_j^{-(n-i+1)}|}. \tag{24.18}$$

The dimension of Γ_λ is easily worked out from Corollary 24.6:

$$\begin{aligned}\dim(\Gamma_\lambda) &= \prod_{i<j}\frac{(l_i - l_j)}{(j-i)}\cdot\prod_{i\le j}\frac{(l_i + l_j)}{(2n+2-i-j)} \\ &= \prod_{i<j}\frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)}\cdot\prod_i\frac{l_i}{m_i},\end{aligned} \tag{24.19}$$

where $l_i = \lambda_i + n - i + 1$ and $m_i = n - i + 1$.

Exercise 24.20. Show that, setting $l_i' = \lambda_i + n - i$,

$$\dim(\Gamma_\lambda) = \frac{\prod_{i<j}(l_i' - l_j')(l_i' + l_j' + 2)\cdot\prod_i(l_i' + 1)}{(2n-1)!\cdot(2n-3)!\cdot\ldots\cdot 1!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$, where the ω_i are the fundamental weights, using the relation $\lambda_i = a_i + \cdots + a_n$.

Exercise 24.21. Use Exercise 24.20 to verify that for $\lambda = L_1 + \cdots + L_k$, the dimension of Γ_λ is $2n$ if $k = 1$, and $\binom{2n}{k} - \binom{2n}{k-2}$ if $k \ge 2$. Use this to give another proof that the kernel of the contraction from $\bigwedge^k V$ to $\bigwedge^{k-2} V$ is irreducible.

The first determinantal formula for the symplectic group goes as follows. Let

$$J_d(x_1, \ldots, x_n) = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}),$$

where H_d is the dth complete symmetric polynomial in $2n$ variables. In other words, J_d is the character of the representation $\operatorname{Sym}^d(\mathbb{C}^{2n})$ of $\mathfrak{sp}_{2n}\mathbb{C}$. From Proposition A.50 of Appendix A we have

Proposition 24.22. *If $\lambda = (\lambda_1 \ge \cdots \ge \lambda_r > 0)$, the character of Γ_λ is the determinant of the $r \times r$ matrix whose ith row is*

$$(J_{\lambda_i-i+1} \quad J_{\lambda_i-i+2} + J_{\lambda_i-i} \quad J_{\lambda_i-i+3} + J_{\lambda_i-i-1} \quad \cdots \quad J_{\lambda_i-i+r} + J_{\lambda_i-i-r+2}).$$

For example, for $\lambda = (d)$, i.e., $\lambda = dL_1$, we have $\operatorname{Char}(\Gamma_{(d)}) = J_d$, which is the character of $\operatorname{Sym}^d(\mathbb{C}^{2n})$. In particular, this verifies that the kth *symmetric powers* $\operatorname{Sym}^k(\mathbb{C}^{2n})$ *of the standard representation are all irreducible.* (This, of course, is a special case of the general description given in §17.3, since all the contraction maps vanish on the symmetric powers.)

Exercise 24.23. (i) Find the character of the representation of $\mathfrak{sp}_4\mathbb{C}$ with highest weight $\omega_1 + \omega_2 = 2L_1 + L_2$, verifying that the multiplicities are as we found in §16.2. (ii) Find the character of the representation of $\mathfrak{sp}_6\mathbb{C}$ with highest weight $\omega_1 + \omega_2$, thus verifying the assertion of Exercise 17.4.

The second Giambelli formula in the symplectic case expresses Γ_λ in terms of the basic representations

$$\Gamma_{\omega_k} = \mathrm{Ker}(\Lambda^k(\mathbb{C}^{2n}) \to \Lambda^{k-2}(\mathbb{C}^{2n}))$$

which are the kernels of the contractions. The character of Γ_{ω_k} is E'_k, where $E'_0 = 1$, $E'_1 = E_1 = x_1 + \cdots + x_n + x_1^{-1} + \cdots + x_n^{-1}$, and

$$E'_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) - E_{k-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$$

for $k \geq 2$, where E_k is the kth elementary symmetric polynomial. The formula is

Corollary 24.24. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose* ith *row is*

$$(E'_{\mu_i-i+1} \quad E'_{\mu_i-i+2} + E'_{\mu_i-i} \quad E'_{\mu_i-i+3} + E'_{\mu_i-i-1} \quad \cdots \quad E'_{\mu_i-i+l} + E'_{\mu_i-i-l+2}).$$

PROOF. This follows from the proposition and Proposition A.44, which equates the two determinants before specializing the variables. □

There is also a simple formula for the character in terms of the characters E_k of $\Lambda^k(\mathbb{C}^{2n})$, which also follows from Proposition A.44:

$$\mathrm{Char}(\Gamma_\lambda) = |E_{\mu_i-i+j} - E_{\mu_i-i-j}|. \tag{24.25}$$

Note that $E_{n+k} = E_{n-k}$ (corresponding to the isomorphism $\Lambda^{n+k}\mathbb{C}^{2n} \cong \Lambda^{n-k}\mathbb{C}^{2n}$) and $E'_{n+k} = -E'_{n-k+2}$. In particular, Corollary 24.24 expresses $\mathrm{Char}(\Gamma_\lambda)$ as a polynomial in the characters of the basic representations Γ_{ω_1}, $\ldots, \Gamma_{\omega_n}$.

The Odd Orthogonal Case

For $\mathfrak{so}_{2n+1}\mathbb{C}$ the weights are $\sum \mu_i L_i$, $\mu = (\mu_1, \ldots, \mu_n)$, with all μ_i integers or all half-integers. The positive roots are $\{L_i - L_j\}_{i<j}$, $\{L_i + L_j\}_{i<j}$, and $\{L_i\}$, so ρ is $\frac{1}{2}(L_1 + \cdots + L_n)$ less than in the case for $\mathfrak{sp}_{2n}$:

$$\rho = \sum (n + \tfrac{1}{2} - i)L_i, \tag{24.26}$$

or

$$\rho = (n - \tfrac{1}{2}, n - \tfrac{3}{2}, \ldots, \tfrac{1}{2}).$$

With $x_i^{\pm 1} = e(\pm L_i)$ and $x_i^{\pm 1/2} = e(\pm L_i/2)$, we have the same formula as before [(24.14)] for A_μ.

Exercise 24.27*. Show that

$$A_\rho = |x_j^{n-i+1/2} - x_j^{-(n-i+1/2)}|$$
$$= \Delta(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1})\cdot(x_1^{1/2} - x_1^{-1/2})\cdot\ldots\cdot(x_n^{1/2} - x_n^{-1/2}).$$

If Γ_λ is the irreducible representation with highest weight $\lambda = \sum \lambda_i L_i$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, then the character formula can be written

$$\mathrm{Char}(\Gamma_\lambda) = \frac{|x_j^{\lambda_i+n-i+1/2} - x_j^{-(\lambda_i+n-i+1/2)}|}{|x_j^{n-i+1/2} - x_j^{-(n-i+1/2)}|}. \tag{24.28}$$

Similarly,

$$\dim(\Gamma_\lambda) = \prod_{i<j}\frac{(l_i - l_j)}{(j-i)}\cdot\prod_{i\leq j}\frac{(l_i + l_j)}{(2n+1-i-j)}$$
$$= \prod_{i<j}\frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)}\cdot\prod_i \frac{l_i}{m_i}, \tag{24.29}$$

where $l_i = \lambda_i + n - i + \frac{1}{2}$, and $m_i = n - i + \frac{1}{2}$.

Exercise 24.30. Show that, with $l_i' = \lambda_i + n - i$,

$$\dim(\Gamma_\lambda) = \frac{\prod_{i<j}(l_i' - l_j')(l_i' + l_j' + 1)\cdot\prod_i(2l_i' + 1)}{(2n-1)!\cdot(2n-3)!\cdot\ldots\cdot 1!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$, where the ω_i are the fundamental weights, using the equations

$$\lambda_i = a_i + \cdots + a_{n-1} + \tfrac{1}{2}a_n.$$

Exercise 24.31. Use the dimension formula to verify that for $\lambda = L_1 + \cdots + L_k$, the dimension of Γ_λ is $\binom{2n+1}{k}$. Use this to give another proof that $\wedge^k V$ is irreducible for $1 \leq k \leq n$. Verify that the dimension of the spin representation is 2^n, thus reproving that it is irreducible.

Exercise 24.32. Use the dimension formula to verify that the kernel of the contraction

$$\mathrm{Sym}^d(\mathbb{C}^{2n+1}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n+1})$$

is an irreducible representation with highest weight dL_1.

In case the representation is a representation of $\mathrm{SO}_{2n+1}\mathbb{C}$, i.e., the λ_i are all integral, there is a first determinantal formula that expresses Γ_λ in terms of the kernels of the contractions

$$\mathrm{Ker}(\mathrm{Sym}^d(\mathbb{C}^{2n+1}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n+1})).$$

Let K_d denote the character of this kernel, so $K_0 = 1$, $K_1 = x_1 + \cdots + x_n + x_1^{-1} + \cdots + x_n^{-1} + 1$, and

$$K_d = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1) - H_{d-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1),$$

where H_d is the dth complete symmetric polynomial. From Proposition A.60 we have

Proposition 24.33. *If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$, with the λ_i integral, then the character of Γ_λ is the determinant of the $r \times r$ matrix whose ith row is*

$$(K_{\lambda_i-i+1} \quad K_{\lambda_i-i+2} + K_{\lambda_i-i} \quad K_{\lambda_i-i+3} + K_{\lambda_i-i-1} \quad \cdots \quad K_{\lambda_i-i+r} + K_{\lambda_i-i-r+2}).$$

In particular, for $\lambda = (d)$, the character is K_d, which verifies that the kernel of $\mathrm{Sym}^d(\mathbb{C}^{2n+1}) \to \mathrm{Sym}^{d-2}(\mathbb{C}^{2n+1})$ is irreducible.

Exercise 24.34. Use the character formula to verify that the multiplicities of the representation $\Gamma_{2L_1+L_2}$ of $\mathfrak{so}_5\mathbb{C}$ are as specified in Exercise 18.9.

The second determinantal formula for $\mathrm{SO}_{2n+1}\mathbb{C}$ writes Γ_λ in terms of the representations $\bigwedge^k(\mathbb{C}^{2n+1})$, whose characters are

$$E_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}, 1).$$

Applying Proposition 24.33 with Corollary A.46, we have

Corollary 24.35. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose ith row is*

$$(E_{\mu_i-i+1} \quad E_{\mu_i-i+2} + E_{\mu_i-i} \quad \cdots \quad E_{\mu_i-i+l} + E_{\mu_i-i-l+2}).$$

Since $E_{n+k} = E_{n+1-k}$ (corresponding to the isomorphism $\bigwedge^{n+k}\mathbb{C}^{2n+1} \cong \bigwedge^{n+1-k}\mathbb{C}^{2n+1}$), this expresses $\mathrm{Char}(\Gamma_\lambda)$ as a polynomial in $E_1, \ldots, E_n$, with $E_d = \mathrm{Char}(\bigwedge^d\mathbb{C}^{2n+1})$.

The Even Orthogonal Case

For $\mathfrak{so}_{2n}\mathbb{C}$ the weights are the same as in the preceding case. This time the $\{L_i\}$ are not positive roots, however, so ρ is $\frac{1}{2}(L_1 + \cdots + L_n)$ less than in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$, or $L_1 + \cdots + L_n$ less than in the case of $\mathfrak{sp}_{2n}\mathbb{C}$:

$$\rho = \sum (n-i)L_i, \tag{24.36}$$

or

$$\rho = (n-1, n-2, \ldots, 0).$$

The calculation of A_μ is similar, but using only those ε of positive sign. This time

$$\sum_{\varepsilon}(-1)^{\varepsilon}e\left(\sum_{i=1}^{n}\varepsilon_i\mu_i L_{\sigma(i)}\right)=\frac{1}{2}\left[\prod_{i=1}^{n}(x_{\sigma(i)}^{\mu_i}+x_{\sigma(i)}^{-\mu_i})+\prod_{i=1}^{n}(x_{\sigma(i)}^{\mu_i}-x_{\sigma(i)}^{-\mu_i})\right].$$

This leads to

$$A_\mu=\tfrac{1}{2}(|x_j^{\mu_i}+x_j^{-\mu_i}|+|x_j^{\mu_i}-x_j^{-\mu_i}|). \tag{24.37}$$

Note that the second determinant term vanishes when any μ_i is zero. In particular,

$$A_\rho=\tfrac{1}{2}|x_j^{n-i}+x_j^{-(n-i)}|. \tag{24.38}$$

From (24.14) or Exercise A.66,

$$A_\rho=\Delta(x_1+x_1^{-1},\ldots,x_n+x_n^{-1}). \tag{24.39}$$

This gives, with Γ_λ the irreducible representation with highest weight $\lambda=\sum\lambda_i L_i$, $\lambda_1\geq\cdots\geq|\lambda_n|\geq 0$,

$$\mathrm{Char}(\Gamma_\lambda)=\frac{|x_j^{l_i}+x_j^{-l_i}|+|x_j^{l_i}-x_j^{-l_i}|}{|x_j^{n-i}+x_j^{-(n-i)}|}, \tag{24.40}$$

where $l_i=\lambda_i+n-i$. As before,

$$\begin{aligned}\dim(\Gamma_\lambda)&=\prod_{i<j}\frac{(l_i-l_j)}{(j-i)}\cdot\frac{(l_i+l_j)}{(2n-i-j)}\\&=\prod_{i<j}\frac{(l_i^2-l_j^2)}{(m_i^2-m_j^2)},\end{aligned} \tag{24.41}$$

where $l_i=\lambda_i+n-i$ and $m_i=n-i$. Note that, as expected, the two representations with weights $(\lambda_1,\ldots,\lambda_{n-1},\pm\lambda_n)$ have the same dimensions.

Exercise 24.42. Show that

$$\dim(\Gamma_\lambda)=2^{n-1}\frac{\prod_{i<j}(l_i-l_j)(l_i+l_j)}{(2n-2)!\cdot(2n-4)!\cdot\ldots\cdot 2!}.$$

These formulas give the dimension of the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1\omega_1+\cdots+a_n\omega_n$, where the ω_i are the fundamental weights, using the equations

$$\lambda_i=a_i+\cdots+a_{n-2}+\tfrac{1}{2}(a_{n-1}+a_n),\quad 1\leq i\leq n-2,$$
$$\lambda_{n-1}=\tfrac{1}{2}(a_{n-1}+a_n),\qquad \lambda_n=\tfrac{1}{2}(-a_{n-1}+a_n).$$

Exercise 24.43. Use the dimension formula to verify that for $\omega=L_1+\cdots+L_k$, $k<n$, the dimension of Γ_ω is $\binom{2n}{k}$, so $\wedge^k(\mathbb{C}^{2n})$ is irreducible. For $\lambda=L_1+\cdots+L_{n-1}\pm L_n$, the dimension is $\frac{1}{2}\binom{2n}{k}$, so $\wedge^n(\mathbb{C}^{2n})$ is the sum of the two corresponding irreducible representations. Verify that the dimension of the two spin representations are 2^{n-1}, proving irreducibility again.

Note that the second term in the numerator in (24.40) changes sign when λ_n is replaced by $-\lambda_n$; in particular, it vanishes when $\lambda_n = 0$. When $\lambda_n = 0$, the representation Γ_λ is a representation of the orthogonal group $O_{2n}\mathbb{C}$. When $\lambda_n \neq 0$, the direct sum of the two representations with highest weights $(\lambda_1, \ldots, \pm\lambda_n)$ is an irreducible representation of $O_{2n}\mathbb{C}$. (See Exercises 23.19 and 23.37.)

Let L_d be the character of $\operatorname{Ker}(\operatorname{Sym}^d(\mathbb{C}^{2n}) \to \operatorname{Sym}^{d-2}(\mathbb{C}^{2n}))$, i.e., $L_d = H_d(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n}) - H_{d-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n})$. In either case, Proposition A.64 applies to give the first determinantal formula:

Proposition 24.44. *Given integers $\lambda_1 \geq \cdots \geq \lambda_r > 0$, the character of the irreducible representation of $O_{2n}\mathbb{C}$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the determinant of the $r \times r$ matrix whose* i*th row is*

$$(L_{\lambda_i-i+1} \quad L_{\lambda_i-i+2} + L_{\lambda_i-i} \quad \cdots \quad L_{\lambda_i-i+r} + L_{\lambda_i-i-r+2}).$$

Again, for $\lambda = (d)$, this verifies that the kernel of the contraction from $\operatorname{Sym}^d(\mathbb{C}^{2n})$ to $\operatorname{Sym}^{d-2}(\mathbb{C}^{2n})$ is irreducible.

The second determinantal formula is the same as in the odd case, but with $E_k = E_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-n})$:

Corollary 24.45. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be the conjugate partition to λ. The character of Γ_λ is equal to the determinant of the $l \times l$ matrix whose* i*th row is*

$$(E_{\mu_i-i+1} \quad E_{\mu_i-i+2} + E_{\mu_i-i} \quad \cdots \quad E_{\mu_i-i+l} + E_{\mu_i-i-l+2}).$$

Using the fact that $E_{n+k} = E_{n-k}$, this expresses $\operatorname{Char}(\Gamma_\lambda)$ as a polynomial in $E_1, \ldots, E_n$, with $E_d = \operatorname{Char}(\wedge^d \mathbb{C}^{2n})$.

Exercise 24.46*. For each of the orthogonal groups $O_m\mathbb{C}$, show that the character of the irreducible representation with highest weight λ can be written in the form

$$\operatorname{Char}(\Gamma_\lambda) = |h_{\lambda_i-i+j} - h_{\lambda_i-i-j}|,$$

where h_k is the character of $\operatorname{Sym}^k(\mathbb{C}^m)$. Another formula for the dimension of Γ_λ is obtained by substituting $\binom{m}{k}$ for h_k in this determinant.

There are other formulas expressing the characters of general representations in terms of simpler ones. Abramsky, Jahn, and King [A-J-K] give one that can be expressed by the *same* formula for the general linear, symplectic, and orthogonal groups. The general irreducible representations are given by partitions λ or Young diagrams, and in their formula the simpler representations are those corresponding to hooks. To express it, let $(a * b)$ denote the hook with horizontal leg of length $a + 1$ and vertical leg of length $b + 1$, i.e., the partition $(a + 1, 1, \ldots, 1)$, with b 1's. More generally, given $\mathbf{a} = (a_1 > \cdots > a_r \geq 0)$ and $\mathbf{b} = (b_1 > \cdots > b_r \geq 0)$ with a_r or b_r nonzero, let $(\mathbf{a} * \mathbf{b})$ denote the partition whose Young diagram has legs of these lengths to the right of and below the r diagonal boxes (cf. Frobenius's notation, Exercise 4.17). Let $\chi_{(\mathbf{a}*\mathbf{b})}$ denote the character of the corresponding irreducible representation. Their formula is

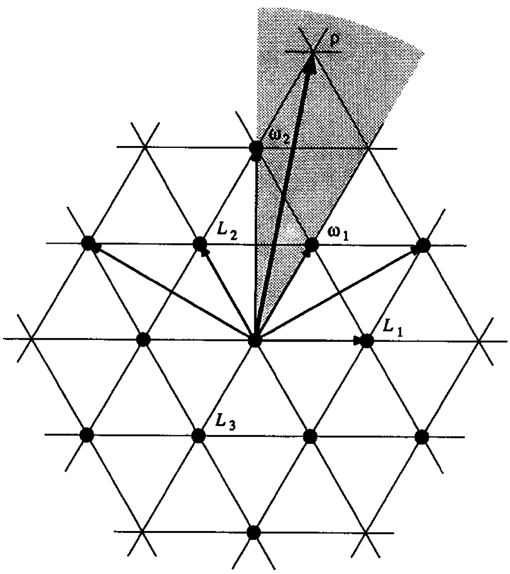
$$\chi_{(\mathbf{a}*\mathbf{b})} = |\chi_{(a_i*b_j)}|_{1\le i,j\le r}. \tag{24.47}$$

Taking the degree of both sides gives new formulas for the dimensions of the irreducible representations. These formulas are particularly useful if the rank r of the partition is small.

Exceptional Cases

We will, as a last example, work out the Weyl character formula for the exceptional Lie algebra $\mathfrak{g}_2$, and thereby verify some of the analysis of its representations given in Lecture 22. The remaining four exceptional Lie algebras we will leave as exercises.

To begin with, the value of ρ is easily seen to be $2L_1 + 3L_2$, in terms of the basis L_1, L_2 for the weight lattice introduced in Lecture 22.



Now, for any weight $\mu = pL_1 + qL_2 + rL_3$, we have

$$\begin{aligned} A_\mu &= \sum_{\sigma\in\mathfrak{S}_3} x_{\sigma(1)}^p\cdot x_{\sigma(2)}^q\cdot x_{\sigma(3)}^r - \sum_{\sigma\in\mathfrak{S}_3} x_{\sigma(1)}^{-p}\cdot x_{\sigma(2)}^{-q}\cdot x_{\sigma(3)}^{-r} \\ &= \Delta(x)\cdot S_{p,q,r}(x) - \Delta(x^{-1})\cdot S_{p,q,r}(x^{-1}), \end{aligned}$$

where we write x for (x_1, x_2, x_3) and x^{-1} for $(x_1^{-1}, x_2^{-1}, x_3^{-1})$, Δ is the discriminant, and $S_{p,q,r}$ the Schur function. Using the relation $\prod x_i = 1$ we can also write this as

$$= \Delta(x)\cdot(S_{p,q,r}(x) - S_{m-p,m-q,m-r}(x))$$

for any $m \geq \max(p, q, r)$. To make this notation agree with the standard notation for Schur polynomials from Appendix (A.4), note that $S_{p,q,r}$ is the Schur polynomial $S_{(s,t)}$ for the partition (s, t), $s \geq t$, where s is two less than the difference between the largest and smallest of p, q, and r, while t is one less than the difference between the second largest and the smallest; if p, q, and r are not distinct, $S_{p,q,r} = 0$. Thus, for example,

$$\begin{aligned} A_\rho &= \Delta(x)\cdot(S_{(1,1)}(x) - S_{(1)}(x)) \\ &= \Delta(x)\cdot(x_1x_2 + x_1x_3 + x_2x_3 - x_1 - x_2 - x_3). \end{aligned}$$

Now, any irreducible representation Γ_λ of $\mathfrak{g}_2$ has highest weight $\lambda = a\omega_1 + b\omega_2$, where $\omega_1 = L_1 + L_2$ and $\omega_2 = L_1 + 2L_2$ are the two fundamental weights, and a and b are non-negative integers. Then $\lambda + \rho = (a + b + 2)L_1 + (a + 2b + 3)L_2$. The Weyl character formula in this case becomes

Proposition 24.48. *The character of the representation of $\mathfrak{g}_2$ with highest weight $a\omega_1 + b\omega_2$ is*

$$\operatorname{Char}(\Gamma_{a,b}) = \frac{S_{(a+2b+1,a+b+1)} - S_{(a+2b+1,b)}}{S_{(1,1)} - S_{(1)}}.$$

Exercise 24.49. In the case of the standard representation $\Gamma_{1,0}$, the adjoint representation $\Gamma_{0,1}$, and the representation $\Gamma_{2,0}$, use this formula to verify the multiplicities found in Lecture 22.

We can also work out the dimension formula explicitly in this case. The two fundamental weights ω_1 and ω_2 have inner products

$$(\omega_1, \omega_1) = 1, \qquad (\omega_1, \omega_2) = 3/2, \qquad \text{and} \qquad (\omega_2, \omega_2) = 3;$$

ω_1 and ω_2 are among the positive roots of $\mathfrak{g}_2$, and in terms of these the remaining positive roots are $2\omega_1 - \omega_2$, $3\omega_1 - \omega_2$, $\omega_2 - \omega_1$, and $2\omega_2 - 3\omega_1$. The weight ρ is the sum of the fundamental weights ω_1 and ω_2, so that for an arbitrary weight $\lambda = a\omega_1 + b\omega_2$ we have the following table of inner products:

	$(\cdot, \rho)$	$(\cdot, \lambda)$	$(\cdot, \lambda + \rho)$
$2\omega_1 - \omega_2$	$1/2$	$a/2$	$(a + 1)/2$
$3\omega_1 - \omega_2$	3	$3a/2 + 3b/2$	$3(a + b + 2)/2$
ω_1	$5/2$	$a + 3b/2$	$(2a + 3b + 5)/2$
ω_2	$9/2$	$3a/2 + 3b$	$3(a + 2b + 3)/2$
$-\omega_1 + \omega_2$	2	$a/2 + 3b/2$	$(a + 3b + 4)/2$
$-3\omega_1 + 2\omega_2$	$3/2$	$3b/2$	$3(b + 1)/2$

We conclude that *the dimension of the irreducible representation $\Gamma_{a,b}$ of $\mathfrak{g}_2$ with highest weight $\lambda = a\omega_1 + b\omega_2$ is*

$$\dim(\Gamma_{a,b}) = \frac{(a+1)(a+b+2)(2a+3b+5)(a+2b+3)(a+3b+4)(b+1)}{120}.$$

We can check this in the cases $a = 1$, $b = 0$ and $a = 0$, $b = 1$, getting the dimensions 7 and 14 of the standard and adjoint representations, respectively. In case $a = 2$, $b = 0$ we may verify the result of the explicit calculation in Lecture 22, finding that

$$\dim(\Gamma_{2,0}) = 27$$

and, therefore, deducing that $\wedge^3 V = \Gamma_{2,0} \oplus V \oplus \mathbb{C}$ and $\mathrm{Sym}^2 V = \Gamma_{2,0} \oplus \mathbb{C}$.

Exercise 24.50. Show that $\mathrm{Sym}^a V = \bigoplus_{k=0}^{[a/2]} \Gamma_{a-2k,0}$.

We leave the analogous computations for the remaining four Lie algebras as exercises, using the description of the root systems found in Exercise 21.16. Since we have not said much about the Weyl group in the exceptional cases the formula (WCF) cannot be used directly—not to mention the fact that the orders of these Weyl groups are: $2^7 \cdot 3^2 = 1152$ for $\mathfrak{f}_4$; $2^7 \cdot 3^4 \cdot 5 = 51{,}840$ for $\mathfrak{e}_6$, $2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2{,}903{,}040$ for $\mathfrak{e}_7$, and $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696{,}729{,}600$ for $\mathfrak{e}_8$. However, the dimension formula is available.

Exercise 24.51*. For each of the four remaining exceptional Lie algebras, compute ρ = half the sum of the positive roots. For each of the fundamental weights ω, at least for $\mathfrak{f}_4$, compute the dimension of the irreducible representation with highest weight ω. In particular, find the nontrivial representation of minimal dimension. Use this to verify that (E_6) is not isomorphic to (B_6) or (C_6), i.e., that $\mathfrak{e}_6$ is not isomorphic to $\mathfrak{so}_{13}\mathbb{C}$ or $\mathfrak{sp}_{12}\mathbb{C}$.

Exercise 24.52*. List all irreducible representations V of simple Lie algebras $\mathfrak{g}$ such that $\dim V \le \dim \mathfrak{g}$. Note that these include all cases where the corresponding group representation has a Zariski dense orbit, or a finite number of orbits.

LECTURE 25

More Character Formulas

In this lecture we give two more formulas for the multiplicities of an irreducible representation of a semisimple Lie algebra or group. First, Freudenthal's formula (§25.1) gives a straightforward way of calculating the multiplicity of a given weight once we know the multiplicity of all higher ones. This in turn allows us to prove in §25.2 the Weyl character formula, as well as another multiplicity formula due to Kostant. Finally, in §25.3 we give Steinberg's formula for the decomposition of the tensor product of two arbitrary irreducible representations of a semisimple Lie algebra, and also give formulas for some pairs $\mathfrak{h} \subset \mathfrak{g}$ for the decomposition of the restriction to $\mathfrak{h}$ of irreducible representations of $\mathfrak{g}$.

§25.1. Freudenthal's Multiplicity Formula

Freudenthal's formula gives a general way of computing the multiplicities of a representation, i.e., the dimensions of its weight spaces, by working down successively from the highest weight. The result is similar to (but more complicated than) what we did for $\mathfrak{sl}_3\mathbb{C}$ in Lecture 13, where we found the multiplicities along successive concentric hexagons in the weight diagram.

Let Γ_λ be the irreducible representation with highest weight λ, which will be fixed throughout this discussion. Let $n_\mu = n_\mu(\Gamma_\lambda)$ be the dimension of the weight space[1] of weight μ in Γ_λ, i.e., $\operatorname{Char}(\Gamma_\lambda) = \sum n_\mu e(\mu)$. Freudenthal gives a formula for n_μ in terms of multiplicities of weights that are higher than μ.

[1] In the literature, these multiplicities n_μ are often referred to as "inner multiplicities."

Proposition 25.1 (Freudenthal's Multiplicity Formula). *With the above notation,*

$$c(\mu)\cdot n_\mu(\Gamma_\lambda) = 2 \sum_{\alpha \in R^+} \sum_{k \geq 1} (\mu + k\alpha, \alpha) n_{\mu+k\alpha},$$

where $c(\mu) = \|\lambda + \rho\|^2 - \|\mu + \rho\|^2$.

Here $\|\beta\|^2 = (\beta, \beta)$, $(\ ,\ )$ is the Killing form, and ρ is half the sum of the positive roots.

Exercise 25.2*. Verify that $c(\mu)$ is positive if $\mu \neq \lambda$ and $n_\mu > 0$.

The proof of Freudenthal's formula uses a *Casimir operator*, denoted C. This is an endomorphism of any representation V of the semisimple Lie algebra $\mathfrak{g}$, and is constructed as follows. Take any basis $U_1, \ldots, U_r$ for $\mathfrak{g}$, and let $U_1', \ldots, U_r'$ be the dual basis with respect to the Killing form on $\mathfrak{g}$. Set

$$C = U_1 U_1' + \cdots + U_r U_r',$$

i.e., for any $v \in V$, $C(v) = \sum U_i \cdot (U_i' \cdot v)$.

Exercise 25.3. Verify that C is independent of the choice of basis[2].

The key fact is

Exercise 25.4*. Show that C commutes with every operation in $\mathfrak{g}$, i.e.,

$$C(X \cdot v) = X \cdot C(v) \quad \text{for all } X \in \mathfrak{g},\ v \in V.$$

The idea is to use a special basis for the construction of C, so that each term $U_i U_i'$ will act as multiplication by a constant on any weight space, and this constant can be calculated in terms of multiplicities. Then Schur's lemma can be applied to know that, in case V is irreducible, C itself is multiplication by a scalar. Taking traces will lead to a relation among multiplicities, and a little algebraic manipulation will give Freudenthal's formula.

The basis for $\mathfrak{g}$ to use is a natural one: Choose the basis $H_1, \ldots, H_n$ for the Cartan subalgebra $\mathfrak{h}$, where $H_i = H_{\alpha_i}$ corresponds to the simple root α_i, and let H_i' be the dual basis for the restriction of the Killing form to $\mathfrak{h}$. For each root α, choose a nonzero $X_\alpha \in \mathfrak{g}_\alpha$. The dual basis will then have X_α' in $\mathfrak{g}_{-\alpha}$. In fact, if we let $Y_\alpha \in \mathfrak{g}_{-\alpha}$ be the usual element so that X_α, Y_α, and $H_\alpha = [X_\alpha, Y_\alpha]$ are the canonical basis for the subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$ that they span, then

$$X_\alpha' = ((\alpha, \alpha)/2)\, Y_\alpha. \tag{25.5}$$

Exercise 25.6*. Verify (25.5) by showing that $(X_\alpha, Y_\alpha) = 2/(\alpha, \alpha)$.

[2] In fancy language, C is an element of the universal enveloping algebra of $\mathfrak{g}$, but we do not need this.

Now we have the Casimir operator

$$C = \sum H_i H_i' + \sum_{\alpha \in R} X_\alpha X_\alpha',$$

and we analyze the action of C on the weight space V_μ corresponding to weight μ for any representation V. Let $n_\mu = \dim(V_\mu)$. First we have

$$\sum H_i H_i' \text{ acts on } V_\mu \text{ by multiplication by } (\mu, \mu) = \|\mu\|^2. \tag{25.7}$$

Indeed, $H_i H_i'$ acts by multiplication by $\mu(H_i)\mu(H_i')$. If we write $\mu = \sum r_i \omega_i$, where the ω_i are the fundamental weights, then $\mu(H_i) = r_i$, and if $\mu = \sum r_i' \omega_i'$, with ω_i' the dual basis to ω_i, then similarly $\mu(H_i') = r_i'$. Hence $\sum \mu(H_i)\mu(H_i') = \sum r_i r_i' = (\mu, \mu)$, as asserted.

Now consider the action of $X_\alpha X_\alpha' = ((\alpha, \alpha)/2) X_\alpha Y_\alpha$ on V_μ. Restricting to the subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ and to the subrepresentation $\bigoplus_i V_{\mu + i\alpha}$ corresponding to the α-string through μ, we are in a situation which we know very well. Suppose this string is

$$V_\beta \oplus V_{\beta - \alpha} \oplus \cdots \oplus V_{\beta - m\alpha},$$

so $m = \beta(H_\alpha)$ [cf. (14.10)], and let k be the integer such that $\mu = \beta - k\alpha$. We assume for now that $k \le m/2$.

On the first term V_β, $X_\alpha Y_\alpha$ acts by multiplication by $m = \beta(H_\alpha) = 2(\beta, \alpha)/(\alpha, \alpha)$, so $X_\alpha X_\alpha'$ acts by multiplication by (β, α). In general, on the part of $V_{\beta - k\alpha}$ which is the image of V_β by multiplication by $(Y_\alpha)^k$, we know [cf. (11.5)] that $X_\alpha Y_\alpha$ acts by multiplication by $(k + 1)(m - k)$. This gives us a subspace of V_μ of dimension n_β on which $X_\alpha X_\alpha'$ acts by multiplication by

$$(k + 1)((\beta, \alpha) - k(\alpha, \alpha)/2) = (k + 1)((\mu, \alpha) + k(\alpha, \alpha)/2).$$

Now peel off the subrepresentation (over $\mathfrak{s}_\alpha$) of V spanned by V_β, and apply the same reasoning to what is left. We have a subspace of $V_{\beta - \alpha}$ of dimension $n_{\beta - \alpha} - n_\beta$ to which the same analysis can be made. From this we get a subspace of V_μ of dimension $n_{\beta - \alpha} - n_\beta$ on which $X_\alpha X_\alpha'$ acts by multiplication by

$$(k)((\mu, \alpha) + (k - 1)(\alpha, \alpha)/2).$$

Continuing to peel off subrepresentations, the space V_μ is decomposed into pieces on which $X_\alpha X_\alpha'$ acts by multiplication by a scalar. The trace of $X_\alpha X_\alpha'$ on V_μ is therefore the sum

$$\begin{aligned} n_\beta \cdot (k + 1)((\mu, \alpha) + k(\alpha, \alpha)/2) + (n_{\beta - \alpha} - n_\beta) \cdot (k)((\mu, \alpha) + (k - 1)(\alpha, \alpha)/2) \\ + \cdots + ((n_{\beta - k\alpha} - n_{\beta - (k-1)\alpha}) \cdot (1)((\mu, \alpha) + (0)(\alpha, \alpha)/2). \end{aligned}$$

Canceling in successive terms, this simplifies to

$$\operatorname{Trace}(X_\alpha X_\alpha'|_{V_\mu}) = \sum_{i=0}^{k} (\mu + i\alpha, \alpha) n_{\mu + i\alpha}. \tag{25.8}$$

One pleasant fact about this sum is that it may be extended to all $i \ge 0$, since $n_{\mu + i\alpha} = 0$ for $i > k$.

In case $k \geq m/2$, the computation is similar, peeling off representations from the other end, starting with $V_{\beta-m\alpha}$. The only difference is that the action of $X_\alpha Y_\alpha$ on $V_{\beta-m\alpha}$ is zero. The result is

$$\operatorname{Trace}(X_\alpha X'_\alpha|_{V_\mu}) = -\sum_{i=1}^{\infty} (\mu - i\alpha, \alpha) n_{\mu-i\alpha}. \tag{25.9}$$

Exercise 25.10. Show that $X_\alpha X'_\alpha = X_{-\alpha} X'_{-\alpha} + ((\alpha, \alpha)/2) H_\alpha$, and deduce (25.9) directly from (25.8) by replacing α by $-\alpha$.

In fact, (25.8) is valid for all μ and α, as we see from the identity

$$\sum_{i=-\infty}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} = 0. \tag{25.11}$$

Exercise 25.12*. Verify (25.11) by using the symmetry of the α-string through β.

Now we add the assumption that V is irreducible, so C is multiplication by some scalar c. Taking the trace of C on V_μ and adding, we get

$$c n_\mu = (\mu, \mu) n_\mu + \sum_{\alpha \in R} \sum_{i \geq 0} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}. \tag{25.13}$$

Note that when $i = 0$ the two terms for α and $-\alpha$ cancel each other, so the summation can begin at $i = 1$ instead. Rewriting this in terms of the positive weights, and using (25.11) the sums become

$$\sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} + \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu - i\alpha, \alpha) n_{\mu-i\alpha}$$

$$= n_\mu \sum_{\alpha \in R^+} (\mu, \alpha) + 2 \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}.$$

Summarizing, and observing that $\sum_{\alpha \in R^+} (\mu, \alpha) = (\mu, 2\rho)$, we have

$$c n_\mu = ((\mu, \mu) + (\mu, 2\rho)) n_\mu + 2 \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}.$$

Note that $(\mu, \mu) + (\mu, 2\rho) = (\mu + \rho, \mu + \rho) - (\rho, \rho) = \|\mu + \rho\|^2 - \|\rho\|^2$. To evaluate the constant we evaluate on the highest weight space V_λ, where $n_\lambda = 1$ and $n_{\lambda+i\alpha} = 0$ for $i > 0$. Hence,

$$c = (\lambda, \lambda) + (\lambda, 2\rho) = \|\lambda + \rho\|^2 - \|\rho\|^2. \tag{25.14}$$

Combining the preceding two equations yields Freudenthal's formula. □

Exercise 25.15. Apply Freudenthal's formula to the representations of $\mathfrak{sl}_3\mathbb{C}$ considered in §13.2, verifying again that the multiplicities are as prescribed on the hexagons and triangles.

Exercise 25.16. Use Freudenthal's formula to calculate multiplicities for the representations $\Gamma_{1,0}$, $\Gamma_{0,1}$, and $\Gamma_{2,0}$ of $(\mathfrak{g}_2)$.

§25.2. Proof of (WCF); the Kostant Multiplicity Formula

It is not unreasonable to anticipate that Weyl's character formula can be deduced from Freudenthal's inductive formula, but some algebraic manipulation is certainly required. Let

$$\chi_\lambda = \operatorname{Char}(\Gamma_\lambda) = \sum n_\mu e(\mu)$$

be the character of the irreducible representation with highest weight λ. Freudenthal's formula, in form (25.13), reads[3]

$$c \cdot \chi_\lambda = \sum_\mu (\mu, \mu) n_\mu e(\mu) + \sum_\mu \sum_{\alpha \in R} \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) n_{\mu + i\alpha} e(\mu),$$

where $c = \|\lambda + \rho\|^2 - \|\rho\|^2$. To get this to look anything like Weyl's formula, we must get rid of the inside sums over i. If α is fixed, they will disappear if we multiply by $e(\alpha) - 1$, as successive terms cancel:

$$(e(\alpha) - 1) \cdot \sum_\mu \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) n_{\mu + i\alpha} e(\mu) = \sum_\mu (\mu, \alpha) n_\mu e(\mu + \alpha).$$

Let $P = \prod_{\alpha \in R} (e(\alpha) - 1) = (e(\alpha) - 1) \cdot P_\alpha$, where $P_\alpha = \prod_{\beta \neq \alpha} (e(\beta) - 1)$. The preceding two formulas give

$$c \cdot P \cdot \chi_\lambda = P \cdot \sum_\mu (\mu, \mu) n_\mu e(\mu) + \sum_{\mu, \alpha} (\mu, \alpha) P_\alpha n_\mu e(\mu + \alpha). \tag{25.17}$$

Note also that

$$P = (-1)^r A_\rho \cdot A_\rho,$$

where r is the number of positive roots, so at least the formula now involves the ingredients that go into (WCF).

We want to prove (WCF): $A_\rho \cdot \chi_\lambda = A_{\lambda + \rho}$. We have seen in §24.1 that both sides of this equation are alternating, and that both have highest weight term $e(\lambda + \rho)$, with coefficient 1. On the right-hand side the only terms that appear are those of the form $\pm e(W(\lambda + \rho))$, for W in the Weyl group. To prove (WCF), it suffices to prove that the only terms appearing with nonzero coefficients in $A_\rho \cdot \chi_\lambda$ are these same $e(W(\lambda + \rho))$, for then the alternating property and the knowledge of the coefficient of $e(\lambda + \rho)$ determine all the coefficients. This can be expressed as:

[3] In this section we work in the ring $\mathbb{C}[\Lambda]$ of finite sums $\sum m_\mu e(\mu)$ with complex coefficients m_μ.

Claim. *The only terms $e(\nu)$ occurring in $A_\rho \cdot \chi_\lambda$ with nonzero coefficient are those with $\|\nu\| = \|\lambda + \rho\|$.*

To see that this is equivalent, note that by definition of A_ρ and χ_λ, the terms in $A_\rho \cdot \chi_\lambda$ are all of the form $\pm e(\nu)$, where $\nu = \mu + W(\rho)$, for μ a weight of Γ_λ and W in the Weyl group. But if $\|\mu + W(\rho)\| = \|\lambda + \rho\|$, since the metric is invariant by the Weyl group, this gives $\|W^{-1}(\mu) + \rho\| = \|\lambda + \rho\|$. But we saw in Exercise 25.2 that this cannot happen unless $\mu = W(\lambda)$, as required.

We are thus reduced to proving the claim. This suggests looking at the "Laplacian" operator that maps $e(\mu)$ to $\|\mu\|^2 e(\mu)$, that is, the map

$$\Delta\colon \mathbb{C}[\Lambda] \to \mathbb{C}[\Lambda]$$

defined by

$$\Delta\left(\sum m_\mu e(\mu)\right) = \sum (\mu, \mu) m_\mu e(\mu).$$

The claim is equivalent to the assertion that $F = A_\rho \cdot \chi_\lambda$ satisfies the "differential equation"

$$\Delta(F) = \|\lambda + \rho\|^2 F.$$

From the definition $\Delta(\chi_\lambda) = \sum (\mu, \mu) n_\mu e(\mu)$. And $\Delta(A_\rho) = \|\rho\|^2 A_\rho$. In general, since $\|W(\alpha)\| = \|\alpha\|$ for all $W \in \mathfrak{W}$,

$$\Delta(A_\alpha) = \sum (-1)^W \|W(\alpha)\|^2 e(W(\alpha)) = \|\alpha\|^2 A_\alpha.$$

So we would be in good shape if we had a formula for Δ of a product of two functions. One expects such a formula to take the form

$$\Delta(fg) = \Delta(f)g + 2(\nabla f, \nabla g) + f\Delta(g), \tag{25.18}$$

where ∇ is a "gradient," and $(\, , \,)$ is an "inner product." Taking $f = e(\mu)$, $g = e(\nu)$, we see that we need to have $(\nabla e(\mu), \nabla e(\nu)) = (\mu, \nu) e(\mu + \nu)$. There is indeed such a gradient and inner product. Define a homomorphism

$$\nabla\colon \mathbb{C}[\Lambda] \to \mathfrak{h}^* \otimes \mathbb{C}[\Lambda] = \operatorname{Hom}(\mathfrak{h}, \mathbb{C}[\Lambda])$$

by the formula $\nabla(e(\mu)) = \mu \cdot e(\mu)$, and define the bilinear form $(\, , \,)$ on $\mathfrak{h}^* \otimes \mathbb{C}[\Lambda]$ by the formula $(\alpha e(\mu), \beta e(\nu)) = (\alpha, \beta) e(\mu + \nu)$, where (α, β) is the Killing form on $\mathfrak{h}^*$.

Exercise 25.19. With these definitions, verify that (25.18) is satisfied, as well as the Leibnitz rule

$$\nabla(fg) = \nabla(f)g + f\nabla(g).$$

For example, $\nabla(\chi_\lambda) = \sum_\mu n_\mu \mu \cdot e(\mu)$, and, by the Leibnitz rule,

$$\nabla(P) = \sum_{\alpha \in R} P_\alpha \alpha \cdot e(\alpha).$$

But now look at formula (25.17). This reads

$$c \cdot P\chi_\lambda = P\Delta(\chi_\lambda) + (\nabla P, \nabla\chi_\lambda).$$

Since, also by the exercise, $\nabla(P) = 2(-1)^r A_\rho \nabla(A_\rho)$, we may cancel $(-1)^r A_\rho$ from each term in the equation, getting

$$c \cdot A_\rho \chi_\lambda = A_\rho \Delta(\chi_\lambda) + 2(\nabla A_\rho, \nabla\chi_\lambda).$$

By the identity (25.18), the right-hand side of this equation is

$$\Delta(A_\rho\chi_\lambda) - \Delta(A_\rho)\chi_\lambda = \Delta(A_\rho\chi_\lambda) - \|\rho\|^2 A_\rho \chi_\lambda.$$

Since $c = \|\lambda + \rho\|^2 - \|\rho\|^2$, this gives $\|\lambda + \rho\|^2 A_\rho \chi_\lambda = \Delta(A_\rho\chi_\lambda)$, which finishes the proof. □

We conclude this section with a proof of another general multiplicity formula, discovered by Kostant. It gives an elegant closed formula for the multiplicities, but at the expense of summing over the entire Weyl group (although as we will indicate below, there are many interesting cases where all but a few terms of the sum vanish). It also involves a kind of partition counting function. For each weight μ, let $P(\mu)$ be the number of ways to write μ as a sum of positive roots; set $P(0) = 1$. Equivalently,

$$\prod_{\alpha \in R^+} \frac{1}{1 - e(\alpha)} = \sum_\mu P(\mu) e(\mu). \tag{25.20}$$

Proposition 25.21. (Kostant's Multiplicity Formula). *The multiplicity $n_\mu(\Gamma_\lambda)$ of weight μ in the irreducible representation Γ_λ is given by*

$$n_\mu(\Gamma_\lambda) = \sum_{W \in \mathfrak{W}} (-1)^W P(W(\lambda + \rho) - (\mu + \rho)),$$

where ρ is half the sum of the positive roots.

Proof. Write $(A_\rho)^{-1} = e(-\rho)/\prod(1 - e(-\alpha)) = \sum_\nu P(\nu) e(-\nu - \rho)$. By (WCF),

$$\begin{aligned}\chi_\lambda = A_{\lambda+\rho}(A_\rho)^{-1} &= \sum_{W,\nu} (-1)^W e(W(\lambda + \rho)) P(\nu) e(-\nu - \rho) \\ &= \sum_{W,\nu} (-1)^W P(\nu) e(W(\lambda + \rho) - (\nu + \rho)) \\ &= \sum_{W,\mu} (-1)^W P(W(\lambda + \rho) - (\mu + \rho)) e(\mu),\end{aligned}$$

as seen by writing $\mu = W(\lambda + \rho) - (\nu + \rho)$. □

In fact, the proof shows that Kostant's formula is equivalent to Weyl's formula, cf. [Cart].

One way to interpret Kostant's formula, at least for weights μ close to the highest weight λ of Γ_λ, is as a sort of converse to Proposition 14.13(ii). Recall that this says that Γ_λ will be generated by the images of its highest weight vector v under successive applications of the generators of the negative root spaces; in practice, we used this fact to bound from above the multiplicities of various weights μ close to λ by counting the number of ways of getting from λ to μ by adding negative roots. The problem in making this precise was always that we did not know how many relations there were among these images, if any. Kostant's formula gives an answer: for example, if the difference $\lambda - \mu$ is small relative to λ, we see that the only nonzero term in the sum is the principle term, corresponding to $W = 1$; in this case the answer is that there are no relations other than the trivial ones $X(Y(v)) - Y(X(v)) = [X, Y](v)$. When μ gets somewhat smaller, other terms appear corresponding to single reflections W in the walls of the Weyl chamber for which $W(\lambda + \rho)$ is higher than $\mu + \rho$; we can think of these terms, which all appear with sign -1, as correction terms indicating the presence of relations. As μ gets smaller still, of course, more terms appear of both signs, and this viewpoint breaks down.

To see how this works in practice, the reader can for example carry out the analysis of the example at the end of §13.1.

Exercise 25.22* (Kostant). Prove the following formula for the function P, which can be used to calculate it inductively: $P(0) = 1$, and, for $\mu \neq 0$,

$$P(\mu) = -\sum_{W \neq 1} (-1)^W P(\mu + W(\rho) - \rho).$$

Exercise 25.23* (Racah). Deduce from Kostant's formula and the preceding exercise the following inductive formula for the multiplicities n_μ of μ in Γ_λ: $n_\mu = 1$ if $\mu = \lambda$, and if μ is any other weight of Γ_λ, then

$$n_\mu = -\sum_{W \neq 1} (-1)^W n_{\mu+\rho-W(\rho)}.$$

Show, in fact, that for any weight μ

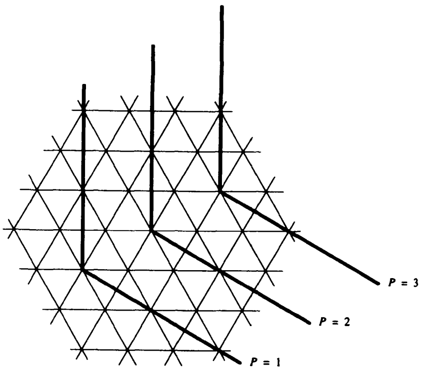
$$\sum_{W \in \mathfrak{W}} (-1)^W n_{\mu+\rho-W(\rho)} = \sum_{W'} (-1)^{W'},$$

where the second sum is over those $W' \in \mathfrak{W}$ such that $W'(\lambda + \rho) = \mu + \rho$.

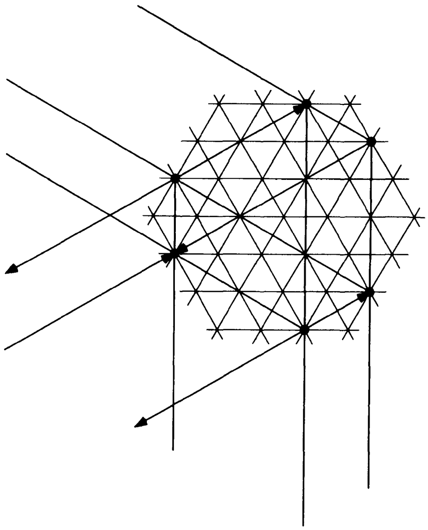
Note that Kostant's formula, more than any of the others, shows us directly the pattern of multiplicities in the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$. For one thing, it is easy to represent the function P diagrammatically: in the weight lattice of $\mathfrak{sl}_3\mathbb{C}$, the function $P(\mu)$ will be a constant 1 on the rays $\{aL_2 - aL_1\}_{a \geq 0}$ and $\{aL_3 - aL_2\}_{a \geq 0}$ through the origin in the direction of the two simple positive roots $L_2 - L_1$ and $L_3 - L_2$. It will have value 2 on the translates $\{aL_2 - (a+3)L_1\}_{a \geq -1}$ and $\{aL_3 - (a-3)L_2\}_{a \geq 2}$ of these two rays by the third positive root $L_3 - L_1$: for example, the first of these can be written as

$$\begin{aligned} aL_2 - (a+3)L_1 &= (a+1)\cdot(L_2 - L_1) + L_3 - L_1 \\ &= (a+2)\cdot(L_2 - L_1) + L_3 - L_2; \end{aligned}$$

and correspondingly its value will increase by 1 on each successive translate of these rays by $L_3 - L_1$. The picture is thus



Now, the prescription given in the Kostant formula for the multiplicities is to take six copies of this function flipped about the origin, translated so that the vertex of the outer shell lies at the points $w(\lambda + \rho) - \rho$ and take their alternating sum. Superimposing the six pictures we arrive at



which shows us clearly the hexagonal pattern of the multiplicities.

Exercise 25.24*. A nonzero dominant weight λ of a simple Lie algebra is called *minuscule* if $\lambda(H_\alpha) = 0$ or 1 for each positive root α.

(a) Show that if λ is minuscule, then every weight space of Γ_λ is one dimensional.

(b) Show that λ is minuscule if and only if all the weights of Γ_λ are conjugate under the Weyl group.
(c) Show that a minuscule weight must be one of the fundamental weights. Find the minuscule weights for each simple Lie algebra.

§25.3. Tensor Products and Restrictions To Subgroups

In the case of the general or special linear groups, we saw general formulas for describing how the tensor product $\Gamma_\lambda \otimes \Gamma_\mu$ of two irreducible representations decomposes:

$$\Gamma_\lambda \otimes \Gamma_\mu = \bigoplus_\nu N_{\lambda\mu\nu}\Gamma_\nu.$$

In these cases the multiplicities $N_{\lambda\mu\nu}$ can be described by a combinatorial formula: the Littlewood–Richardson rule. In general, such a decomposition is equivalent to writing

$$\chi_\lambda\chi_\mu = \sum_\nu N_{\lambda\mu\nu}\chi_\nu \tag{25.25}$$

in $\mathbb{Z}[\Lambda]$, where $\chi_\lambda = \mathrm{Char}(\Gamma_\lambda)$ denotes the character.[4] By Weyl's character formula, these multiplicities $N_{\lambda\mu\nu}$ are determined by the identity

$$A_{\lambda+\rho}\cdot A_{\mu+\rho} = \sum_\nu N_{\lambda\mu\nu}A_\rho\cdot A_{\nu+\rho}. \tag{25.26}$$

This formula gives an effective procedure for calculating the coefficients $N_{\lambda\mu\nu}$, if one that is tedious in practice: we can peel off highest weights, i.e., successively subtract from $A_{\lambda+\rho}\cdot A_{\mu+\rho}$ multiples of $A_\rho\cdot A_{\nu+\rho}$ for the highest ν that appears.

There are some explicit formulas for the other classical groups. R. C. King [Ki2] has showed that for both the symplectic or orthogonal groups, the multiplicities $N_{\lambda\mu\nu}$ are given by the formula

$$N_{\lambda\mu\nu} = \sum_{\zeta,\sigma,\tau} M_{\zeta\sigma\lambda}\cdot M_{\zeta\tau\mu}\cdot M_{\sigma\tau\nu}, \tag{25.27}$$

where the M's denote the Littlewood–Richardson multiplicities, i.e., the corresponding numbers for the general linear group, and the sum is over all partitions ζ, σ, τ. For other formulas for the classical groups, see [Mur1], [Wel, p. 230].

Exercise 25.28*. For $\mathfrak{so}_4\mathbb{C}$, show that all the nonzero multiplicities $N_{\lambda\mu\nu}$ are 1's, and these occur for ν in a rectangle with sides making 45° angles to the axes. Describe this rectangle.

[4] In the literature these multiplicities $N_{\lambda\mu\nu}$ are often called "outer multiplicities," and the problem of finding them, or decomposing the tensor product, the "Clebsch–Gordan" problem.

Steinberg has also given a general formula for the multiplicities $N_{\lambda\mu\nu}$. Since it involves a double summation over the Weyl group, using it in a concrete situation may be a challenge.

Proposition 25.29 (Steinberg's Formula). *The multiplicity of* Γ_ν *in* $\Gamma_\lambda \otimes \Gamma_\mu$ *is*

$$N_{\lambda\mu\nu} = \sum_{W,W'} (-1)^{WW'} P(W(\lambda + \rho) + W'(\mu + \rho) - \nu - 2\rho),$$

where the sum is over pairs $W, W' \in \mathfrak{W}$, *and* P *is the counting function appearing in Kostant's multiplicity formula.*

Exercise 25.30*. Prove Steinberg's formula by multiplying (25.25) by A_ρ, using (WCF) to get $\chi_\lambda A_{\mu+\rho} = \sum N_{\lambda\mu\nu} A_{\nu+\rho}$. Write out both sides, using Kostant's formula for χ_λ, and compute the coefficient of the term $e(\beta + \rho)$ on each side, for any β. This gives

$$\sum_{W,W'} (-1)^{WW'} P(W(\lambda + \rho) + W'(\mu + \rho) - \beta - 2\rho) = \sum_{W} (-1)^{W} N_{\lambda,\mu,W(\beta+\rho)-\rho}.$$

Show that for $\beta = \nu$ all the terms on the right are zero but $N_{\lambda\mu\nu}$.

Exercise 25.31 (Racah). Use the Steinberg and Kostant formulas to show that

$$N_{\lambda\mu\nu} = \sum_{W} (-1)^{W} n_{\nu+\rho-W(\mu+\rho)}(\Gamma_\lambda).$$

The following is the generalization of something we have seen several times:

Exercise 25.32. If λ and μ are dominant weights, and α is a simple root with $\lambda(H_\alpha)$ and $\mu(H_\alpha)$ not zero, show that $\lambda + \mu - \alpha$ is a dominant weight and $\Gamma_\lambda \otimes \Gamma_\mu$ contains the irreducible representation $\Gamma_{\lambda+\mu-\alpha}$ with multiplicity one. So

$$\Gamma_\lambda \otimes \Gamma_\mu = \Gamma_{\lambda+\mu} \oplus \Gamma_{\lambda+\mu-\alpha} \oplus \text{others}.$$

In case $\mu = \lambda$, with $\lambda(H_\alpha) \neq 0$, $\mathrm{Sym}^2(\Gamma_\lambda)$ contains $\Gamma_{\lambda+\mu}$, while $\wedge^2(\Gamma_\lambda)$ contains $\Gamma_{\lambda+\mu-\alpha}$.

Exercise 25.33. If $\lambda + \zeta$ is a dominant weight for each weight ζ of Γ_μ, show that the irreducible representations appearing in $\Gamma_\lambda \otimes \Gamma_\mu$ are exactly the $\Gamma_{\lambda+\zeta}$. In fact, with no assumptions, every component of $\Gamma_\lambda \otimes \Gamma_\mu$ always has this form. One can show that $N_{\lambda\mu\nu}$ is the dimension of

$$\{v \in (\Gamma_\lambda)_{\nu-\mu} \colon H_i^{l_i+1}(v) = 0, 1 \leq i \leq n, l_i = \mu(H_i)\}.$$

For this, see [Žel, §131].

For other general formulas for the multiplicities $N_{\lambda\mu\nu}$ see [Kem], [K-N], [Li], and [Kum1], [Kum2].

We have seen in Exercise 6.12 a formula for decomposing the representation Γ_λ of $GL_m\mathbb{C}$ when restricted to the subgroup $GL_{m-1}\mathbb{C}$. In this case the multiplicities of the irreducible components again have a simple combinatorial description. There are similar formulas for other classical groups. In the literature, such formulas are often called "branching formulas," or "modification rules." We will just state the analogues of this formula for the symplectic and orthogonal cases:

For $\mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{so}_{2n+1}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, the restriction is

$$\operatorname{Res}^{\mathfrak{so}_{2n+1}\mathbb{C}}_{\mathfrak{so}_{2n}\mathbb{C}}(\Gamma_\lambda) = \bigoplus \Gamma_{\bar{\lambda}}, \tag{25.34}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$ with

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1} \geq \lambda_n \geq |\bar{\lambda}_n|,$$

with the $\bar{\lambda}_i$ and λ_i simultaneously all integers or all half integers.

For $\mathfrak{so}_{2n-1}\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{so}_{2n}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq |\lambda_n|)$,

$$\operatorname{Res}^{\mathfrak{so}_{2n}\mathbb{C}}_{\mathfrak{so}_{2n-1}\mathbb{C}}(\Gamma_\lambda) = \bigoplus \Gamma_{\bar{\lambda}}, \tag{25.35}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1})$ with

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1} \geq |\lambda_n|,$$

with the $\bar{\lambda}_i$ and λ_i simultaneously all integers or all half integers.

For $\mathfrak{sp}_{2n-2}\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, the restriction is

$$\operatorname{Res}^{\mathfrak{sp}_{2n}\mathbb{C}}_{\mathfrak{sp}_{2n-2}\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.36}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1})$ with $\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_{n-1} \geq 0$, and the multiplicity $N_{\lambda\bar{\lambda}}$ is the number of sequences $p_1, \ldots, p_n$ of integers satisfying

$$\lambda_1 \geq p_1 \geq \lambda_2 \geq p_2 \geq \cdots \geq \lambda_n \geq p_n \geq 0$$

and

$$p_1 \geq \bar{\lambda}_1 \geq p_2 \cdots \geq p_{n-1} \geq \bar{\lambda}_{n-1} \geq p_n.$$

As in the case of $GL_n\mathbb{C}$, these formulas are equivalent to identities among symmetric polynomials. The reader may enjoy trying to work them out from this point of view, cf. Exercise 23.43 and [Boe]. A less computational approach is given in [Žel].

As we saw in the case of the general linear group, these branching rules can be used inductively to compute the dimensions of the weight spaces. For example, for $\mathfrak{so}_m\mathbb{C}$ consider the chain

$$\mathfrak{so}_m\mathbb{C} \supset \mathfrak{so}_{m-1}\mathbb{C} \supset \mathfrak{so}_{m-2}\mathbb{C} \supset \cdots \supset \mathfrak{so}_3\mathbb{C}.$$

Decomposing a representation successively from one layer to the next will finally write it as a sum of one-dimensional weight spaces, and the dimension can be read off from the number of "partitions" in chains that start with the given λ. The representations can be constructed from these chains, as described by Gelfand and Zetlin, cf. [Žel, §10].

Similarly, one can ask for formulas for decomposing restrictions for other inclusions, such as the natural embeddings: $Sp_{2n}\mathbb{C} \subset SL_{2n}\mathbb{C}$, $SO_m\mathbb{C} \subset SL_m\mathbb{C}$, $GL_m\mathbb{C} \times GL_n\mathbb{C} \subset GL_{m+n}\mathbb{C}$, $GL_m\mathbb{C} \times GL_n\mathbb{C} \subset GL_{mn}\mathbb{C}$, $SL_n\mathbb{C} \subset Sp_{2n}\mathbb{C}$, $SL_n\mathbb{C} \subset SO_{2n+1}\mathbb{C}$, $SL_n\mathbb{C} \subset SO_{2n}\mathbb{C}$, to mention just a few. Such formulas are determined in principle by computing what happens to generators of the representation rings, which is not hard: one need only decompose exterior or symmetric products of standard representations, cf. Exercise 23.31. A few closed formulas for decomposing more general representations can also be found in the literature. We state what happens when the irreducible representations of $GL_m\mathbb{C}$ are restricted to the orthogonal or symplectic subgroups, referring to [Lit3] for the proofs:

For $O_m\mathbb{C} \subset GL_m\mathbb{C}$, with $m = 2n$ or $2n+1$, given $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$,

$$\operatorname{Res}^{GL_m\mathbb{C}}_{O_m\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.37}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_n \geq 0)$, where

$$N_{\lambda\bar{\lambda}} = \sum_{\delta} N_{\delta\bar{\lambda}\lambda},$$

with $N_{\delta\bar{\lambda}\lambda}$ the Littlewood–Richardson coefficient, and the sum over all $\delta = (\delta_1 \geq \delta_2 \geq \cdots)$ with all δ_i even.

Exercise 23.38. Show that the representation $\Gamma_{(2,2)}$ of $GL_m\mathbb{C}$ restricts to the direct sum

$$\Gamma_{(2,2)} \oplus \Gamma_{(2)} \oplus \Gamma_{(0)}$$

over $O_m\mathbb{C}$. (This decomposition is important in differential geometry: the *Riemann–Christoffel* tensor has type (2, 2), and the above three components of its decomposition are the *conformal curvature* tensor, the *Ricci* tensor, and the *scalar* curvature, respectively.)

Similarly for $Sp_{2n}\mathbb{C} \subset GL_{2n}\mathbb{C}$,

$$\operatorname{Res}^{GL_{2n}\mathbb{C}}_{Sp_{2n}\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.39}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_n \geq 0)$, where

$$N_{\lambda\bar{\lambda}} = \sum_{\eta} N_{\eta\bar{\lambda}\lambda},$$

$N_{\eta\bar{\lambda}\lambda}$ is the Littlewood–Richardson coefficient, and the sum is over all $\eta = (\eta_1 = \eta_2 \geq \eta_3 = \eta_4 \geq \cdots)$ with each part occurring an even number of times.

It is perhaps worth pointing out the the decomposition of tensor products is a special case of the decomposition of restrictions: the exterior tensor product $\Gamma_\lambda \boxtimes \Gamma_\mu$ of two irreducible representations of G is an irreducible representation of $G \times G$, and the restriction of this to the diagonal embedding of G in $G \times G$ is the usual tensor product $\Gamma_\lambda \otimes \Gamma_\mu$.

There are also some general formulas, valid whenever $\mathfrak{g}$ is a semisimple Lie subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Assume that the Cartan subalgebra $\bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{h}$, so we have a restriction from $\mathfrak{h}^*$ to $\bar{\mathfrak{h}}^*$, and we assume the half-spaces determining positive roots are compatible. We write $\bar{\mu}$ for weights of $\bar{\mathfrak{g}}$, and we write $\mu \downarrow \bar{\mu}$ to mean that a weight μ of $\mathfrak{g}$ restricts to $\bar{\mu}$. Similarly write $\bar{W}$ for a typical element of the Weyl group of $\bar{\mathfrak{g}}$, and $\bar{\rho}$ for half the sum of its positive weights. If λ (resp. $\bar{\lambda}$) is a dominant weight for $\mathfrak{g}$ (resp. $\bar{\mathfrak{g}}$), let $N_{\lambda\bar{\lambda}}$ denote the multiplicity with which $\Gamma_{\bar{\lambda}}$ appears in the restriction of Γ_λ to $\bar{\mathfrak{g}}$, i.e.,

$$\operatorname{Res}(\Gamma_\lambda) = \bigoplus\nolimits_{\bar{\lambda}} N_{\lambda\bar{\lambda}} \Gamma_{\bar{\lambda}}.$$

Exercise 25.40*. Show that, for any dominant weight λ of $\mathfrak{g}$ and any weight $\bar{\mu}$ of $\bar{\mathfrak{g}}$,

$$\sum_{\mu \downarrow \bar{\mu}} n_\mu(\Gamma_\lambda) = \sum_{\bar{\lambda}} N_{\lambda\bar{\lambda}} n_{\bar{\mu}}(\Gamma_{\bar{\lambda}}).$$

Exercise 25.41* (Klimyk). Show that

$$N_{\lambda\bar{\lambda}} = \sum_{\bar{W}} (-1)^{\bar{W}} \sum_{\mu \downarrow \bar{\lambda} + \bar{\rho} - \bar{W}(\bar{\rho})} n_\mu(\Gamma_\lambda).$$

Exercise 25.42. Show that if the formula of the preceding exercise is applied to the diagonal embedding of $\mathfrak{g}$ in $\mathfrak{g} \times \mathfrak{g}$, then the Racah formula of Exercise 25.31 results.

For additional formulas of a similar vein, as well as discussions of how they can be implemented on a computer, there are several articles in *SIAM J. Appl. Math.* **25**, 1973.

Finally, we note that it is possible, for any semisimple Lie algebra $\mathfrak{g}$, to make the direct sum of all its irreducible representations into a commutative algebra, generalizing constructions we saw in Lectures 15, §17, and §19. Let $\Gamma_{\omega_1}, \ldots, \Gamma_{\omega_n}$ be the irreducible representations corresponding to the fundamental weights $\omega_1, \ldots, \omega_n$. Let

$$A^\bullet = \operatorname{Sym}^\bullet(\Gamma_{\omega_1} \oplus \cdots \oplus \Gamma_{\omega_n}).$$

This is a commutative graded algebra, the direct sum of pieces

$$A^{\mathbf{a}} = \bigoplus_{a_1, \ldots, a_n} \operatorname{Sym}^{a_1}(\Gamma_{\omega_1}) \otimes \cdots \otimes \operatorname{Sym}^{a_n}(\Gamma_{\omega_n}),$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ is an n-tuple of non-negative integers. Then $A^{\mathbf{a}}$ is the direct sum of the irreducible representation Γ_λ whose highest weight is $\lambda = \sum a_i \omega_i$, and a sum $J^{\mathbf{a}}$ of representations whose highest weight is strictly smaller. As before, weight considerations show that $J^\bullet = \bigoplus_{\mathbf{a}} J^{\mathbf{a}}$ is an ideal in $A^\bullet$, so the quotient

$$A^\bullet/J^\bullet = \bigoplus_{\lambda} \Gamma_\lambda$$

is the direct sum of all the irreducible representations. The product

$$\Gamma_\lambda \otimes \Gamma_\mu \to \Gamma_{\lambda+\mu}$$

in this ring is often called *Cartan multiplication*; note that the fact that $\Gamma_{\lambda+\mu}$ occurs once in the tensor product determines such a projection, but only up to multiplication by a scalar.

Using ideas of §25.1, it is possible to give generators for the ideal $J^\bullet$. If C is the Casimir operator, we know that C acts on all representations and is multiplication by the constant $c_\lambda = (\lambda, \lambda) + (2\lambda, \rho)$ on the irreducible representation with highest weight λ. Therefore, if $\lambda = \sum a_i\omega_i$, the endomorphism $C - c_\lambda I$ of $A^\bullet$ vanishes on the factor Γ_λ, and on each of the representations Γ_μ of lower weight μ it is multiplication by $c_\mu - c_\lambda \neq 0$ [cf. (25.2)]. It follows that

$$J^\bullet = \operatorname{Image}(C - c_\lambda I\colon A^\bullet \to A^\bullet).$$

Exercise 25.43*. Write $C = \sum U_i U_i'$ as in §25.1. Show that for $v_1, \ldots, v_m$ vectors in the fundamental weight spaces, with $v_j \in \Gamma_{\alpha_j}$ and $\sum \alpha_j = \sum a_i\omega_i$, the element $(C - c_\lambda I)(v_1 \cdot v_2 \cdot \ldots \cdot v_m)$ is the sum over all pairs j, k, with $1 \leq j < k \leq m$, of the terms

$$\left(\sum_i (U_i(v_j)\cdot U_i'(v_k) + U_i'(v_j)\cdot U_i(v_k)) - 2(\alpha_j, \alpha_k)v_j\cdot v_k\right)\cdot \prod_{l\neq j,k} v_l.$$

From this exercise follows a theorem of Kostant: $J^\bullet$ is generated by the elements

$$\sum_i (U_i(v)\cdot U_i'(w) + U_i'(v)\cdot U_i(w)) - 2(\alpha, \beta)v\cdot w$$

for $v \in \Gamma_\alpha$, $w \in \Gamma_\beta$, with α and β fundamental roots. For the classical Lie algebras, this formula can be used to find concrete realizations of the ring. If one wants a similar ring for a semisimple Lie group, one has the same ring, of course, when the group is simply connected; this leads to the ring described in Lectures 15 and 17 for $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$. For $\mathrm{SO}_m\mathbb{C}$, little change is needed when m is odd, but there is more work for m even. Details can be found in [L-T].

LECTURE 26

Real Lie Algebras and Lie Groups

In this lecture we indicate how to complete the last step in the process outlined at the beginning of Part II: to take our knowledge of the classification and representation theory of complex algebras and groups and deduce the corresponding statements in the real case. We do this in the first section, giving a list of the simple classical real Lie algebras and saying a few words about the corresponding groups and their (complex) representations. The existence of a compact group whose Lie algebra has as complexification a given semisimple complex Lie algebra makes it possible to give another (indeed, the original) way to prove the Weyl character formula; we sketch this in §26.2. Finally, we can ask in regard to real Lie groups G a question analogous to one asked for the representations of finite groups in §3.5: which of the complex representations V of G actually come from real ones. We answer this in the most commonly encountered cases in §26.3. In this final lecture, proofs, when we attempt them, are generally only sketched and may require more than the usual fortitude from the reader.

§26.1. Classification of Real Simple Lie Algebras and Groups

Having described the semisimple complex Lie algebras, we now address the analogous problem for real Lie algebras. Since the complexification $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ of a semisimple real Lie algebra $\mathfrak{g}_0$ is a semisimple complex Lie algebra and we have classified those, we are reduced to the problem of describing the *real forms* of the complex semisimple Lie algebras: that is, for a given complex Lie algebra $\mathfrak{g}$, finding all real Lie algebras $\mathfrak{g}_0$ with

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \underset{\iota}{\cong} \mathfrak{g}.$$

We saw many of the real forms of the classical complex Lie groups and algebras back in Lectures 7 and 8. In this section we will indicate one way to approach the question systematically, but we will only include sketches of proofs.

To get the idea of what to expect, let us work out real forms of $\mathfrak{sl}_2\mathbb{C}$ in detail. To do this, suppose $\mathfrak{g}_0$ is any real Lie subalgebra of $\mathfrak{sl}_2\mathbb{C}$, with $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2\mathbb{C}$. The natural thing to do is to try to carry out our analysis of semisimple Lie algebras for the real Lie algebra $\mathfrak{g}_0$: that is, find an element $H \in \mathfrak{g}_0$ such that $\mathrm{ad}(H)$ acts semisimply on $\mathfrak{g}_0$, decompose $\mathfrak{g}_0$ into eigenspaces, and so on. The first part of this presents no problem: since the subset of $\mathfrak{sl}_2\mathbb{C}$ of non-semisimple matrices is a proper algebraic subvariety, it cannot contain the real subspace $\mathfrak{g}_0 \subset \mathfrak{sl}_2\mathbb{C}$, so that we can certainly find a semisimple $H \in \mathfrak{g}_0$.

The next thing is to consider the eigenspaces of $\mathrm{ad}(H)$ acting on $\mathfrak{g}$. Of course, $\mathrm{ad}(H)$ has one eigenvalue 0, corresponding to the eigenspace $\mathfrak{h}_0 = \mathbb{R}\cdot H$ spanned by H. The remaining two eigenvalues must then sum to zero, which leaves just two possibilities:

(i) $\mathrm{ad}(H)$ has eigenvalues λ and $-\lambda$, for λ a nonzero real number; multiplying H by a real scalar, we can take $\lambda = 2$. In this case we obtain a decomposition of the vector space $\mathfrak{g}_0$ into one-dimensional eigenspaces

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}.$$

We can then choose $X \in \mathfrak{g}_2$ and $Y \in \mathfrak{g}_{-2}$; the standard argument then shows that the bracket $[X, Y]$ is a nonzero multiple of H, which we may take to be 1 by rechoosing X and Y. We thus have the real form $\mathfrak{sl}_2\mathbb{R}$, with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(ii) $\mathrm{ad}(H)$ has eigenvalues $i\lambda$ and $-i\lambda$ for λ some nonzero real number; again, adjusting H by a real scalar we may take $\lambda = 1$. In this case, of course, there are no real eigenvectors for the action of $\mathrm{ad}(H)$ on $\mathfrak{g}_0$; but we can decompose $\mathfrak{g}_0$ into the direct sum of $\mathfrak{h}_0$ and the two-dimensional subspace $\mathfrak{g}_{\{i,-i\}}$ corresponding to the pair of eigenvalues i and $-i$. We may then choose a basis B and C for $\mathfrak{g}_{\{i,-i\}}$ with

$$[H, B] = C \quad \text{and} \quad [H, C] = -B.$$

The commutator $[B, C]$ will then be a nonzero multiple of H, which we may take to be either H or $-H$ (we can multiply B and C simultaneously by a scalar μ, which multiplies the commutator $[B, C]$ by μ^2). In the latter case, we see that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}_2\mathbb{R}$ again: these are the relations we get if we take as basis for $\mathfrak{sl}_2\mathbb{C}$ the three vectors

$$H = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, if the commutator $[B, C] = H$, we do get a new example: $\mathfrak{g}_0$ is in this case isomorphic to the algebra

$$\mathfrak{su}_2 = \{A\colon {}^t\bar{A} = -A \text{ and } \operatorname{trace}(A) = 0\} \subset \mathfrak{sl}_2\mathbb{C},$$

which has as basis

$$H = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}.$$

Exercise 26.1. Carry out this analysis for the real Lie algebras $\mathfrak{so}_3\mathbb{R}$ and $\mathfrak{so}_{2,1}\mathbb{R}$. In particular, give an isomorphism of each with either $\mathfrak{sl}_2\mathbb{R}$ or $\mathfrak{su}_2$.

This completes our analysis of the real forms of $\mathfrak{sl}_2\mathbb{C}$. In the general case, we can try to apply a similar analysis, and indeed at least one aspect generalizes: given a real form $\mathfrak{g}_0 \subset \mathfrak{g}$ of the complex semisimple Lie algebra $\mathfrak{g}$, we can find a real subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ such that $\mathfrak{h}_0 \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$; this is called a *Cartan subalgebra* of $\mathfrak{g}_0$. There is a further complication in the case of Lie algebras of rank 2 or more: the values on $\mathfrak{h}_0$ of a root $\alpha \in R$ of $\mathfrak{g}$ need not be either all real or all purely imaginary. We, thus, need to consider the root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{\bar{\alpha}}$, $\mathfrak{g}_{-\alpha}$, and $\mathfrak{g}_{-\bar{\alpha}}$, and the subalgebra they generate, at the same time. Moreover, as we saw in the above example, whether the values of the roots $\alpha \in R$ of $\mathfrak{g}$ on the real subspace $\mathfrak{h}_0$ are real, purely imaginary, or neither will in general depend on the choice of $\mathfrak{h}_0$.

Exercise 26.2*. In the case of $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{R} \subset \mathfrak{g} = \mathfrak{sl}_3\mathbb{C}$, suppose we choose as Cartan subalgebra $\mathfrak{h}_0$ the space spanned over $\mathbb{R}$ by the elements

$$H_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Show that this is indeed a Cartan subalgebra, and find the decomposition of $\mathfrak{g}$ into eigenspaces for the action of $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$. In particular, find the roots of $\mathfrak{g}$ as linear functions on $\mathfrak{h}$, and describe the corresponding decomposition of $\mathfrak{g}_0$.

Judging from these examples, it is probably prudent to resist the temptation to try to carry out an analysis of real semisimple Lie algebras via an analogue of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$ in this case. Rather, in the present book, we will do two things. First, we will give the statement of the classification theorem for the real forms of the classical algebras—that is, we will list all the simple real Lie algebras whose complexifications are classical algebras. Second, we will focus on two distinguished real forms possessed by any real semisimple Lie algebra, the *split form* and the *compact form*. These are the two forms that you see most often; and the existence of the latter in particular will be essential in the following section.

For the first, it turns out to be enough to work out the complexifications $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0 \oplus i \cdot \mathfrak{g}_0$ of the real Lie algebras $\mathfrak{g}_0$ we know. The list is:

Real Lie algebra	Complexification
$\mathfrak{sl}_n\mathbb{R}$	$\mathfrak{sl}_n\mathbb{C}$
$\mathfrak{sl}_n\mathbb{C}$	$\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_n\mathbb{C}$
$\mathfrak{sl}_n\mathbb{H} = \mathfrak{gl}_n\mathbb{H}/\mathbb{R}$	$\mathfrak{sl}_{2n}\mathbb{C}$
$\mathfrak{so}_{p,q}\mathbb{R}$	$\mathfrak{so}_{p+q}\mathbb{C}$
$\mathfrak{so}_n\mathbb{C}$	$\mathfrak{so}_n\mathbb{C} \times \mathfrak{so}_n\mathbb{C}$
$\mathfrak{sp}_{2n}\mathbb{R}$	$\mathfrak{sp}_{2n}\mathbb{C}$
$\mathfrak{sp}_{2n}\mathbb{C}$	$\mathfrak{sp}_{2n}\mathbb{C} \times \mathfrak{sp}_{2n}\mathbb{C}$
$\mathfrak{su}_{p,q}$	$\mathfrak{sl}_{p+q}\mathbb{C}$
$\mathfrak{u}_{p,q}\mathbb{H}$	$\mathfrak{sp}_{2(p+q)}\mathbb{C}$
$\mathfrak{u}_n^*\mathbb{H}$	$\mathfrak{so}_{2n}\mathbb{C}$

The last two in the left-hand column are the Lie algebras of the groups $U_{p,q}\mathbb{H}$ and $U_n^*\mathbb{H}$ of automorphisms of a quaternionic vector space preserving a Hermitian form with signature (p, q), and a skew-symmetric Hermitian form, respectively.

We should first verify that the algebras on the right are indeed the complexifications of those on the left. Some are obvious, such as the complexification

$$(\mathfrak{sl}_n\mathbb{R})_{\mathbb{C}} = \mathfrak{sl}_n\mathbb{R} \oplus i \cdot \mathfrak{sl}_n\mathbb{R} = \mathfrak{sl}_n\mathbb{C}.$$

The same goes for $\mathfrak{so}_{p,q}\mathbb{R}$ and $\mathfrak{sp}_{2n}\mathbb{R}$.

Next, consider the complexification of

$$\mathfrak{su}_n = \{A \in \mathfrak{sl}_n\mathbb{C} : {}^t\bar{A} = -A\}.$$

To see that $\mathfrak{sl}_n\mathbb{C} = \mathfrak{su}_n \oplus i \cdot \mathfrak{su}_n$, let $M \in \mathfrak{sl}_n\mathbb{C}$, and write

$$M = \tfrac{1}{2}(M - {}^t\overline{M}) + \tfrac{1}{2}(M + {}^t\overline{M}) = \tfrac{1}{2}A + \tfrac{1}{2}B;$$

then $A \in \mathfrak{su}_n$, $iB \in \mathfrak{su}_n$, and $M = \frac{1}{2}A - i(i/2)B$.

The general case of $\mathfrak{su}_{p,q} \subset \mathfrak{sl}_{p+q}\mathbb{C}$ is similar: if the form is given by $(x, y) = {}^t\bar{x}Qy$, then $\mathfrak{su}_{p,q} = \{A : {}^t\bar{A}Q = -QA\}$. Writing $M \in \mathfrak{sl}_{p+q}\mathbb{C}$ in the form

$$M = \tfrac{1}{2}(M - Q \cdot {}^t\overline{M} \cdot Q) - i \cdot (\tfrac{1}{2}(iM + iQ \cdot {}^t\overline{M} \cdot Q))$$

and using $\bar{Q} = {}^tQ = Q^{-1} = Q$, one sees that $M \in \mathfrak{su}_{p,q} \oplus i \cdot \mathfrak{su}_{p,q}$.

For the complexification of $\mathfrak{sl}_m\mathbb{C}$, embed $\mathfrak{sl}_m\mathbb{C}$ in $\mathfrak{sl}_m\mathbb{C} \times \mathfrak{sl}_m\mathbb{C}$ by $A \mapsto (A, \bar{A})$. Given any pair (B, C), write

$$\begin{aligned}(B, C) &= \tfrac{1}{2}(B + \bar{C}, \bar{B} + C) + \tfrac{1}{2}(B - \bar{C}, -\bar{B} + C)\\ &= \tfrac{1}{2}(B + \bar{C}, \bar{B} + C) - i \cdot (\tfrac{1}{2}(iB + \overline{iC}, \overline{iB} + iC).\end{aligned}$$

For the quaternionic Lie algebra, from the description of $\mathrm{GL}_n\mathbb{H}$ we saw in Lecture 7, we have

$$\mathfrak{gl}_n\mathbb{H} = \{A \in \mathfrak{gl}_{2n}\mathbb{C} : AJ = J\bar{A}\},$$

with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. As before, for $M \in \mathfrak{gl}_{2n}\mathbb{C}$, we can write

$$M = \tfrac{1}{2}(M - J \cdot \overline{M} \cdot J) - i \cdot (\tfrac{1}{2}(iM + iJ \cdot \overline{iM} \cdot J))$$

to see that $\mathfrak{gl}_n\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{gl}_{2n}\mathbb{C}$.

Exercise 26.3. Verify the rest of the list.

The theorem, which also goes back to Cartan, is that *this includes the complete list of simple real Lie algebras associated to the classical complex types* (A_n)–(D_n). In fact, there are an additional 17 simple real Lie algebras associated with the five exceptional Lie algebras. The proof of this theorem is rather long, and we refer to the literature (cf. [H-S], [Hel], [Ar]) for it.

Split Forms and Compact Forms

Rather than try to classify in general the real forms $\mathfrak{g}_0$ of a semisimple Lie algebra $\mathfrak{g}$, we would like to focus here on two particular forms that are possessed by every semisimple Lie algebra and that are by far the most commonly dealt with in practice: the *split form* and the *compact form.*

These represent the two extremes of behavior of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$ with respect to the real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$. To begin with, the *split form* of $\mathfrak{g}$ is a form $\mathfrak{g}_0$ such that there exists a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ (that is, a subalgebra whose complexification $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$) whose action on $\mathfrak{g}_0$ has all real eigenvalues—i.e., such that all the roots $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}$) assume all real values on the subspace $\mathfrak{h}_0$. In this case we have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus (\bigoplus \mathfrak{i}_\alpha)$$

of $\mathfrak{g}_0$ into $\mathfrak{h}_0$ and one-dimensional eigenspaces $\mathfrak{i}_\alpha$ for the action of $\mathfrak{h}_0$ (each $\mathfrak{i}_\alpha$ will just be the intersection of the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$ with $\mathfrak{g}_0$); each pair $\mathfrak{i}_\alpha$ and $\mathfrak{i}_{-\alpha}$ will generate a subalgebra isomorphic to $\mathfrak{sl}_2\mathbb{R}$. As we will see momentarily, this uniquely characterizes the real form $\mathfrak{g}_0$ of $\mathfrak{g}$.

By contrast, in the *compact form* all the roots $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}$) assume all purely imaginary values on the subspace $\mathfrak{h}_0$. We accordingly have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus (\bigoplus \mathfrak{l}_\alpha)$$

of $\mathfrak{g}_0$ into $\mathfrak{h}_0$ and two-dimensional spaces on which $\mathfrak{h}_0$ acts by rotation (each $\mathfrak{l}_\alpha$ will just be the intersection of the root space $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with $\mathfrak{g}_0$); each $\mathfrak{l}_\alpha$ will generate a subalgebra isomorphic to $\mathfrak{su}_2$.

The existence of the split form of a semisimple complex Lie algebra was already established in Lecture 21: one way to construct a real—even rational

—form $\mathfrak{g}_0$ of a semisimple Lie algebra $\mathfrak{g}$ is by starting with any generator X_{α_i} for the root space for each positive simple root α_i, completing it to standard basis X_{α_i}, Y_{α_i}, and $H_i = [X_{\alpha_i}, Y_{\alpha_i}]$ for the corresponding $\mathfrak{s}_{\alpha_i} = \mathfrak{sl}_2\mathbb{C}$, and taking $\mathfrak{g}_0$ to be the real subalgebra generated by these elements. Choosing a way to write each positive root as a sum of simple roots even determined a basis $\{H_i \in \mathfrak{h}, X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}\}$ for $\mathfrak{g}_0$, as in (21.20). The Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is the real span of these H_i. Note that once $\mathfrak{h}$ is fixed for $\mathfrak{g}$, the real subalgebra $\mathfrak{h}_0$ is uniquely determined as the span of the H_α for all roots α. The algebra $\mathfrak{g}_0$ is determined up to isomorphism; it is sometimes called the *natural* real form of $\mathfrak{g}$. Note that this also demonstrates the uniqueness of the split form: it is the only real form $\mathfrak{g}_0$ of $\mathfrak{g}$ that has a Cartan subalgebra $\mathfrak{h}_0$ acting on $\mathfrak{g}_0$ with all real eigenvalues.

As for the compact form of a semisimple Lie algebra, it owes much of its significance (as well as its name) to the last condition in

Proposition 26.4. *Suppose $\mathfrak{g}$ is any complex semisimple Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{g}$ a real form of $\mathfrak{g}$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$ the corresponding Cartan subalgebra of $\mathfrak{g}$. The following are equivalent*:

(i) *Each root $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ assumes purely imaginary values on $\mathfrak{h}_0$, and for each root α the subalgebra of $\mathfrak{g}_0$ generated by the intersection $\mathfrak{l}_\alpha$ of $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ with $\mathfrak{g}_0$ is isomorphic to $\mathfrak{su}_2$;*
(ii) *The restriction to $\mathfrak{g}_0$ of the Killing form of $\mathfrak{g}$ is negative definite;*
(iii) *The real Lie group G_0 with Lie algebra $\mathfrak{g}_0$ is compact.*

In (iii), G_0 can be taken to be the adjoint form of $\mathfrak{g}_0$. However, a theorem of Weyl ensures that the fundamental group of any such G_0 is finite, so the condition is independent of the choice of G_0. Note also that, by the equivalence with (ii) and (iii), the condition (i) must be independent of the choice of Cartan subalgebra $\mathfrak{h}_0$. This is in contrast with the split case, where we require only that there exist a Cartan subalgebra whose action on $\mathfrak{g}$ has all real eigenvalues; as we saw in the case of $\mathfrak{sl}_2\mathbb{R}$, in the split case a different $\mathfrak{h}_0$ may have imaginary eigenvalues.

Proof. We start by showing that the first condition implies the second; this will follow from direct observation. To begin with, the value of the Killing form on $H \in \mathfrak{h}_0$ is visibly

$$B(H, H) = \sum (\alpha(H))^2 < 0.$$

Next, the subspaces $\mathfrak{l}_\alpha$ are orthogonal to one another with respect to B, so it remains only to verify $B(Z, Z) < 0$ for a general member $Z \in \mathfrak{l}_\alpha$. To do this, let X and Y be generators of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$ respectively, chosen so as to form, together with their commutator $H = [X, Y]$ a standard basis for $\mathfrak{sl}_2\mathbb{C}$. By the analysis of real forms of $\mathfrak{sl}_2\mathbb{C}$ above, we may take as generators of the algebra generated by $\mathfrak{l}_\alpha$ the elements iH, $U = X - Y$ and $V = iX + iY$. If we set

$$Z = aU + bV = (a + ib)\cdot X + (-a + ib)\cdot Y,$$

then we have

$$\begin{aligned}\mathrm{ad}(Z) \circ \mathrm{ad}(Z) = (a + ib)^2\, &\mathrm{ad}(X) \circ \mathrm{ad}(X)\\ &- (a^2 + b^2)(\mathrm{ad}(X) \circ \mathrm{ad}(Y) + \mathrm{ad}(Y) \circ \mathrm{ad}(X))\\ &+ (a - ib)^2\, \mathrm{ad}(Y) \circ \mathrm{ad}(Y).\end{aligned}$$

Now, $\mathrm{ad}(X) \circ \mathrm{ad}(X)$ and $\mathrm{ad}(Y) \circ \mathrm{ad}(Y)$ have no trace, so we can write

$$\mathrm{trace}(\mathrm{ad}(Z) \circ \mathrm{ad}(Z)) = -2 \cdot (a^2 + b^2) \cdot \mathrm{trace}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)). \tag{26.5}$$

By direct examination, in the representation $\mathrm{Sym}^n V$ of $\mathfrak{sl}_2\mathbb{C}$, $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ acts by multiplication by $(n - \lambda)(n + \lambda - 2)/4 \geq 0$ on the λ-eigenspace for H, from which we deduce that the right-hand side of (26.5) is negative.

Next, we show that the second condition implies the third. This is immediate: the adjoint form G_0 is the connected component of the identity of the group $\mathrm{Aut}(\mathfrak{g}_0)$. In particular, it is a closed subgroup of the adjoint group of $\mathfrak{g}$, and it acts faithfully on the real vector space $\mathfrak{g}_0$, preserving the bilinear form B. If B is negative definite it follows that G_0 is a closed subgroup of the orthogonal group $\mathrm{SO}_m\mathbb{R}$, which is compact.

Finally, if we know that G_0 is compact, by averaging we can construct a positive definite inner product on $\mathfrak{g}_0$ invariant under the action of G_0. For any X in $\mathfrak{g}_0$, $\mathrm{ad}(X)$ is represented by a skew-symmetric matrix $A = (a_{i,j})$ with respect to an orthonormal basis of $\mathfrak{g}_0$ (cf. (14.23)), so $B(X, X) = \mathrm{Tr}(A \circ A) = \sum_{i,j} a_{i,j}a_{j,i} = -\sum a_{i,j}^2 \leq 0$. In particular, the eigenvalues of $\mathrm{ad}(X)$ must be purely imaginary. Therefore $\alpha(\mathfrak{h}_0) \subset i\mathbb{R}$ and $\bar{\alpha} = -\alpha$ for any root α, from which (i) follows. □

We now claim that *every semisimple complex Lie algebra has a unique compact form.* To see this we need an algebraic notion which is, in fact, crucial to the classification theorem mentioned above: that of *conjugate linear involution.* If $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of a real Lie algebra $\mathfrak{g}_0$, there is a map $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ which takes $x \otimes z$ to $x \otimes \bar{z}$ for $x \in \mathfrak{g}_0$ and $z \in \mathbb{C}$; it is conjugate linear, preserves Lie brackets, and σ^2 is the identity. The real algebra $\mathfrak{g}_0$ is the fixed subalgebra of σ, and conversely, given such a conjugate linear involution σ of a complex Lie algebra $\mathfrak{g}$, its fixed algebra $\mathfrak{g}^\sigma$ is a real form of $\mathfrak{g}$. To prove the claim, we start with the split, or natural form, as constructed in Lecture 21 and referred to above. With a basis for $\mathfrak{g}$ chosen as in this construction, it is not hard to show that there is a unique Lie algebra automorphism φ of $\mathfrak{g}$ that takes each element of $\mathfrak{h}$ to its negative and takes each X_α to Y_α (this follows from Claim 21.25). This automorphism φ is a complex linear involution which preserves the real subalgebra $\mathfrak{g}_0$. This automorphism commutes with the associated conjugate linear σ. The composite $\sigma\varphi = \varphi\sigma$ is a conjugate linear involution, from which it follows that its fixed part $\mathfrak{g}_c = \mathfrak{g}^{\sigma\varphi}$ is another real form of $\mathfrak{g}$. This has Cartan subalgebra $\mathfrak{h}_c = \mathfrak{h}^{\sigma\varphi} = i \cdot \mathfrak{h}_0$. We have seen that the restriction of the Killing form to $\mathfrak{h}_0$ is positive definite. It follows that its restriction to $\mathfrak{h}_c$ is negative definite, and hence that $\mathfrak{g}_c$ is a compact form of $\mathfrak{g}$. Finally, this construction of $\mathfrak{g}_c$ from $\mathfrak{g}_0$ is reversible, and from this one can deduce the uniqueness of the compact form.

We may see directly from this construction that

$$\mathfrak{g}_c = \mathfrak{h}_c \oplus \bigoplus_{\alpha \in R^+} \mathfrak{l}_\alpha,$$

where $\mathfrak{l}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})^{\sigma\varphi}$ is a real plane with $\mathfrak{l}_\alpha \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ and $[\mathfrak{h}_c, \mathfrak{l}_\alpha] \subset \mathfrak{l}_\alpha$.

Exercise 26.6. Verify that $\{A_j = i \cdot H_j \colon 1 \le j \le n\}$ is a basis for $\mathfrak{h}_c$, $\{B_\alpha = X_\alpha - Y_\alpha, C_\alpha = i \cdot (X_\alpha + Y_\alpha)\}$ is a basis for $\mathfrak{l}_\alpha$, and the action is given by

$$[A_j, B_\alpha] = p \cdot C_\alpha \quad \text{and} \quad [A_j, C_\alpha] = -p \cdot B_\alpha,$$

where p is the integer $\alpha(H_j)$. In particular, $\mathfrak{h}_c$ acts by rotations on the planes $\mathfrak{l}_\alpha$.

Our classical Lie algebras $\mathfrak{g}$ all came equipped with a natural real form $\mathfrak{g}_0$, and with a basis of the above type. These split forms are:

Complex simple Lie algebra	Split form
$\mathfrak{sl}_{n+1}\mathbb{C}$	$\mathfrak{sl}_{n+1}\mathbb{R}$
$\mathfrak{so}_{2n+1}\mathbb{C}$	$\mathfrak{so}_{n+1,n}$
$\mathfrak{sp}_{2n}\mathbb{C}$	$\mathfrak{sp}_{2n}\mathbb{R}$
$\mathfrak{so}_{2n}\mathbb{C}$	$\mathfrak{so}_{n,n}$

Exercise 26.7. For each of these split forms, find the corresponding compact form $\mathfrak{g}_c$.

Exercise 26.8. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. Show that a subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is a Cartan subalgebra if and only if it is a maximal abelian subalgebra and the adjoint action on $\mathfrak{g}_0$ is semisimple.

Exercise 26.9*. Starting with a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ with associated conjugation σ, show that one can always find a compact form $\mathfrak{g}_c$ of $\mathfrak{g}$ such that $\sigma(\mathfrak{g}_c) = \mathfrak{g}_c$, and such that

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_c$, and $\mathfrak{p} = \mathfrak{g}_0 \cap (i \cdot \mathfrak{g}_c)$. Such a decomposition is called a *Cartan decomposition* of $\mathfrak{g}_0$. It is unique up to inner automorphism.

Exercise 26.10*. For any real form $\mathfrak{g}_0$ of $\mathfrak{g}$, given by a conjugation σ, show that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is preserved by σ, so $\mathfrak{g}_0 \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0$.

Naturally, the various special isomorphisms between complex Lie algebras ($\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C} \cong \mathfrak{sp}_2\mathbb{C}$, etc.) give rise to special isomorphisms among their real forms. For example, we have already seen that

$$\mathfrak{sl}_2\mathbb{R} \cong \mathfrak{su}_{1,1} \cong \mathfrak{so}_{2,1} \cong \mathfrak{sp}_2\mathbb{R},$$

while

$$\mathfrak{su}_2 \cong \mathfrak{so}_3\mathbb{R} \cong \mathfrak{sl}_1\mathbb{H} \cong \mathfrak{u}_1\mathbb{H}$$

(cf. Exercise 26.1). Similarly, each of the remaining three special isomorphisms of complex semisimple Lie algebras gives rise to isomorphisms between their real forms, as follows:

(i) $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$
compact forms: $\mathfrak{so}_4\mathbb{R} \cong \mathfrak{su}_2 \times \mathfrak{su}_2$
split forms: $\mathfrak{so}_{2,2} \cong \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$
others: $\mathfrak{so}_{3,1} \cong \mathfrak{sl}_2\mathbb{C}$, $\mathfrak{u}_2^*\mathbb{H} \cong \mathfrak{su}_2 \times \mathfrak{sl}_2\mathbb{R}$.

(ii) $\mathfrak{sp}_4\mathbb{C} \cong \mathfrak{so}_5\mathbb{C}$
compact forms: $\mathfrak{u}_2\mathbb{H} \cong \mathfrak{so}_5\mathbb{R}$
split forms: $\mathfrak{sp}_4\mathbb{R} \cong \mathfrak{so}_{3,2}$
other: $\mathfrak{u}_{1,1}\mathbb{H} \cong \mathfrak{so}_{4,1}$.

(iii) $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$
compact forms: $\mathfrak{su}_4 \cong \mathfrak{so}_6\mathbb{R}$
split forms: $\mathfrak{sl}_4\mathbb{R} \cong \mathfrak{so}_{3,3}$
others: $\mathfrak{su}_{2,2} \cong \mathfrak{so}_{4,2}$; $\mathfrak{su}_{3,1} \cong \mathfrak{u}_3^*\mathbb{H}$; $\mathfrak{sl}_2\mathbb{H} \cong \mathfrak{so}_{5,1}$.

In addition, the extra automorphism of $\mathfrak{so}_8\mathbb{C}$ coming from triality gives rise to an isomorphism $\mathfrak{u}_4^*\mathbb{H} \cong \mathfrak{so}_{6,2}$.

Exercise 26.11. Verify some of the isomorphisms above. (Of course, in the case of compact and split forms, these are implied by the corresponding isomorphisms of complex Lie algebras, but it is worthwhile to see them directly in any case.)

Real Groups

We turn now to problem of describing the real Lie groups with these Lie algebras. Let G be the adjoint form of the semisimple complex Lie algebra $\mathfrak{g}$. If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, the associated conjugate linear involution σ of $\mathfrak{g}$ that fixes $\mathfrak{g}_0$ lifts to an involution $\tilde{\sigma}$ of G. (This follows from the functorial nature of the adjoint form, noting that G is regarded now as a real Lie group.) The fixed points $G^{\tilde{\sigma}}$ of this involution then form a closed subgroup of G; its connected component of the identity G_0 is a real Lie group whose Lie algebra is $\mathfrak{g}_0$. G is called the *complexification* of G_0.

We have seen in §23.1 that if $\Gamma = \Gamma_W$ is the lattice of those elements in $\mathfrak{h}$ on which all roots take integral values, then $2\pi i\Gamma$ is the kernel of the exponential mapping $\exp\colon \mathfrak{h} \to G$ to the adjoint form. If $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, $T = \exp(\mathfrak{h}_0)$ will be compact precisely when the intersection of $\mathfrak{h}_0$ with the kernel $2\pi i\Gamma$ is a lattice of maximal rank. In this case, T will be a product of n copies of the circle S^1, $n = \dim(\mathfrak{h})$, and, since the Killing form on $\mathfrak{h}_0$ is negative definite, the corresponding real group G_0 will also be compact. Such a G_0 will be a maximal compact subgroup of G.

When $G_0 \subset G$ is a maximal compact subgroup, they have the same irreducible complex representations. Indeed, for any complex group G', each complex homomorphism from G to G' is the extension of a unique real homomorphism from G_0 to G'. This follows from the corresponding fact for Lie algebras and the fact that G_0 and G have the same fundamental group. This is another general fact, which implies the finiteness of the fundamental group of G_0; we omit the proof, noting only that it can be seen directly in the classical cases:

Exercise 26.12*. Prove that $\pi_1(G_0) \to \pi_1(G)$ is an isomorphism for each of the classical adjoint groups.

Exercise 26.13*. The special isomorphisms of real Lie algebras listed above give rise to special isomorphisms of real Lie groups. Can you find these?

It is another general fact that any compact (connected) Lie group is a quotient

$$(G_1 \times G_2 \times \cdots \times G_r \times T)/Z,$$

where the G_i are simple compact Lie groups, $T \cong (S^1)^k$ is a torus, and Z is a discrete subgroup of the center. In particular, its Lie algebra is the direct sum of a semisimple compact Lie algebra and an abelian Lie algebra. This provides another reason why the classification of irreducible representations in the real compact case and the semisimple complex case are essentially the same.

Representations of Real Lie Algebras

Finally, we should say a word here about the irreducible representations (always here in complex vector spaces!) of simple real Lie algebras. In some cases these are easily described in terms of the complex case: for example, the irreducible representations of $\mathfrak{su}_m$ or $\mathfrak{sl}_m\mathbb{R}$ are the same as those for $\mathfrak{sl}_m\mathbb{C}$, i.e., they are the restrictions of the irreducible representations $\Gamma_\lambda = \mathbb{S}_\lambda\mathbb{C}^m$ corresponding to partitions or Young diagrams λ. This is the situation in general whenever the complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ of the real Lie algebra $\mathfrak{g}_0$ is still simple: the representations of $\mathfrak{g}_0$ on complex vector spaces are exactly the representations of $\mathfrak{g}$. The situation is slightly different when we have a simple real Lie algebra whose complexification is not simple: for example, the irreducible representations of $\mathfrak{sl}_m\mathbb{C}$, regarded as a real Lie algebra, are of the form $\Gamma_\lambda \otimes \overline{\Gamma}_\mu$, where $\overline{\Gamma}_\mu$ is the conjugate representation of Γ_μ. The situation in general is expressed in the following

Exercise 26.14. Show that if $\mathfrak{g}_0$ is a simple real Lie algebra whose complexification $\mathfrak{g}$ is simple, its irreducible representations are the restrictions of (uniquely determined) irreducible representations of $\mathfrak{g}$. If $\mathfrak{g}_0$ is the underlying real algebra of a simple complex Lie algebra, show that the irreducible representations of $\mathfrak{g}_0$ are of the form $V \otimes \overline{W}$, where V and W are (uniquely determined) irreducible representations of the complex Lie algebra.

§26.2. Second Proof of Weyl's Character Formula

The title of this section is perhaps inaccurate: what we will give here is actually a sketch of the first proof of the Weyl character formula. Weyl, in his original proof, used what he called the "unitarian trick," which is to say he introduces the compact form of a given semisimple Lie algebra and uses integration on the corresponding compact group G. (This trick was already described in §9.3, in the context of proving complete reducibility of representations of a semisimple algebra.)

Indeed, the main reason for including this section (which is, after all, logically unnecessary) is to acquaint the reader with the "classical" treatment of Lie groups via their compact forms. This treatment follows very much the same lines as the representation theory of finite groups. To begin with, we replace the average $(1/|G|)\sum_{g \in G} f(g)$ by the integral $\int_G f(g)\,d\mu$, the volume element $d\mu$ chosen to be translation invariant and such that $\int_G d\mu = 1$. If $\rho\colon G \to \mathrm{Aut}(V)$ is a finite-dimensional representation, with character

$$\chi_V(g) = \mathrm{Trace}(\rho(g)),$$

then $\int_G \rho(g)\,d\mu \in \mathrm{Hom}(V, V)$ is idempotent, and it is the projection onto the invariant subspace V^G. So $\int_G \chi_V(g)\,d\mu = \dim(V^G)$. Applied to $\mathrm{Hom}(V, W)$ as before, since $\chi_{\mathrm{Hom}(V,W)} = \bar{\chi}_V \chi_W$, it follows that

$$\int_G \bar{\chi}_V \chi_W \,d\mu = \dim(\mathrm{Hom}_G(V, W)).$$

So if V and W are irreducible,

$$\int_G \bar{\chi}_V \chi_W \,d\mu = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

Up to now, everything is completely analogous to the case of finite groups, and is proved in exactly the same way. The last general fact, analogous to the basic Proposition 2.30, is harder in the compact case:

Peter–Weyl Theorem. The characters of irreducible representations span a dense subspace of the space of continuous class functions.

It is, moreover, the case that the coordinate functions of the irreducible matrix representations span a dense subspace of all continuous (or L^2) functions on G. For the proof of these statements we refer to [Ad] or [B-tD]. Given the fundamental role that (2.30) played in the analysis of representations of finite groups, it is not surprising that the Peter–Weyl theorem is the cornerstone of most treatments of compact groups, even though it has played no role so far in this book.

We now proceed to indicate how the original proof of the Weyl character formula went in this setting. In this section, G will denote a fixed compact group, whose Lie algebra $\mathfrak{g}$ is a real form of the semisimple complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We have seen that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{l}_\alpha,$$

compatible with the usual decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ when complexified. The real Cartan algebra $\mathfrak{h}$ acts by rotations on the planes $\mathfrak{l}_\alpha$.

Now let $T = \exp(\mathfrak{h}) \subset G$. As before we have chosen $\mathfrak{h}$ so that it contains the lattice $2\pi i\Gamma$ which is the kernel of the exponential map from $\mathfrak{h}_{\mathbb{C}}$ to the simply-connected form of $\mathfrak{g}_{\mathbb{C}}$, so $T \cong (S^1)^n$ is a compact torus.

In this compact case we can realize the Weyl group on the group level again:

Claim 26.15. $N(T)/T \cong \mathfrak{W}$.

Proof. For each pair of roots α, $-\alpha$, we have a subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}_{\mathbb{C}}$, with a corresponding $\mathfrak{su}_2 \subset \mathfrak{g}$. Exponentiating gives a subgroup $\mathrm{SU}(2) \subset G$. The element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts by Ad, taking H to $-H$, X to Y, and Y to X. It is in $N(T)$, and, with B as in the preceding section, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp\left(\frac{1}{2}\pi i B\right)$. Then $\exp\left(\frac{1}{2}\pi i B\right) \in \mathfrak{g}$ acts by reflection in the hyperplane $\alpha^\perp \subset \mathfrak{h}$. □

Note that $\mathfrak{W}$ acting on $\mathfrak{h}$ takes the lattice $2\pi i\Gamma$ to itself, so $\mathfrak{W}$ acts on $T = \mathfrak{h}/2\pi i\Gamma$ by conjugation.

Theorem 26.16. *Every element of G is conjugate to an element of T. A general element is conjugate to $|\mathfrak{W}|$ such elements of T.*

Sketch of a proof: Note that G acts by left multiplication on the left coset space $X = G/T$. For any $z \in G$, consider the map $f_z\colon X \to X$ which takes yT to zyT. The claim is that f_z must have a fixed point, i.e., there is a y such that $y^{-1}zy \in T$. Since all f_z are homotopic, and X is compact, the Lefschetz number of f_z is the topological Euler characteristic of X. The first statement follows from the claim that this Euler characteristic is not zero. This is a good exercise for the classical groups; see [Bor2] for a general proof. For another proof see Remark 26.20 below.

For the second assertion, check first that any element that commutes with every element of T is in T. Take an "irrational" element x in T so that its multiples are dense in T. Then for any $y \in G$, $yxy^{-1} \in T \Leftrightarrow yTy^{-1} = T$, and $yxy^{-1} = x \Leftrightarrow y \in T$. This gives precisely $|\mathfrak{W}|$ conjugates of x that are in T.

Corollary 26.17. *The class functions on G are the $\mathfrak{W}$-invariant functions on T.*

Suppose G is a real form of the complex semisimple group $G_{\mathbb{C}}$, i.e., G is a real analytic closed subgroup of $G_{\mathbb{C}}$, and the Lie algebra of $G_{\mathbb{C}}$ is $\mathfrak{g}_{\mathbb{C}}$. The characters on $G_{\mathbb{C}}$ can be written $\sum n_\mu e^{2\pi i\mu}$, the sum over μ in the weight lattice Λ; they are invariant under the Weyl group. From what we have seen, they can be identified with $\mathfrak{W}$-invariant functions on the torus T. Let us work this out for the classical groups:

Case (A_n): $G = \mathrm{SU}(n+1)$. The Lie algebra $\mathfrak{su}_{n+1}$ consists of skew-Hermitian matrices,

$$\mathfrak{h} = \mathfrak{su}_{n+1} \cap \mathfrak{sl}_{n+1}\mathbb{R} = \{\text{imaginary diagonal matrices of trace } 0\},$$

and $T = \{\mathrm{diag}(e^{2\pi i\vartheta_1}, \ldots, e^{2\pi i\vartheta_{n+1}}): \sum \vartheta_j = 0\}$. In this case, the Weyl group $\mathfrak{W}$ is the symmetric group $\mathfrak{S}_{n+1}$, represented by permutation matrices (with one entry ± 1 on each row and column, other entries 0) modulo T. Let $z_i\colon T \to S^1$ correspond to the ith diagonal entry $e^{2\pi i\vartheta_i}$. So characters on T are symmetric polynomials in $z_1, \ldots, z_{n+1}$ modulo the relation $z_1 \cdot \ldots \cdot z_{n+1} = 1$. Therefore, characters on $\mathrm{SU}(n+1)$ are symmetric polynomials in $z_1, \ldots, z_{n+1}$.

Case (B_n): $G = \mathrm{SO}(2n+1)$. $\mathfrak{h}$ consists of matrices with n 2×2 blocks of the form

$$\begin{pmatrix} \cos(2\pi\vartheta_i) & -\sin(2\pi\vartheta_i) \\ \sin(2\pi\vartheta_i) & \cos(2\pi\vartheta_i) \end{pmatrix}$$

along the diagonal, and one 1 in the lower right corner. Again we see that $T = (S^1)^n$. This time $N(T)$ will have block permutations to interchange the blocks, and also matrices with some blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the squares along the diagonal, with the other blocks 2×2 identity matrices, with a ± 1 in the corner to make the determinant positive; these take ϑ_i to $-\vartheta_i$ for each i where a block is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This again realizes the Weyl group as a semidirect product of $\mathfrak{S}_n$ and $(\mathbb{Z}/2)^n$. With z_i identified with $e^{2\pi i\vartheta_i}$ again, we see that the characters are the symmetric polynomials in the variables $z_i + z_i^{-1}$, i.e., in $\cos(2\pi\vartheta_1), \ldots, \cos(2\pi\vartheta_n)$.

Case (D_n): $G = \mathrm{SO}(2n)$. $\mathfrak{h}$ is as in the preceding case, but with no lower corner. Since we have no corner to put a -1 in, there can be only an even number of blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, reflecting the fact that $\mathfrak{W}$ is a semidirect product of $(\mathbb{Z}/2)^{n-1}$ and $\mathfrak{S}_n$. This time the invariants are symmetric polynomials in the $z_i + z_i^{-1}$, and one additional $\Pi_i(z_i - z_i^{-1})$.

Case (C_n): $G = \mathrm{Sp}(2n)$. $\mathfrak{h}$ consists of imaginary diagonal matrices, T consists of diagonal matrices with entries $e^{2\pi i\vartheta_i}$. The Weyl group in generated by permutation matrices and diagonal matrices with entries which are 1's and quaternionic j's: $\mathfrak{W}$ is a semidirect product of $(\mathbb{Z}/2)^n$ and $\mathfrak{S}_n$. The invariants are symmetric polynomials in the $z_i + z_i^{-1}$.

The key to Weyl's analysis is to calculate the integral of a class function f on G as a suitable integral over the torus T. For this, consider the map

$$\pi\colon G/T \times T \to G, \qquad \pi(xT, y) = xyx^{-1}.$$

By what we said earlier, π is a generically finite-sheeted covering, with $|\mathfrak{W}|$ sheets. It follows that

$$\int_G f\,d\mu = \frac{1}{|\mathfrak{W}|}\int_{G/T\times T} \pi^*(f)\pi^* d\mu.$$

Now $\pi^*(f)(xT, y) = f(y)$ since f is a class function. To calculate $\pi^* d\mu$, consider the induced map on tangent spaces

$$\pi_* = d\pi\colon \mathfrak{g}/\mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}.$$

At the point $(x_0 T, y_0) \in G/T \times T$,

$$(x_0 e^{tx}T, y_0 e^{ty}) \mapsto x_0 e^{tx} y_0 e^{ty} e^{-tx} x_0^{-1}.$$

We want to calculate

$$\frac{d}{dt}(x_0 e^{tx} y_0 e^{ty} e^{-tx} x_0^{-1})|_{t=0}(x_0 y_0 x_0^{-1})^{-1},$$

which is

$$x_0(xy_0 + y_0 y - y_0 x)x_0^{-1}(x_0 y_0^{-1} x_0^{-1}) = x_0(x + y_0 y y_0^{-1} - y_0 x y_0^{-1})x_0^{-1}.$$

Now $y_0 y y_0^{-1} = y$ since $y_0 \in T$ and $y \in \mathfrak{h}$. To calculate the determinant of π_* we can ignore the volume-preserving transformation $x_0(\ )x_0^{-1}$. If we identify $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{h} \times \mathfrak{h}$, the matrix becomes

$$\begin{pmatrix} I - \mathrm{Ad}(y_0) & 0 \\ 0 & I \end{pmatrix}.$$

So the determinant of π_* is $\det(I - \mathrm{Ad}(y_0))$. Now $(\mathfrak{g}/\mathfrak{h})_{\mathbb{C}} = \bigoplus \mathfrak{g}_\alpha$, and $\mathrm{Ad}(y_0)$ acts as $e^{2\pi i\alpha(y_0)}$ on $\mathfrak{g}_\alpha$. Hence

$$\det(\pi_*) = \prod_{\alpha\in R}(1 - e^{2\pi i\alpha}), \tag{26.18}$$

as a function on T alone, independent of the factor G/T. This gives *Weyl's integration formula*:

$$\int_G f\,d\mu_G = \frac{1}{|\mathfrak{W}|}\int_T f(y)\prod_{\alpha\in R}(1 - e^{2\pi i\alpha(y)})\,d\mu_T. \tag{26.19}$$

Remark 26.20. The same argument gives another proof of the theorem that G is covered by conjugates of T. This amounts to the assertion that the map

$\pi\colon G/T \times T \to G$ of compact manifolds is surjective. By what we saw above, for a generic point $y_0 \in T$ there are exactly $|\mathfrak{W}|$ points in $\pi^{-1}(y_0)$, and at each of these the Jacobian determinant is the same (nonzero) number. It follows that the topological degree of the map π is $|\mathfrak{W}|$, so the map must be surjective.

Now $(1 - e^{2\pi i\alpha})(1 - e^{-2\pi i\alpha}) = (e^{\pi i\alpha} - e^{-\pi i\alpha})(\overline{e^{\pi i\alpha} - e^{-\pi i\alpha}})$, so if we set

$$\Delta = \prod_{\alpha \in R^+} (e^{\pi i\alpha} - e^{-\pi i\alpha}),$$

then $\det(\pi_*) = \Delta\bar{\Delta}$. As we saw in Lemma 24.3, $\Delta = A_\rho$, where ρ is half the sum of the positive roots and, for any weight μ,

$$A_\mu = \sum_{W \in \mathfrak{W}} (-1)^W e^{2\pi i W(\mu)}.$$

Now we can complete the second proof of Weyl's character formula: the character of the representation with highest weight λ is $A_{\lambda+\rho}/A_\rho$. Since we saw in §24.1 that $A_{\lambda+\rho}/A_\rho$ has highest weight λ and (see Corollary 24.6) its value at the identity is positive, it suffices to show that the integral of $\int_G \chi\bar{\chi} = 1$, where $\chi = A_{\lambda+\rho}/A_\rho$. By Weyl's integration formula,

$$\begin{aligned}\int_G \chi\bar{\chi} &= \frac{1}{|\mathfrak{W}|}\int_T \chi\bar{\chi}\Delta\bar{\Delta} = \frac{1}{|\mathfrak{W}|}\int_T A_{\lambda+\rho}\overline{A_{\lambda+\rho}} \\ &= \frac{1}{|\mathfrak{W}|}\int_T \sum_{W \in \mathfrak{W}} (-1)^W e^{2\pi i W(\lambda+\rho)} \cdot \sum_{W \in \mathfrak{W}} (-1)^W e^{-2\pi i W(\lambda+\rho)} = 1,\end{aligned}$$

which concludes the proof.

§26.3. Real, Complex, and Quaternionic Representations

The final topic we want to take up is the classification of irreducible complex representations of semisimple Lie groups or algebras into those of real, quaternionic, or complex type. To define our terms, given a real semisimple Lie group G_0 or its Lie algebra $\mathfrak{g}_0$ and a representation of G_0 or $\mathfrak{g}_0$ on a complex vector space V we say that the representation V is *real*, or of *real type*, if it comes from a representation of G_0 or $\mathfrak{g}_0$ on a real vector space V_0 by extension of scalars ($V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$); this is equivalent to saying that it has a conjugate linear endomorphism whose square is the identity. It is *quaternionic* if it comes from a quaternionic representation by restriction of scalars, or equivalently if it has a conjugate linear endomorphism whose square is minus the identity. Finally, we say that the representation is *complex* if it is neither of these. (Compare with Theorem 3.37 for finite groups.)

Having completely classified the irreducible representations of the classical complex Lie algebras, and having described all the real forms of these Lie algebras, we have a clear-cut problem: to detemine the type of the restriction of each representation to each real form. Rather than try to answer this in every case, however, we will instead mention some of the ideas that allow us to answer this question, and then focus on the cases of the split forms (where the answer is easy) and the compact forms (where the answer is more interesting, and where we have more tools to play with). We assume the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ is simple, so irreducible representations of $\mathfrak{g}_0$ are restrictions of unique irreducible representations of $\mathfrak{g}$ (cf. (26.14)); in particular, we have the classification of irreducible representations by dominant weights.

To begin with, the tensor products of two real, or two quaternionic, or of a pair of complex conjugate representations is always real; and exterior powers of real and quaternionic representations are equally easy to analyze, as for finite groups (see Exercise 3.43). Such tensor and exterior powers may not be irreducible, but the following criterion can often be used to describe an irreducible component of highest weight that occurs inside them:

Exercise 26.21*. Suppose W is a representation of a semisimple group G that is real or quaternionic, and suppose W has a highest weight λ that occurs with multiplicity 1. Show that the irreducible representations Γ_λ with highest weight λ has the same type as W.

We may apply this in particular to the tensor product $\Gamma_\lambda \otimes \Gamma_\mu$ of the irreducible representations of $\mathfrak{g}$ with highest weights λ and μ; since the irreducible representation $\Gamma_{\lambda+\mu}$ with highest weight $\lambda + \mu$ appears once in this tensor product, we deduce

Exercise 26.22*. (i) If Γ_λ and Γ_μ are both real or both quaternionic, then $\Gamma_{\lambda+\mu}$ is real. (ii) If Γ_λ is real and Γ_μ is quaternionic, then $\Gamma_{\lambda+\mu}$ is quaternionic. (iii) If Γ_λ and Γ_μ are complex and conjugate, then $\Gamma_{\lambda+\mu}$ is real.

The last two exercises almost completely answer the question of the representations of the split forms of the classical groups: we have

Proposition 26.23. *Every irreducible representation of the split forms $\mathfrak{sl}_{n+1}\mathbb{R}$, $\mathfrak{so}_{n+1,n}\mathbb{R}$, $\mathfrak{sp}_{2n}\mathbb{R}$, and $\mathfrak{so}_{n,n}\mathbb{R}$ of the classical Lie algebras is real.*

PROOF. In each of these cases, the standard representation V is real, from which it follows that the exterior powers $\wedge^k V$ are real, from which it follows that the symmetric powers $\mathrm{Sym}^{a_k}(\wedge^k V)$ are real. Now, in the cases of $\mathfrak{sl}_{n+1}\mathbb{R}$ and $\mathfrak{sp}_{2n}\mathbb{R}$, we have seen that the highest weights ω_k of the representations $\wedge^k V$ for $k = 1, \dots, n$ form a set of fundamental weights: that is, every irreducible representation Γ has highest weight $\sum a_k \cdot \omega_k$ for some non-negative integers $a_1, \dots, a_n$. It follows that Γ appears once in the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_n}(\wedge^n V)$$

and so is real. (Alternatively, Weyl's construction produces real representations when applied to real vector spaces.)

The only difference in the orthogonal case is that some of the exterior powers $\bigwedge^k V$ of the standard representation must be replaced in this description by the spin representation(s). That the spin representations are real follows from the construction in Lecture 20, cf. Exercise 20.23; the result in this case then follows as before. □

The Compact Case

We turn now to the compact forms of the classical Lie algebras. In this case, the theory behaves very much like that of finite groups, discussed in Lecture 5. Specifically, any action of a compact group G_0 on a complex vector space V preserves a nondegenerate Hermitian inner product (obtained, for example, by choosing one arbitrarily and averaging its translates under the action of G_0). It follows that the dual of V is isomorphic to its conjugate, so that V will be either real or quaternionic exactly when it is isomorphic to its dual V^*. (In terms of characters, this says that the character $\mathrm{Char}(V)$ is invariant under the automorphism of $\mathbb{Z}[\Lambda]$ which takes $e(\mu)$ to $e(-\mu)$; for groups, this says the character is real.) More precisely, an irreducible representation of a compact group/Lie algebra will be real (resp. quaternionic) if and only if it has an invariant nondegenerate symmetric (resp. skew-symmetric) bilinear form. In other words, the classification of an irreducible V is determined by whether

$$V \otimes V = \mathrm{Sym}^2 V \oplus \textstyle\bigwedge^2 V$$

contains the trivial representation, and, if so, in which factor. So determining which type a representation belongs to is a very special case of the general plethysm problem of decomposing such representations.

With this said, we consider in turn the algebras $\mathfrak{su}_n$, $\mathfrak{u}_n\mathbb{H}$, and $\mathfrak{so}_m\mathbb{R}$.

Let Γ_λ be the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $\lambda = \sum a_i \cdot \omega_i$, where $\omega_i = L_1 + \cdots + L_i$, $i = 1, \ldots, n-1$ are the fundamental weights of $\mathfrak{sl}_n\mathbb{C}$. The dual of Γ will have highest weight $\sum a_{n-i} \cdot \omega_i$, so that Γ will be real or quaternionic if and only if $a_i = a_{n-i}$ for all i. We now distinguish three cases:

(i) If n is odd, then the sublattice of weights $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i is freely generated by the sums $\omega_i + \omega_{n-i}$ for $i = 1, \ldots, (n-1)/2$. Now, ω_i is the highest weight of the exterior power $\bigwedge^i V$, so that the irreducible representation with highest weight $\omega_i + \omega_{n-i}$ will appear once in the tensor product

$$\textstyle\bigwedge^i V \otimes \bigwedge^{n-i} V = (\bigwedge^i V) \otimes (\bigwedge^i V)^*,$$

which by Exercise 26.21 above is real. It follows that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i, the irreducible representation Γ_λ is real.

(iia) If $n = 2k$ is even, then the sublattice of weights $\lambda = \sum a_i \cdot \omega_i$ with

$a_i = a_{n-i}$ for all i is freely generated by the sums $\omega_i + \omega_{n-i}$ for $i = 1, \ldots, k-1$, together with the weight ω_k. As before, the irreducible representations with highest weight $\omega_i + \omega_{n-i}$ are all real. Moreover, in case n is divisible by 4 the representation $\wedge^k V$ is real as well, since $\wedge^k V$ admits a symmetric bilinear form

$$\wedge^k V \otimes \wedge^k V \to \wedge^{2k} V = \mathbb{C}$$

given by wedge product. It follows then as before that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i, the irreducible representation Γ_λ is real.

(iib) In case n is congruent to 2 mod 4, the analysis is similar to the last case except that wedge product gives a skew-symmetric bilinear pairing on $\wedge^k V$. The representation $\wedge^k V$ is thus quaternionic, and it follows that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i, the irreducible representation Γ_λ is real if a_k is even, quaternionic if a_k is odd. In sum, then, we have

Proposition 26.24. *For any weight $\lambda = \sum a_i \cdot \omega_i$ of $\mathfrak{su}_n$, the irreducible representation Γ_λ with highest weight λ is: complex if $a_i \neq a_{n-i}$ for any i; real if $a_i = a_{n-i}$ for all i and n is odd, or $n = 4k$, or $n = 4k + 2$ and a_{2k+1} is even; and quaternionic if $a_i = a_{n-i}$ for all i and $n = 4k + 2$ and a_{2k+1} is odd.*

Next, we consider the case of the compact form $\mathfrak{u}_n\mathbb{H}$ of $\mathfrak{sp}_{2n}\mathbb{C}$. To begin with, we note that since the restriction to $\mathfrak{u}_n\mathbb{H}$ of the standard representation of $\mathfrak{sp}_{2n}\mathbb{C}$ on $V \cong \mathbb{C}^{2n}$ is quaternionic, the exterior power $\wedge^k V$ is real for k even and quaternionic for k odd. Since the highest weights ω_k of $\wedge^k V$ for $k = 1, \ldots, n$ form a set of fundamental weights, this completely determines the type of the irreducible representations of $\mathfrak{u}_n\mathbb{H}$: we have

Proposition 26.25. *For any weight $\lambda = \sum a_i \cdot \omega_i$ of $\mathfrak{u}_n\mathbb{H}$, the irreducible representation Γ_λ with highest weight λ is real if a_i is even for all odd i, and quaternionic if a_i is odd for any odd i.*

Next, we consider the odd orthogonal algebras. Part of this is easy: since the restriction to $\mathfrak{so}_{2n+1}\mathbb{R}$ of the standard representation V of $\mathfrak{so}_{2n+1}\mathbb{C}$ is real, so are all its exterior powers; and it follows that any representation of $\mathfrak{so}_{2n+1}\mathbb{R}$ whose highest weight lies in the sublattice of index two generated by the highest weights of these exterior powers is real. It remains, then, to describe the type of the spin representation; the answer, whose verification we leave as Exercise 26.28 below, is that the spin representation Γ_α of $\mathfrak{so}_{2n+1}\mathbb{C}$ (that is, the irreducible representation whose highest weight is one-half the highest weight of $\wedge^n V$) is real when $n \equiv 0$ or 3 mod 4, and quaternionic if $n \equiv 1$ or 2 mod 4. This yields

Proposition 26.26. *Let ω_i be the highest weight of the representation $\wedge^i V$ of $\mathfrak{so}_{2n+1}\mathbb{C}$. For any weight $\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1} + a_n\omega_n/2$ of $\mathfrak{so}_{2n+1}\mathbb{R}$, the irreducible representation Γ_λ with highest weight λ is real if a_n is even, or if n is* *congruent to* 0 *or* 3 mod 4; *if* a_n *is odd and* $n \equiv 1$ *or* 2 mod 4, *then* Γ_λ *is quaternionic.*

(Note that, in each of the last two cases, the fact that every representation is either real or quaternionic follows from the observation that the Weyl group action on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ includes multiplication by -1.)

Finally, we have the even orthogonal Lie algebras. As before, the exterior powers of the standard representation V are all real, but we now have two spin representations to deal with, with highest vectors (in the notation of Lecture 19) $\alpha = (L_1 + \cdots + L_n)/2$ and $\beta = (L_1 + \cdots + L_{n-1} - L_n)/2$. The first question is whether these two are self-conjugate or conjugate to each other. In case n is even, as in the case of the symplectic and odd orthogonal algebras, the Weyl group action on the Cartan subalgebra contains multiplication by -1 (the Weyl group contains the automorphism of $\mathfrak{h}^*$ reversing the sign of any even number of the basis elements L_i), so that Γ_α and Γ_β will be isomorphic to their duals; if n is odd, on the other hand, we see that Γ_α will have $-\beta$ as a weight, so that Γ_α and Γ_β will be complex representations dual to each other. We consider these cases in turn.

(i) Suppose first that n is odd, and say λ is any weight, written as

$$\lambda = a_1\omega_1 + \cdots + a_{n-2}\omega_{n-2} + a_{n-1}\beta + a_n\alpha.$$

If $a_{n-1} \neq a_n$, the representation Γ_λ with highest weight λ will not be isomorphic to its dual, and so will be complex. On the other hand, $\Gamma_{\alpha+\beta}$ appears once in $\Gamma_\alpha \otimes \Gamma_\beta = \mathrm{End}(\Gamma_\alpha)$, and so is real; thus, if $a_{n-1} = a_n$, the representation Γ_λ will be real.

(ii) If, by contrast, n is even then all representations of $\mathfrak{so}_{2n}\mathbb{R}$ will be either real or quaternionic. The half-spin representations Γ_α and Γ_β are real if $n \equiv 0$ (mod 4), quaternionic if $n \equiv 2$ (mod 4), a fact that we leave as Exercise 26.28. It follows that, with λ as above, Γ_λ will be real if either n is divisible by 4, or if $a_{n-1} + a_n$ is even; if $n \equiv 2$ mod 4 and $a_{n-1} + a_n$ is odd, Γ_λ will be quaternionic. In sum, then, we have

Proposition 26.27. *The representation* Γ_λ *of* $\mathfrak{so}_{2n}\mathbb{R}$ *with highest weight* $\lambda = a_1\omega_1 + \cdots + a_{n-2}\omega_{n-2} + a_{n-1}\beta + a_n\alpha$ *will be complex if* n *is odd and* $a_{n-1} \neq a_n$; *it will be quaternionic if* $n \equiv 2$ mod 4 *and* $a_{n-1} + a_n$ *is odd*; *and it will be real otherwise.*

Exercise 26.28*. Verify the statements made above about the types of the spin representation Γ_α of the orthogonal Lie algebras, i.e., that the spin representation Γ_α of $\mathfrak{so}_{2n+1}\mathbb{R}$ is real when $n \equiv 0$ or 3 (mod 4); and quaternionic if $n \equiv 1$ or 2 (mod 4), and that the half-spin representations of $\mathfrak{so}_{2n}\mathbb{R}$ are real if $n \equiv 0$ (mod 4) and quaternionic if $n \equiv 2$ (mod 4). Show, in fact, that the even Clifford algebras $C_m^{\text{even}} \subset C_m = C(0, m)$ are products of one or two copies of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, with $\mathbb{R}$ occurring for $m \equiv 0$ or ± 1 mod 8, $\mathbb{C}$ occurring for $m \equiv \pm 2$ mod 8, and $\mathbb{H}$ for $m \equiv \pm 3$ or 4 mod 8.

Exercise 26.29. Show that for a representation V of a compact group G,

$$\int_G \chi_V(g^2) = \begin{cases} 0 & \text{if } V \text{ is complex} \\ 1 & \text{if } V \text{ is real} \\ -1 & \text{if } V \text{ is quaternionic.} \end{cases}$$

Exercise 26.30*. Show that for a representation V of a compact group, the number of irreducible real components it contains, minus the number of quaternionic representations, is the number of times the trivial representation occurs in $\psi^2 V$ in the representation ring, where ψ^2 is the Adams operation (cf. Exercise 23.39).

APPENDICES

These appendices contain proofs of some of the general Lie algebra facts that were postponed during the course, as well as some results from algebra and invariant theory which were used particularly in the "Weyl construction–Schur functor" descriptions of representations.

The first appendix is a fairly serious excursion in polynomial algebra. It proves some basic facts about symmetric functions, especially the Schur polynomials, which occur as characters of representations of GL_n or SL_n, and gives determinantal formulas for them in terms of other basic symmetric polynomials. The last section of Appendix A includes some new identities among symmetric polynomials, which, when the variables are specialized, express characters of representations of Sp_{2n} and SO_m as determinants in the characters of basic representations.

Appendix B gives a short summary of some basic multilinear facts about exterior and symmetric powers. The first two sections can be used as a reference for the conventions and notations we have followed; the third contains a general discussion of constructions such as contractions, many special cases of which were discussed in the main text.

The next three appendices conclude our discussion of the theory of Lie algebras, which began in Lectures 9, 14, and 21. Proofs are given, by standard methods, of the promised general results on semisimplicity, the theorem on conjugacy of Cartan subalgebras, facts about the Weyl group, Ado's theorem that every Lie algebra has a faithful representation, and Levi's theorem that splits the map from a Lie algebra to its semisimple quotient.

The last appendix develops just enough classical invariant theory to find the polynomial invariants for $SL_n\mathbb{C}$, $Sp_{2n}\mathbb{C}$, and $SO_n\mathbb{C}$. This was the key to our proof that Weyl's construction gives the irreducible representations of the symplectic and orthogonal groups.

APPENDIX A

On Symmetric Functions

§A.1. Basic Symmetric Polynomials and Relations among Them

The vector space of homogeneous symmetric polynomials of degree d in k variables $x_1, \ldots, x_k$ has several important bases, usually indexed by the partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ of d into at most k parts, or by Young diagrams with at most k rows (see §4.1). We list four of these bases, which are all valid for polynomials with integer coefficients, or coefficients in any commutative ring.

First we have the monomials in the complete symmetric polynomials:

$$H_\lambda = H_{\lambda_1} \cdot H_{\lambda_2} \cdot \ldots \cdot H_{\lambda_k}, \tag{A.1}$$

where H_j is the jth *complete symmetric polynomial*, i.e., the sum of all distinct monomials of degree j; equivalently,

$$\prod_{i=1}^{k} \frac{1}{1 - x_i t} = \sum_{j=0}^{\infty} H_j t^j.$$

For example, with three variables,

$$H_{(1,1)} = (x_1 + x_2 + x_3)^2,$$
$$H_{(2,0)} = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3.$$

Next are the *monomial symmetric polynomials*:

$$M_\lambda = \sum X^\alpha, \tag{A.2}$$

the sum over all distinct permutations $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $(\lambda_1, \ldots, \lambda_k)$; here $X^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k}$. For example,

$$M_{(1,1)} = x_1x_2 + x_1x_3 + x_2x_3,$$
$$M_{(2,0)} = x_1^2 + x_2^2 + x_3^2.$$

The third are the monomials in the elementary symmetric functions. Unlike the first two, these are parametrized by partitions μ of d in integers no larger than k, i.e., $k \geq \mu_1 \geq \cdots \geq \mu_l \geq 0$. These are exactly the partitions that are conjugate to a partition of d into at most k parts. (The *conjugate* to a partition λ is the partition whose Young diagram is obtained from that of λ by interchanging rows and columns. We denote the conjugate of λ by λ', although the notation $\tilde{\lambda}$ is also common.) For such μ set

$$E_\mu = E_{\mu_1} \cdot E_{\mu_2} \cdot \ldots \cdot E_{\mu_l}, \tag{A.3}$$

where E_j is the jth *elementary symmetric polynomial*, i.e.,

$$E_j = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdot \ldots \cdot x_{i_j}, \qquad \prod_{i=1}^{k} (1 + x_i t) = \sum_{j=0}^{\infty} E_j t^j.$$

For example,

$$E_{(1,1)} = (x_1 + x_2 + x_3)^2,$$
$$E_{(2,0)} = x_1x_2 + x_1x_3 + x_2x_3.$$

The fourth are the *Schur polynomials*, which may be the most important, although they are less often met in modern algebra courses:

$$S_\lambda = \frac{|x_j^{\lambda_i + k - i}|}{|x_j^{k-i}|} = \frac{|x_j^{\lambda_i + k - i}|}{\Delta}, \tag{A.4}$$

where $\Delta = \prod_{i<j}(x_i - x_j)$ is the discriminant, and $|a_{i,j}|$ denotes the determinant of a $k \times k$ matrix. For example,

$$S_{(1,1)} = x_1x_2 + x_1x_3 + x_2x_3,$$
$$S_{(2,0)} = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3.$$

The first task of this appendix is to describe some relations among these symmetric polynomials. For example, one sees quickly that

$$S_{(1,1)} = E_{(2,0)} = H_1^2 - H_2,$$
$$S_{(2,0)} = H_{(2,0)} = E_1^2 - E_2,$$
$$S_{(1,0)} \cdot S_{(1,0)} = S_{(1,1)} + S_{(2,0)}.$$

These are special cases of three important formulas involving Schur polynomials, which we state next. The first two are known as *determinantal* *formulas.* The first is also known as the *Jacobi–Trudy identity.* From geometry, the first two are sometimes called *Giambelli's formulas,* and the third is *Pieri's formula.* The proofs will be given in the next section.

$$S_\lambda = |H_{\lambda_i+j-i}| = \begin{vmatrix} H_{\lambda_1} & H_{\lambda_1+1} \cdots H_{\lambda_1+k-1} \\ H_{\lambda_2-1} & H_{\lambda_2} \cdots & \\ \vdots & & \\ H_{\lambda_k-k+1} \cdots & & H_{\lambda_k} \end{vmatrix}. \tag{A.5}$$

Note that if $\lambda_{p+1} = \cdots = \lambda_k = 0$, the determinant on the right is the same as the determinant of the upper left $p \times p$ corner. The second is

$$S_\lambda = |E_{\mu_i+j-i}| = \begin{vmatrix} E_{\mu_1} & E_{\mu_1+1} \cdots E_{\mu_1+l-1} \\ E_{\mu_2-1} & E_{\mu_2} \cdots & \\ \vdots & & \\ E_{\mu_l-l+1} \cdots & & E_{\mu_l} \end{vmatrix}, \tag{A.6}$$

where $\mu = (\mu_1, \ldots, \mu_l)$ is the conjugate partition to λ.

The third "Pieri" formula tells how to multiply a Schur polynomial S_λ by a basic Schur polynomial $S_{(m)} = H_m$[1]:

$$S_\lambda S_{(m)} = \sum S_\nu, \tag{A.7}$$

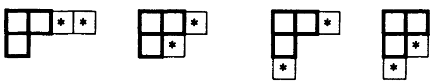
the sum over all ν whose Young diagram can be obtained from that of λ by adding a total of m boxes to the rows, but with no two boxes in the same column, i.e., those $\nu = (\nu_1, \ldots, \nu_k)$ with

$$\nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_k \geq \lambda_k \geq 0,$$

and $\sum \nu_j = \sum \lambda_j + m = d + m$. For example, the identity

$$S_{(2,1)} \cdot S_{(2)} = S_{(4,1)} + S_{(3,2)} + S_{(3,1,1)} + S_{(2,2,1)}$$

can be seen from the pictures



One can use the Pieri and determinantal formulas to multiply any two Schur polynomials, but there is a more direct formula, which generalizes Pieri's formula. This *Littlewood–Richardson rule* gives a combinatorial formula for the coefficients $N_{\lambda\mu\nu}$ in the expansion of a product as a linear combination of Schur polynomials:

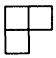
[1] When k is fixed, we often omit zeros at the end of partitions, so (m) denotes the partition $(m, 0, \ldots, 0)$.

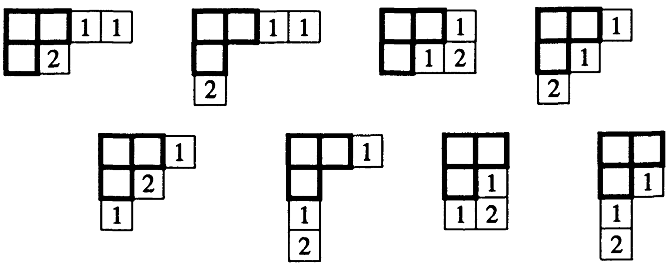
$$S_\lambda \cdot S_\mu = \sum N_{\lambda\mu\nu} S_\nu. \tag{A.8}$$

Here λ is a partition of d, μ a partition of m, and the sum is over all partitions ν of $d + m$ (each with at most k parts). The Littlewood–Richardson rule says that $N_{\lambda\mu\nu}$ *is the number of ways the Young diagram for* λ *can be expanded to the Young diagram for* ν *by a strict* μ*-expansion.* If $\mu = (\mu_1, \ldots, \mu_k)$, a μ*-expansion* of a Young diagram is obtained by first adding μ_1 boxes, according to the above description in Pieri's formula, and putting the integer 1 in each of these μ_1 boxes; then adding similarly μ_2 boxes with a 2, continuing until finally μ_k boxes are added with the integer k. The expansion is called *strict* if, when the integers in the boxes are listed from right to left, starting with the top row and working down, and one looks at the first t entries in this list (for any t between 1 and $\mu_1 + \cdots + \mu_k$), each integer p between 1 and $k - 1$ occurs at least as many times as the next integer $p + 1$.

For example, the equation

$$S_{(2,1)} \cdot S_{(2,1)} = S_{(4,2)} + S_{(4,1,1)} + S_{(3,3)} + 2S_{(3,2,1)}$$
$$+ S_{(3,1,1,1)} + S_{(2,2,2)} + S_{(2,2,1,1)}$$

can be seen by listing the strict (2, 1)-expansions of the Young diagram :



A proof of the Littlewood–Richardson rule can be found in [Mac, §I.9]; for the other results of this appendix we can get by without using it.

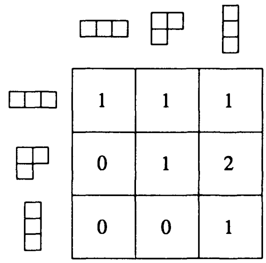
Formula (A.7), applied inductively, yields

$$H_\lambda = S_{(\lambda_1)} \cdot S_{(\lambda_2)} \cdot \ldots \cdot S_{(\lambda_k)} = \sum K_{\mu\lambda} S_\mu, \tag{A.9}$$

where $K_{\mu\lambda}$ *is the number of ways one can fill the boxes of the Young diagram of* μ *with* λ_1 1's, λ_2 2's, *up to* λ_k k's, *in such a way that the entries in each row are nondecreasing, and those in each column are strictly increasing.* Such a tableau is called a *semistandard tableau on* μ *of type* λ. These integers $K_{\mu\lambda}$ are all non-negative, with

$$K_{\lambda\lambda} = 1 \quad \text{and} \quad K_{\mu\lambda} = 0 \quad \text{if } \lambda > \mu, \tag{A.10}$$

i.e., if the first nonvanishing $\lambda_i - \mu_i$ is positive; in addition, $K_{\mu\lambda} = 0$ if λ has more nonzero terms than μ. For example, if $k = 3$, $(K_{\mu\lambda})$ is given by the matrix



	1	1	1
	0	1	2
	0	0	1

The integers $K_{\mu\lambda}$ are called *Kostka numbers.*

Exercise A.11. Show that $K_{\mu\lambda}$ is nonzero if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i \geq 1$.

When $\lambda = (1, 1, \ldots, 1)$, $K_{\mu(1,\ldots,1)}$ *is the number of standard tableaux on the diagram of* μ, where a *standard tableau* is a numbering of the d boxes of a Young diagram by the integers 1 through d, increasing in both rows and columns.

We need one more formula involving Schur polynomials, which comes from an identity of Cauchy. Let $y_1, \ldots, y_k$ be another set of indeterminates, and write $P(x)$ and $P(y)$ for the same polynomial P expressed in terms of variables $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, respectively. The formula we need is

$$\det\left|\frac{1}{1 - x_i y_j}\right| = \frac{\Delta(x)\Delta(y)}{\prod\limits_{i,j}(1 - x_i y_j)}. \tag{A.12}$$

The proof is by induction on k. To compute the determinant, first subtract the first row from each of the other rows, noting that

$$\frac{1}{1 - x_i y_j} - \frac{1}{1 - x_1 y_j} = \frac{x_i - x_1}{1 - x_1 y_j} \cdot \frac{y_j}{1 - x_i y_j}$$

and factor out common factors. Then subtract the first column from each of the other columns, this time using the equation

$$\frac{y_j}{1 - x_i y_j} - \frac{y_1}{1 - x_i y_1} = \frac{y_j - y_1}{1 - x_i y_1} \cdot \frac{1}{1 - x_i y_j}$$

to factor out common factors. One is left with a matrix whose first row is $(1\ 0 \ldots 0)$, and whose lower right square has the original entries. The formula follows by induction (cf. [Wel, p. 202]). □

Another form of Cauchy's identity is

$$\frac{1}{\prod_{i,j}(1-x_iy_j)} = \sum_{\lambda} S_\lambda(x)S_\lambda(y), \tag{A.13}$$

the sum over all partitions λ with at most k terms. To prove this, expand the determinant whose i, j entry is $(1 - x_iy_j)^{-1} = 1 + x_iy_j + x_i^2y_j^2 + \cdots$. One sees that for any $l_1 > \cdots > l_k$ the coefficient of $y_1^{l_1}y_2^{l_2}\cdot\ldots\cdot y_k^{l_k}$ is the determinant $|x_j^{l_i}|$. By symmetry of the x and y variables we have

$$\det\left|\frac{1}{1-x_iy_j}\right| = \sum_{l} |x_j^{l_i}|\cdot|y_j^{l_i}|. \tag{A.14}$$

Combining (A.12) with (A.4) gives (A.13). □

Expansion of the left-hand side of (A.13) gives

$$\frac{1}{\prod_{i,j}(1-x_iy_j)} = \prod_{j}\left(\sum_{m=0}^{\infty} H_m(x)y_j^m\right) = \sum_{\lambda} H_\lambda(x)M_\lambda(y). \tag{A.15}$$

Since the polynomials H_λ as well as the M_μ form a basis for the symmetric polynomials, one can define a bilinear form $\langle\ ,\ \rangle$ on the space of homogeneous symmetric polynomials of degree d in k variables, by requiring that

$$\langle H_\lambda, M_\mu\rangle = \delta_{\lambda,\mu}, \tag{A.16}$$

where $\delta_{\lambda,\mu}$ is 1 if $\lambda = \mu$ and 0 otherwise. The basic fact here is that *the Schur polynomials form an orthonormal basis for this pairing*:

$$\langle S_\lambda, S_\mu\rangle = \delta_{\lambda,\mu}. \tag{A.17}$$

In particular, this implies that the pairing $\langle\ ,\ \rangle$ is *symmetric*. Equation (A.17) is easily deduced from the preceding equations, as follows. Write $S_\lambda = \sum a_{\lambda\gamma}H_\gamma = \sum b_{\gamma\lambda}M_\gamma$, for some integer matrices $a_{\lambda\gamma}$ and $b_{\gamma\lambda}$. Then

$$\langle S_\lambda, S_\mu\rangle = \sum_{\gamma} a_{\lambda\gamma}b_{\gamma\mu}. \tag{A.18}$$

In order that

$$\sum_{\lambda} S_\lambda(x)S_\lambda(y) = \sum_{\lambda,\gamma,\rho} a_{\lambda\gamma}H_\gamma(x)b_{\rho\lambda}M_\rho(y)$$

be equal to $\sum_\gamma H_\gamma(x)M_\gamma(y)$, which it must by (A.13) and (A.15), we must have

$$\sum_{\lambda} b_{\rho\lambda}a_{\lambda\gamma} = \delta_{\rho,\gamma}.$$

This is equivalent to the equation $\sum_\gamma a_{\lambda\gamma}b_{\gamma\mu} = \delta_{\lambda,\mu}$, which by (A.18) implies (A.17).

Because of this duality, formula (A.9) is equivalent to the equation

$$S_\mu = \sum_{\lambda} K_{\mu\lambda}M_\lambda. \tag{A.19}$$

This gives another formula for these Kostka numbers: $K_{\mu\lambda}$ *is the coefficient of* X^λ *in* S_μ, where $X^\lambda = x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$.

The identities (A.9) and (A.19) for the basic symmetric polynomials allow us to relate the coefficients of X^λ in any symmetric polynomial P with the coefficients expanding P as a linear combination of the Schur polynomials. If P is any homogeneous symmetric polynomial of degree d in k variables, and λ is any partition of d into at most k parts, define numbers $\psi_\lambda(P)$ and $\omega_\lambda(P)$ by

$$\psi_\lambda(P) = [P]_\lambda, \tag{A.20}$$

where $[P]_\lambda$ denotes the coefficient of $X^\lambda = x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$ in P, and

$$\omega_\lambda(P) = [\Delta \cdot P]_l, \quad l = (\lambda_1 + k - 1, \lambda_2 + k - 2, \ldots, \lambda_k); \tag{A.21}$$

here $\Delta = \prod_{i<j}(x_i - x_j)$. We want to compare these two collections of numbers, as λ varies over the partitions.

The first numbers $\psi_\lambda(P)$ are the coefficients in the expression

$$P = \sum \psi_\lambda(P) M_\lambda \tag{A.22}$$

for P as a linear combination of the monomial symmetric polynomials M_λ. The integers $\omega_\lambda(P)$ have a similar interpretation in terms of Schur polynomials:

$$P = \sum \omega_\lambda(P) S_\lambda. \tag{A.23}$$

Note from the definition that the coefficient of X^l in $\Delta \cdot S_\lambda$ is 1, and that no other monomial with strictly decreasing exponents appears in $\Delta \cdot S_\lambda$; from this, formula (A.23) is evident. In this terminology we may rewrite (A.19) and (A.9) as

$$K_{\mu\lambda} = \psi_\lambda(S_\mu) = [S_\mu]_\lambda = \text{coefficient of } X^\lambda \text{ in } S_\mu \tag{A.24}$$

and

$$K_{\mu\lambda} = \omega_\mu(H_\lambda) = [\Delta \cdot H_\lambda]_{(\lambda_1 + k - 1, \ldots, \lambda_k)}. \tag{A.25}$$

Lemma A.26. *For any symmetric polynomial P of degree d in k variables,*

$$\psi_\lambda(P) = \sum_\mu K_{\mu\lambda} \cdot \omega_\mu(P).$$

Proof. We have

$$\sum_\lambda \psi_\lambda(P) M_\lambda = P = \sum_\mu \omega_\mu(P) S_\mu = \sum_{\lambda,\mu} \omega_\mu(P) K_{\mu\lambda} M_\lambda$$
$$= \sum_\lambda \left(\sum_\mu K_{\mu\lambda} \omega_\mu(P) \right) M_\lambda,$$

and the result follows, since the M_λ are independent. □

We want to apply the preceding discussion when the polynomial P is a product of sums of powers of the variables. Let $P_j = x_1^j + \cdots + x_k^j$, and for

$\mathbf{i} = (i_1, \ldots, i_d)$, a d-tuple of non-negative integers with $\sum \alpha i_\alpha = d$, set

$$P^{(\mathbf{i})} = P_1^{i_1} \cdot P_2^{i_2} \cdot \ldots \cdot P_d^{i_d}.$$

These *Newton* or *power sum* polynomials form a basis for the symmetric functions with rational coefficients, but not with integer coefficients. Let

$$\omega_\lambda(\mathbf{i}) = \omega_\lambda(P^{(\mathbf{i})}).$$

Equivalently,

$$P^{(\mathbf{i})} = \sum \omega_\lambda(\mathbf{i}) S_\lambda. \tag{A.27}$$

For the proof of Frobenius's formula in Lecture 4 we need a formal lemma about these coefficients $\omega_\lambda(\mathbf{i})$:

Lemma A.28. *For partitions λ and μ of d,*

$$\sum_{\mathbf{i}} \frac{1}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!} \omega_\lambda(\mathbf{i}) \omega_\mu(\mathbf{i}) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We will use Cauchy's formula (A.13). Note that

$$\log\left(\prod_{i,j} (1 - x_i y_j)^{-1}\right) = \sum_{j=1}^{\infty} \frac{1}{j} P_j(x) P_j(y),$$

so

$$\begin{aligned} \frac{1}{\prod (1 - x_i y_j)} &= \prod_j \exp\left(\frac{1}{j} P_j(x) P_j(y)\right) \\ &= \sum_{\mathbf{i}} \frac{1}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!} P^{(\mathbf{i})}(x) P^{(\mathbf{i})}(y) \\ &= \sum_{\mathbf{i}} \frac{1}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!} \sum_\lambda \omega_\lambda(\mathbf{i}) S_\lambda(x) \sum_\mu \omega_\mu(\mathbf{i}) S_\mu(y). \end{aligned}$$

Comparing with (A.13), the conclusion follows. □

Exercise A.29*. Using the pairing $\langle\ ,\ \rangle$ of (A.16), the coefficients $\omega_\lambda(\mathbf{i}) = \omega_\lambda(P^{(\mathbf{i})})$ can be written $\omega_\lambda(\mathbf{i}) = \langle S_\lambda, P^{(\mathbf{i})} \rangle$.

(a) Show that the Newton polynomials are orthogonal for this pairing, and

$$\langle P^{(\mathbf{i})}, P^{(\mathbf{i})} \rangle = 1^{i_1} i_1! 2^{i_2} i_2! \cdot \ldots \cdot d^{i_d} i_d!.$$

Equivalently,

$$S_\lambda = \sum \frac{1}{z(\mathbf{i})} \omega_\lambda(\mathbf{i}) P^{(\mathbf{i})},$$

where the sum is over all partitions $\mathbf{i} = (i_1, \ldots, i_d)$ with $\sum \alpha i_\alpha = d$, and $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \ldots \cdot i_d! d^{i_d}$.

(b) Show that $\omega_\lambda(\mathbf{i}) = \sum_\nu \langle S_\lambda, M_\nu \rangle \cdot \langle H_\nu, P^{(\mathbf{i})} \rangle$.

We should remark that we have chosen to write our formulas for a fixed number k of variables, since that often simplifies computations when k is small. It is more usual to require the number of variables to be large, at least as large as the numbers being partitioned—or in the limiting ring with an infinite number of variables, cf. Exercise A.32; the formulas for smaller k are then recovered by setting the variables $x_i = 0$ for $i > k$. For example, if $k \geq 2$ we have $S_{(1)}^2 = S_{(2)} + S_{(1,1)}$, which reduces to $S_{(1)}^2 = S_{(2)}$ when $k = 1$.

The next two exercises give formulas for the value of the Schur polynomials when the variables x_i are all set equal to 1; these numbers are the dimensions of the corresponding representations. For a formula for $S_\lambda(1, \ldots, 1)$ involving hook lengths of the Young diagram of λ, see Exercise 6.4.

Exercise A.30*. When $x_i = x^{i-1}$, the numerators in (A.4) are van der Monde determinants, leading to

(i) $$S_\lambda(1, x, x^2, \ldots, x^{k-1}) = x^k \prod_{i<j} \frac{x^{\lambda_i - \lambda_j + j - i} - 1}{x^{j-i} - 1}.$$

Taking the limit as $x \to 1$, one finds

(ii) $$S_\lambda(1, \ldots, 1) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

By (A.5) and (A.6) we have also the following two formulas:

(iii) $$S_\lambda(1, \ldots, 1) = |h_{\lambda_i + j - i}|, \quad \text{where } \sum h_j t^j = \frac{1}{(1-t)^k}.$$

(iv) $$S_\lambda(1, \ldots, 1) = \left| \binom{k}{\mu_i + j - i} \right|, \quad \text{where } (\mu_1, \ldots, \mu_r) = \lambda'.$$

Exercise A.31*. (a) Show that

$$S_\mu = \sum K_{\mu a} X^a,$$

the sum over all monomials $X^a = x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}$, where, for *any* k-tuple a of non-negative integers, $K_{\mu a}$ is the number of ways to number the boxes of the Young diagram of μ with a_1 1's, a_2 2's, $\ldots$, a_k k's, with nondecreasing rows and strictly increasing columns. In particular, the right-hand side is a symmetric polynomial, a fact which is not obvious from the definition.

(b) Deduce that $S_\mu(1, \ldots, 1)$ is the number of ways to number the boxes of the Young diagram of μ with integers from 1 to k, with nondecreasing rows and strictly increasing columns (i.e., the number of semistandard tableaux).

Exercise A.32*. The idea of considering symmetric polynomials in an arbitrarily large number of variables can be formalized by working in the ring $\Lambda = \varprojlim \Lambda(k)$, where $\Lambda(k)$ denotes the ring of symmetric polynomials in k variables. Then

$$\Lambda = \mathbb{Z}[H_1, \ldots H_k, \ldots] = \mathbb{Z}[E_1, \ldots E_k, \ldots]$$

is a graded polynomial ring, with H_i and E_i of degree i. A ring homomorphism $\vartheta\colon \Lambda \to \Lambda$ can be defined by requiring

$$\vartheta(E_i) = H_i \qquad \text{for all } i.$$

(i) Show that ϑ is an involution: $\vartheta^2 = \vartheta$. Equivalently,

$$\vartheta(H_i) = E_i.$$

(ii) If λ' is the conjugate partition to λ, show that

$$\vartheta(S_\lambda) = S_{\lambda'}.$$

(iii) If $P_j = x_1^j + \cdots + x_k^j$ is the jth power sum, show that

$$\vartheta(P_j) = (-1)^{j-1} P_j.$$

(iv) Deduce the formula

$$E_\lambda = \sum_\mu K_{\mu\lambda} \Delta_{\mu'}.$$

(v) Deduce a dual form of (A.7):

$$S_\lambda \cdot S_{(1,\ldots,1)} = S_\lambda \cdot E_m = \sum S_\pi,$$

the sum over all partitions π whose Young diagram can be obtained from that of λ by adding m boxes, with no two in any row.

(vi) Show that

$$H_m = \sum \frac{1}{z(\mathbf{i})} P^{(\mathbf{i})}, \qquad E_m = \sum \frac{(-1)^{\sum(i_j - 1)}}{z(\mathbf{i})} P^{(\mathbf{i})},$$

where the sums are over all $\mathbf{i} = (i_1, \ldots, i_d)$ with $\sum \alpha i_\alpha = d$, and $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \ldots \cdot i_d! d^{i_d}$. Note that

$$\sum (-1)^i H_i t^i = (\sum E_i t^i)^{-1}.$$

§A.2. Proofs of the Determinantal Identities

To prove the Jacobi–Trudi identity (A.5), note the identities

$$x_j^p - E_1 x_j^{p-1} + E_2 x_j^{p-1} - \cdots + (-1)^k E_k x_j^{p-k} = 0, \tag{A.33}$$

for any $1 \le j \le k$, $p \ge k$. And for any $0 \le m < k$ and $p \ge k$,

$$H_{p-m} - E_1 H_{p-m-1} + E_2 H_{p-m-2} + \cdots + (-1)^k E_k H_{p-m-k} = 0. \tag{A.34}$$

Both of these follow immediately from the defining power series for the E_j and H_j. Since these two recursion relations are the same, there are universal polynomials $A(p, q)$ in the variables $E_1, \ldots, E_k$ such that

$$\begin{aligned} x_j^p &= A(p, 1)x_j^{k-1} + A(p, 2)x_j^{k-2} + \cdots + A(p, k), \\ H_{p-m} &= A(p, 1)H_{k-m-1} + A(p, 2)H_{k-m-2} + \cdots + A(p, k)H_{-m}. \end{aligned} \tag{A.35}$$

For any integers $\lambda_1, \ldots, \lambda_k$ this leads to matrix identities

$$\begin{aligned} (x_j^{\lambda_i+k-i})_{ij} &= (A(\lambda_i + \mathrm{k} - \mathrm{i}, r))_{ir} \cdot (x_j^{k-r})_{rj}, \\ (H_{\lambda_i+j-i})_{ij} &= (A(\lambda_i + \mathrm{k} - \mathrm{i}, r))_{ir} \cdot (H_{j-r})_{rj}, \end{aligned} \tag{A.36}$$

where $(\ )_{pq}$ denotes the $k \times k$ matrix whose p, q entry is specified between the parentheses. The relations (A.34) also imply:

Lemma (A.37). *The matrices* (H_{q-p}) *and* $((-1)^{q-p}E_{q-p})$ *are lower-triangular matrices with* 1's *along the diagonal, and are inverses of each other.*

The identities (A.36) therefore combine to give

$$(x_j^{\lambda_i+k-i})_{ij} = (H_{\lambda_i+p-i})_{ip} \cdot ((-1)^{q-p}E_{q-p})_{pq} \cdot (x_j^{k-q})_{qj} \tag{A.38}$$

Taking determinants gives (A.5), since the determinant of the matrix in the middle is 1.

Exercise A.39*. Prove the identity

$$|x_j^{l_i}| \cdot \prod_{j=1}^{k} (1 - x_j)^{-1} = \sum |x_j^{m_i}|,$$

the sum over all k-tuples $(m_1, \ldots, m_k)$ of non-negative integers with $m_1 \geq l_1 > m_2 \geq \cdots > m_k \geq l_k$, and deduce Pieri's formula (A.7).

To complete the proofs of the assertions in §A.1, we show that the two determinants appearing in the Giambelli formulas (A.5) and (A.6) are equal, i.e., if $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ are conjugate partitions, then

$$|H_{\lambda_i+j-i}| = |E_{\mu_i+j-i}|. \tag{A.40}$$

Here the H_i and E_i can be any elements (in a commutative ring) satisfying the identity $(\sum H_i t^i) \cdot (\sum (-1)^i E_i t^i) = 1$, with $H_0 = E_0 = 1$ and $H_i = E_i = 0$ for $i < 0$. To prove it, we need a combinatorial characterization of the conjugacy of partitions:

Exercise A.41*. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ conjugate partitions, show that the sets

$$\{\lambda_i + n + 1 - i \colon 1 \leq i \leq k\} \quad \text{and} \quad \{n + j - \mu_j \colon 1 \leq j \leq l\}$$

form a disjoint union of the set $\{1, \ldots, k + l\}$.

We also need a basic matrix identity which relates minors of a matrix to minors of its inverse (or matrix of cofactors). If $A = (a_{ij})$ is an $r \times r$ matrix, and $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ are two sequences of k distinct integers from $\{1, \ldots, r\}$, let $A_{S,T}$ denote the corresponding minor: $A_{S,T}$ is the determinant of the $k \times k$ matrix whose i, j entry is a_{s_i, t_j}.

Lemma A.42. *Let A and B be $r \times r$ matrices whose product is a scalar matrix $c \cdot I_r$. Let (S, S') and (T, T') be permutations of the sequence $(1, \ldots, r)$, where S and T consists of k integers, S' and T' of $r - k$. Then*

$$c^{r-k} \cdot A_{S,T} = \varepsilon \cdot \det(A) \cdot B_{T',S'},$$

where ε is the product of the signs of the two permutations.

Proof. By permuting the rows and columns of A, multiplying on the left and right by permutation matrices P and Q corresponding to the two permutations of $(1, \ldots, r)$, we may take the (S, T) minor to the upper left corner:

$$PAQ = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad A_{S,T} = \det A_1.$$

Then

$$Q^{-1}BP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \qquad B_{T',S'} = \det B_4.$$

Now taking determinants in the identity

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \cdot \begin{pmatrix} I_k & B_2 \\ 0 & B_4 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ A_3 & cI_{r-k} \end{pmatrix}$$

gives the equation $\det(PAQ) \cdot \det(B_4) = \det(A_1) \cdot c^{r-k}$. Since ε is the product of the determinants of P and Q, the lemma follows. □

Proof of (A.40). Apply the lemma to $A = (H_{q-p})$ and $B = ((-1)^{q-p}E_{q-p})$, with $r = k + l$, and

$$S = (\lambda_1 + k, \lambda_2 + k - 1, \ldots, \lambda_k + 1),$$
$$S' = (k + 1 - \mu_1, k + 2 - \mu_2, \ldots, k + l - \mu_l),$$
$$T = (k, k - 1, \ldots, 1),$$
$$T' = (k + 1, k + 2, \ldots, k + l).$$

Then

$$A_{S,T} = \det(H_{(\lambda_i + k + 1 - i) - (k + 1 - j)}) = |H_{\lambda_i + j - i}|.$$

Similarly,

$$B_{T',S'} = |(-1)^{\mu_j + i - j} E_{\mu_j + i - j}| = (-1)^{\sum (\mu_j - j)} (-1)^{\sum i} |E_{\mu_j + i - j}|$$
$$= (-1)^d |E_{\mu_j + i - j}|,$$

with $d = \sum \mu_j = \sum \lambda_i$. Since $\varepsilon = (-1)^d$, (A.40) follows. □

§A.3. Other Determinantal Identities

In this final section we prove some variations of these formulas which are useful for calculating characters of symplectic and orthogonal groups. We want to compare minors, not of $H = (H_{i-j})$ and $E = ((-1)^{i-j}E_{i-j})$, but of matrices H^+ and E^- constructed from them by the following procedures:

For an $r \times r$ matrix $H = (H_{i,j})$, and a fixed integer k between 1 and r, H^+ denotes the $r \times r$ matrix obtained from H by folding H along the kth column, and adding each column to the right of the kth column to the column the same distance to the left. That is,

$$H^+_{i,j} = \begin{cases} H_{i,j} + H_{i,2k-j} & \text{if } j < k \\ H_{i,j} & \text{if } j \geq k \end{cases}$$

(with the convention that $H_{p,q} = 0$ if p or q is not between 1 and r). The matrix E^- is obtained by folding E along its kth row, and subtracting rows above this row from those below:

$$E^-_{i,j} = \begin{cases} E_{i,j} - E_{2k-i,j} & \text{if } i > k \\ E_{i,j} & \text{if } i \leq k. \end{cases}$$

Lemma A.43. *If H and E are lower-triangular matrices with 1's along the diagonal, that are inverse to each other, then the same is true for H^+ and E^-.*

Proof. This is a straightforward calculation: the i, j entry of the matrix $H^+ \cdot E^-$ is

$$\sum_{p=1}^{k-1} (H_{i,p} + H_{i,2k-p})E_{p,j} + H_{i,k}E_{k,j} + \sum_{p=k+1}^{r} H_{i,p}(E_{p,j} - E_{2k-p,j})$$
$$= \sum_{p=1}^{r} H_{i,p}E_{p,j} + \sum_{p=1}^{k-1} H_{i,2k-p}E_{p,j} - \sum_{q=k+1}^{r} H_{i,q}E_{2k-q,j}.$$

The first sum is $\delta_{i,j}$, and the others cancel term by term. □

Proposition A.44. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be conjugate partitions. Set*

$$E'_i = E_i \quad \text{for } i \leq 1, \qquad \text{and} \qquad E'_i = E_i - E_{i-2} \quad \text{for } i \geq 2.$$

Then the determinant of the $k \times k$ matrix whose ith row is

$$(H_{\lambda_i-i+1} \quad H_{\lambda_i-i+2} + H_{\lambda_i-i} \quad H_{\lambda_i-i+3} + H_{\lambda_i-i-1} \quad \cdots \quad H_{\lambda_i-i+k} + H_{\lambda_i-i-k+2})$$

is equal to the determinant of the $l \times l$ matrix whose ith row is

$$(E'_{\mu_i-i+1} \quad E'_{\mu_i-i+2} + E'_{\mu_i-i} \quad E'_{\mu_i-i+3} + E'_{\mu_i-i-1} \quad \cdots \quad E'_{\mu_i-i+l} + E'_{\mu_i-i-l+2}).$$

Each of these determinants is equal to the determinant

$$|E_{\mu_i-i+j} - E_{\mu_i-i-j}|$$

and to the determinant

$$|H''_{\lambda_i-i+j} - H''_{\lambda_i-i-j}|,$$

where $H''_i = H_i$ *for* $i \leq 1$, *and for* $i \geq 2$

$$H''_i = H_i + H_{i-2} + H_{i-4} + \cdots + \begin{cases} H_1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

PROOF. With $H = (H_{i-j})$ and $E = ((-1)^{q-p}E_{q-p})$ we can apply the basic lemma (A.42) to the new matrices $A = H^+$ and $B = E^-$, and the same permutations (S, S') and (T, T') used in the proof of (A.40). This time

$$A_{S,T} = \det(H^+_{\lambda_i+k+1-i,k-j+1}),$$

and

$$H^+_{\lambda_i+k+1-i,k-j+1} = \begin{cases} H_{\lambda_i-i+j} + H_{\lambda_i-i-j+2} & \text{if } j = 2, \dots, k \\ H_{\lambda_i-i+1} & \text{if } j = 1. \end{cases}$$

Similarly,

$$B_{T',S'} = \det(E^-_{k+i,k+j-\mu_j}),$$

with

$$E^-_{k+i,k+j-\mu_j} = (-1)^{\mu_j+i-j}(E_{\mu_j+i+j} - E_{\mu_j+i-j}).$$

As before, Lemma A.42 implies that the determinant of the first displayed matrix of the proposition is equal to that of the third. Noting that

$$E_{\mu_j+i+j} - E_{\mu_j+i-j} = E'_{\mu_j+i+j} + E'_{\mu_j+i+j-2} + \cdots + E'_{\mu_j+i-j+2},$$

one can do elementary column operations on the third matrix, subtracting the first column from the third, then the second by the fourth, etc., to see that the second and third determinants are equal. Since $H_i = H''_i - H''_{i-2}$, the same argument shows the equality of the first and fourth determinants. □

Note that in these four formulas, as in the determinantal formulas for Schur polynomials, if a partition has p nonzero terms, only the upper left $p \times p$ subdeterminant needs to be calculated. We denote by $S_{\langle\lambda\rangle}$ the determinant of the proposition:

$$S_{\langle\lambda\rangle} = |H_{\lambda_i-i+1} \quad H_{\lambda_i-i+2} + H_{\lambda_i-i} \quad \dots \quad H_{\lambda_i-i+k} + H_{\lambda_i-i-k+2}|. \tag{A.45}$$

Dually, set $H'_i = H_i - H_{i-2}$ and $E''_i = E_i + E_{i-2} + E_{i-4} + \cdots$.

Corollary A.46. *The following determinants are equal:*

(i) $$|H'_{\lambda_i-i+1} \quad H'_{\lambda_i-i+2} + H'_{\lambda_i-i} \quad \dots \quad H'_{\lambda_i-i+k} + H'_{\lambda_i-i-k+2}|,$$

(ii) $$|E_{\mu_i-i+1} \quad E_{\mu_i-i+2} + E_{\mu_i-i} \quad \dots \quad E_{\mu_i-i+l} + E_{\mu_i-i-l+2}|,$$

(iii) $$|E''_{\mu_i-i+j} - E''_{\mu_i-i-j}|,$$

(iv) $$|H_{\lambda_i-i+j} - H_{\lambda_i-i-j}|.$$

Define $S_{[\lambda]}$ to be the determinant of this corollary:

$$S_{[\lambda]} = |H'_{\lambda_i-i+1} \quad H'_{\lambda_i-i+2} + H'_{\lambda_i-i} \quad \cdots \quad H'_{\lambda_i-i+k} + H'_{\lambda_i-i-k+2}|. \tag{A.47}$$

Exercise A.48*. Let Λ be the ring of symmetric polynomials, $\vartheta\colon \Lambda \to \Lambda$ the involution of Exercise A.32. Show that

$$\vartheta(S_{\langle\lambda\rangle}) = S_{[\mu]}$$

when λ and μ are conjugate partitions.

For applications to symplectic and orthogonal characters we need to specialize the variables $x_1, \ldots, x_k$. First (for the symplectic group Sp_{2n}) take $k = 2n$, let $z_1, \ldots, z_n$ be independent variables, and specialize

$$x_1 \mapsto z_1, \ \ldots, \ x_n \mapsto z_n, \ x_{n+1} \mapsto z_1^{-1}, \ \ldots, \ x_{2n} \mapsto z_n^{-1}.$$

Set

$$J_j = H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}) \tag{A.49}$$

in the field $\mathbb{Q}(z_1, \ldots, z_n)$ of rational functions.

Proposition A.50. *Given integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, we have*

$$\frac{|z_j^{\lambda_i+n-i+1} - z_j^{-(\lambda_i+n-i+1)}|}{|z_j^{n-i+1} - z_j^{-(n-i+1)}|} = |J_\lambda|,$$

where J_λ denotes the $n \times n$ matrix whose ith *row is*

$$(J_{\lambda_i-i+1} \quad J_{\lambda_i-i+2} + J_{\lambda_i-i} \quad \cdots \quad J_{\lambda_i-i+n} + J_{\lambda_i-i-n+2}).$$

From Proposition A.44 we obtain three other formulas for the right-hand side, e.g.,

$$|J_\lambda| = |e_{\mu_i-i+j} - e_{\mu_i-i-j}|, \tag{A.51}$$

where $e_j = E_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})$, and μ is the conjugate partition to λ.

Exercise A.52. Calculate the denominator of the left-hand side:

$$|z_j^{n-i+1} - z_j^{-(n-i+1)}| = \Delta(\xi_1, \ldots, \xi_n) \cdot \zeta_1 \cdot \ldots \cdot \zeta_n,$$

where $\xi_j = z_j + z_j^{-1}$ and $\zeta_j = z_j - z_j^{-1}$.

Proof of Proposition A.50. Set

$$\zeta_j(p) = z_j^p - z_j^{-p}, \qquad \xi_j(p) = z_j^p + z_j^{-p}. \tag{A.53}$$

By the same argument that proved the Jacobi–Trudy formula (A.5) via (A.38), the proposition follows from the following lemma:

Lemma A.54. *For $1 \leq j \leq n$ and any integer $l \geq 0$, $\zeta_j(l)$ is the product of the $1 \times n$, $n \times n$, and $n \times 1$ matrices*

$$(J_{l-n} \quad J_{l-n+1} + J_{l-n-1} \quad \cdots \quad J_{l-1} + J_{l-2n+1}) \cdot ((-1)^{q-p} e_{q-p}) \cdot \begin{pmatrix} \zeta_j(n) \\ \zeta_j(n-1) \\ \vdots \\ \vdots \\ \zeta_j(1) \end{pmatrix}$$

PROOF. From (A.37) we can calculate z_j^l and z_j^{-l}, and subtracting gives

$$\zeta_j(l) = \sum_{p=1}^{2n} J_{l-2n+p} s_p, \tag{A.55}$$

where $s_p = \sum_{q=p}^{2n} (-1)^{q-p} e_{q-p} \zeta_j(2n-q)$. Multiplying (A.33) by z_j^{-p} and subtracting we find

$$\begin{aligned} &\zeta_j(p) - e_1 \zeta_j(p-1) + \cdots + (-1)^{p-1} e_p \zeta_j(1) \\ &\quad = (-1)^{p+1} e_{p+1} \zeta_j(1) + (-1)^{p+2} e_{p+2} \zeta_j(2) + \cdots + e_{2n} \zeta_j(2n-p). \end{aligned} \tag{A.56}$$

Note also that

$$(-1)^p e_p = (-1)^{2n-p} e_{2n-p}, \tag{A.57}$$

since $\sum (-1)^p e_p t^p = \prod (1 - z_i t)(1 - z_i^{-1} t) = \prod (1 - \xi_i t + t^2)$. From (A.56) and (A.57) follows

$$s_{2n-p} = s_p = r_{n-p+1}, \tag{A.58}$$

where $r_p = \sum_{q=p}^{n} (-1)^{q-p} e_{q-p} \zeta_j(n+1-q)$. Combining (A.55) and (A.58) concludes the proof. □

Next (for the odd orthogonal groups O_{2n+1}) let $k = 2n+1$, and specialize the variables $x_1, \ldots, x_{2n}$ as above, and $x_{2n+1} \mapsto 1$. We introduce variables $z_j^{1/2}$ and $z_j^{-1/2}$, square roots of the variables just considered, and we work in the field $\mathbb{Q}(z_1^{1/2}, \ldots, z_n^{1/2})$. Set

$$\begin{aligned} K_j &= H_j'(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1) \\ &= H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1) - H_{j-2}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1), \end{aligned} \tag{A.59}$$

where H_j is the jth complete symmetric polynomial in $2n+1$ variables.

Proposition A.60. *Given integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, we have*

$$\frac{|z_j^{\lambda_i+n-i+1/2} - z_j^{-(\lambda_i+n-i+1/2)}|}{|z_j^{n-i+1/2} - z_j^{-(n-i+1/2)}|} = |K_\lambda|,$$

where K_λ *is the* $n \times n$ *matrix whose* ith *row is*

$$(K_{\lambda_i-i+1} \quad K_{\lambda_i-i+2}+K_{\lambda_i-i} \quad \cdots \quad K_{\lambda_i-i+n}+K_{\lambda_i-i-n+2}).$$

Corollary A.46 gives three alternative expressions for this determinant, e.g.,

$$|K_\lambda| = |h_{\lambda_i-i+j} - h_{\lambda_i-i-j}|, \tag{A.61}$$

where $h_j = H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1)$.

Exercise A.62. Calculate the denominator of the left-hand side:

$$|z_j^{n-i+1/2} - z_j^{-(n-i+1/2)}| = \Delta(\xi_1, \ldots, \xi_n)\cdot\zeta_1(\tfrac{1}{2})\cdot\ldots\cdot\zeta_n(\tfrac{1}{2}).$$

Proof of Proposition A.60. We have $\zeta_j(l) = z_j^l - z_j^{-l}$ and $\xi_j(l) = z_j^l + z_j^{-l}$ in $\mathbb{Q}(z_1^{1/2}, \ldots, z_n^{1/2})$ for l an integer or a half integer. First note that

$$\xi_j(\tfrac{1}{2})\cdot\zeta_j(l) = \zeta_j(l+\tfrac{1}{2}) + \zeta_j(l-\tfrac{1}{2}).$$

Multiplying the numerator and denominator of the left-hand side of the statement of the proposition by $\xi_1(\frac{1}{2})\cdot\ldots\cdot\xi_n(\frac{1}{2})$, the numerator becomes $|\zeta_j(\lambda_i+n-i+1)+\zeta_j(\lambda_i+n-i)|$, and the denominator becomes $|\zeta_j(n-i+1)+\zeta_j(n-i)| = |\zeta_j(n-i+1)|$. We can, therefore, apply Lemma A.54 to calculate the ratio, getting the determinant of a matrix whose entries are sums of certain J_j's. Note that by direct calculation $K_j = J_j + J_{j-1}$, so the terms can be combined, and the ratio is the determinant of the displayed matrix K_λ. □

Finally (for the even orthogonal groups O_{2n}), let $k = 2n$, and specialize the variables $x_1, \ldots, x_{2n}$ as above. Set

$$\begin{aligned} L_j &= H_j'(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}) \\ &= H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}) - H_{j-2}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}), \end{aligned} \tag{A.63}$$

with H_j the complete symmetric polynomial in $2n$ variables.

Proposition A.64. *Given integers* $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, *we have*

$$\frac{|z_j^{\lambda_i+n-i} + z_j^{-(\lambda_i+n-i)}|}{|z_j^{n-i} + z_j^{-(n-i)}|} = \begin{cases} \frac{1}{2}|L_\lambda| & \text{if } \lambda_n > 0 \\ |L_\lambda| & \text{if } \lambda_n = 0, \end{cases}$$

where L_λ *is the* $n \times n$ *matrix whose* ith *row is*

$$(L_{\lambda_i-i+1} \quad L_{\lambda_i-i+2}+L_{\lambda_i-i} \quad \cdots \quad L_{\lambda_i-i+n}+L_{\lambda_i-i-n+2}).$$

As before, there are other expressions for these determinants, e.g.,

$$|L_\lambda| = |h_{\lambda_i-i+j} - h_{\lambda_i-i-j}|, \tag{A.65}$$

where $h_j = H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})$.

Exercise A.66. Calculate the denominator of the left-hand side:

$$|z_j^{n-i} + z_j^{-(n-i)}| = 2 \cdot \Delta(\xi_1, \ldots, \xi_n).$$

Proof of Proposition A.64. Note that $\zeta_j \cdot \xi_j(l) = \xi_j(l+1) - \xi_j(l-1)$. Multiplying the numerator and denominator by $\zeta_1 \cdot \ldots \cdot \zeta_n$, the numerator becomes $|\zeta_j(\lambda_i + n - i + 1) - \zeta_j(\lambda_i + n - i - 1)|$ and the denominator becomes

$$|\zeta_j(n-i+1) - \zeta_j(n-i-1)| = 2|\zeta_j(n-i+1)|;$$

this is seen by noting that the bottom row of the matrix on the left is $(\zeta_j(1) - \zeta_j(-1)) = (2\zeta_j(1))$, and performing row reductions starting from the bottom row. The rest of the proof is the same as in the preceding proposition. The only change is when $\lambda_n = 0$, in which case the bottom row in the numerator matrix is the same as that in the denominator. □

Exercise A.67*. Find a similar formula for

$$\frac{|z_j^{\lambda_i+n-i} - z_j^{-(\lambda_i+n-i)}|}{|z_j^{n-i} + z_j^{-(n-i)}|}.$$

APPENDIX B

On Multilinear Algebra

In this appendix we state the basic facts about tensor products and exterior and symmetric powers that are used in the text. It is hoped that a reader with some linear algebra background can fill in details of the proofs.

§B.1. Tensor Products

The *tensor product* of two vector spaces V and W over a field is a vector space $V \otimes W$ equipped with a bilinear map

$$V \times W \to V \otimes W, \qquad v \times w \mapsto v \otimes w,$$

which is universal: for any bilinear map β: $V \times W \to U$ to a vector space U, there is a unique linear map from $V \otimes W$ to U that takes $v \otimes w$ to $\beta(v, w)$. This universal property determines the tensor product up to canonical isomorphism. If the ground field K needs to be mentioned, the tensor product is denoted $V \otimes_K W$.

If $\{e_i\}$ and $\{f_j\}$ are bases for V and W, the elements $\{e_i \otimes f_j\}$ form a basis for $V \otimes W$. This can be used to construct $V \otimes W$. The construction is functorial: linear maps $V \to V'$ and $W \to W'$ determine a linear map from $V \otimes W$ to $V' \otimes W'$.

Similarly one has the tensor product $V_1 \otimes \cdots \otimes V_n$ of n vector spaces, with its universal multilinear map

$$V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n,$$

taking $v_1 \times \cdots \times v_n$ to $v_1 \otimes \cdots \otimes v_n$. (Recall that a map from the Cartesian product to a vector space U is *multilinear* if, when all but one of the factors V_i are fixed, the resulting map from V_i to U is linear.) The construction of tensor products is commutative:

$$V \otimes W \cong W \otimes V, \qquad v \otimes w \mapsto w \otimes v;$$

distributive:

$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W);$$

and associative:

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W,$$

by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$.

In particular, there are *tensor powers* $V^{\otimes n} = V \otimes \cdots \otimes V$ of a fixed space V. By convention, $V^{\otimes 0}$ is the ground field.

If A is an algebra over the ground field, and V is a right A-module, and W a left A-module, there is a tensor product denoted $V \otimes_A W$, which can be constructed as the quotient of $V \otimes W$ by the subspace generated by all $(v \cdot a) \otimes w - v \otimes (a \cdot w)$ for all $v \in V$, $w \in W$, and $a \in A$. The resulting map from $V \times W$ to $V \otimes_A W$ is universal for bilinear maps β from $V \times W$ to vector spaces U that satisfy the property that $\beta(v \cdot a, w) = \beta(v, a \cdot w)$. This tensor product is also distributive.

§B.2. Exterior and Symmetric Powers

The *exterior powers* $\bigwedge^n V$ of a vector space V, sometimes denoted $\mathrm{Alt}^n V$, come equipped with an alternating multilinear map

$$V \times \cdots \times V \to \textstyle\bigwedge^n V, \qquad v_1 \times \cdots \times v_n \mapsto v_1 \wedge \cdots \wedge v_n,$$

that is universal: for $\beta\colon V \times \cdots \times V \to U$ an alternating multilinear map, there is a unique linear map from $\bigwedge^n V$ to U which takes $v_1 \wedge \cdots \wedge v_n$ to $\beta(v_1, \ldots, v_n)$. Recall that a multilinear map β is *alternating* if $\beta(v_1, \ldots, v_n) = 0$ whenever two of the vectors v_i are equal. This implies that $\beta(v_1, \ldots, v_n)$ changes sign when two of the vectors are interchanged.[1] It follows that

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \mathrm{sgn}(\sigma)\beta(v_1, \ldots, v_n) \quad \text{for all } \sigma \in \mathfrak{S}_n.$$

The exterior power can be constructed as the quotient space of $V^{\otimes n}$ by the subspace generated by all $v_1 \otimes \cdots \otimes v_n$ with two of the vectors equal. We let

$$\pi\colon V^{\otimes n} \to \textstyle\bigwedge^n V, \qquad \pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$$

[1] This follows from the standard polarization.: for two factors, $\beta(v + w, v + w) - \beta(v, v) - \beta(w, w) = \beta(v, w) + \beta(w, v)$.

denote the projection. If $\{e_i\}$ is a basis for V, then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} : i_1 < i_2 < \cdots < i_n\}$$

is a basis for $\wedge^n V$. Define $\wedge^0 V$ to be the ground field.

If V and W are vector spaces, there is a canonical linear map from $\wedge^a V \otimes \wedge^b W$ to $\wedge^{a+b}(V \oplus W)$, which takes $(v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_b)$ to $v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$. This determines an isomorphism

$$\wedge^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \wedge^a V \otimes \wedge^{n-a} W. \tag{B.1}$$

(From this isomorphism the assertion about bases of $\wedge^n V$ follows by induction on the dimension.)

The *symmetric powers* $\mathrm{Sym}^n V$, sometimes denoted $S^n V$, comes with a universal symmetric multilinear map

$$V \times \cdots \times V \to \mathrm{Sym}^n V, \qquad v_1 \times \cdots \times v_n \mapsto v_1 \cdot \ldots \cdot v_n.$$

Recall that a multilinear map $\beta: V \times \cdots \times V \to U$ is *symmetric* if it is unchanged when any two factors are interchanged, or

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \beta(v_1, \ldots, v_n) \quad \text{for all } \sigma \in \mathfrak{S}_n.$$

The symmetric power can be constructed as the quotient space of $V^{\otimes n}$ by the subspace generated by all $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, or by those in which σ permutes two successive factors. Again we let

$$\pi: V^{\otimes n} \to \mathrm{Sym}^n V, \qquad \pi(v_1 \otimes \cdots \otimes v_n) = v_1 \cdot \ldots \cdot v_n,$$

denote the projection. If $\{e_i\}$ is a basis for V, then

$$\{e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_n} : i_1 \le i_2 \le \cdots \le i_n\}$$

is a basis for $\mathrm{Sym}^n V$. So $\mathrm{Sym}^n V$ can be regarded as the space of homogeneous polynomials of degree n in the variables e_i. Define $\mathrm{Sym}^0 V$ to be the ground field. As before, there are canonical isomorphisms

$$\mathrm{Sym}^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \mathrm{Sym}^a V \otimes \mathrm{Sym}^{n-a} W. \tag{B.2}$$

The exterior powers $\wedge^n V$ and symmetric powers $\mathrm{Sym}^n V$ can also be realized as subspaces of $V^{\otimes n}$, assuming, as we have throughout, that the ground field has characteristic 0. We will denote the inclusions by ι, so we have

$$V^{\otimes n} \xrightarrow{\pi} \wedge^n V \xrightarrow{\iota} V^{\otimes n}, \qquad V^{\otimes n} \xrightarrow{\pi} \mathrm{Sym}^n V \xrightarrow{\iota} V^{\otimes n}.$$

The imbedding $\iota: \wedge^n V \to V^{\otimes n}$ is defined by

$$\iota(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{B.3}$$

(This is well defined since the right-hand side is alternating.) The image of ι is the space of anti-invariants of the right action of $\mathfrak{S}_n$ on $V^{\otimes n}$:

$$(v_1 \otimes \cdots \otimes v_n)\cdot\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad v_i \in V, \sigma \in \mathfrak{S}_n. \tag{B.4}$$

(The anti-invariants are the vectors $z \in V^{\otimes n}$ such that $z\cdot\sigma = \mathrm{sgn}(\sigma)z$ for all $\sigma \in \mathfrak{S}_n$.) Moreover, if $A = \iota \circ \pi$, then $(1/n!)A$ is the projection onto this anti-invariant subspace.[2] (Often the coefficient $1/n!$ is put in front of the formula for ι; this makes no essential difference, but leads to awkward formulas for contractions.)

Similarly we have $\iota\colon \mathrm{Sym}^n V \to V^{\otimes n}$ by

$$\iota(v_1 \cdot \ldots \cdot v_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{B.5}$$

The image of ι is the space of invariants of the right action of $\mathfrak{S}_n$ on $V^{\otimes n}$. If $A = \iota \circ \pi$, then $(1/n!)A$ is the projection onto this invariant subspace.

The wedge product $\wedge$ determines a product

$$\textstyle\bigwedge^m V \otimes \bigwedge^n V \xrightarrow{\wedge} \bigwedge^{m+n} V, \tag{B.6}$$

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n},$$

which is associative and skew-commutative. This product is compatible with the projection from the tensor powers onto the exterior powers, but care must be taken for the inclusion of exterior in tensor powers, since for example $v \wedge w$ is sent to $v \otimes w - w \otimes v$ [not to $\frac{1}{2}(v \otimes w - w \otimes v)$] by ι. In general, the diagram

$$\begin{tikzcd} \bigwedge^m V \otimes \bigwedge^n V \arrow[r, "\wedge"] \arrow[d, hook, "\iota\otimes\iota"'] & \bigwedge^{m+n} V \arrow[d, hook, "\iota"'] \\ V^{\otimes m} \otimes V^{\otimes n} \arrow[r] & V^{\otimes(m+n)} \end{tikzcd} \tag{B.7}$$

commutes when the bottom horizontal map is defined by the formula

$$\begin{aligned} &(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_{m+n}) \\ &\quad\mapsto \sum \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \otimes v_{\sigma(m+1)} \otimes \cdots \otimes v_{\sigma(m+n)}, \end{aligned} \tag{B.8}$$

the sum over all "shuffles," i.e., permutations σ of $\{1, \ldots, m+n\}$ that preserve the order of the subsets $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$.

Similarly the symmetric powers have a commutative product $(v_1 \cdot \ldots \cdot v_m) \otimes (v_{m+1} \cdot \ldots \cdot v_{m+n}) \mapsto v_1 \cdot \ldots \cdot v_m \cdot v_{m+1} \cdot \ldots \cdot v_{m+n}$, with a similar compatibility. Note that $v^2 \in \mathrm{Sym}^2 V$ is sent to $2v \otimes v$ in $V \otimes V$, $v^n \in \mathrm{Sym}^n V$ to $n!(v \otimes \cdots \otimes v)$ in $V^{\otimes n}$, and generally one has the analogue of (B.7), changing each "sgn(σ)" to "1" in formula (B.8).

All these mappings are compatible with linear maps of vector spaces $V \to W$, and in particular commute with the left actions of the general linear group $\mathrm{GL}(V) = \mathrm{Aut}(V)$ of automorphisms, or the algebra $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ of endomorphisms, on $V^{\otimes n}$, $\bigwedge^n V$, and $\mathrm{Sym}^n V$.

[2] It is this factor which limits our present discussion to vector spaces over fields of characteristic 0.

It is sometimes convenient to make algebras out of the direct sum of all of the tensor, exterior, or symmetric powers. The *tensor algebra* $T^{\cdot}V$ is the sum $\bigoplus_{n\geq 0} V^{\otimes n}$, with product determined by the canonical isomorphism $V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes(n+m)}$. The *exterior algebra* $\bigwedge^{\cdot}V$ is the sum $\bigoplus_{n\geq 0} \bigwedge^n V$, which is the quotient of $T^{\cdot}V$ by the two-sided ideal generated by all $v \otimes v$ in $V^{\otimes 2}$. The *symmetric algebra* $\mathrm{Sym}^{\cdot}V$ is the sum $\bigoplus_{n\geq 0} \mathrm{Sym}^n V$, which is the quotient of $T^{\cdot}V$ by the two-sided ideal generated by all $v \otimes w - w \otimes v$ in $V^{\otimes 2}$.

Exercise B.9. The algebra $\mathrm{Sym}^{\cdot}V$ is a commutative, graded algebra, which satisfies the universal property that any linear map from V to the first graded piece C^1 of a commutative graded algebra $C^{\cdot}$ determines a homomorphism $\mathrm{Sym}^{\cdot}V \to C^{\cdot}$ of graded algebras. Use this to show that $\mathrm{Sym}^{\cdot}(V \oplus W) \cong \mathrm{Sym}^{\cdot}V \otimes \mathrm{Sym}^{\cdot}W$, and deduce the isomorphism (B.2). Prove the analogous assertions for $\bigwedge^{\cdot}V$, in the category of skew-commutative graded algebras. In particular, construct an isomorphism $\bigwedge^{\cdot}(V \oplus W) \cong \bigwedge^{\cdot}V \mathbin{\hat{\otimes}} \bigwedge^{\cdot}W$, where $\hat{\otimes}$ denotes the skew-commutative tensor product: it is the usual tensor product additively, but the product has $(a \otimes b)\cdot(c \otimes d) = (-1)^{\deg(b)\deg(c)}(a \cdot b) \otimes (c \cdot d)$ for homogeneous elements a and c in the first algebra, and b and d in the second. In particular, this proves (B.1).

§B.3. Duals and Contractions

Although only a few simple contractions are used in the lectures, and most of these are written out by hand where needed, it may be useful to see the general picture.

If V^* denotes the dual space to V, there are contraction maps

$$c_j^i\colon V^{\otimes p} \otimes (V^*)^{\otimes q} \to V^{\otimes(p-1)} \otimes (V^*)^{\otimes(q-1)},$$

for any $1 \leq i \leq p$ and $1 \leq j \leq q$, determined by evaluating the jth coordinate of $(V^*)^{\otimes q}$ on the ith coordinate of $V^{\otimes p}$:

$$\begin{aligned} &c_j^i(v_1 \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_q) \\ &\quad = \varphi_j(v_i) v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \hat{\varphi}_j \otimes \cdots \otimes \varphi_q. \end{aligned} \tag{B.10}$$

More generally if $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ are two sequences of n distinct indices from $\{1, \ldots, p\}$ and $\{1, \ldots, q\}$, respectively, there is a contraction

$$c_J^I\colon V^{\otimes p} \otimes (V^*)^{\otimes q} \to V^{\otimes(p-n)} \otimes (V^*)^{\otimes(q-n)} \tag{B.11}$$

which takes $v_1 \wedge \cdots \wedge v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_q$ to

$$\prod_{\alpha=1}^{n} \varphi_{j_\alpha}(v_{i_\alpha}) v_1 \otimes \cdots \otimes \hat{v}_{i_1} \otimes \cdots \otimes \hat{v}_{i_2} \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \hat{\varphi}_{j_1} \otimes \cdots \otimes \varphi_q.$$

For example, if $p = q = n$ and $I = J = (1, \ldots, n)$, this contraction $V^{\otimes n} \otimes (V^*)^{\otimes n} \to \mathbb{C}$ identifies $(V^*)^{\otimes n}$ with the dual space of $V^{\otimes n}$.

Now $(V^{\otimes n})^*$ consists of n-multilinear forms on V, and $(\wedge^n V)^*$ consists of alternating n multilinear forms on V; in particular, $(\wedge^n V)^*$ is a subspace of $(V^{\otimes n})^*$; this is the inclusion via π^*. The composite

$$\wedge^n(V^*) \to (V^*)^{\otimes n} \to (V^{\otimes n})^*,$$

where the first map is the inclusion ι and the second is the isomorphism of the preceding paragraph, maps $\wedge^n(V^*)$ isomorphically onto the subspace $(\wedge^n V)^*$. Explicitly,

$$\wedge^n(V^*) \xrightarrow{\cong} (\wedge^n V)^*,$$

$$\varphi_1 \wedge \cdots \wedge \varphi_n \mapsto [v_1 \wedge \cdots \wedge v_n \mapsto \sum \operatorname{sgn}(\sigma)\varphi_{\sigma(1)}(v_1)\cdot \ldots \cdot \varphi_{\sigma(n)}(v_n)$$
$$= \det(\varphi_j(v_i))].$$

This dual pairing $\wedge^n V \otimes \wedge^n(V^*) \to K$ is often denoted $\langle\ ,\ \rangle$.

There is a similar isomorphism of $\operatorname{Sym}^n(V^*)$ with $\operatorname{Sym}^n(V)^*$, but without the signs "$\operatorname{sgn}(\sigma)$."

Exercise B.12. If $e_1, \ldots, e_m$ is a basis for V, with e_i^* the dual basis for V^*, then $\{e_{i_1} \wedge \cdots \wedge e_{i_n}: 1 \le i_1 < \cdots < i_n \le m\}$ is a basis for $\wedge^n V$, and $\{e_1^{i_1}\cdot \ldots \cdot e_m^{i_m}: i_\alpha \ge 0, \sum i_\alpha = n\}$ is a basis for $\operatorname{Sym}^n V$. Show that, via the above isomorphisms, the dual bases for $\wedge^n(V^*)$ and $\operatorname{Sym}^n(V^*)$ are

$$\{e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*\} \quad \text{and} \quad \left\{\frac{1}{\prod_\alpha (i_\alpha!)}(e_1^*)^{i_1}\cdot \ldots \cdot (e_m^*)^{i_m}\right\}.$$

There are related contractions, sometimes called internal products, and denoted $\lrcorner$ and $\llcorner$, on exterior and symmetric powers. For the exterior powers they are maps:

$$\begin{aligned} &\wedge^p V \otimes \wedge^{p+q}(V^*) \to \wedge^q(V^*), \qquad x \otimes \alpha \mapsto x \lrcorner \alpha; \\ &\wedge^{p+q} V \otimes \wedge^p(V^*) \to \wedge^q(V), \qquad x \otimes \alpha \mapsto x \llcorner \alpha. \end{aligned} \tag{B.13}$$

These can be defined most simply as transposes of wedge products, i.e., they are determined by the identities

$$\langle z, x \lrcorner \alpha\rangle = \langle z \wedge x, \alpha\rangle \quad \text{for } z \in \wedge^q V$$

and

$$\langle x \llcorner \alpha, \beta\rangle = \langle x, \alpha \wedge \beta\rangle \quad \text{for } \beta \in \wedge^q(V^*).$$

(The relation of this definition to the contraction maps c_J^I above is expressed in Exercise B.16.) Note that when $q = 0$, these contractions reduce to the previous duality pairing between $\wedge^p V$ and $\wedge^p(V^*)$.

For symmetric powers, the internal products are defined similarly:

$$\begin{aligned} &\operatorname{Sym}^p V \otimes \operatorname{Sym}^{p+q}(V^*) \to \operatorname{Sym}^q(V^*), \qquad x \otimes \alpha \mapsto x \lrcorner \alpha; \\ &\operatorname{Sym}^{p+q} V \otimes \operatorname{Sym}^p(V^*) \to \operatorname{Sym}^q(V), \qquad x \otimes \alpha \mapsto x \llcorner \alpha. \end{aligned} \tag{B.14}$$

Exercise B.15. For $v, w \in V$, and $\varphi, \psi \in V^*$, show that

$$v \lrcorner (\varphi \wedge \psi) = \psi(v)\varphi - \varphi(v)\psi \quad \text{and} \quad (v \wedge w) \llcorner \varphi = \varphi(v)w - \varphi(w)v.$$

More generally, for if $x = v_1 \wedge \cdots \wedge v_p$ and $\alpha = \varphi_1 \wedge \cdots \wedge \varphi_{p+q}$, with $v_i \in V$ and $\varphi_j \in V^*$, then

(i) $$x \lrcorner \alpha = \sum \operatorname{sgn}(\sigma)\varphi_{\sigma(q+1)}(v_1)\cdot \ldots \cdot \varphi_{\sigma(q+p)}(v_p)\cdot \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(q)},$$

the sum over all permutations σ of $\{1, \ldots, p+q\}$ that preserve the order of $\{1, \ldots, q\}$. If $x = v_1 \wedge \cdots \wedge v_{p+q}$ and $\alpha = \varphi_1 \wedge \cdots \wedge \varphi_p$, then

(ii) $$x \llcorner \alpha = \sum \operatorname{sgn}(\sigma)\varphi_1(v_{\sigma(1)})\cdot \ldots \cdot \varphi_p(v_{\sigma(p)})\cdot v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(p+q)},$$

the sum over all permutations that preserve the order of $\{p+1, \ldots, p+q\}$. Verify these formulas and use them to give formulas for these internal products in terms of standard bases. State and verify analogous formulas for symmetric powers. In particular, for $v, w \in V$, $\varphi, \psi \in V^*$,

$$v \lrcorner (\varphi \cdot \psi) = \psi(v)\varphi + \varphi(v)\psi \quad \text{and} \quad (v \cdot w) \llcorner \varphi = \varphi(v)w + \varphi(w)v.$$

For example, $v \lrcorner (\varphi^2) = 2\varphi(v)\varphi$ and $(v^2) \llcorner \varphi = 2\varphi(v)v$.

Exercise B.16. Using formula (ii) of the preceding exercise, show that the contraction map $\llcorner$ may be given as $1/p!q!$ times the composition of the maps

$$\textstyle\bigwedge^{p+q} V \otimes \bigwedge^p(V^*) \to V^{\otimes(p+q)} \otimes (V^*)^{\otimes p} \to V^{\otimes q} \to \bigwedge^q V,$$

where the middle map is the contraction map c_J^I of (B.11), with $I = J = \{1, \ldots, p\}$, and the other maps come from ι and π. Prove the same formulas (with the same scalar factor) for the other internal products.

Exercise B.17. In the situation of formula (ii), suppose the v_i are independent, and let W be the $(p+q)$-dimensional subspace of V that they span; suppose the φ_i are independent, and let Z be the p-codimensional subspace of V of the common zeros of the φ_i. Show that $x \llcorner \alpha = 0$ if $\dim(W \cap Z) > q$, and otherwise $x \llcorner \alpha = u_1 \wedge \cdots \wedge u_q$ for some vectors u_i that span $W \cap Z$.

Exercise B.18. Prove the formulas

$$(x \wedge y) \lrcorner \alpha = x \lrcorner (y \lrcorner \alpha) \quad \text{and} \quad x \llcorner (\alpha \wedge \beta) = (x \llcorner \alpha) \llcorner \beta.$$

State and verify the analogous formulas for symmetric powers.

For a detailed development of these ideas, see [Bour, *Algebra*, Chap. 3].

APPENDIX C

On Semisimplicity

§C.1. The Killing Form and Cartan's Criterion

We recall first the Jordan decomposition of a linear transformation X of a finite-dimensional complex vector space V as a sum of its semisimple and nilpotent parts: $X = X_s + X_n$, where X_s is the semisimple part of X, and X_n the nilpotent part. It is uniquely characterized by the fact that X_s is semisimple (diagonalizable), X_n is nilpotent, and X_s and X_n commute with each other. In fact, X_s and X_n can be written as polynomials in X, so any endomorphism that commutes with X automatically commutes with X_s and X_n. One case of the invariance of Jordan decomposition is an easy calculation:

Exercise C.1*. For any $X \in \mathfrak{gl}(V)$, the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{gl}(V)$ satisfies

$$\mathrm{ad}(X)_s = \mathrm{ad}(X_s) \quad \text{and} \quad \mathrm{ad}(X)_n = \mathrm{ad}(X_n).$$

There is a Killing form B_V defined on $\mathfrak{gl}(V)$ by the formula

$$B_V(X, Y) = \mathrm{Tr}(X \circ Y), \tag{C.2}$$

where Tr is the trace and $\circ$ denotes composition of transformations. As in (14.23), the identity

$$B_V(X, [Y, Z]) = B_V([X, Y], Z) \tag{C.3}$$

holds for all X, Y, Z in $\mathfrak{gl}(V)$.

The Killing form B on a Lie algebra $\mathfrak{g}$ is that of Exercise C.1 for the adjoint representation: $B(X, Y) = B_{\mathfrak{g}}(\mathrm{ad}(X), \mathrm{ad}(Y))$. This was introduced in Lecture 14, where a few of its properties were proved. Here we use the Killing form to characterize solvability and semisimplicity of the Lie algebra.

If $\mathfrak{g}$ is solvable, by Lie's theorem its adjoint representation can be put in upper-triangular form. It follows that $\mathscr{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ acts by strictly upper-triangular matrices. So if X is in $\mathscr{D}\mathfrak{g}$ and Y in $\mathfrak{g}$, then $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is strictly upper triangular; in particular its trace $B(X, Y)$ is zero. Cartan's criterion is that this characterizes solvability:

Proposition C.4. *The Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $B(\mathfrak{g}, \mathscr{D}\mathfrak{g}) = 0$.

We will prove something that looks a little weaker, but will turn out to be a little stronger. We prove:

Theorem C.5 (Cartan's criterion). *If* $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{gl}(V)$ *and* $B_V(X, Y) = 0$ *for all* X *and* Y *in* $\mathfrak{g}$, *then* $\mathfrak{g}$ *is solvable.*

For this, it suffices to show that every element of $\mathscr{D}\mathfrak{g}$ is nilpotent, for then by Engel's theorem $\mathscr{D}\mathfrak{g}$ must be a nilpotent ideal, and therefore $\mathfrak{g}$ is solvable.

So take $X \in \mathscr{D}\mathfrak{g}$, and let $\lambda_1, \ldots, \lambda_r$ be its eigenvalues (counted with multiplicity) for X as an endomorphism of V. We must show the λ_i are all zero. These eigenvalues satisfy some obvious relations; for example, $\sum \lambda_i \lambda_i = \mathrm{Tr}(X \circ X)) = B_V(X, X) = 0$. What we need to show is

$$\bar{\lambda}_1 \lambda_1 + \cdots + \bar{\lambda}_r \lambda_r = 0. \tag{C.6}$$

To prove this, take a basis for V so that X is in Jordan canonical form, with $\lambda_1, \ldots, \lambda_r$ down the diagonal; the semisimple part $D = X_s$ of X is this diagonal transformation. Let $\bar{D}$ be the endomorphism of V given by the diagonal matrix with $\bar{\lambda}_1, \ldots, \bar{\lambda}_r$ down the diagonal. Since $\mathrm{Tr}(\bar{D} \circ X) = \sum \bar{\lambda}_i \lambda_i$, it suffices to prove

$$\mathrm{Tr}(\bar{D} \circ X) = 0. \tag{C.7}$$

Since X is a sum of commutators $[Y, Z]$, with Y and Z in $\mathfrak{g}$, $\mathrm{Tr}(\bar{D} \circ X)$ is a sum of terms of the form $\mathrm{Tr}(\bar{D} \circ [Y, Z]) = \mathrm{Tr}([\bar{D}, Y] \circ Z)$. So we will be done if we know that $[\bar{D}, Y]$ belongs to $\mathfrak{g}$, for our hypothesis is that $\mathrm{Tr}(\mathfrak{g} \circ \mathfrak{g}) \equiv 0$. That is, we are reduced to showing

$$\mathrm{ad}(\bar{D})(\mathfrak{g}) \subset \mathfrak{g}. \tag{C.8}$$

For this it suffices to prove that $\mathrm{ad}(\bar{D})$ can be written as a polynomial in $\mathrm{ad}(X)$, for we know that $\mathrm{ad}(X)^k(Y)$ is in $\mathfrak{g}$ if X and Y are in $\mathfrak{g}$. Since $\mathrm{ad}(D) = \mathrm{ad}(X_s) = \mathrm{ad}(X)_s$ is a polynomial in $\mathrm{ad}(X)$, it suffices to show that $\mathrm{ad}(\bar{D})$ can be written as a polynomial in $\mathrm{ad}(D)$. This is a simple computation: using the usual basis $\{E_{ij}\}$ for $\mathfrak{gl}(V)$, $\mathrm{ad}(D)$ and $\mathrm{ad}(\bar{D})$ are complex conjugate diagonal matrices, and any such are polynomials in each other. □

We can prove now that if $\mathfrak{g}$ is a Lie algebra for which $B(\mathscr{D}\mathfrak{g}, \mathscr{D}\mathfrak{g}) \equiv 0$, then $\mathfrak{g}$ is solvable, which certainly implies Proposition C.4. By what we just proved, the image of $\mathscr{D}\mathfrak{g}$ by the adjoint representation in $\mathfrak{gl}(\mathfrak{g})$ is solvable. Since the kernel of the adjoint map is abelian, this makes $\mathscr{D}\mathfrak{g}$ solvable (cf. Exercise 9.8), and by definition this makes $\mathfrak{g}$ solvable. □

Exercise C.9. Show that a Lie algebra $\mathfrak{g}$ is solvable if and only if $B(\mathrm{ad}(X), \mathrm{ad}(X)) = 0$ for all X in $\mathfrak{g}$.

It is easy to deduce from Cartan's criterion a criterion for semisimplicity—part of which we saw in Lecture 14, but there assuming some facts we had not proved yet:

Proposition C.10. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if its Killing form* B *is nondegenerate.*

PROOF. By (C.3) the null-space $\mathfrak{s} = \{X \in \mathfrak{g}\colon B(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$ is an ideal. Suppose $\mathfrak{g}$ is semisimple. By Cartan's criterion, the image $\mathrm{ad}(\mathfrak{s}) \subset \mathfrak{gl}(\mathfrak{g})$ is solvable; as in the preceding proof, $\mathfrak{s}$ is then solvable, so $\mathfrak{s} = 0$ by the definition of semisimple. Conversely, if B is nondegenerate, we must show that any abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$ must be zero. If $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$, then $A = \mathrm{ad}(X) \circ \mathrm{ad}(Y)$ maps $\mathfrak{g}$ into $\mathfrak{a}$ and $\mathfrak{a}$ to 0, so $\mathrm{Tr}(A) = 0$. This implies that $\mathfrak{a} \subset \mathfrak{s} = 0$, as required. □

Corollary C.11. *A semisimple Lie algebra is a direct product of simple Lie algebras.*

PROOF. For any ideal $\mathfrak{h}$ of $\mathfrak{g}$, the annihilator

$$\mathfrak{h}^{\perp} = \{X \in \mathfrak{g}\colon B(X, Y) = 0 \text{ for all } Y \in \mathfrak{h}\}$$

is an ideal, by (C.3) again. By Cartan's criterion, $\mathfrak{h} \cap \mathfrak{h}^{\perp}$ is solvable, hence zero, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. The decomposition follows by a simple induction. □

It follows that $\mathfrak{g} = \mathscr{D}\mathfrak{g}$, and that all ideals and images of $\mathfrak{g}$ are semisimple. In fact:

Exercise C.12*. Show that if $\mathfrak{g}$ is a direct product of simple Lie algebras, the only ideals in $\mathfrak{g}$ are sums of some of the factors. In particular, the decomposition into simple factors is unique (not just up to isomorphism).

Exercise C.13*. Show that if $\mathfrak{g}$ is semisimple, the adjoint map ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$ onto the algebra $\mathrm{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$.

Exercise C.14. Show that if $\mathfrak{g}$ is nilpotent then its Killing form is identically zero, and find a counterexample to the converse.

§C.2. Complete Reducibility and the Jordan Decomposition

We repeat that this section is optional, since the results can be deduced from the existence of a compact group such that the complexification of its Lie algebra is a given semisimple Lie algebra. We include here the standard algebraic approach. A finite-dimensional representation of a Lie algebra $\mathfrak{g}$ will be called a $\mathfrak{g}$-module, and a $\mathfrak{g}$-invariant subspace a submodule.

Proposition C.15. *Let V be a representation of the semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ a submodule. Then there exists a submodule $W' \subset V$ complementary to W.*

PROOF. Since the image of $\mathfrak{g}$ by the representation is semisimple, we may assume $\mathfrak{g} \subset \mathfrak{gl}(V)$. We will require a slight generalization of the Casimir operator $C_V \in \operatorname{End}(V)$ which was used in §25.1 in the proof of Freudenthal's formula. We take a basis $U_1, \ldots, U_r$ for $\mathfrak{g}$, and a dual basis $U_1', \ldots, U_r'$, but this time with respect to the Killing form B_V defined in Exercise C.1: $B_V(X, Y) = \operatorname{Tr}(X \circ Y)$. (Note by Cartan's criterion that B_V is nondegenerate.) Then C_V is defined by the formula $C_V(v) = \sum U_i \cdot (U_i' \cdot v)$.

As before, a simple calculation shows that C_V is an endomorphism of V that commutes with the action of $\mathfrak{g}$. Its trace is

$$\operatorname{Tr}(C_V) = \sum \operatorname{Tr}(U_i \circ U_i') = \sum B_V(U_i, U_i') = \dim(\mathfrak{g}). \tag{C.16}$$

We note also that since C_V maps any submodule W to itself, and since it commutes with $\mathfrak{g}$, its kernel $\operatorname{Ker}(C_V)$ and image are submodules.

Note first that all one-dimensional representations of a semisimple $\mathfrak{g}$ are trivial, since $\mathscr{D}\mathfrak{g}$ must act trivially on a one-dimensional representation, and $\mathfrak{g} = \mathscr{D}\mathfrak{g}$.

We proceed to the proof itself. As should be familiar from Lecture 9, the basic case to prove is when $W \subset V$ is an irreducible invariant subspace of codimension one. Then C_V maps W into itself, and C_V acts trivially on V/W. But now by Schur's lemma, since W is irreducible, C_V is multiplication by a scalar on W. This scalar is not zero, or (C.16) would be contradicted. Hence $V = W \oplus \operatorname{Ker}(C_V)$, which finishes this special case.

It follows easily by induction on the dimension that the same is true whenever $W \subset V$ has codimension one. For if W is not irreducible, let Z be a nonzero submodule, and find a complement to $W/Z \subset V/Z$ (by induction), say Y/Z. Since Y/Z is one dimensional, find (by induction) U so that $Y = Z \oplus U$. Then $V = W \oplus U$.

By the same argument, it suffices to prove the statement of the theorem when W is irreducible. Consider the restriction map

$$\rho\colon \operatorname{Hom}(V, W) \to \operatorname{Hom}(W, W),$$

a homomorphism of $\mathfrak{g}$-modules. The second contains the one-dimensional submodule $\mathrm{Hom}_{\mathfrak{g}}(W, W)$. By the preceding case, there is a one-dimensional submodule of $\rho^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W)) \subset \mathrm{Hom}(V, W)$ which maps onto $\mathrm{Hom}_{\mathfrak{g}}(W, W)$ by ρ. Since one-dimensional modules are trivial, this means there is a $\mathfrak{g}$-invariant ψ in $\mathrm{Hom}(V, W)$ such that $\rho(\psi) = 1$. But this means that ψ is a $\mathfrak{g}$-invariant projection of V onto W, so $V = W \oplus \mathrm{Ker}(\psi)$, as required. □

We will apply this to prove the invariance of Jordan decomposition (Theorem 9.20). The essential point is:

Proposition C.17. *Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Then for any element $X \in \mathfrak{g}$, the semisimple part X_s and the nilpotent part X_n are also in $\mathfrak{g}$.*

PROOF. The idea is to write $\mathfrak{g}$ as an intersection of Lie subalgebras of $\mathfrak{gl}(V)$ for which the conclusion of the theorem is easy to prove. For example, we know $\mathfrak{g} \subset \mathfrak{sl}(V)$ since $\mathfrak{g} = \mathscr{D}\mathfrak{g}$, and clearly X_s and X_n are traceless if X is. Similarly, if V is not irreducible, for any submodule W of V, let

$$\mathfrak{s}_W = \{Y \in \mathfrak{gl}(V): Y(W) \subset W \text{ and } \mathrm{Tr}(Y|_W) = 0\}.$$

Then $\mathfrak{g}$ is also a subalgebra of $\mathfrak{s}_W$, and X_s and X_n are also in $\mathfrak{s}_W$.

Since $[X, \mathfrak{g}] \subset \mathfrak{g}$, it follows that $[p(X), \mathfrak{g}] \subset \mathfrak{g}$ for any polynomial $p(T)$. Hence $[X_s, \mathfrak{g}] \subset \mathfrak{g}$ and $[X_n, \mathfrak{g}] \subset \mathfrak{g}$. In other words, X_s and X_n belong to the Lie subalgebra $\mathfrak{n}$ of $\mathfrak{gl}(V)$ consisting of those endomorphisms A such that $[A, \mathfrak{g}] \subset \mathfrak{g}$. So $\mathfrak{n}$ gives us another subalgebra to work with. Now we claim that $\mathfrak{g}$ is the intersection of $\mathfrak{n}$ and all the algebras $\mathfrak{s}_W$ for all submodules W of V. This claim, as we saw, will finish the proof. Let $\mathfrak{g}'$ be the intersection of all these Lie algebras. Then $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$ since $\mathfrak{g}' \subset \mathfrak{n}$.

By the complete reducibility theorem we can find a submodule U of $\mathfrak{g}'$ so that $\mathfrak{g}' = \mathfrak{g} \oplus U$. Since $[\mathfrak{g}, \mathfrak{g}'] \subset \mathfrak{g}$, we must have $[\mathfrak{g}, U] = 0$. To show that U is 0, it suffices to show that for any $Y \in U$ its restriction to any irreducible submodule W of V is zero (noting that Y preserves W since $Y \in \mathfrak{s}_W$, and that V is a sum of irreducible submodules). But since Y commutes with $\mathfrak{g}$, Schur's lemma implies that the restriction of Y to W is multiplication by a scalar, and the assumption that $Y \in \mathfrak{s}_W$ means that $\mathrm{Tr}(Y|_W) = 0$, so $Y|_W = 0$, as required. □

Now if $\mathfrak{g}$ is a semisimple algebra, the adjoint representation ad embeds $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$. For any X in $\mathfrak{g}$ the theorem implies that the semisimple and nilpotent parts of $\mathrm{ad}(X)$ are in $\mathfrak{g}$. We write these X_s and X_n. The decomposition $X = X_s + X_n$ may be called the *absolute* Jordan decomposition. Note that $[X_s, X_n] = 0$. It follows easily from the definition that if $\rho\colon \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism from one semisimple Lie algebra onto another, then $\rho(X_s) = \rho(X)_s$ and $\rho(X_n) = \rho(X)_n$. (This follows for example from the fact that $\mathfrak{g}'$ is obtained from $\mathfrak{g}$ by factoring out some of its simple ideals.) In fact, the absolute decomposition determines all others:

Corollary C.18. *If $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is any representation of a semisimple Lie algebra $\mathfrak{g}$, then $\rho(X_s)$ is the semisimple part of $\rho(X)$ and $\rho(X_n)$ is the nilpotent part of $\rho(X)$.*

PROOF. We just saw that $\rho(X_s)$ and $\rho(X_n)$ are the semisimple and nilpotent parts of $\rho(X)$ as regarded in the semisimple Lie algebra $\mathfrak{g}' = \rho(\mathfrak{g})$. Apply the theorem to $\mathfrak{g}' \subset \mathfrak{gl}(V)$. □

It follows that an element X in a semisimple Lie algebra that is semisimple in one faithful representation is semisimple in all representations.

§C.3. On Derivations

In this final section we collect a few facts relating the Killing form, solvability, and nilpotency with derivations of Lie algebras, mainly for use in Appendix E. We first prove a couple of lemmas related to the Lie–Engel theory of Lecture 9. For these $\mathfrak{g}$ is any Lie algebra, $\mathfrak{r} = \mathrm{Rad}(\mathfrak{g})$ denotes its radical, and $\mathscr{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Lemma C.19. *For any representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$, every element of $\rho(\mathscr{D}\mathfrak{g} \cap \mathfrak{r})$ is a nilpotent endomorphism.*

PROOF. It suffices to treat the case where the representation V is irreducible, for if W were a proper subrepresentation, we would know the result by induction on the dimension for W and V/W, which implies it for V. We may replace $\mathfrak{g}$ by its image, so we may assume ρ is injective. In this case we show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{r} = 0$. We may assume $\mathfrak{r} \neq 0$. Consider the largest integer k such that $\mathfrak{a} = \mathscr{D}^k\mathfrak{r}$ is not zero. This $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$. It suffices to show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{a} = 0$, for if $k > 0$, then $\mathfrak{a} \subset \mathscr{D}\mathfrak{g}$.

We need three facts:

(i) If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is an irreducible representation and $\mathfrak{b}$ is any ideal of $\mathfrak{g}$ that consists of nilpotent transformations of V, then $\mathfrak{b} = 0$. (Indeed, by Engel's theorem,

$$W = \{v \in V: X(v) = 0 \text{ for all } X \in \mathfrak{b}\}$$

is nonzero, and by Lemma 9.13, W is preserved by $\mathfrak{g}$. Since V is irreducible, $W = V$, which says that $\mathfrak{b} = 0$.)

(ii) A transformation X is nilpotent exactly when $\mathrm{Tr}(X^n) = 0$ for all positive integers n. (This is seen by writing X in Jordan canonical form.)

(iii) $\mathrm{Tr}([X, Y] \cdot Z) = 0$ whenever $[Y, Z] = 0$. (This follows from the identity (C.3): $\mathrm{Tr}([X, Y] \cdot Z) = \mathrm{Tr}(X \cdot [Y, Z])$.)

Next we can see that $[\mathfrak{g}, \mathfrak{a}] = 0$. For if $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$, then $[X, Y] \in \mathfrak{a}$; since $\mathfrak{a}$ is abelian, Y commutes with $[X, Y]$ and hence with powers of $[X, Y]$.

Applying (iii) with $Z = [X, Y]^{n-1}$ gives $\mathrm{Tr}([X, Y]^n) = 0$ for $n > 0$, and (ii) and (i) imply that $[\mathfrak{g}, \mathfrak{a}] = 0$.

Finally we show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{a} = 0$. If $X, Y \in \mathfrak{g}$ and $[X, Y] \in \mathfrak{a}$, then $[Y, [X, Y]] = 0$ by the preceding step, so again Y commutes with powers of $[X, Y]$, and the same argument shows that $\mathrm{Tr}([X, Y]^n) = 0$, and (ii) and (i) again show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{a} = 0$. □

Lemma C.20. *For any Lie algebra* $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{r}]$ *is nilpotent.*

Proof. Look at the images $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{r}}$ of $\mathfrak{g}$ and $\mathfrak{r}$ by the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. By Lemma C.19 and Engel's theorem, $[\bar{\mathfrak{g}}, \bar{\mathfrak{r}}]$ is a nilpotent ideal of $\bar{\mathfrak{g}}$. Since the kernel of the adjoint representation is the center of $\mathfrak{g}$, it follows that the quotient of $[\mathfrak{g}, \mathfrak{r}]$ by a central ideal is nilpotent, which implies that $[\mathfrak{g}, \mathfrak{r}]$ itself is nilpotent. □

An ideal $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is called *characteristic* if any derivation of $\mathfrak{g}$ maps $\mathfrak{a}$ into itself. Note that an ideal is just a subspace that is preserved by all inner derivations $D_X = \mathrm{ad}(X)$. It follows from the definitions that if $\mathfrak{a}$ is any ideal in $\mathfrak{g}$, then any characteristic ideal in $\mathfrak{a}$ is automatically an ideal in $\mathfrak{g}$.

The following simple construction is useful for turning questions about general derivations into questions about inner derivations. Given any Lie algebra $\mathfrak{g}$ and a derivation D of $\mathfrak{g}$, let $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}$, and define a bracket on $\mathfrak{g}'$ by

$$[(X, \lambda), (Y, \mu)] = ([X, Y] + \lambda D(Y) - \mu D(X), 0).$$

It is easy to verify that $\mathfrak{g}'$ is a Lie algebra containing $\mathfrak{g} = \mathfrak{g} \oplus 0$ as an ideal, and that, setting $\xi = (0, 1)$, the restriction of $D_\xi = \mathrm{ad}(\xi)$ to $\mathfrak{g}$ is the given derivation D.

As a simple application of this construction, if B is the Killing form on $\mathfrak{g}$, we have the identity

$$B(D(X), Y) + B(X, D(Y)) = 0 \tag{C.21}$$

for any derivation D of $\mathfrak{g}$, and any X and Y in $\mathfrak{g}$. Indeed, if B' is the Killing form on $\mathfrak{g}'$, (C.3) gives $B'([\xi, X], Y) + B'(X, [\xi, Y]) = 0$; since $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$, B is the restriction of B' to $\mathfrak{g}$, and (C.21) follows.

From (C.21) it follows that if $\mathfrak{a}$ is a characteristic ideal of $\mathfrak{g}$, then its orthogonal complement with respect to the Killing form is also a characteristic ideal of $\mathfrak{g}$.

Proposition C.22. *For any Lie algebra* $\mathfrak{g}$, $\mathrm{Rad}(\mathfrak{g})$ *is the orthogonal complement to* $\mathscr{D}\mathfrak{g}$ *with respect to the Killing form.*

Proof. To see that $\mathfrak{r} = \mathrm{Rad}(\mathfrak{g})$ is contained in $\mathscr{D}\mathfrak{g}^\perp$, i.e., that $\mathscr{D}\mathfrak{g}$ is perpendicular to $\mathfrak{r}$, let $X, Y \in \mathfrak{g}$ and $Z \in \mathfrak{r}$. Recalling that $B([X, Y], Z) = B(X, [Y, Z])$, it suffices to show that $B(X, [Y, Z]) = 0$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{r}$ and X. Then $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{r}$, so $\mathfrak{h}$ is solvable, so by Lie's theorem, under the adjoint action, $\mathfrak{h}$ acts on $\mathfrak{g}$ by upper-triangular matrices. By Lemma C.19, $[Y, Z]$ acts on $\mathfrak{g}$ by nilpotent transformations. It follows that $X \circ [Y, Z]$ also acts nilpotently on $\mathfrak{g}$, from which it follows that $B(X, [Y, Z]) = \mathrm{Tr}(X \circ [Y, Z]) = 0$, as required.

Since $\mathscr{D}\mathfrak{g}$ is a characteristic ideal, $(\mathscr{D}\mathfrak{g})^{\perp}$ is an ideal. It is solvable by Cartan's criterion (Proposition C.4), since

$$B(\mathscr{D}\mathfrak{g}^{\perp}, \mathscr{D}(\mathscr{D}\mathfrak{g}^{\perp})) \subset B(\mathscr{D}\mathfrak{g}^{\perp}, \mathscr{D}\mathfrak{g}) = 0.$$

It follows that $\mathscr{D}\mathfrak{g}^{\perp} \subset \mathfrak{r}$, which concludes the proof. □

Corollary C.23. *If* $\mathfrak{a}$ *is an ideal in a Lie algebra* $\mathfrak{g}$, *then*

$$\mathrm{Rad}(\mathfrak{a}) = \mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{a}.$$

PROOF. Since $\mathrm{Rad}(\mathfrak{a})$ is a characteristic ideal of an ideal, it is an ideal of $\mathfrak{g}$. Since it is solvable, it must be contained in the radical of $\mathfrak{g}$. This shows the inclusion $\subset$; the opposition inclusion is clear since $\mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{a}$ is a solvable ideal in $\mathfrak{a}$. □

Proposition C.24. *If D is a derivation of a Lie algebra* $\mathfrak{g}$, *then* $D(\mathrm{Rad}(\mathfrak{g}))$ *is contained in a nilpotent ideal of* $\mathfrak{g}$.

PROOF. Construct $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}$ as before, with $\xi = (0, 1)$. Since $\mathrm{Rad}(\mathfrak{g}) \subset \mathrm{Rad}(\mathfrak{g}')$, we have

$$D(\mathrm{Rad}(\mathfrak{g})) = [\xi, \mathrm{Rad}(\mathfrak{g})] \subset [\mathfrak{g}', \mathrm{Rad}(\mathfrak{g}')] \cap \mathfrak{g}.$$

By Lemma C.20, $[\mathfrak{g}', \mathrm{Rad}(\mathfrak{g}')]$ is a nilpotent ideal in $\mathfrak{g}'$, so its intersection with $\mathfrak{g}$ is also nilpotent. □

Just as with the notion of solvability, any Lie algebra $\mathfrak{g}$ contains a largest nilpotent ideal, usually called the *nil radical* of $\mathfrak{g}$, and denoted $\mathrm{Nil}(\mathfrak{g})$ or $\mathfrak{n}$. Proposition C.24 says that any derivation maps $\mathfrak{r}$ into $\mathfrak{n}$, which includes the result of Lemma C.20 that $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$. The existence of this ideal follows from:

Lemma C.25. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are nilpotent ideals in a Lie algebra* $\mathfrak{g}$, *then* $\mathfrak{a} + \mathfrak{b}$ *is also a nilpotent ideal.*

PROOF. An ideal $\mathfrak{a}$ is nilpotent iff there is a positive integer k so that all k-fold brackets $[X_1, [X_2, [\ldots, [X_{k-1}, X_k] \ldots]]]$ are zero when each X_i is in $\mathfrak{a}$. Equivalently, all m-fold brackets of m elements of $\mathfrak{g}$ are zero if at least k of them are in $\mathfrak{a}$. If k is chosen to work for $\mathfrak{a}$ and for $\mathfrak{b}$, it is easy to verify that $2k$ works for the sum $\mathfrak{a} + \mathfrak{b}$, since any bracket of $2k$ elements, each from $\mathfrak{a}$ or from $\mathfrak{b}$, contains at least k elements from $\mathfrak{a}$ or from $\mathfrak{b}$. □

Since $\mathrm{Nil}(\mathfrak{g}) \subset \mathrm{Rad}(\mathfrak{g})$, it follows from Proposition C.24 that $\mathrm{Nil}(\mathfrak{g})$ is a characteristic ideal of $\mathfrak{g}$. The same reasoning as in Corollary C.23 gives:

Corollary C.26. *If* $\mathfrak{a}$ *is an ideal in a Lie algebra* $\mathfrak{g}$, *then*

$$\mathrm{Nil}(\mathfrak{a}) = \mathrm{Nil}(\mathfrak{g}) \cap \mathfrak{a}.$$

If $\mathfrak{g}$ is a Lie algebra, its *universal enveloping algebra* $U = U(\mathfrak{g})$ is the quotient of the tensor algebra of $\mathfrak{g}$ modulo the two-sided ideal generated by all $X \otimes Y - Y \otimes X - [X, Y]$ for all X, Y in $\mathfrak{g}$. It is an associative algebra, with a map $\imath\colon \mathfrak{g} \to U$ such that

$$\imath([X, Y]) = [\imath(X), \imath(Y)] = \imath(X)\imath(Y) - \imath(Y)\imath(X),$$

and satisfying the universal property: for any linear map φ from $\mathfrak{g}$ to an associative algebra A such that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ for all X, Y, there is a unique homomorphism of algebras $\tilde{\varphi}\colon U \to A$ such that $\varphi = \tilde{\varphi} \circ \imath$. For example, a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ determines an algebra homomorphism $\tilde{\rho}\colon U(\mathfrak{g}) \to \mathrm{End}(V)$. Conversely, any representation arises in this way.

We will need the following easy lemma:

Lemma C.27. *For any derivation* D *of a Lie algebra* $\mathfrak{g}$, *there is a unique derivation* $\tilde{D}$ *of the associative algebra* $U(\mathfrak{g})$ *such that* $\tilde{D} \circ \imath = \imath \circ D$.

Proof. Define an endomorphism of the tensor algebra of $\mathfrak{g}$ which is zero on the zeroth tensor power, and on the nth tensor power is

$$X_1 \otimes \cdots \otimes X_n \mapsto DX_1 \otimes X_2 \otimes \cdots \otimes X_n + X_1 \otimes DX_2 \otimes \cdots \otimes X_n + \cdots$$
$$+ X_1 \otimes X_2 \otimes \cdots \otimes DX_n.$$

This is well defined, since it is multilinear in each factor, and it is easily checked to be a derivation of the tensor algebra; denote it by D'. To see that D' passes to the quotient $U(\mathfrak{g})$ one checks routinely that it vanishes on generators for the ideal of relations. □

Exercise C.28. If D is an inner derivation by an element X in $\mathfrak{g}$, verify that $\tilde{D}$ is the inner derivation by the element $\imath(X)$.

It is a fact that the canonical map $\imath$ embeds $\mathfrak{g}$ in $U(\mathfrak{g})$. The *Poincaré–Birkhoff–Witt* theorem asserts that, in fact, if $U(\mathfrak{g})$ is filtered with the nth piece generated by all products of at most n products of elements of $\imath(\mathfrak{g})$, then the associated graded ring is the symmetric algebra on $\mathfrak{g}$. Equivalently, if $X_1, \ldots, X_r$ is a basis for $\mathfrak{g}$, then the monomials $X_1^{i_1} \cdot \ldots \cdot X_r^{i_r}$ form a basis for $U(\mathfrak{g})$. We do not need this theorem, but we will use the fact that these monomials generate $U(\mathfrak{g})$; this follows by a simple induction, using the equations $X_i \cdot X_j - X_j \cdot X_i = [X_i, X_j]$ to rearrange the order in products.

APPENDIX D

Cartan Subalgebras

Our task here is to prove the basic general facts that were stated in Lecture 14 about the decomposition of a semisimple Lie algebra $\mathfrak{g}$ into a Cartan algebra $\mathfrak{h}$ and a sum of root spaces $\mathfrak{g}_\alpha$, including the existence of such $\mathfrak{h}$ and its uniqueness up to conjugation.

§D.1. The Existence of Cartan Subalgebras

Note that if we have a decomposition as in Lecture 14, and H is any element of $\mathfrak{h}$ such that $\alpha(H) \neq 0$ for all roots α, then $\mathfrak{h}$ is determined by H: $\mathfrak{h} = \mathfrak{c}(H)$, where

$$\mathfrak{c}(H) = \{X \in \mathfrak{g}: [H, X] = 0\}. \tag{D.1}$$

The elements of $\mathfrak{h}$ with this property are called *regular*. They form a Zariski open subset of $\mathfrak{h}$: the complement of the union of the hyperplanes defined by the equations $\alpha = 0$. In particular, regular elements are dense in $\mathfrak{h}$. If $H \in \mathfrak{h}$ is not regular, then $\mathfrak{c}(H)$ is larger than $\mathfrak{h}$, since it contains other root spaces. Note that all elements of $\mathfrak{h}$ are also semisimple, i.e., they are equal to their semisimple parts.

Of course, this discussion depends on knowing the decomposition which we are trying to prove. But it suggests one way to construct and characterize

Cartan subalgebras: they should be subalgebras of the form $\mathfrak{c}(H)$ for some semisimple element H, that are minimal in some sense. We can measure this minimality simply by dimension.

Definition D.2. The *rank* n of a semisimple Lie algebra $\mathfrak{g}$ is the minimum of the dimension of $\mathfrak{c}(H)$ as H varies over all semisimple elements of $\mathfrak{g}$. A semisimple element H is called *regular* if $\mathfrak{c}(H)$ has dimension n. A *Cartan subalgebra* of $\mathfrak{g}$ is an abelian subalgebra all of whose elements are semisimple, and that is not contained in any larger such subalgebra. Our first main goal is

Proposition D.3. *If H is regular, then $\mathfrak{c}(H)$ is a Cartan subalgebra.*

For any semisimple element H, $\mathfrak{g}$ decomposes into eigenspaces for the adjoint action of H:

$$\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_\lambda(H) = \mathfrak{c}(H) \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda(H), \tag{D.4}$$

where $\mathfrak{g}_\lambda(H) = \{X \in \mathfrak{g}\colon [H, X] = \lambda X\}$, and $\mathfrak{c}(H) = \mathfrak{g}_0(H)$. There is a similar decomposition even if H (or $\mathfrak{g}$) is not semisimple, but replacing the eigenspace by $\mathfrak{g}_\lambda(H) = \{X \in \mathfrak{g}\colon (\mathrm{ad}(H) - \lambda I)^k(X) = 0 \text{ for large } k\}$.

Exercise D.5. Without assuming that H is semisimple, show that $[\mathfrak{g}_\lambda(H), \mathfrak{g}_\mu(H)] \subset \mathfrak{g}_{\lambda+\mu}(H)$, by proving the identity

$$\begin{aligned}(\mathrm{ad}(H) &- (\lambda + \mu)I)^k([X, Y]) \\ &= \sum_{j=0}^{k} \binom{k}{j} [(\mathrm{ad}(H) - \lambda I)^j(X), (\mathrm{ad}(H) - \mu I)^{k-j}(Y)]\end{aligned}$$

Let us (temporarily) call an arbitrary element $H \in \mathfrak{g}$ regular if $\dim(\mathfrak{g}_0(H)) \leq \dim(\mathfrak{g}_0(X))$ for all $X \in \mathfrak{g}$.

Lemma D.6. *If H is regular, then $\mathfrak{g}_0(H)$ is abelian.*

Proof. Consider how the Killing form B respects the decomposition (D.4)—again knowing what to expect from Lecture 14. If Y is in $\mathfrak{g}_\lambda(H)$ with $\lambda \neq 0$, then $\mathrm{ad}(Y)$ maps each eigenspace to a different eigenspace (by Exercise D.5), as does $\mathrm{ad}(Y) \circ \mathrm{ad}(X)$ for $X \in \mathfrak{g}_0(H)$. The trace of such an endomorphism is zero, i.e., $B(X, Y) = 0$ for such X and Y.

Because $\mathfrak{g}$ is semisimple, B is nondegenerate. Since we have shown that $\mathfrak{g}_0(H)$ is perpendicular to the other weight spaces, it follows that the restriction of B to $\mathfrak{g}_0(H)$ is nondegenerate.

Consider the Jordan decomposition $X = X_s + X_n$ of an element X in $\mathfrak{g}_0(H)$. Since $\mathrm{ad}(X_n) = \mathrm{ad}(X)_n$ is nilpotent, X_n belongs to $\mathfrak{g}_0(H)$, so $X_s = X - X_n$ does also. Then $\mathrm{ad}(X_s) = \mathrm{ad}(X)_s$ is nilpotent and semisimple on $\mathfrak{g}_0(H)$, so it vanishes there. But this already shows that $\mathrm{ad}(X) = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$ is a nilpotent endomorphism of $\mathfrak{g}_0(H)$ for any $X \in \mathfrak{g}_0(H)$. Hence, by Engel's theorem, $\mathfrak{g}_0(H)$ is nilpotent, so by Lie's theorem $\mathfrak{g}$ has a basis in which the endomorphisms $\mathrm{ad}(X)$ are upper-triangular for all $X \in \mathfrak{g}_0(H)$. It follows that for any elements in $\mathfrak{g}_0(H)$, the trace of products of their adjoint actions on $\mathfrak{g}$ is independent of the order of composition. In particular, for $X, Y, Z \in \mathfrak{g}_0(H)$, the trace of $\mathrm{ad}([X, Y]) \circ \mathrm{ad}(Z)$ on $\mathfrak{g}$ is zero, i.e., $B([X, Y], Z]) \equiv 0$. But since B is nondegenerate on $\mathfrak{g}_0(H)$, $[X, Y] = 0$, so $\mathfrak{g}_0(H)$ is abelian. □

It follows immediately that $\mathfrak{g}_0(H)$ is not contained in any larger abelian subalgebra, since any element that commutes with H is in $\mathfrak{g}_0(H)$ by definition. To finish the proof of the proposition we must prove the following lemma, which also shows that the temporary definition of regular agrees with the first one:

Lemma D.7. *If H is regular, then any element of $\mathfrak{g}_0(H)$ is semisimple.*

Proof. We saw that if X is in $\mathfrak{g}_0(H)$ then X_n is also. Using the same basis as in the preceding proof, we see that $\mathrm{ad}(X_n)$ has a strictly upper-triangular matrix. Hence, $B(X_n, Y) = \mathrm{Tr}(\mathrm{ad}(X_n) \circ \mathrm{ad}(Y)) = 0$ for all Y in $\mathfrak{g}_0(H)$. By the nondegeneracy again, $X_n = 0$, as required. □

It follows from Lemma D.6 that if H is regular, and X is in $\mathfrak{g}_0(H)$, then $\mathfrak{g}_0(X)$ contains $\mathfrak{g}_0(H)$, and they are equal exactly when X is also regular.

Problem D.8*. Prove that if H is regular in any Lie algebra, then $\mathfrak{g}_0(H)$ is a nilpotent Lie algebra.

Exercise D.9. Show that a subalgebra is a Cartan subalgebra if and only if it consists entirely of semisimple elements and is contained in no larger subalgebra with this property.

§D.2. On the Structure of Semisimple Lie Algebras

Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Under the adjoint representation it consists of commuting semisimple endomorphisms. It is then a standard linear algebra fact that this action is simultaneously diagonalizable:

$$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha, \tag{D.10}$$

where the eigenspaces are parametrized by some set of linear forms $\alpha \in \mathfrak{h}^*$, including $\alpha = 0$, and where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}\colon [H, X] = \alpha(H) \cdot X \text{ for all } H \in \mathfrak{h}\}.$$

In particular, $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. The nonzero α are called *roots*.

Lemma D.11. $\mathfrak{h} = \mathfrak{g}_0$.

Proof. Since $\mathfrak{h}$ is abelian, $\mathfrak{h}$ is contained in $\mathfrak{g}_0$. If $\mathfrak{h}$ corresponds to a regular element H, i.e., $\mathfrak{h} = \mathfrak{g}_0(H)$, anything that commutes with H must be in $\mathfrak{h}$, so $\mathfrak{g}_0$ is contained in $\mathfrak{h}$. □

If $\mathfrak{h}$ is constructed from the regular element H, then by definition $\mathfrak{g}_\lambda(H)$ is the direct sum of those $\mathfrak{g}_\alpha$ for which $\alpha(H) = \lambda$. Note that the decomposition (D.10) may be finer than (D.4), but that if H is chosen to be an element of $\mathfrak{h}$ such that the $\alpha(H)$ are distinct for distinct roots α, then the decompositions coincide.

Our next task is to study the other eigenspaces $\mathfrak{g}_\alpha$. As before, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. It follows that if $\alpha + \beta \neq 0$, and if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, then $\operatorname{ad}(X) \circ \operatorname{ad}(Y)$ is nilpotent, so its trace is zero, i.e.,

$$\textit{If } \alpha + \beta \neq 0, \textit{ then } B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0. \tag{D.12}$$

Now for any root α, if $-\alpha$ were not a root, this implies $\mathfrak{g}_\alpha$ is perpendicular to all $\mathfrak{g}_\beta$ (including $\beta = 0$), which would contradict the nondegeneracy of B. So we get one of the facts asserted in Lecture 14:

$$\textit{If } \alpha \textit{ is a root, then } -\alpha \textit{ is also a root.} \tag{D.13}$$

Moreover, the pairing $B\colon \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is nondegenerate. Another fact also follows easily:

$$\textit{The roots } \alpha \textit{ span } \mathfrak{h}^*. \tag{D.14}$$

For if not there would be a nonzero $X \in \mathfrak{h}$ with $\alpha(X) = 0$ for all roots α, which means that $[X, Y] = 0$ for all Y in all $\mathfrak{g}_\alpha$. But then X is in the center of $\mathfrak{g}$, which is zero by semisimplicity of $\mathfrak{g}$.

Now let α be a root, let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, and take any $H \in \mathfrak{h}$. Then

$$B(H, [X, Y]) = B([H, X], Y) = \alpha(H)B(X, Y). \tag{D.15}$$

This cannot be zero for all H, X, and Y without contradicting what we have just proved. In particular,

$$\textit{For any root } \alpha, [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0. \tag{D.16}$$

Let $T_\alpha \in \mathfrak{h}$ be the element dual to α via the pairing B on $\mathfrak{h}$, i.e., characterized by the identity $B(T_\alpha, H) = \alpha(H)$ for all H in $\mathfrak{h}$. We claim next that

$$[X, Y] = B(X, Y)T_\alpha \quad \text{for all } X \in \mathfrak{g}_\alpha,\ Y \in \mathfrak{g}_{-\alpha}. \tag{D.17}$$

To see it, pair both sides with an arbitrary element H of $\mathfrak{h}$. Using (D.15), we have

$$B(H, B(X, Y)T_\alpha) = B(H, T_\alpha)B(X, Y) = \alpha(H)B(X, Y) = B(H, [X, Y]),$$

as required. Next we show that

$$\alpha(T_\alpha) \neq 0. \tag{D.18}$$

Suppose this were false. Choose $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ such that $B(X, Y) = c \neq 0$. Then $[X, Y] = cT_\alpha$, so X, Y, and T_α span a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. If $\alpha(T_\alpha) = 0$, $\mathfrak{s}$ is solvable. Since $[X, Y] \in \mathscr{D}\mathfrak{s}$, it follows that $\mathrm{ad}([X, Y])$ is a nilpotent endomorphism of $\mathfrak{g}$. But then T_α is nilpotent; but all elements of $\mathfrak{h}$ are semisimple, so $T_\alpha = 0$, a contradiction. This gives another claim from Lecture 14:

$$\textit{For any root } \alpha,\ [[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0. \tag{D.19}$$

For with X and Y as above, $[[X, Y], X] = c\cdot[T_\alpha, X] = c\cdot\alpha(T_\alpha)X \neq 0$.

The last remaining fact about root spaces left unproved from Lecture 14 is

$$\textit{For any root } \alpha,\ \mathfrak{g}_\alpha \textit{ is one-dimensional.} \tag{D.20}$$

By what we have seen, we can find $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, so that $H = [X, Y] \neq 0$, and $\alpha(H) \neq 0$. Adjusting by scalars, they generate a subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$, with standard basis H, X, Y, so in particular $\alpha(H) = 2$. Consider the adjoint action of $\mathfrak{s}$ on the sum $V = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_{k\alpha}$, the sum over all nonzero complex multiples $k\alpha$ of α. From what we know about the weights of representations of $\mathfrak{s}$, the only k that can occur are integral multiples of $\frac{1}{2}$.

Now $\mathfrak{s}$ acts trivially on $\mathrm{Ker}(\alpha) \subset \mathfrak{h} \subset V$, and it acts irreducibly on $\mathfrak{s} \subset V$. Together these cover the zero weight space $\mathfrak{h}$, since H is not in $\mathrm{Ker}(\alpha)$. So the only even weights occurring can be 0 and ± 2. In particular,

$$2\alpha \textit{ cannot be a root.} \tag{D.21}$$

But this implies that $\frac{1}{2}\alpha$ cannot be a root, which says that 1 is not a weight occurring in V. But then there can be no other representations occurring in V, i.e., $V = \mathrm{Ker}(\alpha) \oplus \mathfrak{s}$, which proves (D.20). □

§D.3. The Conjugacy of Cartan Subalgebras

We show that any two Cartan subalgebras are conjugate by an inner automorphism of the adjoint subgroup of $\mathrm{Aut}(\mathfrak{g})$. Fix one Cartan subalgebra $\mathfrak{h}$, and consider the decomposition (D.10). For any element X in a root space $\mathfrak{g}_\alpha$, $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent, as we have seen, so its exponential $\exp(\mathrm{ad}(X)) \in GL(\mathfrak{g})$ is just a finite polynomial in $\mathrm{ad}(X)$. Set

$$e(X) = \exp(\mathrm{ad}(X)).$$

Let $E(\mathfrak{h})$ be the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by all such $e(X)$. We want to prove now that this group is independent of the choice of $\mathfrak{h}$, and that all Cartan subalgebras are conjugate by elements in this group. (We will see in the next section that $E(\mathfrak{h})$ is the connected component of $\mathrm{Aut}(\mathfrak{g})$, i.e., that it is the adjoint group.) The proof will be a kind of complex algebraic analogue of the corresponding argument for compact tori that was sketched in Lecture 26.

Theorem D.22. *Let* $\mathfrak{h}$ *and* $\mathfrak{h}'$ *be two Cartan subalgebras of* $\mathfrak{g}$. *Then* (i) $E(\mathfrak{h}) = E(\mathfrak{h}')$, *and* (ii) *there is an element* $g \in E = E(\mathfrak{h})$ *so that* $g(\mathfrak{h}) = \mathfrak{h}'$.

Proof. Fix a Cartan subalgebra $\mathfrak{h}$. Let $\alpha_1, \ldots, \alpha_r$ be its roots. Consider the mapping

$$F: \mathfrak{g}_{\alpha_1} \times \cdots \times \mathfrak{g}_{\alpha_r} \times \mathfrak{h} \to \mathfrak{g}$$

defined by $F(X_1, \ldots, X_r, H) = e(X_1) \circ \cdots \circ e(X_r)(H)$. Note that F is a polynomial mapping from one complex vector space to another of the same dimension. We want to show that not only is the image of F dense, but that, if $\mathfrak{h}_{\mathrm{reg}}$ denotes the set of regular elements in $\mathfrak{h}$, then

$$F(\mathfrak{g}_{\alpha_1} \times \cdots \times \mathfrak{g}_{\alpha_r} \times \mathfrak{h}_{\mathrm{reg}}) \textit{ contains a Zariski open set,} \tag{D.23}$$

i.e., it contains the complement of a hypersurface defined by a polynomial equation.

Suppose that this claim is proved. It follows that for any other Cartan subalgebra $\mathfrak{h}'$, the corresponding image also contains a Zariski open set. But two nonempty Zariski open sets always meet. In this case this means $E(\mathfrak{h}) \cdot \mathfrak{h}_{\mathrm{reg}}$ meets $E(\mathfrak{h}') \cdot \mathfrak{h}'_{\mathrm{reg}}$. That is, there are $g \in E(\mathfrak{h})$, $H \in \mathfrak{h}_{\mathrm{reg}}$, $g' \in E(\mathfrak{h}')$, $H' \in \mathfrak{h}'_{\mathrm{reg}}$ such that $g(H) = g'(H')$. But then since H and H' are regular,

$$g(\mathfrak{h}) = g(\mathfrak{g}_0(H)) = \mathfrak{g}_0(g(H)) = \mathfrak{g}_0(g'(H')) = g'(\mathfrak{g}_0(H')) = g'(\mathfrak{h}').$$

This proves the conjugacy of $\mathfrak{h}$ and $\mathfrak{h}'$. And since

$$E(\mathfrak{h}) = gE(\mathfrak{h})g^{-1} = E(g(\mathfrak{h})) = E(g'(\mathfrak{h}')) = g'E(\mathfrak{h}')(g')^{-1} = E(\mathfrak{h}'),$$

both statements of the theorem are proved. □

To prove (D.23), we use a special case of a very general fact from basic algebraic geometry: if $F: \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial mapping whose derivative $dF_*|_P$ is invertible at some point P, then for any nonempty Zariski open set $U \subset \mathbb{C}^N$, $F(U)$ contains a nonempty Zariski open set. For the proof we refer to any basic algebraic geometry text, e.g., [Ha], or to [Bour, VI, App. A]. So it suffices to show that $dF_*|_P$ is surjective at a point $P = (0, \ldots, 0, H)$, where $H \in \mathfrak{h}_{\mathrm{reg}}$. This is a simple calculation:

Exercise D.24*. Show that $dF_*|_P(0, \ldots, 0, Z) = Z$ for $Z \in \mathfrak{h}$, and that $dF_*|_P(0, \ldots, 0, Y, 0, \ldots, 0, 0) = \mathrm{ad}(Y)(H) = -\mathrm{ad}(H)(Y)$ for $Y \in \mathfrak{g}_{\alpha_i}$. Conclude that the image of $dF_*|_P$ contains $\mathfrak{h}$ and each root space, so $dF_*|_P$ is surjective. □

We remark that although this section, like the preceding appendix, was written for complex Lie algebras, a simple "base change" argument shows that the results extend to Lie algebras over any algebraically closed field of characteristic zero. Some, such as Cartan's criterion, then follow over any field of characteristic zero, by extending to an algebraic closure.

§D.4. On the Weyl Group

In this section we complete the proofs of some of the general facts about the Weyl group that were stated in Lectures 14 and 21. The notation will be as in those sections: $\mathbb{E}$ is the real space generated by the roots R; $\mathfrak{W}$ is the Weyl group, generated by the involutions W_α of $\mathbb{E}$ determined by

$$W_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha,$$

where (,) denotes the Killing form (or any inner product invariant for the Weyl group). We consider a decomposition

$$R = R^+ \cup R^-$$

into positive and negative roots, given by some $l\colon \mathbb{E} \to \mathbb{R}$ as in Lecture 14, and we let $S \subset R^+$ be the set of simple roots for this decomposition. Note that for any W in the Weyl group,

$$R = W(R^+) \cup W(R^-)$$

is the decomposition into positive and negative roots for the linear map $l \circ W^{-1}$. We want to show that every decomposition arises this way. To prove this we need some simple variations of the ideas in §21.1.

Lemma D.25. *If α is a simple root, then W_α permutes all the other positive roots, i.e., W_α maps $R^+ \setminus \{\alpha\}$ to itself.*

Proof. This follows from the expression of positive roots as sums $\beta = \sum m_i\alpha_i$, with the α_i simple, and the m_i non-negative integers. If $\alpha = \alpha_i$, $W_\alpha(\beta)$ differs from β only by an integral multiple of α_i. If $\beta \neq \alpha_i$, $W_\alpha(\beta)$ still has some positive coefficients, so it must be a positive root. □

Let $\mathfrak{W}_0$ be the subgroup of $\mathfrak{W}$ generated by the W_α, as α varies over the simple roots. (We will soon see that $\mathfrak{W}_0 = \mathfrak{W}$.)

Lemma D.26. *Any root β can be written in the form $\beta = W(\alpha)$ for some $\alpha \in S$ and $W \in \mathfrak{W}_0$. In particular, $R = \mathfrak{W}(S)$.*

Proof. It suffices to do this for positive roots, since $\mathfrak{W}_0(\alpha) = \mathfrak{W}_0 W_\alpha(\alpha) = -\mathfrak{W}_0(\alpha)$ for any $\alpha \in S$. If β is positive but not simple, write $\beta = \sum m_i\alpha_i$ as above, and induct on the level $\sum m_i$. As in the previous lemma, there is a simple root γ so that $W_\gamma(\beta)$ is a positive root of lower level. By induction, $W_\gamma(\beta) = W(\alpha)$ for $\alpha \in S$ and $W \in \mathfrak{W}_0$, so $\beta = W_\gamma W(\alpha)$, as required. □

Lemma D.27. *The Weyl group is generated by the reflections in the simple roots, i.e., $\mathfrak{W} = \mathfrak{W}_0$.*

Proof. Given a root β, we must show that W_β is in $\mathfrak{W}_0$. By the preceding lemma, write $\beta = U(\alpha)$ for some $U \in \mathfrak{W}_0$, $\alpha \in S$. Then

$$W_\beta = W_{U(\alpha)} = U \cdot W_\alpha \cdot U^{-1}, \tag{D.28}$$

since both sides act the same on β and $\beta^\perp$. □

Proposition D.29. *The Weyl group acts simply transitively on the set of decompositions of R into positive and negative roots.*

Proof. For the transitivity, suppose $R = Q^+ \cup Q^-$ is another decomposition. We induct on the number of roots that are in R^+ but not in Q^+. If this number is zero, then $R^+ = Q^+$. Otherwise there must be some simple root α that is not in Q^+. It suffices to prove that $W_\alpha(Q^+)$ has more roots in common with R^+ than Q^+ does, for then by induction we can write $W_\alpha(Q^+) = W(R^+)$ for some $W \in \mathfrak{W}$, so $Q^+ = W_\alpha W(R^+)$, as required. In fact, we have by Lemma D.25,

$$W_\alpha(Q^+) \cap R^+ \supset W_\alpha(Q^+ \cap R^+) \cup \{\alpha\} = W_\alpha(Q^+ \cap R^+ \cup \{-\alpha\}),$$

and this proves the assertion.

For simple transitivity, we must show that if an element W in the Weyl group takes R^+ to itself, then it must be the identity. If not, write W as a product of reflections in simple roots,

$$W = W_1 \cdot \ldots \cdot W_r,$$

with r minimal, with W_i the reflection in the simple root β_i. Let $\alpha = \beta_r$. It suffices to show that

$$W_1 \cdot \ldots \cdot W_r = W_1 \cdot \ldots \cdot W_{s-1} W_{s+1} \cdot \ldots \cdot W_{r-1}$$

for some s, $1 \le s \le r-2$. Let $U_s = W_{s+1} \cdot \ldots \cdot W_{r-1}$. This equation is equivalent to the equation $W_s U_s W_r = U_s$, or $U_s W_r U_s^{-1} = W_s$, or $U_s(\alpha) = \beta_s$ (since by (D.28), $W_{U(\alpha)} = U W_\alpha U^{-1}$).

To finish the proof we must find an s so that $U_s(\alpha) = \beta_s$. Note that $U_{r-2}(\alpha) = W_{r-1}(\alpha)$ is a positive root (by Lemma D.25, since $\beta_{r-1} \neq \alpha$). On the other hand, the hypothesis implies that

$$U_0(\alpha) = W_1 \cdot \ldots \cdot W_{r-1}(\alpha) = W_1 \cdot \ldots \cdot W_r(-\alpha) = -W(\alpha)$$

is a negative root. So there must be some s with $1 \le s \le r-2$ such that $U_s(\alpha)$ is positive and $U_{s-1}(\alpha)$ is negative. This means that W_s takes the positive root $U_s(\alpha)$ to the negative root $U_{s-1}(\alpha)$. But by Lemma D.25 again, this can happen only if W_s is the reflection in the root $U_s(\alpha)$, i.e., $\beta_s = \mathrm{U}_s(\alpha)$. □

The simple roots S for a decomposition $R = R^+ \cup R^-$ are called a *basis* for the roots. Since S and R^+ determine each other, the proposition is equivalent to the assertion that *the Weyl group acts simply transitively on the set of bases.*

Exercise D.30. For $W \in \mathfrak{W}$, set $l(W) = \#(R^+ \cap W(R^-))$. Show that W can be written as a product of $l(W)$ reflections in simple roots, but no fewer.

If Ω_α denotes the hyperplane in $\mathbb{E}$ perpendicular to the root α, the (closed) *Weyl chambers* are the closures of the connected components of the complement $\mathbb{E}\setminus\bigcup\Omega_\alpha$ of these hyperplanes. For a decomposition $R = R^+ \cup R^-$ with simple roots S, the set

$$\mathscr{W} = \{\beta \in \mathbb{E}: (\beta, \alpha) \geq 0, \forall\alpha \in R^+\} = \{\beta \in \mathbb{E}: (\beta, \alpha) \geq 0, \forall\alpha \in S\}$$

is one of these Weyl chambers. The fact that every Weyl chamber arises this way follows from

Lemma D.31. *For any β in $\mathbb{E}$ there is some $W \in \mathfrak{W}$ such that $(W(\beta), \alpha) \geq 0$ for all $\alpha \in S$.*

Proof. Let ρ be half the sum of the positive roots. It follows from Lemma D.25 that $W_\alpha(\rho) = \rho - \alpha$ for any simple root α. Take W in $\mathfrak{W}$ to maximize the inner product $(W(\beta), \rho)$. Then for all $\alpha \in S$,

$$(W_\alpha W(\beta), \rho) = (W(\beta), W_\alpha \rho) = (W(\beta), \rho - \alpha) = (W(\beta), \rho) - (W(\beta), \alpha)$$

cannot be larger than $(W(\beta), \rho)$, so $(W(\beta), \alpha) \leq 0$. □

Thus, the orbit of one Weyl chamber by the Weyl group covers $\mathbb{E}$, so all Weyl chambers are conjugate to each other by the action of the Weyl group. So all arise by partitioning R into positive and negative roots. This partitioning is uniquely determined by the Weyl chamber. In fact, the walls of a Weyl chamber are the hyperplanes Ω_α as α varies over the n corresponding simple roots, $n = \dim(\mathbb{E})$. From the proposition we have:

Corollary D.32. *The Weyl group acts simply transitively on Weyl chambers.*

Exercise D.33*. Let $\mathfrak{G}$ be the group of automorphisms of $\mathbb{E}$ that map R to itself.

(i) Show that $\mathfrak{W}$ is a normal subgroup of $\mathfrak{G}$.
(ii) Let $\mathfrak{R}$ be the automorphisms in $\mathfrak{G}$ which map a given set of simple roots S to itself. Show that $\mathfrak{G}$ is a semidirect product of $\mathfrak{W}$ and $\mathfrak{R}$.
(iii) Show that $\mathfrak{R}$ is isomorphic to the group of automorphisms of the Dynkin diagram.
(iv) Compute $\mathfrak{R}$ for each of the simple groups.

Our next goal is to show that the lattice $\mathbb{Z}\{H_\alpha: \alpha \in R\} \subset \mathfrak{h}$ has a basis of elements H_α where α varies over the simple roots. This is analogous to the statement we have proved that the root lattice Λ_R in $\mathfrak{h}^*$ is generated by simple roots. The first statement can be deduced from the second, using the Killing form to map $\mathfrak{h}$ to $\mathfrak{h}^*$, $H \mapsto (H, -)$, where $(\ ,\ )$ is the Killing form. We saw in Lecture 14 that this map takes H_α to $\alpha' = (2/(\alpha, \alpha))\alpha$. Given a root system R in a Euclidean space $\mathbb{E}$, to each root α one can define its *coroot* α' in $\mathbb{E}$ by the formula

$$\alpha' = \frac{2}{(\alpha, \alpha)}\alpha.$$

Let $R' = \{\alpha': \alpha \in R\}$ be the set of coroots. For any $0 \neq \alpha \in \mathfrak{h}$, set $\alpha' = (2/(\alpha, \alpha))\alpha$, and for any $\alpha, \beta \in \mathfrak{h}^*$, set $n_{\beta\alpha} = 2(\beta, \alpha)/(\alpha, \alpha)$. Let $R = R^+ \cup R^-$ be a decomposition of R into positive and negative roots, and let S be the corresponding set of simple positive roots.

Lemma D.34. (i) *The set R' of coroots forms a root system in $\mathbb{E}$.*

(ii) *The set $S' = \{\alpha': \alpha \in S\}$ is a set of simple roots for R'.*

(iii) *For $\alpha, \beta \in S$, $n_{\beta'\alpha'} = n_{\alpha\beta}$.*

PROOF. It is a straightforward calculation that $n_{\beta'\alpha'} = n_{\alpha\beta}$. It follows by another short calculation that if W_α denotes the reflection in the hyperplane perpendicular to α, then $W_{\alpha'}(\beta') = (W_\alpha(\beta))'$. The four defining properties of a root system specified in §21.1 follow immediately from this. It is clear that if R^+ is the set of roots in R that are positive for a functional l on $\mathbb{E}$, then $(R^+)' = \{\alpha': \alpha \in R^+\}$ is the corresponding set of positive roots for R'. Roots in R^+ are those that can be written as a nonnegative linear combinations of roots in S, and this property characterizes S. Since α' is a positive multiple of α for any α, it follows that roots in $(R^+)'$ are those that can be written as non-negative linear combinations of roots in S', which proves (ii). □

The root system R' is called the *dual* of R.

Exercise D.35. Find the dual of each type of simple root system.

Proposition D.36. (i) *The elements H_α for $\alpha \in S$ generate the lattice $\mathbb{Z}\{H_\alpha: \alpha \in R\}$.*

(ii) *If $\omega_\alpha \in \mathfrak{h}$ are defined by the property that $\omega_\alpha(H_\beta) = \delta_{\alpha,\beta}$, then the elements ω_α generate the weight lattice Λ_W.*

(iii) *The nonnegative integral linear combinations of the fundamental weights ω_α are precisely the weights in $\mathscr{W} \cap \Lambda_W$, where $\mathscr{W}$ is the closed Weyl chamber corresponding to R^+.*

PROOF. The isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ given by the Killing form takes H_α to the coroot α'. By the lemma and the fact that all positive roots are sums of simple roots, the set $\{\alpha': \alpha \in S\}$ spans the same lattice as $\{\alpha': \alpha \in R\}$. This proves (i), and it follows that the weights are precisely those elements in $\mathfrak{h}$ that take integral values on the set $\{H_\alpha: \alpha \in S\}$. The rest of the proposition follows, noting that

$$\begin{aligned}\mathscr{W} &= \{\beta \in \mathbb{E}: \beta(H_\alpha) \geq 0 \text{ for all } \alpha \in R^+\} \\ &= \{\beta \in \mathbb{E}: \beta(H_\alpha) \geq 0 \text{ for all } \alpha \in S\}.\end{aligned}$$ □

If we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of the Killing form, we can regard $\mathfrak{W}$ as a group of automorphisms of $\mathfrak{h}$. By means of this, the reflection W_α corresponding to a root α becomes the automorphism of $\mathfrak{h}$ which takes an element H to $H - \alpha(H)\cdot H_\alpha$. We have a last debt (Fact 14.11) to pay about the Weyl group:

Proposition D.37. *Every element of the Weyl group is induced by an automorphism of* $\mathfrak{g}$ *which maps* $\mathfrak{h}$ *to itself.*

PROOF. It suffices to produce the generating involutions W_α in this way. The claim is that if X_α and Y_α are generators of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ as usual, then $\vartheta_\alpha = e(X_\alpha)e(-Y_\alpha)e(X_\alpha)$ is such an automorphism, where, as in the preceding section, we write $e(X)$ for $\exp(\mathrm{ad}(X))$. We must show that $\vartheta_\alpha(H) = H - \alpha(H)\cdot H_\alpha$ for all H in $\mathfrak{h}$. It suffices to do this for H with $\alpha(H) = 0$, and for $H = H_\alpha$, since such together span $\mathfrak{h}$. If $\alpha(H) = 0$, then $[X_\alpha, H] = [Y_\alpha, H] = 0$, so $\vartheta_\alpha(H) = H$, which takes care of this case. For $H = H_\alpha$, it suffices to calculate on the subalgebra $\mathfrak{s}_\alpha = \mathbb{C}\{H_\alpha, X_\alpha, Y_\alpha\} \cong \mathfrak{sl}_2\mathbb{C}$, and this is a simple calculation:

Exercise D.38. (a) For $\mathfrak{sl}_2\mathbb{C}$ with its standard basis, show that $\vartheta = e(X)e(Y)e(X)$ maps H to $-H$, X to $-Y$, and Y to $-X$.

(b) Show that if G is a Lie group with Lie algebra $\mathfrak{g}$, then ϑ_α is induced by the element $\exp(\frac{1}{2}\pi(X_\alpha - Y_\alpha))$ of G.

We need a refinement of the preceding calculation. For a root α and a nonzero complex number t, define two automorphisms of $\mathfrak{g}$:

$$\vartheta_\alpha(t) = e(t\cdot X_\alpha) \circ e(-(t)^{-1}\cdot Y_\alpha) \circ e(t\cdot X_\alpha)$$

and

$$\Phi_\alpha(t) = \vartheta_\alpha(t) \circ \vartheta_\alpha(-1).$$

Lemma D.39. *The automorphism* $\Phi_\alpha(t)$ *is the identity on* $\mathfrak{h}$, *and for any root* β, *it is multiplication by* $t^{\beta(H_\alpha)}$ *on* $\mathfrak{g}_\beta$.

PROOF. Look first in $\mathfrak{sl}_2$, with $X = X_\alpha$, $Y = Y_\alpha$. It is simplest to calculate in the covering $\mathrm{SL}_2\mathbb{C}$ of the adjoint group. Here $\vartheta_\alpha(t)$ lifts to

$$\begin{aligned}\exp(tX)\cdot\exp(-t^{-1}Y)\cdot\exp(tX) &= \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ -t^{-1} & 1\end{pmatrix}\cdot\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}0 & t\\ -t^{-1} & 0\end{pmatrix},\end{aligned}$$

so $\Phi_\alpha(t)$ lifts to

$$\begin{pmatrix}0 & t\\ -t^{-1} & 0\end{pmatrix}\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}.$$

To see how $\Phi_\alpha(t)$ acts on $\mathfrak{g}_\beta$, for $\beta \neq \pm\alpha$, it suffices to consider the action of the $\mathrm{SL}_2\mathbb{C}$ corresponding to $\mathfrak{s}_\alpha = \mathbb{C}\{H_\alpha, X_\alpha, Y_\alpha\}$ on the α-string through β, i.e., on $\bigoplus \mathfrak{g}_{\beta+k\alpha}$. We know that this is an irreducible representation of $\mathrm{SL}_2\mathbb{C}$, and the weight of $\mathfrak{g}_\alpha$ is $\beta(H_\alpha)$. It follows that $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ acts by multiplication by $t^{\beta(H_\alpha)}$. Similarly on $\mathfrak{h}$ it acts by multiplication by $t^0 = 1$. □

Putting the preceding results together, we can give a description of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. Let $E = E(\mathfrak{h})$ be the subgroup generated by elements $\exp(\mathrm{ad}(Z))$, as Z varies over root spaces $\mathfrak{g}_\alpha$, $\alpha \neq 0$, as in §D.3. Let G be the adjoint form of $\mathfrak{g}$, so we have

$$E \subset G \subset \mathrm{Aut}^0(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g}),$$

where $\mathrm{Aut}^0(\mathfrak{g})$ is the connected component of the identity.

Proposition D.40. *We have* $E = G = \mathrm{Aut}^0(\mathfrak{g})$, *and* $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$ *is isomorphic to the automorphism group of the Dynkin diagram.*

PROOF. Fix the Cartan algebra $\mathfrak{h}$ and positive roots R^+. Let $\mathrm{Aut}(\mathfrak{g})'$ be the group of automorphisms of $\mathfrak{g}$ that map $\mathfrak{h}$ to itself, and similarly denote by primes the intersections of subgroups with $\mathrm{Aut}(\mathfrak{g})'$. We leave it to the reader to construct a finite subgroup K of $\mathrm{Aut}(\mathfrak{g})'$ which maps isomorphically onto the automorphism group of the Dynkin diagram, and which meets G only in the identity element (see Exercise 22.25 for a direct case-by-case approach, or use (21.25)). It then suffices to prove that $\mathrm{Aut}(\mathfrak{g})$ is a semidirect product of E and K, i.e., that $\mathrm{Aut}(\mathfrak{g}) = E \cdot K$.

To see this, start with any element σ in $\mathrm{Aut}(\mathfrak{g})$. By Theorem D.22, there is a $\tau_1 \in E$ with $\sigma(\mathfrak{h}) = \tau_1(\mathfrak{h})$. Then $\sigma_1 = \tau_1^{-1} \cdot \sigma$ is in $\mathrm{Aut}(\mathfrak{g})'$. By Proposition D.29 and the proof of Proposition D.37 there is a $\tau_2 \in E'$ so that $\sigma_2 = \tau_2^{-1} \cdot \sigma_1$ maps R^+ to R^+. This element may permute the simple roots, but there is some $k \in K$ so that $\sigma_3 = \sigma_2 \cdot k^{-1}$ is the identity on the set of simple roots. Now σ_3 is the identity on $\mathfrak{h}$ and it is multiplication by some nonzero scalar c_β on each $\mathfrak{g}_\beta$. By the nonsingularity of the Cartan matrix there is some nonzero complex number t and some $\lambda \in \Lambda_R$ so that $c_\beta = t^{\lambda(H_\beta)}$ for every simple root β. From Lemma D.39 it follows that there is a τ in E' so that τ and σ_3 agree on each $\mathfrak{g}_\beta$ for each simple root β, and both are the identity on $\mathfrak{h}$. But it then follows from the uniqueness theorem (Claim 21.25) that $\sigma_3 = \tau$. Hence

$$\sigma = \tau_1 \cdot \tau_2 \cdot \sigma_3 \cdot k \in E \cdot K,$$

as required. □

Exercise D.41. Show that any two Borel subalgebras of a semisimple Lie algebra are conjugate.

APPENDIX E

Ado's and Levi's Theorems

§E.1. Levi's Theorem

The object of this section is to prove Levi's theorem:

Theorem E.1. *Let* $\mathfrak{g}$ *be a Lie algebra with radical* $\mathfrak{r}$. *Then there is a subalgebra* $\mathfrak{l}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{l}$.

Proof. There are several simple reductions. First, we may assume there is no nonzero ideal of $\mathfrak{g}$ that is properly contained in $\mathfrak{r}$. For if $\mathfrak{a}$ were such an ideal, by induction on the dimension of $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{a}$ would have a subalgebra complementary to $\mathfrak{r}/\mathfrak{a}$, and this subalgebra has the form $\mathfrak{l}/\mathfrak{a}$, with $\mathfrak{l}$ as required. In particular, we may assume $\mathfrak{r}$ is abelian, since otherwise $\mathscr{D}\mathfrak{r}$ is a proper ideal in $\mathfrak{r}$ which is an ideal in $\mathfrak{g}$ by Corollary C.23. We may also assume that $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$, for if $[\mathfrak{g}, \mathfrak{r}] = 0$ then the adjoint representation factors through $\mathfrak{g}/\mathfrak{r}$, and since $\mathfrak{g}/\mathfrak{r}$ is semisimple, the submodule $\mathfrak{r} \subset \mathfrak{g}$ has a complement, which is the required $\mathfrak{l}$.

Now $V = \mathfrak{gl}(\mathfrak{g})$ is a $\mathfrak{g}$-module via the adjoint representation: for $X \in \mathfrak{g}$ and $\varphi \in V$,

$$X \cdot \varphi = [\mathrm{ad}(X), \varphi] = \mathrm{ad}(X) \circ \varphi - \varphi \circ \mathrm{ad}(X).$$

In other words, for $X, Y \in \mathfrak{g}$ and $\varphi \in V$,

$$(X \cdot \varphi)(Y) = [X, \varphi(Y)] - \varphi([X, Y]). \tag{E.2}$$

The trick is to consider the following subspaces of V:

$$\begin{aligned} C &= \{\varphi \in V\colon \varphi(\mathfrak{g}) \subset \mathfrak{r} \text{ and } \varphi|_{\mathfrak{r}} \text{ is multiplication by a scalar}\} \\ &\cup \\ B &= \{\varphi \in V\colon \varphi(\mathfrak{g}) \subset \mathfrak{r} \text{ and } \varphi(\mathfrak{r}) = 0\} \\ &\cup \\ A &= \{\mathrm{ad}(X)\colon X \in \mathfrak{r}\}. \end{aligned}$$

These are easily checked to be $\mathfrak{g}$-submodules of V, included in each other as indicated. And C/B is a trivial $\mathfrak{g}$-module of rank 1, i.e. $C/B = \mathbb{C}$, by taking φ in C to the scalar λ such that $\varphi|_{\mathfrak{r}} = \lambda \cdot I$. (Note that $C/B \neq 0$ since one can find an endomorphism of the vector space $\mathfrak{g}$ which is the identity on $\mathfrak{r}$ and zero on a vector space complement to $\mathfrak{r}$.) We claim also that

$$\mathfrak{g} \cdot C \subset B \quad \text{and} \quad \mathfrak{r} \cdot C \subset A. \tag{E.3}$$

To prove these let $\varphi \in C$, and assume the restriction of φ to $\mathfrak{r}$ is multiplication by the scalar c. If $X \in \mathfrak{g}$ and $Y \in \mathfrak{r}$, then by (E.2),

$$(X \cdot \varphi)(Y) = [X, cY] - c[X, Y] = 0,$$

so $X \cdot \varphi \in B$; this proves the first inclusion. If $X \in \mathfrak{r}$, and $Y \in \mathfrak{g}$, then $[X, \varphi(Y)] \in [\mathfrak{r}, \mathfrak{r}] = 0$, so

$$(X \cdot \varphi)(Y) = -\varphi([X, Y]) = [-cX, Y],$$

and $X \cdot \varphi = \mathrm{ad}(-cX)$ is in A, which proves the second inclusion.

This means that the map $C/A \to C/B = \mathbb{C}$ is a surjection of $\mathfrak{g}/\mathfrak{r}$-modules, which must split since $\mathfrak{g}/\mathfrak{r}$ is semisimple. In other words, there is an element φ in C such that $\varphi|_{\mathfrak{r}} = \mathrm{id}_{\mathfrak{r}}$ and $\mathfrak{g} \cdot \varphi$ is contained in A. Now let

$$\mathfrak{l} = \{X \in \mathfrak{g}\colon X \cdot \varphi = 0\}.$$

It is easy to check that $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$. We must verify: (i) $\mathfrak{l} \cap \mathfrak{r} = 0$; and (ii) $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$. For the first, if X is a nonzero element of the intersection, then, as we saw above, $X \cdot \varphi = \mathrm{ad}(-X)$, so $\mathrm{ad}(X) = 0$. Hence $[\mathfrak{g}, X] = 0$, so $\mathbb{C} \cdot X$ is a nonzero ideal in $\mathfrak{r}$, contradicting our assumptions. For (ii), let $X \in \mathfrak{g}$. Then $X \cdot \varphi$ is in A, so $X \cdot \varphi = \mathrm{ad}(Y)$ for some Y in $\mathfrak{r}$. We saw that $\mathrm{ad}(Y) = -Y \cdot \varphi$, so $(X + Y) \cdot \varphi = 0$, i.e., $X + Y$ belongs to $\mathfrak{l}$. Hence $X = (X + Y) - Y$ is in the sum of $\mathfrak{l}$ and $\mathfrak{r}$. □

This proves the existence of Levi subalgebras $\mathfrak{l}$ of any Lie algebra. We have no need to prove the companion fact that any two Levi subalgebras are conjugate, cf. [Bour, I, §6.8].

§E.2. Ado's Theorem

The goal is Ado's theorem that every Lie algebra is linear, i.e., is a subalgebra of $\mathfrak{gl}(V)$ for some vector space V, which is the same as saying it has a finite-dimensional faithful representation. As in the previous section, there are some easy steps, and then a clever argument is needed to create an appropriate representation.

We start, of course, with the adjoint representation, which is about the only representation we have for an abstract Lie algebra $\mathfrak{g}$. Since the kernel of the adjoint representation is the center $\mathfrak{c}$ of $\mathfrak{g}$, it suffices to find a representation of $\mathfrak{g}$ which is faithful on $\mathfrak{c}$. For then the sum of this representation and the adjoint representation is a faithful representation of $\mathfrak{g}$.

The abelian Lie algebra $\mathfrak{c}$ has a faithful representation by nilpotent matrices. For example, when $\mathfrak{c} = \mathbb{C}$ is one dimensional, one can take the representation $\lambda \mapsto \left(\begin{smallmatrix} 0 & \lambda \\ 0 & 0 \end{smallmatrix}\right)$; in general a direct sum of such representations will suffice.

We can choose a sequence of subalgebras

$$\mathfrak{c} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_p = \mathfrak{n} \subset \mathfrak{g}_{p+1} \subset \cdots \subset \mathfrak{g}_q = \mathfrak{r} \subset \mathfrak{g}_{q+1} = \mathfrak{g},$$

each an ideal in the next, with $\mathfrak{n} = \mathrm{Nil}(\mathfrak{g})$ the largest nilpotent ideal of $\mathfrak{g}$, and $\mathfrak{r} = \mathrm{Rad}(\mathfrak{g})$ the largest solvable ideal; as in §9.1 we may assume $\dim(\mathfrak{g}_i/\mathfrak{g}_{i-1}) = 1$ for $i \leq q$. The plan is to start with a faithful representation of $\mathfrak{g}_0$, and construct successively representations of each $\mathfrak{g}_i$ which are faithful on $\mathfrak{c}$. The conditions we will need to make this step are that $\mathfrak{g}_i = \mathfrak{g}_{i-1} \oplus \mathfrak{h}_i$ with $\mathfrak{g}_{i-1}$ a solvable ideal in $\mathfrak{g}_i$ and $\mathfrak{h}_i$ a subalgebra of $\mathfrak{g}_i$. We can achieve this by taking $\mathfrak{h}_i$ to be any one-dimensional vector space complementary to $\mathfrak{g}_{i-1}$ for $i \leq q$. Similarly to go from $\mathfrak{r}$ to $\mathfrak{g}$, use Levi's theorem to write $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{h}$ for a subalgebra $\mathfrak{h}$.

Call a representation ρ of a Lie algebra $\mathfrak{g}$ a *nilrepresentation* if $\rho(X)$ is a nilpotent endomorphism for every X in $\mathrm{Nil}(\mathfrak{g})$. A stronger version of Ado's theorem is:

Theorem E.4. *Every Lie algebra has a faithful finite-dimensional nilrepresentation.*

The crucial step is:

Proposition E.5. *Let $\mathfrak{g}$ be a Lie algebra which is a direct sum of a solvable ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$. Let σ be a nilrepresentation of $\mathfrak{a}$. Then there is a representation ρ of $\mathfrak{g}$ such that*

$$\mathfrak{h} \cap \mathrm{Ker}(\rho) \subset \mathrm{Ker}(\sigma).$$

If $\mathrm{Nil}(\mathfrak{g}) = \mathrm{Nil}(\mathfrak{a})$ or $\mathrm{Nil}(\mathfrak{g}) = \mathfrak{g}$, then ρ may be taken to be a nilrepresentation.

Ado's theorem follows readily from this proposition. Starting with a faithful representation ρ_0 of $\mathfrak{c} = \mathfrak{g}_0$ by nilpotent matrices, one uses the proposition to construct successively nilrepresentations ρ_i of $\mathfrak{g}_i$. The displayed condition assures that they are all faithful on $\mathfrak{c}$. Note that if $i \leq p$, $\mathrm{Nil}(\mathfrak{g}_i) = \mathfrak{g}_i$, while if $i > p$ we have $\mathrm{Nil}(\mathfrak{g}_i) = \mathrm{Nil}(\mathfrak{g}_{i-1}) = \mathfrak{n}$ by Corollary C.26, so the hypotheses assure that all representations can be taken to be nilrepresentations. □

Suppose $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ is a Lie algebra which is a direct sum of an ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$. Let $U = U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. Any

Y in $\mathfrak{a}$ determines a linear endomorphism L_Y of U, which is simply left multiplication by the image of Y in U. Any X in $\mathfrak{g}$ determines an inner derivation $Y \mapsto [X, Y]$ of $\mathfrak{a}$; let D_X be the corresponding derivation of U, cf. Lemma C.27. For each X in $\mathfrak{g}$ we define a linear mapping $T_X: U \to U$ by writing $X = Y + Z$ with Y in $\mathfrak{a}$ and Z in $\mathfrak{h}$, and setting

$$T_X = L_Y + D_Z.$$

A straightforward calculation shows that

$$T_{[X_1, X_2]} = T_{X_1} \circ T_{X_2} - T_{X_2} \circ T_{X_1}. \tag{E.6}$$

If $\mathfrak{gl}(U)$ denotes the infinite-dimensional Lie algebra of endomorphisms of U, with the usual bracket $[A, B] = A \circ B - B \circ A$, this means that the mapping $\mathfrak{a} \to \mathfrak{gl}(U)$, $X \mapsto T_X$, is a homomorphism of Lie algebras.

Suppose $\sigma: \mathfrak{a} \to \mathfrak{gl}(V)$ is a finite-dimensional representation of $\mathfrak{a}$. Let $\tilde{\sigma}: U \to \operatorname{End}(V)$ be the corresponding homomorphism of algebras, as in §C.3, and let I be the kernel of $\tilde{\sigma}$. The basic step is:

Lemma E.7. *Assume that* $\mathfrak{a}$ *is solvable. Suppose* I *is an ideal of* $U = U(\mathfrak{a})$ *satisfying the following two properties*: (i) U/I *is finite dimensional*; (ii) *the image of every element in* Nil($\mathfrak{a}$) *in* U/I *is nilpotent. Then there is an ideal* $J \subset I$ *of* U *satisfying properties* (i) *and* (ii), *and also* (iii) *for every derivation* D *of* $\mathfrak{a}$, *the corresponding derivation of* U *maps* J *into itself.*

Granting this lemma, we prove Proposition E.5 as follows. From the representation σ we constructed an ideal I in $U = U(\mathfrak{a})$, with $U/I \subset \operatorname{End}(V)$, so condition (i) is satisfied; the fact that σ is a nilrepresentation implies that condition (ii) also holds. Let J be an ideal whose existence is asserted in the lemma. Because of (iii), each of the endomorphisms T_X of U maps J into itself, and so determines an endomorphism $\overline{T_X}$ of U/J. By (E.6), the mapping $X \mapsto \overline{T_X}$ is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{gl}(U/J)$. This is the representation ρ required in the proposition.

We first verify that $\operatorname{Ker}(\rho) \cap \mathfrak{a} \subset \operatorname{Ker}(\sigma)$. Note that if X is in $\mathfrak{a}$, then $\overline{T_X}$ is just left multiplication by X on U/J, so if $\rho(X)$ vanishes, the image of X in U must be in J; since $J \subset I$, X maps to zero in $U/I \subset \operatorname{End}(V)$, so $\sigma(X) = 0$, as required.

It remains to show that, under either of the additional hypotheses, ρ is a nilrepresentation. Note first that each X in $\mathfrak{a}$ acts on U/J by left multiplication, and if X is in Nil($\mathfrak{a}$), by (ii) its image in U/J is nilpotent. Thus $\rho(X)$ is nilpotent for every X in Nil($\mathfrak{a}$). In particular, this shows that ρ is a nilrepresentation when Nil($\mathfrak{g}$) = Nil($\mathfrak{a}$).

In the other case, $\mathfrak{g}$ is nilpotent, so $\mathfrak{a}$ is also nilpotent, and the preceding shows that $\rho(Y)$ is nilpotent for every Y in $\mathfrak{a}$. We need a slightly stronger assertion than this. Let $A \subset \operatorname{End}(U/J)$ be the associative algebra (with unit) generated by $\rho(\mathfrak{g})$, and let $P \subset A$ be the two-sided ideal generated by $\rho(\mathfrak{a})$. The claim is that P is a nilpotent ideal, i.e., that $P^k = P \cdot \ldots \cdot P = 0$ for some k. To see this, note that there is a k such that every product of k elements of $\rho(\mathfrak{a})$ is zero; this follows from Engel's theorem, putting the action in strictly upper-triangular form. To show that $P^k = 0$, we must show that any product of elements in $\rho(\mathfrak{g})$ which contains at least k members from $\rho(\mathfrak{a})$ is zero. But if x is in $\rho(\mathfrak{g})$ and y is in $\rho(\mathfrak{a})$, we have

$$x \cdot y = y \cdot x + [x, y],$$

and $[x, y]$ is in $\rho(\mathfrak{a})$, so terms from $\rho(\mathfrak{a})$ can be successively moved to the left until the product is a sum of products each beginning with k terms from $\rho(\mathfrak{a})$.

Now if $\mathfrak{g}$ is nilpotent, for any Z in $\mathfrak{h}$ (or in $\mathfrak{g}$), $\mathrm{ad}(Z)$ is a nilpotent endomorphism of $\mathfrak{g}$, and hence of $\mathfrak{a}$. By the Leibnitz rule for derivations, it follows that the corresponding derivation D_Z of U is nilpotent on any element, although the power required to annihilate an element may be unbounded. However, since U/J is finite dimensional, it follows readily that the induced derivation of U/J is nilpotent. In other words, $\rho(Z)$ is nilpotent for every Z in $\mathfrak{h}$. Given X in $\mathfrak{g}$, write $X = Y + Z$ with $Y \in \mathfrak{a}$ and $Z \in \mathfrak{h}$. Choose k as in the preceding paragraph, and choose l so that $\rho(Z)^l = 0$. It follows that $\rho(X)^{kl} = (\rho(Y) + \rho(Z))^{kl}$ vanishes, since, when the latter is expanded, each summand either has $\rho(Y)$ occurring at least k times, or else $\rho(Z)^l$ occurs somewhere in the product. □

To finish, we must prove Lemma E.7. Let Q be the two-sided ideal in the algebra U/I generated by the image of $\mathrm{Nil}(\mathfrak{a})$. Since U/I is generated by the image of $\mathfrak{a}$, the same argument as in the paragraph before last shows that $Q^k = 0$ for some k. Write $Q = K/I$ for an ideal K of U, and set $J = K^k$. Clearly $J \subset I$, and we claim that J satisfies the conditions (i)–(iii) of the lemma.

To see that J has finite codimension, let $x_1, \ldots, x_n$ be a basis for the image of $\mathfrak{a}$ in U, and choose monic polynomials p_i such that $p_i(x_i)$ is in K; this is possible since U/K is finite dimensional. Therefore, $p_i(x_i)^k$ is in J, so the images of the x_i satisfy monic equations in U/J. Since U is generated by the monomials $x_1^{i_1} \cdot \ldots \cdot x_r^{i_r}$, it follows readily that U/J is spanned by a finite number of these elements.

Property (ii) is clear from the construction, for if $x \in U$ is the image of an element of $\mathrm{Nil}(\mathfrak{a})$, some power x^p is in I by assumption, so x^{pk} is in $I^k \subset K^k = J$.

For (iii), if D is a derivation of $\mathfrak{a}$, since $\mathfrak{a}$ is solvable, it follows from Proposition C.24 that D maps $\mathfrak{a}$ into $\mathrm{Nil}(\mathfrak{a})$. The corresponding derivation of U therefore maps U into K, from which it follows that it maps $J = K^k$ to itself. □

As before, the results of this section also apply to real Lie algebras: if $\mathfrak{g}$ is real, a faithful representation (complex) representation of $\mathfrak{g} \otimes \mathbb{C}$ is automatically a faithful real representation, and embeds $\mathfrak{g}$ is some $\mathfrak{gl}_n\mathbb{R}$.

APPENDIX F

Invariant Theory for the Classical Groups

The object is to derive just enough invariant theory for the classical groups to verify the claims made in the text. We follow a classical, constructive approach, using an identity of Capelli.

§F.1. The Polynomial Invariants

Let $V = \mathbb{C}^n$, regarded as the standard representation of $\mathrm{GL}_n\mathbb{C}$, so of any of the subgroups $G = \mathrm{SL}_n\mathbb{C}$, $\mathrm{O}_n\mathbb{C}$, $\mathrm{SO}_n\mathbb{C}$, or $\mathrm{Sp}_n\mathbb{C}$ (for n even); $e_1, \ldots, e_n$ denotes a standard basis for V, compatible with one of the standard realizations of G. The goal is to find those polynomials $F(x^{(1)}, \ldots, x^{(m)})$ of m variables on V which are invariant by G. For example, if $Q\colon V \otimes V \to \mathbb{C}$ is the bilinear form determining the orthogonal or symplectic group, the polynomials $Q(x^{(i)}, x^{(j)})$ are invariants. In addition, if G is a subgroup of $\mathrm{SL}(V)$, the *bracket* $[x^{(1)}\, x^{(2)} \ldots x^{(n)}]$, given by the determinant,

$$[x^{(1)} \quad x^{(2)} \quad \ldots \quad x^{(n)}] = \det(x_j^{(i)}), \tag{F.1}$$

is an invariant of G. The *first fundamental theorem* of invariant theory for these groups asserts that any invariant is a polynomial function of these basic invariants. This is the goal of this appendix.

We denote by S^d the homogeneous polynomial functions of degree d on V, i.e., $S^d = \mathrm{Sym}^d(V^*)$. For an m-tuple $\mathbf{d} = (d_1, \ldots, d_m)$ of non-negative integers, let $S^{\mathbf{d}} = S^{d_1} \otimes \cdots \otimes S^{d_m}$ be the polynomials on $V^{\oplus m}$ which are homogeneous of degree d_i in the ith variable. Note that

$$\mathrm{Sym}^k(V^{\oplus m})^* = \bigoplus S^{\mathbf{d}},$$

the sum over all $\mathbf{d}$ with $d_1 + d_2 + \cdots + d_m = k$, which identifies elements of $S^{\mathbf{d}}$ with functions of m-tuples in V. We write $F(x^{(1)}, \ldots, x^{(m)})$ for such a polynomial, with usual abbreviations to $F(x)$ for $m = 1$, $F(x, y)$ for $m = 2$, $F(x, y, z)$ for $m = 3$.

When $m = 1$ we have already found the invariants: for $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_n\mathbb{C}$ all symmetric powers S^d are irreducible, so there are no invariants unless $d = 0$; for $\mathrm{SO}_n\mathbb{C}$ the kernel of the map $S^d \to S^{d-2}$ (contracting with the given quadratic form Q) is irreducible, so by induction one sees that there are no invariants if d is odd, whereas if d is even, the invariants are scalar multiples of the polynomial $Q(x, x)^{d/2}$. (These results will be proved again below.)

In theory one could follow procedures outlined in the text to decompose the tensor products of the known representations S^{d_i} to find out how the trivial representation occurs in $S^{\mathbf{d}}$. Except in small degrees and dimensions, however, this is rather impractical.

To describe the G-invariant polynomials in $S^{\mathbf{d}}$, we will carry out an induction, first with respect to the total degree $\sum d_i$, then with respect to the individual multidegrees ordered *antilexicographically*: $\mathbf{d}' < \mathbf{d}$ means that either $\sum d_i' < \sum d_i$ or $\sum d_i' = \sum d_i$ and the largest i for which d_i' and d_i differ has $d_i' < d_i$.

For integers i and j between 1 and m there is a canonical "polarization" map D_{ij} which takes a polynomial F of m variables to the polynomial

$$D_{ij}(F) = \sum_{k=1}^{n} x_k^{(i)} \frac{\partial F}{\partial x_k^{(j)}}. \tag{F.2}$$

This operator lowers the jth degree by 1, while it increases the ith degree by 1, i.e., it maps $S^{\mathbf{d}}$ to $S^{\mathbf{d}'}$, where $\mathbf{d}'$ is the same sequence of multi-indices as $\mathbf{d}$, but with $d_j' = d_j - 1$ and $d_i' = d_i + 1$; if $d_j = 0$ set $S^{\mathbf{d}'} = 0$. When $j = i$, note that by *Euler's formula*, D_{ii} is multiplication by d_i. Note also that these D_{ij} are derivations:

$$D_{ij}(F_1 \cdot F_2) = D_{ij}(F_1) \cdot F_2 + F_1 \cdot D_{ij}(F_2). \tag{F.3}$$

These maps may be described intrinsically in terms of the multilinear algebra of Appendix B, as follows. Since only two factors are involved, it suffices to look at the map D_{12} when there are only two factors. In this case the map

$$D_{12}\colon S^d \otimes S^e \to S^{d+1} \otimes S^{e-1}$$

is defined by

$$u_1 \cdot \ldots \cdot u_d \otimes w_1 \cdot \ldots \cdot w_e \mapsto \sum_{i=1}^{e} u_1 \cdot \ldots \cdot u_d \cdot w_i \otimes w_1 \cdot \ldots \cdot \hat{w}_i \cdot \ldots \cdot w_e.$$

Equivalently, D_{12} is the composite

$$S^d \otimes S^e \to S^d \otimes (S^1 \otimes S^{e-1}) = (S^d \otimes S^1) \otimes S^{e-1} \to S^{d+1} \otimes S^{e-1},$$

where the second is determined by the product $S^d \otimes S^1 \to S^{d+1}$ of symmetric powers, and the first by the dual map $S^e \to S^1 \otimes S^{e-1}$ (which takes $F(x)$ to $\sum_k x_k \otimes \partial F/\partial x_k$). This shows, if there were any doubt, that the D_{ij} are maps of $\mathrm{GL}(V)$-modules, i.e., that they are independent of choice of coordinates.

Note that $D_{ji} \circ D_{ij}$ maps $S^{\mathbf{d}}$ to itself. Explicitly, for $\mathbf{d} = (d, e)$,

$$\begin{aligned} D_{21} \circ D_{12}(F) &= \sum_k y_k \frac{\partial}{\partial x_k}\left(\sum_l x_l \frac{\partial F}{\partial y_l}\right) \\ &= \sum_k y_k \frac{\partial F}{\partial y_k} + \sum_{k,l} y_k x_l \frac{\partial^2 F}{\partial y_l \partial x_k} \\ &= e \cdot F + \sum_{k,l} y_k x_l \frac{\partial^2 F}{\partial y_l \partial x_k}. \end{aligned}$$

A first idea is that, if F is an invariant by a group $G \subset \mathrm{GL}(V)$, then $D_{ij}(F)$ will also be an invariant, and these invariants will be known by induction if $i < j$, so one can describe the possible $D_{ji} \circ D_{ij}(F)$ that arise. If one also knew the second term in the above expression for this, one could determine $e \cdot F$, which suffices to determine F, provided e is not zero.

In general, it is not evident how to proceed, but in case $\dim V = 2$, and $\mathbf{d} = (d, e)$, this can idea can be carried through as follows. Some of the terms in the second term also occur in the expression

$$[xy] \cdot \Omega(F) = (x_1 y_2 - x_2 y_1) \cdot \left(\frac{\partial^2 F}{\partial x_1 \partial y_2} - \frac{\partial^2 F}{\partial x_2 \partial y_1}\right).$$

The rest occur in

$$de \cdot F = d \cdot \left(\sum y_l \frac{\partial F}{\partial y_l}\right) = \sum x_k y_l \frac{\partial^2 F}{\partial x_k \partial y_l}.$$

Comparing the preceding three formulas gives the identity

$$(d+1)e \cdot F = D_{21} \circ D_{12}(F) + [xy] \cdot \Omega(F). \tag{F.4}$$

From this identity it is easy to find all invariants for one of our subgroups of $\mathrm{GL}_2\mathbb{C}$ and for functions of two variables. We will do it for $G = \mathrm{SO}_2\mathbb{C}$, as it illustrates the ideas of the general case—even though G is not semisimple, and the results can be seen directly by identifying G with $\mathbb{C}^*$. We assume the simple case of functions of one variable has been checked: only multiples of $Q(x, x)^{d/2}$ are invariant. Suppose $F \in S^d \otimes S^e$ is an invariant of $G = \mathrm{SO}_2\mathbb{C}$, with $e > 0$. We claim that F is a polynomial in the bracket function $[xy]$ and the polynomials $Q(x, y)$, $Q(x, x)$, and $Q(y, y)$. Either directly or from the above identity one sees that $\Omega(F)$ is also an $\mathrm{SO}_2\mathbb{C}$-invariant, and by induction it is a polynomial in these basic polynomials. Similarly by the antilexicographic induction we know that $D_{12}(F)$ is a polynomial in the the basic invariants. It therefore suffices to verify that D_{21} preserves polynomials in the four basic invariants. By the derivation property (F.3) it is enough to compute the effect of D_{21} on the basic invariants, and this is easy:

$$D_{21}[xy] = 0, \qquad D_{21}Q(x, y) = Q(y, y),$$
$$D_{21}Q(x, x) = 2Q(x, y), \qquad D_{21}Q(y, y) = 0.$$

By (F.4) we conclude that $(d + 1)e \cdot F$ is a polynomial in the basic invariants, which concludes the proof.

This plan of attack, in fact, extends to find all polynomial invariants of all the classical subgroups of $\mathrm{GL}(V)$. What is needed is an appropriate generalization of the identity (F.4). About a century ago Capelli found such an identity. The clue is to write (F.4) in the more suggestive form

$$\begin{vmatrix} D_{11} + 1 & D_{12} \\ D_{21} & D_{22} \end{vmatrix} (F) = [xy] \cdot \Omega(F),$$

where the determinant on the left is evaluated by expanding as usual, but being careful to read the composition of operators from left to right, since they do not commute.

This is the formula which generalizes. If F is a function of m variables from V, and $\dim V = m$, define, following Cayley,

$$\Omega(F) = \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \frac{\partial^m F}{\partial x^{(1)}_{\sigma(1)} \cdot \ldots \cdot \partial x^{(m)}_{\sigma(m)}}; \tag{F.5}$$

in symbols, Ω is given by the determinant

$$\begin{vmatrix} \dfrac{\partial}{\partial x_1^{(1)}} & \dfrac{\partial}{\partial x_1^{(2)}} & \cdots & \dfrac{\partial}{\partial x_1^{(m)}} \\ \dfrac{\partial}{\partial x_2^{(1)}} & & \cdots & \dfrac{\partial}{\partial x_2^{(m)}} \\ \vdots & & & \vdots \\ \dfrac{\partial}{\partial x_m^{(1)}} & & \cdots & \dfrac{\partial}{\partial x_m^{(m)}} \end{vmatrix}.$$

The *Capelli identity* is the formula:

$$\begin{vmatrix} D_{11} + m - 1 & D_{12} & \cdots & D_{1m} \\ D_{21} & D_{22} + m - 2 & \cdots & D_{2m} \\ \vdots & & & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mm} \end{vmatrix} = [x^{(1)}\, x^{(2)} \ldots x^{(m)}] \cdot \Omega. \tag{F.6}$$

This is an identity of operators acting on functions $F = F(x^{(1)}, \ldots, x^{(m)})$ of m variables, with $m = n = \dim V$, and as always the determinant is expanded with compositions of operators reading from left to right. Note the important corollary: if the number of variables is greater than the dimension, $m > n$, then

$$\begin{vmatrix} D_{11}+m-1 & D_{12} & \dots & D_{1m} \\ D_{21} & D_{22}+m-2 & \dots & D_{2m} \\ \vdots & & & \vdots \\ D_{m1} & D_{m2} & \dots & D_{mm} \end{vmatrix} (F) = 0. \qquad \text{(F.7)}$$

This follows by regarding F as a function on $\mathbb{C}^m$ which is independent of the last $m - n$ coordinates. Since $\Omega(F) = 0$ for such a function, (F.7) follows from (F.6).

We will prove Capelli's identity in §F.3. Now we use it to compute invariants. Let K denote the operator on the left-hand side of these Capelli identities. The expansion of K has a main diagonal term, the product of the diagonal entries $D_{ii} + m - i$, which are scalars on multihomogeneous functions. Note that in any other product of the expansion, the last nondiagonal term which occurs is one of the D_{ij} with $i < j$. Since the diagonal terms commute with the others, we can group the products that precede a given D_{ij} into one operator, so we can write, for $F \in S^{\mathbf{d}}$,

$$K(F) = \rho \cdot F - \sum_{i<j} P_{ij} D_{ij}(F),$$

where $\rho = (d_1 + m - 1)\cdot(d_2 + m - 2)\cdot \ldots \cdot(d_m)$, and each P_{ij} is a linear combination of compositions of various D_{ab}. Capelli's identities say that

$$\rho \cdot F = \sum_{i<j} P_{ij} D_{ij}(F) \qquad \text{if } m > n; \qquad \text{(F.8)}$$

$$\rho \cdot F = \sum_{i<j} P_{ij} D_{ij}(F) + [x^{(1)} \ldots x^{(m)}] \cdot \Omega(F) \quad \text{if } m = n. \qquad \text{(F.9)}$$

Just as in the above special case, if F is an invariant of a group G, each $D_{ij}(F)$ is also an invariant in a $S^{\mathbf{d}'}$ where we will know all such invariants by induction. If G is a subgroup of $\mathrm{SL}(V)$, and $m = n$, then $\Omega(F)$ is also an invariant, as follows from the definition or Capelli's identity.

Invariants for $\mathrm{SL}_n\mathbb{C}$.

Let $F \in S^{\mathbf{d}}$ be an invariant of the group $\mathrm{SL}_n\mathbb{C}$. We must show that F can be written as a polynomial in the basic bracket polynomials. In particular, if $m < n$, we must verify that there are no invariants except the constants in $S^0 = \mathbb{C}$. This is a simple consequence of the fact that for a dense open set of m-tuples of vectors—namely, those which are linearly independent—there is an automorphism of $\mathrm{SL}_n\mathbb{C}$ taking them to a fixed m-tuple of independent vectors, say $e_1, \ldots, e_m$. So an invariant function must take the same value on all such m-tuples. By the density, it must be constant.

For $m \geq n$, we proceed by induction as indicated above. All $D_{ij}F$ are known to be invariants (for $i < j$), as is $\Omega(F)$, so these are polynomials in the brackets.

To complete the proof, by Capelli's identities (F.8) and (F.9), it suffices to see that the operators D_{ab} all take brackets to scalar multiples of brackets. This is an obvious calculation: D_{ab} takes a bracket $[x^{(i_1)} x^{(i_2)} \dots x^{(i_n)}]$ to zero if b does not appear as one of the superscripts, or to the bracket with the variable $x^{(b)}$ replaced by $x^{(a)}$ if $x^{(b)}$ does occur; the latter is zero if $x^{(a)}$ also occurs and is a bracket otherwise. To avoid repeats, one needs only consider brackets where the superscripts are increasing. This completes the proof of

Proposition F.10. *Polynomial invariants $F(x^{(1)}, \dots, x^{(m)})$ of $\mathrm{SL}_n\mathbb{C}$ can be written as polynomials in the brackets*

$$[x^{(i_1)} x^{(i_2)} \dots x^{(i_n)}], \quad 1 \le i_1 < i_2 < \dots < i_n \le m.$$

Exercise F.11. Show that the only polynomial invariants of $\mathrm{GL}_n\mathbb{C}$ are the constants.

Invariants for $\mathrm{Sp}_n\mathbb{C}$

Let $r = n/2$, and let Q be the skew form defining the symplectic group $\mathrm{Sp}_n\mathbb{C}$, e.g. $Q(x, y) = \sum_{i=1}^{r} x_i y_{r+i} - x_{r+i} y_i$ in standard coordinates. Note first that the brackets are not needed:

Exercise F.12*. Show that the bracket $[x^{(1)} x^{(2)} \dots x^{(n)}]$ is equal to

$$\sum \operatorname{sgn}(\sigma) Q(x^{(\sigma(1))}, x^{\sigma(2)}) \cdot Q(x^{(\sigma(3))}, x^{\sigma(4)}) \cdot \dots \cdot Q(x^{(\sigma(n-1))}, x^{\sigma(n)}),$$

where the sum is over all permutations σ of $\{1, \dots, n\}$ such that $\sigma(2i-1) < \sigma(2i)$ for $1 \le i \le r$ and $\sigma(i-1) < \sigma(i)$ for $2 \le i \le r$.

Let T_n^m be the assertion that any $\mathrm{Sp}_n\mathbb{C}$-invariant polynomial in m variables from $\mathbb{C}^n$ can be written as a polynomial in the basic polynomials $Q(x^{(i)}, x^{(j)})$. The antilexicographic induction using the Capelli identities is the same as before, and gives the implications

$$T_n^{n-1} \Rightarrow T_n^n \Rightarrow T_n^m \quad \text{for all } m > n.$$

The only variation here is to verify that the operators D_{ab} preserve polynomials in the basic invariants, and $D_{ab}Q(x^{(i)}, x^{(j)})$ is again zero or another basic invariant.

The situation where $m < n$ is a little more complicated than that for the special linear group, however—which is hardly surprising since there are nontrivial invariants for $\mathrm{Sp}_n\mathbb{C}$ in this range. Note that T_n^m implies $T_n^{m'}$ for $m' < m$, so it suffices to prove T_n^{n-1}. This is done by induction on $r = n/2$, i.e., by proving the implication $T_{n-2}^{n-1} \Rightarrow T_n^{n-1}$. To prove this, consider the restriction F' of an invariant polynomial F on $V = \mathbb{C}^n$ to the subspace $V' = \mathbb{C}^{n-2}$ perpendicular to the plane spanned by e_r and e_n. This restriction is an invariant of the group $\mathrm{Sp}_{n-2}\mathbb{C}$. By induction, F' is a polynomial in the basic invariants. Since $Q(x^{(i)}, x^{(j)})$ restricts to the corresponding invariant on V', there is a polynomial in these $Q(x^{(i)}, x^{(j)})$ such that F and this polynomial have the same restriction to V'. Subtracting, it suffices to prove that if an invariant F restricts to zero on V', then F is zero.

We show first that the restriction of F to the larger subspace $W = V' \oplus \mathbb{C}e_r$ must be zero. Fix $y^{(1)}, \ldots, y^{(m)}$ in V', and consider the function of m complex variables.

$$h(t_1, \ldots, t_m) = F(y^{(1)} + t_1 e_r, \ldots, y^{(m)} + t_m e_r).$$

The fact that F is invariant by automorphisms in $\mathrm{Sp}_n\mathbb{C}$ which fix V' and send e_r to $\alpha \cdot e_r$ and e_n to $\alpha^{-1} \cdot e_n$ shows that

$$h(\lambda t_1, \ldots, \lambda t_m) = h(t_1, \ldots, t_m) \quad \text{for all } \lambda \neq 0.$$

Since h is a polynomial, it must be constant, so $h(t_1, \ldots, t_m) = h(0, \ldots, 0) = 0$, as required.

Since F is invariant, it follows that the restriction of F to any hyperplane of the form $g \cdot W$, for any $g \in \mathrm{Sp}_n\mathbb{C}$ is zero. It is not hard to verify that every hyperplane in $\mathbb{C}^n$ has this form. So any $n - 1$ vectors lie in such an hyperplane, and so F is identically zero. This finishes the proof for the symplectic group:

Proposition F.13. *Polynomial invariants $F(x^{(1)}, \ldots, x^{(m)})$ of $\mathrm{Sp}_n\mathbb{C}$ can be written as polynomials in functions*

$$Q(x^{(i)}, x^{(j)}), \quad 1 \leq i < j \leq m.$$

Invariants for $\mathrm{SO}_n\mathbb{C}$

This time brackets may be needed, as well as the functions given by the symmetric form Q, but products of brackets are not required:

Exercise F.14. Prove the identity

$$[x^{(1)}\, x^{(2)} \ldots x^{(n)}] \cdot [y^{(1)}\, y^{(2)} \ldots y^{(n)}] = |Q(x^{(i)}, y^{(j)})|_{1 \leq i, j \leq n}$$

for any variables $x^{(1)}, \ldots, x^{(n)}, y^{(1)}, \ldots, y^{(n)}$.

Let T_n^m be the assertion that any $\mathrm{SO}_n\mathbb{C}$-invariant polynomial in m variables can be written as a polynomial in the brackets and the invariants $Q(x^{(i)}, x^{(j)})$, where we take $Q(x, y) = \sum_{i=1}^n x_i y_i$ to be the form determining the orthogonal group. The proofs of the implications $T_n^{n-1} \Rightarrow T_n^n \Rightarrow T_n^m$ for $m > n$ are exactly as in the preceding cases, and require no further comment. As before, it remains to prove T_n^{n-1}, and, by induction on n, it suffices to prove the implication $T_{n-1}^{n-1} \Rightarrow T_n^{n-1}$.

Let $V' = \mathbb{C}^{n-1}$ be the orthogonal complement to e_n. The restriction F' to V' of an $\mathrm{SO}_n\mathbb{C}$-invariant polynomial F is $\mathrm{SO}_{n-1}\mathbb{C}$-invariant, and by induction we know it is a polynomial in the restrictions of the basic polynomials $Q(x^{(i)}, x^{(j)})$ and in the bracket $[x^{(1)} \dots x^{(n-1)}]$. An apparent snag is met here, however, since this bracket is not the restriction of an invariant on V. By Exercise F.14, we can write

$$F' = A + B \cdot [x^{(1)} \dots x^{(n-1)}],$$

where A and B are polynomials in the Q's alone. In particular, A and B are *even*, i.e., they are invariants of the full orthogonal group $\mathrm{O}_{n-1}\mathbb{C}$. But F' is also even, since any element of $\mathrm{O}_{n-1}\mathbb{C}$ is the restriction of some element in $\mathrm{SO}_n\mathbb{C}$ (mapping e_n to $\pm e_n$). Since the bracket is taken to minus itself by automorphisms of determinant -1, we must have $F' = A$. This means that we can subtract a polynomial in the invariants $Q(x^{(i)}, x^{(j)})$ from F, so we can assume $F' = 0$. Therefore, the restriction of F to any hyperplane of the form $g \cdot V'$, $g \in \mathrm{SO}_n\mathbb{C}$, is zero. But it is easy to verify that $(n-1)$-tuples in such hyperplanes form an open dense subset of all $(n-1)$-tuples in $\mathbb{C}^n$ (the condition is that there be an orthogonal vector e with $Q(e \cdot e) \neq 0$). This proves:

Proposition F.15. *Polynomial invariants $F(x^{(1)}, \dots, x^{(m)})$ of $\mathrm{SO}_n\mathbb{C}$ can be written as polynomials in functions*

$$Q(x^{(i)}, x^{(j)}) \quad and \quad [x^{(i_1)} x^{(i_2)} \dots x^{(i_n)}],$$

with $1 \le i \le j \le m$, $1 \le i_1 < i_2 < \cdots < i_n \le m$.

Exercise F.16*. Show that the polynomial invariants of $\mathrm{O}_n\mathbb{C}$ can be written as polynomials in the functions $Q(x^{(i)}, x^{(j)})$, $1 \le i < j \le m$. Show that *odd* polynomial invariants of $\mathrm{O}_n\mathbb{C}$, i.e., polynomials F which are taken to $\det(g) \cdot F$ by g in $\mathrm{O}_n\mathbb{C}$, can be written as linear combinations of even invariants times brackets.

§F.2. Applications to Symplectic and Orthogonal Groups

We consider the symplectic group $\mathrm{Sp}_n\mathbb{C}$ and the orthogonal group $\mathrm{O}_n\mathbb{C}$ together, letting Q denote the corresponding skew or symmetric form. The results in the first section, applied to the case $\mathbf{d} = (1, \dots, 1)$, say that the invariants in $(V^*)^{\otimes m}$ are all polynomials in the polynomials $Q(x^{(i)}, x^{(j)})$, and by degree considerations m must be even, and they are all linear combinations of products

$$Q(x^{(\sigma(1))}, x^{(\sigma(2))}) \cdot Q(x^{(\sigma(3))}, x^{(\sigma(4))}) \cdot \dots \cdot Q(x^{(\sigma(m-1))}, x^{(\sigma(m))}) \qquad \text{(F.17)}$$

for permutations σ of $\{1, \ldots, m\}$ such that $\sigma(2i-1) < \sigma(2i)$ for $1 \leq i \leq m/2$. Regarding $Q \in V^* \otimes V^*$, these are obtained from the invariant $Q \otimes \cdots \otimes Q$ ($m/2$ times) by permuting the factors. In other words, one pairs off the m components, and inserts Q in the place indicated by each pair.

The form Q gives an isomorphism of V with V^*, which takes v to $Q(v, -)$. Using this we can find all invariants of tensor products $(V^*)^{\otimes k} \otimes (V)^{\otimes l}$, via the isomorphism

$$(V^*)^{\otimes(k+l)} = (V^*)^{\otimes k} \otimes (V^*)^{\otimes l} \cong (V^*)^{\otimes k} \otimes (V)^{\otimes l}.$$

They are linear combinations of the images of the above invariants under this identification. To see what they are, we just need to see what happens to Q under the isomorphisms $V^* \otimes V^* \cong V^* \otimes V$ and $V^* \otimes V^* \cong V \otimes V$:

Exercise F.18. (i) Verify that under the canonical isomorphism

$$V^* \otimes V^* \cong V^* \otimes V = \operatorname{Hom}(V, V) = \operatorname{End}(V)$$

Q maps to the identity endomorphism. (ii) Let ψ be the image of Q under the canonical isomorphism $V^* \otimes V^* \cong V \otimes V$. Verify that

$$\psi = \sum_{i=1}^{r} e_i \otimes e_{r+i} - e_{r+i} \otimes e_i \quad \text{for } G = \mathrm{Sp}_n\mathbb{C},\ n = 2r;$$

$$\psi = \sum_{i=1}^{n} e_i \otimes e_i \quad \text{for } G = \mathrm{O}_n\mathbb{C}.$$

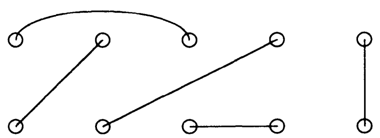
For the applications in Lectures 17 and 19, we need only the case $l = k$, but we want to reinterpret these invariants by way of the canonical isomorphism

$$(V^*)^{\otimes 2d} \cong (V^*)^{\otimes d} \otimes (V)^{\otimes d} \cong \operatorname{Hom}(V^{\otimes d}, V^{\otimes d}) = \operatorname{End}(V^{\otimes d}). \qquad \text{(F.19)}$$

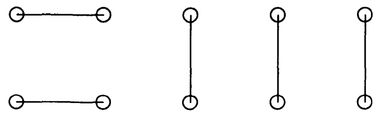
In §§17.3 and 19.5 we defined endomorphisms $\vartheta_I \in \operatorname{End}(V^{\otimes d})$ for each pair I of integers from $\{1, \ldots, d\}$; for I the first pair,

$$\vartheta_I(v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_d) = Q(v_1, v_2) \cdot \psi \otimes v_3 \otimes \cdots \otimes v_d;$$

the case for general I is a permutation of this. We claim that an invariant in $(V^*)^{\otimes 2d}$ of the form (F.17) is taken by the isomorphism (F.19) to a composition of operators ϑ_I and permutations σ in $\mathfrak{S}_d$. This is simply a matter of unraveling the definitions, which may be simpler to follow pictorially than notationally. The invariant in (F.17) is described by pairing the integers from 1 to $2d$. These pairs are either from the first d, the last d, or one of each. For example, if $d = 5$ the pairings could be as indicated:



for the pairs $\{1, 3\}$, $\{8, 9\}$, $\{2, 6\}$, $\{4, 7\}$, $\{5, 10\}$. Composing before and after with permutations, this can be changed to



The corresponding endomorphism of $V^{\otimes 5}$ becomes ϑ_I, $I = \{1, 2\}$. The general invariant one gets can be expressed in the form

$$\sigma \circ \vartheta_{I_1} \circ \vartheta_{I_2} \circ \cdots \circ \vartheta_{I_p} \circ \tau,$$

where σ and τ permute the d factors, and the pairs I_j are the first p pairs: $I_j = \{2j - 1, 2j\}$.

Now let A be the subalgebra of the ring $\operatorname{End}(V^{\otimes d})$ generated by all $g \otimes \cdots \otimes g$ for g in the group $G = \mathrm{Sp}_n\mathbb{C}$ (or $\mathrm{O}_n\mathbb{C}$). By the simplicity of the group, we know that A is a semisimple algebra of endomorphisms. We have just computed that the ring B of commutators of A is the ring generated by all permutations in $\mathfrak{S}_d$ and the operators ϑ_I. By the general theory of semisimple algebras, cf. §6.2, A must be the commutator algebra of B. In English, *any endomorphism of* $V^{\otimes d}$ *which commutes with permutations and with the operators* ϑ_I *must be a finite linear combination of operators of the form* $g \otimes \cdots \otimes g$ *for* g *in* G. This is precisely the fact from invariant theory that was used in the text.

We remark that a similar procedure can be used for $\mathrm{SL}_n\mathbb{C}$, but since in this case V and V^* are not isomorphic, to do this one must first do some more work to compute invariants in tensor products of covariant and contravariant factors. The idea is simple enough: use the canonical isomorphism $V \cong \bigwedge^{n-1}(V^*)$ to turn each V factor into several V^* factors. Tracing through the invariants by this procedure is rather complicated, however, and we refer to [Wel, II.8] for details. We did not need this analysis, because it was easy to work the commutator story the other way around, showing that the commutator of $\mathbb{C}[\mathfrak{S}_d]$ is the algebra generated by all $g \otimes \cdots \otimes g$ for g in $\mathrm{SL}_n\mathbb{C}$ (or $\mathrm{GL}_n\mathbb{C}$). This can, in turn, be run backwards:

Exercise F.20*. Use the fact that the the $\mathrm{GL}_n\mathbb{C}$-invariants of $\operatorname{End}(V^{\otimes d})$ are generated by permutations to show that the $\mathrm{GL}_n\mathbb{C}$-invariants of $(V^*)^{\otimes d} \otimes V^{\otimes d}$ are obtained by pairing off the factors and contracting. There are no $\mathrm{GL}_n\mathbb{C}$-invariants in $(V^*)^{\otimes k} \otimes V^{\otimes l}$ if $k \neq l$. For $\mathrm{SL}_n\mathbb{C}$-invariants, one also has determinant factors when $k - l$ is a multiple of the dimension.

We also omit any discussion of the *second fundamental theorems*, which describe the relations among the generators of the rings of invariants (but see the discussions at the ends of Lectures 17 and 19). These results can also be found in [Wel].

§F.3. Proof of Capelli's Identity

The proof is not essentially different from the case $m = 2$, once one has a good notational scheme to keep track of the algebraic manipulations which come about because the basic operators D_{ij} do not commute with each other. A convenient way to do this is as follows. For indices $i_1, j_1, \ldots, i_p, j_p$ between 1 and m, define an operator $\Delta_{i_1 j_1}\Delta_{i_2 j_2} \ldots \Delta_{i_p j_p}$ which takes a function F of m variables $x^{(1)}, \ldots, x^{(m)}$ to the function

$$\Delta_{i_1 j_1} \ldots \Delta_{i_p j_p}(F) = \sum_{k_1, \ldots, k_p = 1}^{n} x_{k_1}^{(i_1)} \cdot \ldots \cdot x_{k_p}^{(i_p)} \cdot \frac{\partial^p F}{\partial x_{k_1}^{(j_1)} \cdot \ldots \cdot \partial x_{k_p}^{(j_p)}}.$$

For $p = 1$, Δ_{ij} is just the operator D_{ij}, but for $p > 1$, this is *not* the composition of the operators $\Delta_{i_k j_k}$. Note that the order of the terms in the expression $\Delta_{i_1 j_1} \ldots \Delta_{i_p j_p}$ is unimportant.

We can form determinants of $p \times p$ matrices with entries these Δ_{ij}, which act on functions by expanding the determinant as usual, with each of the $p!$ products operating as above. For example, for the $m \times m$ matrix (Δ_{ij}),

$$|\Delta_{ij}|(F) = \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \cdot \Delta_{1\sigma(1)}\Delta_{2\sigma(2)} \cdots . \Delta_{m\sigma(m)}(F).$$

The matrix (Δ_{ij}) is a product of matrices $(x_k^{(i)}) \cdot (\partial/\partial x_k^{(j)})$, and taking determinants gives the

Lemma F.21. *For* $m = n$, $|\Delta_{ij}|(F) = [x^{(1)} \ldots x^{(m)}] \cdot \Omega(F)$.

To prove Capelli's identity (F.6), then, we must prove the following identity of operators on functions $F(x^{(1)}, \ldots, x^{(m)})$:

$$\begin{vmatrix} D_{11} + m - 1 & D_{12} & \cdots & D_{1m} \\ D_{21} & D_{22} + m - 2 & \cdots & D_{2m} \\ \vdots & & & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mm} \end{vmatrix} = \begin{vmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1m} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2m} \\ \vdots & & & \vdots \\ \Delta_{m1} & \Delta_{m2} & \cdots & \Delta_{mm} \end{vmatrix} \tag{F.22}$$

This is a formal identity, based on the simple identities:

$$D_{qp} \circ D_{ab} = D_{qp}\Delta_{ab} = \Delta_{qp}\Delta_{ab} \quad \text{if } p \neq a;$$

$$D_{qp} \circ D_{ab} = \Delta_{qp}\Delta_{ab} + D_{qb} \qquad \text{if } p = a.$$

Similarly, if $p \neq a_k$ for all k, then

$$D_{qp} \circ \Delta_{a_1 b_1} \ldots \Delta_{a_r b_r} = \Delta_{qp}\Delta_{a_1 b_1} \ldots \Delta_{a_r b_r}; \tag{F.23}$$

while if there is just one k with $p = a_k$, then

$$D_{qp} \circ \Delta_{a_1 b_1} \ldots \Delta_{a_r b_r} = \Delta_{qp}\Delta_{a_1 b_1} \ldots \Delta_{a_r b_r} + \Delta_{a_1 b_1} \ldots \Delta_{q b_k} \ldots \Delta_{a_r b_r} \tag{F.24}$$

where in the last term the $\Delta_{q b_k}$ replaces $\Delta_{a_k b_k}$.

We prove (F.22) by showing inductively that all $r \times r$ minors of the two matrices of (F.22) which are taken from the last r columns are equal (as operators on functions F as always). This is obvious when $r = 1$. We suppose it has been proved for $r = m - p$, and show it for $r + 1$. By induction, we may replace the last r columns of the matrix on the left by the last r columns of the matrix on the right. The difference of a minor on the left and the corresponding minor on the right will then be a maximal minor of the matrix

$$\begin{vmatrix} D_{1p} - \Delta_{1p} & \Delta_{1p+1} & \cdots & \Delta_{1m} \\ D_{2p} - \Delta_{2p} & \Delta_{2p+1} & \cdots & \Delta_{2m} \\ \vdots & & & \vdots \\ D_{pp} - \Delta_{pp} + r & \Delta_{pp+1} & \cdots & \Delta_{pm} \\ \vdots & & & \vdots \\ D_{mp} - \Delta_{mp} & \Delta_{mp+1} & \cdots & \Delta_{mm} \end{vmatrix},$$

so we must show that all maximal minors of this matrix are zero. Suppose the minor chosen is that using the q_ith rows, for $1 \le q_0 < q_1 < \cdots < q_r \le m$. Expanding along the left column, this determinant is

$$E_0 M_0 - E_1 M_1 + E_2 M_2 - \cdots + (-1)^r E_r M_r, \qquad \text{(F.25)}$$

where $E_k = D_{q_k p} - \Delta_{q_k p}$ if $q_k \neq p$, and $E_k = D_{pp} - \Delta_{pp} + r$ if $q_k = p$, and M_k is the corresponding cofactor ($r \times r$) determinant:

$$\sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) \Delta_{q_0 p + \sigma(1)} \cdots \Delta_{q_{k-1} p + \sigma(k)} \Delta_{q_{k+1} p + \sigma(k+1)} \cdots \Delta_{q_r p + \sigma(r)}. \qquad \text{(F.26)}$$

To show that (F.25) is zero, there are two cases. In the first case, the pth row is not included in the minor, i.e., $q_i \neq p$ for all i. In this case each term $E_i M_i$ is zero, since $E_i = D_{q_i p} - \Delta_{q_i p}$, and all the products in the expansion of M_i are of the form $\Delta_{a_1 b_1} \ldots \Delta_{a_r b_r}$ with all $a_i \neq p$, and the assertion follows from (F.23).

In the second case, the pth row is included, i.e., $q_k = p$ for some k. As in the first case, $(D_{pp} - \Delta_{pp}) M_k = 0$, and since $E_k = D_{pp} - \Delta_{pp} + r$, we have

$$E_k M_k = r \cdot M_k.$$

We claim that each of the other terms $E_i M_i$, for $i \neq k$, is equal to $(-1)^{k-i+1} M_k$, from which it follows that the alternating sum in (F.25) is zero. When M_i is written out as in (F.26), and it is multiplied by $E_i = D_{q_i p} - \Delta_{q_i p}$, an application of (F.24) shows that one gets the same determinant as (F.26), but expanded with the q_ith row moved between the q_{k-1}th and the q_{k+1}th rows. This transposition of rows accounts for the sign $(-1)^{k-i+1}$, yielding $E_i M_i = (-1)^{k-i+1} M_k$, as required. □

Exercise F.27. Find a $GL(V)$-linear surjection from $S^{d_1} \otimes \cdots \otimes S^{d_n}$ onto $\bigwedge^n V^* \otimes S^{d_1 - 1} \otimes \cdots \otimes S^{d_n - 1}$ that realizes the map $F \mapsto [x^{(1)} \ldots x^{(n)}] \cdot \Omega(F)$.

Hints, Answers, and References

Note: Usually answers or references are given only for more theoretical exercises, or those which may be referred to elsewhere.

Lecture 1

(1.3) The hypotheses ensure that $\wedge^n V$ is trivial, and the bilinear map $\wedge^k V \otimes \wedge^{n-k} V \to \wedge^n V = \mathbb{C}$ is a perfect pairing, i.e., it makes each space the dual of the other, cf. §B.3.

(1.4) For (b), take the function α to the function α', where $\alpha'(g) = \alpha(g^{-1})$.

(1.13) Yes. See Exercise 6.18.

(1.14) If H is a Hermitian inner product on V, let $\tilde{H}\colon V \to V^*$ be the conjugate linear map given by $v \mapsto H(v, \cdot)$. If H' is another, the composite $(\tilde{H}')^{-1} \circ \tilde{H}$ is linear, and a G-homomorphism if H and H' are G-invariant. Apply Schur's lemma.

Lecture 2

(2.3) For a general formula expressing complete symmetric polynomials and elementary symmetric polynomials in terms of sums of powers, see Exercise A.32(vi).

(2.4) Look at the induced action on $\wedge^k V$.

(2.7) $V^{\otimes n} = U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, with $a = b = \frac{1}{3}(2^{n-1} + (-1)^n)$, and $c = \frac{1}{3}(2^n + (-1)^{n-1})$.

(2.25) Answers: (i) $U \oplus V \oplus U' \oplus V'$; (ii) $U \oplus V^{\oplus 2} \oplus V' \oplus W$.

(2.29) The regular representation will do.

(2.33) For (c) use characters or the isomorphism

$$\mathrm{Hom}_G(V \otimes W, U) \cong \mathrm{Hom}_G(W, V^* \otimes U).$$

(2.34) Schur's lemma applies to L.

(2.35) Apply the preceding exercise, with L_0 given by a matrix of indeterminates. For details, see [Se2, §2.2].

(2.36) Show that $(\chi, \chi) = 1$, and compute the sum of the squares of these representations. Reference: [Se2, §3.2].

(2.37) If φ is the character of an irreducible representation, and χ is the character of V, let $a_n = (\varphi, \chi^n)$, and consider the power series

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{|G|} \sum_{n=0}^{\infty} \sum_C |C| \overline{\varphi(C)} \chi(C)^n t^n = \frac{1}{|G|} \sum_C \frac{|C| \overline{\varphi(C)}}{(1 - \chi(C)t)}.$$

Here C runs over conjugacy classes. Since $\chi(C) = \dim(V)$ only for $C = [e]$, the right-hand side is a nontrivial rational function; in particular a_n cannot be zero for all positive n.

(2.38) This is another theorem of Burnside. If C is a conjugacy class in G, $\varphi = \sum_{g \in C} g\colon V \to V$ is a G-map, so multiplication by a scalar λ_C, and $\lambda_C \cdot \dim V = \mathrm{Trace}(\varphi) = |C| \cdot \chi_V(C)$. The λ_C are algebraic integers, since the elements $\sum_{g \in C} e_g$, as C varies over the conjugacy classes, generate the center of the group ring $\mathbb{Z}[G]$, which is a finitely generated abelian group. Now

$$\sum_C |C| \cdot \overline{\chi_V(C)} \chi_V(C) = |G|,$$

so $|G|/\dim V = \sum_C \lambda_C \cdot \overline{\chi_V(C)}$ is an algebraic integer. In fact, the dimension of V divides the index of the center of G, cf. [Se2, p. 53].

(2.39) In case the character χ is $\mathbb{Z}$-valued, the equation $\sum |\chi(g)|^2 = |G|$ shows that $|G|$ is the sum of $|G|$ non-negative integers, one of which, $|\chi(e)|^2$, is greater than 1, so at least one must be 0. In general, the values of χ are algebraic integers, since they are sums of roots of unity. Let $\chi_1, \dots \chi_m$ be the characters obtained from χ by the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (or $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$) on χ; these characters are also characters of irreducible representations of G. Now if $\chi(g) \neq 0$, then $\prod_i \chi_i(g)$ is a nonzero integer, so $|\prod_i \chi_i(g)|^2 \geq 1$. Since the arithmetic mean is at least the geometric mean, $\sum_i |\chi_i(g)|^2 \geq m$. Therefore,

$$m|G| = \sum_{i=1}^{m} \sum_{g \in G} |\chi_i(g)|^2 \geq m|G|,$$

and we must have equality for every $g \in G$. In particular, if d is the degree of the representation, $md^2 = \sum_i |\chi_i(e)|^2 = m$, so $d = 1$.

Lecture 3

(3.5) Use the fact that $g = (12345)$ is conjugate to its inverse, so $\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$ is real.

(3.25) See §5.1.

(3.26) If $H \subset G$ is the subgroup of order 7, there are three one-dimensional representations from G/H, and two three-dimensional representations induced from H. For generalizations, see [Se2, §8.2].

(3.30) W is embedded in the space of W-valued functions on G by sending $w \in W$ to the function which takes $h \in H$ to $h \cdot w$ and all other cosets to zero. Note that if $\{g_\sigma\}$ is a set of coset representatives, the map $f \mapsto \sum g_\sigma \otimes f(g_\sigma^{-1})$ gives an isomorphism from $\mathrm{Hom}_H(\mathbb{C}G, W)$ to $\mathbb{C}G \otimes_{\mathbb{C}H} W$.

(3.32) For (b), identify the right-hand side with the trace of an endomorphism of $\mathbb{C}G$. For (c), take φ to be the characteristic function of an element g and apply (b).

(3.33) F is the determinant of left multiplication by the element $a = \sum x_g e_g \in \mathbb{C}G$ on the regular representation, and F_ρ is the determinant of left multiplication by a on the irreducible $\mathbb{C}G$-module V_ρ corresponding to ρ. The factorization of F follows from the decomposition of the regular representation. The irreducibility of F_ρ follows from the irreducibility of a matrix whose entries are indeterminates, using Proposition 3.29. Fixing g in G, set the variables $x_e = 1$ and $x_h = 0$ for $h \neq g$; the coefficient of x_g in the determinant of left multiplication by $1 + x_g e_g$ on V_ρ is $\chi_\rho(g)$.

(3.34) See Exercises 3.8 and 3.9.

(3.38) V can be replaced by V^*; $V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V$ contains at most one copy of the trivial representation. If $\mathrm{Sym}^2 V$ contains the trivial representation, then

$$|G| = \sum_{g \in G} \chi_{\mathrm{Sym}^2 V}(g) = \tfrac{1}{2}\left(\sum \chi_V(g)^2 + \sum \chi_V(g^2)\right).$$

Otherwise, the right-hand side is zero; similarly for $\wedge^2 V$. Note that if χ_V is real, then $\sum \chi_V(g)^2 = |G|$.

(3.41) Reference: [Se2, §13.2].

(3.42) Reference: [Ja-Ke, p. 12].

(3.43) Consider the endomorphism $J \otimes J$ of $V \otimes W$.

(3.44) For $G = \mathbb{Z}/3$, the rank of $R_{\mathbb{R}}(G)$ is 2, whereas that of $R(G)$ is 3.

(3.45) See [Se2, §12] for details.

Lecture 4

(4.4) Right multiplication by a gives a map $Aab \to Aba$, and right multiplication by b gives a map back. The composites are multiplications by nonzero scalars. More generally, if $A = \mathbb{C}G$ is a group algebra, call an element $a = \sum a_g e_g$ *Hermitian* if $\hat{a} = \bar{a}$, i.e., $a_{g^{-1}} = \overline{a_g}$. If a and b are idempotents which are Hermitian, then $Aab \cong Aba$.

(4.6) A basis for $V_{(d-1,1)} = \mathbb{C}\mathfrak{S}_d \cdot c_\lambda$ is $v_2, \ldots, v_d$, where

$$v_j = \sum_{g(d)=j} e_g - \sum_{h(1)=j} e_h.$$

Note that $v_d = c_\lambda$, $v_1 + \cdots + v_d = 0$, and $g \cdot v_j = v_i$ if $g(j) = i$. A basis for $V \subset \mathbb{C}^d$ is $v_2, \ldots, v_d$, where $v_j = e_j - e_{j-1}$. For the case $s > 1$, use (4.10) or see (4.43).

(4.13) Note that the hook lengths of the boxes in the first column are the numbers $l_1, \ldots, l_k$. Induct from the diagram obtained by omitting the first column.

(4.14) Induct as in the preceding exercise by removing the first column, considering separately the cases when the remaining diagram is one of the exceptions.

(4.15) Frobenius [Fro1] gives these and analogous formulas for $\lambda = (d-3, 3)$, $(d-3, 1, 1, 1)$, $(d-4, 4)$, ...

(4.16) Using Frobenius's formula, the coefficient of $x_1^{l_1} \cdot \ldots \cdot x_k^{l_k}$ in $\Delta \cdot (x_1^d + \cdots + x_k^d)$ can be nonzero only if $l_1 = d$, so λ has the prescribed form; the coefficient of $x_2^{k-1} x_3^{k-2} \cdot \ldots \cdot x_k$ in $\Delta(0, x_2, \ldots, x_k)$ is $(-1)^{k-1}$.

(4.19) See Exercise 4.51 for a general procedure for decomposing tensor products.

(4.20) Use Frobenius's formula as in Exercise 4.16 to show that $\chi_\lambda(g) = (-1)^{k-1}\chi_\mu(h)$, where $\mu = (\lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_k - 1)$ and $h \in \mathfrak{S}_{d-q_1}$ is the product of cycles of lengths $q_2, \ldots, q_r$.

(4.24) If $\lambda < \mu$ use the anti-involution $\hat{}$ of A induced by the map $g \mapsto g^{-1}$, $g \in \mathfrak{S}_d$, noting that $\hat{c}_\lambda = (a_\lambda b_\lambda)\hat{} = \hat{b}_\lambda \hat{a}_\lambda = b_\lambda a_\lambda$, so $(c_\lambda \cdot x \cdot c_\mu)\hat{} = \hat{c}_\mu \cdot \hat{x} \cdot \hat{c}_\lambda = b_\mu \cdot (a_\mu \cdot \hat{x} \cdot b_\lambda) \cdot a_\lambda = 0$.

(4.40) Note that the ψ_λ's are related to the χ_λ's by the same equations as the symmetric polynomials H_λ's to the Schur polynomials S_λ's, cf. (A.9) in the appendix. The equation (A.5) for the S_λ's in terms of the H_λ's therefore implies the determinantal formula.

(4.43) Use Frobenius reciprocity and (4.42) to prove the general formula. To prove that $V_{(d-s, 1, \ldots, 1)} \cong \wedge^s V$, argue by induction on d. Note that the restriction of $\wedge^s V$ splits into a sum of two exterior powers of the standard representation, and from anything but a hook one can remove at least three boxes.

(4.44) The induced representation of V_λ by the inclusion of $\mathfrak{S}_d$ in $\mathfrak{S}_{d+m}$ is $V_\lambda \circ V_{(m)}$. Use the transitivity of induction, Exercise 3.16(b).

(4.45) For (a), see [Jam, pp. 79–83]. For (b), using (4.33), the coefficient of X^a in $(x_1^m + \cdots + x_k^m) \cdot P^{(\mathbf{i})}$ is the sum of the coefficients of $X^a x_i^{-m}$ in $P^{(\mathbf{i})}$, summing over those i for which $a_i \geq m$. Use the determinantal formula to write $\chi_\lambda(g)$ as a sum $\Sigma \pm \chi_\mu(h)$, and show that the μ which occur are those obtained by removing skew hooks. Reference: [Boe, pp. 192–196].

(4.46) See Exercise A.11. In fact, this condition is equivalent to the condition that $K_{\rho\lambda} \leq K_{\rho\mu}$ for all ρ, or to the condition that U_λ is isomorphic to $U_\mu \oplus W$, for some representation W, cf. [L-V].

(4.47) References: For the first construction see [Jam], [Ja-Ke]; for the second, see [Pe2].

(4.48) There are several ways to do this: (i) Use the methods of this lecture to show that the value of the character of U'_λ on the class $C_\mathbf{i}$ is $[\vartheta(P^{(\mathbf{i})})]_{\lambda'}$, where ϑ is the involution defined in Exercise A.32. Then apply Lemma A.26. (ii) Show that $U'_\lambda \otimes U'$ is isomorphic to $U_{\lambda'}$ and use Corollary 4.39. (iii) Use Exercise 4.40 or 4.44.

(4.49) Use Exercise A.32(v).

(4.51) (a) Note that $\chi_\lambda = \sum_\mathbf{i} \omega_\lambda(\mathbf{i})\xi_{(\mathbf{i})}$, and $\xi_{(\mathbf{i})} = (1/z(\mathbf{i}))\sum_\nu \omega_\nu(\mathbf{i})\chi_\nu$, where $\xi_{(\mathbf{i})}$ is the characteristic function of the conjugacy class $C_{(\mathbf{i})}$. Therefore,

$$\chi_\lambda \chi_\mu = \sum_\mathbf{i} \omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i})\xi_{(\mathbf{i})},$$

from which the required formula follows. For other procedures and tables for small d see [Ja-Ke], [Co], and [Ham].

(b) $V_\lambda \otimes V_{(d)} = V_\lambda$, and $V_\lambda \otimes V_{(1,\dots,1)} = V_{\lambda'}$, which prove the corresponding results for $C_{\lambda(d)\mu}$ and $C_{\lambda(1,\dots,1)\mu}$. Use (a) to permute the subscripts.

(4.52) For (a), the described map from Λ to R is surjective by the determinantal formula of Exercise 4.40; it is an isomorphism since R_n and Λ_n are free of the same rank. For (f), note that $P^{(\mathbf{i})}$ corresponds to the character $\sum_\lambda \chi_\lambda(C_{(\mathbf{i})})\chi_\lambda$, which by Exercise 2.21 is the class function which is zero outside the conjugacy class $C_{(\mathbf{i})}$, and whose value on $C_{(\mathbf{i})}$ is $z(\mathbf{i})$.

For more on this correspondence, see [Bu], [Di2], [Mac]. In [Kn] a λ-ring structure on this ring is related to representation theory. In [Liu] this Hopf algebra is used to derive many of the facts about representations of $\mathfrak{S}_d$ from scratch. In [Ze] a similar approach is also used for representations of $\mathrm{GL}_n(\mathbb{F}_q)$.

More about representations of the symmetric groups can also be found in [Foa] and [J-L].

Lecture 5

(5.2) Consider the class functions on H which are invariant by conjugation by an element not in H.

(5.4) *Step* 1. (i) Inverses of elements of c' are conjugate to elements of c' if m is even, and to elements of c'' if m is odd; $\chi(g^{-1}) = \overline{\chi(g)}$. (ii) (ϑ, ϑ) is

$$\frac{2}{d!}(\#c' \cdot |u - v|^2 + \#c'' \cdot |v - u|^2) = \frac{2}{d!}\frac{d!}{q_1 \cdot \ldots \cdot q_r}|u - v|^2.$$

(iii) If λ corresponded to $p \neq q$, the values of χ'_λ and χ''_λ on the corresponding conjugacy classes $c'(p)$ and $c''(p)$ would be the same number, say w, and Exercise 4.20 implies that $2w = \pm 1$. Since w is an algebraic integer, this is impossible. Therefore, λ corresponds to q, and now from Exercise 4.20 we get the additional equation $u + v = (-1)^m$.

Step 2. (ii) Information about the characters χ' and χ'' of X' and X'' is easily determined from Exercise 3.19, and the fact that the characters of the factors are known by induction. In particular, since $c'(q)$ and $c''(q)$ each decomposes into two conjugacy classes in H, we have

$$\begin{aligned}\chi'(c'(q)) &= \frac{\varepsilon_1 + \sqrt{\varepsilon_1 q_1}}{2}\cdot\frac{\varepsilon' + \sqrt{\varepsilon' q'}}{2} + \frac{\varepsilon_1 - \sqrt{\varepsilon_1 q_1}}{2}\cdot\frac{\varepsilon' - \sqrt{\varepsilon' q'}}{2}\\ &= \frac{\varepsilon + \sqrt{\varepsilon q_1 \cdot \ldots \cdot q_r}}{2},\end{aligned}$$

where $\varepsilon_1 = (-1)^{(q_1-1)/2}$, $\varepsilon' = (-1)^{(d-q_1-r+1)/2}$, $\varepsilon = \varepsilon_1 \cdot \varepsilon'$, and $q' = q_2 \cdot \ldots \cdot q_r$; and similarly for the other values. (iv) The character of Y takes equal values on each pair of conjugate classes. (Reference: [Fro2], [Boe]).

(5.5) Reference: [Ja-Ke].

(5.9) If N is a normal subgroup properly between $\{\pm 1\}$ and $\mathrm{SL}_2(\mathbb{F}_q)$, one of the nontrivial characters χ must take the value $\chi(1)$ identically on N.

(5.11) Reference: [Ste1].

Lecture 6

(6.4) Compare (1) of the theorem with formulas (4.11) and (4.12). For a procedure to construct a basis of $\mathbb{S}_\lambda V$, see Exercise 6.28.

(6.10) By (4.41), there is an isomorphism of $\mathbb{C}\mathfrak{S}_{d+m}$-modules:

$$\mathbb{C}\mathfrak{S}_{d+m} \otimes_{\mathbb{C}(\mathfrak{S}_d \times \mathfrak{S}_m)} (V_\lambda \boxtimes V_\mu) \cong \bigoplus\nolimits_\nu N_{\lambda\mu\nu} V_\nu.$$

Tensoring on the left with the right $\mathbb{C}\mathfrak{S}_{d+m}$-module $V^{\otimes(d+m)} = V^{\otimes d} \otimes_{\mathbb{C}} V^{\otimes m}$, and noting that $\mathbb{C}(\mathfrak{S}_d \times \mathfrak{S}_m) = \mathbb{C}\mathfrak{S}_d \otimes \mathbb{C}\mathfrak{S}_m$,

$$(V^{\otimes d} \otimes_{\mathbb{C}} V^{\otimes m}) \otimes_{\mathbb{C}\mathfrak{S}_d \otimes \mathbb{C}\mathfrak{S}_m} (V_\lambda \otimes V_\mu) \cong \bigoplus\nolimits_\nu N_{\lambda\mu\nu} \mathbb{S}_\nu V.$$

(This also uses the general fact: if $A \to B$ is a ring homomorphism, N a left A-module, and M a right B-module, then $M \otimes_B (B \otimes_A N) \cong M \otimes_A N$.) The left-hand side of the displayed equation is

$$(V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda) \otimes_{\mathbb{C}} (V^{\otimes m} \otimes_{\mathbb{C}\mathfrak{S}_m} V_\mu) \cong \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V),$$

which concludes the proof.

(6.11) (a) The key observation is that

$$(V \oplus W)^{\otimes d} = \bigoplus (V^{\otimes a} \otimes W^{\otimes b}) \otimes_{\mathbb{C}(\mathfrak{S}_a \times \mathfrak{S}_b)} \mathbb{C}(\mathfrak{S}_d),$$

the sum over all a, b with $a + b = d$. Tensoring this on the right with the $\mathbb{C}(\mathfrak{S}_d)$-module V_ν one gets

$$(V \oplus W)^{\otimes d} = \bigoplus (V^{\otimes a} \otimes W^{\otimes b}) \otimes_{\mathbb{C}(\mathfrak{S}_a \times \mathfrak{S}_b)} \mathrm{Res}_{a,b} V_\nu,$$

where $\mathrm{Res}_{a,b}$ denotes the restriction to $\mathfrak{S}_a \times \mathfrak{S}_b$. Then use Exercise 4.43 to decompose this restriction.

(b) By Frobenius reciprocity, the representation induced by V_ν via the diagonal embedding of $\mathfrak{S}_d$ in $\mathfrak{S}_d \times \mathfrak{S}_d$ is $\bigoplus C_{\lambda\mu\nu} V_\lambda \boxtimes V_\mu$. With $A = \mathbb{C}\mathfrak{S}_d$, this says

$$(A \otimes A) \otimes_A Ac_\nu = \bigoplus C_{\lambda\mu\nu}(Ac_\lambda \otimes Ac_\mu).$$

Tensor this with the right $(A \otimes A)$-module $(V \otimes W)^{\otimes d} = V^{\otimes d} \otimes W^{\otimes d}$. The special case follow from Exercise 4.51(b).

(6.13) Use Exercise A.32(iv), or write the left side as $V^{\otimes d} \otimes A \cdot b_\lambda$ and use Exercise 4.48.

(6.14) These come from the realizations of the representation $V_\lambda = Ac_\lambda$ as the image of the maps $Ab_\lambda \to Aa_\lambda$ given by right multiplication by a_λ, and similarly $Aa_\lambda \to Ab_\lambda$ by right multiplication by b_λ.

(6.15) It is clear that if one allows T to vary over all tableaux with strictly increasing columns but no conditions on the rows, then the corresponding v_T span the first space $\bigotimes_i (\wedge^{\mu_i} V)$; to show that the v_T for T semistandard span the image the key point is to show how to interchange elements in successive rows. Once it is checked that the elements span, the independence can be deduced from the fact that the number of semistandard tableaux is the same as the dimension. For a direct proof of both spanning and independence, see [A-B-W]—but note that their partitions are all the conjugates of ours. See also Proposition 15.55.

(6.16) Use Exercise 6.14 to realize each $\mathbb{S}_\lambda V$ which occurs as the image in $V^{\otimes d} \otimes V^{\otimes d}$ of a symmetrizing map, and check whether this image is invariant or anti-invariant by the map which permutes the two factors.

(6.17) (a) Identifying the dm elements on which $\mathfrak{S}_{dm}$ acts with the set of pairs $\{(i, j) | 1 \leq i \leq d, 1 \leq j \leq m\}$ determines embeddings of the groups $\mathfrak{S}_d \times \cdots \times \mathfrak{S}_d$ (m factors) and $\mathfrak{S}_m$ in $\mathfrak{S}_{dm}$. Let

$$c' = c_\lambda \otimes \cdots \otimes c_\lambda \in \mathbb{C}\mathfrak{S}_d \otimes \cdots \otimes \mathbb{C}\mathfrak{S}_d = \mathbb{C}(\mathfrak{S}_d \times \cdots \times \mathfrak{S}_d) \subset \mathbb{C}\mathfrak{S}_{dm},$$

$$c'' = c_\mu \in \mathbb{C}\mathfrak{S}_m \subset \mathbb{C}\mathfrak{S}_{dm}.$$

Then $c = c' \cdot c''$ is the required element of $\mathbb{C}\mathfrak{S}_{dm}$. For a combinatorial description of plethysm see [Mac, §I.8].

(b) The answers are

$$\mathrm{Sym}^2(\mathbb{S}_{(2,2)}V) = \mathbb{S}_{(4,4)}V \oplus \mathbb{S}_{(4,2,2)}V \oplus \mathbb{S}_{(3,3,1,1)}V \oplus \mathbb{S}_{(2,2,2,2)}V;$$

$$\Lambda^2(\mathbb{S}_{(2,2)}V) = \mathbb{S}_{(4,3,2)}V \oplus \mathbb{S}_{(3,2,2,1)}V.$$

Reference: [Lit2, p. 278].

(6.18) Their characters are the same. In fact, if x and y are eigenvalues of an endomorphism of V, the trace on the left-hand side is $\sum f(k)x^k y^{pq-k}$, where $f(k)$ is the number of partitions of k into at most p integers each at most q. This number is symmetric in p and q, by conjugating partitions.

(6.19) The facts about skew Schur polynomials are straightforward generalizations of corresponding facts for regular Schur polynomials given in Appendix A; proofs of (i)–(iv) can be found in [Mac]. To see that the two descriptions of $V_{\lambda/\mu}$ agree see the hint for Exercise 4.4(a). Skew Schur functors are discussed in [A-B-W], where the construction of a basis is given; from this the character formula (viii) follows. Then (iv) implies (v) and (ix).

(6.20) References, with proofs of similar statements in arbitrary characteristic (where the results, however, are weaker), are [Pe1] and [Jam].

(6.21) References: [A-B-W] and [P-W].

(6.29) A reference for the general theory of semisimple algebras and its applications to group theory is [C-R, §26].

Lecture 7

(7.1) One way to show that a symplectic transformation has determinant 1, cf. [Di1], is to show that the group $\mathrm{Sp}_{2n}\mathbb{C}$ is generated by those which fix a hyperplane, i.e., transformations of the form $v \mapsto v + \lambda Q(v, u)u$ for some vector u and scalar λ. Another, cf. Exercise F.12, is to write the determinant as a polynomial expression in terms of the form Q.

(7.2) Consider the action on the quadric $Q(v, v) = 1$.

(7.11) For any y, the image of the map $x \mapsto xyx^{-1}y^{-1}$ is discrete only if y is central.

(7.13) $\mathrm{PGL}_n\mathbb{C}$ acts by conjugation on $n \times n$ matrices.

Lecture 8

(8.10) (b) $\operatorname{ad}[X, Y](Z) = [[X, Y], Z]$, and $[\operatorname{ad} X, \operatorname{ad} Y](Z) = (\operatorname{ad} X \circ \operatorname{ad} Y - \operatorname{ad} Y \circ \operatorname{ad} X)(Z) = [X, [Y, Z]] - [Y, [X, Z]]$.

(8.16) The kernel of Ad is the center $Z(G)$, cf. Exercise 7.11.

(8.17) Use statement (ii), noting that W is G-invariant if it is $\tilde{G}$-invariant, $\tilde{G}$ the universal covering of G.

(8.24) With A, B, C, D $n \times n$ matrices,

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^tAC = {}^tCA, {}^tBD = {}^tDB, {}^tAD - {}^tCB = \mathrm{I}\right\}.$$

$$\mathfrak{sp}_{2n}\mathbb{R} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| {}^tB = B, {}^tC = C, {}^tA = -D\right\}.$$

(8.28) The automorphisms of $G = \tilde{G}/C$ are the automorphisms of $\tilde{G}$ which preserve C.

(8.29) The point is that the commutator of two vector fields is again a vector field, which can be checked in local coordinates.

(8.35) Both signs are plus.

(8.38) Reference: [Ho1].

(8.42) For $h \in H$, $H_0 \cdot h$ gives a coordinate neighborhood of h. For another approach to Proposition 8.41, with more details, see [Hel, §II.2].

(8.43) For an example, take any simply connected group which contains a torus of dimension greater than one, say SU(3), and take an irrational line in the torus.

Lecture 9

(9.7) If H is an abelian subgroup of G, and the claim holds for G/H, show that it holds for G. Or, if G is realized as a group of nilpotent matrices, apply Campbell–Hausdorff.

(9.10) If each $\operatorname{ad}(X)$ is nilpotent, the theorem gives a flag $\mathfrak{g} = V_0 \supset V_1 \supset \cdots \supset V_k = 0$, with $[\mathfrak{g}, V_i] \subset V_{i+1}$, from which it follows that $\mathscr{D}_i\mathfrak{g} \subset V_i$.

(9.21) If $\mathfrak{g}$ had an abelian ideal $\mathfrak{a}$, semisimplicity of the adjoint representation would mean that there is a surjection $\mathfrak{g} \to \mathfrak{a}$ of Lie algebras. But an abelian Lie algebra has lots of representations that are not semisimple.

(9.24) For the last statement, note that the adjoint representation is semisimple. Or see Corollary C.11.

(9.25) Reference: [Bour, I] for this (as well as for details for many other statements in Lecture 9).

(9.27) the adjoint representation is semisimple.

Lecture 10

(10.1) Any holomorphic map from E to $\mathbb{C}$ must be constant.

(10.2) An isomorphism $G_n \cong G_m$ would lift to a map $G \to G$; show that this map would have to be an isomorphism.

(10.4) By hypothesis, the Lie algebra $\mathfrak{g}$ of G has an ideal $\mathfrak{h}$ with abelian quotient; use the corresponding exact sequence of groups, with the corresponding long exact homotopy sequence (cf. §23.1), and an induction on the dimension of G.

Lecture 11

(11.11) Verify the combinatorial formula

$$\left(\sum_{i=0}^{a} x^{a-2i}\right)\left(\sum_{j=0}^{b} x^{b-2j}\right) = \sum_{k=0}^{a}\left(\sum_{l=0}^{a+b-2k} x^{a+b-2k-2l}\right).$$

Reference: [B-tD, p. 87]

(11.19) Given two points on C there is a 2-dimensional vector space of quadrics containing C and the chord between the points.

(11.20) Answer: it is the subspace of the space of quadrics spanned by the squares of the osculating planes to the twisted cubic curve.

(11.23) Answer: the cones over the curve, with vertex a varying point in $\mathbb{P}^3$.

(11.25) Look at the chordal variety of the rational normal curve in $\mathbb{P}^4$.

(11.32) The sum for $\alpha \geq k$ corresponds to the quadrics containing the osculating $(k-1)$-planes to the curve.

(11.34) See Exercise 6.18.

(11.35) Reference: [Mur1, §15].

Lecture 13

(13.3) For V standard, $\mathbb{S}_{(a+b,b)}V \cong \Gamma_{a,b}$. See §15.3 for details.

(13.8) If $a, b > 0$, $V \otimes \Gamma_{a,b} = \Gamma_{a+1,b} \oplus \Gamma_{a-1,b+1} \oplus \Gamma_{a,b-1}$, cf. §15.3.

(13.20) Warning: writing out the eigenvalue diagram and performing the algorithm above is probably not the way to do this.

(13.22) The tangent planes to the Veronese surface should span a subrepresentation.

(13.24) See §23.3 for a general description of these closed orbits.

More applications of representation theory to geometry can be found in [Don] and [Gre].

Lecture 14

(14.15) The fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ is proved in Claim 21.19.

(14.33) See the proof of Proposition 14.31.

(14.34) If $\mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{g}_\alpha \neq 0$, then $\mathrm{Rad}(\mathfrak{g}) \supset \mathfrak{s}_\alpha \cong \mathfrak{sl}_2$, which is not solvable. If $\mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{h} \ni H$, and $\alpha(H) \neq 0$, then $\mathfrak{g}_\alpha = [H, \mathfrak{g}_\alpha] \subset \mathrm{Rad}(\mathfrak{g})$. Use the fact that $[\mathfrak{h}, \mathrm{Rad}(\mathfrak{g})] \subset \mathrm{Rad}(\mathfrak{g})$ to conclude that $\mathrm{Rad}(\mathfrak{g}) = \mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{h} + \sum \mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{g}_\alpha = 0$. For a stronger theorem, see [Va, §4.4].

(14.35) If $\mathfrak{b}' \supset \mathfrak{b}$, then $\mathfrak{b}' \supset \mathfrak{h}$, so $\mathfrak{b}'$ is a direct sum of $\mathfrak{h}$ and some root spaces $\mathfrak{g}_\alpha$ for $\alpha \in T$, $T \supsetneqq R^+$. Then T contains some $-\alpha$ together with α, so $\mathfrak{b}' \supset \mathfrak{s}_\alpha \cong \mathfrak{sl}_2$, which is not solvable.

(14.36) For $\mathfrak{sl}_m\mathbb{C}$, $B(X, Y) = 2m\,\mathrm{Tr}(X \circ Y)$. For $\mathfrak{so}_m\mathbb{C}$, the coefficient is $(m-2)$, and for $\mathfrak{sp}_m\mathbb{C}$, the coefficient is $(m+2)$.

Lecture 15

(15.19) See also Exercise 6.20.

(15.20) See Pieri's formulas (6.9), (6.8).

(15.21) Use the dimension formula (15.17).

(15.31) See Exercise 6.20.

(15.32) This is Exercise 6.16 in another notation (and restricted to the special linear group).

(15.33) See Exercise 6.16.

(15.51) Use Weyl's unitary trick with the group $\mathrm{U}(n)$.

(15.52) See Exercise 6.18.

(15.54) Show by induction on r that $r!$ times the difference is an integral linear combination of generators for $I^{\bullet}$. For details see [Tow2].

(15.57) The analogue of (15.53) is valid for these products of minors, and that can be used as in Proposition 15.55 to show that the e_T for semistandard T generate D_λ. The same e_T as in Proposition 15.55 is a highest weight vector. For more on this construction, see [vdW]; we learned it from J. Towber.

For other realizations of the representations of $\mathrm{GL}_n\mathbb{C}$, see [N-S].

Lecture 16

(16.7) With $v = (e_1 \wedge e_2)^2$, calculate as in §13.1; the two vectors $X_{2,1}V_2X_{2,1}V_2v$ and $X_{2,1}X_{2,1}V_2V_2v$ are proportional, and $V_2X_{2,1}X_{2,1}V_2v$ is independent of them.

Lecture 17

(17.18) (i) Note that $\Psi_{\{1,2\}}\colon \wedge^{s-2}V \to \wedge^s V$ is surjective if $s > n$. See Exercise 6.14 for the second statement. (ii) This can be done by direct calculation, as in [Wel, p. 155] for the harder case of the orthogonal group. Or, show that $\mathbb{S}_\lambda(V)$ has a highest weight vector with weight λ, and this cannot occur in any $\Psi_I(V^{(d-2)})$.

(17.22) This follows from the theorem and the corresponding result for the general linear groups. Or see Exercise 6.30.

Lecture 19

(19.3)

$$V_{p,q}(v_I) = \begin{cases} 0 & \text{if } \{p, q\} \cap I = \varnothing \text{ or } \{p, q\} \subset I \\ \pm v_{I\setminus\{q\}\cup\{p\}} & \text{if } p \notin I \text{ and } q \in I \\ \pm v_{I\setminus\{p\}\cup\{q\}} & \text{if } q \notin I \text{ and } p \in I. \end{cases}$$

The first assertion follows readily. If $w = \sum a_I v_I$, with the fewest number of nonzero coefficients, and a_J and a_K are nonzero, choose $q \in J\setminus K$, $p \notin J \cup K$ (possible since $2k < m$); then $V_{p,q}(v_J) \neq 0$, $V_{p,q}(v_K) = 0$, and so $V_{p,q}(w)$ is a nonzero vector with fewer nonzero coefficients.

(19.4) The multiplicity of $L_1 + \cdots + L_a - L_{n-b} - \cdots - L_n$ in $\wedge^k V$ is $\binom{2r}{r}$ if $k - a - b = 2r$. For $\Gamma_{2\alpha}$ or $\Gamma_{2\beta}$ the multiplicity is $\frac{1}{2}\binom{2r}{r}$ if r is positive, by symmetry under replacing any Γ_p by $-\Gamma_p$. For Γ_α the weights are $\frac{1}{2}(\varepsilon_1 L_1 + \cdots + \varepsilon_n L_n)$, with $\varepsilon_i = \pm 1$, and $\prod \varepsilon_i = 1$; the multiplicities are all one since these are conjugate under the Weyl group; similarly for Γ_β but with $\prod \varepsilon_i = -1$.

(19.21) For generalizations, see §23.2.

Lecture 20

(20.17) If f spans $\wedge^n W'$, and u_0 spans U with $Q(u_0, u_0) = 1$, then $f\cdot(1 + (-1)^n u_0)$ is such a generator. See Exercise 20.12.

(20.21) If x is in the center, take an orthogonal basis $\{v_i\}$, write out $x = \sum a_I v_I$ in terms of the basis, and look at the equations $x \cdot v_j = v_j \cdot x$ for all j. Note that $v_I \cdot v_j = (-1)^{|I|} v_j \cdot v_I$ if $j \notin I$, whereas $v_I \cdot v_j = (-1)^{|I|-1} v_j \cdot v_I$ if $j \in I$. Conclude that $a_I = 0$ if $|I|$ is odd and there is some $j \notin I$ or if $|I|$ is even and there is some $j \in I$. A similar argument works if x is odd. Reference [A-B-S, p. 7].

(20.22) If $X = a \wedge b$, $[X, v] = \frac{1}{2}(a\cdot b\cdot v - b\cdot a\cdot v - v\cdot a\cdot b + v\cdot b\cdot a)$, which is

$$\tfrac{1}{2}(2Q(b, v)a - a\cdot v\cdot b - 2Q(a, v)b + b\cdot v\cdot a - 2Q(a, v)b + a\cdot v\cdot b + 2Q(b, v)a - b\cdot v\cdot a)$$

$$= 2Q(b, v)a - 2Q(a, v)b = \varphi_{a\wedge b}(v).$$

(20.23) Reference: [Por], but note that his $C(p, q)$ is our $C(q, p)$. See also [A-B-S].

(20.32) If $Q(v-w, v-w) \neq 0$, then $R_{v-w}(v) = w$. Otherwise, $R_{v+w}(v) = -w$, and $R_w(-w) = w$. For (b) compose a given element of $O(Q)$ with an element constructed by (a) to get one fixed on a line, and write, by induction on the dimension, the restriction to the perpendicular hyperplane as a product of reflections.

(20.33) By Exercise 20.22, $X \cdot v = [X, v]$. See also Exercise 8.24.

(20.36) If v_i are a basis for V with $Q(v_i, v_j) = -\delta_{i,j}$, then $\omega = v_1 \cdot \ldots \cdot v_m$. If $m \equiv 2$ (4), the center is cyclic of order four, while if $m \equiv 0$ (4), it is the Klein four group.

(20.37) Show that $\mathfrak{so}(Q)$ acts by traceless endomorphisms. For example, the trace of H_i on S^+ is the number of $I \subset \{1, \ldots, n\}$ such that $|I|$ is even and $i \in I$, minus the number with $i \notin I$.

(20.38) For the first statement of (a), choose f spanning $\wedge^n W'$ so that, for the chosen generator of $\wedge^n W$, $\tau(f) \cdot e \cdot f = f$. For the second, when m is even, $x(s)f = x \cdot s \cdot f$ by Exercise 20.12, so $\beta(x(s), x(t))f = \tau(x \cdot s \cdot f) \cdot (x \cdot t \cdot f) = \tau(s \cdot f) \cdot \tau(x) \cdot x \cdot (t \cdot f) = \tau(s \cdot f) \cdot (t \cdot f) = \beta(s, t)$. The odd case can be reduced to the even case by imbedding $C(Q)$ into a larger Clifford algebra as in Exercise 20.40.

(20.43) Reference: [Por].

(20.44) For example, the transposition of α_1 and α_4 is achieved by the matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(20.50) Reference: [Ch2, §4.3].

(20.51) Reference: [Ch2, §4.2–4.5], [Jac1].

Other references include [L-M], [Ca1], [B-tD], [Hus], [P-S].

Lecture 21

(21.9) If $\alpha_1, \ldots, \alpha_r$ are the vectors, and we have a nontrivial relation

$$v = \sum_{i \leq k} n_i \alpha_i = \sum_{j > k} n_j \alpha_j,$$

with non-negative coefficients, then $(v, v) = \sum_{i,j} n_i n_j (\alpha_i, \alpha_j) \leq 0$, so $v = 0$. But v lies on the same side of the hyperplane.

(21.15) The first is ruled out by considering

$$u = e_2, \qquad v = (3e_3 + 2e_4 + e_5)/\sqrt{6}, \qquad w = (3e_6 + 2e_7 + e_8)/\sqrt{6},$$

with $1 > (e_1, u)^2 + (e_1, v)^2 + (e_1, w)^2 = 1/4 + 3/8 + 3/8 = 1$. For the second, use

$$u = e_2, \qquad v = (2e_3 + e_4)/\sqrt{3}, \qquad w = (5e_5 + 4e_6 + 3e_7 + 2e_8 + e_9)/\sqrt{15},$$

with $(e_1, u)^2 + (e_1, v)^2 + (e_1, w)^2 = 1/4 + 1/3 + 5/12 = 1$.

(21.16) Using the characterization that $\omega_i(H_{\alpha_j}) = \delta_{i,j}$, one can write the fundamental weights ω_i in terms of the basis L_i. The tables in [Bour, Ch. 6] also express them in terms of the simple roots.

(E6): $\omega_1 = 2\frac{\sqrt{3}}{3}L_6,$

$\omega_2 = \frac{1}{2}(L_1 + L_2 + L_3 + L_4 + L_5) + \frac{\sqrt{3}}{2}L_6,$

$\omega_3 = \frac{1}{2}(-L_1 + L_2 + L_3 + L_4 + L_5) + 5\frac{\sqrt{3}}{6}L_6,$

$\omega_4 = L_3 + L_4 + L_5 + \sqrt{3}L_6,$

$\omega_5 = L_4 + L_5 + 2\frac{\sqrt{3}}{3}L_6,$

$\omega_6 = L_5 + \frac{\sqrt{3}}{3}L_6;$

(E_7): $\omega_1 = \sqrt{2}L_7,$

$\omega_2 = \frac{1}{2}(L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + \sqrt{2}L_7,$

$\omega_3 = \frac{1}{2}(-L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + 3\frac{\sqrt{2}}{2}L_7,$

$\omega_4 = L_3 + L_4 + L_5 + L_6 + 2\sqrt{2}L_7,$

$\omega_5 = L_4 + L_5 + L_6 + 3\frac{\sqrt{2}}{2}L_7,$

$\omega_6 = L_5 + L_6 + \sqrt{2}L_7,$

$\omega_7 = L_6 + \frac{\sqrt{2}}{2}L_7;$

(E_8) $\omega_1 = 2L_8,$

$\omega_2 = \frac{1}{2}(L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + 5L_8),$

$\omega_3 = \frac{1}{2}(-L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + 7L_8),$

$\omega_4 = L_3 + L_4 + L_5 + L_6 + L_7 + 5L_8,$

$\omega_5 = L_4 + L_5 + L_6 + L_7 + 4L_8,$

$\omega_6 = L_5 + L_6 + L_7 + 3L_8,$

$\omega_7 = L_6 + L_7 + 2L_8,$

$\omega_8 = L_7 + L_8;$

(F_4): $\omega_1 = L_1 + L_2,$

$\omega_2 = 2L_1 + L_2 + L_3,$

$\omega_3 = \frac{1}{2}(3L_1 + L_2 + L_3 + L_4),$

$\omega_4 = L_1;$

(G_2) $\omega_1 = \frac{1}{2}(L_1 + \sqrt{3}L_2) = 2\alpha_1 + \alpha_2,$

$\omega_2 = \sqrt{3}L_2 = 3\alpha_1 + 2\alpha_2.$

(21.17) The only cases of the same rank that have the same number of roots are (B_n) and (C_n) for all n, and (B_6), (C_6), and (E_6); (B_n) has n roots shorter than the others, (C_n) n roots longer, and in (E_6) all the roots are the same length.

(21.18) For the matrices see [Bour, Ch. 6] or [Hul, p. 59]. The determinants are:

$n+1$ for (A_n); 2 for (B_n), (C_n) and (E_7); 4 for (D_n); 3 for (E_6); and 1 for (G_2), (F_4), and (E_8).

(21.23) See Lecture 22.

The proof of Lemma 21.20 is from [Jacl, p. 124], where details can be found.

For more on Dynkin diagrams and classification, see [Ch3], [Dem], [Dy-O], [LIE], and [Til].

Lecture 22

(22.5) Use the fact that $B(Y, Z) = 6\operatorname{Tr}(Y \circ Z)$ on $\mathfrak{sl}_3\mathbb{C}$, and the formula $[e_i, e_j^*] = 3 \cdot E_{i,j} - \delta_{i,j} \cdot I$, giving

$$B([e_i, e_j^*], Z) = 6 \cdot \operatorname{Tr}((3 \cdot E_{i,j} - \delta_{i,j} \cdot I) \circ Z) = 18 \cdot \operatorname{Tr}(E_{i,j} \circ Z) = 18 \cdot e_j^*(Z \cdot e_i).$$

(22.13) Hint: use the dihedral group symmetry.

(22.15) Answer: $\mathfrak{sl}_3\mathbb{C} \times \mathfrak{sl}_3\mathbb{C}$.

(22.20) For (b), apply ψ to both sides of (22.17), and evaluate both sides of (22.18) on w. Note that $\psi((v \wedge w) \wedge \varphi) = (v \wedge w)(\varphi \wedge \psi) = ((\varphi \wedge \psi) \wedge v)(w)$.

(22.21) For a triple $J = \{p < q < r\} \subset \{1, \ldots, 9\}$, let $e_J = e_p \wedge e_q \wedge e_r$ and similarly for φ_J. For triples J and K the essential calculation (see Exercise 22.5) is to verify that $e_J * \varphi_K$ is 1/18 times

$$\begin{array}{ll} 0 & \text{if } \# J \cap K \leq 1; \\ \pm E_{m,n} & \text{if } K = \{p, q, n\},\ J = \{p, q, m\},\ m \neq n; \\ E_{p,p} + E_{q,q} + E_{r,r} - \frac{1}{3}I & \text{if } K = J = \{p, q, r\}; \end{array}$$

the sign in front of $E_{m,n}$ is the product of the signs of the permutations that put the two sets in order. Verify that $(v \wedge w) \wedge \varphi = 18((w * \varphi) \cdot v - (v * \varphi) \cdot w)$. For Freudenthal's construction, see [Fr2], [H-S].

(22.24) For $\mathfrak{sl}_{n+1}\mathbb{C}$, such an involution takes $E_{i,j}$ to $(-1)^{j-i+1}E_{n+2-j,n+2-i}$; the fixed algebra is $\{X : {}^tXM = -MX\}$, where $M = (m_{ij})$, with $m_{ij} = 0$ if $i + j \neq n + 2$, and otherwise $m_{ij} = (-1)^i$. This M is symmetric if n is even, skew if n is odd, so the fixed subalgebra for (A_{2m}) is the Lie algebra $\mathfrak{so}_{2m+1}\mathbb{C}$ of (B_m), and that for (A_{2m-1}) is the Lie algebra $\mathfrak{sp}_{2m}\mathbb{C}$ of (C_m). For (D_n), the fixed algebra is $\mathfrak{so}_{2n-1}\mathbb{C}$, corresponding to (B_{n-1}), while for the rotation of (D_4), the fixed algebra is $\mathfrak{g}_2$. For a description of possible automorphisms of simple Lie algebras, see [Jacl, §IX].

(22.25) Answer: For $\mathfrak{sl}_{n+1}\mathbb{C}$, $X \mapsto -X^t$. For $\mathfrak{so}_{2n}\mathbb{C}$, $n \geq 5$, $X \mapsto PXP^{-1}$, where P is the automorphism of $\mathbb{C}^{2n}$ that interchanges e_n and e_{2n} and preserves the other basic vectors. For the other automorphisms of $\mathfrak{so}_8\mathbb{C}$, see Exercise 20.44.

(22.27) References: [Her], [Jac3, p. 777], [Pos], [Hul, §19.3].

(22.38) Reference [Ch2, §4.5], [Jac4, p. 131], [Jacl], [Lo, p. 104].

Lecture 23

(23.3) The map takes $z = x + iy$ to (u, v) with $u = x/\|x\|$, $v = y$.

(23.10) Since $\rho(\exp(\sum a_j H_j)) = (e^{a_1}, \ldots, e^{a_n}, e^{-a_1}, \ldots)$, to be in the kernel we must have $a_j = 2\pi i \cdot n_j$, and then $\exp(\sum a_j H_j) = (-1)^{\sum n_j}$.

(23.11) Note that the surjectivity of the fundamental groups is equivalent to the connectedness of $\pi^{-1}(H)$ when $\pi\colon \tilde{G} \to G$ is the universal covering, which is equivalent to the Cartan subgroup of $\tilde{G}$ containing the center of $\tilde{G}$.

(23.17) Note that $\Gamma(G) = \pi_1(H)$ surjects onto $\pi_1(G)$, and there is an exact sequence

$$0 \to \pi_1(G) \to \mathrm{Center}(\tilde{G}) \to \mathrm{Center}(G) \to 0.$$

(23.19) When m is odd, the representations are the representations of $\mathrm{SO}_m\mathbb{C}$, and the products of those by the one-dimensional alternating (determinant) representation. When $m = 2n$, the representations of $\mathrm{SO}_m\mathbb{C}$ with highest weights $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1, \ldots, -\lambda_n)$ are conjugate, so that, if $\lambda_n \neq 0$, they correspond to one irreducible representation of $\mathrm{O}_{2n}\mathbb{C}$, whose underlying space can be identified with $\Gamma_{(\lambda_1, \ldots, \lambda_n)} \oplus \Gamma_{(\lambda_1, \ldots, -\lambda_n)}$. If $\lambda_n = 0$, then Γ_λ is an irreducible representation of $\mathrm{O}_m\mathbb{C}$. In either case, the representations correspond to partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. See §19.5 for an argument.

(23.31) See Exercises 19.6, 19.7, and 19.16.

(23.36) For (b), consider $(D^+)^2 \cdot (D^-)^2 = (D^+ \cdot D^-)^2$.

(23.37) Reference: [B-tD, VI §7].

(23.38) For $\mathfrak{sl}_{n+1}\mathbb{C}$, $\Gamma_\lambda^* = \Gamma_{(\lambda_1, \lambda_1 - \lambda_n, \ldots, \lambda_1 - \lambda_2)}$; for $\mathfrak{so}_{2n}\mathbb{C}$, n odd, $\Gamma_\lambda^* = \Gamma_{(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, -\lambda_n)}$.

(23.39) Reference: [Bour, VIII §7, Exer. 11].

(23.42) Compute highest weight vectors in the (external) tensor product of two irreducible representations, to verify that it is irreducible with highest weight the sum of the two weights.

(23.43) See Exercise 20.40 and Theorems 17.5 and 19.2.

(23.51) An isotropic $(n-1)$-plane is automatically contained in an isotropic n-plane. These are two-step flag varieties, corresponding to omitting two nodes.

(23.62) For (b), use the fact that $B \cdot n' \cdot B$ is open in G. For (c), if μ is a weight, $f(x^{-1}wy) = \mu(x)\lambda(y)f(w)$ for x and y in B, so with $x \in H$ and $w = n'$,

$$\mu(x)f(w) = f(x^{-1}w) = f(wx) = \lambda(x)f(w).$$

Other references on homogeneous spaces include [B-G-G], [Hel], and [Hi].

Lecture 24

(24.4) (a) is proved in Lemma D.25, and (b) follows. For (c), note that by the definition of ρ as half the sum of the positive roots, $\rho - W(\rho)$ is the sum of those positive β such that $W(\beta)$ is negative.

(24.27) This is Exercise A.62.

(24.46) This follows from formulas (A.61) and (A.65).

(24.51) In the following the fundamental weights are numbered as in the answer to Exercise 21.16:

(F_4): $$\rho = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4;$$
$$\dim(\Gamma_{\omega_i}), (i = 1, 2, 3, 4):\quad 52,\ 1274,\ 273,\ 26.$$

(E_6): $$\rho = L_2 + 2L_3 + 3L_4 + 4L_5 + 4\sqrt{3}L_6$$
$$= 8\alpha_1 + 11\alpha_2 + 15\alpha_3 + 21\alpha_4 + 15\alpha_5 + 8\alpha_6;$$
$$\dim(\Gamma_{\omega_i}), (i = 1, \dots, 6):\quad 27,\ 78,\ 351,\ 2925,\ 351,\ 27.$$

(E_7): $$\rho = L_2 + 2L_3 + 3L_4 + 4L_5 + 5L_6 + 17\sqrt{2}/2L_7$$
$$= \tfrac{1}{2}(34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7);$$
$$\dim(\Gamma_{\omega_i}), (i = 1, \dots, 7):\quad 133,\ 912,\ 8645,\ 365750,\ 27664,\ 1539,\ 56.$$

(E_8): $$\rho = L_2 + 2L_3 + 3L_4 + 4L_5 + 5L_6 + 6L_7 + 23L_8$$
$$= 46\alpha_1 + 68\alpha_2 + 91\alpha_3 + 135\alpha_4 + 110\alpha_5 + 84\alpha_6 + 57\alpha_7 + 29\alpha_8;$$
$$\dim(\Gamma_{\omega_i}), (i = 1, \dots, 8):\quad 3875,\ 147250,\ 6696000,\ 6899079264,\ 146325270,$$
$$2450240,\ 30380,\ 248.$$

(24.52) Using the dimension formula as in Exercise 24.9, it suffices to check which fundamental weights correspond to small representations, and then which sums of these are still small. The results are:

(A) $n \geq 1$; $\dim G = n^2 + 2n$; $\dim \Gamma_{\omega_k} = \binom{n+1}{k}$;

the dominant weights whose representations have dimension at most $\dim G$ are:

$\omega_1, \omega_2, \omega_{n-1}, \omega_n$;

$2\omega_1, 2\omega_n$, of dimension $\binom{n+2}{2}$;

$\omega_1 + \omega_n$, of dimension $n^2 + 2n$;

ω_3 for $n = 5$; ω_3, ω_4 for $n = 6$; ω_3, ω_5 for $n = 7$.

(B_n) $n \geq 2$; $\dim G = 2n^2 + n$; $\dim \Gamma_{\omega_k} = \binom{2n+1}{k}$ for $k < n$, and $\dim \Gamma_{\omega_n} = 2^n$, giving:

ω_1, ω_2;

ω_n for $n = 3, 4, 5, 6$;

$2\omega_2$, of dimension 10, for $n = 2$.

(C_n) $n \geq 3$; $\dim G = 2n^2 + n$; $\dim \Gamma_{\omega_k} = \binom{2n}{k} - \binom{2n}{k-2}$, giving:

ω_1, ω_2;

$2\omega_1$, of dimension $2n^2 + n$;

ω_3 for $n = 3$.

(D_n) $n \geq 4$; $\dim G = 2n^2 - n$; $\dim \Gamma_{\omega_k} = \binom{2n}{k}$ for $k \leq n-2$, and

$\dim \Gamma_{\omega_{n-1}} = \dim \Gamma_{\omega_n} = 2^{n-1}$, giving:

ω_1, ω_2;

ω_{n-1}, ω_n for $n = 4, 5, 6, 7$.

(E_6) $\dim G = 78$; $\omega_1, \omega_2, \omega_6$.

(E_7) $\dim G = 133$; ω_1, ω_7.

(E_8) $\dim G = 248$; ω_8.

(F_4) $\dim G = 52$; ω_1, ω_4.

(G_2) $\dim G = 14$; ω_1, ω_2.

For irreducible representations of general Lie groups with this property, see [S-K].

Other references with character formulas include [ES-K], [Ki1], [Ki2], [Kl], [Mur2], and [Ra].

Lecture 25

(25.2) Changing μ by an element of the Weyl group, one can assume μ is also dominant and $\lambda - \mu$ is a sum of positive roots. Then $\|\lambda\| > \|\mu\|$, and $c(\mu) = (\lambda, \lambda) - (\mu, \mu) + (\lambda - \mu, 2\rho) > 0$.

(25.4) A direct calculation gives

$$C(X \cdot v) - X \cdot C(v) = \sum U_i \cdot [U_i', X] \cdot v + \sum [U_i, X] \cdot U_i' \cdot v.$$

To see that this is zero, write $[U_i, X] = \sum \alpha_{ij} U_j$; then by (14.23), $\alpha_{ij} = ([U_i, X], U_j') = -([U_j', X], U_i)$, so $[U_j', X] = -\sum \alpha_{ij} U_i'$. The terms in the above sums then cancel in pairs.

(25.6) By (14.25), $(H_\alpha, H_\alpha) = \alpha(H_\alpha)(X_\alpha, Y_\alpha) = 2(X_\alpha, Y_\alpha)$. Use Exercise 14.28.

(25.12) The symmetry gives

$$(\beta - i\alpha, \alpha) n_{\beta - i\alpha} + (\beta - (m - i)\alpha, \alpha) n_{\beta - (m-i)\alpha} = (2\beta - m\alpha, \alpha) n_{\beta - i\alpha} = 0$$

since $2(\beta, \alpha) = m(\alpha, \alpha)$, so the terms cancel in pairs.

(25.22) We have

$$\sum_{W, \mu} (-1)^W P(\mu + W(\rho) - \rho) e(-\mu) = \sum_W (-1)^W (e(W(\rho) - \rho)) / \prod_{\alpha \in R^+} (1 - e(-\alpha)),$$

and the right-hand side is 1 by Lemma 24.3.

(25.23) We have

$$\begin{aligned}\sum_W (-1)^W n_{\mu + \rho - W(\rho)} &= \sum_{W, W'} (-1)^{WW'} P(W'(\lambda + \rho) - ((\mu + \rho - W(\rho)) + \rho)) \\ &= \sum_{W'} (-1)^{W'} \sum_W (-1)^W P((W'(\lambda + \rho) - \mu - \rho) + W(\rho) - \rho),\end{aligned}$$

and the inner sum is zero unless $W'(\lambda + \rho) = \mu + \rho$. Note that if μ is a root of Γ_λ, this happens only if $\mu = \lambda$ by Exercise 25.2.

(25.24) The minuscule weights are:

(A_n):	$\omega_1, \ldots, \omega_n$,
(B_n):	ω_1,
(C_n):	ω_n,
(D_n):	$\omega_1, \omega_{n-1}, \omega_n$,
(E_6):	ω_1, ω_6,
(E_7):	ω_7.

Reference: [Bour, VIII, §7.3].

(25.28) One easy way is to use the isomorphism $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$.

(25.30) $N_{\lambda\mu\gamma}$ is zero by definition when γ is not in the closed positive Weyl chamber $\mathscr{W}$, and $W(\nu + \rho) - \rho$ is not in $\mathscr{W}$ if $W \neq 1$. Reference: [Hu1].

(25.40) The weight space of the restriction of Γ_λ corresponding to $\bar{\mu}$ is the direct sum of the weight spaces of Γ_λ corresponding to those μ which restrict to $\bar{\mu}$.

(25.41) Use the preceding exercise and Exercise 25.23.

(25.43) Using the action of a Lie algebra on a tensor product, the action of C on $v_1 \cdot \ldots \cdot v_m$ is a sum over terms where U_i and U_i' act on different elements or the same element. Grouping the terms accordingly leads to the displayed formula. See [L-T, I, pp. 19–20].

Lecture 26

(26.2) In terms of the basis L_1, L_2 of $\mathfrak{h}^*$ dual to $\{H_1, H_2\}$, eigenvalues are $\pm iL_2$ and $\pm 3L_1 \pm iL_2$.

(26.9) Reference: [Hel, §III.7].

(26.10) Constructing $\mathfrak{h} = \mathfrak{g}_0(H)$ as in Appendix D, take H so that $\sigma(H) = H$.

(26.12) See Exercise 23.6.

(26.13) Reference: [Hel, §X.6.4].

(26.21) If a conjugate linear endomorphism $\varphi\colon W \to W$ did not map Γ_λ to itself, there would be another factor U of W and an isomorphism of Γ_λ with U^*; the highest weight of $(\Gamma_\lambda)^*$ cannot be lower than λ.

(26.22) See Exercise 3.43 and Exercise 26.21.

(26.28) References: [A-B-S], [Hus], [Por]. See also Exercise 20.38.

(26.30) Use the identity $\psi^2[V] = [V \otimes V] - 2[\wedge^2 V]$.

Other references on real forms are [Gi1], [B-tD], [Va].

Appendix A

(A.29) (b) Use $P^{(i)} = \sum_\nu \langle H_\nu, P^{(i)} \rangle M_\nu$.

(A.30) Some of these formulas also follow from Weyl's character formula.

(A.31) For part (a), when $a_1 \geq a_2 \geq \cdots \geq a_k$, this is (A.19). The proof of (A.9) shows that for any $a = (a_1, \ldots, a_k)$,

$$H_{a_1} \cdot H_{a_2} \cdot \ldots \cdot H_{a_k} = \sum K_{\mu a} S_\mu,$$

which shows that the $K_{\mu a}$ are unchanged when the a_i's are reordered. For a purely combinatorial proof see [Sta, §10].

(A.32) For (i) compare the generating functions $E(t) = \sum E_i t^i = \prod(1 + x_i t)$ and $H(t) = \sum H_i t^i = 1/E(-t)$; (ii) follows from (A.5) and (A.6). For (iii), note that $P(t) = \sum P_j t^j = \sum x_i t/(1 - x_i t) = tH'(t)/H(t)$. Exponentiate this to get (vi). For details and more on this involution, see [Mac] or [Sta], where it is used to derive basic identities among symmetric polynomials.

(A.39) References: [Mac], [Sta], [Fu, §A.9.4].

(A.41) See [Mac, p. 33] or [Fu, p. 420].

(A.48) Since $\vartheta(E_i') = H_i'$ and $\vartheta(E_i'') = H_i''$,

$$\vartheta(S_{\langle\lambda\rangle}) = \vartheta(|H''_{\lambda_i - i + j} - H''_{\lambda_i - i - j}|) = |E''_{\mu_i - i + j} - E''_{\mu_i - i - j}| = S_{[\mu]}.$$

(A.67) Answer: $\frac{1}{2}\zeta_1 \cdot \ldots \cdot \zeta_n$ times the determinant of the matrix whose ith row is

$$(J_{\lambda_i - i} \quad J_{\lambda_i - i + 1} + J_{\lambda_i - i - 1} \quad \ldots \quad J_{\lambda_i - i + n - 1} + J_{\lambda_i - i - n + 1}).$$

More on symmetric polynomials can be found in [Mac], [Sta], [L-S], and references listed in these sources. Some of the identities in §A.3 are new, although results along these lines can be found in [We1], [Lit1], [Lit2] and [Ko-Te]; other identities involving the determinants discussed in §A.3 can be found in [Mac, §I.5]. Discussions of Schur functions and representation theory can be found in [Di2] and [Lit2].

Appendix C

(C.1) Take a basis in which X has Jordan canonical form, and compute using the corresponding basis E_{ij} for $\mathfrak{gl}(V)$.

(C.12) If $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, and $\mathfrak{h}$ is a simple ideal, $\mathfrak{h} = [\mathfrak{g}, \mathfrak{h}] = \bigoplus [\mathfrak{g}_i, \mathfrak{h}]$, so $\mathfrak{h}$ is contained in some $\mathfrak{g}_i$.

(C.13) Since for $\delta \in \mathrm{Der}(\mathfrak{g})$ and $X \in \mathfrak{g}$, $\mathrm{ad}(\delta(X)) = [\delta, \mathrm{ad}(X)]$, $\mathrm{ad}(\mathfrak{g})$ is an ideal in the Lie algebra $\mathrm{Der}(\mathfrak{g})$. Therefore, $[\mathrm{ad}(\mathfrak{g})^\perp, \mathrm{ad}(\mathfrak{g})] = 0$; in particular, if $\delta \in \mathrm{ad}(\mathfrak{g})^\perp$ and $X \in \mathfrak{g}$, then $\mathrm{ad}(\delta(X)) = [\delta, \mathrm{ad}(X)] = 0$. So $\mathrm{ad}(\mathfrak{g})^\perp = 0$ and $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

Appendix D

(D.8) To show $\mathrm{ad}(X)$ is nilpotent on $\mathfrak{g}_0(H)$ for X in $\mathfrak{g}_0(H)$, consider the complex line from H to X: set $H(z) = (1 - z)H + zX$. Then $\mathrm{ad}(H(z))$ preserves each eigenspace $\mathfrak{g}_\lambda(H)$. By continuity, for z sufficiently near 0, $\mathrm{ad}(H(z))$ is a nonsingular transformation of $\mathfrak{g}_\lambda(H)$ for $\lambda \neq 0$, which implies that $\mathfrak{g}_0(H(z))$ is contained in $\mathfrak{g}_0(H)$, and by the regularity of H, $\mathfrak{g}_0(H(z)) = \mathfrak{g}_0(H)$ for small z.

This means that there is an integer k so that $\mathrm{ad}(H(z))^k(Y) = 0$ for all $Y \in \mathfrak{g}_0(H)$ and all small z. But $\mathrm{ad}(H(z))^k(Y)$ is a polynomial function of z, so it must vanish identically. Hence, setting $z = 1$, $\mathrm{ad}(X)^k$ vanishes on $\mathfrak{g}_0(H)$, as asserted.

(D.24) See [Bour, VII, §3] for details.

(D.33) References: [Se3, §V.11], [Hu1, §12.2].

Appendix E

Proofs of both of these theorems can be found, together with many other related results, in [Bour I]. See also [Se3], [Pos], [Va], [Jac1].

Appendix F

(F.12) Check that the right-hand side is multilinear, alternating, and takes the value 1 on a standard basis. Or see [We1, §VI.1].

(F.16) $\mathrm{SO}_n\mathbb{C}$-invariants can be written in the form $A + \sum A_i B_i$ where A and the A_i are polynomials in the $Q(x^{(i)}, x^{(j)})$ and the B_i are brackets. Such is taken to $A + \det(g)\sum A_i B_i$ by g in $\mathrm{O}_n\mathbb{C}$. For an odd (resp. even) invariant the first (resp. the second) term must vanish.

(F.20) Reference: [We1, II.6], or [Br, p. 866].

There are many elementary references for invariant theory, such as [D-C], [Pr], [Sp1], and [Ho2]; the last contains a proof of Capelli's formula. There are also many modern approaches to invariant theory, some which can be found in [DC-P], [Sch] and [Vu] and references described therein; some of these also contain some invariant theory for exceptional groups. For a more conceptual and representation-theoretic approach to Capelli's identity, see [Ho3]. Weyl's book [We1] remains an excellent reference for invariant theory of the orthogonal and symplectic groups together with the related [Br], [We2].

Bibliography

[A-B] M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes: II. Applications, *Ann. Math.* **9** (1968), 451–491.

[A-B-S] M. F. Atiyah, R. Bott, and A. Shapiro, Clifford modules, *Topology* **3**, Supp. 1 (1964), 3–38.

[A-B-W] K. Akin, D. A. Buchsbaum, and J. Weyman, Schur functors and Schur complexes, *Adv. Math.* **44** (1982), 207–278.

[Ad] J. F. Adams, *Lectures on Lie Groups*, W. A. Benjamin, Inc., New York, 1969.

[Ahl] L. V. Ahlfors, *Complex Analysis, Second Edition*, McGraw-Hill, New York 1966.

[A-J-K] Y. J. Abramsky, H. A. Jahn, and R. C. King, Frobenius symbols and the groups S_s, GL(n), O(n), and Sp(n), *Can. J. Math.* **25** (1973), 941–959.

[And] G. E. Andrews, The Theory of Partitions, *Encyclopedia of Mathematics and Its Applications*, vol. 2, Addison-Wesley, Reading, MA, 1976.

[Ar] S. K. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, *J. Math. Osaka City Univ.* **13** (1963), 1–34.

[A-T] M. Atiyah and D. O. Tall, Group representations, λ-rings, and the J-homomorphism, *Topology* **8** (1969), 253–297.

[B-G-G] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, Schubert cells and cohomology of the spaces G/P, *Russ. Math. Surv.* **28** (1973), 1–26.

[Boe] H. Boerner, *Representations of Groups*, Elsevier North-Holland, Amsterdam, 1970.

[Bor1] A. Borel, *Linear Algebraic Groups*, W. A. Benjamin, 1969 and (GTM 126), Springer-Verlag, New York, 1991.

[Bor2] A. Borel, Topology of Lie groups and characteristic classes, *Bull. Amer. Math. Soc.* **61** (1955), 397–432.

[Bot] R. Bott, On induced representations, in *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math Vol. 48, American Mathematical Society, Providence, RI 1988, pp. 1–13.

[Bour] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 1–3*, Springer-Verlag, New York, 1989; *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Masson, Paris, 1981; *Groupes et algèbres de Lie, Chapitres 7 et 8*, Diffusion C.C.L.S., Paris, 1975; *Algebra 1*, Chapter 3, Springer-Verlag, New York, 1989.

[Br] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. Math.* **38** (1937), 857–872.

[B-tD] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, New York, 1985.

[Bu] J. Burroughs, Operations in Grothendieck rings and the symmetric group, *Can. J. Math.* **26** (1974), 543–550.

[Ca1] E. Cartan, *The Theory of Spinors*, Hermann, Paris, 1966, and Dover Publications, 1981.

[Ca2] E. Cartan, Le principe de dualité et la théorie des groupes simples et semi-simples, Bull. Sci. Math. **49** (1925), 361–374.

[Cart] P. Cartier, On H. Weyl's character formula, *Bull. Amer. Math. Soc.* **67** (1961), 228–230.

[Ch1] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, Princeton, NJ, 1946.

[Ch2] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[Ch3] Séminaire C. Chevalley 1956–1958, *Classification des Groupes de Lie Algébriques, Secrétariat mathématique*, Paris, 1958.

[Ch-S] C. Chevalley and R. D. Schafer, The exceptional simple Lie algebras F_4 and E_6, *Proc. Natl. Acad. Sci. USA.* **36** (1950), 137–141.

[Co] A. J. Coleman, *Induced Representations with Applications to S_n and* $\mathrm{GL}(n)$, *Queens Papers Pure Appl. Math.* **4** (1966).

[C-R] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York, 1962.

[D-C] J. Dieudonné and J. Carrell, *Invariant Theory, Old and New*, Academic Press, New York, 1971.

[DC-P] C. De Concini and C. Procesi, A characteristic free approach to invariant theory, *Adv. Math.* **21** (1976), 330–354.

[Dem] M. Demazure, *A, B, C, D, E, F*, etc., *Springer Lecture Notes* 777, Springer-Verlag, Heidelberg, 1980, pp. 221–227.

[D'H] E. D'Hoker, Decompositions of representations into basis representations for the classical groups, J. Math. Physics **25** (1984), 1–12.

[Di1] J. Dieudonné, *Sur les Groupes Classiques*, Hermann, Paris, 1967.

[Di2] J. Dieudonné, Schur functions and group representations, in *Young tableaux and Schur functors in algebra and geometry*, *Astérisque* **87–88** (1981), 7–19.

[Dia] P. Diaconis, *Group Representations in Probability and Statistics*, Institute of Mathematical Statistics, Hayward, CA, 1988.

[Don] R. Donagi, On the geometry of Grassmannians, *Duke Math. J.* **44** (1977), 795–837.

[Dor] L. Dornhoff, *Group Representation Theory*, Parts *A* and *B*, Marcel Dekker, New York, 1971, 1972.

[Dr] D. Drucker, Exceptional Lie algebras and the structure of hermitian symmetric spaces, *Mem. Amer. Math. Soc.* **208** (1978).

[Dy-O] E. B. Dynkin and A. L. Oniščik, Compact global Lie groups, *Amer. Math. Soc. Transl., Series 2* **21** (1962), 119–192.

[ES-K] N. El Samra and R. C. King, Reduced determinantal forms for characters of the classical Lie groups, *J. Phys. A: Math. Gen.* **12** (1979), 2305–2315.

[Foa] D. Foata (ed.), *Combinatoire et Représentation du Groupe Symétrique, Strasbourg 1976*, Springer Lecture Notes 579, Springer-Verlag, Heidelberg, 1977.

[Fr1] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Mathematisch Instituut der Rijksuniversiteit te Utrecht, Utrecht, 1951, 1960.

[Fr2] H. Freudenthal, Lie groups in the foundations of geometry, *Adv. Math.* **1** (1964), 145–190.

[Fr-dV] H. Freudenthal and H. de Vries, *Linear Lie Groups*, Academic Press, New York, 1969.

[Fro1] F. G. Frobenius, Über die Charaktere der symmetrischen Gruppe, *Sitz. König. Preuss. Akad. Wissen.* (1900), 516–534; Gesammelte Abhandlungen III, Springer-Verlag, Heidelberg, 1968, pp. 148–166.

[Fro2] F. G. Frobenius, Über die Charaktere der alternirenden Gruppe, *Sitz. König. Preuss. Akad. Wissen.* (1901), 303–315; Gesammelte Abhandlungen III, Springer-Verlag, Heidelberg, 1968, pp. 167–179.

[Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, New York, 1984.

[G-H] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Gil] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Wiley, New York, 1974.

[G-N-W] C. Greene, A. Nijenhuis, and H. S. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. Math.* **31** (1979), 104–109.

[Gr] J. A. Green, The characters of the finite general linear group, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[Gre] M. L. Green, Koszul cohomology and the geometry of projective varieties, I, II, *J. Diff. Geom.* **19** (1984), 125–171; **20** (1984), 279–289.

[Grie] R. L. Griess, Automorphisms of extra special groups and nonvanishing degree 2 cohomology, *Pacific J. Math.* **48** (1973), 403–422.

[Ha] J. Harris, *Algebraic Geometry*, Springer-Verlag, New York, to appear.

[Ham] M. Hamermesh, *Group Theory and its Application to Physical Problems*, Addison-Wesley, Reading, MA, 1962 and Dover, 1989.

[Har] G. H. Hardy, *Ramanujan*, Cambridge University Press, Cambridge, MA, 1940.

[Hel] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[Her] R. Hermann, *Spinors, Clifford and Cayley Algebras*, Interdisciplinary Mathematics Volume VII, 1974.

[Hi] H. Hiller, *Geometry of Coxeter Groups*, Pitman, London, 1982.

[Ho1] R. Howe, Very basic Lie theory, *Amer. Math. Monthly* **90** (1983), 600–623; **91** (1984), 247.

[Ho2] R. Howe, The classical groups and invariants of binary forms, *Proc. Symp. Pure Math.* **48** (1988), 133–166.

[Ho3] R. Howe, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), 539–569.

[H-P] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, vols. 1 and 2, Cambridge University Press, Cambridge, MA, 1947, 1952; 1968.

[H-S] M. Hausner and J. T. Schwarz, *Lie groups*; *Lie algebras*, Gordon and Breach, New York, 1968.

[Hu1] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972, 1980.

[Hu2] J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1975, 1981.

[Hu3] J. E. Humphreys, Representations of SL(2, p), *Amer. Math. Monthly* **82** (1975), 21–39.

[Hur] A. Hurwitz, Über die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **55** (1902), 53–66.

[Hus] D. Husemoller, *Fibre Bundles*, second edition, Springer-Verlag, New York, 1975.

[In] R. E. Ingram, Some characters of the symmetric group, *Proc. Amer. Math. Soc.* **1** (1950), 358–369.

[Iv] B. Iversen, The geometry of algebraic groups, *Adv. Math.* **20** (1976), 57–85.

[Jac1] N. Jacobson, *Lie Algebras*, Wiley, New York 1962, and Dover, 1979.

[Jac2] N. Jacobson, *Exceptional Lie Algebras*, Marcel Dekker, New York 1971.

[Jac3] N. Jacobson, Cayley numbers and simple Lie algebras of type G, *Duke Math. J.* **5** (1939), 775–783.

[Jac4] N. Jacobson, Triality and Lie algebras of type D_4, *Rend. Circ. Mat. Palermo* (2) **13** (1964), 129–153.

[Ja-Ke] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and Its Applications, vol. 16, Addison-Wesley, Reading, MA, 1981.

[Jam] G. D. James, *The Representation Theory of the Symmetric Groups*, Springer Lecture Notes 682, Springer-Verlag, Heidelberg, 1978.

[J-L] T. Jósefiak and A. Lascoux (eds.), *Young Tableaux and Schur Functors in Algebra and Geometry, Toruń, Poland* 1980, *Astérisque* 87–88, 1981.

[Ke] A. Kerber, *Representations of Symmetric Groups I*, Springer Lecture Notes 240, Springer-Verlag, Heidelberg, 1971.

[Kem] G. Kempf, Tensor products of representations, *Amer. J. Math.* **109** (1987), 401–415.

[Ki1] R. C. King, The dimensions of irreducible tensor representations of the orthogonal and symplectic groups, *Can. J. Math.* **23** (1971), 176–188.

[Ki2] R. C. King, Modification rules and products of irreducible representations of the unitary, orthogonal and symplectic groups, *J. Math. Phys.* **12** (1971), 1588–1598.

[Kir] A. A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, New York, 1976.

[Kl] A. U. Klymyk, Multiplicities of weights of representations and multiplicities of representations of semisimple Lie algebras, *Sov. Math. Dokl.* **8** (1967), 1531–1534.

[K-N] G. Kempf and L. Ness, Tensor products of fundamental representations, *Can. J. Math.* **40** (1988), 633–648.

[Kn] D. Knutson, *λ-Rings and the Representation Theory of the Symmetric Group*, Springer Lecture Notes 308, Springer-Verlag, Heidelberg 1973.

[Kos] B. Kostant, A formula for the multiplicity of a weight, *Trans. Amer. Math. Soc.* **93** (1959), 53–73.

[Ko-Te] K. Koike and I. Terada, Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, and D_n, *J. Algebra* **107** (1987), 466–511.

[Ku] J. P. S. Kung (ed.), *Young Tableaux in Combinatorics, Invariant Theory, and Algebra*, Academic Press, New York, 1982.

[Kum1] S. Kumar, Proof of the Parthasarathy–Ranga Rao–Varadarajan conjecture, *Invent. Math.* **93** (1988), 117–130.

[Kum2] S. Kumar, A refinement of the PRV conjecture, *Invent. Math.* **97** (1989), 305–311.

[Le] W. Ledermann, *Introduction to Group Characters*, Cambridge University Press, Cambridge, MA, 1977.

[Li] P. Littelmann, A Littlewood–Richardson rule for classical groups, *C. R. Acad. Sci. Paris* **306** (1988), 299–303.

[LIE] Séminaire Sophus LIE 1954/1955, *Théorie des Algèbres de Lie, Topologie des groupes de Lie*, École Normale Supérieure, Paris, 1955.

[Lit1] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, second ed., Oxford University Press, Oxford, 1950.

[Lit2] D. E. Littlewood, *A University Algebra*, William Heinemann Ltd, London, 1950; second ed., 1958, and Dover, 1970.

[Lit3] D. E. Littlewood, On invariants under restricted groups, *Philos. Trans. Roy. Soc. A* **239** (1944), 387–417.

[Liu] A. Liulevicius, Arrows, symmetries and representation rings, *J. Pure Appl. Algebra* **19** (1980), 259–273.

[L-M] H. B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, NJ, 1989.

[L-M-S] V. Lakshmibai, C. Musili, and C. S. Seshadri, Geometry of G/P, *Bull. Amer. Math. Soc.* **1** (1979), 432–435.

[Lo] Loos, *Symmetric Spaces*, W. A. Benjamin, New York, 1969.

[L-S] A. Lascoux and M. P. Schützenberger, *Formulaire raisonné de fonctions symmetriques*, U. E. Maths, Paris VII, L.A. 248, 1985.

[L-T] G. Lancaster and J. Towber, Representation-functors and flag-algebras for the classical groups I, II, *J. Algebra* **59** (1979), 16–38; **94** (1985), 265–316.

[L-VdV] R. Lazarsfeld and A. Van de Ven, *Topics in the Geometry of Projective Space*, DMV Seminar Band 4, Birkhäuser, Boston, MA, 1984.

[L-V] R. A. Liebler and M. R. Vitale, Ordering the partition characters of the symmetric group, *J. Algebra* **25** (1973), 487–489.

[Mac] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.

[Mack] G. W. Mackey, *Introduction to The Racah–Wigner Algebra in Quantum Theory*, by L. C. Biedenharn and J. D. Louck, Encyclopedia of Mathematics and Its Applications, vol. 9, Addison-Wesley, Reading, MA, 1981.

[M-S] G. Musili and C. S. Seshadri, *Standard monomial theory*, Springer Lecture Notes **867** (1981), 441–476.

[Mur1] F. D. Murnaghan, *The Theory of Group Representations*, The Johns Hopkins Press, Baltimore, 1938.

[Mur2] F. D. Murnaghan, *The Unitary and Rotation Groups*, Spartan Books, Washington, DC, 1962.

[No] K. Nomizu, *Lie Groups and Differential Geometry*, Mathematics Society of Japan, Tokyo, 1956.

[N-S] M. A. Naimark and A. I. Stern, *Theory of Group Representations*, Springer-Verlag, New York, 1982.

[Pe1] M. H. Peel, Hook representations of symmetric groups, *Glasgow Math. J.* **12** (1971), 136–149.

[Pe2] M. H. Peel, Specht modules and the symmetric groups, *J. Algebra* **36** (1975), 88–97.

[Por] I. R. Porteous, *Topological Geometry*, second edition, Cambridge University Press, Cambridge, MA, 1981.

[Pos] M. Postnikov, *Lie Groups and Lie Algebras*, MIR, Moscow 1986.

[Pr] C. Procesi, *A Primer of Invariant Theory*, Brandeis Lecture Notes 1, 1982.

[P-S] A. Pressley and G. Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.

[P-W] P. Pragacz and J. Weyman, On the construction of resolutions of determinantal ideals: a survey, Springer Lecture Notes 1220 (1986), 73–92.

[Qu] D. Quillen, The mod 2 cohomology rings of extra-special 2-groups and the spinor groups, *Math. Ann.* **194** (1971), 197–212.

[Ra] G. Racah, Lectures on Lie groups, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Gordon and Breach, New York, 1964, 1–36.

[Sc] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.

[Sch] G. W. Schwarz, On classical invariant theory and binary cubics, *Ann. Inst. Fourier* **37** (1987), 191–216.

[Se1] J-P. Serre, *Lie Algebras and Lie Groups*, W. A. Benjamin, New York, 1965.

[Se2] J-P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, New York, 1977.

[Se3] J-P. Serre, *Complex Semi-simple Lie Algebras*, Springer-Verlag, New York, 1987.

[S-K] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.* **65** (1977), 1–155.

[Sp] T. A. Springer, *Invariant Theory*, Springer Lecture Notes 565, Springer-Verlag, Heidelberg, 1977.

[Sta] R. P. Stanley, Theory and Application of Plane Partitions, Parts 1 and 2, *Studies Appl. Math.* **1** (1971), 167–188, 259–279.

[Ste1] R. Steinberg, The representations of GL(3, q), GL(4, q), PGL(3, q), and PGL(4, q), *Can. J. Math.* **3** (1951), 225–235.

[Ste2] R. Steinberg, *Conjugacy classes in Algebraic Groups*, Springer Lecture Notes 366, Springer-Verlag, Heidelberg, 1974.

[S-W] D. H. Sattinger and O. L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, Springer-Verlag, New York, 1986.

[Ti1] J. Tits, Groupes simples et géométries associées, *Proc. Intern. Cong. Math. Stockholm* (1962), 197–221.

[Ti2] J. Tits, Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples, *Publ. Math.* I.H.E.S. **31** (1965), 21–58.

[To] M. L. Tomber, Lie algebras of type *F*, *Proc. Amer. Math. Soc.* **4** (1953), 759–768.

[Tow1] J. Towber, Two new functors from modules to algebras, *J. Algebra* **47** (1977), 80–104.

[Tow2] J. Towber, Young symmetry, the flag manifold, and representations of GL(n), *J. Algebra* **61** (1979), 414–462.

[Va] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer-Verlag, New York, 1974, 1984.

[vdW] B. L. van der Waerden, Reihenentwicklungen und Überschiebungen in der Invariantentheorie, insbesondere im quarternären Giebiet, *Math. Ann.* **113** (1936), 14–35.

[Vu] T. Vust, Sur la théorie des invariants des groupes classiques, *Ann. Inst. Fourier* **26** (1976), 1–31.

[Wa] Z-X. Wan, *Lie Algebras*, Pergamon Press, New York, 1975.

[We1] H. Weyl, *Classical Groups*, Princeton University Press, Princeton, NJ, 1939; second edition, 1946.

[We2] H. Weyl, Über Algebren, die mit der Komplexgruppe in Zusammenhang stehen, und ihre Darstellungen, *Math. Zeit.* **35** (1932), 300–320.

[Ze] A. V. Zelevinsky, *Representations of Finite Classical Groups*, Springer Lecture Notes 869, Springer-Verlag, Heidelberg 1981.

[Žel] D. P. Želobenko, *Compact Lie Groups and Their Representations*, Translations of Mathematical Monographs, vol. 40, American Mathematical Society, Providence, RI, 1973.

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