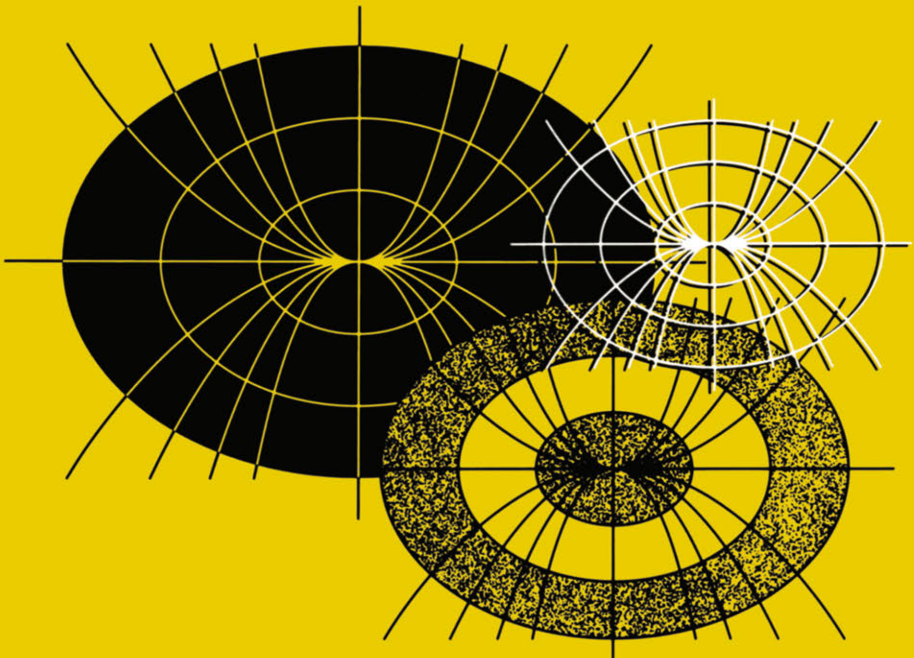
Martin Braun

Differential Equations and Their Applications

Fourth Edition



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Martin Braun

Differential Equations and Their Applications

An Introduction to Applied Mathematics

Fourth Edition

With 68 Illustrations

Springer

Martin Braun
Department of Mathematics
Queens College
City University of New York
Flushing, NY 11367
USA

Series Editors

Jerrold E. Marsden
Control and Dynamical Systems, 107-81
California Institute of Technology
Pasadena, CA 91125
USA

L. Sirovich
Division of Applied Mathematics
Brown University
Providence, RI 02912
USA

M. Golubitsky
Department of Mathematics
University of Houston
Houston, TX 77204-3476
USA

W. Jäger
Department of Applied Mathematics
Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg, Germany

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To four beautiful people:

Zelda Lee

Adeena Rachelle, I. Nasanayl, and Shulamit

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface to the Fourth Edition

There are two major changes in the Fourth Edition of *Differential Equations and Their Applications*. The first concerns the computer programs in this text. In keeping with recent trends in computer science, we have replaced all the APL programs with Pascal and C programs. The Pascal programs appear in the text in place of the APL programs, where they are followed by the Fortran programs, while the C programs appear in Appendix C.

The second change, in response to many readers' suggestions, is the inclusion of a new chapter (Chapter 6) on Sturm-Liouville boundary value problems. Our goal in this chapter is not to present a whole lot of technical material. Rather it is to show that the theory of Fourier series presented in Chapter 5 is not an isolated theory but is part of a much more general and beautiful theory which encompasses many of the key ideas of linear algebra.

To accomplish this goal we have included some additional material from linear algebra. In particular, we have introduced the notions of inner product spaces and self-adjoint matrices, proven that the eigenvalues of a self-adjoint matrix are real, and shown that all self-adjoint matrices possess an orthonormal basis of eigenvectors. These results are at the heart of Sturm-Liouville theory.

I wish to thank Robert Giresi for writing the Pascal and C programs.

New York City
May, 1992

Martin Braun

Preface to the Third Edition

There are three major changes in the Third Edition of *Differential Equations and Their Applications*. First, we have completely rewritten the section on singular solutions of differential equations. A new section, 2.8.1, dealing with Euler equations has been added, and this section is used to motivate a greatly expanded treatment of singular equations in sections 2.8.2 and 2.8.3.

Our second major change is the addition of a new section, 4.9, dealing with bifurcation theory, a subject of much current interest. We felt it desirable to give the reader a brief but nontrivial introduction to this important topic.

Our third major change is in Section 2.6, where we have switched to the metric system of units. This change was requested by many of our readers.

In addition to the above changes, we have updated the material on population models, and have revised the exercises in this section. Minor editorial changes have also been made throughout the text.

New York City
November, 1982

Martin Braun

Preface to the First Edition

This textbook is a unique blend of the theory of differential equations and their exciting application to "real world" problems. First, and foremost, it is a rigorous study of ordinary differential equations and can be fully understood by anyone who has completed one year of calculus. However, in addition to the traditional applications, it also contains many exciting "real life" problems. These applications are completely self contained. First, the problem to be solved is outlined clearly, and one or more differential equations are derived as a model for this problem. These equations are then solved, and the results are compared with real world data. The following applications are covered in this text.

1. In Section 1.3 we prove that the beautiful painting "Disciples of Emmaus" which was bought by the Rembrandt Society of Belgium for \$170,000 was a modern forgery.

2. In Section 1.5 we derive differential equations which govern the population growth of various species, and compare the results predicted by our models with the known values of the populations.

3. In Section 1.6 we derive differential equations which govern the rate at which farmers adopt new innovations. Surprisingly, these same differential equations govern the rate at which technological innovations are adopted in such diverse industries as coal, iron and steel, brewing, and railroads.

4. In Section 1.7 we try to determine whether tightly sealed drums filled with concentrated waste material will crack upon impact with the ocean floor. In this section we also describe several tricks for obtaining information about solutions of a differential equation that cannot be solved explicitly.

5. In Section 2.7 we derive a very simple model of the blood glucose regulatory system and obtain a fairly reliable criterion for the diagnosis of diabetes.

6. Section 4.5 describes two applications of differential equations to arms races and actual combat. In Section 4.5.1 we discuss L. F. Richardson's theory of the escalation of arms races and fit his model to the arms race which led eventually to World War I. This section also provides the reader with a concrete feeling for the concept of stability. In Section 4.5.2 we derive two Lanchestrian combat models, and fit one of these models, with astonishing accuracy, to the battle of Iwo Jima in World War II.

7. In Section 4.10 we show why the predator portion (sharks, skates, rays, etc.) of all fish caught in the port of Fiume, Italy rose dramatically during the years of World War I. The theory we develop here also has a spectacular application to the spraying of insecticides.

8. In Section 4.11 we derive the "principle of competitive exclusion," which states, essentially, that no two species can earn their living in an identical manner.

9. In Section 4.12 we study a system of differential equations which govern the spread of epidemics in a population. This model enables us to prove the famous "threshold theorem of epidemiology," which states that an epidemic will occur only if the number of people susceptible to the disease exceeds a certain threshold value. We also compare the predictions of our model with data from an actual plague in Bombay.

10. In Section 4.13 we derive a model for the spread of gonorrhea and prove that either this disease dies out, or else the number of people who have gonorrhea will ultimately approach a fixed value.

This textbook also contains the following important, and often unique features.

1. In Section 1.10 we give a complete proof of the existence–uniqueness theorem for solutions of first-order equations. Our proof is based on the method of Picard iterates, and can be fully understood by anyone who has completed one year of calculus.

2. In Section 1.11 we show how to solve equations by iteration. This section has the added advantage of reinforcing the reader's understanding of the proof of the existence–uniqueness theorem.

3. Complete Fortran and APL programs are given for every computer example in the text. Computer problems appear in Sections 1.13–1.17, which deal with numerical approximations of solutions of differential equations; in Section 1.11, which deals with solving the equations $x=f(x)$ and $g(x)=0$; and in Section 2.8, where we show how to obtain a power-series solution of a differential equation even though we cannot explicitly solve the recurrence formula for the coefficients.

4. A self-contained introduction to the computing language APL is presented in Appendix C. Using this appendix we have been able to teach our students APL in just two lectures.

5. Modesty aside, Section 2.12 contains an absolutely super and unique treatment of the Dirac delta function. We are very proud of this section because it eliminates all the ambiguities which are inherent in the traditional exposition of this topic.

6. All the linear algebra pertinent to the study of systems of equations is presented in Sections 3.1–3.7. One advantage of our approach is that the reader gets a concrete feeling for the very important but extremely abstract properties of linear independence, spanning, and dimension. Indeed, many linear algebra students sit in on our course to find out what's really going on in their course.

Differential Equations and Their Applications can be used for a one- or two-semester course in ordinary differential equations. It is geared to the student who has completed two semesters of calculus. Traditionally, most authors present a "suggested syllabus" for their textbook. We will not do so here, though, since there are already more than twenty different syllabi in use. Suffice it to say that this text can be used for a wide variety of courses in ordinary differential equations.

I greatly appreciate the help of the following people in the preparation of this manuscript: Douglas Reber who wrote the Fortran programs, Eleanor Addison who drew the original figures, and Kate MacDougall, Sandra Spinacci, and Miriam Green who typed portions of this manuscript.

I am grateful to Walter Kaufmann-Bühler, the mathematics editor at Springer-Verlag, and Elizabeth Kaplan, the production editor, for their extensive assistance and courtesy during the preparation of this manuscript. It is a pleasure to work with these true professionals.

Finally, I am especially grateful to Joseph P. LaSalle for the encouragement and help he gave me. Thanks again, Joe.

New York City
July, 1976

Martin Braun

Contents

1 First-order differential equations

1.1 Introduction

This book is a study of differential equations and their applications. A differential equation is a relationship between a function of time and its derivatives. The equations

$$\frac{dy}{dt} = 3y^2\sin(t+y) \tag{i}$$

and

$$\frac{d^3y}{dt^3} = e^{-y} + t + \frac{d^2y}{dt^2} \tag{ii}$$

are both examples of differential equations. The order of a differential equation is the order of the highest derivative of the function y that appears in the equation. Thus (i) is a first-order differential equation and (ii) is a third-order differential equation. By a solution of a differential equation we will mean a continuous function $y(t)$ which together with its derivatives satisfies the relationship. For example, the function

$$y(t) = 2\sin t - \tfrac{1}{3}\cos 2t$$

is a solution of the second-order differential equation

$$\frac{d^2y}{dt^2} + y = \cos 2t$$

since

$$\frac{d^2}{dt^2}\left(2\sin t - \tfrac{1}{3}\cos 2t\right) + \left(2\sin t - \tfrac{1}{3}\cos 2t\right)$$
$$= \left(-2\sin t + \tfrac{4}{3}\cos 2t\right) + 2\sin t - \tfrac{1}{3}\cos 2t = \cos 2t.$$

Differential equations appear naturally in many areas of science and the humanities. In this book, we will present serious discussions of the applications of differential equations to such diverse and fascinating problems as the detection of art forgeries, the diagnosis of diabetes, the increase in the percentage of sharks present in the Mediterranean Sea during World War I, and the spread of gonorrhea. Our purpose is to show how researchers have used differential equations to solve, or try to solve, *real life* problems. And while we will discuss some of the great success stories of differential equations, we will also point out their limitations and document some of their failures.

1.2 First-order linear differential equations

We begin by studying first-order differential equations and we will assume that our equation is, or can be put, in the form

$$\frac{dy}{dt}=f(t,y). \tag{1}$$

The problem before us is this: Given $f(t,y)$ find all functions $y(t)$ which satisfy the differential equation (1). We approach this problem in the following manner. A fundamental principle of mathematics is that the way to solve a new problem is to reduce it, in some manner, to a problem that we have already solved. In practice this usually entails successively simplifying the problem until it resembles one we have already solved. Since we are presently in the business of solving differential equations, it is advisable for us to take inventory and list all the differential equations we can solve. If we assume that our mathematical background consists of just elementary calculus then the very sad fact is that the only first-order differential equation we can solve at present is

$$\frac{dy}{dt}=g(t) \tag{2}$$

where g is any integrable function of time. To solve Equation (2) simply integrate both sides with respect to t, which yields

$$y(t)=\int g(t)\,dt+c.$$

Here c is an arbitrary constant of integration, and by $\int g(t)\,dt$ we mean an anti-derivative of g, that is, a function whose derivative is g. Thus, to solve any other differential equation we must somehow reduce it to the form (2). As we will see in Section 1.9, this is impossible to do in most cases. Hence, we will not be able, without the aid of a computer, to solve most differential equations. It stands to reason, therefore, that to find those differential equations that we *can* solve, we should start with very simple equations and not ones like

$$\frac{dy}{dt} = e^{\sin(t - 37\sqrt{|y|}\,)}$$

(which incidentally, cannot be solved exactly). Experience has taught us that the "simplest" equations are those which are *linear* in the dependent variable y.

Definition. The general first-order linear differential equation is

$$\frac{dy}{dt} + a(t)y = b(t). \tag{3}$$

Unless otherwise stated, the functions $a(t)$ and $b(t)$ are assumed to be continuous functions of time. We single out this equation and call it linear because the dependent variable y appears by itself, that is, no terms such as e^{-y}, y^3 or $\sin y$ etc. appear in the equation. For example $dy/dt = y^2 + \sin t$ and $dy/dt = \cos y + t$ are both *nonlinear* equations because of the y^2 and $\cos y$ terms respectively.

Now it is not immediately apparent how to solve Equation (3). Thus, we simplify it even further by setting $b(t) = 0$.

Definition. The equation

$$\frac{dy}{dt} + a(t)y = 0 \tag{4}$$

is called the ***homogeneous*** first-order linear differential equation, and Equation (3) is called the ***nonhomogeneous*** first-order linear differential equation for $b(t)$ not identically zero.

Fortunately, the homogeneous equation (4) can be solved quite easily. First, divide both sides of the equation by y and rewrite it in the form

$$\frac{\frac{dy}{dt}}{y} = -a(t).$$

Second, observe that

$$\frac{\frac{dy}{dt}}{y} \equiv \frac{d}{dt}\ln|y(t)|$$

where by $\ln|y(t)|$ we mean the natural logarithm of $|y(t)|$. Hence Equation (4) can be written in the form

$$\frac{d}{dt}\ln|y(t)| = -a(t). \tag{5}$$

But this is Equation (2) "essentially" since we can integrate both sides of (5) to obtain that

$$\ln|y(t)| = -\int a(t)\,dt + c_1$$

where c_1 is an arbitrary constant of integration. Taking exponentials of both sides yields

$$|y(t)| = \exp\left(-\int a(t)\,dt + c_1\right) = c\exp\left(-\int a(t)\,dt\right)$$

or

$$\left|y(t)\exp\left(\int a(t)\,dt\right)\right| = c. \tag{6}$$

Now, $y(t)\exp\left(\int a(t)\,dt\right)$ is a continuous function of time and Equation (6) states that its absolute value is constant. But if the absolute value of a continuous function $g(t)$ is constant then g itself must be constant. To prove this observe that if g is not constant, then there exist two different times t_1 and t_2 for which $g(t_1) = c$ and $g(t_2) = -c$. By the intermediate value theorem of calculus g must achieve all values between $-c$ and $+c$ which is impossible if $|g(t)| = c$. Hence, we obtain the equation $y(t)\exp\left(\int a(t)\,dt\right) = c$ or

$$y(t) = c\exp\left(-\int a(t)\,dt\right). \tag{7}$$

Equation (7) is said to be the *general solution* of the homogeneous equation since every solution of (4) must be of this form. Observe that an arbitrary constant c appears in (7). This should not be too surprising. Indeed, we will always expect an arbitrary constant to appear in the general solution of any first-order differential equation. To wit, if we are given dy/dt and we want to recover $y(t)$, then we must perform an integration, and this, of necessity, yields an arbitrary constant. Observe also that Equation (4) has infinitely many solutions; for each value of c we obtain a distinct solution $y(t)$.

Example 1. Find the general solution of the equation $(dy/dt) + 2ty = 0$.

Solution. Here $a(t) = 2t$ so that $y(t) = c\exp\left(-\int 2t\,dt\right) = ce^{-t^2}$.

Example 2. Determine the behavior, as $t \to \infty$, of all solutions of the equation $(dy/dt) + ay = 0$, a constant.

Solution. The general solution is $y(t) = c\exp\left(-\int a\,dt\right) = ce^{-at}$. Hence if $a < 0$, all solutions, with the exception of $y = 0$, approach infinity, and if $a > 0$, all solutions approach zero as $t \to \infty$.

In applications, we are usually not interested in all solutions of (4). Rather, we are looking for the *specific* solution $y(t)$ which at some initial time t_0 has the value y_0. Thus, we want to determine a function $y(t)$ such that

$$\frac{dy}{dt}+a(t)y=0, \qquad y(t_0)=y_0. \tag{8}$$

Equation (8) is referred to as an initial-value problem for the obvious reason that of the totality of all solutions of the differential equation, we are looking for the one solution which initially (at time t_0) has the value y_0. To find this solution we integrate both sides of (5) between t_0 and t. Thus

$$\int_{t_0}^{t}\frac{d}{ds}\ln|y(s)|\,ds=-\int_{t_0}^{t}a(s)\,ds$$

and, therefore

$$\ln|y(t)|-\ln|y(t_0)|=\ln\left|\frac{y(t)}{y(t_0)}\right|=-\int_{t_0}^{t}a(s)\,ds.$$

Taking exponentials of both sides of this equation we obtain that

$$\left|\frac{y(t)}{y(t_0)}\right|=\exp\left(-\int_{t_0}^{t}a(s)\,ds\right)$$

or

$$\left|\frac{y(t)}{y(t_0)}\exp\left(\int_{t_0}^{t}a(s)\,ds\right)\right|=1.$$

The function inside the absolute value sign is a continuous function of time. Thus, by the argument given previously, it is either identically $+1$ or identically -1. To determine which one it is, evaluate it at the point t_0; since

$$\frac{y(t_0)}{y(t_0)}\exp\left(\int_{t_0}^{t_0}a(s)\,ds\right)=1$$

we see that

$$\frac{y(t)}{y(t_0)}\exp\left(\int_{t_0}^{t}a(s)\,ds\right)=1.$$

Hence

$$y(t)=y(t_0)\exp\left(-\int_{t_0}^{t}a(s)\,ds\right)=y_0\exp\left(-\int_{t_0}^{t}a(s)\,ds\right).$$

Example 3. Find the solution of the initial-value problem

$$\frac{dy}{dt}+(\sin t)y=0, \qquad y(0)=\tfrac{3}{2}.$$

Solution. Here $a(t)=\sin t$ so that

$$y(t)=\tfrac{3}{2}\exp\left(-\int_0^t \sin s\,ds\right)=\tfrac{3}{2}e^{(\cos t)-1}.$$

Example 4. Find the solution of the initial-value problem

$$\frac{dy}{dt}+e^{t^2}y=0, \qquad y(1)=2.$$

Solution. Here $a(t)=e^{t^2}$ so that

$$y(t)=2\exp\left(-\int_1^t e^{s^2}\,ds\right).$$

Now, at first glance this problem would seem to present a very serious difficulty in that we cannot integrate the function e^{s^2} directly. However, this solution is equally as valid and equally as useful as the solution to Example 3. The reason for this is twofold. First, there are very simple numerical schemes to evaluate the above integral to any degree of accuracy with the aid of a computer. Second, even though the solution to Example 3 is given explicitly, we still cannot evaluate it at any time t without the aid of a table of trigonometric functions and some sort of calculating aid, such as a slide rule, electronic calculator or digital computer.

We return now to the nonhomogeneous equation

$$\frac{dy}{dt}+a(t)y=b(t).$$

It should be clear from our analysis of the homogeneous equation that the way to solve the nonhomogeneous equation is to express it in the form

$$\frac{d}{dt}(\text{“something”})=b(t)$$

and then to integrate both sides to solve for “something”. However, the expression $(dy/dt)+a(t)y$ does not appear to be the derivative of some simple expression. The next logical step in our analysis therefore should be the following: Can we make the left hand side of the equation to be d/dt of “something”? More precisely, we can multiply both sides of (3) by any continuous function $\mu(t)$ to obtain the equivalent equation

$$\mu(t)\frac{dy}{dt}+a(t)\,\mu(t)y=\mu(t)b(t). \tag{9}$$

(By equivalent equations we mean that every solution of (9) is a solution of (3) and vice-versa.) Thus, can we *choose* $\mu(t)$ so that $\mu(t)(dy/dt)+a(t)\mu(t)y$ is the derivative of some simple expression? The answer to this question is yes, and is obtained by observing that

$$\frac{d}{dt}\mu(t)y=\mu(t)\frac{dy}{dt}+\frac{d\mu}{dt}y.$$

Hence, $\mu(t)(dy/dt)+a(t)\mu(t)y$ will be equal to the derivative of $\mu(t)y$ if and only if $d\mu(t)/dt=a(t)\mu(t)$. But this is a first-order linear homogeneous equation for $\mu(t)$, i.e. $(d\mu/dt)-a(t)\mu=0$ which we already know how to solve, and since we only need one such function $\mu(t)$ we set the constant c in (7) equal to one and take

$$\mu(t)=\exp\left(\int a(t)\,dt\right).$$

For this $\mu(t)$, Equation (9) can be written as

$$\frac{d}{dt}\mu(t)y=\mu(t)b(t). \tag{10}$$

To obtain the general solution of the nonhomogeneous equation (3), that is, to find all solutions of the nonhomogeneous equation, we take the indefinite integral (anti-derivative) of both sides of (10) which yields

$$\mu(t)y=\int\mu(t)b(t)\,dt+c$$

or

$$y=\frac{1}{\mu(t)}\left(\int\mu(t)b(t)\,dt+c\right)=\exp\left(-\int a(t)\,dt\right)\left(\int\mu(t)b(t)\,dt+c\right). \tag{11}$$

Alternately, if we are interested in the specific solution of (3) satisfying the initial condition $y(t_0)=y_0$, that is, if we want to solve the initial-value problem

$$\frac{dy}{dt}+a(t)y=b(t), \qquad y(t_0)=y_0$$

then we can take the definite integral of both sides of (10) between t_0 and t to obtain that

$$\mu(t)y-\mu(t_0)y_0=\int_{t_0}^{t}\mu(s)b(s)\,ds$$

or

$$y=\frac{1}{\mu(t)}\left(\mu(t_0)y_0+\int_{t_0}^{t}\mu(s)b(s)\,ds\right). \tag{12}$$

Remark 1. Notice how we used our knowledge of the solution of the homogeneous equation to find the function $\mu(t)$ which enables us to solve the nonhomogeneous equation. This is an excellent illustration of how we use our knowledge of the solution of a simpler problem to solve a harder problem.

Remark 2. The function $\mu(t)=\exp\left(\int a(t)\,dt\right)$ is called an *integrating factor* for the nonhomogeneous equation since after multiplying both sides by $\mu(t)$ we can immediately integrate the equation to find all solutions.

Remark 3. The reader should not memorize formulae (11) and (12). Rather, we will solve all nonhomogeneous equations by first multiplying both sides by $\mu(t)$, by writing the new left-hand side as the derivative of $\mu(t)y(t)$, and then by integrating both sides of the equation.

Remark 4. An alternative way of solving the initial-value problem $(dy/dt)+a(t)y=b(t)$, $y(t_0)=y_0$ is to find the general solution (11) of (3) and then use the initial condition $y(t_0)=y_0$ to evaluate the constant c. If the function $\mu(t)b(t)$ cannot be integrated directly, though, then we must take the definite integral of (10) to obtain (12), and this equation is then approximated numerically.

Example 5. Find the general solution of the equation $(dy/dt)-2ty=t$.
Solution. Here $a(t)=-2t$ so that

$$\mu(t)=\exp\left(\int a(t)\,dt\right)=\exp\left(-\int 2t\,dt\right)=e^{-t^2}.$$

Multiplying both sides of the equation by $\mu(t)$ we obtain the equivalent equation

$$e^{-t^2}\left(\frac{dy}{dt}-2ty\right)=te^{-t^2} \quad\text{or}\quad \frac{d}{dt}e^{-t^2}y=te^{-t^2}.$$

Hence,

$$e^{-t^2}y=\int te^{-t^2}\,dt+c=\frac{-e^{-t^2}}{2}+c$$

and

$$y(t)=-\tfrac{1}{2}+ce^{t^2}.$$

Example 6. Find the solution of the initial-value problem

$$\frac{dy}{dt}+2ty=t, \qquad y(1)=2.$$

Solution. Here $a(t)=2t$ so that

$$\mu(t)=\exp\left(\int a(t)\,dt\right)=\exp\left(\int 2t\,dt\right)=e^{t^2}.$$

Multiplying both sides of the equation by $\mu(t)$ we obtain that

$$e^{t^2}\left(\frac{dy}{dt}+2ty\right)=te^{t^2} \quad\text{or}\quad \frac{d}{dt}\left(e^{t^2}y\right)=te^{t^2}.$$

Hence,

$$\int_1^t \frac{d}{ds} e^{s^2} y(s)\, ds = \int_1^t s e^{s^2}\, ds$$

so that

$$e^{s^2} y(s)\Big|_1^t = \frac{e^{s^2}}{2}\Big|_1^t.$$

Consequently,

$$e^{t^2} y - 2e = \frac{e^{t^2}}{2} - \frac{e}{2}$$

and

$$y = \frac{1}{2} + \frac{3e}{2} e^{-t^2} = \tfrac{1}{2} + \tfrac{3}{2} e^{1-t^2}.$$

Example 7. Find the solution of the initial-value problem

$$\frac{dy}{dt} + y = \frac{1}{1+t^2}, \quad y(2) = 3.$$

Solution. Here $a(t) = 1$, so that

$$\mu(t) = \exp\left(\int a(t)\, dt\right) = \exp\left(\int 1\, dt\right) = e^t.$$

Multiplying both sides of the equation by $\mu(t)$ we obtain that

$$e^t\left(\frac{dy}{dt} + y\right) = \frac{e^t}{1+t^2} \quad \text{or} \quad \frac{d}{dt} e^t y = \frac{e^t}{1+t^2}.$$

Hence

$$\int_2^t \frac{d}{ds} e^s y(s)\, ds = \int_2^t \frac{e^s}{1+s^2}\, ds,$$

so that

$$e^t y - 3e^2 = \int_2^t \frac{e^s}{1+s^2}\, ds$$

and

$$y = e^{-t}\left[3e^2 + \int_2^t \frac{e^s}{1+s^2}\, ds\right].$$

Exercises

In each of Problems 1–7 find the general solution of the given differential equation.

1. $\dfrac{dy}{dt} + y\cos t = 0$

2. $\dfrac{dy}{dt} + y\sqrt{t}\, \sin t = 0$

3. $\dfrac{dy}{dt}+\dfrac{2t}{1+t^2}y=\dfrac{1}{1+t^2}$

4. $\dfrac{dy}{dt}+y=te^t$

5. $\dfrac{dy}{dt}+t^2y=1$

6. $\dfrac{dy}{dt}+t^2y=t^2$

7. $\dfrac{dy}{dt}+\dfrac{t}{1+t^2}y=1-\dfrac{t^3}{1+t^4}y$

In each of Problems 8–14, find the solution of the given initial-value problem.

8. $\dfrac{dy}{dt}+\sqrt{1+t^2}\,y=0, \qquad y(0)=\sqrt{5}$

9. $\dfrac{dy}{dt}+\sqrt{1+t^2}\,e^{-t}y=0, \qquad y(0)=1$

10. $\dfrac{dy}{dt}+\sqrt{1+t^2}\,e^{-t}y=0, \qquad y(0)=0$

11. $\dfrac{dy}{dt}-2ty=t, \qquad y(0)=1$

12. $\dfrac{dy}{dt}+ty=1+t, \qquad y(\tfrac{3}{2})=0$

13. $\dfrac{dy}{dt}+y=\dfrac{1}{1+t^2}, \qquad y(1)=2$

14. $\dfrac{dy}{dt}-2ty=1, \qquad y(0)=1$

15. Find the general solution of the equation

$$(1+t^2)\frac{dy}{dt}+ty=(1+t^2)^{5/2}.$$

(*Hint*: Divide both sides of the equation by $1+t^2$.)

16. Find the solution of the initial-value problem

$$(1+t^2)\frac{dy}{dt}+4ty=t, \qquad y(1)=\tfrac{1}{4}.$$

17. Find a continuous solution of the initial-value problem

$$y'+y=g(t), \qquad y(0)=0$$

where

$$g(t)=\begin{cases} 2, & 0\leqslant t\leqslant 1 \\ 0, & t>1 \end{cases}.$$

18. Show that every solution of the equation $(dy/dt)+ay=be^{-ct}$ where a and c are positive constants and b is any real number approaches zero as t approaches infinity.

19. Given the differential equation $(dy/dt)+a(t)y=f(t)$ with $a(t)$ and $f(t)$ continuous for $-\infty<t<\infty$, $a(t)\geqslant c>0$, and $\lim_{t\to\infty}f(t)=0$, show that every solution tends to zero as t approaches infinity.

When we derived the solution of the nonhomogeneous equation we tacitly assumed that the functions $a(t)$ and $b(t)$ were continuous so that we could perform the necessary integrations. If either of these functions was discontinuous at a point t_1, then we would expect that our solutions might be discontinuous at $t=t_1$. Problems 20–23 illustrate the variety of things that may happen. In Problems 20–22 determine the behavior of all solutions of the given differential equation as $t \to 0$, and in Problem 23 determine the behavior of all solutions as $t \to \pi/2$.

20. $\dfrac{dy}{dt} + \dfrac{1}{t}y = \dfrac{1}{t^2}$

21. $\dfrac{dy}{dt} + \dfrac{1}{\sqrt{t}}y = e^{\sqrt{t}/2}$

22. $\dfrac{dy}{dt} + \dfrac{1}{t}y = \cos t + \dfrac{\sin t}{t}$

23. $\dfrac{dy}{dt} + y\tan t = \sin t \cos t.$

1.3 The Van Meegeren art forgeries

After the liberation of Belgium in World War II, the Dutch Field Security began its hunt for Nazi collaborators. They discovered, in the records of a firm which had sold numerous works of art to the Germans, the name of a banker who had acted as an intermediary in the sale to Goering of the painting "Woman Taken in Adultery" by the famed 17th century Dutch painter Jan Vermeer. The banker in turn revealed that he was acting on behalf of a third rate Dutch painter H. A. Van Meegeren, and on May 29, 1945 Van Meegeren was arrested on the charge of collaborating with the enemy. On July 12, 1945 Van Meegeren startled the world by announcing from his prison cell that he had never sold "Woman Taken in Adultery" to Goering. Moreover, he stated that this painting and the very famous and beautiful "Disciples at Emmaus", as well as four other presumed Vermeers and two de Hooghs (a 17th century Dutch painter) were his own work. Many people, however, thought that Van Meegeren was only lying to save himself from the charge of treason. To prove his point, Van Meegeren began, while in prison, to forge the Vermeer painting "Jesus Amongst the Doctors" to demonstrate to the skeptics just how good a forger of Vermeer he was. The work was nearly completed when Van Meegeren learned that a charge of forgery had been substituted for that of collaboration. He, therefore, refused to finish and age the painting so that hopefully investigators would not uncover his secret of aging his forgeries. To settle the question an international panel of distinguished chemists, physicists and art historians was appointed to investigate the matter. The panel took x-rays of the paintings to determine whether other paintings were underneath them. In addition, they analyzed the pigments (coloring materials) used in the paint, and examined the paintings for certain signs of old age.

Now, Van Meegeren was well aware of these methods. To avoid detection, he scraped the paint from old paintings that were not worth much, just to get the canvas, and he tried to use pigments that Vermeer would have used. Van Meegeren also knew that old paint was extremely hard, and impossible to dissolve. Therefore, he very cleverly mixed a chemical, phenoformaldehyde, into the paint, and this hardened into bakelite when the finished painting was heated in an oven.

However, Van Meegeren was careless with several of his forgeries, and the panel of experts found traces of the modern pigment cobalt blue. In addition, they also detected the phenoformaldehyde, which was not discovered until the turn of the 19th century, in several of the paintings. On the basis of this evidence Van Meegeren was convicted, of forgery, on October 12, 1947 and sentenced to one year in prison. While in prison he suffered a heart attack and died on December 30, 1947.

However, even following the evidence gathered by the panel of experts, many people still refused to believe that the famed "Disciples at Emmaus" was forged by Van Meegeren. Their contention was based on the fact that the other alleged forgeries and Van Meegeren's nearly completed "Jesus Amongst the Doctors" were of a very inferior quality. Surely, they said, the creator of the beautiful "Disciples at Emmaus" could not produce such inferior pictures. Indeed, the "Disciples at Emmaus" was certified as an authentic Vermeer by the noted art historian A. Bredius and was bought by the Rembrandt Society for $170,000. The answer of the panel to these skeptics was that because Van Meegeren was keenly disappointed by his lack of status in the art world, he worked on the "Disciples at Emmaus" with the fierce determination of proving that he was better than a third rate painter. After producing such a masterpiece his determination was gone. Moreover, after seeing how easy it was to dispose of the "Disciples at Emmaus" he devoted less effort to his subsequent forgeries. This explanation failed to satisfy the skeptics. They demanded a thoroughly scientific and conclusive proof that the "Disciples at Emmaus" was indeed a forgery. This was done recently in 1967 by scientists at Carnegie Mellon University, and we would now like to describe their work.

The key to the dating of paintings and other materials such as rocks and fossils lies in the phenomenon of radioactivity discovered at the turn of the century. The physicist Rutherford and his colleagues showed that the atoms of certain "radioactive" elements are unstable and that within a given time period a fixed proportion of the atoms spontaneously disintegrates to form atoms of a new element. Because radioactivity is a property of the atom, Rutherford showed that the radioactivity of a substance is directly proportional to the number of atoms of the substance present. Thus, if $N(t)$ denotes the number of atoms present at time t, then dN/dt, the number of atoms that disintegrate per unit time is proportional to N, that is,

$$\frac{dN}{dt} = -\lambda N. \tag{1}$$

The constant λ which is positive, is known as the decay constant of the substance. The larger λ is, of course, the faster the substance decays. One measure of the rate of disintegration of a substance is its ***half-life*** which is defined as the time required for half of a given quantity of radioactive atoms to decay. To compute the half-life of a substance in terms of λ, assume that at time t_0, $N(t_0) = N_0$. Then, the solution of the initial-value problem $dN/dt = -\lambda N$, $N(t_0) = N_0$ is

$$N(t) = N_0 \exp\left(-\lambda \int_{t_0}^{t} ds\right) = N_0 e^{-\lambda(t-t_0)}$$

or $N/N_0 = \exp(-\lambda(t-t_0))$. Taking logarithms of both sides we obtain that

$$-\lambda(t-t_0) = \ln\frac{N}{N_0}. \tag{2}$$

Now, if $N/N_0 = \frac{1}{2}$ then $-\lambda(t-t_0) = \ln\frac{1}{2}$ so that

$$(t-t_0) = \frac{\ln 2}{\lambda} = \frac{0.6931}{\lambda}. \tag{3}$$

Thus, the half-life of a substance is $\ln 2$ divided by the decay constant λ. The dimension of λ, which we suppress for simplicity of writing, is reciprocal time. If t is measured in years then λ has the dimension of reciprocal years, and if t is measured in minutes, then λ has the dimension of reciprocal minutes. The half-lives of many substances have been determined and recorded. For example, the half-life of carbon-14 is 5568 years and the half-life of uranium-238 is 4.5 billion years.

Now the basis of "radioactive dating" is essentially the following. From Equation (2) we can solve for $t - t_0 = 1/\lambda \ln(N_0/N)$. If t_0 is the time the substance was initially formed or manufactured, then the age of the substance is $1/\lambda \ln(N_0/N)$. The decay constant λ is known or can be computed, in most instances. Moreover, we can usually evaluate N quite easily. Thus, if we knew N_0 we could determine the age of the substance. But this is the real difficulty of course, since we usually do not know N_0. In some instances though, we can either determine N_0 indirectly, or else determine certain suitable ranges for N_0, and such is the case for the forgeries of Van Meegeren.

We begin with the following well-known facts of elementary chemistry. Almost all rocks in the earth's crust contain a small quantity of uranium. The uranium in the rock decays to another radioactive element, and that one decays to another and another, and so forth (see Figure 1) in a series of elements that results in lead, which is not radioactive. The uranium (whose half-life is over four billion years) keeps feeding the elements following it in the series, so that as fast as they decay, they are replaced by the elements before them.

Now, all paintings contain a small amount of the radioactive element lead-210 (^{210}Pb), and an even smaller amount of radium-226 (^{226}Ra), since these elements are contained in white lead (lead oxide), which is a pigment that artists have used for over 2000 years. For the analysis which follows, it is important to note that white lead is made from lead metal, which, in turn, is extracted from a rock called lead ore, in a process called smelting. In this process, the lead-210 in the ore goes along with the lead metal. However, 90–95% of the radium and its descendants are removed with

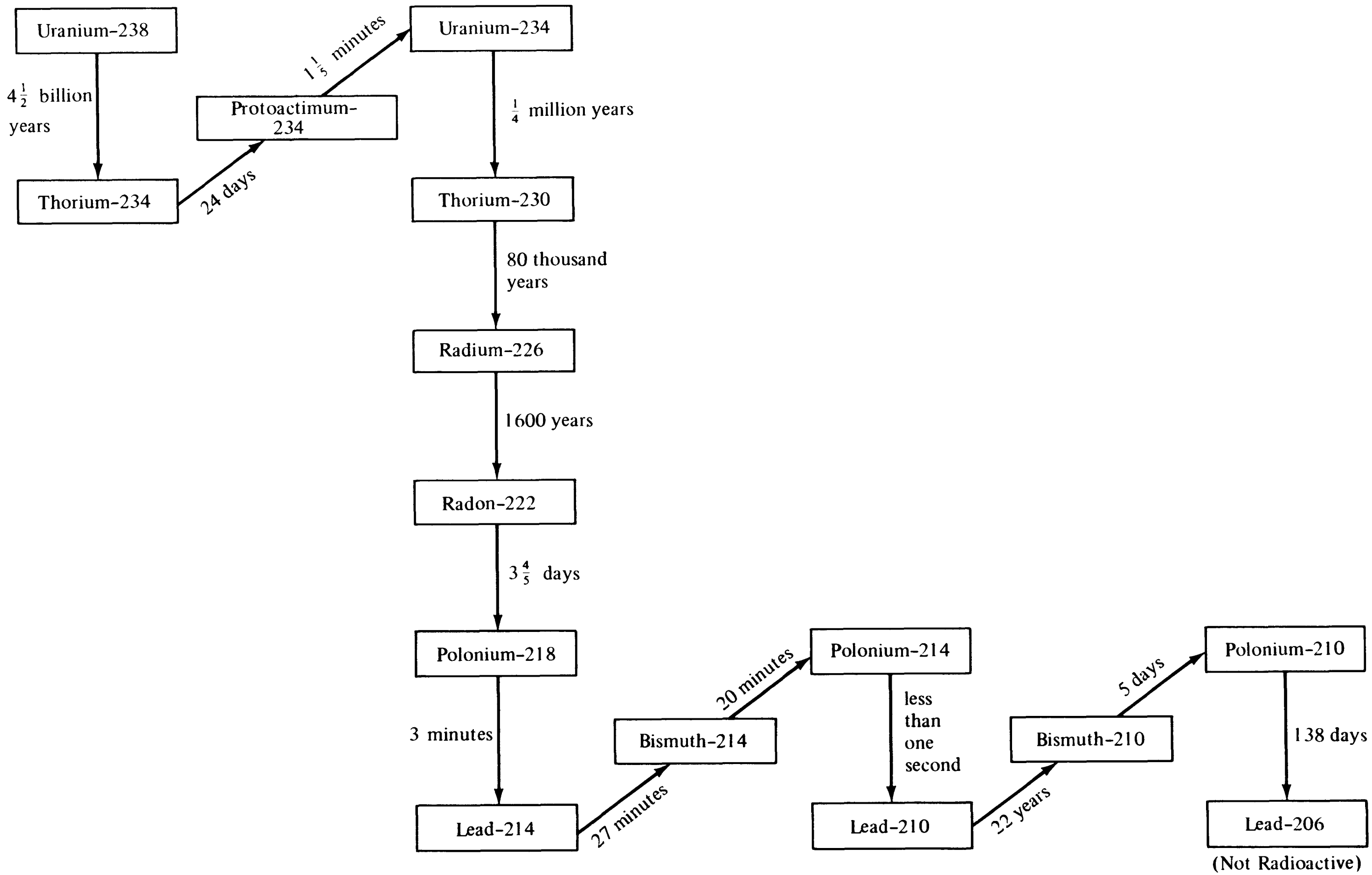

Figure 1. The Uranium series. (The times shown on the arrows are the half-lives of each step.)

other waste products in a material called slag. Thus, most of the supply of lead-210 is cut off and it begins to decay very rapidly, with a half-life of 22 years. This process continues until the lead-210 in the white lead is once more in radioactive equilibrium with the small amount of radium present, i.e. the disintegration of the lead-210 is exactly balanced by the disintegration of the radium.

Let us now use this information to compute the amount of lead-210 present in a sample in terms of the amount originally present at the time of manufacture. Let $y(t)$ be the amount of lead-210 per gram of white lead at time t, y_0 the amount of lead-210 per gram of white lead present at the time of manufacture t_0, and $r(t)$ the number of disintegrations of radium-226 per minute per gram of white lead, at time t. If λ is the decay constant for lead-210, then

$$\frac{dy}{dt} = -\lambda y + r(t), \qquad y(t_0) = y_0. \tag{4}$$

Since we are only interested in a time period of at most 300 years we may assume that the radium-226, whose half-life is 1600 years, remains constant, so that $r(t)$ is a constant r. Multiplying both sides of the differential equation by the integrating factor $\mu(t) = e^{\lambda t}$ we obtain that

$$\frac{d}{dt} e^{\lambda t} y = r e^{\lambda t}.$$

Hence

$$e^{\lambda t} y(t) - e^{\lambda t_0} y_0 = \frac{r}{\lambda}\left(e^{\lambda t} - e^{\lambda t_0}\right)$$

or

$$y(t) = \frac{r}{\lambda}\left(1 - e^{-\lambda(t-t_0)}\right) + y_0 e^{-\lambda(t-t_0)}. \tag{5}$$

Now $y(t)$ and r can be easily measured. Thus, if we knew y_0 we could use Equation (5) to compute $(t-t_0)$ and consequently, we could determine the age of the painting. As we pointed out, though, we cannot measure y_0 directly. One possible way out of this difficulty is to use the fact that the original quantity of lead-210 was in radioactive equilibrium with the larger amount of radium-226 in the ore from which the metal was extracted. Let us, therefore, take samples of different ores and count the number of disintegrations of the radium-226 in the ores. This was done for a variety of ores and the results are given in Table 1 below. These numbers vary from 0.18 to 140. Consequently, the number of disintegrations of the lead-210 per minute per gram of white lead at the time of manufacture will vary from 0.18 to 140. This implies that y_0 will also vary over a very large interval, since the number of disintegrations of lead-210 is proportional to the amount present. Thus, we cannot use Equation (5) to obtain an accurate, or even a crude estimate, of the age of a painting.

Table 1. Ore and ore concentrate samples. All disintegration rates are per gram of white lead.

Description and Source		Disintegrations per minute of ^{226}Ra
Ore concentrate	(Oklahoma–Kansas)	4.5
Crushed raw ore	(S.E. Missouri)	2.4
Ore concentrate	(S.E. Missouri)	0.7
Ore concentrate	(Idaho)	2.2
Ore concentrate	(Idaho)	0.18
Ore concentrate	(Washington)	140.0
Ore concentrate	(British Columbia)	1.9
Ore concentrate	(British Columbia)	0.4
Ore concentrate	(Bolivia)	1.6
Ore concentrate	(Australia)	1.1

However, we can still use Equation (5) to distinguish between a 17th century painting and a modern forgery. The basis for this statement is the simple observation that if the paint is very old compared to the 22 year half-life of lead, then the amount of radioactivity from the lead-210 in the paint will be nearly equal to the amount of radioactivity from the radium in the paint. On the other hand, if the painting is modern (approximately 20 years old, or so) then the amount of radioactivity from the lead-210 will be much greater than the amount of radioactivity from the radium.

We make this argument precise in the following manner. Let us assume that the painting in question is either very new or about 300 years old. Set $t-t_0=300$ years in (5). Then, after some simple algebra, we see that

$$\lambda y_0=\lambda y(t)e^{300\lambda}-r(e^{300\lambda}-1). \tag{6}$$

If the painting is indeed a modern forgery, then λy_0 will be absurdly large. To determine what is an absurdly high disintegration rate we observe (see Exercise 1) that if the lead-210 decayed originally (at the time of manufacture) at the rate of 100 disintegrations per minute per gram of white lead, then the ore from which it was extracted had a uranium content of approximately 0.014 per cent. This is a fairly high concentration of uranium since the average amount of uranium in rocks of the earth's crust is about 2.7 parts per million. On the other hand, there are some very rare ores in the Western Hemisphere whose uranium content is 2-3 per cent. To be on the safe side, we will say that a disintegration rate of lead-210 is certainly absurd if it exceeds 30,000 disintegrations per minute per gram of white lead.

To evaluate λy_0, we must evaluate the present disintegration rate, $\lambda y(t)$, of the lead-210, the disintegration rate r of the radium-226, and $e^{300\lambda}$. Since the disintegration rate of polonium-210 (^{210}Po) equals that of lead-210 after several years, and since it is easier to measure the disintegration rate of polonium-210, we substitute these values for those of lead-210. To compute

$e^{300\lambda}$, we observe from (3) that $\lambda = (\ln 2/22)$. Hence

$$e^{300\lambda} = e^{(300/22)\ln 2} = 2^{(150/11)}.$$

The disintegration rates of polonium-210 and radium-226 were measured for the "Disciples at Emmaus" and various other alleged forgeries and are given in Table 2 below.

Table 2. Paintings of questioned authorship. All disintegration rates are per minute, per gram of white lead.

Description	^{210}Po disintegration	^{226}Ra disintegration
"Disciples at Emmaus"	8.5	0.8
"Washing of Feet"	12.6	0.26
"Woman Reading Music"	10.3	0.3
"Woman Playing Mandolin"	8.2	0.17
"Lace Maker"	1.5	1.4
"Laughing Girl"	5.2	6.0

If we now evaluate λy_0 from (6) for the white lead in the painting "Disciples at Emmaus" we obtain that

$$\begin{aligned}\lambda y_0 &= (8.5)2^{150/11} - 0.8(2^{150/11} - 1)\\ &= 98{,}050\end{aligned}$$

which is unacceptably large. Thus, this painting must be a modern forgery. By a similar analysis, (see Exercises 2–4) the paintings "Washing of Feet", "Woman Reading Music" and "Woman Playing Mandolin" were indisputably shown to be faked Vermeers. On the other hand, the paintings "Lace Maker" and "Laughing Girl" cannot be recently forged Vermeers, as claimed by some experts, since for these two paintings, the polonium-210 is very nearly in radioactive equilibrium with the radium-226, and no such equilibrium has been observed in any samples from 19th or 20th century paintings.

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Exercises

1. In this exercise we show how to compute the concentration of uranium in an ore from the dpm/(g of Pb) of the lead-210 in the ore.

(a) The half-life of uranium-238 is 4.51×10^9 years. Since this half-life is so large, we may assume that the amount of uranium in the ore is constant over a period of two to three hundred years. Let $N(t)$ denote the number of atoms of ^{238}U per gram of ordinary lead in the ore at time t. Since the lead-210 is in radioactive equilibrium with the uranium-238 in the ore, we know that $dN/dt = -\lambda N = -100$ dpm/g of Pb at time t_0. Show that there are 3.42×10^{17} atoms of uranium-238 per gram of ordinary lead in the ore at time t_0. (Hint: 1 year = 525,600 minutes.)

(b) Using the fact that one mole of uranium-238 weighs 238 grams, and that there are 6.02×10^{23} atoms in a mole, show that the concentration of uranium in the ore is approximately 0.014 percent.

For each of the paintings 2, 3, and 4 use the data in Table 2 to compute the disintegrations per minute of the original amount of lead-210 per gram of white lead, and conclude that each of these paintings is a forged Vermeer.

2. "Washing of Feet"

3. "Woman Reading Music"

4. "Woman Playing Mandolin"

5. The following problem describes a very accurate derivation of the age of uranium.

(a) Let $N_{238}(t)$ and $N_{235}(t)$ denote the number of atoms of ^{238}U and ^{235}U at time t in a given sample of uranium, and let $t=0$ be the time this sample was created. By the radioactive decay law,

$$\frac{d}{dt}N_{238}(t) = \frac{-\ln 2}{(4.5)10^9}N_{238}(t),$$

$$\frac{d}{dt}N_{235}(t) = \frac{-\ln 2}{0.707(10)^9}N_{235}(t).$$

Solve these equations for $N_{238}(t)$ and $N_{235}(t)$ in terms of their original numbers $N_{238}(0)$ and $N_{235}(0)$.

(b) In 1946 the ratio of $^{238}U/^{235}U$ in any sample was 137.8. Assuming that equal amounts of ^{238}U and ^{235}U appeared in any sample at the time of its creation, show that the age of uranium is 5.96×10^9 years. This figure is universally accepted as the age of uranium.

6. In a samarskite sample discovered recently, there was 3 grams of Thorium (^{232}TH). Thorium decays to lead-208 (^{208}Pb) through the reaction $^{232}Th \rightarrow {}^{208}Pb + 6(4\,{}^4He)$. It was determined that 0.0376 of a gram of lead-208 was produced by the disintegration of the original Thorium in the sample. Given that the half-life of Thorium is 13.9 billion years, derive the age of this samarskite sample. (Hint: 0.0376 grams of ^{208}Pb is the product of the decay of $(232/208)\times$ 0.0376 grams of Thorium.)

One of the most accurate ways of dating archaeological finds is the method of carbon-14 (^{14}C) dating discovered by Willard Libby around 1949. The basis of this method is delightfully simple: The atmosphere of the earth is continuously bombarded by cosmic rays. These cosmic rays produce neutrons in the earth's atmosphere, and these neutrons combine with nitrogen to produce ^{14}C, which is usually called radiocarbon, since it decays radioactively. Now, this radiocarbon is incorporated in carbon dioxide and thus moves through the atmosphere to be absorbed by plants. Animals, in turn, build radiocarbon into their tissues by eating the plants. In living tissue, the rate of ingestion of ^{14}C exactly balances the rate of disintegration of ^{14}C. When an organism dies, though, it ceases to ingest carbon-14 and thus its ^{14}C concentration begins to decrease through disintegration of the ^{14}C present. Now, it is a fundamental assumption of physics that the rate of bombardment of the earth's atmosphere by cosmic rays has always been constant. This implies that the original rate of disintegration of the ^{14}C in a sample such as charcoal is the same as the rate measured today.* This assumption enables us to determine the age of a sample of charcoal. Let $N(t)$ denote the amount of carbon-14 present in a sample at time t, and N_0 the amount present at time $t=0$ when the sample was formed. If λ denotes the decay constant of ^{14}C (the half-life of carbon-14 is 5568 years) then $dN(t)/dt = -\lambda N(t)$, $N(0)=N_0$. Consequently, $N(t)=N_0e^{-\lambda t}$. Now the present rate $R(t)$ of disintegration of the ^{14}C in the sample is given by $R(t)=\lambda N(t)= \lambda N_0 e^{-\lambda t}$ and the original rate of disintegration is $R(0)=\lambda N_0$. Thus, $R(t)/R(0) = e^{-\lambda t}$ so that $t=(1/\lambda)\ln[R(0)/R(t)]$. Hence if we measure $R(t)$, the present rate of disintegration of the ^{14}C in the charcoal, and observe that $R(0)$ must equal the rate of disintegration of the ^{14}C in a comparable amount of living wood, then we can compute the age t of the charcoal. The following two problems are real life illustrations of this method.

7. Charcoal from the occupation level of the famous Lascaux Cave in France gave an average count in 1950 of 0.97 disintegrations per minute per gram. Living wood gave 6.68 disintegrations. Estimate the date of occupation and hence the probable date of the remarkable paintings in the Lascaux Cave.

8. In the 1950 excavation at Nippur, a city of Babylonia, charcoal from a roof beam gave a count of 4.09 disintegrations per minute per gram. Living wood gave 6.68 disintegrations. Assuming that this charcoal was formed during the time of Hammurabi's reign, find an estimate for the likely time of Hamurabi's succession.

*Since the mid 1950's the testing of nuclear weapons has significantly increased the amount of radioactive carbon in our atmosphere. Ironically this unfortunate state of affairs provides us with yet another extremely powerful method of detecting art forgeries. To wit, many artists' materials, such as linseed oil and canvas paper, come from plants and animals, and so will contain the same concentration of carbon-14 as the atmosphere at the time the plant or animal dies. Thus linseed oil (which is derived from the flax plant) that was produced during the last few years will contain a much greater concentration of carbon-14 than linseed oil produced before 1950.

1.4 Separable equations

We solved the first-order linear homogeneous equation

$$\frac{dy}{dt}+a(t)y=0 \tag{1}$$

by dividing both sides of the equation by $y(t)$ to obtain the equivalent equation

$$\frac{1}{y(t)}\frac{dy(t)}{dt}=-a(t) \tag{2}$$

and observing that Equation (2) can be written in the form

$$\frac{d}{dt}\ln|y(t)|=-a(t). \tag{3}$$

We then found $\ln|y(t)|$, and consequently $y(t)$, by integrating both sides of (3). In an exactly analogous manner, we can solve the more general differential equation

$$\frac{dy}{dt}=\frac{g(t)}{f(y)} \tag{4}$$

where f and g are continuous functions of y and t. This equation, and any other equation which can be put into this form, is said to be separable. To solve (4), we first multiply both sides by $f(y)$ to obtain the equivalent equation

$$f(y)\frac{dy}{dt}=g(t). \tag{5}$$

Then, we observe that (5) can be written in the form

$$\frac{d}{dt}F(y(t))=g(t) \tag{6}$$

where $F(y)$ is any anti-derivative of $f(y)$; i.e., $F(y)=\int f(y)dy$. Consequently,

$$F(y(t))=\int g(t)dt+c \tag{7}$$

where c is an arbitrary constant of integration, and we solve for $y=y(t)$ from (7) to find the general solution of (4).

Example 1. Find the general solution of the equation $dy/dt=t^2/y^2$.
Solution. Multiplying both sides of this equation by y^2 gives

$$y^2\frac{dy}{dt}=t^2, \quad \text{or} \quad \frac{d}{dt}\frac{y^3(t)}{3}=t^2.$$

Hence, $y^3(t)=t^3+c$ where c is an arbitrary constant, and $y(t)=(t^3+c)^{1/3}$.

Example 2. Find the general solution of the equation

$$e^{y}\frac{dy}{dt}-t-t^{3}=0.$$

Solution. This equation can be written in the form

$$\frac{d}{dt}e^{y(t)}=t+t^{3}$$

and thus $e^{y(t)}=t^2/2+t^4/4+c$. Taking logarithms of both sides of this equation gives $y(t)=\ln(t^2/2+t^4/4+c)$.

In addition to the differential equation (4), we will often impose an initial condition on $y(t)$ of the form $y(t_0)=y_0$. The differential equation (4) together with the initial condition $y(t_0)=y_0$ is called an initial-value problem. We can solve an initial-value problem two different ways. Either we use the initial condition $y(t_0)=y_0$ to solve for the constant c in (7), or else we integrate both sides ot (6) between t_0 and t to obtain that

$$F(y(t))-F(y_0)=\int_{t_0}^{t}g(s)\,ds. \tag{8}$$

If we now observe that

$$F(y)-F(y_0)=\int_{y_0}^{y}f(r)\,dr, \tag{9}$$

then we can rewrite (8) in the simpler form

$$\int_{y_0}^{y}f(r)\,dr=\int_{t_0}^{t}g(s)\,ds. \tag{10}$$

Example 3. Find the solution $y(t)$ of the initial-value problem

$$e^{y}\frac{dy}{dt}-(t+t^{3})=0,\qquad y(1)=1.$$

Solution. Method (i). From Example 2, we know that the general solution of this equation is $y=\ln(t^2/2+t^4/4+c)$. Setting $t=1$ and $y=1$ gives $1=\ln(3/4+c)$, or $c=e-3/4$. Hence, $y(t)=\ln(e-3/4+t^2/2+t^4/4)$.
Method (ii). From (10),

$$\int_{1}^{y}e^{r}\,dr=\int_{1}^{t}(s+s^{3})\,ds.$$

Consequently,

$$e^{y}-e=\frac{t^2}{2}+\frac{t^4}{4}-\frac{1}{2}-\frac{1}{4},\quad\text{and}\quad y(t)=\ln(e-3/4+t^2/2+t^4/4).$$

Example 4. Solve the initial-value problem $dy/dt=1+y^2$, $y(0)=0$.
Solution. Divide both sides of the differential equation by $1+y^2$ to obtain

the equivalent equation $1/(1+y^2)dy/dt=1$. Then, from (10)

$$\int_0^y \frac{dr}{1+r^2}=\int_0^t ds.$$

Consequently, $\arctan y=t$, and $y=\tan t$.

The solution $y=\tan t$ of the above problem has the disturbing property that it goes to $\pm\infty$ at $t=\pm\pi/2$. And what's even more disturbing is the fact that there is nothing at all in this initial-value problem which even hints to us that there is any trouble at $t=\pm\pi/2$. The sad fact of life is that solutions of perfectly nice differential equations can go to infinity in finite time. Thus, solutions will usually exist only on a finite open interval $a<t<b$, rather than for all time. Moreover, as the following example shows, different solutions of the same differential equation usually go to infinity at different times.

Example 5. Solve the initial-value problem $dy/dt=1+y^2$, $y(0)=1$.
Solution. From (10)

$$\int_1^y \frac{dr}{1+r^2}=\int_0^t ds.$$

Consequently, $\arctan y-\arctan 1=t$, and $y=\tan(t+\pi/4)$. This solution exists on the open interval $-3\pi/4<t<\pi/4$.

Example 6. Find the solution $y(t)$ of the initial-value problem

$$y\frac{dy}{dt}+(1+y^2)\sin t=0, \qquad y(0)=1.$$

Solution. Dividing both sides of the differential equation by $1+y^2$ gives

$$\frac{y}{1+y^2}\frac{dy}{dt}=-\sin t.$$

Consequently,

$$\int_1^y \frac{r\,dr}{1+r^2}=\int_0^t -\sin s\,ds,$$

so that

$$\tfrac{1}{2}\ln(1+y^2)-\tfrac{1}{2}\ln 2=\cos t-1.$$

Solving this equation for $y(t)$ gives

$$y(t)=\pm\left(2e^{-4\sin^2 t/2}-1\right)^{1/2}.$$

To determine whether we take the plus or minus branch of the square root, we note that $y(0)$ is positive. Hence,

$$y(t)=\left(2e^{-4\sin^2 t/2}-1\right)^{1/2}$$

This solution is only defined when

$$2e^{-4\sin^2 t/2} \geqslant 1$$

or

$$e^{4\sin^2 t/2} \leqslant 2. \tag{11}$$

Since the logarithm function is monotonic increasing, we may take logarithms of both sides of (11) and still preserve the inequality. Thus, $4\sin^2 t/2 \leqslant \ln 2$, which implies that

$$\left|\frac{t}{2}\right| \leqslant \arcsin\frac{\sqrt{\ln 2}}{2}$$

Therefore, $y(t)$ only exists on the open interval $(-a,a)$ where

$$a = 2\arcsin\left[\sqrt{\ln 2}\,/2\right].$$

Now, this appears to be a new difficulty associated with nonlinear equations, since $y(t)$ just "disappears" at $t = \pm a$, without going to infinity. However, this apparent difficulty can be explained quite easily, and moreover, can even be anticipated, if we rewrite the differential equation above in the standard form

$$\frac{dy}{dt} = -\frac{(1+y^2)\sin t}{y}.$$

Notice that this differential equation is not defined when $y=0$. Therefore, if a solution $y(t)$ achieves the value zero at some time $t=t^*$, then we cannot expect it to be defined for $t>t^*$. This is exactly what happens here, since $y(\pm a)=0$.

Example 7. Solve the initial-value problem $dy/dt=(1+y)t,\ y(0)=-1$.
Solution. In this case, we cannot divide both sides of the differential equation by $1+y$, since $y(0)=-1$. However, it is easily seen that $y(t)=-1$ is one solution of this initial-value problem, and in Section 1.10 we show that it is the only solution. More generally, consider the initial-value problem $dy/dt=f(y)g(t)$, $y(t_0)=y_0$, where $f(y_0)=0$. Certainly, $y(t)=y_0$ is one solution of this initial-value problem, and in Section 1.10 we show that it is the only solution if $\partial f/\partial y$ exists and is continuous.

Example 8. Solve the initial-value problem

$$(1+e^y)dy/dt = \cos t, \qquad y(\pi/2)=3.$$

Solution. From (10),

$$\int_3^y (1+e^r)\,dr = \int_{\pi/2}^t \cos s\,ds$$

so that $y+e^y = 2+e^3+\sin t$. This equation cannot be solved explicitly for y

as a function of t. Indeed, most separable equations cannot be solved explicitly for y as a function of t. Thus, when we say that

$$y+e^{y}=2+e^{3}+\sin t$$

is the solution of this initial-value problem, we really mean that it is an implicit, rather than an explicit solution. This does not present us with any difficulties in applications, since we can always find $y(t)$ numerically with the aid of a digital computer (see Section 1.11).

Example 9. Find all solutions of the differential equation $dy/dt=-t/y$.
Solution. Multiplying both sides of the differential equation by y gives $y\,dy/dt=-t$. Hence

$$y^2+t^2=c^2. \tag{12}$$

Now, the curves (12) are *closed*, and we cannot solve for y as a *single-valued* function of t. The reason for this difficulty, of course, is that the differential equation is not defined when $y=0$. Nevertheless, the circles $t^2+y^2=c^2$ are perfectly well defined, even when $y=0$. Thus, we will call the circles $t^2+y^2=c^2$ *solution curves* of the differential equation

$$dy/dt=-t/y.$$

More generally, we will say that any curve defined by (7) is a solution curve of (4).

Exercises

In each of Problems 1–5, find the general solution of the given differential equation.

1. $(1+t^2)\dfrac{dy}{dt}=1+y^2$. *Hint*: $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$.

2. $\dfrac{dy}{dt}=(1+t)(1+y)$

3. $\dfrac{dy}{dt}=1-t+y^2-ty^2$

4. $\dfrac{dy}{dt}=e^{t+y+3}$

5. $\cos y\sin t\dfrac{dy}{dt}=\sin y\cos t$

In each of Problems 6–12, solve the given initial-value problem, and determine the interval of existence of each solution.

6. $t^2(1+y^2)+2y\dfrac{dy}{dt}=0,\quad y(0)=1$

7. $\dfrac{dy}{dt}=\dfrac{2t}{y+yt^2},\quad y(2)=3$

8. $(1+t^2)^{1/2}\dfrac{dy}{dt}=ty^3(1+t^2)^{-1/2},\quad y(0)=1$

9. $\dfrac{dy}{dt}=\dfrac{3t^2+4t+2}{2(y-1)},\quad y(0)=-1$

10. $\cos y\dfrac{dy}{dt}=\dfrac{-t\sin y}{1+t^2},\quad y(1)=\pi/2$

11. $\dfrac{dy}{dt}=k(a-y)(b-y),\quad y(0)=0,\ a,b>0$

12. $3t\dfrac{dy}{dt}=y\cos t,\quad y(1)=0$

13. Any differential equation of the form $dy/dt=f(y)$ is separable. Thus, we can solve all those first-order differential equations in which time does not appear explicitly. Now, suppose we have a differential equation of the form $dy/dt=f(y/t)$, such as, for example, the equation $dy/dt=\sin(y/t)$. Differential equations of this form are called homogeneous equations. Since the right-hand side only depends on the single variable y/t, it suggests itself to make the substitution $y/t=v$ or $y=tv$.

(a) Show that this substitution replaces the equation $dy/dt=f(y/t)$ by the equivalent equation $t\,dv/dt+v=f(v)$, which is separable.

(b) Find the general solution of the equation $dy/dt=2(y/t)+(y/t)^2$.

14. Determine whether each of the following functions of t and y can be expressed as a function of the single variable y/t.

(a) $\dfrac{y^2+2ty}{y^2}$ (b) $\dfrac{y^3+t^3}{yt^2+y^3}$

(c) $\dfrac{y^3+t^3}{t^2+y^3}$ (d) $\ln y-\ln t+\dfrac{t+y}{t-y}$

(e) $\dfrac{e^{t+y}}{e^{t-y}}$ (f) $\ln\sqrt{t+y}-\ln\sqrt{t-y}$

(g) $\sin\dfrac{t+y}{t-y}$ (h) $\dfrac{(t^2+7ty+9y^2)^{1/2}}{3t+5y}$.

15. Solve the initial-value problem $t(dy/dt)=y+\sqrt{t^2+y^2}$, $y(1)=0$.

Find the general solution of the following differential equations.

16. $2ty\dfrac{dy}{dt}=3y^2-t^2$

17. $(t-\sqrt{ty}\,)\dfrac{dy}{dt}=y$

18. $\dfrac{dy}{dt}=\dfrac{t+y}{t-y}$.

19. $e^{t/y}(y-t)\dfrac{dy}{dt}+y(1+e^{t/y})=0$

$$\left[\textit{Hint: } \int\frac{v-1}{ve^{-1/v}+v^2}\,dv=\ln(1+ve^{1/v})\right]$$

20. Consider the differential equation

$$\frac{dy}{dt} = \frac{t+y+1}{t-y+3}. \qquad (*)$$

We could solve this equation if the constants 1 and 3 were not present. To eliminate these constants, we make the substitution $t = T + h$, $y = Y + k$.

(a) Determine h and k so that (*) can be written in the form $dY/dT = (T+Y)/(T-Y)$.

(b) Find the general solution of (*). (See Exercise 18).

21. (a) Prove that the differential equation

$$\frac{dy}{dt} = \frac{at+by+m}{ct+dy+n}$$

where a, b, c, d, m, and n are constants, can always be reduced to $dy/dt = (at+by)/(ct+dy)$ if $ad - bc \neq 0$.

(b) Solve the above equation in the special case that $ad = bc$.

Find the general solution of the following equations.

22. $(1+t-2y)+(4t-3y-6)dy/dt=0$

23. $(t+2y+3)+(2t+4y-1)dy/dt=0$

1.5 Population models

In this section we will study first-order differential equations which govern the growth of various species. At first glance it would seem impossible to model the growth of a species by a differential equation since the population of any species always changes by integer amounts. Hence the population of any species can never be a differentiable function of time. However, if a given population is very large and it is suddenly increased by one, then the change is very small compared to the given population. Thus, we make the approximation that large populations change continuously and even differentiably with time.

Let $p(t)$ denote the population of a given species at time t and let $r(t,p)$ denote the difference between its birth rate and its death rate. If this population is isolated, that is, there is no net immigration or emigration, then dp/dt, the rate of change of the population, equals $rp(t)$. In the most simplistic model we assume that r is constant, that is, it does not change with either time or population. Then, we can write down the following differential equation governing population growth:

$$\frac{dp(t)}{dt} = ap(t), \qquad a = \text{constant}.$$

This is a linear equation and is known as the Malthusian law of population growth. If the population of the given species is p_0 at time t_0, then $p(t)$ satisfies the initial-value problem $dp(t)/dt = ap(t)$, $p(t_0) = p_0$. The solution of this initial-value problem is $p(t) = p_0 e^{a(t-t_0)}$. Hence any species satisfying the Malthusian law of population growth grows exponentially with time.

Now, we have just formulated a very simple model for population growth; so simple, in fact, that we have been able to solve it completely in a few lines. It is important, therefore, to see if this model, with its simplicity, has any relationship at all with reality. Let $p(t)$ denote the human population of the earth at time t. It was estimated that the earth's human population was increasing at an average rate of 2% per year during the period 1960–1970. Let us start in the middle of this decade on January 1, 1965, at which time the U.S. Department of Commerce estimated the earth's population to be 3.34 billion people. Then, $t_0 = 1965$, $p_0 = 3.34 \times 10^9$ and $a = .02$, so that

$$p(t) = (3.34)10^9 e^{.02(t-1965)}.$$

One way of checking the accuracy of this formula is to compute the time required for the population of the earth to double, and then compare it to the observed value of 35 years. Our formula predicts that the population of the earth will double every T years, where

$$e^{.02T} = 2.$$

Taking logarithms of both sides of this equation gives $.02T = \ln 2$, so that

$$T = 50\ln 2 \simeq 34.6 \text{ years.}$$

This is in excellent agreement with the observed value. On the other hand, though, let us look into the distant future. Our equation predicts that the earth's population will be 200,000 billion in the year 2515, 1,800,000 billion in the year 2625, and 3,600,000 billion in the year 2660. These are astronomical numbers whose significance is difficult to gauge. The total surface of this planet is approximately 1,860,000 billion square feet. Eighty percent of this surface is covered by water. Assuming that we are willing to live on boats as well as land, it is easy to see that by the year 2515 there will be only 9.3 square feet per person; by 2625 each person will have only one square foot on which to stand; and by 2660 we will be standing two deep on each other's shoulders.

It would seem therefore, that this model is unreasonable and should be thrown out. However, we cannot ignore the fact that it offers exceptional agreement in the past. Moreover, we have additional evidence that populations do grow exponentially. Consider the Microtus Arvallis Pall, a small rodent which reproduces very rapidly. We take the unit of time to be a month, and assume that the population is increasing at the rate of 40% per month. If there are two rodents present initially at time $t=0$, then $p(t)$, the number of rodents at time t, satisfies the initial-value problem

$$dp(t)/dt = 0.4p, \qquad p(0)=2.$$

Consequently,

$$p(t) = 2e^{0.4t}. \tag{1}$$

Table 1 compares the observed population with the population calculated from Equation (1).

Table 1. The growth of Microtus Arvallis Pall.

Months	0	2	6	10
p Observed	2	5	20	109
p Calculated	2	4.5	22	109.1

As one can see, there is excellent agreement.

Remark. In the case of the Microtus Arvallis Pall, p observed is very accurate since the pregnancy period is three weeks and the time required for the census taking is considerably less. If the pregnancy period were very short then p observed could not be accurate since many of the pregnant rodents would have given birth before the census was completed.

The way out of our dilemma is to observe that linear models for population growth are satisfactory *as long as* the population is not too large. When the population gets extremely large though, these models cannot be very accurate, since they do not reflect the fact that individual members are now competing with each other for the limited living space, natural resources and food available. Thus, we must add a competition term to our linear differential equation. A suitable choice of a competition term is $-bp^2$, where b is a constant, since the statistical average of the number of encounters of two members per unit time is proportional to p^2. We consider, therefore, the modified equation

$$\frac{dp}{dt} = ap - bp^2.$$

This equation is known as the logistic law of population growth and the numbers a,b are called the vital coefficients of the population. It was first introduced in 1837 by the Dutch mathematical-biologist Verhulst. Now, the constant b, in general, will be very small compared to a, so that if p is not too large then the term $-bp^2$ will be negligible compared to ap and the population will grow exponentially. However, when p is very large, the term $-bp^2$ is no longer negligible, and thus serves to slow down the rapid rate of increase of the population. Needless to say, the more industrialized a nation is, the more living space it has, and the more food it has, the smaller the coefficient b is.

Let us now use the logistic equation to predict the future growth of an isolated population. If p_0 is the population at time t_0, then $p(t)$, the population at time t, satisfies the initial-value problem

$$\frac{dp}{dt}=ap-bp^2, \qquad p(t_0)=p_0.$$

This is a separable differential equation, and from Equation (10), Section 1.4,

$$\int_{p_0}^{p}\frac{dr}{ar-br^2}=\int_{t_0}^{t}ds=t-t_0.$$

To integrate the function $1/(ar-br^2)$ we resort to partial fractions. Let

$$\frac{1}{ar-br^2}\equiv\frac{1}{r(a-br)}=\frac{A}{r}+\frac{B}{a-br}.$$

To find A and B, observe that

$$\frac{A}{r}+\frac{B}{a-br}=\frac{A(a-br)+Br}{r(a-br)}=\frac{Aa+(B-bA)r}{r(a-br)}.$$

Therefore, $Aa+(B-bA)r=1$. Since this equation is true for all values of r, we see that $Aa=1$ and $B-bA=0$. Consequently, $A=1/a$, $B=b/a$, and

$$\int_{p_0}^{p}\frac{dr}{r(a-br)}=\frac{1}{a}\int_{p_0}^{p}\left(\frac{1}{r}+\frac{b}{a-br}\right)dr$$

$$=\frac{1}{a}\left[\ln\frac{p}{p_0}+\ln\left|\frac{a-bp_0}{a-bp}\right|\right]=\frac{1}{a}\ln\frac{p}{p_0}\left|\frac{a-bp_0}{a-bp}\right|.$$

Thus,

$$a(t-t_0)=\ln\frac{p}{p_0}\left|\frac{a-bp_0}{a-bp}\right|. \tag{2}$$

Now, it is a simple matter to show (see Exercise 1) that

$$\frac{a-bp_0}{a-bp(t)}$$

is always positive. Hence,

$$a(t-t_0)=\ln\frac{p}{p_0}\frac{a-bp_0}{a-bp}.$$

Taking exponentials of both sides of this equation gives

$$e^{a(t-t_0)} = \frac{p}{p_0}\frac{a-bp_0}{a-bp},$$

or

$$p_0(a-bp)e^{a(t-t_0)} = (a-bp_0)p.$$

Bringing all terms involving p to the left-hand side of this equation, we see that

$$\left[a-bp_0+bp_0e^{a(t-t_0)}\right]p(t) = ap_0e^{a(t-t_0)}.$$

Consequently,

$$p(t) = \frac{ap_0e^{a(t-t_0)}}{a-bp_0+bp_0e^{a(t-t_0)}} = \frac{ap_0}{bp_0+(a-bp_0)e^{-a(t-t_0)}}. \tag{3}$$

Let us now examine Equation (3) to see what kind of population it predicts. Observe that as $t\to\infty$,

$$p(t)\to\frac{ap_0}{bp_0} = \frac{a}{b}.$$

Thus, *regardless of its initial value, the population always approaches the limiting value* a/b. Next, observe that $p(t)$ is a monotonically increasing function of time if $0<p_0<a/b$. Moreover, since

$$\frac{d^2p}{dt^2} = a\frac{dp}{dt} - 2bp\frac{dp}{dt} = (a-2bp)p(a-bp),$$

we see that dp/dt is increasing if $p(t)<a/2b$, and that dp/dt is decreasing if $p(t)>a/2b$. Hence, if $p_0<a/2b$, the graph of $p(t)$ must have the form given in Figure 1. Such a curve is called a logistic, or S-shaped curve. From its shape we conclude that the time period before the population reaches half its limiting value is a period of accelerated growth. After this point, the rate of growth decreases and in time reaches zero. This is a period of diminishing growth.

These predictions are borne out by an experiment on the protozoa Paramecium caudatum performed by the mathematical biologist G. F. Gause. Five individuals of Paramecium were placed in a small test tube containing 0.5 cm^3 of a nutritive medium, and for six days the number of individuals in every tube was counted daily. The Paramecium were found to increase at a rate of 230.9% per day when their numbers were low. The number of individuals increased rapidly at first, and then more slowly, until towards the fourth day it attained a maximum level of 375, saturating the test tube. From this data we conclude that if the Paramecium caudatum grow according to the logistic law $dp/dt = ap - bp^2$, then $a = 2.309$ and

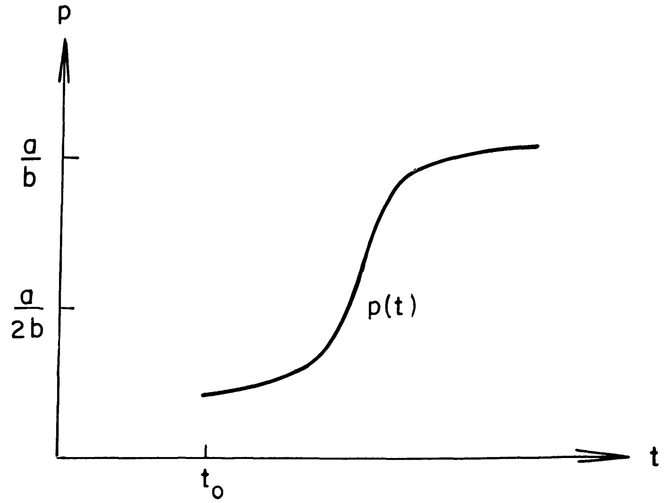

Figure 1. Graph of $p(t)$

$b=2.309/375$. Consequently, the logistic law predicts that

$$p(t)=\frac{(2.309)5}{\frac{(2.309)5}{375}+\left(2.309-\frac{(2.309)5}{375}\right)e^{-2.309t}}$$
$$=\frac{375}{1+74e^{-2.309t}}. \tag{4}$$

(We have taken the initial time t_0 to be 0.) Figure 2 compares the graph of $p(t)$ predicted by Equation (4) with the actual measurements, which are denoted by o. As can be seen, the agreement is remarkably good.

In order to apply our results to predict the future human population of the earth, we must estimate the vital coefficients a and b in the logistic equation governing its growth. Some ecologists have estimated that the natural value of a is 0.029. We also know that the human population was increasing at the rate of 2% per year when the population was $(3.34)10^9$. Since $(1/p)(dp/dt)=a-bp$, we see that

$$0.02=a-b\,(3.34)10^9.$$

Consequently, $b=2.695\times10^{-12}$. Thus, according to the logistic law of population growth, the human population of the earth will tend to the limiting value of

$$\frac{a}{b}=\frac{0.029}{2.695\times10^{-12}}=10.76\text{ billion people}$$

Note that according to this prediction, we were still on the accelerated

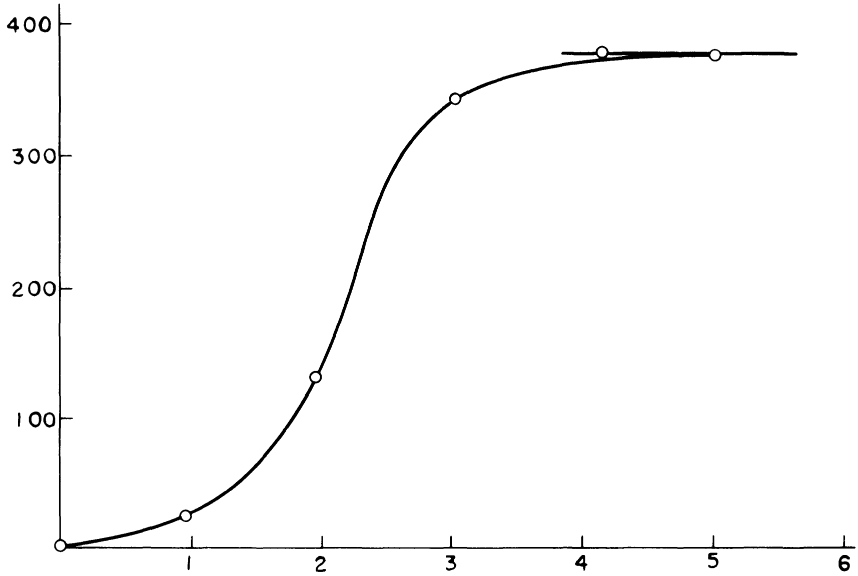

Figure 2. The growth of paramecium

growth portion of the logistic curve in 1965, since we had not yet attained half the limiting population predicted for us.

Remark. A student of mine once suggested that we use Equation (3) to find the time t at which $p(t)=2$, and then we can deduce how long ago mankind appeared on earth. On the surface this seems like a fantastic idea. However, we cannot travel that far backwards into the past, since our model is no longer accurate when the population is small.

As another verification of the validity of the logistic law of population growth, we consider the equation

$$p(t)=\frac{197{,}273{,}000}{1+e^{-0.03134(t-1913.25)}} \tag{5}$$

which was introduced by Pearl and Reed as a model of the population growth of the United States. First, using the census figures for the years 1790, 1850, and 1910, Pearl and Reed found from (3) (see Exercise 2a) that $a=0.03134$ and $b=(1.5887)10^{-10}$. Then (see Exercise 2b), Pearl and Reed calculated that the population of the United States reached half its limiting population of $a/b=197{,}273{,}000$ in April 1913. Consequently (see Exercise 2c), we can rewrite (3) in the simpler form (5).

Table 2 below compares Pearl and Reed's predictions with the observed values of the population of the United States. These results are remarkable,

Table 2. Population of the U.S. from 1790–1950. (The last 4 entries were added by the Dartmouth College Writing Group.)

	Actual	Predicted	Error	%
1790	3,929,000	3,929,000	0	0.0
1800	5,308,000	5,336,000	28,000	0.5
1810	7,240,000	7,228,000	−12,000	−0.2
1820	9,638,000	9,757,000	119,000	1.2
1830	12,866,000	13,109,000	243,000	1.9
1840	17,069,000	17,506,000	437,000	2.6
1850	23,192,000	23,192,000	0	0.0
1860	31,443,000	30,412,000	−1,031,000	−3.3
1870	38,558,000	39,372,000	814,000	2.1
1880	50,156,000	50,177,000	21,000	0.0
1890	62,948,000	62,769,000	−179,000	−0.3
1900	75,995,000	76,870,000	875,000	1.2
1910	91,972,000	91,972,000	0	0.0
1920	105,711,000	107,559,000	1,848,000	1.7
1930	122,775,000	123,124,000	349,000	0.3
1940	131,669,000	136,653,000	4,984,000	3.8
1950	150,697,000	149,053,000	−1,644,000	−1.1

especially since we have not taken into account the large waves of immigration into the United States, and the fact that the United States was involved in five wars during this period.

In 1845 Verhulst prophesied a maximum population for Belgium of 6,600,000, and a maximum population for France of 40,000,000. Now, the population of Belgium in 1930 was already 8,092,000. This large discrepancy would seem to indicate that the logistic law of population growth is very inaccurate, at least as far as the population of Belgium is concerned. However, this discrepancy can be explained by the astonishing rise of industry in Belgium, and by the acquisition of the Congo which secured for the country sufficient additional wealth to support the extra population. Thus, after the acquisition of the Congo, and the astonishing rise of industry in Belgium, Verhulst should have lowered the vital coefficient b.

On the other hand, the population of France in 1930 was in remarkable agreement with Verhulst's forecast. Indeed, we can now answer the following tantalizing paradox: Why was the population of France increasing extremely slowly in 1930 while the French population of Canada was increasing very rapidly? After all, they are the same people! The answer to this paradox, of course, is that the population of France in 1930 was very near its limiting value and thus was far into the period of diminishing growth, while the population of Canada in 1930 was still in the period of accelerated growth.

Remark 1. It is clear that technological developments, pollution considerations and sociological trends have significant influence on the vital coefficients a and b. Therefore, they must be re-evaluated every few years.

Remark 2. To derive more accurate models of population growth, we should not consider the population as made up of one homogeneous group of individuals. Rather, we should subdivide it into different age groups. We should also subdivide the population into males and females, since the reproduction rate in a population usually depends more on the number of females than on the number of males.

Remark 3. Perhaps the severest criticism leveled at the logistic law of population growth is that some populations have been observed to fluctuate periodically between two values, and any type of fluctuation is ruled out in a logistic curve. However, some of these fluctuations can be explained by the fact that when certain populations reach a sufficiently high density, they become susceptible to epidemics. The epidemic brings the population down to a lower value where it again begins to increase, until when it is large enough, the epidemic strikes again. In Exercise 10 we derive a model to describe this phenomenon, and we apply this model in Exercise 11 to explain the sudden appearance and disappearance of hordes of small rodents.

Epilog. The following article appeared in the New York Times on March 26, 1978, and was authored by Nick Eberstadt.

The gist of the following article is that it is very difficult, using statistical methods alone, to make accurate population projections even 30 years into the future. In 1970, demographers at the United Nations projected the population of the earth to be 6.5 billion people by the year 2000. Only six years later, this forecast was revised downward to 5.9 billion people.

Let us now use Equation (3) to predict the population of the earth in the year 2000. Setting $a=.029$, $b=2.695\times10^{-12}$, $p_0=3.34\times10^9$, $t_0=1965$, and $t=2{,}000$ gives

$$\begin{aligned} p(2000) &= \frac{(.029)(3.34)10^9}{.009+(.02)e^{-(.029)35}} \\ &= \frac{29(3.34)}{9+20e^{-1.015}}10^9 \\ &= 5.96 \text{ billion people!} \end{aligned}$$

This is another spectacular application of the logistic equation.

World Population Figures Are Misleading

The rate of world population growth has risen fairly steadily for most of man's history, but within the last decade it has peaked, and now appears to be declining. How has this happened, and why?

The "how" is fairly easy. It is not because famines and ecological catastrophes have elevated the death rates. Rather, a large and generally unexpected decrease in fertility in the less developed countries has taken place. From 1970 to 1977 birth rates in the less developed world (excluding China) fell from about 42 to nearly 36 per thousand. This is still higher than the 17 per thousand in developed countries, but the rate of fertility decline appears to be accelerating: the six-point drop of the past seven years compares with a two-point decline for the previous twenty.

This fertility decline has been a very uneven process. The average birth-rate drop of about 13 percent since 1970 in poor world birth rates reflects a very rapid decline in certain countries, while a great many others have remained almost totally unaffected.

Why fertility has dropped so rapidly in the past decade—and why it has dropped so dramatically in some places, but not in others—is far more difficult to explain. Demographers and sociologists offer explanations having to do with social change in the poor world. Unfortunately, these partial explanations are more often theory than tested fact, and there seems to be an exception for almost every rule.

The debate over family planning is characteristic. Surveys show that in some nations as many as a fifth of the children were "mistakes" who presumably would not have been born if parents had had better contraceptives. Family planning experts such as Parker Mauldin of the Population Council have pointed out that no poor nation without an active family planning program has significantly lowered its fertility. On the other hand, such sociologists as William Petersen of Ohio State University attribute the population decline in these nations to social and economic development rather than increased contraceptive use, arguing that international "population control" programs have usually been clumsy and insensitive (or worse), and that in any event even a well-received change in contraceptive "technology" does not necessarily influence parents to want fewer children.

The effects of income distribution are less vociferously debated, but are almost as mysterious. James Kocher and Robert Repetto, both of Harvard, have argued that more equitable income distribution in less developed countries contributes to fertility decline. They have pointed out that such countries as Sri Lanka, South Korea, Cuba and China have seen their fertility rates fall as their income distribution became more nearly equal. Improving a nation's income distribution, however, appears to be neither a necessary nor a sufficient condition for inducing a fertility drop. Income distribution in Burma, for example, has presumably equalized somewhat under 30 years of homemade socialism, but birth rates have hardly fallen at all, while Mexico and Colombia, with highly unequal income distributions, have found their birth rates plummeting in recent years.

One key to changes in fertility levels may be the economic costs and benefits from children. In peasant societies, where children are afforded few amenities and start work young, they may become economic assets early: A recent study in Bangladesh by Mead Cain of the Population Council put the age for boys at 12. Furthermore, children (or more precisely, sons) may also serve as social security and unemployment insurance when parents become too old and weak to work, or when work is unavailable. Where birth rates in poor countries are dropping, social and economic development may be making children less necessary as sources of income and security, but so little work has been done in this area that this is still just a reasonable speculation.

Some of the many other factors whose effects on fertility have been studied are urbanization, education, occupational structure, public health and the status of women. One area which population experts seem to have shied away from, however, is the non-quantifiable realm of attitudes, beliefs and values which may have had much to do with the recent changes in the decisions of hundreds of millions of couples. Cultural differences, ethnic conflicts, psychological, ideological and even political changes could clearly have effects on fertility. As Maris Vonovskis of the House Select Committee on Population has said, just because you can't measure something doesn't mean it isn't important.

What does the decline in fertility mean about future levels of population? Obviously, if the drop continues, population growth will be slower than previously anticipated, and world population will eventually stabilize at a lower level. Only five years ago the United Nations "medium variant" projection for world population in the year 2000 was 6.5 billion; last year this was dropped more than 200 million, and recent work by Gary Littman and Nathan Keyfitz at the Harvard Center for Population Studies shows that in the light of recent changes, one might easily drop it 400 million more.

Population projections, however, are a very tricky business. To begin with, the figures for today's population, upon which tomorrow's projections must be based, contain large margins of error. For example, estimates for China's population run from 750 million to over 950 million. By the account of John Durand of the University of Pennsylvania, the margins of error for world population add up to over 200 million; historian Fernand Braudel puts the margin of error at 10 percent, which, given the world's approximate present population, means about 400 million people.

Population projections inspire even less confidence than population estimates, for they hinge on predicting birth and death rates for the future. These can change rapidly and unexpectedly: two extreme examples are Sri Lanka's 34 percent drop in the death rate in just two years, and Japan's 50 percent drop in the birth rate in 10. "Medium variant" U.N. projections computed just 17 years before 1975 overestimate Russia's population by 10 to 20 million, and underestimate India's by 50 million. Even projections for the United States done in 1966 overestimate its population only nine years later by over 10 million. Enormous as that gap may sound, it seems quite small next to those of the 1930's estimates which extrapolated low Depression era birth rates into an American population peaking at 170 million in the late 1970's (the population now is over 220 million), and then declining!

Could birth rates in the less developed world, which now appear to be declining at an accelerating pace, suddenly stabilize, or even rise again? This could theoretically happen. Here are four of the many reasons: 1) The many countries where fertility has as yet been unaffected by the decline might simply continue to be unaffected far into the future. 2) Since sterility and infertility are widespread in many of the poorest and most disease-ridden areas of the world, improvements in health and nutrition there could raise birth rates. 3) The Gandhi regime's cold-hearted and arbitrary mass sterilization regimen may have hardened that sixth of the world against future family limitation messages. 4) If John Aird of the Department of Commerce and others are correct that China's techniques of political mobilization and social persuasion have induced many parents to have fewer children than they actually want, a relaxation of these rules for whatever reasons might make the birth rate of China's enormous population rise. One of the only long-term rules about population projections which has held up is that within their limits of accuracy (about five years in the future) they can tell nothing interesting, and when they start giving interesting results, they are no longer accurate.

Nick Eberstadt is an affiliate of the Harvard Center for Population Studies.

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Exercises

1. Prove that $(a-bp_0)/(a-bp(t))$ is positive for $t_0<t<\infty$. *Hint*: Use Equation (2) to show that $p(t)$ can never equal a/b if $p_0 \neq a/b$.

2. (a) Choose 3 times t_0, t_1, and t_2, with $t_1-t_0=t_2-t_1$. Show that (3) determines a and b uniquely in terms of t_0, $p(t_0)$, t_1, $p(t_1)$, t_2, and $p(t_2)$.

(b) Show that the period of accelerated growth for the United States ended in April, 1913.

(c) Let a population $p(t)$ grow according to the logistic law (3), and let $\bar{t}$ be the time at which half the limiting population is achieved. Show that

$$p(t)=\frac{a/b}{1+e^{-a(t-\bar{t})}}.$$

3. In 1879 and 1881 a number of yearling bass were seined in New Jersey, taken across the continent in tanks by train, and planted in San Francisco Bay. A total of only 435 Striped Bass survived the rigors of these two trips. Yet, in 1899, the commercial net catch alone was 1,234,000 pounds. Since the growth of this population was so fast, it is reasonable to assume that it obeyed the Malthusian law $dp/dt=ap$. Assuming that the average weight of a bass fish is three pounds, and that in 1899 every tenth bass fish was caught, find a lower bound for a.

4. Suppose that a population doubles its original size in 100 years, and triples it in 200 years. Show that this population cannot satisfy the Malthusian law of population growth.

5. Assume that $p(t)$ satisfies the Malthusian law of population growth. Show that the increases in p in successive time intervals of equal duration form the terms of a geometric progression. This is the source of Thomas Malthus' famous dictum "Population when unchecked increases in a geometrical ratio. Subsistence increases only in an arithmetic ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison of the second."

6. A population grows according to the logistic law, with a limiting population of 5×10^8 individuals. When the population is low it doubles every 40 minutes. What will the population be after two hours if initially it is (a) 10^8, (b) 10^9?

7. A family of salmon fish living off the Alaskan Coast obeys the Malthusian law of population growth $dp(t)/dt=0.003p(t)$, where t is measured in minutes. At time $t=0$ a group of sharks establishes residence in these waters and begins attacking the salmon. The rate at which salmon are killed by the sharks is $0.001p^2(t)$, where $p(t)$ is the population of salmon at time t. Moreover, since an undesirable element has moved into their neighborhood, 0.002 salmon per minute leave the Alaskan waters.

(a) Modify the Malthusian law of population growth to take these two factors into account.

(b) Assume that at time $t=0$ there are one million salmon. Find the population $p(t)$. What happens as $t\to\infty$?
(c) Show that the above model is really absurd. Hint: Show, according to this model, that the salmon population decreases from one million to about one thousand in one minute.

8. The population of New York City would satisfy the logistic law

$$\frac{dp}{dt}=\frac{1}{25}p-\frac{1}{(25)10^6}p^2,$$

where t is measured in years, if we neglected the high emigration and homicide rates.
(a) Modify this equation to take into account the fact that 9,000 people per year move from the city, and 1,000 people per year are murdered.
(b) Assume that the population of New York City was 8,000,000 in 1970. Find the population for all future time. What happens as $t\to\infty$?

9. An initial population of 50,000 inhabits a microcosm with a carrying capacity of 100,000. After five years, the population has increased to 60,000. Show that the natural growth rate a for this population is $(1/5)\ln 3/2$.

10. We can model a population which becomes susceptible to epidemics in the following manner. Assume that our population is originally governed by the logistic law

$$\frac{dp}{dt}=ap-bp^2 \tag{i}$$

and that an epidemic strikes as soon as p reaches a certain value Q, with Q less than the limiting population a/b. At this stage the vital coefficients become $A<a$, $B<b$, and Equation (i) is replaced by

$$\frac{dp}{dt}=Ap-Bp^2. \tag{ii}$$

Suppose that $Q>A/B$. The population then starts decreasing. A point is reached when the population falls below a certain value $q>A/B$. At this moment the epidemic ceases and the population again begins to grow following Equation (i), until the incidence of a fresh epidemic. In this way there are periodic fluctuations of p between q and Q. We now indicate how to calculate the period T of these fluctuations.
(a) Show that the time T_1 taken by the first part of the cycle, when p increases from q to Q is given by

$$T_1=\frac{1}{a}\ln\frac{Q(a-bq)}{q(a-bQ)}.$$

(b) Show that the time T_2 taken by the second part of the cycle, when p decreases from Q to q is given by

$$T_2=\frac{1}{A}\ln\frac{q(QB-A)}{Q(qB-A)}.$$

Thus, the time for the entire cycle is T_1+T_2.

11. It has been observed that plagues appear in mice populations whenever the population becomes too large. Further, a local increase of density attracts predators in large numbers. These two factors will succeed in destroying 97–98% of a population of small rodents in two or three weeks, and the density then falls to a level at which the disease cannot spread. The population, reduced to 2% of its maximum, finds its refuges from the predators sufficient, and its food abundant. The population therefore begins to grow again until it reaches a level favorable to another wave of disease and predation. Now, the speed of reproduction in mice is so great that we may set $b=0$ in Equation (i) of Exercise 7. In the second part of the cycle, on the contrary, A is very small in comparison with B, and it may be neglected therefore in Equation (ii).

(a) Under these assumptions, show that

$$T_1=\frac{1}{a}\ln\frac{Q}{q} \quad \text{and} \quad T_2=\frac{Q-q}{qQB}.$$

(b) Assuming that T_1 is approximately four years, and Q/q is approximately fifty, show that a is approximately one. This value of a, incidentally, corresponds very well with the rate of multiplication of mice in natural circumstances.

12. There are many important classes of organisms whose birth rate is *not* proportional to the population size. Suppose, for example, that each member of the population requires a partner for reproduction, and that each member relies on chance encounters for meeting a mate. If the expected number of encounters is proportional to the product of the numbers of males and females, and if these are equally distributed in the population, then the number of encounters, and hence the birthrate too, is proportional to p^2. The death rate is still proportional to p. Consequently, the population size $p(t)$ satisfies the differential equation

$$\frac{dp}{dt}=bp^2-ap, \quad a,b>0.$$

Show that $p(t)$ approaches 0 as $t\to\infty$ if $p_0<a/b$. Thus, once the population size drops below the critical size a/b, the population tends to extinction. Thus, a species is classified endangered if its current size is perilously close to its critical size.

1.6 The spread of technological innovations

Economists and sociologists have long been concerned with how a technological change, or innovation, spreads in an industry. Once an innovation is introduced by one firm, how soon do others in the industry come to adopt it, and what factors determine how rapidly they follow? In this section we construct a model of the spread of innovations among farmers, and then show that this same model also describes the spread of innovations in such diverse industries as bituminous coal, iron and steel, brewing, and railroads.

Assume that a new innovation is introduced into a fixed community of N farmers at time $t=0$. Let $p(t)$ denote the number of farmers who have adopted at time t. As in the previous section, we make the approximation that $p(t)$ is a continuous function of time, even though it obviously changes by integer amounts. The simplest realistic assumption that we can make concerning the spread of this innovation is that a farmer adopts the innovation only after he has been told of it by a farmer who has already adopted. Then, the number of farmers Δp who adopt the innovation in a small time interval Δt is directly proportional to the number of farmers p who have already adopted, and the number of farmers $N-p$ who are as yet unaware. Hence, $\Delta p = cp(N-p)\Delta t$ or $\Delta p/\Delta t = cp(N-p)$ for some positive constant c. Letting $\Delta t \to 0$, we obtain the differential equation

$$\frac{dp}{dt} = cp(N-p). \tag{1}$$

This is the logistic equation of the previous section if we set $a = cN$, $b = c$. Assuming that $p(0) = 1$; i.e., one farmer has adopted the innovation at time $t = 0$, we see that $p(t)$ satisfies the initial-value problem

$$\frac{dp}{dt} = cp(N-p), \qquad p(0) = 1. \tag{2}$$

The solution of (2) is

$$p(t) = \frac{Ne^{cNt}}{N-1+e^{cNt}} \tag{3}$$

which is a logistic function (see Section 1.5). Hence, our model predicts that the adoption process accelerates up to that point at which half the community is aware of the innovation. After this point, the adoption process begins to decelerate until it eventually reaches zero.

Let us compare the predictions of our model with data on the spread of two innovations through American farming communities in the middle 1950's. Figure 1 represents the cumulative number of farmers in Iowa during 1944–1955 who adopted 2,4-D weed spray, and Figure 2 represents the cumulative percentage of corn acreage in hybrid corn in three American states during the years 1934–1958. The circles in these figures are the actual measurements, and the graphs were obtained by connecting these measurements with straight lines. As can be seen, these curves have all the properties of logistic curves, and on the whole, offer very good agreement with our model. However, there are two discrepancies. First, the actual point at which the adoption process ceases to accelerate is not always when fifty per cent of the population has adopted the innovation. As can be seen from Figure 2, the adoption process for hybrid corn began to decelerate in Alabama only after nearly sixty per cent of the farmers had adopted the innovation. Second, the agreement with our model is much better in the later stages of the adoption process than in the earlier stages.

The source of the second discrepancy is our assumption that a farmer only learns of an innovation through contact with another farmer. This is not entirely true. Studies have shown that mass communication media such

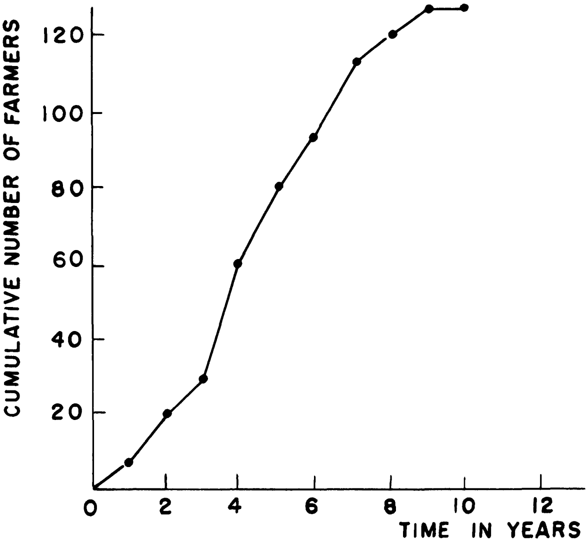


Figure 1. Cumulative number of farmers who adopted 2,4-D weed spray in Iowa

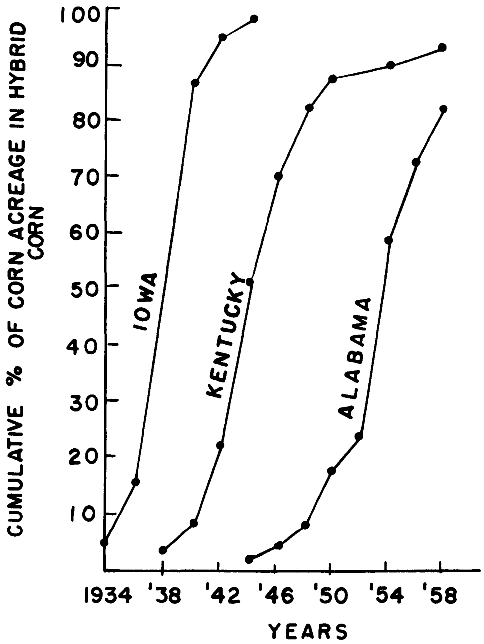


Figure 2. Cumulative percentage of corn acreage in hybrid corn in three American states

as radio, television, newspapers and farmers' magazines play a large role in the early stages of the adoption process. Therefore, we must add a term to the differential equation (1) to take this into account. To compute this term, we assume that the number of farmers Δp who learn of the innovation through the mass communication media in a short period of time Δt is proportional to the number of farmers who do not yet know; i.e.,

$$\Delta p = c'(N-p)\Delta t$$

for some positive constant c'. Letting $\Delta t \to 0$, we see that $c'(N-p)$ farmers, per unit time, learn of the innovation through the mass communication media. Thus, if $p(0)=0$, then $p(t)$ satisfies the initial-value problem

$$\frac{dp}{dt} = cp(N-p) + c'(N-p), \qquad p(0)=0. \tag{4}$$

The solution of (4) is

$$p(t) = \frac{Nc'\left[e^{(c'+cN)t}-1\right]}{cN + c'e^{(c'+cN)t}}, \tag{5}$$

and in Exercises 2 and 3 we indicate how to determine the shape of the curve (5).

The corrected curve (5) now gives remarkably good agreement with Figures 1 and 2, for suitable choices of c and c'. However, (see Exercise 3c) it still doesn't explain why the adoption of hybrid corn in Alabama only began to decelerate after sixty per cent of the farmers had adopted the innovation. This indicates, of course, that other factors, such as the time interval that elapses between when a farmer first learns of an innovation and when he actually adopts it, may play an important role in the adoption process, and must be taken into account in any model.

We would now like to show that the differential equation

$$dp/dt = cp(N-p)$$

also governs the rate at which firms in such diverse industries as bituminous coal, iron and steel, brewing, and railroads adopted several major innovations in the first part of this century. This is rather surprising, since we would expect that the number of firms adopting an innovation in one of these industries certainly depends on the profitability of the innovation and the investment required to implement it, and we haven't mentioned these factors in deriving Equation (1). However, as we shall see shortly, these two factors are incorporated in the constant c.

Let n be the total number of firms in a particular industry who have adopted an innovation at time t. It is clear that the number of firms Δp who adopt the innovation in a short time interval Δt is proportional to the number of firms $n-p$ who have not yet adopted; i.e., $\Delta p = \lambda(n-p)\Delta t$. Letting $\Delta t \to 0$, we see that

$$\frac{dp}{dt} = \lambda(n-p).$$

The proportionality factor λ depends on the profitability π of installing this innovation relative to that of alternative investments, the investment s required to install this innovation as a percentage of the total assets of the firm, and the percentage of firms who have already adopted. Thus,

$$\lambda = f(\pi, s, p/n).$$

Expanding f in a Taylor series, and dropping terms of degree more than two, gives

$$\lambda = a_1 + a_2\pi + a_3 s + a_4 \frac{p}{n} + a_5\pi^2 + a_6 s^2 + a_7 \pi s$$
$$+ a_8\pi\left(\frac{p}{n}\right) + a_9 s\left(\frac{p}{n}\right) + a_{10}\left(\frac{p}{n}\right)^2.$$

In the late 1950's, Edwin Mansfield of Carnegie Mellon University investigated the spread of twelve innovations in four major industries. From his exhaustive studies, Mansfield concluded that $a_{10}=0$ and

$$a_1 + a_2\pi + a_3 s + a_5\pi^2 + a_6 s^2 + a_7\pi s = 0.$$

Thus, setting

$$k = a_4 + a_8\pi + a_9 s, \tag{6}$$

we see that

$$\frac{dp}{dt} = k\frac{p}{n}(n-p).$$

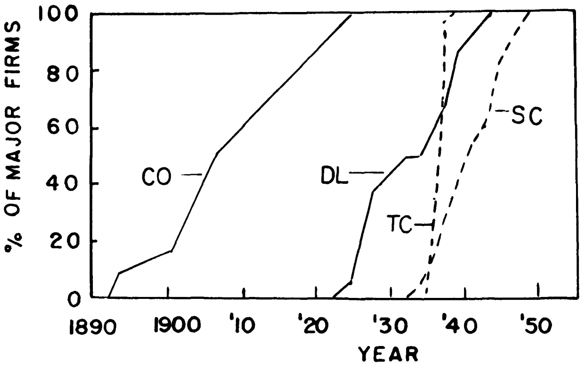
(This is the equation obtained previously for the spread of innovations among farmers, if we set $k/n = c$.) We assume that the innovation is first adopted by one firm in the year t_0. Then, $p(t)$ satisfies the initial-value problem

$$\frac{dp}{dt} = \frac{k}{n}p(n-p), \qquad p(t_0) = 1 \tag{7}$$

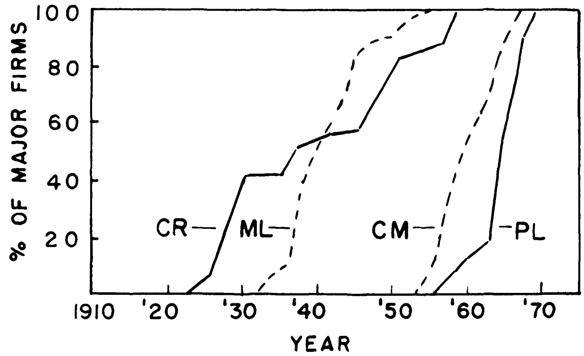
and this implies that

$$p(t) = \frac{n}{1 + (n-1)e^{-k(t-t_0)}}.$$

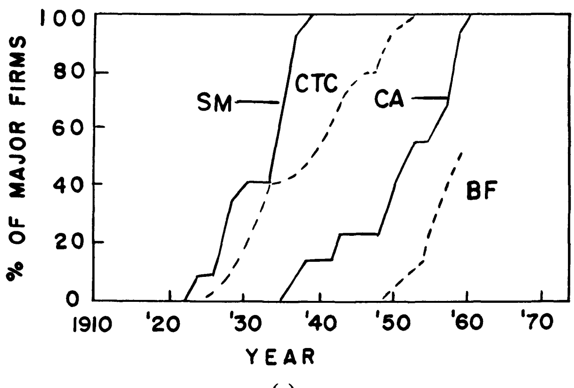
Mansfield studied how rapidly the use of twelve innovations spread from enterprise to enterprise in four major industries—bituminous coal, iron and steel, brewing, and railroads. The innovations are the shuttle car, trackless mobile loader, and continuous mining machine (in bituminous coal); the by-product coke oven, continuous wide strip mill, and continuous annealing line for tin plate (in iron and steel); the pallet-loading machine, tin container, and high speed bottle filler (in brewing); and the diesel locomotive, centralized traffic control, and car retarders (in railroads). His results are described graphically in Figure 3. For all but the by-product coke oven and tin container, the percentages given are for every two years from the year of initial introduction. The length of the interval for the by-product coke oven is about six years, and for the tin container, it is six months. Notice how all these curves have the general appearance of a logistic curve.



(a)



(b)



(c)

Figure 3. Growth in the percentage of major firms that introduced twelve innovations; bituminous coal, iron and steel, brewing, and railroad industries, 1890–1958; (a) By-product coke oven (CO), diesel locomotive (DL), tin container (TC), and shuttle car (SC); (b) Car retarder (CR), trackless mobile loader (ML), continuous mining machine (CM), and pallet-loading machine (PL); (c) Continuous wide-strip mill (SM), centralized traffic control (CTC), continuous annealing (CA), and highspeed bottle filler (BF).

Table 1.

Innovation	n	t_0	a_4	a_8	a_9	π	s
Diesel locomotive	25	1925	−0.59	0.530	−0.027	1.59	0.015
Centralized traffic control	24	1926	−0.59	0.530	−0.027	1.48	0.024
Car retarders	25	1924	−0.59	0.530	−0.027	1.25	0.785
Continuous wide strip mill	12	1924	−0.52	0.530	−0.027	1.87	4.908
By-product coke oven	12	1894	−0.52	0.530	−0.027	1.47	2.083
Continuous annealing	9	1936	−0.52	0.530	−0.027	1.25	0.554
Shuttle car	15	1937	−0.57	0.530	−0.027	1.74	0.013
Trackless mobile loader	15	1934	−0.57	0.530	−0.027	1.65	0.019
Continuous mining machine	17	1947	−0.57	0.530	−0.027	2.00	0.301
Tin container	22	1935	−0.29	0.530	−0.027	5.07	0.267
High speed bottle filler	16	1951	−0.29	0.530	−0.027	1.20	0.575
Pallet-loading machine	19	1948	−0.29	0.530	−0.027	1.67	0.115

For a more detailed comparison of the predictions of our model (7) with these observed results, we must evaluate the constants n, k, and t_0 for each of the twelve innovations. Table 1 gives the value of n, t_0, a_4, a_5, a_9, π, and s for each of the twelve innovations; the constant k can then be computed from Equation (6). As the answers to Exercises 5 and 6 will indicate, our model (7) predicts with reasonable accuracy the rate of adoption of these twelve innovations.

Reference

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Exercises

1. Solve the initial-value problem (2).

2. Let $c=0$ in (5). Show that $p(t)$ increases monotonically from 0 to N, and has no points of inflection.

3. Here is a heuristic argument to determine the behavior of the curve (5). If $c'=0$, then we have a logistic curve, and if $c=0$, then we have the behavior described in Exercise 2. Thus, if c is large relative to c', then we have a logistic curve, and if c is small relative to c' then we have the behavior illustrated in Exercise 2.

(a) Let $p(t)$ satisfy (4). Show that

$$\frac{d^2p}{dt^2} = (N-p)(cp+c')(cN-2cp-c').$$

(b) Show that $p(t)$ has a point of inflection, at which dp/dt achieves a maximum, if, and only if, $c'/c < N$.

(c) Assume that $p(t)$ has a point of inflection at $t = t^*$. Show that $p(t^*) \leqslant N/2$.

4. Solve the initial-value problem (7).

5. It seems reasonable to take the time span between the date when 20% of the firms had introduced the innovation and the date when 80% of the firms had introduced the innovation, as the rate of imitation.

(a) Show from our model that this time span is $4(\ln 2)/k$.

(b) For each of the twelve innovations, compute this time span from the data in Table 1, and compare with the observed value in Figure 3.

6. (a) Show from our model that $(1/k)\ln(n-1)$ years elapse before 50% of the firms introduce an innovation.

(b) Compute this time span for each of the 12 innovations and compare with the observed values in Figure 3.

1.7 An atomic waste disposal problem

For several years the Atomic Energy Commission (now known as the Nuclear Regulatory Commission) had disposed of concentrated radioactive waste material by placing it in tightly sealed drums which were then dumped at sea in fifty fathoms (300 feet) of water. When concerned ecologists and scientists questioned this practice, they were assured by the A.E.C. that the drums would never develop leaks. Exhaustive tests on the drums proved the A.E.C. right. However, several engineers then raised the question of whether the drums could crack from the impact of hitting the ocean floor. "Never," said the A.E.C. "We'll see about that," said the engineers. After performing numerous experiments, the engineers found that the drums could crack on impact if their velocity exceeded forty feet per second. The problem before us, therefore, is to compute the velocity of the drums upon impact with the ocean floor. To this end, we digress briefly to study elementary Newtonian mechanics.

Newtonian mechanics is the study of Newton's famous laws of motion and their consequences. Newton's first law of motion states that an object will remain at rest, or move with constant velocity, if no force is acting on it. A force should be thought of as a push or pull. This push or pull can be exerted directly by something in contact with the object, or it can be exerted indirectly, as the earth's pull of gravity is.

Newton's second law of motion is concerned with describing the motion of an object which is acted upon by several forces. Let $y(t)$ denote the position of the center of gravity of the object. (We assume that the object moves in only one direction.) Those forces acting on the object, which tend to increase y, are considered positive, while those forces tending to decrease y are considered negative. The resultant force F acting on an object is defined to be the sum of all positive forces minus the sum of all negative forces. Newton's second law of motion states that the acceleration d^2y/dt^2 of an object is proportional to the resultant force F acting on it; i.e.,

$$\frac{d^2y}{dt^2}=\frac{1}{m}F. \tag{1}$$

The constant m is the mass of the object. It is related to the weight W of the object by the relation $W=mg$, where g is the acceleration of gravity. Unless otherwise stated, we assume that the weight of an object and the acceleration of gravity are constant. We will also adopt the English system of units, so that t is measured in seconds, y is measured in feet, and F is measured in pounds. The units of m are then slugs, and the gravitational acceleration g equals 32.2 ft/s^2.

Remark. We would prefer to use the mks system of units, where y is measured in meters and F is measured in newtons. The units of m are then kilograms, and the gravitational acceleration equals 9.8 m/s^2. In the third edition of this text, we have changed from the English system of units to the mks system in Section 2.6. However, changing to the mks system in this section would have caused undue confusion to the users of the first and second editions. This is because of the truncation error involved in converting from feet to meters and pounds to newtons.

We return now to our atomic waste disposal problem. As a drum descends through the water, it is acted upon by three forces W, B, and D. The force W is the weight of the drum pulling it down, and in magnitude, $W=527.436$ lb. The force B is the buoyancy force of the water acting on the drum. This force pushes the drum up, and its magnitude is the weight of the water displaced by the drum. Now, the Atomic Energy Commission used 55 gallon drums, whose volume is 7.35 ft^3. The weight of one cubic foot of salt water is 63.99 lb. Hence $B=(63.99)(7.35)=470.327$ lb.

The force D is the drag force of the water acting on the drum; it resists the motion of the drum through the water. Experiments have shown that any medium such as water, oil, and air resists the motion of an object through it. This resisting force acts in the direction opposite the motion, and is usually directly proportional to the velocity V of the object. Thus, $D=cV$, for some positive constant c. Notice that the drag force increases as V increases, and decreases as V decreases. To calculate D, the engineers conducted numerous towing experiments. They concluded that the orientation of the drum had little effect on the drag force, and that

$$D=0.08\,V\frac{(\text{lb})(\text{s})}{\text{ft}}.$$

Now, set $y=0$ at sea level, and let the direction of increasing y be downwards. Then, W is a positive force, and B and D are negative forces. Consequently, from (1),

$$\frac{d^2y}{dt^2}=\frac{1}{m}(W-B-cV)=\frac{g}{W}(W-B-cV).$$

We can rewrite this equation as a first-order linear differential equation for $V=dy/dt$; i.e.,

$$\frac{dV}{dt}+\frac{cg}{W}V=\frac{g}{W}(W-B). \tag{2}$$

Initially, when the drum is released in the ocean, its velocity is zero. Thus, $V(t)$, the velocity of the drum, satisfies the initial-value problem

$$\frac{dV}{dt}+\frac{cg}{W}V=\frac{g}{W}(W-B), \qquad V(0)=0, \tag{3}$$

and this implies that

$$V(t)=\frac{W-B}{c}\left[1-e^{(-cg/W)t}\right]. \tag{4}$$

Equation (4) expresses the velocity of the drum as a function of time. In order to determine the impact velocity of the drum, we must compute the time t at which the drum hits the ocean floor. Unfortunately, though, it is impossible to find t as an explicit function of y (see Exercise 2). Therefore, we cannot use Equation (4) to find the velocity of the drum when it hits the ocean floor. However, the A.E.C. can use this equation to try and prove that the drums do not crack on impact. To wit, observe from (4) that $V(t)$ is a monotonic increasing function of time which approaches the limiting value

$$V_T=\frac{W-B}{c}$$

as t approaches infinity. The quantity V_T is called the terminal velocity of the drum. Clearly, $V(t)\leqslant V_T$, so that the velocity of the drum when it hits the ocean floor is certainly less than $(W-B)/c$. Now, if this terminal velocity is less than 40 ft/s, then the drums could not possibly break on impact. However,

$$\frac{W-B}{c}=\frac{527.436-470.327}{0.08}=713.86 \text{ ft/s},$$

and this is way too large.

It should be clear now that the only way we can resolve the dispute between the A.E.C. and the engineers is to find $v(y)$, the velocity of the drum as a function of position. The function $v(y)$ is very different from the function $V(t)$, which is the velocity of the drum as a function of time. However, these two functions are related through the equation

$$V(t)=v(y(t))$$

if we express y as a function of t. By the chain rule of differentiation, $dV/dt=(dv/dy)(dy/dt)$. Hence

$$\frac{W}{g}\frac{dv}{dy}\frac{dy}{dt}=W-B-cV.$$

But $dy/dt=V(t)=v(y(t))$. Thus, suppressing the dependence of y on t, we see that $v(y)$ satisfies the first-order differential equation

$$\frac{W}{g}v\frac{dv}{dy}=W-B-cv,\quad\text{or}\quad\frac{v}{W-B-cv}\frac{dv}{dy}=\frac{g}{W}.$$

Moreover,

$$v(0)=v(y(0))=V(0)=0.$$

Hence,

$$\int_0^v\frac{r\,dr}{W-B-cr}=\int_0^y\frac{g}{W}ds=\frac{gy}{W}.$$

Now,

$$\begin{aligned}\int_0^v\frac{r\,dr}{W-B-cr}&=\int_0^v\frac{r-(W-B)/c}{W-B-cr}dr+\frac{W-B}{c}\int_0^v\frac{dr}{W-B-cr}\\&=-\frac{1}{c}\int_0^v dr+\frac{W-B}{c}\int_0^v\frac{dr}{W-B-cr}\\&=-\frac{v}{c}-\frac{(W-B)}{c^2}\ln\frac{|W-B-cv|}{W-B}.\end{aligned}$$

We know already that $v<(W-B)/c$. Consequently, $W-B-cv$ is always positive, and

$$\frac{gy}{W}=-\frac{v}{c}-\frac{(W-B)}{c^2}\ln\frac{W-B-cv}{W-B}.\tag{5}$$

At this point, we are ready to scream in despair since we cannot find v as an explicit function of y from (5). This is not an insurmountable difficulty, though. As we show in Section 1.11, it is quite simple, with the aid of a digital computer, to find $v(300)$ from (5). We need only supply the computer with a good approximation of $v(300)$ and this is obtained in the following manner. The velocity $v(y)$ of the drum satisfies the initial-value problem

$$\frac{W}{g}v\frac{dv}{dy}=W-B-cv,\qquad v(0)=0.\tag{6}$$

Let us, for the moment, set $c=0$ in (6) to obtain the new initial-value problem

$$\frac{W}{g}u\frac{du}{dy}=W-B,\qquad u(0)=0.\tag{6'}$$

(We have replaced v by u to avoid confusion later.) We can integrate (6′) immediately to obtain that

$$\frac{W}{g}\frac{u^2}{2}=(W-B)y,\quad\text{or}\quad u(y)=\left[\frac{2g}{W}(W-B)y\right]^{1/2}.$$

In particular,

$$u(300)=\left[\frac{2g}{W}(W-B)300\right]^{1/2}=\left[\frac{2(32.2)(57.109)(300)}{527.436}\right]^{1/2}$$
$$\cong\sqrt{2092}\cong 45.7 \text{ ft/s}.$$

We claim, now, that $u(300)$ is a very good approximation of $v(300)$. The proof of this is as follows. First, observe that the velocity of the drum is always greater if there is no drag force opposing the motion. Hence,

$$v(300)<u(300).$$

Second, the velocity v increases as y increases, so that $v(y)\leqslant v(300)$ for $y\leqslant 300$. Consequently, the drag force D of the water acting on the drum is always less than $0.08\times u(300)\cong 3.7$ lb. Now, the resultant force $W-B$ pulling the drum down is approximately 57.1 lb, which is very large compared to D. It stands to reason, therefore, that $u(y)$ should be a very good approximation of $v(y)$. And indeed, this is the case, since we find numerically (see Section 1.11) that $v(300)=45.1$ ft/s. Thus, the drums can break upon impact, and the engineers were right.

Epilog. The rules of the Atomic Energy Commission now expressly forbid the dumping of low level atomic waste at sea. This author is uncertain though, as to whether Western Europe has also forbidden this practice.

Remark. The methods introduced in this section can also be used to find the velocity of any object which is moving through a medium that resists the motion. We just disregard the buoyancy force if the medium is not water. For example, let $V(t)$ denote the velocity of a parachutist falling to earth under the influence of gravity. Then,

$$\frac{W}{g}\frac{dV}{dt}=W-D$$

where W is the weight of the man and the parachute, and D is the drag force exerted by the atmosphere on the falling parachutist. The drag force on a bluff object in air, or in any fluid of small viscosity is usually very nearly proportional to V^2. Proportionality to V is the exceptional case, and occurs only at very low speeds. The criterion as to whether the square or the linear law applies is the "Reynolds number"

$$R=\rho VL/\mu.$$

L is a representative length dimension of the object, and ρ and μ are the density and viscosity of the fluid. If $R<10$, then $D\sim V$, and if $R>10^3$, $D\sim V^2$. For $10<R<10^3$, neither law is accurate.

Exercises

1. Solve the initial-value problem (3).

2. Solve for $y=y(t)$ from (4), and then show that the equation $y=y(t)$ cannot be solved explicitly for $t=t(y)$.

3. Show that the drums of atomic waste will not crack upon impact if they are dropped into L feet of water with $(2g(W-B)L/W)^{1/2}<40$.

4. Fat Richie, an enormous underworld hoodlum weighing 400 lb, was pushed out of a penthouse window 2800 feet above the ground in New York City. Neglecting air resistance find (a) the velocity with which Fat Richie hit the ground; (b) the time elapsed before Fat Richie hit the ground.

5. An object weighing 300 lb is dropped into a river 150 feet deep. The volume of the object is 2 ft^3, and the drag force exerted by the water on it is 0.05 times its velocity. The drag force may be considered negligible if it does not exceed 5% of the resultant force pulling the drum down. Prove that the drag force is negligible in this case. (Here $B=2(62.4)=124.8$.)

6. A 400 lb sphere of volume $4\pi/3$ and a 300 lb cylinder of volume π are simultaneously released from rest into a river. The drag force exerted by the water on the falling sphere and cylinder is λV_s and λV_c, respectively, where V_s and V_c are the velocities of the sphere and cylinder, and λ is a positive constant. Determine which object reaches the bottom of the river first.

7. A parachutist falls from rest toward earth. The combined weight of man and parachute is 161 lb. Before the parachute opens, the air resistance equals $V/2$. The parachute opens 5 seconds after the fall begins; and the air resistance is then $V^2/2$. Find the velocity $V(t)$ of the parachutist after the parachute opens.

8. A man wearing a parachute jumps from a great height. The combined weight of man and parachute is 161 lb. Let $V(t)$ denote his speed at time t seconds after the fall begins. During the first 10 seconds, the air resistance is $V/2$. Thereafter, while the parachute is open, the air resistance is $10V$. Find an explicit formula for $V(t)$ at any time t greater than 10 seconds.

9. An object of mass m is projected vertically downward with initial velocity V_0 in a medium offering resistance proportional to the square root of the magnitude of the velocity.
   (a) Find a relation between the velocity V and the time t if the drag force equals $c\sqrt{V}$.
   (b) Find the terminal velocity of the object. *Hint*: You can find the terminal velocity even though you cannot solve for $V(t)$.

10. A body of mass m falls from rest in a medium offering resistance proportional to the square of the velocity; that is, $D=cV^2$. Find $V(t)$ and compute the terminal velocity V_T.

11. A body of mass m is projected upward from the earth's surface with an initial velocity V_0. Take the y-axis to be positive upward, with the origin on the surface of the earth. Assuming there is no air resistance, but taking into

account the variation of the earth's gravitational field with altitude, we obtain that

$$m\frac{dV}{dt} = -\frac{mgR^2}{(y+R)^2}$$

where R is the radius of the earth.

(a) Let $V(t)=v(y(t))$. Find a differential equation satisfied by $v(y)$.

(b) Find the smallest initial velocity V_0 for which the body will not return to earth. This is the so-called escape velocity. *Hint*: The escape velocity is found by requiring that $v(y)$ remain strictly positive.

12. It is not really necessary to find $v(y)$ explicitly in order to prove that $v(300)$ exceeds 40 ft/s. Here is an alternate proof. Observe first that $v(y)$ increases as y increases. This implies that y is a monotonic increasing function of v. Therefore, if y is less than 300 ft when v is 40 ft/s, then v must be greater than 40 ft/s when y is 300 ft. Substitute $v=40$ ft/s in Equation (5), and show that y is less than 300 ft. Conclude, therefore, that the drums can break upon impact.

1.8 The dynamics of tumor growth, mixing problems and orthogonal trajectories

In this section we present three very simple but extremely useful applications of first-order equations. The first application concerns the growth of solid tumors; the second application is concerned with "mixing problems" or "compartment analysis"; and the third application shows how to find a family of curves which is orthogonal to a given family of curves.

(a) *The dynamics of tumor growth*

It has been observed experimentally, that "free living" dividing cells, such as bacteria cells, grow at a rate proportional to the volume of dividing cells at that moment. Let $V(t)$ denote the volume of dividing cells at time t. Then,

$$\frac{dV}{dt} = \lambda V \tag{1}$$

for some positive constant λ. The solution of (1) is

$$V(t) = V_0 e^{\lambda(t-t_0)} \tag{2}$$

where V_0 is the volume of dividing cells at the initial time t_0. Thus, free living dividing cells grow *exponentially* with time. One important consequence of (2) is that the volume of cells keeps doubling (see Exercise 1) every time interval of length $\ln 2/\lambda$.

On the other hand, solid tumors do not grow exponentially with time. As the tumor becomes larger, the doubling time of the total tumor volume continuously increases. Various researchers have shown that the data for many solid tumors is fitted remarkably well, over almost a 1000 fold increase in tumor volume, by the equation

$$V(t) = V_0 \exp\left(\frac{\lambda}{\alpha}(1 - \exp(-\alpha t))\right) \tag{3}$$

where $\exp(x) \equiv e^x$, and λ and α are positive constants.

Equation (3) is usually referred to as a Gompertzian relation. It says that the tumor grows more and more slowly with the passage of time, and that it ultimately approaches the limiting volume $V_0 e^{\lambda/\alpha}$. Medical researchers have long been concerned with explaining this deviation from simple exponential growth. A great deal of insight into this problem can be gained by finding a differential equation satisfied by $V(t)$. Differentiating (3) gives

$$\begin{aligned} \frac{dV}{dt} &= V_0 \lambda \exp(-\alpha t) \exp\left(\frac{\lambda}{\alpha}(1 - \exp(-\alpha(t)))\right) \\ &= \lambda e^{-\alpha t} V. \end{aligned} \tag{4}$$

Two conflicting theories have been advanced for the dynamics of tumor growth. They correspond to the two arrangements

$$\frac{dV}{dt} = (\lambda e^{-\alpha t})V \tag{4a}$$

$$\frac{dV}{dt} = \lambda(e^{-\alpha t}V) \tag{4b}$$

of the differential equation (4). According to the first theory, the retarding effect of tumor growth is due to an increase in the mean generation time of the cells, without a change in the proportion of reproducing cells. As time goes on, the reproducing cells mature, or age, and thus divide more slowly. This theory corresponds to the bracketing (a).

The bracketing (b) suggests that the mean generation time of the dividing cells remains constant, and the retardation of growth is due to a loss in reproductive cells in the tumor. One possible explanation for this is that a *necrotic region* develops in the center of the tumor. This necrosis appears at a critical size for a particular type of tumor, and thereafter the necrotic "core" increases rapidly as the total tumor mass increases. According to this theory, a necrotic core develops because in many tumors the supply of blood, and thus of oxygen and nutrients, is almost completely confined to the surface of the tumor and a short distance beneath it. As the tumor grows, the supply of oxygen to the central core by diffusion becomes more and more difficult resulting in the formation of a necrotic core.

(b) *Mixing problems*

Many important problems in biology and engineering can be put into the following framework. A solution containing a fixed concentration of substance x flows into a tank, or compartment, containing the substance x and possibly other substances, at a specified rate. The mixture is stirred together very rapidly, and then leaves the tank, again at a specified rate. Find the concentration of substance x in the tank at any time t.

Problems of this type fall under the general heading of "mixing problems," or compartment analysis. The following example illustrates how to solve these problems.

Example 1. A tank contains S_0 lb of salt dissolved in 200 gallons of water. Starting at time $t=0$, water containing $\frac{1}{2}$ lb of salt per gallon enters the tank at the rate of 4 gal/min, and the well stirred solution leaves the tank at the same rate. Find the concentration of salt in the tank at any time $t>0$.

Solution. Let $S(t)$ denote the amount of salt in the tank at time t. Then, $S'(t)$, which is the rate of change of salt in the tank at time t, must equal the rate at which salt enters the tank minus the rate at which it leaves the tank. Obviously, the rate at which salt enters the tank is

$$\tfrac{1}{2}\text{ lb/gal times 4 gal/min} = 2\text{ lb/min}.$$

After a moment's reflection, it is also obvious that the rate at which salt leaves the tank is

$$4\text{ gal/min times } \frac{S(t)}{200}.$$

Thus

$$S'(t)=2-\frac{S(t)}{50}, \qquad S(0)=S_0,$$

and this implies that

$$S(t)=S_0e^{-0.02t}+100(1-e^{-0.02t}). \tag{5}$$

Hence, the concentration $c(t)$ of salt in the tank is given by

$$c(t)=\frac{S(t)}{200}=\frac{S_0}{200}e^{-0.02t}+\tfrac{1}{2}(1-e^{-0.02t}). \tag{6}$$

Remark. The first term on the right-hand side of (5) represents the portion of the original amount of salt remaining in the tank at time t. This term becomes smaller and smaller with the passage of time as the original solution is drained from the tank. The second term on the right-hand side of (5) represents the amount of salt in the tank at time t due to the action of the flow process. Clearly, the amount of salt in the tank must ultimately approach the limiting value of 100 lb, and this is easily verified by letting t approach ∞ in (5).

(c) *Orthogonal trajectories*

In many physical applications, it is often necessary to find the orthogonal trajectories of a given family of curves. (A curve which intersects each member of a family of curves at right angles is called an orthogonal trajectory of the given family.) For example, a charged particle moving under the influence of a magnetic field always travels on a curve which is perpendicular to each of the magnetic field lines. The problem of computing orthogonal trajectories of a family of curves can be solved in the following manner. Let the given family of curves be described by the relation

$$F(x,y,c)=0. \tag{7}$$

Differentiating this equation yields

$$F_x+F_y y'=0, \quad \text{or} \quad y'=-\frac{F_x}{F_y}. \tag{8}$$

Next, we solve for $c=c(x,y)$ from (7) and replace every c in (8) by this value $c(x,y)$. Finally, since the slopes of curves which intersect orthogonally are negative reciprocals of each other, we see that the orthogonal trajectories of (7) are the solution curves of the equation

$$y'=\frac{F_y}{F_x}. \tag{9}$$

Example 2. Find the orthogonal trajectories of the family of parabolas

$$x=cy^2.$$

Solution. Differentiating the equation $x=cy^2$ gives $1=2cyy'$. Since $c=x/y^2$, we see that $y'=y/2x$. Thus, the orthogonal trajectories of the family

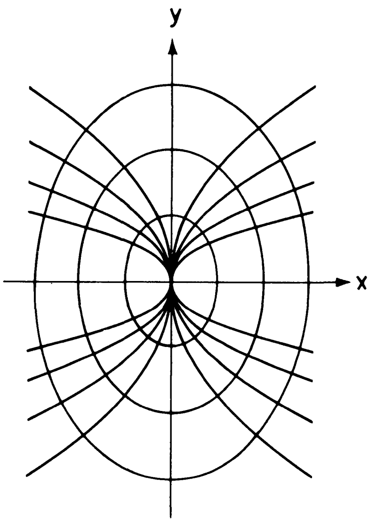

Figure 1. The parabolas $x=cy^2$ and their orthogonal trajectories

of parabolas $x=cy^2$ are the solution curves of the equation

$$y'=-\frac{2x}{y}. \tag{10}$$

This equation is separable, and its solution is

$$y^2+2x^2=k^2. \tag{11}$$

Thus, the family of ellipses (11) (see Figure 1) are the orthogonal trajectories of the family of parabolas $x=cy^2$.

Reference

Burton, Alan C., Rate of growth of solid tumors as a problem of diffusion, *Growth*, 1966, vol. 30, pp. 157–176.

Exercises

1. A given substance satisfies the exponential growth law (1). Show that the graph of $\ln V$ versus t is a straight line.

2. A substance x multiplies exponentially, and a given quantity of the substance doubles every 20 years. If we have 3 lb of substance x at the present time, how many lb will we have 7 years from now?

3. A substance x decays exponentially, and only half of the given quantity of x remains after 2 years. How long does it take for 5 lb of x to decay to 1 lb?

4. The equation $p'=ap^{\alpha}$, $\alpha>1$, is proposed as a model of the population growth of a certain species. Show that $p(t)\to\infty$ in finite time. Conclude, therefore, that this model is not accurate over a reasonable length of time.

5. A cancerous tumor satisfies the Gompertzian relation (3). Originally, when it contained 10^4 cells, the tumor was increasing at the rate of 20% per unit time. The numerical value of the retarding constant α is 0.02. What is the limiting number of cells in this tumor?

6. A *tracer dose* of radioactive iodine ^{131}I is injected into the blood stream at time $t=0$. Assume that the original amount Q_0 of iodine is distributed evenly in the entire blood stream before any loss occurs. Let $Q(t)$ denote the amount of iodine in the blood at time $t>0$. Part of the iodine leaves the blood and enters the urine at the rate k_1Q. Another part of the iodine enters the thyroid gland at the rate k_2Q. Find $Q(t)$.

7. Industrial waste is pumped into a tank containing 1000 gallons of water at the rate of 1 gal/min, and the well-stirred mixture leaves the tank at the same rate. (a) Find the concentration of waste in the tank at time t. (b) How long does it take for the concentration to reach 20%?

8. A tank contains 300 gallons of water and 100 gallons of pollutants. Fresh water is pumped into the tank at the rate of 2 gal/min, and the well-stirred mixture leaves at the same rate. How long does it take for the concentration of pollutants in the tank to decrease to 1/10 of its original value?

9. Consider a tank containing, at time $t=0$, Q_0 lb of salt dissolved in 150 gallons of water. Assume that water containing $\frac{1}{2}$ lb of salt per gallon is entering the tank at a rate of 3 gal/min, and that the well-stirred solution is leaving the tank at the same rate. Find an expression for the concentration of salt in the tank at time t.

10. A room containing 1000 cubic feet of air is originally free of carbon monoxide. Beginning at time $t=0$, cigarette smoke containing 4 percent carbon monoxide is blown into the room at the rate of 0.1 ft^3/min, and the well-circulated mixture leaves the room at the same rate. Find the time when the concentration of carbon monoxide in the room reaches 0.012 percent. (Extended exposure to this concentration of carbon monoxide is dangerous.)

11. A 500 gallon tank originally contains 100 gallons of fresh water. Beginning at time $t=0$, water containing 50 percent pollutants flows into the tank at the rate of 2 gal/min, and the well-stirred mixture leaves at the rate of 1 gal/min. Find the concentration of pollutants in the tank at the moment it overflows.

In Exercises 12–17, find the orthogonal trajectories of the given family of curves.

12. $y=cx^2$

13. $y^2-x^2=c$

14. $y=c\sin x$

15. $x^2+y^2=cx$ (see Exercise 13 of Section 1.4)

16. $y=ce^x$

17. $y=e^{cx}$

18. The presence of toxins in a certain medium destroys a strain of bacteria at a rate jointly proportional to the number of bacteria present and to the amount of toxin. Call the constant of proportionality a. If there were no toxins present, the bacteria would grow at a rate proportional to the amount present. Call this constant of proportionality b. Assume that the amount T of toxin is increasing at a constant rate c, that is, $dT/dt=c$, and that the production of toxins begins at time $t=0$. Let $y(t)$ denote the number of living bacteria present at time t.
(a) Find a first-order differential equation satisfied by $y(t)$.
(b) Solve this differential equation to obtain $y(t)$. What happens to $y(t)$ as t approaches ∞?

19. Many savings banks now advertise continuous compounding of interest. This means that the amount of money $P(t)$ on deposit at time t, satisfies the differential equation $dP(t)/dt=rP(t)$ where r is the annual interest rate and t is measured in years. Let P_0 denote the original principal.
(a) Show that $P(1)=P_0e^r$.
(b) Let $r=0.0575$, 0.065, 0.0675, and 0.075. Show that $e^r=1.05919$, 1.06716, 1.06983, and 1.07788, respectively. Thus, the effective annual yield on interest rates of $5\frac{3}{4}$, $6\frac{1}{2}$, $6\frac{3}{4}$, and $7\frac{1}{2}$% should be 5.919, 6.716, 6.983, and 7.788%, respectively. Most banks, however, advertise effective annual yields of 6, 6.81, 7.08, and 7.9%, respectively. The reason for this discrepancy is that banks calculate a daily rate of interest based on 360 days, and they pay interest for each day money is on deposit. For a year, one gets five extra

days. Thus, we must multiply the annual yields of 5.919, 6.716, 6.983, and 7.788% by 365/360, and then we obtain the advertised values.

(c) It is interesting to note that the Old Colony Cooperative Bank in Rhode Island advertises an effective annual yield of 6.72% on an annual interest rate of $6\frac{1}{2}\%$ (the lower value), and an effective annual yield of 7.9% on an annual interest rate of $7\frac{1}{2}\%$. Thus they are inconsistent.

1.9 Exact equations, and why we cannot solve very many differential equations

When we began our study of differential equations, the only equation we could solve was $dy/dt = g(t)$. We then enlarged our inventory to include all linear and separable equations. More generally, we can solve all differential equations which are, or can be put, in the form

$$\frac{d}{dt}\phi(t,y) = 0 \tag{1}$$

for some function $\phi(t,y)$. To wit, we can integrate both sides of (1) to obtain that

$$\phi(t,y) = \text{constant} \tag{2}$$

and then solve for y as a function of t from (2).

Example 1. The equation $1+\cos(t+y)+\cos(t+y)(dy/dt)=0$ can be written in the form $(d/dt)[t+\sin(t+y)]=0$. Hence,

$$\phi(t,y) = t+\sin(t+y) = c, \quad \text{and} \quad y = -t+\arcsin(c-t).$$

Example 2. The equation $\cos(t+y)+[1+\cos(t+y)]dy/dt=0$ can be written in the form $(d/dt)[y+\sin(t+y)]=0$. Hence,

$$\phi(t,y) = y+\sin(t+y) = c.$$

We must leave the solution in this form though, since we cannot solve for y explicitly as a function of time.

Equation (1) is clearly the most general first-order differential equation that we can solve. Thus, it is important for us to be able to recognize when a differential equation can be put in this form. This is not as simple as one might expect. For example, it is certainly not obvious that the differential equation

$$2t+y-\sin t+(3y^2+\cos y+t)\frac{dy}{dt} = 0$$

can be written in the form $(d/dt)(y^3+t^2+ty+\sin y+\cos t)=0$. To find all those differential equations which can be written in the form (1), observe, from the chain rule of partial differentiation, that

$$\frac{d}{dt}\phi(t,y(t)) = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}.$$

Hence, the differential equation $M(t,y)+N(t,y)(dy/dt)=0$ can be written in the form $(d/dt)\phi(t,y)=0$ if and only if there exists a function $\phi(t,y)$ such that $M(t,y)=\partial\phi/\partial t$ and $N(t,y)=\partial\phi/\partial y$.

This now leads us to the following question. Given two functions $M(t,y)$ and $N(t,y)$, does there exist a function $\phi(t,y)$ such that $M(t,y) = \partial\phi/\partial t$ and $N(t,y)=\partial\phi/\partial y$? Unfortunately, the answer to this question is almost always no as the following theorem shows.

Theorem 1. *Let $M(t,y)$ and $N(t,y)$ be continuous and have continuous partial derivatives with respect to t and y in the rectangle R consisting of those points (t,y) with $a<t<b$ and $c<y<d$. There exists a function $\phi(t,y)$ such that $M(t,y)=\partial\phi/\partial t$ and $N(t,y)=\partial\phi/\partial y$ if, and only if,*

$$\partial M/\partial y = \partial N/\partial t$$

in R.

PROOF. Observe that $M(t,y)=\partial\phi/\partial t$ for some function $\phi(t,y)$ if, and only if,

$$\phi(t,y) = \int M(t,y)\,dt + h(y) \tag{3}$$

where $h(y)$ is an arbitrary function of y. Taking partial derivatives of both sides of (3) with respect to y, we obtain that

$$\frac{\partial\phi}{\partial y} = \int \frac{\partial M(t,y)}{\partial y}dt + h'(y).$$

Hence, $\partial\phi/\partial y$ will be equal to $N(t,y)$ if, and only if,

$$N(t,y) = \int \frac{\partial M(t,y)}{\partial y}dt + h'(y)$$

or

$$h'(y) = N(t,y) - \int \frac{\partial M(t,y)}{\partial y}dt. \tag{4}$$

Now $h'(y)$ is a function of y alone, while the right-hand side of (4) appears to be a function of both t and y. But a function of y alone cannot be equal to a function of both t and y. Thus Equation (4) makes sense only if the right-hand side is a function of y alone, and this is the case if, and only if,

$$\frac{\partial}{\partial t}\left[N(t,y) - \int \frac{\partial M(t,y)}{\partial y}dt\right] = \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0.$$

Hence, if $\partial N/\partial t \neq \partial M/\partial y$, then there is no function $\phi(t,y)$ such that $M = \partial\phi/\partial t$, $N = \partial\phi/\partial y$. On the other hand, if $\partial N/\partial t = \partial M/\partial y$ then we can solve for

$$h(y) = \int\left[N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt\right]dy.$$

Consequently, $M = \partial\phi/\partial t$, and $N = \partial\phi/\partial y$ with

$$\phi(t,y) = \int M(t,y)\,dt + \int\left[N(t,y) - \int \frac{\partial M(t,y)}{\partial y}\,dt\right]dy. \qquad \square \tag{5}$$

Definition. The differential equation

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \tag{6}$$

is said to be *exact* if $\partial M/\partial y = \partial N/\partial t$.

The reason for this definition, of course, is that the left-hand side of (6) is the exact derivative of a known function of t and y if $\partial M/\partial y = \partial N/\partial t$.

Remark 1. It is not essential, in the statement of Theorem 1, that $\partial M/\partial y = \partial N/\partial t$ in a rectangle. It is sufficient if $\partial M/\partial y = \partial N/\partial t$ in any region R which contains no "holes". That is to say, if C is any closed curve lying entirely in R, then its interior also lies entirely in R.

Remark 2. The differential equation $dy/dt = f(t,y)$ can always be written in the form $M(t,y) + N(t,y)(dy/dt) = 0$ by setting $M(t,y) = -f(t,y)$ and $N(t,y) = 1$.

Remark 3. It is customary to say that the solution of an exact differential equation is given by $\phi(t,y) = \text{constant}$. What we really mean is that the equation $\phi(t,y) = c$ is to be solved for y as a function of t and c. Unfortunately, most exact differential equations cannot be solved explicitly for y as a function of t. While this may appear to be very disappointing, we wish to point out that it is quite simple, with the aid of a computer, to compute $y(t)$ to any desired accuracy (see Section 1.11).

In practice, we do not recommend memorizing Equation (5). Rather, we will follow one of three different methods to obtain $\phi(t,y)$.
First Method: The equation $M(t,y) = \partial\phi/\partial t$ determines $\phi(t,y)$ up to an arbitrary function of y alone, that is,

$$\phi(t,y) = \int M(t,y)\,dt + h(y).$$

The function $h(y)$ is then determined from the equation

$$h'(y)=N(t,y)-\int\frac{\partial M(t,y)}{\partial y}dt.$$

Second Method: If $N(t,y)=\partial\phi/\partial y$, then, of necessity,

$$\phi(t,y)=\int N(t,y)\,dy+k(t)$$

where $k(t)$ is an arbitrary function of t alone. Since

$$M(t,y)=\frac{\partial\phi}{\partial t}=\int\frac{\partial N(t,y)}{\partial t}dy+k'(t)$$

we see that $k(t)$ is determined from the equation

$$k'(t)=M(t,y)-\int\frac{\partial N(t,y)}{\partial t}dy.$$

Note that the right-hand side of this equation (see Exercise 2) is a function of t alone if $\partial M/\partial y=\partial N/\partial t$.
Third Method: The equations $\partial\phi/\partial t=M(t,y)$ and $\partial\phi/\partial y=N(t,y)$ imply that

$$\phi(t,y)=\int M(t,y)\,dt+h(y)\quad\text{and}\quad\phi(t,y)=\int N(t,y)\,dy+k(t).$$

Usually, we can determine $h(y)$ and $k(t)$ just by inspection.

Example 3. Find the general solution of the differential equation

$$3y+e^t+(3t+\cos y)\frac{dy}{dt}=0.$$

Solution. Here $M(t,y)=3y+e^t$ and $N(t,y)=3t+\cos y$. This equation is exact since $\partial M/\partial y=3$ and $\partial N/\partial t=3$. Hence, there exists a function $\phi(t,y)$ such that

$$\text{(i) } 3y+e^t=\frac{\partial\phi}{\partial t}\quad\text{and}\quad\text{(ii) } 3t+\cos y=\frac{\partial\phi}{\partial y}.$$

We will find $\phi(t,y)$ by each of the three methods outlined above.

First Method: From (i), $\phi(t,y)=e^t+3ty+h(y)$. Differentiating this equation with respect to y and using (ii) we obtain that

$$h'(y)+3t=3t+\cos y.$$

Thus, $h(y)=\sin y$ and $\phi(t,y)=e^t+3ty+\sin y$. (Strictly speaking, $h(y)=\sin y+\text{constant}$. However, we already incorporate this constant of integration into the solution when we write $\phi(t,y)=c$.) The general solution of the differential equation must be left in the form $e^t+3ty+\sin y=c$ since we cannot find y explicitly as a function of t from this equation.
Second Method: From (ii), $\phi(t,y)=3ty+\sin y+k(t)$. Differentiating this

expression with respect to t, and using (i) we obtain that

$$3y+k'(t)=3y+e^t.$$

Thus, $k(t)=e^t$ and $\phi(t,y)=3ty+\sin y+e^t$.
Third Method: From (i) and (ii)

$$\phi(t,y)=e^t+3ty+h(y) \quad \text{and} \quad \phi(t,y)=3ty+\sin y+k(t).$$

Comparing these two expressions for the *same* function $\phi(t,y)$ it is obvious that $h(y)=\sin y$ and $k(t)=e^t$. Hence

$$\phi(t,y)=e^t+3ty+\sin y.$$

Example 4. Find the solution of the initial-value problem

$$3t^2y+8ty^2+(t^3+8t^2y+12y^2)\frac{dy}{dt}=0, \qquad y(2)=1.$$

Solution. Here $M(t,y)=3t^2y+8ty^2$ and $N(t,y)=t^3+8t^2y+12y^2$. This equation is exact since

$$\frac{\partial M}{\partial y}=3t^2+16ty \quad \text{and} \quad \frac{\partial N}{\partial t}=3t^2+16ty.$$

Hence, there exists a function $\phi(t,y)$ such that

$$\text{(i)}\ 3t^2y+8ty^2=\frac{\partial\phi}{\partial t} \quad \text{and} \quad \text{(ii)}\ t^3+8t^2y+12y^2=\frac{\partial\phi}{\partial y}.$$

Again, we will find $\phi(t,y)$ by each of three methods.

First Method: From (i), $\phi(t,y)=t^3y+4t^2y^2+h(y)$. Differentiating this equation with respect to y and using (ii) we obtain that

$$t^3+8t^2y+h'(y)=t^3+8t^2y+12y^2.$$

Hence, $h(y)=4y^3$ and the general solution of the differential equation is $\phi(t,y)=t^3y+4t^2y^2+4y^3=c$. Setting $t=2$ and $y=1$ in this equation, we see that $c=28$. Thus, the solution of our initial-value problem is defined implicitly by the equation $t^3y+4t^2y^2+4y^3=28$.
Second Method: From (ii), $\phi(t,y)=t^3y+4t^2y^2+4y^3+k(t)$. Differentiating this expression with respect to t and using (i) we obtain that

$$3t^2y+8ty^2+k'(t)=3t^2y+8ty^2.$$

Thus $k(t)=0$ and $\phi(t,y)=t^3y+4t^2y^2+4y^3$.
Third Method: From (i) and (ii)

$$\phi(t,y)=t^3y+4t^2y^2+h(y) \quad \text{and} \quad \phi(t,y)=t^3y+4t^2y^2+4y^3+k(t).$$

Comparing these two expressions for the same function $\phi(t,y)$ we see that $h(y)=4y^3$ and $k(t)=0$. Hence, $\phi(t,y)=t^3y+4t^2y^2+4y^3$.

In most instances, as Examples 3 and 4 illustrate, the third method is the simplest to use. However, if it is much easier to integrate N with respect to y than it is to integrate M with respect to t, we should use the second method, and vice-versa.

Example 5. Find the solution of the initial-value problem

$$4t^3e^{t+y}+t^4e^{t+y}+2t+(t^4e^{t+y}+2y)\frac{dy}{dt}=0, \qquad y(0)=1.$$

Solution. This equation is exact since

$$\frac{\partial}{\partial y}(4t^3e^{t+y}+t^4e^{t+y}+2t)=(t^4+4t^3)e^{t+y}=\frac{\partial}{\partial t}(t^4e^{t+y}+2y).$$

Hence, there exists a function $\phi(t,y)$ such that

$$\text{(i)}\quad 4t^3e^{t+y}+t^4e^{t+y}+2t=\frac{\partial\phi}{\partial t}$$

and

$$\text{(ii)}\quad t^4e^{t+y}+2y=\frac{\partial\phi}{\partial y}.$$

Since it is much simpler to integrate $t^4e^{t+y}+2y$ with respect to y than it is to integrate $4t^3e^{t+y}+t^4e^{t+y}+2t$ with respect to t, we use the second method. From (ii), $\phi(t,y)=t^4e^{t+y}+y^2+k(t)$. Differentiating this expression with respect to t and using (i) we obtain

$$(t^4+4t^3)e^{t+y}+k'(t)=4t^3e^{t+y}+t^4e^{t+y}+2t.$$

Thus, $k(t)=t^2$ and the general solution of the differential equation is $\phi(t,y)=t^4e^{t+y}+y^2+t^2=c$. Setting $t=0$ and $y=1$ in this equation yields $c=1$. Thus, the solution of our initial-value problem is defined implicitly by the equation $t^4e^{t+y}+t^2+y^2=1$.

Suppose now that we are given a differential equation

$$M(t,y)+N(t,y)\frac{dy}{dt}=0 \tag{7}$$

which is not exact. Can we make it exact? More precisely, can we find a function $\mu(t,y)$ such that the equivalent differential equation

$$\mu(t,y)M(t,y)+\mu(t,y)N(t,y)\frac{dy}{dt}=0 \tag{8}$$

is exact? This question is simple, in principle, to answer. The condition that (8) be exact is that

$$\frac{\partial}{\partial y}(\mu(t,y)M(t,y))=\frac{\partial}{\partial t}(\mu(t,y)N(t,y))$$

or

$$M\frac{\partial\mu}{\partial y}+\mu\frac{\partial M}{\partial y}=N\frac{\partial\mu}{\partial t}+\mu\frac{\partial N}{\partial t}. \tag{9}$$

(For simplicity of writing, we have suppressed the dependence of μ, M and N on t and y in (9).) Thus, Equation (8) is exact if and only if $\mu(t,y)$ satisfies Equation (9).

Definition. A function $\mu(t,y)$ satisfying Equation (9) is called an *integrating factor* for the differential equation (7).

The reason for this definition, of course, is that if μ satisfies (9) then we can write (8) in the form $(d/dt)\phi(t,y)=0$ and this equation can be integrated immediately to yield the solution $\phi(t,y)=c$. Unfortunately, though, there are only two special cases where we can find an explicit solution of (9). These occur when the differential equation (7) has an integrating factor which is either a function of t alone, or a function of y alone. Observe that if μ is a function of t alone, then Equation (9) reduces to

$$N\frac{d\mu}{dt}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)\quad\text{or}\quad\frac{d\mu}{dt}=\frac{\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial t}\right)}{N}\mu.$$

But this equation is meaningless unless the expression

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial t}}{N}$$

is a function of t alone, that is,

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial t}}{N}=R(t).$$

If this is the case then $\mu(t)=\exp\left(\int R(t)\,dt\right)$ is an integrating factor for the differential equation (7).

Remark. It should be noted that the expression

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial t}}{N}$$

is almost always a function of both t and y. Only for very special pairs of functions $M(t,y)$ and $N(t,y)$ is it a function of t alone. A similar situation occurs if μ is a function of y alone (see Exercise 17). It is for this reason that we cannot solve very many differential equations.

Example 6. Find the general solution of the differential equation

$$\frac{y^2}{2}+2ye^t+(y+e^t)\frac{dy}{dt}=0.$$

Solution. Here $M(t,y)=(y^2/2)+2ye^t$ and $N(t,y)=y+e^t$. This equation is not exact since $\partial M/\partial y = y+2e^t$ and $\partial N/\partial t = e^t$. However,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial t}\right)=\frac{y+e^t}{y+e^t}=1.$$

Hence, this equation has $\mu(t)=\exp\left(\int 1\,dt\right)=e^t$ as an integrating factor. This means, of course, that the equivalent differential equation

$$\frac{y^2}{2}e^t+2ye^{2t}+\left(ye^t+e^{2t}\right)\frac{dy}{dt}=0$$

is exact. Therefore, there exists a function $\phi(t,y)$ such that

$$\text{(i)}\ \frac{y^2}{2}e^t+2ye^{2t}=\frac{\partial\phi}{\partial t}$$

and

$$\text{(ii)}\ ye^t+e^{2t}=\frac{\partial\phi}{\partial y}.$$

From Equations (i) and (ii),

$$\phi(t,y)=\frac{y^2}{2}e^t+ye^{2t}+h(y)$$

and

$$\phi(t,y)=\frac{y^2}{2}e^t+ye^{2t}+k(t).$$

Thus, $h(y)=0$, $k(t)=0$ and the general solution of the differential equation is

$$\phi(t,y)=\frac{y^2}{2}e^t+ye^{2t}=c.$$

Solving this equation for y as a function of t we see that

$$y(t)=-e^t\pm\left[e^{2t}+2ce^{-t}\right]^{1/2}.$$

Example 7. Use the methods of this section to find the general solution of the linear equation $(dy/dt)+a(t)y=b(t)$.

Solution. We write this equation in the form $M(t,y)+N(t,y)(dy/dt)=0$ with $M(t,y)=a(t)y-b(t)$ and $N(t,y)=1$. This equation is not exact since $\partial M/\partial y=a(t)$ and $\partial N/\partial t=0$. However, $((\partial M/\partial y)-(\partial N/\partial t))/N=a(t)$. Hence, $\mu(t)=\exp\left(\int a(t)\,dt\right)$ is an integrating factor for the first-order linear equation. Therefore, there exists a function $\phi(t,y)$ such that

$$\text{(i)}\ \mu(t)\left[a(t)y-b(t)\right]=\frac{\partial\phi}{\partial t}$$

and

$$\text{(ii)}\quad \mu(t) = \frac{\partial \phi}{\partial y}.$$

Now, observe from (ii) that $\phi(t,y) = \mu(t)y + k(t)$. Differentiating this equation with respect to t and using (i) we see that

$$\mu'(t)y + k'(t) = \mu(t)a(t)y - \mu(t)b(t).$$

But, $\mu'(t) = a(t)\mu(t)$. Consequently, $k'(t) = -\mu(t)b(t)$ and

$$\phi(t,y) = \mu(t)y - \int \mu(t)b(t)\,dt.$$

Hence, the general solution of the first-order linear equation is

$$\mu(t)y - \int \mu(t)b(t)\,dt = c,$$

and this is the result we obtained in Section 1.2.

Exercises

1. Use the theorem of equality of mixed partial derivatives to show that $\partial M/\partial y = \partial N/\partial t$ if the equation $M(t,y) + N(t,y)(dy/dt) = 0$ is exact.

2. Show that the expression $M(t,y) - \int (\partial N(t,y)/\partial t)\,dy$ is a function of t alone if $\partial M/\partial y = \partial N/\partial t$.

In each of Problems 3–6 find the general solution of the given differential equation.

3. $2t\sin y + y^3e^t + (t^2\cos y + 3y^2e^t)\dfrac{dy}{dt} = 0$

4. $1 + (1 + ty)e^{ty} + (1 + t^2e^{ty})\dfrac{dy}{dt} = 0$

5. $y\sec^2 t + \sec t\tan t + (2y + \tan t)\dfrac{dy}{dt} = 0$

6. $\dfrac{y^2}{2} - 2ye^t + (y - e^t)\dfrac{dy}{dt} = 0$

In each of Problems 7–11, solve the given initial-value problem.

7. $2ty^3 + 3t^2y^2\dfrac{dy}{dt} = 0, \qquad y(1) = 1$

8. $2t\cos y + 3t^2y + (t^3 - t^2\sin y - y)\dfrac{dy}{dt} = 0, \qquad y(0) = 2$

9. $3t^2 + 4ty + (2y + 2t^2)\dfrac{dy}{dt} = 0, \qquad y(0) = 1$

10. $y(\cos 2t)e^{ty} - 2(\sin 2t)e^{ty} + 2t + (t(\cos 2t)e^{ty} - 3)\dfrac{dy}{dt} = 0, \qquad y(0) = 0$

11. $3ty+y^2+(t^2+ty)\dfrac{dy}{dt}=0, \qquad y(2)=1$

In each of Problems 12–14, determine the constant a so that the equation is exact, and then solve the resulting equation.

12. $t+ye^{2ty}+ate^{2ty}\dfrac{dy}{dt}=0$

13. $\dfrac{1}{t^2}+\dfrac{1}{y^2}+\dfrac{(at+1)}{y^3}\dfrac{dy}{dt}=0$

14. $e^{at+y}+3t^2y^2+(2yt^3+e^{at+y})\dfrac{dy}{dt}=0$

15. Show that every separable equation of the form $M(t)+N(y)dy/dt=0$ is exact.

16. Find all functions $f(t)$ such that the differential equation

$$y^2\sin t+yf(t)(dy/dt)=0$$

is exact. Solve the differential equation for these $f(t)$.

17. Show that if $((\partial N/\partial t)-(\partial M/\partial y))/M=Q(y)$, then the differential equation $M(t,y)+N(t,y)dy/dt=0$ has an integrating factor $\mu(y)=\exp\left(\int Q(y)\,dy\right)$.

18. The differential equation $f(t)(dy/dt)+t^2+y=0$ is known to have an integrating factor $\mu(t)=t$. Find all possible functions $f(t)$.

19. The differential equation $e^t\sec y-\tan y+(dy/dt)=0$ has an integrating factor of the form $e^{-at}\cos y$ for some constant a. Find a, and then solve the differential equation.

20. The Bernoulli differential equation is $(dy/dt)+a(t)y=b(t)y^n$. Multiplying through by $\mu(t)=\exp\left(\int a(t)\,dt\right)$, we can rewrite this equation in the form $d/dt(\mu(t)y)=b(t)\mu(t)y^n$. Find the general solution of this equation by finding an appropriate integrating factor. *Hint*: Divide both sides of the equation by an appropriate function of y.

1.10 The existence–uniqueness theorem; Picard iteration

Consider the initial-value problem

$$\frac{dy}{dt}=f(t,y), \qquad y(t_0)=y_0 \tag{1}$$

where f is a given function of t and y. Chances are, as the remarks in Section 1.9 indicate, that we will be unable to solve (1) explicitly. This leads us to ask the following questions.

1. How are we to know that the initial-value problem (1) actually has a solution if we can't exhibit it?

2. How do we know that there is only one solution $y(t)$ of (1)? Perhaps there are two, three, or even infinitely many solutions.
3. Why bother asking the first two questions? After all, what's the use of determining whether (1) has a unique solution if we won't be able to explicitly exhibit it?

The answer to the third question lies in the observation that it is never necessary, in applications, to find the solution $y(t)$ of (1) to more than a finite number of decimal places. Usually, it is more than sufficient to find $y(t)$ to four decimal places. As we shall see in Sections 1.13–17, this can be done quite easily with the aid of a digital computer. In fact, we will be able to compute $y(t)$ to eight, and even sixteen, decimal places. Thus, the knowledge that (1) has a unique solution $y(t)$ is our hunting license to go looking for it.

To resolve the first question, we must establish the existence of a function $y(t)$ whose value at $t=t_0$ is y_0, and whose derivative at any time t equals $f(t,y(t))$. In order to accomplish this, we must find a theorem which enables us to establish the existence of a function having certain properties, without our having to exhibit this function explicitly. If we search through the Calculus, we find that we encounter such a situation exactly once, and this is in connection with the theory of limits. As we show in Appendix B, it is often possible to prove that a sequence of functions $y_n(t)$ has a limit $y(t)$, without our having to exhibit $y(t)$. For example, we can prove that the sequence of functions

$$y_n(t)=\frac{\sin \pi t}{1^2}+\frac{\sin 2\pi t}{2^2}+\dots+\frac{\sin n\pi t}{n^2}$$

has a limit $y(t)$ even though we cannot exhibit $y(t)$ explicitly. This suggests the following algorithm for proving the existence of a solution $y(t)$ of (1).

(a) Construct a sequence of functions $y_n(t)$ which come closer and closer to solving (1).
(b) Show that the sequence of functions $y_n(t)$ has a limit $y(t)$ on a suitable interval $t_0 \leqslant t \leqslant t_0+\alpha$.
(c) Prove that $y(t)$ is a solution of (1) on this interval.

We now show how to implement this algorithm.

(a) *Construction of the approximating sequence $y_n(t)$*

The problem of finding a sequence of functions that come closer and closer to satisfying a certain equation is one that arises quite often in mathematics. Experience has shown that it is often easiest to resolve this problem when our equation can be written in the special form

$$y(t)=L(t,y(t)), \tag{2}$$

where L may depend explicitly on y, and on integrals of functions of y.

For example, we may wish to find a function $y(t)$ satisfying

$$y(t)=1+\sin\left[t+y(t)\right],$$

or

$$y(t)=1+y^2(t)+\int_0^t y^3(s)\,ds.$$

In these two cases, $L(t,y(t))$ is an abbreviation for

$$1+\sin\left[t+y(t)\right]$$

and

$$1+y^2(t)+\int_0^t y^3(s)\,ds,$$

respectively.

The key to understanding what is special about Equation (2) is to view $L(t,y(t))$ as a "machine" that takes in one function and gives back another one. For example, let

$$L(t,y(t))=1+y^2(t)+\int_0^t y^3(s)\,ds.$$

If we plug the function $y(t)=t$ into this machine, (that is, if we compute $1+t^2+\int_0^t s^3\,ds$) then the machine returns to us the function $1+t^2+t^4/4$. If we plug the function $y(t)=\cos t$ into this machine, then it returns to us the function

$$1+\cos^2 t+\int_0^t \cos^3 s\,ds=1+\cos^2 t+\sin t-\frac{\sin^3 t}{3}.$$

According to this viewpoint, we can characterize all solutions $y(t)$ of (2) as those functions $y(t)$ which the machine L leaves unchanged. In other words, if we plug a function $y(t)$ into the machine L, and the machine returns to us this same function, then $y(t)$ is a solution of (2).

We can put the initial-value problem (1) into the special form (2) by integrating both sides of the differential equation $y'=f(t,y)$ with respect to t. Specifically, if $y(t)$ satisfies (1), then

$$\int_{t_0}^t \frac{dy(s)}{ds}\,ds=\int_{t_0}^t f(s,y(s))\,ds$$

so that

$$y(t)=y_0+\int_{t_0}^t f(s,y(s))\,ds. \tag{3}$$

Conversely, if $y(t)$ is continuous and satisfies (3), then $dy/dt=f(t,y(t))$. Moreover, $y(t_0)$ is obviously y_0. Therefore, $y(t)$ is a solution of (1) if, and only if, it is a continuous solution of (3).

Equation (3) is called an integral equation, and it is in the special form (2) if we set

$$L(t,y(t)) = y_0 + \int_{t_0}^{t} f(s,y(s))\,ds.$$

This suggests the following scheme for constructing a sequence of "approximate solutions" $y_n(t)$ of (3). Let us start by guessing a solution $y_0(t)$ of (3). The simplest possible guess is $y_0(t)=y_0$. To check whether $y_0(t)$ is a solution of (3), we compute

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s,y_0(s))\,ds.$$

If $y_1(t)=y_0$, then $y(t)=y_0$ is indeed a solution of (3). If not, then we try $y_1(t)$ as our next guess. To check whether $y_1(t)$ is a solution of (3), we compute

$$y_2(t) = y_0 + \int_{t_0}^{t} f(s,y_1(s))\,ds,$$

and so on. In this manner, we define a sequence of functions $y_1(t)$, $y_2(t),\ldots,$ where

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s,y_n(s))\,ds. \tag{4}$$

These functions $y_n(t)$ are called successive approximations, or Picard iterates, after the French mathematician Picard who first discovered them. Remarkably, these Picard iterates always converge, on a suitable interval, to a solution $y(t)$ of (3).

Example 1. Compute the Picard iterates for the initial-value problem

$$y' = y, \qquad y(0) = 1,$$

and show that they converge to the solution $y(t)=e^t$.

Solution. The integral equation corresponding to this initial-value problem is

$$y(t) = 1 + \int_0^t y(s)\,ds.$$

Hence, $y_0(t)=1$

$$y_1(t) = 1 + \int_0^t 1\,ds = 1+t$$

$$y_2(t) = 1 + \int_0^t y_1(s)\,ds = 1 + \int_0^t (1+s)\,ds = 1 + t + \frac{t^2}{2!}$$

and, in general,

$$y_n(t) = 1 + \int_0^t y_{n-1}(s)\,ds = 1 + \int_0^t \left[1 + s + \dots + \frac{s^{n-1}}{(n-1)!}\right] ds$$
$$= 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!}.$$

Since $e^t = 1 + t + t^2/2! + \dots$, we see that the Picard iterates $y_n(t)$ converge to the solution $y(t)$ of this initial-value problem.

Example 2. Compute the Picard iterates $y_1(t), y_2(t)$ for the initial-value problem $y' = 1 + y^3$, $y(1) = 1$.

Solution. The integral equation corresponding to this initial-value problem is

$$y(t) = 1 + \int_1^t \left[1 + y^3(s)\right] ds.$$

Hence, $y_0(t) = 1$

$$y_1(t) = 1 + \int_1^t (1+1)\,ds = 1 + 2(t-1)$$

and

$$y_2(t) = 1 + \int_1^t \left\{1 + \left[1 + 2(s-1)\right]^3\right\} ds$$
$$= 1 + 2(t-1) + 3(t-1)^2 + 4(t-1)^3 + 2(t-1)^4.$$

Notice that it is already quite cumbersome to compute $y_3(t)$.

(b) *Convergence of the Picard iterates*

As was mentioned in Section 1.4, the solutions of nonlinear differential equations may not exist for all time t. Therefore, we cannot expect the Picard iterates $y_n(t)$ of (3) to converge for all t. To provide us with a clue, or estimate, of where the Picard iterates converge, we try to find an interval in which all the $y_n(t)$ are uniformly bounded (that is, $|y_n(t)| \leq K$ for some fixed constant K). Equivalently, we seek a rectangle R which contains the graphs of all the Picard iterates $y_n(t)$. Lemma 1 shows us how to find such a rectangle.

Lemma 1. *Choose any two positive numbers a and b, and let R be the rectangle: $t_0 \leq t \leq t_0 + a, |y - y_0| \leq b$. Compute*

$$M = \max_{(t,y) \text{ in } R} |f(t,y)|, \quad \textit{and set} \quad \alpha = \min\left(a, \frac{b}{M}\right).$$

Then,

$$|y_n(t) - y_0| \leq M(t - t_0) \tag{5}$$

for $t_0 \leq t \leq t_0 + \alpha$.

Lemma 1 states that the graph of $y_n(t)$ is sandwiched between the lines $y=y_0+M(t-t_0)$ and $y=y_0-M(t-t_0)$, for $t_0 \leqslant t \leqslant t_0+\alpha$. These lines leave the rectangle R at $t=t_0+a$ if $a \leqslant b/M$, and at $t=t_0+b/M$ if $b/M < a$ (see Figures 1a and 1b). In either case, therefore, the graph of $y_n(t)$ is contained in R for $t_0 \leqslant t \leqslant t_0+\alpha$.

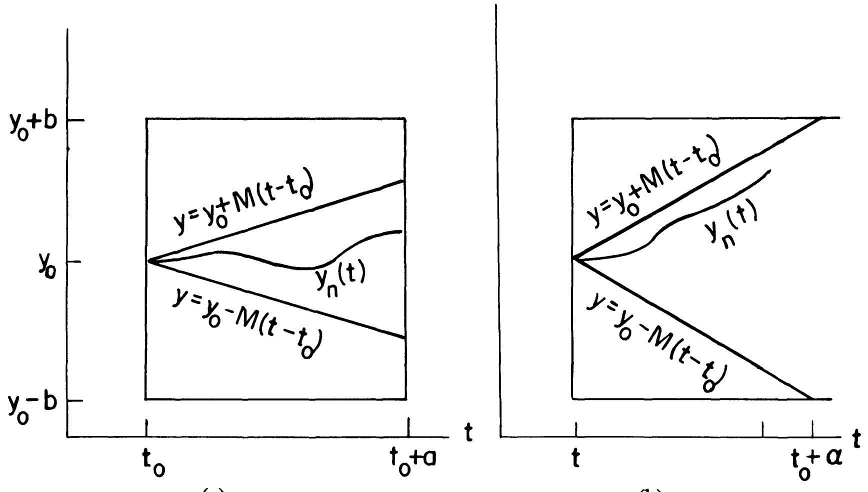

Figure 1. (a) $\alpha=a$; (b) $\alpha=b/M$

PROOF OF LEMMA 1. We establish (5) by induction on n. Observe first that (5) is obviously true for $n=0$, since $y_0(t)=y_0$. Next, we must show that (5) is true for $n=j+1$ if it is true for $n=j$. But this follows immediately, for if $|y_j(t)-y_0| \leqslant M(t-t_0)$, then

$$\begin{aligned}|y_{j+1}(t)-y_0| &= \left|\int_{t_0}^{t} f(s,y_j(s))\,ds\right| \\ &\leqslant \int_{t_0}^{t} |f(s,y_j(s))|\,ds \leqslant M(t-t_0)\end{aligned}$$

for $t_0 \leqslant t \leqslant t_0+\alpha$. Consequently, (5) is true for all n, by induction. □

We now show that the Picard iterates $y_n(t)$ of (3) converge for each t in the interval $t_0 \leqslant t \leqslant t_0+\alpha$, if $\partial f/\partial y$ exists and is continuous. Our first step is to reduce the problem of showing that the sequence of functions $y_n(t)$ converges to the much simpler problem of proving that an infinite series converges. This is accomplished by writing $y_n(t)$ in the form

$$y_n(t)=y_0(t)+\left[y_1(t)-y_0(t)\right]+\ldots+\left[y_n(t)-y_{n-1}(t)\right].$$

Clearly, the sequence $y_n(t)$ converges if, and only if, the infinite series

$$\left[y_1(t)-y_0(t)\right]+\left[y_2(t)-y_1(t)\right]+\ldots+\left[y_n(t)-y_{n-1}(t)\right]+\ldots \tag{6}$$

converges. To prove that the infinite series (6) converges, it suffices to show that

$$\sum_{n=1}^{\infty} |y_n(t)-y_{n-1}(t)| < \infty. \tag{7}$$

This is accomplished in the following manner. Observe that

$$\begin{aligned}|y_n(t)-y_{n-1}(t)| &= \left|\int_{t_0}^{t}\left[f(s,y_{n-1}(s))-f(s,y_{n-2}(s))\right]ds\right| \\ &\leqslant \int_{t_0}^{t}|f(s,y_{n-1}(s))-f(s,y_{n-2}(s))|\,ds \\ &= \int_{t_0}^{t}\left|\frac{\partial f(s,\xi(s))}{\partial y}\right||y_{n-1}(s)-y_{n-2}(s)|\,ds,\end{aligned}$$

where $\xi(s)$ lies between $y_{n-1}(s)$ and $y_{n-2}(s)$. (Recall that $f(x_1)-f(x_2)=f'(\xi)(x_1-x_2)$, where ξ is some number between x_1 and x_2.) It follows immediately from Lemma 1 that the points $(s,\xi(s))$ all lie in the rectangle R for $s<t_0+\alpha$. Consequently,

$$|y_n(t)-y_{n-1}(t)| \leqslant L\int_{t_0}^{t}|y_{n-1}(s)-y_{n-2}(s)|\,ds, \qquad t_0\leqslant t\leqslant t_0+\alpha, \tag{8}$$

where

$$L = \max_{(t,y)\text{ in }R}\left|\frac{\partial f(t,y)}{\partial y}\right|. \tag{9}$$

Equation (9) defines the constant L. Setting $n=2$ in (8) gives

$$\begin{aligned}|y_2(t)-y_1(t)| &\leqslant L\int_{t_0}^{t}|y_1(s)-y_0|\,ds \leqslant L\int_{t_0}^{t}M(s-t_0)\,ds \\ &= \frac{LM(t-t_0)^2}{2}.\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}|y_3(t)-y_2(t)| &\leqslant L\int_{t_0}^{t}|y_2(s)-y_1(s)|\,ds \leqslant ML^2\int_{t_0}^{t}\frac{(s-t_0)^2}{2}\,ds \\ &= \frac{ML^2(t-t_0)^3}{3!}.\end{aligned}$$

Proceeding inductively, we see that

$$|y_n(t)-y_{n-1}(t)| \leqslant \frac{ML^{n-1}(t-t_0)^n}{n!}, \quad \text{for } t_0\leqslant t\leqslant t_0+\alpha. \tag{10}$$

Therefore, for $t_0 \leqslant t \leqslant t_0 + \alpha$,

$$|y_1(t) - y_0(t)| + |y_2(t) - y_1(t)| + \dots$$

$$\begin{aligned}
&\leqslant M(t-t_0) + \frac{ML(t-t_0)^2}{2!} + \frac{ML^2(t-t_0)^3}{3!} + \dots \\
&\leqslant M\alpha + \frac{ML\alpha^2}{2!} + \frac{ML^2\alpha^3}{3!} + \dots \\
&= \frac{M}{L}\left[\alpha L + \frac{(\alpha L)^2}{2!} + \frac{(\alpha L)^3}{3!} + \dots\right] \\
&= \frac{M}{L}(e^{\alpha L} - 1).
\end{aligned}$$

This quantity, obviously, is less than infinity. Consequently, the Picard iterates $y_n(t)$ converge for each t in the interval $t_0 \leqslant t \leqslant t_0 + \alpha$. (A similar argument shows that $y_n(t)$ converges for each t in the interval $t_0 - \beta \leqslant t \leqslant t_0$, where $\beta = \min(a, b/N)$, and N is the maximum value of $|f(t,y)|$ for (t,y) in the rectangle $t_0 - a \leqslant t \leqslant t_0, |y - y_0| \leqslant b$.) We will denote the limit of the sequence $y_n(t)$ by $y(t)$. □

(c) *Proof that $y(t)$ satisfies the initial-value problem* (1)

We will show that $y(t)$ satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds \tag{11}$$

and that $y(t)$ is continuous. To this end, recall that the Picard iterates $y_n(t)$ are defined recursively through the equation

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds. \tag{12}$$

Taking limits of both sides of (12) gives

$$y(t) = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s))\,ds. \tag{13}$$

To show that the right-hand side of (13) equals

$$y_0 + \int_{t_0}^{t} f(s, y(s))\,ds,$$

(that is, to justify passing the limit through the integral sign) we must show that

$$\left|\int_{t_0}^{t} f(s, y(s))\,ds - \int_{t_0}^{t} f(s, y_n(s))\,ds\right|$$

approaches zero as n approaches infinity. This is accomplished in the following manner. Observe first that the graph of $y(t)$ lies in the rectangle R for $t \leqslant t_0 + \alpha$, since it is the limit of functions $y_n(t)$ whose graphs lie in R.

Hence

$$\left|\int_{t_0}^{t} f(s,y(s))\,ds - \int_{t_0}^{t} f(s,y_n(s))\,ds\right|$$

$$\leqslant \int_{t_0}^{t} |f(s,y(s)) - f(s,y_n(s))|\,ds \leqslant L\int_{t_0}^{t} |y(s)-y_n(s)|\,ds$$

where L is defined by Equation (9). Next, observe that

$$y(s)-y_n(s) = \sum_{j=n+1}^{\infty} \left[y_j(s)-y_{j-1}(s)\right]$$

since

$$y(s) = y_0 + \sum_{j=1}^{\infty} \left[y_j(s)-y_{j-1}(s)\right]$$

and

$$y_n(s) = y_0 + \sum_{j=1}^{n} \left[y_j(s)-y_{j-1}(s)\right].$$

Consequently, from (10),

$$|y(s)-y_n(s)| \leqslant M \sum_{j=n+1}^{\infty} L^{j-1}\frac{(s-t_0)^j}{j!}$$

$$\leqslant M \sum_{j=n+1}^{\infty} \frac{L^{j-1}\alpha^j}{j!} = \frac{M}{L} \sum_{j=n+1}^{\infty} \frac{(\alpha L)^j}{j!}, \tag{14}$$

and

$$\left|\int_{t_0}^{t} f(s,y(s))\,ds - \int_{t_0}^{t} f(s,y_n(s))\,ds\right| \leqslant M \sum_{j=n+1}^{\infty} \frac{(\alpha L)^j}{j!}\int_{t_0}^{t} ds$$

$$\leqslant M\alpha \sum_{j=n+1}^{\infty} \frac{(\alpha L)^j}{j!}.$$

This summation approaches zero as n approaches infinity, since it is the tail end of the convergent Taylor series expansion of $e^{\alpha L}$. Hence,

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s,y_n(s))\,ds = \int_{t_0}^{t} f(s,y(s))\,ds,$$

and $y(t)$ satisfies (11).

To show that $y(t)$ is continuous, we must show that for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|y(t+h) - y(t)| < \varepsilon \quad \text{if } |h| < \delta.$$

Now, we cannot compare $y(t+h)$ with $y(t)$ directly, since we do not know $y(t)$ explicitly. To overcome this difficulty, we choose a large integer N and observe that

$$y(t+h)-y(t)=\left[y(t+h)-y_N(t+h)\right]$$
$$+\left[y_N(t+h)-y_N(t)\right]+\left[y_N(t)-y(t)\right].$$

Specifically, we choose N so large that

$$\frac{M}{L}\sum_{j=N+1}^{\infty}\frac{(\alpha L)^j}{j!}<\frac{\varepsilon}{3}.$$

Then, from (14),

$$|y(t+h)-y_N(t+h)|<\frac{\varepsilon}{3} \quad \text{and} \quad |y_N(t)-y(t)|<\frac{\varepsilon}{3},$$

for $t<t_0+\alpha$, and h sufficiently small (so that $t+h<t_0+\alpha$.) Next, observe that $y_N(t)$ is continuous, since it is obtained from N repeated integrations of continuous functions. Therefore, we can choose $\delta>0$ so small that

$$|y_N(t+h)-y_N(t)|<\frac{\varepsilon}{3} \quad \text{for } |h|<\delta.$$

Consequently,

$$|y(t+h)-y(t)|\leqslant|y(t+h)-y_N(t+h)|+|y_N(t+h)-y_N(t)|$$
$$+|y_N(t)-y(t)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

for $|h|<\delta$. Therefore, $y(t)$ is a continuous solution of the integral equation (11), and this completes our proof that $y(t)$ satisfies (1). □

In summary, we have proven the following theorem.

Theorem 2. *Let f and $\partial f/\partial y$ be continuous in the rectangle $R: t_0\leqslant t\leqslant t_0+a$, $|y-y_0|\leqslant b$. Compute*

$$M=\max_{(t,y)\text{ in }R}|f(t,y)|, \quad \textit{and set} \quad \alpha=\min\left(a,\frac{b}{M}\right).$$

Then, the initial-value problem $y'=f(t,y)$, $y(t_0)=y_0$ has at least one solution $y(t)$ on the interval $t_0\leqslant t\leqslant t_0+\alpha$. A similar result is true for $t<t_0$.

Remark. The number α in Theorem 2 depends specifically on our choice of a and b. Different choices of a and b lead to different values of α. Moreover, α doesn't necessarily increase when a and b increase, since an increase in a or b will generally result in an increase in M.

Finally, we turn our attention to the problem of uniqueness of solutions of (1). Consider the initial-value problem

$$\frac{dy}{dt}=(\sin 2t)y^{1/3}, \qquad y(0)=0. \tag{15}$$

One solution of (15) is $y(t)=0$. Additional solutions can be obtained if we ignore the fact that $y(0)=0$ and rewrite the differential equation in the form

$$\frac{1}{y^{1/3}}\frac{dy}{dt}=\sin 2t,$$

or

$$\frac{d}{dt}\frac{3y^{2/3}}{2}=\sin 2t.$$

Then,

$$\frac{3y^{2/3}}{2}=\frac{1-\cos 2t}{2}=\sin^2 t$$

and $y=\pm\sqrt{8/27}\,\sin^3 t$ are two additional solutions of (15).

Now, initial-value problems that have more than one solution are clearly unacceptable in applications. Therefore, it is important for us to find out exactly what is "wrong" with the initial-value problem (15) that it has more than one solution. If we look carefully at the right-hand side of this differential equation, we see that it does not have a partial derivative with respect to y at $y=0$. This is indeed the problem, as the following theorem shows.

Theorem 2′. *Let f and $\partial f/\partial y$ be continuous in the rectangle $R: t_0\leqslant t\leqslant t_0+a$, $|y-y_0|\leqslant b$. Compute*

$$M=\max_{(t,y)\text{ in }R}|f(t,y)|,\quad \text{and set}\quad \alpha=\min\left(a,\frac{b}{M}\right).$$

Then, the initial-value problem

$$y'=f(t,y),\qquad y(t_0)=y_0 \tag{16}$$

has a unique solution $y(t)$ on the interval $t_0\leqslant t\leqslant t_0+\alpha$. In other words, if $y(t)$ and $z(t)$ are two solutions of (16), *then $y(t)$ must equal $z(t)$ for $t_0\leqslant t\leqslant t_0+\alpha$.*

PROOF. Theorem 2 guarantees the existence of at least one solution $y(t)$ of (16). Suppose that $z(t)$ is a second solution of (16). Then,

$$y(t)=y_0+\int_{t_0}^{t}f(s,y(s))\,ds\quad\text{and}\quad z(t)=y_0+\int_{t_0}^{t}f(s,z(s))\,ds.$$

Subtracting these two equations gives

$$\begin{aligned}|y(t)-z(t)|&=\left|\int_{t_0}^{t}\left[f(s,y(s))-f(s,z(s))\right]ds\right|\\&\leqslant\int_{t_0}^{t}|f(s,y(s))-f(s,z(s))|\,ds\\&\leqslant L\int_{t_0}^{t}|y(s)-z(s)|\,ds\end{aligned}$$

where L is the maximum value of $|\partial f/\partial y|$ for (t,y) in R. As Lemma 2 below shows, this inequality implies that $y(t)=z(t)$. Hence, the initial-value problem (16) has a unique solution $y(t)$. □

Lemma 2. *Let $w(t)$ be a nonnegative function, with*

$$w(t)\leqslant L\int_{t_0}^{t} w(s)\,ds. \tag{17}$$

Then, $w(t)$ is identically zero.

FAKE PROOF. Differentiating both sides of (17) gives

$$\frac{dw}{dt}\leqslant Lw(t),\quad \text{or}\quad \frac{dw}{dt}-Lw(t)\leqslant 0.$$

Multiplying both sides of this inequality by the integrating factor $e^{-L(t-t_0)}$ gives

$$\frac{d}{dt}e^{-L(t-t_0)}w(t)\leqslant 0,\quad \text{so that}\quad e^{-L(t-t_0)}w(t)\leqslant w(t_0)$$

for $t\geqslant t_0$. But $w(t_0)$ must be zero if $w(t)$ is nonnegative and satisfies (17). Consequently, $e^{-L(t-t_0)}w(t)\leqslant 0$, and this implies that $w(t)$ is identically zero.

The error in this proof, of course, is that we cannot differentiate both sides of an inequality, and still expect to preserve the inequality. For example, the function $f_1(t)=2t-2$ is less than $f_2(t)=t$ on the interval $[0,1]$, but $f_1'(t)$ is greater than $f_2'(t)$ on this interval. We make this proof "kosher" by the clever trick of setting

$$U(t)=\int_{t_0}^{t} w(s)\,ds.$$

Then,

$$\frac{dU}{dt}=w(t)\leqslant L\int_{t_0}^{t} w(s)\,ds=LU(t).$$

Consequently, $e^{-L(t-t_0)}U(t)\leqslant U(t_0)=0$, for $t\geqslant t_0$, and thus $U(t)=0$. This, in turn, implies that $w(t)=0$ since

$$0\leqslant w(t)\leqslant L\int_{t_0}^{t} w(s)\,ds=LU(t)=0.$$

□

Example 3. Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt}=t^2+e^{-y^2},\qquad y(0)=0$$

exists for $0\leqslant t\leqslant \frac{1}{2}$, and in this interval, $|y(t)|\leqslant 1$.

Solution. Let R be the rectangle $0 \leqslant t \leqslant \frac{1}{2}$, $|y| \leqslant 1$. Computing

$$M = \max_{(t,y) \text{ in } R} t^2 + e^{-y^2} = 1 + \left(\tfrac{1}{2}\right)^2 = \tfrac{5}{4},$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \min\left(\frac{1}{2}, \frac{1}{5/4}\right) = \frac{1}{2},$$

and in this interval, $|y(t)| \leqslant 1$.

Example 4. Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \qquad y(0) = 1$$

exists for $0 \leqslant t \leqslant 1/9$, and in this interval, $0 \leqslant y \leqslant 2$.
Solution. Let R be the rectangle $0 \leqslant t \leqslant \frac{1}{9}$, $0 \leqslant y \leqslant 2$. Computing

$$M = \max_{(t,y) \text{ in } R} e^{-t^2} + y^3 = 1 + 2^3 = 9,$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \min\left(\tfrac{1}{9}, \tfrac{1}{9}\right)$$

and in this interval, $0 \leqslant y \leqslant 2$.

Example 5. What is the largest interval of existence that Theorem 2 predicts for the solution $y(t)$ of the initial-value problem $y' = 1 + y^2$, $y(0) = 0$?
Solution. Let R be the rectangle $0 \leqslant t \leqslant a$, $|y| \leqslant b$. Computing

$$M = \max_{(t,y) \text{ in } R} 1 + y^2 = 1 + b^2,$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \alpha = \min\left(a, \frac{b}{1+b^2}\right).$$

Clearly, the largest α that we can achieve is the maximum value of the function $b/(1+b^2)$. This maximum value is $\frac{1}{2}$. Hence, Theorem 2 predicts that $y(t)$ exists for $0 \leqslant t \leqslant \frac{1}{2}$. The fact that $y(t) = \tan t$ exists for $0 \leqslant t \leqslant \pi/2$ points out the limitation of Theorem 2.

Example 6. Suppose that $|f(t,y)| \leqslant K$ in the strip $t_0 \leqslant t < \infty$, $-\infty < y < \infty$. Show that the solution $y(t)$ of the initial-value problem $y' = f(t,y)$, $y(t_0) = y_0$ exists for all $t \geqslant t_0$.
Solution. Let R be the rectangle $t_0 \leqslant t \leqslant t_0 + a$, $|y - y_0| \leqslant b$. The quantity

$$M = \max_{(t,y) \text{ in } R} |f(t,y)|$$

is at most K. Hence, $y(t)$ exists for

$$t_0 \leqslant t \leqslant t_0 + \min(a, b/K).$$

Now, we can make the quantity $\min(a,b/K)$ as large as desired by choosing a and b sufficiently large. Therefore $y(t)$ exists for $t \geqslant t_0$.

Exercises

1. Construct the Picard iterates for the initial-value problem $y' = 2t(y+1)$, $y(0) = 0$ and show that they converge to the solution $y(t) = e^{t^2} - 1$.

2. Compute the first two Picard iterates for the initial-value problem $y' = t^2 + y^2$, $y(0) = 1$.

3. Compute the first three Picard iterates for the initial-value problem $y' = e^t + y^2$, $y(0) = 0$.

In each of Problems 4–15, show that the solution $y(t)$ of the given initial-value problem exists on the specified interval.

4. $y' = y^2 + \cos t^2, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \frac{1}{2}$

5. $y' = 1 + y + y^2 \cos t, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \frac{1}{3}$

6. $y' = t + y^2, y(0) = 0; 0 \leqslant t \leqslant (\frac{1}{2})^{2/3}$

7. $y' = e^{-t^2} + y^2, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \frac{1}{2}$

8. $y' = e^{-t^2} + y^2, y(1) = 0; 1 \leqslant t \leqslant 1 + \sqrt{e}/2$

9. $y' = e^{-t^2} + y^2, \quad y(0) = 1; \quad 0 \leqslant t \leqslant \dfrac{\sqrt{2}}{1 + (1+\sqrt{2})^2}$

10. $y' = y + e^{-y} + e^{-t}, \quad y(0) = 0; \quad 0 \leqslant t \leqslant 1$

11. $y' = y^3 + e^{-5t}, \quad y(0) = 0.4; \quad 0 \leqslant t \leqslant \frac{3}{10}$

12. $y' = e^{(y-t)^2}, \quad y(0) = 1; \quad 0 \leqslant t \leqslant \dfrac{\sqrt{3}-1}{2} e^{-((1+\sqrt{3})/2)^2}$

13. $y' = (4y + e^{-t^2})e^{2y}, \quad y(0) = 0; \quad 0 \leqslant t \leqslant \dfrac{1}{8\sqrt{e}}$

14. $y' = e^{-t} + \ln(1 + y^2), \quad y(0) = 0; \quad 0 \leqslant t < \infty$

15. $y' = \frac{1}{4}(1 + \cos 4t)y - \frac{1}{800}(1 - \cos 4t)y^2, \quad y(0) = 100; \quad 0 \leqslant t \leqslant 1$

16. Consider the initial-value problem

$$y' = t^2 + y^2, \qquad y(0) = 0, \tag{*}$$

and let R be the rectangle $0 \leqslant t \leqslant a, \ -b \leqslant y \leqslant b$.
(a) Show that the solution $y(t)$ of (*) exists for

$$0 \leqslant t \leqslant \min\left(a, \frac{b}{a^2 + b^2}\right).$$

(b) Show that the maximum value of $b/(a^2+b^2)$, for a fixed, is $1/2a$.
(c) Show that $\alpha=\min(a,\frac{1}{2}a)$ is largest when $a=1/\sqrt{2}$.
(d) Conclude that the solution $y(t)$ of (*) exists for $0\leqslant t\leqslant 1/\sqrt{2}$.

17. Prove that $y(t)=-1$ is the only solution of the initial-value problem

$$y'=t(1+y), \qquad y(0)=-1.$$

18. Find a nontrivial solution of the initial-value problem $y'=ty^a$, $y(0)=0$, $a>1$. Does this violate Theorem 2′? Explain.

19. Find a solution of the initial-value problem $y'=t\sqrt{1-y^2}$, $y(0)=1$, other than $y(t)=1$. Does this violate Theorem 2′? Explain.

20. Here is an alternate proof of Lemma 2. Let $w(t)$ be a nonnegative function with

$$w(t)\leqslant L\int_{t_0}^{t} w(s)\,ds \tag{*}$$

on the interval $t_0\leqslant t\leqslant t_0+\alpha$. Since $w(t)$ is continuous, we can find a constant A such that $0\leqslant w(t)\leqslant A$ for $t_0\leqslant t\leqslant t_0+\alpha$.
(a) Show that $w(t)\leqslant LA(t-t_0)$.
(b) Use this estimate of $w(t)$ in (*) to obtain

$$w(t)\leqslant\frac{AL^2(t-t_0)^2}{2}.$$

(c) Proceeding inductively, show that $w(t)\leqslant AL^n(t-t_0)^n/n!$, for every integer n.
(d) Conclude that $w(t)=0$ for $t_0\leqslant t\leqslant t_0+\alpha$.

1.11 Finding roots of equations by iteration

Suppose that we are interested in finding the roots of an equation having the special form

$$x=f(x). \tag{1}$$

For example, we might want to find the roots of the equation

$$x=\sin x+\tfrac{1}{4}.$$

The methods introduced in the previous section suggest the following algorithm for solving this problem.

1. Try an initial guess x_0, and use this number to construct a sequence of guesses $x_1,x_2,x_3,\ldots$, where $x_1=f(x_0)$, $x_2=f(x_1)$, $x_3=f(x_2)$, and so on.
2. Show that this sequence of iterates x_n has a limit η as n approaches infinity.
3. Show that η is a root of (1); i.e., $\eta=f(\eta)$.

The following theorem tells us when this algorithm will work.

Theorem 3. *Let $f(x)$ and $f'(x)$ be continuous in the interval $a\leqslant x\leqslant b$, with $|f'(x)|\leqslant\lambda<1$ in this interval. Suppose, moreover, that the iterates x_n, de-*

fined recursively by the equation

$$x_{n+1}=f(x_n) \tag{2}$$

all lie in the interval $[a,b]$. *Then, the iterates* x_n *converge to a unique number* η *satisfying* (1).

PROOF. We convert the problem of proving that the sequence x_n converges to the simpler problem of proving that an infinite series converges by writing x_n in the form

$$x_n=x_0+(x_1-x_0)+(x_2-x_1)+\ldots+(x_n-x_{n-1}).$$

Clearly, the sequence x_n converges if, and only if, the infinite series

$$(x_1-x_0)+(x_2-x_1)+\ldots+(x_n-x_{n-1})+\ldots=\sum_{n=1}^{\infty}(x_n-x_{n-1})$$

converges. To prove that this infinite series converges, it suffices to show that

$$|x_1-x_0|+|x_2-x_1|+\ldots=\sum_{n=1}^{\infty}|x_n-x_{n-1}|<\infty.$$

This is accomplished in the following manner. By definition, $x_n=f(x_{n-1})$ and $x_{n-1}=f(x_{n-2})$. Subtracting these two equations gives

$$x_n-x_{n-1}=f(x_{n-1})-f(x_{n-2})=f'(\xi)(x_{n-1}-x_{n-2}),$$

where ξ is some number between x_{n-1} and x_{n-2}. In particular, ξ is in the interval $[a,b]$. Therefore, $|f'(\xi)|\leqslant\lambda$, and

$$|x_n-x_{n-1}|\leqslant\lambda|x_{n-1}-x_{n-2}|. \tag{3}$$

Iterating this inequality $n-1$ times gives

$$\begin{aligned}|x_n-x_{n-1}|&\leqslant\lambda|x_{n-1}-x_{n-2}|\\&\leqslant\lambda^2|x_{n-2}-x_{n-3}|\\&\ \ \vdots\\&\leqslant\lambda^{n-1}|x_1-x_0|.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{n=1}^{\infty}|x_n-x_{n-1}|&\leqslant\sum_{n=1}^{\infty}\lambda^{n-1}|x_1-x_0|\\&=|x_1-x_0|[1+\lambda+\lambda^2+\ldots]=\frac{|x_1-x_0|}{1-\lambda}.\end{aligned}$$

This quantity, obviously, is less than infinity. Therefore, the sequence of iterates x_n has a limit η as n approaches infinity. Taking limits of both sides of (2) gives

$$\eta = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\eta).$$

Hence, η is a root of (1).

Finally, suppose that η is not unique; that is, there exist two solutions η_1 and η_2 of (1) in the interval $[a,b]$. Then,

$$\eta_1 - \eta_2 = f(\eta_1) - f(\eta_2) = f'(\xi)(\eta_1 - \eta_2),$$

where ξ is some number between η_1 and η_2. This implies that $\eta_1 = \eta_2$ or $f'(\xi) = 1$. But $f'(\xi)$ cannot be one, since ξ is in the interval $[a,b]$. Therefore, $\eta_1 = \eta_2$. ☐

Example 1. Show that the sequence of iterates

$$x_0, \quad x_1 = 1 + \tfrac{1}{2}\arctan x_0, \quad x_2 = 1 + \tfrac{1}{2}\arctan x_1, \ldots$$

converge to a unique number η satisfying

$$\eta = 1 + \tfrac{1}{2}\arctan \eta$$

for every initial guess x_0.

Solution. Let $f(x) = 1 + \frac{1}{2}\arctan x$. Computing $f'(x) = \frac{1}{2} 1/(1+x^2)$, we see that $|f'(x)|$ is always less than or equal to $\frac{1}{2}$. Hence, by Theorem 3, the sequence of iterates $x_0, x_1, x_2, \ldots$ converges to the unique root η of the equation $x = 1 + \frac{1}{2}\arctan x$, for every choice of x_0.

There are many instances where we know, a priori, that the equation $x = f(x)$ has a unique solution η in a given interval $[a,b]$. In these instances, we can use Theorem 3 to obtain a very good approximation of η. Indeed, life is especially simple in these instances, since we don't have to check that the iterates x_n all lie in a specified interval. If x_0 is sufficiently close to η, then the iterates x_n will always converge to η, as we now show.

Theorem 4. *Assume that $f(\eta) = \eta$, and that $|f'(x)| \leqslant \lambda < 1$ in the interval $|x - \eta| \leqslant \alpha$. Choose a number x_0 in this interval. Then, the sequence of iterates x_n, defined recursively by the equation $x_{n+1} = f(x_n)$, will always converge to η.*

PROOF. Denote the interval $|x - \eta| \leqslant \alpha$ by I. By Theorem 3, it suffices to show that all the iterates x_n lie in I. To this end, observe that

$$x_{j+1} - \eta = f(x_j) - f(\eta) = f'(\xi)(x_j - \eta)$$

where ξ is some number between x_j and η. In particular, ξ is in I if x_j is in

I. Thus,

$$|x_{j+1}-\eta| \leqslant \lambda |x_j-\eta| < |x_j-\eta| \tag{4}$$

if x_j is in I. This implies that x_{j+1} is in I whenever x_j is in I. By induction, therefore, all the iterates x_n lie in I. □

Equation (4) also shows that x_{n+1} is closer to η than x_n. Specifically, the error we make in approximating η by x_n decreases by at least a factor of λ each time we increase n. Thus, if λ is very small, then the convergence of x_n to η is very rapid, while if λ is close to one, then the convergence is very slow.

Example 2.
(a) Show that the equation

$$x = \sin x + \tfrac{1}{4} \tag{5}$$

has a unique root η in the interval $[\pi/4, \pi/2]$.
(b) Show that the sequence of numbers

$$x_0, \quad x_1 = \sin x_0 + \tfrac{1}{4}, \quad x_2 = \sin x_1 + \tfrac{1}{4}, \ldots$$

will converge to η if $\pi/4 \leqslant x_0 \leqslant \pi/2$.
(c) Write a computer program to evaluate the first N iterates $x_1, x_2, \ldots, x_N$.
Solution.
(a) Let $g(x) = x - \sin x - \frac{1}{4}$, and observe that $g(\pi/4)$ is negative while $g(\pi/2)$ is positive. Moreover, $g(x)$ is a monotonic increasing function of x for $\pi/4 \leqslant x \leqslant \pi/2$, since its derivative is strictly positive in this interval. Therefore, Equation (5) has a unique root $x = \eta$ in the interval $\pi/4 < x < \pi/2$.
(b) Let I denote the interval $\eta - \pi/4 \leqslant x \leqslant \eta + \pi/4$. The left endpoint of this interval is greater than zero, while the right endpoint is less than $3\pi/4$. Hence, there exists a number λ, with $0 < \lambda < 1$, such that

$$|\cos x| = \left| \frac{d}{dx}\left(\sin x + \tfrac{1}{4}\right) \right| \leqslant \lambda$$

for x in I. Clearly, the interval $[\pi/4, \pi/2]$ is contained in I. Therefore, by Theorem 4, the sequence of numbers

$$x_0, \quad x_1 = \sin x_0 + \tfrac{1}{4}, \quad x_2 = \sin x_1 + \tfrac{1}{4}, \ldots$$

will converge to η for every x_0 in the interval $[\pi/4, \pi/2]$.

(c)

Pascal Program

```
Program Iterate (input, output);
var
  X: array[0..199] of real;
  k, N: integer;

begin
  readln(X[0], N);
  page;
  writeln('N':4, 'X[N]':14);
  for k := 0 to N do
    begin
      writeln(K:4, '  ':4, X[k]:17:9);
      X[k+1] := 0.25+sin(X[k]);
    end;
end.
```

Fortran Program

```
      DIMENSION X(200)
      READ (5,10) X0,N
   10 FORMAT (F15.8,I5)
C     COMPUTE X(1) FIRST
      X(1)=0.25+SIN(X0)
      KA=0
      KB=1
      WRITE (6,20) KA,X0,KB,X(1)
   20 FORMAT (1H1,4X,'N',10X,'X'/(1H,3X,I3,4X,F15.9))
C     COMPUTE X(2) THRU X(N)
      D0 40 K=2,N
      X(K)=0.25+SIN(X(K-1))
      WRITE (6,30) K,X(K)
   30 FORMAT (1H,3X,I3,4X,F15.9)
   40 CONTINUE
      CALL EXIT
      END
```

See also C Program 1 in Appendix C for a sample C program.

Table 1

n	x_n	n	x_n
0	1	8	1.17110411
1	1.09147099	9	1.17122962
2	1.13730626	10	1.17122964
3	1.15750531	11	1.17122965
4	1.16580403	12	1.17122965
5	1.16910543	13	1.17122965
6	1.17040121	14	1.17122965
7	1.17090706	15	1.17122965

In many instances, we want to compute a root of the equation $x=f(x)$ to within a certain accuracy. The easiest, and most efficient way of accomplishing this is to instruct the computer to terminate the program at $k=j$ if x_{j+1} agrees with x_j within the prescribed accuracy.

Exercises

1. Let η be the unique root of Equation (5).
(a) Let $x_0=\pi/4$. Show that 20 iterations are required to find η to 8 significant decimal places.
(b) Let $x_0=\pi/2$. Show that 20 iterations are required to find η to 8 decimal places.
(c) Let $x_0=3\pi/8$. Show that 16 iterations are required to find η to 8 decimal places.

2. (a) Determine suitable values of x_0 so that the iterates x_n, defined by the equation

$$x_{n+1}=x_n-\tfrac{1}{4}(x_n^2-2)$$

will converge to $\sqrt{2}$.
(b) Choose $x_0=1.4$. Show that 14 iterations are required to find $\sqrt{2}$ to 8 significant decimal places. ($\sqrt{2}=1.41421356$ to 8 significant decimal places.)

3. (a) Determine suitable values of x_0 so that the iterates x_n, defined by the equation

$$x_{n+1}=x_n-\tfrac{1}{10}(x_n^2-2)$$

will converge to $\sqrt{2}$.
(b) Choose $x_0=1.4$. Show that 30 iterations are required to find $\sqrt{2}$ to 6 significant decimal places.

4. (a) Determine a suitable value of α so that the iterates x_n, defined by the equation

$$x_{n+1}=x_n-\alpha(x_n^2-3), \qquad x_0=1.7$$

will converge to $\sqrt{3}$.
(b) Find $\sqrt{3}$ to 6 significant decimal places.

5. Let η be the unique root of the equation $x=1+\frac{1}{2}\arctan x$. Find η to 5 significant decimal places.

6. (a) Show that the equation $2-x=(\ln x)/4$ has a unique root $x=\eta$ in the interval $0<x<\infty$.
(b) Let

$$x_{n+1}=2-(\ln x_n)/4, \qquad n=0,1,2,\dots$$

Show that $1\leqslant x_n\leqslant 2$ if $1\leqslant x_0\leqslant 2$.
(c) Prove that $x_n\to\eta$ as $n\to\infty$ if $1\leqslant x_0\leqslant 2$.
(d) Compute η to 5 significant decimal places.

7. (a) Show that the equation $x=\cos x$ has a unique root $x=\eta$ in the interval $0\leqslant x\leqslant 1$.
(b) Let $x_{n+1}=\cos x_n$, $n=0,1,2,\dots$, with $0<x_0<1$. Show that $0<x_n<1$. Conclude, therefore, that $x_n\to\eta$ as $n\to\infty$.
(c) Find η to 5 significant decimal places.

1.11.1 Newton's method

The method of iteration which we used to solve the equation $x=f(x)$ can also be used to solve the equation $g(x)=0$. To wit, any solution $x=\eta$ of the equation $g(x)=0$ is also a solution of the equation

$$x=f(x)=x-g(x), \tag{1}$$

and vice-versa. Better yet, any solution $x=\eta$ of the equation $g(x)=0$ is also a solution of the equation

$$x=f(x)=x-\frac{g(x)}{h(x)} \tag{2}$$

for any function $h(x)$. Of course, $h(x)$ must be unequal to zero for x near η.

Equation (2) has an arbitrary function $h(x)$ in it. Let us try and choose $h(x)$ so that (i) the assumptions of Theorem 4, Section 1.11 are satisfied, and (ii) the iterates

$$x_0, \quad x_1=x_0-\frac{g(x_0)}{h(x_0)}, \quad x_2=x_1-\frac{g(x_1)}{h(x_1)},\dots$$

converge as "rapidly as possible" to the desired root η. To this end, we compute

$$f'(x)=\frac{d}{dx}\left[x-\frac{g(x)}{h(x)}\right]=1-\frac{g'(x)}{h(x)}+\frac{h'(x)g(x)}{h^2(x)}$$

and observe that

$$f'(\eta)=1-\frac{g'(\eta)}{h(\eta)}.$$

This suggests that we set $h(x)=g'(x)$, since then $f'(\eta)=0$. Consequently, the iterates x_n, defined recursively by the equation

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)}, \qquad n=0,1,2,\ldots \tag{3}$$

will converge to η if the initial guess x_0 is sufficiently close to η. (If $f'(\eta)=0$, then $|f'(x)|\leqslant\lambda<1$ for $|x-\eta|$ sufficiently small.) Indeed, the choice of $h(x)=f'(x)$ is an *optimal* choice of $h(x)$, since the convergence of x_n to η will be extremely rapid. This follows immediately from the fact that the number λ in Equation 4, Section 1.11 can be taken arbitrarily small, as x_n approaches η.

The iteration scheme (3) is known as Newton's method for solving the equation $g(x)=0$. It can be shown that if $g(\eta)=0$, and x_0 is sufficiently close to η, then

$$|x_{n+1}-\eta|\leqslant c|x_n-\eta|^2,$$

for some positive constant c. In other words, the error we make in approximating η by x_{n+1} is proportional to the square of the error we make in approximating η by x_n. This type of convergence is called quadratic convergence, and it implies that the iterates x_n converge extremely rapidly to η. In many instances, only five or six iterations are required to find η to eight or more significant decimal places.

Example 1. Use Newton's method to compute $\sqrt{2}$.
Solution. The square root of two is a solution of the equation

$$g(x)=x^2-2=0.$$

Hence, Newton's scheme for this problem is

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)}=x_n-\frac{(x_n^{\,2}-2)}{2x_n}$$
$$=\frac{x_n}{2}+\frac{1}{x_n}, \qquad n=0,1,2,\ldots. \tag{4}$$

Sample Pascal and Fortran programs to compute the first N iterates of an initial guess x_0 are given below.

Pascal Program

```
Program Newton (input, output);
var
  X: array[0..199] of real;
  k, N: integer;

begin
  readln(X[0], N);
  page;
```

```
  writeln('N':4, 'X[N]':14);
  for k := 0 to N do
    begin
      writeln(K:4, ' ':4, X[k]:17:9);
      X[k+1] := X[k]/2+1/X[k];
    end;
end.
```

Table 1

n	x_n	n	x_n
0	1.4	3	1.41421356
1	1.41428571	4	1.41421356
2	1.41421356	5	1.41421356

Fortran Program

We need only replace the instructions for computing X(1) and X(K) in the Fortran program of Section 1.11 by

```
X(1) = (X0/2) + 1/X0
```

and

```
X(K) = (X(K-1)/2) + 1/X(K-1)
```

We ran these programs for $x_0 = 1.4$ and $N = 5$, and the results are given in Table 1. Notice that Newton's method requires only 2 iterations to find $\sqrt{2}$ to eight significant decimal places.

See also C Program 2 in Appendix C for a sample C program.

Example 2. Use Newton's method to find the impact velocity of the drums in Section 1.7.

Solution. The impact velocity of the drums satisfies the equation

$$g(v) = v + \frac{300cg}{W} + \frac{W-B}{c}\ln\left[\frac{W-B-cv}{W-B}\right] = 0 \tag{5}$$

where

$$c = 0.08, \quad g = 32.2, \quad W = 527.436, \quad \text{and} \quad B = 470.327.$$

Setting $a = (W-B)/c$ and $d = 300cg/W$ puts (5) in the simpler form

$$g(v) = v + d + a\ln(1 - v/a) = 0. \tag{6}$$

Newton's iteration scheme for this problem is

$$v_{n+1}=v_n-\frac{g(v_n)}{g'(v_n)}=v_n+\frac{(1-v_n/a)\left[v_n+d+a\ln(1-v_n/a)\right]}{v_n/a}$$
$$=v_n+\frac{a-v_n}{v_n}\left[v_n+d+a\ln(1-v_n/a)\right],\qquad n=0,1,2,\dots.$$

Sample Pascal and Fortran programs to compute the first N iterates of v_0 are given below.

Pascal Program

```
Program Newton (input, output);

const
  c =0.08;
  g =32.2;
  W=527.436;
  B =470.327;

var
  V: array[0..199] of real;
  a, d: real;
  k, N: integer;

begin
  readln(V[0], N);
  a:=(W-B)/c;
  d:=300*c*g/W;
  page;
  writeln('N':4, 'V[N]':14);
  for k:=0 to N do
    begin
      writeln(K:4, ' ':4, V[k]:17:9);
      V[k+1]:=V[k]+((a-V[k])/V[k])
                *(V[k]+d+a*ln(1-(V[k]/a)));
    end;
end.
```

Fortran Program

Change every X to V, and replace the instructions for X(1) and X(K) in the Fortran program of Section 1.11 by

```
V(1)=V0+((A-V0)/V0)*(V0+D+A*A LOG(1-(V0/A)))
```

and

```
V(K)=V(K-1)+((A-V(K-1))/V(K-1))* (V(K-1)+D
        +A*A LOG (1-(V(K-1) / A)))
```

(Before running these programs, of course, we must instruct the computer to evaluate the constants $a=(W-B)/c$ and $d=300\ cg/W$.)

As was shown in Section 1.7, $v_0=45.7$ is a very good approximation of v. We set $v_0=45.7$ in the above programs, and the iterates v_n converged very rapidly to $v=45.1$ ft/s. Thus, the drums can indeed break upon impact.

In general, it is not possible to determine, a priori, how many iterations will be required to achieve a certain accuracy. In practice, we usually take N very large, and instruct the computer to terminate the program if one of the iterates agrees with its predecessor to the desired accuracy.

See also C Program 3 in Appendix C for a sample C program.

Exercises

1. Show that the iterates x_n defined by (4) will converge to $\sqrt{2}$ if

$$\sqrt{2/3} < x_0 < \sqrt{2} + \left(\sqrt{2} - \sqrt{2/3}\right).$$

2. Use Newton's method to find the following numbers to 8 significant decimal places. (a) $\sqrt{3}$, (b) $\sqrt{5}$, (c) $\sqrt{7}$.

3. The number π is a root of the equation

$$\tan\frac{x}{4} - \cot\frac{x}{4} = 0.$$

Use Newton's method to find π to 8 significant decimal places.

Show that each of the following equations has a unique solution in the given interval, and use Newton's method to find it to 5 significant decimal places.

4. $2x - \tan x = 0;\quad \pi \leqslant x \leqslant 3\pi/2$

5. $\frac{1}{2} - x + \frac{1}{5}\sin x = 0;\quad \frac{1}{2} \leqslant x \leqslant 1$

6. $\ln x + (x+1)^3 = 0;\quad 0 < x < 1$

7. $2\sqrt{x} = \cos\frac{\pi x}{2};\quad 0 \leqslant x \leqslant 1$

8. $(x-1)^2 - \frac{1}{2}e^x = 0;\quad 0 < x < 1$

9. $x - e^{-x^2} = 1;\quad 0 \leqslant x \leqslant 2.$

1.12 Difference equations, and how to compute the interest due on your student loans

In Sections 1.13–1.16 we will construct various approximations of the solution of the initial-value problem $dy/dt = f(t,y)$, $y(t_0) = y_0$. In determining how good these approximations are, we will be confronted with the following problem: How large can the numbers $E_1,\ldots,E_N$ be if

$$E_{n+1} \leqslant AE_n + B, \qquad n=0,1,\ldots,N-1 \tag{1}$$

for some positive constants A and B, and $E_0=0$? This is a very difficult problem since it deals with *inequalities*, rather than *equalities*. Fortunately, though, we can convert the problem of solving the inequalities (1) into the simpler problem of solving a system of equalities. This is the content of the following lemma.

Lemma 1. *Let $E_1,\ldots,E_N$ satisfy the inequalities*

$$E_{n+1} \leqslant AE_n + B, \qquad E_0=0$$

for some positive constants A and B. Then, E_n is less than or equal to y_n, where

$$y_{n+1}=Ay_n+B, \qquad y_0=0. \tag{2}$$

Proof. We prove Lemma 1 by induction on n. To this end, observe that Lemma 1 is obviously true for $n=0$. Next, we assume that Lemma 1 is true for $n=j$. We must show that Lemma 1 is also true for $n=j+1$. That is to say, we must prove that $E_j \leqslant y_j$ implies $E_{j+1} \leqslant y_{j+1}$. But this follows immediately, for if $E_j \leqslant y_j$ then

$$E_{j+1} \leqslant AE_j + B \leqslant Ay_j + B = y_{j+1}.$$

By induction, therefore, $E_n \leqslant y_n$, $n=0,1,\ldots,N$. □

Our next task is to solve Equation (2), which is often referred to as a difference equation. We will accomplish this in two steps. First we will solve the "simple" difference equation

$$y_{n+1}=y_n+B_n, \qquad y_0=y_0. \tag{3}$$

Then we will reduce the difference equation (2) to the difference equation (3) by a clever change of variables.

Equation (3) is trivial to solve. Observe that

$$\begin{aligned} y_1-y_0 &= B_0\\ y_2-y_1 &= B_1\\ &\vdots\\ y_{n-1}-y_{n-2} &= B_{n-2}\\ y_n-y_{n-1} &= B_{n-1}. \end{aligned}$$

Adding these equations gives

$$(y_n-y_{n-1})+(y_{n-1}-y_{n-2})+\ldots+(y_1-y_0)=B_0+B_1+\ldots+B_{n-1}.$$

Hence,

$$y_n=y_0+B_0+\ldots+B_{n-1}=y_0+\sum_{j=0}^{n-1}B_j.$$

Next, we reduce the difference equation (2) to the simpler equation (3) in the following clever manner. Let

$$z_n = \frac{y_n}{A^n}, \qquad n = 0, 1, \ldots, N.$$

Then, $z_{n+1} = y_{n+1}/A^{n+1}$. But $y_{n+1} = Ay_n + B$. Consequently,

$$z_{n+1} = \frac{y_n}{A^n} + \frac{B}{A^{n+1}} = z_n + \frac{B}{A^{n+1}}.$$

Therefore,

$$z_n = z_0 + \sum_{j=0}^{n-1} \frac{B}{A^{j+1}} = y_0 + \frac{B}{A}\left[\frac{1-\left(\frac{1}{A}\right)^n}{1-\frac{1}{A}}\right]$$

$$= y_0 + \frac{B}{A-1}\left[1-\left(\frac{1}{A}\right)^n\right]$$

and

$$y_n = A^n z_n = A^n y_0 + \frac{B}{A-1}(A^n - 1). \tag{4}$$

Finally, returning to the inequalities (1), we see that

$$E_n \leqslant \frac{B}{A-1}(A^n - 1), \qquad n = 1, 2, \ldots, N. \tag{5}$$

While collecting material for this book, this author was approached by a colleague with the following problem. He had just received a bill from the bank for the first payment on his wife's student loan. This loan was to be repaid in 10 years in 120 equal monthly installments. According to his rough estimate, the bank was overcharging him by at least 20%. Before confronting the bank's officers, though, he wanted to compute exactly the monthly payments due on this loan.

This problem can be put in the following more general framework. Suppose that P dollars are borrowed from a bank at an annual interest rate of $R\%$. This loan is to be repaid in n years in equal monthly installments of x dollars. Find x.

Our first step in solving this problem is to compute the interest due on the loan. To this end observe that the interest I_1 owed when the first payment is due is $I_1 = (r/12)P$, where $r = R/100$. The principal outstanding during the second month of the loan is $(x - I_1)$ less than the principal outstanding during the first month. Hence, the interest I_2 owed during the second month of the loan is

$$I_2 = I_1 - \frac{r}{12}(x - I_1).$$

Similarly, the interest I_{j+1} owed during the $(j+1)$st month is

$$I_{j+1}=I_j-\frac{r}{12}(x-I_j)=\left(1+\frac{r}{12}\right)I_j-\frac{r}{12}x, \tag{6}$$

where I_j is the interest owed during the jth month.

Equation (6) is a difference equation for the numbers

$$I_1=\frac{r}{12}P, I_2, \ldots, I_{12n}.$$

Its solution (see Exercise 4) is

$$I_j=\frac{r}{12}P\left(1+\frac{r}{12}\right)^{j-1}+x\left[1-\left(1+\frac{r}{12}\right)^{j-1}\right]$$

Hence, the total amount of interest paid on the loan is

$$\begin{aligned} I=I_1+I_2+\ldots+I_{12n} &= \sum_{j=1}^{12n} I_j \\ &= \frac{r}{12}P\sum_{j=1}^{12n}\left(1+\frac{r}{12}\right)^{j-1}+12nx-x\sum_{j=1}^{12n}\left(1+\frac{r}{12}\right)^{j-1} \end{aligned}$$

Now,

$$\sum_{j=1}^{12n}\left(1+\frac{r}{12}\right)^{j-1}=\frac{12}{r}\left[\left(1+\frac{r}{12}\right)^{12n}-1\right].$$

Therefore,

$$\begin{aligned} I &= P\left[\left(1+\frac{r}{12}\right)^{12n}-1\right]+12nx-\frac{12x}{r}\left[\left(1+\frac{r}{12}\right)^{12n}-1\right] \\ &= 12nx-P+P\left(1+\frac{r}{12}\right)^{12n}-\frac{12x}{r}\left[\left(1+\frac{r}{12}\right)^{12n}-1\right]. \end{aligned}$$

But, $12nx-P$ must equal I, since $12nx$ is the amount of money paid the bank and P was the principal loaned. Consequently,

$$P\left(1+\frac{r}{12}\right)^{12n}-\frac{12x}{r}\left[\left(1+\frac{r}{12}\right)^{12n}-1\right]=0$$

and this equation implies that

$$x=\frac{\frac{r}{12}P\left(1+\frac{r}{12}\right)^{12n}}{\left(1+\frac{r}{12}\right)^{12n}-1}. \tag{7}$$

Epilog. Using Equation (7), this author computed x for his wife's and his colleague's wife's student loans. In both cases the bank was right—to the penny.

Exercises

1. Solve the difference equation $y_{n+1} = -7y_n + 2, y_0 = 1$.

2. Find y_{37} if $y_{n+1} = 3y_n + 1$, $y_0 = 0$, $n = 0, 1, \ldots, 36$.

3. Estimate the numbers $E_0, E_1, \ldots, E_N$ if $E_0 = 0$ and
(a) $E_{n+1} \leqslant 3E_n + 1$, $n = 0, 1, \ldots, N-1$;
(b) $E_{n+1} \leqslant 2E_n + 2$, $n = 0, 1, \ldots, N-1$.

4. (a) Show that the transformation $y_j = I_{j+1}$ transforms the difference equation

$$I_{j+1} = \left(1 + \frac{r}{12}\right)I_j - \frac{r}{12}x, \qquad I_1 = \frac{r}{12}P$$

into the difference equation

$$y_{j+1} = \left(1 + \frac{r}{12}\right)y_j - \frac{r}{12}x, \qquad y_0 = \frac{r}{12}P.$$

(b) Use Equation (4) to find $y_{j-1} = I_j$.

5. Solve the difference equation $y_{n+1} = a_n y_n + b_n$, $y_1 = \alpha$. *Hint*: Set $z_1 = y_1$ and $z_n = y_n / a_1 \ldots a_{n-1}$ for $n \geqslant 2$. Observe that

$$\begin{aligned} z_{n+1} &= \frac{y_{n+1}}{a_1 \ldots a_n} = \frac{a_n y_n}{a_1 \ldots a_n} + \frac{b_n}{a_1 \ldots a_n} \\ &= z_n + \frac{b_n}{a_1 \ldots a_n}. \end{aligned}$$

Hence, conclude that $z_n = z_1 + \sum_{j=1}^{n-1} b_j / a_1 \ldots a_j$.

6. Solve the difference equation $y_{n+1} - ny_n = 1 - n$, $y_1 = 2$.

7. Find y_{25} if $y_1 = 1$ and $(n+1)y_{n+1} - ny_n = 2^n$, $n = 1, \ldots, 24$.

8. A student borrows P dollars at an annual interest rate of $R\%$. This loan is to be repayed in n years in equal monthly installments of x dollars. Find x if
(a) $P = 4250$, $R = 3$, and $n = 5$;
(b) $P = 5000$, $R = 7$, and $n = 10$.

9. A home buyer takes out a \$30,000 mortgage at an annual interest rate of 9%. This loan is to be repaid over 20 years in 240 equal monthly installments of x dollars.
(a) Compute x.
(b) Find x if the annual interest rate is 10%.

10. The quantity supplied of some commodity in a given week is obviously an increasing function of its price the previous week, while the quantity demanded in a given week is a function of its current price. Let S_j and D_j denote, respectively, the quantities supplied and demanded in the jth week, and let P_j denote the price of the commodity in the jth week. We assume that there exist positive constants a, b, and c such that

$$S_j = aP_{j-1} \quad \text{and} \quad D_j = b - cP_j.$$

(a) Show that $P_j = b/(a+c) + (-a/c)^j (P_0 - b/(a+c))$, if supply always equals demand.

(b) Show that P_j approaches $b/(a+c)$ as j approaches infinity if $a/c<1$.
(c) Show that $P=b/(a+c)$ represents an equilibrium situation. That is to say, if supply always equals demand, and if the price ever reaches the level $b/(a+c)$, then it will always remain at that level.

1.13 Numerical approximations; Euler's method

In Section 1.9 we showed that it is not possible, in general, to solve the initial-value problem

$$\frac{dy}{dt}=f(t,y), \qquad y(t_0)=y_0. \tag{1}$$

Therefore, in order that differential equations have any practical value for us, we must devise ways of obtaining accurate approximations of the solution $y(t)$ of (1). In Sections 1.13–1.16 we will derive algorithms, which can be implemented on a digital computer, for obtaining accurate approximations of $y(t)$.

Now, a computer obviously cannot approximate a function on an entire interval $t_0 \leqslant t \leqslant t_0+a$ since this would require an infinite amount of information. At best it can compute approximate values $y_1,\ldots,y_N$ of $y(t)$ at a finite number of points $t_1,t_2,\ldots,t_N$. However, this is sufficient for our purpose since we can use the numbers $y_1,\ldots,y_N$ to obtain an accurate approximation of $y(t)$ on the entire interval $t_0\leqslant t\leqslant t_0+a$. To wit, let $\hat{y}(t)$ be the function whose graph on each interval $[t_j,t_{j+1}]$ is the straight line connecting the points (t_j,y_j) and (t_{j+1},y_{j+1}) (see Figure 1). We can express $\hat{y}(t)$ analytically by the equation

$$\hat{y}(t)=y_j+\frac{1}{h}(t-t_j)(y_{j+1}-y_j), \qquad t_j\leqslant t\leqslant t_{j+1}.$$

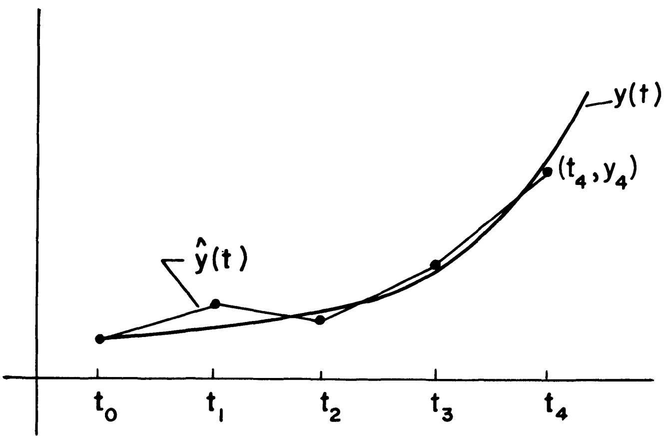

Figure 1. Comparison of $\hat{y}(t)$ and $y(t)$

If $\hat{y}(t)$ is close to $y(t)$ at $t=t_j$; that is, if y_j is close to $y(t_j)$, and if t_{j+1} is close to t_j, then $\hat{y}(t)$ is close to $y(t)$ on the entire interval $t_j \leqslant t \leqslant t_{j+1}$. This follows immediately from the continuity of both $y(t)$ and $\hat{y}(t)$. Thus, we need only devise schemes for obtaining accurate approximations of $y(t)$ at a discrete number of points $t_1,\dots,t_N$ in the interval $t_0 \leqslant t \leqslant t_0+a$. For simplicity, we will require that the points $t_1,\dots,t_N$ be equally spaced. This is achieved by choosing a large integer N and setting $t_k=t_0+k(a/N)$, $k=1,\dots,N$. Alternately, we may write $t_{k+1}=t_k+h$ where $h=a/N$.

Now, the only thing we know about $y(t)$ is that it satisfies a certain differential equation, and that its value at $t=t_0$ is y_0. We will use this information to compute an approximate value y_1 of y at $t=t_1=t_0+h$. Then, we will use this approximate value y_1 to compute an approximate value y_2 of y at $t=t_2=t_1+h$, and so on. In order to accomplish this we must find a theorem which enables us to compute the value of y at $t=t_k+h$ from the knowledge of y at $t=t_k$. This theorem, of course, is Taylor's Theorem, which states that

$$y(t_k+h)=y(t_k)+h\frac{dy(t_k)}{dt}+\frac{h^2}{2!}\frac{d^2y(t_k)}{dt^2}+\dots. \tag{2}$$

Thus, if we know the value of y and its derivatives at $t=t_k$, then we can compute the value of y at $t=t_k+h$. Now, $y(t)$ satisfies the initial-value problem (1). Hence, its derivative, when evaluated at $t=t_k$, must equal $f(t_k,y(t_k))$. Moreover, by repeated use of the chain rule of partial differentiation (see Appendix A), we can evaluate

$$\frac{d^2y(t_k)}{dt^2}=\left[\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}\right](t_k,y(t_k))$$

and all other higher-order derivatives of $y(t)$ at $t=t_k$. Hence, we can rewrite (2) in the form

$$\begin{aligned} y(t_{k+1})=y(t_k)&+hf(t_k,y(t_k))\\ &+\frac{h^2}{2!}\left[\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}\right](t_k,y(t_k))+\dots. \end{aligned} \tag{3}$$

The simplest approximation of $y(t_{k+1})$ is obtained by truncating the Taylor series (3) after the second term. This gives rise to the numerical scheme

$$y_1=y_0+hf(t_0,y_0), \qquad y_2=y_1+hf(t_1,y_1),$$

and, in general,

$$y_{k+1}=y_k+hf(t_k,y_k), \qquad y_0=y(t_0). \tag{4}$$

Notice how we use the initial-value y_0 and the fact that $y(t)$ satisfies the differential equation $dy/dt=f(t,y)$ to compute an approximate value y_1 of $y(t)$ at $t=t_1$. Then, we use this approximate value y_1 to compute an approximate value y_2 of $y(t)$ at $t=t_2$, and so on.

Equation (4) is known as *Euler's scheme*. It is the simplest numerical scheme for obtaining approximate values $y_1,\ldots,y_N$ of the solution $y(t)$ at times $t_1,\ldots,t_N$. Of course, it is also the least accurate scheme, since we have only retained two terms in the Taylor series expansion for $y(t)$. As we shall see shortly, Euler's scheme is not accurate enough to use in many problems. However, it is an excellent introduction to the more complicated schemes that will follow.

Example 1. Let $y(t)$ be the solution of the initial-value problem

$$dy/dt=1+(y-t)^2, \qquad y(0)=\tfrac{1}{2}.$$

Use Euler's scheme to compute approximate values $y_1,\ldots,y_N$ of $y(t)$ at the points $t_1=1/N, t_2=2/N,\ldots,t_N=1$.
Solution. Euler's scheme for this problem is

$$y_{k+1}=y_k+h\left[1+(y_k-t_k)^2\right], \qquad k=0,1,\ldots,N-1, \quad h=1/N$$

with $y_0=\frac{1}{2}$. Sample Pascal and Fortran programs to compute $y_1,\ldots,y_N$ are given below. These programs, as well as all subsequent programs, have variable values for t_0, y_0, a, and N, so that they may also be used to solve the more general initial-value problem $dy/dt=1+(y-t)^2$, $y(t_0)=y_0$ on any desired interval. Moreover, these same programs work even if we change the differential equation; if we change the function $f(t,y)$ then we need only change line 12 in the Pascal program (and line 11 in the C program) and the expressions for Y(1) and Y(K) in Section B of the Fortran program.

Pascal Program

```
Program Euler (input, output);

var
  T, Y: array[0..999] of real;
  a, h: real;
  k, N: integer;

begin
  readln(T[0], Y[0], a, N);
  h := a/N;
  page;
  for k := 0 to N-1 do
    begin
      T[k+1] := T[k]+h;
      Y[k+1] := Y[k]+h * (1+(Y[k]-T[k]) * (Y[k]-T[k]));
    end;
  writeln('T':4, 'Y':16);
  for k := 0 to N do
    writeln(T[k]:10:7, ' ':2, Y[k]:16:9);
end.
```

Fortran Program

```
Section A               DIMENSION T(1000),Y(1000)
Read in data            READ (5,10) T0,Y0,A,N
                 10     FORMAT (3F20.8,I5)
                        H=A/N
Section B               T(1)=T0+H
Do computations         Y(1)=Y0+H*(1+(Y0-T0)**2)
                        DO 20 K=2,N
                        T(K)=T(K-1)+H
                        Y(K)=Y(K-1)+H*(1+(Y(K-1)
                    1   -T(K-1))**2)
                 20     CONTINUE
Section C               WRITE (6,30) T0,Y0,(T(J),Y(J),J=1,N)
Print out        30     FORMAT (1H1,3X,1HT,4X,1HY,/(1H,1X,
results             1   F10.7,2X,F20.9/))
                        CALL EXIT
                        END
```

See also C Program 4 in Appendix C for a sample C program.
Table 1 below gives the results of these computations for $a=1$, $N=10$, $t_0=0$, and $y_0=\frac{1}{2}$. All of these computations, and all subsequent computations, were carried out on an IBM 360 computer using 16 decimal places accuracy. The results have been rounded to 8 significant decimal places.

Table 1

t	y	t	y
0	0.5	0.6	1.29810115
0.1	0.625	0.7	1.44683567
0.2	0.7525625	0.8	1.60261202
0.3	0.88309503	0.9	1.76703063
0.4	1.01709501	1	1.94220484
0.5	1.15517564		

The exact solution of this initial-value problem (see Exercise 7) is

$$y(t)=t+1/(2-t).$$

Thus, the error we make in approximating the value of the solution at $t=1$ by y_{10} is approximately 0.06, since $y(1)=2$. If we run this program for $N=20$ and $N=40$, we obtain that $y_{20}=1.96852339$ and $y_{40}=1.9835109$. Hence, the error we make in approximating $y(1)$ by y_{40} is already less than 0.02.

EXERCISES

Using Euler's method with step size $h=0.1$, determine an approximate value of the solution at $t=1$ for each of the initial-value problems 1–5. Repeat these computations with $h=0.025$ and compare the results with the given value of the solution.

1. $\dfrac{dy}{dt}=1+t-y, \quad y(0)=0; \; (y(t)=t)$

2. $\dfrac{dy}{dt}=2ty, \quad y(0)=2; \; (y(t)=2e^{t^2})$

3. $\dfrac{dy}{dt}=1+y^2-t^2, \quad y(0)=0; \; (y(t)=t)$

4. $\dfrac{dy}{dt}=te^{-y}+\dfrac{t}{1+t^2}, \quad y(0)=0; \; (y(t)=\ln(1+t^2))$

5. $\dfrac{dy}{dt}=-1+2t+\dfrac{y^2}{(1+t^2)^2}, \quad y(0)=1; \; (y(t)=1+t^2)$

6. Using Euler's method with $h=\pi/40$, determine an approximate value of the solution of the initial-value problem
$$\frac{dy}{dt}=2\sec^2 t-(1+y^2), \qquad y(0)=0$$
at $t=\pi/4$. Repeat these computations with $h=\pi/160$ and compare the results with the number one which is the value of the solution $y(t)=\tan t$ at $t=\pi/4$.

7. (a) Show that the substitution $y=t+z$ reduces the initial-value problem $y'=1+(y-t)^2, y(0)=0.5$ to the simpler initial-value problem $z'=z^2, z(0)=0.5$.
(b) Show that $z(t)=1/(2-t)$. Hence, $y(t)=t+1/(2-t)$.

1.13.1 Error analysis for Euler's method

One of the nice features of Euler's method is that it is relatively simple to estimate the error we make in approximating $y(t_k)$ by y_k. Unfortunately, though, we must make the severe restriction that $t_1,\ldots,t_N$ do not exceed $t_0+\alpha$, where α is the number defined in the existence–uniqueness theorem of Section 1.10. More precisely, let a and b be two positive numbers and assume that the functions f, $\partial f/\partial t$, and $\partial f/\partial y$ are defined and continuous in the rectangle $t_0\leqslant t\leqslant t_0+a, y_0-b\leqslant y\leqslant y_0+b$. We will denote this rectangle by R. Let M be the maximum value of $|f(t,y)|$ for (t,y) in R, and set $\alpha=\min(a,b/M)$. We will determine the error committed in approximating $y(t_k)$ by y_k, for $t_k\leqslant t_0+\alpha$.

To this end observe that the numbers $y_0,y_1,\ldots,y_N$ satisfy the difference equation

$$y_{k+1}=y_k+hf(t_k,y_k), \qquad k=0,1,\ldots,N-1 \tag{1}$$

while the numbers $y(t_0),y(t_1),\ldots,y(t_N)$ satisfy the difference equation

$$y(t_{k+1})=y(t_k)+hf(t_k,y(t_k))+\frac{h^2}{2}\left[\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}\right](\xi_k,y(\xi_k)) \tag{2}$$

where ξ_k is some number between t_k and t_{k+1}. Equation (2) follows from the identity

$$\frac{d^2y}{dt^2}=\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}$$

and the fact that

$$y(t+h)=y(t)+h\frac{dy(t)}{dt}+\frac{h^2}{2}\frac{d^2y(\tau)}{dt^2},$$

for some number τ between t and $t+h$. Subtracting Equation (1) from Equation (2) gives

$$\begin{aligned}y(t_{k+1})-y_{k+1}&=y(t_k)-y_k+h\left[f(t_k,y(t_k))-f(t_k,y_k)\right]\\&\quad+\frac{h^2}{2}\left[\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}\right](\xi_k,y(\xi_k)).\end{aligned}$$

Next, observe that

$$f(t_k,y(t_k))-f(t_k,y_k)=\frac{\partial f(t_k,\eta_k)}{\partial y}\left[y(t_k)-y_k\right]$$

where η_k is some number between $y(t_k)$ and y_k. Consequently,

$$\begin{aligned}|y(t_{k+1})-y_{k+1}|&\leqslant|y(t_k)-y_k|+h\left|\frac{\partial f(t_k,\eta_k)}{\partial y}\right||y(t_k)-y_k|\\&\quad+\frac{h^2}{2}\left|\left[\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}\right](\xi_k,y(\xi_k))\right|.\end{aligned}$$

In order to proceed further, we must obtain estimates of the quantities $(\partial f(t_k,\eta_k))/\partial y$ and $[(\partial f/\partial t)+f(\partial f/\partial y)](\xi_k,y(\xi_k))$. To this end observe that the points $(\xi_k,y(\xi_k))$ and (t_k,y_k) all lie in the rectangle R. (It was shown in Section 1.10 that the points $(\xi_k,y(\xi_k))$ lie in R. In addition, a simple induction argument (see Exercise 9) shows that the points (t_k,y_k) all lie in R.) Consequently, the points (t_k,η_k) must also lie in R. Let L and D be two positive numbers such that

$$\max_{(t,y)\text{ in }R}\left|\frac{\partial f}{\partial y}\right|\leqslant L$$

and

$$\max_{(t,y)\text{ in }R}\left|\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}\right|\leqslant D.$$

Such numbers always exist if f, $\partial f/\partial t$, and $\partial f/\partial y$ are continuous in R. Then,

$$|y(t_{k+1})-y_{k+1}|\leqslant|y(t_k)-y_k|+hL|y(t_k)-y_k|+\frac{Dh^2}{2}. \tag{3}$$

Now, set $E_k=|y(t_k)-y_k|$, $k=0,1,\ldots,N$. The number E_k is the error we make at the kth step in approximating $y(t_k)$ by y_k. From (3)

$$E_{k+1}\leqslant(1+hL)E_k+\frac{Dh^2}{2},\qquad k=0,1,\ldots,N-1. \tag{4}$$

Moreover, $E_0=0$ since $y(t_0)=y_0$. Thus, the numbers $E_0,E_1,\ldots,E_N$ satisfy the set of inequalities

$$E_{k+1}\leqslant AE_k+B,\qquad E_0=0$$

with $A=1+hL$ and $B=Dh^2/2$. Consequently, (see Section 1.12)

$$E_k\leqslant\frac{B}{A-1}(A^k-1)=\frac{Dh}{2L}\left[(1+hL)^k-1\right]. \tag{5}$$

We can also obtain an estimate for E_k that is independent of k. Observe that $1+hL\leqslant e^{hL}$. This follows from the fact that

$$\begin{aligned}e^{hL}&=1+hL+\frac{(hL)^2}{2!}+\frac{(hL)^3}{3!}+\ldots\\&=(1+hL)+\text{"something positive"}.\end{aligned}$$

Therefore,

$$E_k\leqslant\frac{Dh}{2L}\left[(e^{hL})^k-1\right]=\frac{Dh}{2L}\left[e^{khL}-1\right].$$

Finally, since $kh\leqslant\alpha$, we see that

$$E_k\leqslant\frac{Dh}{2L}\left[e^{\alpha L}-1\right],\qquad k=1,\ldots,N. \tag{6}$$

Equation (6) says that the error we make in approximating the solution $y(t)$ at time $t=t_k$ by y_k is at most a fixed constant times h. This suggests, as a rule of thumb, that our error should decrease by approximately $\frac{1}{2}$ if we decrease h by $\frac{1}{2}$. We can verify this directly in Example 1 of the previous section where our error at $t=1$ for $h=0.1$, 0.05, and 0.025 is 0.058, 0.032, and 0.017 respectively.

Example 1. Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt}=\frac{t^2+y^2}{2},\qquad y(0)=0.$$

(a) Show that $y(t)$ exists at least for $0\leqslant t\leqslant 1$, and that in this interval, $-1\leqslant y(t)\leqslant 1$.
(b) Let N be a large positive integer. Set up Euler's scheme to find approximate values of y at the points $t_k=k/N$, $k=0,1,\ldots,N$.
(c) Determine a step size $h=1/N$ so that the error we make in approximating $y(t_k)$ by y_k does not exceed 0.0001.

Solution. (a) Let R be the rectangle $0\leqslant t\leqslant 1$, $-1\leqslant y\leqslant 1$. The maximum value that $(t^2+y^2)/2$ achieves for (t,y) in R is 1. Hence, by the existence–uniqueness theorem of Section 1.10, $y(t)$ exists at least for

$$0 \leqslant t \leqslant \alpha = min\left(1, \tfrac{1}{1}\right) = 1,$$

and in this interval, $-1 \leqslant y \leqslant 1$.

(b)
$$y_{k+1} = y_k + h\left(\frac{t_k^2 + y_k^2}{2}\right) = y_k + \frac{1}{2N}\left[\left(\frac{k}{N}\right)^2 + y_k^2\right]$$

with $y_0 = 0$. The integer k runs from 0 to $N-1$.
(c) Let $f(t,y) = (t^2 + y^2)/2$, and compute

$$\frac{\partial f}{\partial y} = y \quad \text{and} \quad \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} = t + \frac{y}{2}(t^2 + y^2).$$

From (6), $|y(t_k) - y_k| \leqslant (Dh/2L)(e^L - 1)$ where L and D are two positive numbers such that

$$\max_{(t,y) \text{ in } R} |y| \leqslant L$$

and

$$\max_{(t,y) \text{ in } R} \left|t + \frac{y}{2}(t^2 + y^2)\right| \leqslant D.$$

Now, the maximum values of the functions $|y|$ and $|t + (y/2)(t^2 + y^2)|$ for (t,y) in R are clearly 1 and 2 respectively. Hence,

$$|y(t_k) - y_k| \leqslant \frac{2h}{2}(e-1) = h(e-1).$$

This implies that the step size h should be smaller than $0.0001/(e-1)$. Equivalently, N should be larger than $(e-1)10^4 = 17{,}183$. Thus, we must iterate the equation

$$y_{k+1} = y_k + \frac{1}{2(17{,}183)}\left[\left(\frac{k}{17{,}183}\right)^2 + y_k^2\right]$$

17,183 times to be sure that $y(1)$ is correct to four decimal places.

Example 2. Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt} = t^2 + e^{-y^2}, \qquad y(0) = 1.$$

(a) Show that $y(t)$ exists at least for $0 \leqslant t \leqslant 1$, and that in this interval, $-1 \leqslant y \leqslant 3$.
(b) Let N be a large positive integer. Set up Euler's scheme to find approximate values of $y(t)$ at the points $t_k = k/N$, $k = 0, 1, \ldots, N$.
(c) Determine a step size h so that the error we make in approximating $y(t_k)$ by y_k does not exceed 0.0001.
Solution.
(a) Let R be the rectangle $0 \leqslant t \leqslant 1$, $|y-1| \leqslant 2$. The maximum value that $t^2 + e^{-y^2}$ achieves for (t,y) in R is 2. Hence, $y(t)$ exists at least for $0 \leqslant t \leqslant \min(1, 2/2) = 1$, and in this interval, $-1 \leqslant y \leqslant 3$.

(b) $y_{k+1}=y_k+h(t_k^{\,2}+e^{-y_k^2})=y_k+(1/N)[(k/N)^2+e^{-y_k^2}]$ with $y_0=1$. The integer k runs from 0 to $N-1$.
(c) Let $f(t,y)=t^2+e^{-y^2}$ and compute

$$\frac{\partial f}{\partial y}=-2ye^{-y^2},\quad \text{and}\quad \frac{\partial f}{\partial t}+f\frac{\partial f}{\partial y}=2t-2y(t^2+e^{-y^2})e^{-y^2}.$$

From (6), $|y(t_k)-y_k|\leqslant(Dh/2L)(e^L-1)$ where L and D are two positive numbers such that

$$\max_{(t,y)\text{ in }R}|-2ye^{-y^2}|\leqslant L$$

and

$$\max_{(t,y)\text{ in }R}|2t-2y(t^2+e^{-y^2})e^{-y^2}|\leqslant D.$$

Now, it is easily seen that the maximum value of $|2ye^{-y^2}|$ for $-1\leqslant y\leqslant 3$ is $\sqrt{2/e}$. Thus, we take $L=\sqrt{2/e}$. Unfortunately, though, it is extremely difficult to compute the maximum value of the function

$$|2t-2y(t^2+e^{-y^2})e^{-y^2}|$$

for (t,y) in R. However, we can still find an acceptable value D by observing that for (t,y) in R,

$$\begin{aligned}\max|2t-2v(t^2+e^{-y^2})e^{-y^2}|&\leqslant\max|2t|+\max|2y(t^2+e^{-y^2})e^{-y^2}|\\&\leqslant\max|2t|+\max|2ye^{-y^2}|\times\max(t^2+e^{-y^2})\\&=2+2\sqrt{2/e}=2\left(1+\sqrt{2/e}\right).\end{aligned}$$

Hence, we may choose $D=2(1+\sqrt{2/e})$. Consequently,

$$|y(t_k)-y_k|\leqslant\frac{2\left(1+\sqrt{2/e}\right)h\left[e^{\sqrt{2/e}}-1\right]}{2\sqrt{2/e}}.$$

This implies that the step size h must be smaller than

$$\frac{\sqrt{2/e}}{1+\sqrt{2/e}}\times\frac{0.0001}{e^{\sqrt{2/e}}-1}.$$

Examples 1 and 2 show that Euler's method is not very accurate since approximately 20,000 iterations are required to achieve an accuracy of four decimal places. One obvious disadvantage of a scheme which requires so many iterations is the cost. The going rate for computer usage at present is about \$1200.00 per hour. A second, and much more serious disadvantage, is that y_k may be very far away from $y(t_k)$ if N is exceptionally large. To wit, a digital computer can never perform a computation exactly since it only retains a finite number of decimal places. Consequently, every time we perform an arithmetic operation on the computer, we must introduce a "round off" error. This error, of course, is small. However, if we perform too many operations then the accumulated round off error may become so large as to make our results meaningless. Exercise 8 gives an illustration of this for Euler's method.

Exercises

1. Determine an upper bound on the error we make in using Euler's method with step size h to find an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt}=\frac{t^2+y^2}{2}, \qquad y(0)=1$$

at any point t in the interval $[0,\frac{2}{5}]$. *Hint*: Let R be the rectangle $0\leqslant t\leqslant 1$, $0\leqslant y\leqslant 2$.

2. Determine an upper bound on the error we make in using Euler's method with step size h to find an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt}=t-y^4, \qquad y(0)=0$$

at any point t in the interval $[0,1]$. *Hint*: Let R be the rectangle $0\leqslant t\leqslant 1$, $-1\leqslant y\leqslant 1$.

3. Determine an upper bound on the error we make in using Euler's method with step size h to find an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt}=t+e^y, \qquad y(0)=0$$

at any point t in the interval $[0,1/(e+1)]$. *Hint*: Let R be the rectangle $0\leqslant t\leqslant 1$, $-1\leqslant y\leqslant 1$.

4. Determine a suitable value of h so that the error we make in using Euler's method with step size h to find an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt}=e^t-y^2, \qquad y(0)=0$$

at any point t in the interval $[0,1/e]$ is at most 0.0001. *Hint*: Let R be the rectangle $0\leqslant t\leqslant 1$, $-1\leqslant y\leqslant 1$.

5. Determine a suitable value of h so that the error we make in using Euler's method with step size h to find an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt}=t^2+\tan^2 y, \qquad y(0)=0$$

at any point t in the interval $[0,\frac{1}{2}]$ is at most 0.00001. *Hint*: Let R be the rectangle $0\leqslant t\leqslant \frac{1}{2}$, $-\pi/4\leqslant y\leqslant \pi/4$.

6. Determine a suitable value of h so that the error we make in using Euler's method with step size h to find an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt} = \frac{1}{1+t^2+y^2}, \qquad y(0)=0$$

at any point t in the interval $[0,1]$ is at most 0.0001. *Hint*: Let R be the rectangle $0 \leqslant t \leqslant 1,\ -1 \leqslant y \leqslant 1$.

7. Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt} = f(t,y), \qquad y(0)=0.$$

Suppose that $|f(t,y)| \leqslant 1$, $|\partial f/\partial y| \leqslant 1$, and $|(\partial f/\partial t)+f(\partial f/\partial y)| \leqslant 2$ in the rectangle $0 \leqslant t \leqslant 1,\ -1 \leqslant y \leqslant 1$. When the Euler scheme

$$y_{k+1} = y_k + hf(t_k,y_k), \qquad h = \frac{1}{N}$$

is used with $N=10$, the value of y_5 is $-0.15[(\frac{11}{10})^5 - 1]$, and the value of y_6 is $0.12[(\frac{11}{10})^6 - 1]$. Prove that $y(t)$ is zero at least once in the interval $(\frac{1}{2}, \frac{3}{5})$.

8. Let $y(t)$ be the solution of the initial-value problem

$$y' = f(t,y), \qquad y(t_0) = y_0.$$

Euler's method for finding approximate values of $y(t)$ is $y_{k+1} = y_k + hf(t_k,y_k)$. However, the quantity $y_k + hf(t_k,y_k)$ is never computed exactly: we always introduce an error ε_k with $|\varepsilon_k| < \varepsilon$. That is to say, the computer computes numbers $\tilde{y}_1, \tilde{y}_2, \ldots$, such that

$$\tilde{y}_{k+1} = \tilde{y}_k + hf(t_k,\tilde{y}_k) + \varepsilon_k$$

with $\tilde{y}_0 = y_0$. Suppose that $|\partial f/\partial y| \leqslant L$ and $|(\partial f/\partial t)+f(\partial f/\partial y)| \leqslant D$ for all t and y.

(a) Show that

$$E_{k+1} \equiv |y(t_{k+1}) - \tilde{y}_{k+1}| \leqslant (1+hL)E_k + \frac{D}{2}h^2 + \varepsilon$$

(b) Conclude from (a) that

$$E_k \leqslant \left[\frac{Dh}{2} + \frac{\varepsilon}{h}\right]\frac{e^{\alpha L}-1}{L}$$

for $kh \leqslant \alpha$.

(c) Choose h so that the error E_k is minimized. Notice that the error E_k may be very large if h is very small.

9. Let $y_1, y_2, \ldots$ satisfy the recursion relation

$$y_{k+1} = y_k + hf(t_k,y_k).$$

Let R be the rectangle $t_0 \leqslant t \leqslant t_0 + a, y_0 - b \leqslant y \leqslant y_0 + b$, and assume that $|f(t,y)| \leqslant M$ for (t,y) in R. Finally, let $\alpha = \min(a, b/M)$.

(a) Prove that $|y_j - \dot{y}_0| \leqslant jhM$, as long as $jh \leqslant \alpha$. *Hint*: Use induction.

(b) Conclude from (a) that the points (t_j,y_j) all lie in R as long as $j \leqslant \alpha/h$.

1.14 The three term Taylor series method

Euler's method was derived by truncating the Taylor series

$$y(t_{k+1}) = y(t_k) + hf(t_k, y(t_k)) + \frac{h^2}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right](t_k, y(t_k)) + \dots \tag{1}$$

after the second term. The most obvious way of obtaining better numerical schemes is to retain more terms in Equation (1). If we truncate this Taylor series after three terms then we obtain the numerical scheme

$$y_{k+1} = y_k + hf(t_k, y_k) + \frac{h^2}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right](t_k, y_k), \qquad k = 0, \dots, N-1 \tag{2}$$

with $y_0 = y(t_0)$.

Equation (2) is called the *three term Taylor series method*. It is obviously more accurate than Euler's method. Hence, for fixed h, we would expect that the numbers y_k generated by Equation (2) are better approximations of $y(t_k)$ than the numbers y_k generated by Euler's scheme. This is indeed the case, for it can be shown that $|y(t_k) - y_k|$ is proportional to h^2 whereas the error we make using Euler's method is only proportional to h. The quantity h^2 is much less than h if h is very small. Thus, the three term Taylor series method is a significant improvement over Euler's method.

Example 1. Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt} = 1 + (y-t)^2, \qquad y(0) = \frac{1}{2}.$$

Use the three term Taylor series method to compute approximate values of $y(t)$ at the points $t_k = k/N$, $k = 1, \dots, N$.

Solution. Let $f(t,y) = 1 + (y-t)^2$. Then,

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} = -2(y-t) + 2(y-t)\left[1 + (y-t)^2\right] = 2(y-t)^3.$$

Hence, the three term Taylor series scheme is

$$y_{k+1} = y_k + h\left[1 + (y_k - t_k)^2\right] + h^2(y_k - t_k)^3$$

with $h = 1/N$ and $y_0 = \frac{1}{2}$. The integer k runs from 0 to $N-1$. Sample Pascal and Fortran programs to compute $y_1, \dots, y_N$ are given below. Again, these programs have variable values for t_0, y_0, a, and N.

Pascal Program

```
Program Taylor (input, output);

var
  T, Y: array[0..999] of real;
  a, h, Temp: real;
  k, N: integer;

begin
  readln(T[0], Y[0], a, N);
  h := a/N;
  page;
  for k := 0 to N-1 do
    begin
      Temp := Y[k] - T[k];
      T[k+1] := T[k] + h;
      Y[k+1] := Y[k] + h * (1 + Temp * Temp)
                + h * h * Temp * Temp * Temp;
    end;
  writeln('T':4, 'Y':16);
  for k := 0 to N do
    writeln(T[k]:10:7, ' ':2, Y[k]:16:9);
end.
```

Fortran Program

Replace Section B of the Fortran program in Section 1.13 by the following:

```
      T(1) = T0 + H
      D2Y = H*(Y0 - T0)**3
      Y(1) = Y0 + H*(D2Y + 1 + (Y0 - T0)**2)
      DO 20 K = 2, N
      T(K) = T(K-1) + H
      D2Y = H*(Y(K-1) - T(K-1))**3
      Y(K) = Y(K-1) + H*(D2Y + 1 + (Y(K-1) - T(K-1))**2)
20    CONTINUE
```

See also C Program 5 in Appendix C for a sample C program.

Table 1 below shows the results of these computations for $a=1$, $N=10$, $t_0=0$, and $y_0=0.5$.

Now Euler's method with $N=10$ predicted a value of 1.9422 for $y(1)$. Notice how much closer the number 1.9957 is to the correct value 2. If we run this program for $N=20$ and $N=40$, we obtain that $y_{20}=1.99884247$ and

Table 1

t	y	t	y
0	0.5	0.6	1.31331931
0.1	0.62625	0.7	1.4678313
0.2	0.7554013	0.8	1.63131465
0.3	0.88796161	0.9	1.80616814
0.4	1.02456407	1	1.99572313
0.5	1.1660084		

$y_{40} = 1.99969915$. These numbers are also much more accurate than the values 1.96852339 and 1.9835109 predicted by Euler's method.

Exercises

Using the three term Taylor series method with $h=0.1$, determine an approximate value of the solution at $t=1$ for each of the initial-value problems 1–5. Repeat these computations with $h=0.025$ and compare the results with the given value of the solution.

1. $dy/dt = 1+t-y,\ \ y(0)=0;\ \ (y(t)=t)$

2. $dy/dt = 2ty,\ \ y(0)=2;\ \ (y(t)=2e^{t^2})$

3. $dy/dt = 1+y^2-t^2,\ \ y(0)=0;\ \ (y(t)=t)$

4. $dy/dt = te^{-y}+t/(1+t^2),\ \ y(0)=0;\ \ (y(t)=\ln(1+t^2))$

5. $dy/dt = -1+2t+y^2/(1+t^2)^2,\ \ y(0)=1;\ \ (y(t)=1+t^2)$

6. Using the three term Taylor series method with $h=\pi/40$, determine an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt} = 2\sec^2 t-(1+y^2), \qquad y(0)=0$$

at $t=\pi/4$. Repeat these computations with $h=\pi/160$ and compare the results with the number one which is the value of the solution $y(t)=\tan t$ at $t=\pi/4$.

1.15 An improved Euler method

The three term Taylor series method is a significant improvement over Euler's method. However, it has the serious disadvantage of requiring us to compute partial derivatives of $f(t,y)$, and this can be quite difficult if the function $f(t,y)$ is fairly complicated. For this reason we would like to derive numerical schemes which do not require us to compute partial derivatives of $f(t,y)$. One approach to this problem is to integrate both sides of the differential equation $y'=f(t,y)$ between t_k and t_k+h to obtain that

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_k+h} f(t,y(t))\,dt. \tag{1}$$

This reduces the problem of finding an approximate value of $y(t_{k+1})$ to the much simpler problem of approximating the area under the curve $f(t,y(t))$ between t_k and t_k+h. A crude approximation of this area is $hf(t_k,y(t_k))$, which is the area of the rectangle R in Figure 1a. This gives rise to the numerical scheme

$$y_{k+1}=y_k+hf(t_k,y_k)$$

which, of course, is Euler's method.

A much better approximation of this area is

$$\frac{h}{2}\left[f(t_k,y(t_k))+f(t_{k+1},y(t_{k+1}))\right]$$

which is the area of the trapezoid T in Figure 1b. This gives rise to the numerical scheme

$$y_{k+1}=y_k+\frac{h}{2}\left[f(t_k,y_k)+f(t_{k+1},y_{k+1})\right]. \tag{2}$$

However, we cannot use this scheme to determine y_{k+1} from y_k since y_{k+1} also appears on the right-hand side of (2). A very clever way of overcoming this difficulty is to replace y_{k+1} in the right-hand side of (2) by the value $y_k+hf(t_k,y_k)$ predicted for it by Euler's method. This gives rise to the numerical scheme

$$y_{k+1}=y_k+\frac{h}{2}\left[f(t_k,y_k)+f(t_k+h,y_k+hf(t_k,y_k))\right],\qquad y_0=y(t_0). \tag{3}$$

Equation (3) is known as the *improved Euler method.* It can be shown that $|y(t_k)-y_k|$ is at most a fixed constant times h^2. Hence, the improved Euler method gives us the same accuracy as the three term Taylor series method without requiring us to compute partial derivatives.

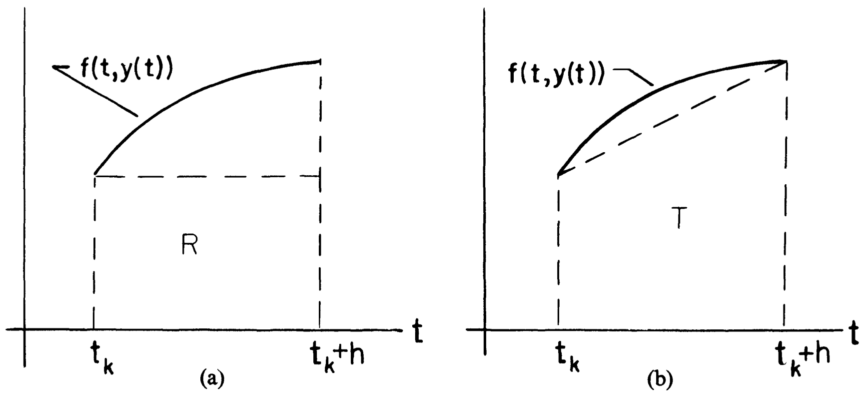

Figure 1

Example 1. Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt}=1+(y-t)^2,\qquad y(0)=\frac{1}{2}.$$

Use the improved Euler method to compute approximate values of $y(t)$ at the points $t_k = k/N$, $k = 1,\ldots,N$.

Solution. The improved Euler scheme for this problem is

$$y_{k+1} = y_k + \frac{h}{2}\left\{1 + (y_k - t_k)^2 + 1 + \left[y_k + h\left(1 + (y_k - t_k)^2\right) - t_{k+1}\right]^2\right\}$$

with $h = 1/N$ and $y_0 = 0.5$. The integer k runs from 0 to $N-1$. Sample Pascal and Fortran programs to compute $y_1,\ldots,y_N$ are given below. Again, these programs have variable values for t_0, y_0, a, and N.

Pascal Program

```
Program Improved (input, output);

var
   T, Y: array[0..999] of real;
   a, h, R: real;
   k, N: integer;

begin
   readln(T[0], Y[0], a, N);
   h := a/N;
   page;
   for k := 0 to N-1 do
      begin
         R := 1 + (Y[k]-T[k]) * (Y[k]-T[k]);
         T[k+1] := T[k] + h;
         Y[k+1] := Y[k] + (h/2) * (R+1
                 + (Y[k] + h * R - T[k+1]) * (Y[k] + h * R - T[k+1]));
      end;
   writeln('T':4, 'Y':16);
   for k := 0 to N do
      writeln(T[k]:10:7, ' ':2, Y[k]:16:9);
end.
```

Fortran Program

Replace Section B of the Fortran program in Example 1 of Section 1.13 by the following:

```
     T(1)=T0+H
     R=1+(Y0-T0)**2
     Y(1)=Y0+(H/2)*(R+1+(Y0+(H*R)-T(1))**2)
     DO 20 K=2,N
     T(K)=T(K-1)+H
     R=1+(Y(K-1)-T(K-1))**2
     Y(K)=Y(K-1)+(H/2)*(R+1+(Y(K-1)+(H*R)-T(K))**2)
20   CONTINUE
```

See also C Program 6 in Appendix C for a sample C program.

Table 1 below shows the results of these computations for $a=1$, $N=10$, $t_0=0$, and $y_0=0.5$. If we run this program for $N=20$ and $N=40$ we obtain that $y_{20}=1.99939944$ and $y_{40}=1.99984675$. Hence the values y_{10}, y_{20}, and y_{40} computed by the improved Euler method are even closer to the correct value 2 than the corresponding values 1.99572313, 1.99884246, and 1.99969915 computed by the three term Taylor series method.

Table 1

t	y	t	y
0	0.5	0.6	1.31377361
0.1	0.62628125	0.7	1.46848715
0.2	0.75547445	0.8	1.63225727
0.3	0.88809117	0.9	1.80752701
0.4	1.02477002	1	1.99770114
0.5	1.16631867		

EXERCISES

Using the improved Euler method with $h=0.1$, determine an approximate value of the solution at $t=1$ for each of the initial-value problems 1–5. Repeat these computations with $h=0.025$ and compare the results with the given value of the solution.

1. $dy/dt=1+t-y,\quad y(0)=0;\quad (y(t)=t)$

2. $dy/dt=2ty,\quad y(0)=2;\quad (y(t)=2e^{t^2})$

3. $dy/dt=1+y^2-t^2,\quad y(0)=0;\quad (y(t)=t)$

4. $dy/dt=te^{-y}+t/(1+t^2),\quad y(0)=0;\quad (y(t)=\ln(1+t^2))$

5. $dy/dt=-1+2t+y^2/(1+t^2)^2,\quad y(0)=1;\quad (y(t)=1+t^2)$

6. Using the improved Euler method with $h=\pi/40$, determine an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt}=2\sec^2 t-(1+y^2), \qquad y(0)=0$$

at $t=\pi/4$. Repeat these computations with $h=\pi/160$ and compare the results with the number one which is the value of the solution $y(t)=\tan t$ at $t=\pi/4$.

1.16 The Runge–Kutta method

We now present, without proof, a very powerful scheme which was developed around 1900 by the mathematicians Runge and Kutta. Because of its simplicity and great accuracy, the Runge–Kutta method is still one of the most widely used numerical schemes for solving differential equations. It is defined by the equation

$$y_{k+1}=y_k+\frac{h}{6}\left[L_{k,1}+2L_{k,2}+2L_{k,3}+L_{k,4}\right], \qquad k=0,1,\dots,N-1$$

where $y_0=y(t_0)$ and

$$L_{k,1}=f(t_k,y_k), \qquad L_{k,2}=f\left(t_k+\tfrac{1}{2}h,y_k+\tfrac{1}{2}hL_{k,1}\right)$$
$$L_{k,3}=f\left(t_k+\tfrac{1}{2}h,y_k+\tfrac{1}{2}hL_{k,2}\right), \qquad L_{k,4}=f(t_k+h,y_k+hL_{k,3}).$$

This formula involves a weighted average of values of $f(t,y)$ taken at different points. Hence the sum $\frac{1}{6}[L_{k,1}+2L_{k,2}+2L_{k,3}+L_{k,4}]$ can be interpreted as an average slope It can be shown that the error $|y(t_k)-y_k|$ is at most a fixed constant times h^4. Thus, the Runge–Kutta method is much more accurate than Euler's method, the three term Taylor series method and the improved Euler method.

Example 1. Let $y(t)$ be the solution of the initial-value problem

$$\frac{dy}{dt}=1+(y-t)^2, \qquad y(0)=\tfrac{1}{2}.$$

Use the Runge–Kutta method to find approximate values $y_1,\dots,y_N$ of y at the points $t_k=k/N$, $k=1,\dots,N$.

Solution. Sample Pascal and Fortran programs to compute $y_1,\dots,y_N$ by the Runge–Kutta method are given below. These programs differ from our previous programs in that they do not compute y_1 separately. Rather, they compute y_1 in the same "loop" as they compute $y_2,\dots,y_N$. This is accomplished by relabeling the numbers t_0 and y_0 as t_1 and y_1 respectively.

Pascal Program

```
Program Runge_Kutta (input, output);

var
  T, Y: array[0..999] of real;
  a, h, LK1, LK2, LK3, LK4: real;
  k, N: integer;

begin
  readln(T[0], Y[0], a, N);
  h := a/N;
  page;
  for k := 0 to N-1 do
    begin
      T[k+1] := T[k]+h;
      LK1 := 1+(Y[k]-T[k])*(Y[k]-T[k]);
      LK2 := 1+((Y[k]+(h/2)*LK1)-(T[k]+h/2))
               *((Y[k]+(h/2)*LK1)-(T[k]+h/2));
```

```
        LK3 := 1 + ((Y[k] + (h/2) * LK2) - (T[k] + h/2))
                  * ((Y[k] + (h/2) * LK2) - (T[k] + h/2));
        LK4 := 1 + ((Y[k] + h * LK3) - (T[k] + h))
                  * ((Y[k] + h * LK3) - (T[k] + h));
        Y[k + 1] := Y[k] + (h/6) * (LK1 + LK4 + 2 * (LK2 + LK3));
      end;
   writeln('T':4, 'Y':16);
   for k := 0 to N do
      writeln(T[k]:10:7, '  ':2, Y[k]:16:9);
end.
```

Fortran Program

```
      DIMENSION T(1000), Y(1000)
      READ (5,10) T(1), Y(1), A, N
10    FORMAT (3F20.8, I5)
      H = A/N
      D0 20 K = 1, N
      T(K + 1) = T(K) + H
      REAL  LK1,  LK2,  LK3,  LK4
      LK1 = 1 + (Y(K) - T(K)) * *2
      LK2 = 1 + ((Y(K) + (H/2) * LK1) - (T(K) + H/2)) * *2
      LK3 = 1 + ((Y(K) + (H/2) * LK2) - (T(K) + H/2)) * *2
      LK4 = 1 + ((Y(K) + H * LK3) - (T(K) + H)) * *2
      Y(K + 1) = Y(K) + (H/6) * (LK1 + LK4 + 2 * (LK2 + LK3))
20    CONTINUE
      NA = N + 1
      WRITE (6,30) (T(J), Y(J), J = 1, NA)
30    FORMAT (1H1, 3X, 1HT, 4X, 1HY, /(1H, 1X, F10.7, 2X, F20.9/))
      CALL EXIT
      END
```

See also C Program 7 in Appendix C for a sample C program.

Table 1 below shows the results of these computations for $a=1$, $N=10$, $t_0=0$, and $y_0=0.5$.

Table 1

t	y	t	y
0	0.5	0.6	1.31428555
0.1	0.62631578	0.7	1.4692305
0.2	0.75555536	0.8	1.6333329
0.3	0.88823526	0.9	1.8090902
0.4	1.02499993	1	1.9999988
0.5	1.16666656		

Notice how much closer the number $y_{10}=1.9999988$ computed by the Runge–Kutta method is to the correct value 2 than the numbers $y_{10}=1.94220484$, $y_{10}=1.99572312$, and $y_{10}=1.99770114$ computed by the Euler, three term Taylor series and improved Euler methods, respectively. If we run this program for $N=20$ and $N=40$, we obtain that $y_{20}=1.99999992$ and $y_{40}=2$. Thus, our approximation of $y(1)$ is already correct to eight decimal places when $h=0.025$. Equivalently, we need only choose $N \geqslant 40$ to achieve eight decimal places accuracy.

To put the accuracy of the various schemes into proper perspective, let us say that we have three different schemes for numerically solving the initial-value problem $dy/dt=f(t,y)$, $y(0)=0$ on the interval $0 \leqslant t \leqslant 1$, and that the error we make in using these schemes is $3h$, $11h^2$, and $42h^4$, respectively. If our problem is such that we require eight decimal places accuracy, then the step sizes h_1, h_2, and h_3 of these three schemes must satisfy the inequalities $3h_1 \leqslant 10^{-8}$, $11h_2^2 \leqslant 10^{-8}$, and $42h_3^4 \leqslant 10^{-8}$. Hence, the number of iterations N_1, N_2, and N_3 of these three schemes must satisfy the inequalities

$$N_1 \geqslant 3\times 10^8 = 300{,}000{,}000, \qquad N_2 \geqslant \sqrt{11}\times 10^4 \approx 34{,}000$$

and

$$N_3 \geqslant (42)^{1/4}\times 10^2 \approx 260.$$

This is a striking example of the difference between the Runge–Kutta method and the Euler, improved Euler and three term Taylor series methods.

Remark. It should be noted that we perform four functional evaluations at each step in the Runge–Kutta method, whereas we only perform one functional evaluation at each step in Euler's method. Nevertheless, the Runge–Kutta method still beats the heck out of Euler's method, the three term Taylor series method, and the improved Euler method.

Exercises

Using the Runge–Kutta method with $h=0.1$, determine an approximate value of the solution at $t=1$ for each of the initial-value problems 1–5. Repeat these computations with $h=0.025$ and compare the results with the given value of the solution.

1. $dy/dt=1+t-y$, $y(0)=0$; $(y(t)=t)$

2. $dy/dt=2ty$, $y(0)=2$; $(y(t)=2e^{t^2})$

3. $dy/dt=1+y^2-t^2$, $y(0)=0$; $(y(t)=t)$

4. $dy/dt=te^{-y}+t/(1+t^2)$, $y(0)=0$; $(y(t)=\ln(1+t^2))$

5. $dy/dt=-1+2t+y^2/((1+t^2)^2)$, $y(0)=1$; $(y(t)=1+t^2)$

6. Using the Runge–Kutta method with $h=\pi/40$, determine an approximate value of the solution of the initial-value problem

$$\frac{dy}{dt}=2\sec^2 t-(1+y^2), \qquad y(0)=0$$

at $t=\pi/4$. Repeat these computations with $h=\pi/160$ and compare the results with the number one which is the value of the solution $y(t)=\tan t$ at $t=\pi/4$.

1.17 What to do in practice

In this section we discuss some of the practical problems which arise when we attempt to solve differential equations on the computer. First, and foremost, is the problem of estimating the error that we make. It is not too difficult to show that the error we make using Euler's method, the three term Taylor series method, the improved Euler method and the Runge–Kutta method with step size h is at most c_1h, c_2h^2, c_3h^2, and c_4h^4, respectively. With one exception, though, it is practically impossible to find the constants c_1, c_2, c_3, and c_4. The one exception is Euler's method where we can explicitly estimate (see Section 1.13.1) the error we make in approximating $y(t_k)$ by y_k. However, this estimate is not very useful, since it is only valid for t_k sufficiently close to t_0, and we are usually interested in the values of y at times t much larger than t_0. Thus, we usually do not know, a priori, how small to choose the step size h so as to achieve a desired accuracy. We only know that the approximate values y_k that we compute get closer and closer to $y(t_k)$ as h gets smaller and smaller.

One way of resolving this difficulty is as follows. Using one of the schemes presented in the previous section, we choose a step size h and compute numbers $y_1,\dots,y_N$. We then repeat the computations with a step size $h/2$ and compare the results. If the changes are greater than we are willing to accept, then it is necessary to use a smaller step size. We keep repeating this process until we achieve a desired accuracy. For example, suppose that we require the solution of the initial-value problem $y'=f(t,y)$, $y(0)=y_0$ at $t=1$ to four decimal places accuracy. We choose a step size $h=1/100$, say, and compute $y_1,\dots,y_{100}$. We then repeat these computations with $h=1/200$ and obtain new approximations $z_1,\dots,z_{200}$. If y_{100} and z_{200} agree in their first four decimal places then we take z_{200} as our approximation of $y(1)$.* If y_{100} and z_{200} do not agree in their first four decimal places, then we repeat our computations with step size $h=1/400$.

Example 1. Find the solution of the initial-value problem

*This does not guarantee that z_{200} agrees with $y(1)$ to four decimal places. As an added precaution, we might halve the step size again. If the first four decimal places still remain unchanged, then we can be reasonably certain that z_{200} agrees with $y(1)$ to four decimal places.

$$\frac{dy}{dt}=y(1+e^{-y})+e^{t}, \qquad y(0)=0$$

at $t=1$ to four decimal places accuracy.

Solution. We illustrate how to try and solve this problem using Euler's method, the three term Taylor series method, the improved Euler method, and the Runge–Kutta method.

(i) *Euler's method*:

Pascal Program

```
Program Euler (input, output);

var
  T, Y: array[0..999] of real;
  a, h: real;
  k, N: integer;

begin
  readln(T[0], Y[0], a, N);
  h := a/N;
  page;
  for k := 0 to N-1 do
    begin
      T[k+1] := T[k]+h;
      Y[k+1] := Y[k]+h*(Y[k]*(1+exp(-Y[k]))+exp(T[k]));
    end;
  writeln('N':4, 'h':10, 'Y[N]':20);
  writeln(N:4, ' ':2, h:10:7, ' ':2, Y[N]:18:10);
end.
```

Fortran Program

```
Section A                 DIMENSION T(1000), Y(1000)
Read in data              READ (5,10) T(1),Y(1),A,N,
                  10      FORMAT (3F20.8,I5)
                          H=A/N

Section B                 DO20 K=1,N
Do computations           T(K+1)=T(K)+H
                          Y(K+1)=Y(K)+H*(Y(K)*(1+EXP(-Y(K)))
                      1           +EXP(T(K)))
                  20      CONTINUE

Section C                 WRITE (6,30) N,H,Y(N+1)
Print out         30      FORMAT (1H,1X,I5,2X,F10.7,4X,F20.9)
results                   CALL EXIT
                          END
```

See also C Program 8 in Appendix C for a sample C program.
We set $A=1$, $T[0]=0$, $Y[0]=0$ ($T(1)=Y(1)=0$ in the Fortran program) and ran these programs for $N=10$, 20, 40, 80, 160, 320, and 640. The results of these computations are given in Table 1. Notice that even with a step size h

Table 1

N	h	y_N
10	0.1	2.76183168
20	0.05	2.93832741
40	0.025	3.03202759
80	0.0125	3.08034440
160	0.00625	3.10488352
320	0.003125	3.11725009
640	0.0015625	3.12345786

as small as $1/640$, we can only guarantee an accuracy of one decimal place. This points out the limitation of Euler's method. Since N is so large already, it is wiser to use a more accurate scheme than to keep choosing smaller and smaller step sizes h for Euler's method.

(ii) *The three term Taylor series method*

Pascal Program

```
Program Taylor (input, output);

var
  T, Y: array[0..999] of real;
  a, h, DY1, DY2: real;
  k, N: integer;

begin
  readln(T[0], Y[0], a, N);
  h := a/N;
  page;
  for k := 0 to N-1 do
    begin
      T[k+1] := T[k]+h;
      DY1 := 1+(1-Y[k]) * exp(-Y[k]);
      DY2 := Y[k] * (1+exp(-Y[k]))+exp(T[k]);
      Y[k+1] := Y[k]+h * DY2+(h * h/2) * (exp(T[k])+DY1 * DY2);
    end;
  writeln('N':4, 'h':10, 'Y[N]':20);
  writeln(N:4, ' ':2, h:10:7, ' ':2, Y[N]:18:10);
end.
```

Fortran Program

Replace Section B of the previous Fortran program by

```
      DO20 K=1,N
      T(K+1)=T(K)+H
      DY1=1+(1-Y(K))*EXP(-Y(K))
      DY2=Y(K)*(1+EXP(-Y(K)))+EXP(T(K))
      Y(K+1)=Y(K)+H*DY2+(H*H/2)*(EXP(T(K))+DY1*DY2)
20    CONTINUE
```

See also C Program 9 in Appendix C for a sample C program.
We set $A=1$, $T[0]=0$, and $Y[0]=0$ ($T(1)=0$, and $Y(1)=0$ in the Fortran program) and ran these programs for $N=10$, 20, 40, 60, 80, 160, and 320. The results of these computations are given in Table 2. Observe that y_{160} and y_{320} agree in their first four decimal places. Hence the approximation $y(1)=3.12966689$ is correct to four decimal places.

Table 2

N	h	y_N
10	0.1	3.11727674
20	0.05	3.12645293
40	0.025	3.12885845
80	0.0125	3.12947408
160	0.00625	3.12962979
320	0.003125	3.12966689

(iii) *The improved Euler method*

Pascal Program

```
Program Improved (input, output);

var
  T, Y: array[0..999] of real;
  a, h, R1, R2: real;
  k, N: integer;

begin
  readln(T[0], Y[0], a, N);
  h:=a/N;
  page;
  for k:=0 to N-1 do
    begin
      T[k+1]:=T[k]+h;
```

```
      R1:=Y[k]*(1+exp(-Y[k]))+exp(T[k]);
      R2:=(Y[k]+h*R1)*(1+exp(-(Y[k]+h*R1)))+exp(T[k+1]);
      Y[k+1]:=Y[k]+(h/2)*(R1+R2);
    end;
  writeln('N':4, 'h':10, 'Y[N]':20);
  writeln(N:4, ' ':2, h:10:7, ' ':2, Y[N]:18:10);
end.
```

Fortran Program

Replace Section B of the first Fortran program in this section by

```
      DO 20 K=1,N
      T(K+1)=T(K)+H
      R1=Y(K)*(1+EXP(-Y(K)))+EXP(T(K))
      R2=(Y(K)+H*R1)*(1+EXP(-(Y(K)+H*R1)))+EXP(T(K+1))
      Y(K+1)=Y(K)+(H/2)*(R1+R2)
20    CONTINUE
```

See also C Program 10 in Appendix C for a sample C program.
We set $A=1$, $T[0]=0$ and $Y[0]=0$ ($T(1)=0$ and $Y(1)=0$ in the Fortran program) and ran these programs for $N=10$, 20, 40, 80, 160, and 320. The results of these computations are given in Table 3. Observe that y_{160} and y_{320}

Table 3

N	h	y_N
10	0.1	3.11450908
20	0.05	3.12560685
40	0.025	3.1286243
80	0.0125	3.12941247
160	0.00625	3.12961399
320	0.003125	3.12964943

agree in their first four decimal places. Hence the approximation $y(1)=3.12964943$ is correct to four decimal places.

(iv) *The method of Runge–Kutta*

Pascal Program

```
Program Runge_Kutta (input, output);

var
  T, Y: array[0..999] of real;
  a, h, LK1, LK2, LK3, LK4: real;
  k, N: integer;
```

```
begin
  readln(T[0], Y[0], a, N);
  h := a/N;
  page;
  for k := 0 to N - 1 do
    begin
      T[k+1] := T[k] + h;
      LK1 := (Y[k] * (1 + exp(-Y[k])) + exp(T[k]);
      LK2 := (Y[k] + (h/2) * LK1) * (1 + exp(-(Y[k] + (h/2) * LK1)))
            + exp(T[k] + (h/2));
      LK3 := (Y[k] + (h/2) * LK2) * (1 + exp(-(Y[k] + (h/2) * LK2)))
            + exp(T[k] + (h/2));
      LK4 := (Y[k] + h * LK3) * (1 + exp(-(Y[k] + h * LK3)))
            + exp(T[k+1]);
      Y[k+1] := Y[k] + (h/6) * (LK1 + 2 * LK2 + 2 * LK3 + LK4);
    end;
  writeln('N':4, 'h':10, 'Y[N]':20);
  writeln(N:4, ' ':2, h:10:7, ' ':2, Y[N]:18:10);
0 0 1 10
```

Fortran Program

Replace Section B of the first Fortran program in this section by

```
      DO 20 K = 1, N
      T(K + 1) = T(K) + H
      LK1 = Y(K) * (1 + EXP( - Y(K)) + EXP(T(K))
      LK2 = (Y(K) + (H/2) * LK1) * (1 + EXP( - (Y(K) + (H/2) * LK1)))
     1 + EXP(T(K) + (H/2))
      LK3 = (Y(K) + (H/2) * LK2) * (1 + EXP( - (Y(K) + (H/2) * LK2)))
     1 + EXP(T(K) + (H/2))
      LK4 = (Y(K) + H * LK3) * (1 + EXP( - (Y(K) + H * LK3)))
     1 + EXP(T(K + 1))
      Y(K + 1) = Y(K) + (H/6) * (LK1 + 2 * LK2 + 2 * LK3 + LK4)
   20 CONTINUE
```

See also C Program 11 in Appendix C for a sample C program.

We set $A=1$, $T[0]=0$ and $Y[0]=0$ ($T(1)=0$ and $Y(1)=0$ in the Fortran program) and ran these programs for $N=10$, 20, 40, 80, 160, and 320. The results of these computations are given in Table 4. Notice that our approximation of $y(1)$ is already correct to four decimal places with $h=0.1$, and that it is already correct to eight decimal places with $h=0.00625 (N=160)$. This example again illustrates the power of the Runge–Kutta method.

We conclude this section with two examples which point out some additional difficulties which may arise when we solve initial-value problems on a digital computer.

Table 4

N	h	y_N
10	0.1	3.1296517
20	0.05	3.12967998
40	0.025	3.1296819
80	0.0125	3.12968203
160	0.00625	3.12968204
320	0.003125	3.12968204

Example 2. Use the Runge–Kutta method to find approximate values of the solution of the initial-value problem

$$\frac{dy}{dt} = t^2 + y^2, \qquad y(0) = 1$$

at the points $t_k = k/N$, $k = 1, \ldots, N$.
Solution.

Pascal Program

```
Program Runge_Kutta (input, output);

var
  T, Y: array[0..999] of real;
  a, h, LK1, LK2, LK3, LK4: real;
  k, N: integer;

begin
  readln(T[0], Y[0], a, N);
  h := a/N;
  page;
  for k := 0 to N - 1 do
    begin
      T[k + 1] := T[k] + h;
      LK1 := T[k] * T[k] + Y[k] * Y[k];
      LK2 := (T[k] + h/2) * (T[k] + h/2)
            + (Y[k] + (h/2) * LK1) * (Y[k] + (h/2) * LK1);
      LK3 := (T[k] + h/2) * (T[k] + h/2)
            + (Y[k] + (h/2) * LK2) * (Y[k] + (h/2) * LK2);
      LK4 := (T[k] + h) * (T[k] + h)
            + (Y[k] + h * LK3) * (Y[k] + h * LK3);
      Y[k + 1] := Y[k] + (h/6) * (LK1 + 2 * LK2 + 2 * LK3 + LK4);
    end;
  writeln('T':4, 'Y':15);
  for k := 0 to N do
    writeln(T[k]:10:7, ' ':2, Y[k]:16:9);
end.
```

Fortran Program

Replace Sections B and C of the first Fortran program in this section by

```
Section B            |      DO20 K=1,N
Do computa-          |      T(K+1)=T(K)+H
  tions              |      LK1=T(K)**2+Y(K)**2
                     |      LK2=(T(K)+(H/2))**2+(Y(K)+(H/2)*LK1)**2
                     |      LK3=(T(K)+(H/2))**2+(Y(K)+(H/2)*LK2)**2
                     |      LK4=(T(K)+H)**2+(Y(K)+H*LK3)**2
                     |      Y(K+1)=Y(K)+(H/6)*(LK1+2*LK2+2*LK3+LK4)
                     | 20   CONTINUE

Section C            |      NA=N+1
Print out            |      WRITE (6,30) (T(J),Y(J),J=1,NA)
results              | 30   FORMAT (1H1,3X,1HT,4X,1HY/(1H,1X,F9.7,
                     |     1 2X,F20.9/))
                     |      CALL EXIT
                     |      END
```

See also C Program 12 in Appendix C for a sample C program.

We attempted to run these programs with $A=1$, $T[0]=0$, $Y[0]=1$ ($T(1)=0$, and $Y(1)=1$ in the Fortran program) and $N=10$, but we received an error message that the numbers being computed exceeded the domain of the computer. That is to say, they were larger than 10^{38}. This indicates that the solution $y(t)$ goes to infinity somewhere in the interval $[0,1]$. We can prove this analytically, and even obtain an estimate of where $y(t)$ goes to infinity, by the following clever argument. Observe that for $0 \leqslant t \leqslant 1$, $y(t)$ is never less than the solution $\phi_1(t)=1/(1-t)$ of the initial-value problem

$$\frac{dy}{dt}=y^2, \qquad y(0)=1.$$

In addition, $y(t)$ never exceeds the solution $\phi_2(t)=\tan(t+\pi/4)$ of the initial-value problem $dy/dt=1+y^2$, $y(0)=1$. Hence, for $0 \leqslant t \leqslant 1$,

$$\frac{1}{1-t} \leqslant y(t) \leqslant \tan(t+\pi/4).$$

This situation is described graphically in Figure 1. Since $\phi_1(t)$ and $\phi_2(t)$ become infinite at $t=1$ and $t=\pi/4$ respectively, we conclude that $y(t)$ becomes infinite somewhere between $\pi/4$ and 1.

The solutions of most initial-value problems which arise in physical and biological applications exist for all future time. Thus, we need not be overly concerned with the problem of solutions going to infinity in finite time or the problem of solutions becoming exceedingly large. On the other hand, though, there are several instances in economics where this problem is of paramount importance. In these instances, we are often interested in

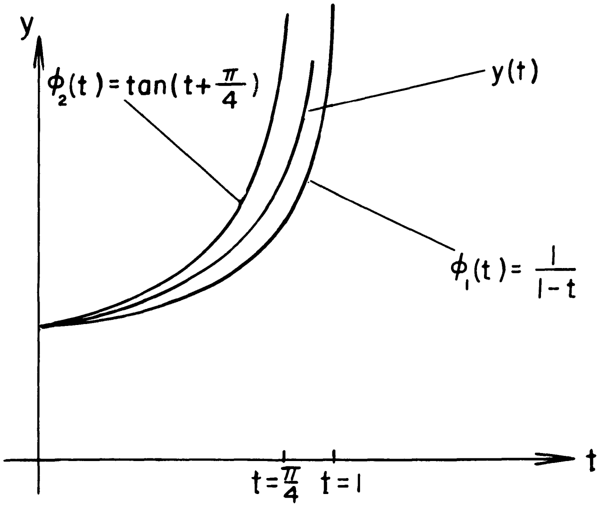

Figure 1

determining whether certain differential equations can accurately model a given economic phenomenon. It is often possible to eliminate several of these equations by showing that they allow solutions which are unrealistically large.

Example 3. Use Euler's method to determine approximate values of the solution of the initial-value problem

$$\frac{dy}{dt} = y|y|^{-3/4} + t\sin\frac{\pi}{t}, \qquad y(0) = 0 \tag{1}$$

at the points $1/N, 2/N, \ldots, 2$.

Solution. The programming for this problem is simplified immensely by observing that

$$y|y|^{-3/4} = (\operatorname{sgn} y)|y|^{1/4}, \quad \text{where} \quad \operatorname{sgn} y = \begin{cases} 1, & y > 0 \\ 0, & y = 0 \\ -1, & y < 0 \end{cases}$$

Pascal Program

```
Program Euler (input, output);

const
  PI = 3.141592654;

var
  T, Y: array[0..999] of real;
  h: real;
  k, N: integer;
```

```
begin
  readln(N);
  page;
  h:=2/N;
  T[1]:=h;
  Y[1]:=0;
  for k:=1 to N-1 do
    begin
      T[k+1]:=T[k]+h;
      if Y[k]=0 then Y[k+1]:=h*T[k]*sin(PI/T[k]) else
      Y[k+1]:=Y[k]+h*(Y[k]*exp((-3/4)*ln(abs(Y[k])))
                     +T[k]*sin(PI/T[k]));
    end;
  writeln('T':4,'Y':15);
  for k:=1 to N do
    writeln(T[k]:10:7,' ':2,Y[k]:16:9);
end.
```

Fortran Program

```
      DIMENSION T(1000),Y(1000)
      READ (5,10) N
10    FORMAT (I5)
      H=2/N
      T(1)=H
      Y(1)=0
      DO20 K=2,N
      T(K)=T(K-1)+H
      Y(K)=Y(K-1)+H*(SIGN(Y(K-1))*ABS(Y(K-1))**0.25
     1     +T(K-1)*SIN(3.141592654/T(K-1)))
20    CONTINUE
      WRITE(6,30) 0,0,(T(J),Y(J),J=1,N)
30    FORMAT (1H1,3X,1HT,4X,1HY/(1H,1X,F10.7,2X,F20.9/))
      CALL EXIT
      END
```

See also C Program 13 in Appendix C for a sample C program. When we set $N=25$ we obtained the value 2.4844172 for $y(2)$, but when we set $N=27$, we obtained the value -0.50244575 for $y(2)$. Moreover, all the y_k were positive for $N=25$ and negative for $N=27$. We repeated these computations with $N=89$ and $N=91$ and obtained the values 2.64286349 and -0.6318074 respectively. In addition, all the y_k were again positive for $N=89$ and negative for $N=91$. Indeed, it is possible, but rather difficult, to prove that all the y_k will be positive if $N=1,5,9,13,17,\ldots$ and negative if $N=3,7,11,15,\ldots$. This suggests that the solution of the initial-value problem (1) is not unique. We cannot prove this analytically, since we cannot solve the differential equation explicitly. It should be noted though, that the existence–uniqueness theorem of Section 1.10 does not apply here, since the partial derivative with respect to y of the function $|y|^{-3/4}y+t\sin\pi/t$ does not exist at $y=0$.

Most of the initial-value problems that arise in applications have unique solutions. Thus, we need not be overly concerned with the problem of non-uniqueness of solutions. However, we should always bear in mind that initial-value problems which do not obey the hypotheses of the existence–uniqueness theorem of Section 1.10 might possess more than one solution, for the consequences of picking the wrong solution in these rare instances can often be catastrophic.

Exercises

In each of Problems 1–5, find the solution of the given initial-value problem at $t=1$ to four decimal places accuracy.

1. $\dfrac{dy}{dt}=y+e^{-y}+2t,\quad y(0)=0$

2. $\dfrac{dy}{dt}=1-t+y^2,\quad y(0)=0$

3. $\dfrac{dy}{dt}=\dfrac{t^2+y^2}{1+t+y^2},\quad y(0)=0$

4. $\dfrac{dy}{dt}=e^ty^2-2y,\quad y(0)=1$

5. $\dfrac{dy}{dt}=ty^3-y,\quad y(0)=1$

Second-order linear differential equations

2

2.1 Algebraic properties of solutions

A second-order differential equation is an equation of the form

$$\frac{d^2y}{dt^2}=f\left(t,y,\frac{dy}{dt}\right). \tag{1}$$

For example, the equation

$$\frac{d^2y}{dt^2}=\sin t+3y+\left(\frac{dy}{dt}\right)^2$$

is a second-order differential equation. A function $y=y(t)$ is a solution of (1) if $y(t)$ satisfies the differential equation; that is

$$\frac{d^2y(t)}{dt^2}=f\left(t,y(t),\frac{dy(t)}{dt}\right).$$

Thus, the function $y(t)=\cos t$ is a solution of the second-order equation $d^2y/dt^2=-y$ since $d^2(\cos t)/dt^2=-\cos t$.

Second-order differential equations arise quite often in applications. The most famous second-order differential equation is Newton's second law of motion (see Section 1.7)

$$m\frac{d^2y}{dt^2}=F\left(t,y,\frac{dy}{dt}\right)$$

which governs the motion of a particle of mass m moving under the influence of a force F. In this equation, m is the mass of the particle, $y=y(t)$ is its position at time t, dy/dt is its velocity, and F is the total force acting on the particle. As the notation suggests, the force F may depend on the position and velocity of the particle, as well as on time.

In addition to the differential equation (1), we will often impose initial conditions on $y(t)$ of the form

$$y(t_0)=y_0, \qquad y'(t_0)=y'_0. \tag{1'}$$

The differential equation (1) together with the initial conditions (1′) is referred to as an initial-value problem. For example, let $y(t)$* denote the position at time t of a particle moving under the influence of gravity. Then, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2}=-g; \qquad y(t_0)=y_0, \quad y'(t_0)=y'_0,$$

where y_0 is the initial position of the particle and y'_0 is the initial velocity of the particle.

Second-order differential equations are extremely difficult to solve. This should not come as a great surprise to us after our experience with first-order equations. We will only succeed in solving the special differential equation

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=g(t). \tag{2}$$

Fortunately, though, many of the second-order equations that arise in applications are of this form.

The differential equation (2) is called a second-order linear differential equation. We single out this equation and call it linear because both y and dy/dt appear by themselves. For example, the differential equations

$$\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+(\sin t)y=e^t$$

and

$$\frac{d^2y}{dt^2}+e^t\frac{dy}{dt}+2y=1$$

are linear, while the differential equations

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+\sin y=t^3$$

and

$$\frac{d^2y}{dt^2}+\left(\frac{dy}{dt}\right)^2=1$$

are both nonlinear, due to the presence of the $\sin y$ and $(dy/dt)^2$ terms, respectively.

We consider first the second-order linear homogeneous equation

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{3}$$

*The positive direction of y is taken upwards.

which is obtained from (2) by setting $g(t)=0$. It is certainly not obvious at this point how to find all the solutions of (3), or how to solve the initial-value problem

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0; \qquad y(t_0)=y_0, \quad y'(t_0)=y_0'. \tag{4}$$

Therefore, before trying to develop any elaborate procedures for solving (4), we should first determine whether it actually has a solution. This information is contained in the following theorem, whose proof will be indicated in Chapter 4.

Theorem 1. (Existence–uniqueness Theorem). *Let the functions $p(t)$ and $q(t)$ be continuous in the open interval $\alpha<t<\beta$. Then, there exists one, and only one function $y(t)$ satisfying the differential equation* (3) *on the entire interval $\alpha<t<\beta$, and the prescribed initial conditions $y(t_0)=y_0$, $y'(t_0)=y_0'$. In particular, any solution $y=y(t)$ of* (3) *which satisfies $y(t_0)=0$ and $y'(t_0)=0$ at some time $t=t_0$ must be identically zero.*

Theorem 1 is an extremely important theorem for us. On the one hand, it is our hunting license to find the unique solution $y(t)$ of (4). And, on the other hand, we will actually use Theorem 1 to help us find all the solutions of (3).

We begin our analysis of Equation (3) with the important observation that the left-hand side

$$y''+p(t)y'+q(t)y \qquad \left(y'=\frac{dy}{dt}, y''=\frac{d^2y}{dt^2}\right)$$

of the differential equation can be viewed as defining a "function of a function": with each function y having two derivatives, we associate another function, which we'll call $L[y]$, by the relation

$$L[y](t)=y''(t)+p(t)y'(t)+q(t)y(t).$$

In mathematical terminology, L is an operator which operates on functions; that is, there is a prescribed recipe for associating with each function y a new function $L[y]$.

Example 1. Let $p(t)=0$ and $q(t)=t$. Then,

$$L[y](t)=y''(t)+ty(t).$$

If $y(t)=\cos t$, then

$$L[y](t)=(\cos t)''+t\cos t=(t-1)\cos t,$$

and if $y(t)=t^3$, then

$$L[y](t)=(t^3)''+t(t^3)=t^4+6t.$$

Thus, the operator L assigns the function $(t-1)\cos t$ to the function $\cos t$, and the function $6t+t^4$ to the function t^3.

The concept of an operator acting on functions, or a "function of a function" is analogous to that of a function of a single variable t. Recall the definition of a function f on an interval I: with each number t in I we associate a new number called $f(t)$. In an exactly analogous manner, we associate with each function y having two derivatives a new function called $L[y]$. This is an extremely sophisticated mathematical concept, because in a certain sense, we are treating a function exactly as we do a point. Admittedly, this is quite difficult to grasp. It's not surprising, therefore, that the concept of a "function of a function" was not developed till the beginning of this century, and that many of the "high powered" theorems of mathematical analysis were proved only after this concept was mastered.

We now derive several important properties of the operator L, which we will use to great advantage shortly.

Property 1. $L[cy]=cL[y]$, *for any constant* c.

PROOF.
$$\begin{aligned}L[cy](t)&=(cy)''(t)+p(t)(cy)'(t)+q(t)(cy)(t)\\&=cy''(t)+cp(t)y'(t)+cq(t)y(t)\\&=c\left[y''(t)+p(t)y'(t)+q(t)y(t)\right]\\&=cL[y](t).\end{aligned}$$
□

The meaning of Property 1 is that the operator L assigns to the function (cy) c times the function it assigns to y. For example, let

$$L[y](t)=y''(t)+6y'(t)-2y(t).$$

This operator L assigns the function

$$(t^2)''+6(t^2)'-2(t^2)=2+12t-2t^2$$

to the function t^2. Hence, L must assign the function $5(2+12t-2t^2)$ to the function $5t^2$.

Property 2. $L[y_1+y_2]=L[y_1]+L[y_2]$.

PROOF.
$$\begin{aligned}L[y_1+y_2](t)&=(y_1+y_2)''(t)+p(t)(y_1+y_2)'(t)+q(t)(y_1+y_2)(t)\\&=y_1''(t)+y_2''(t)+p(t)y_1'(t)+p(t)y_2'(t)+q(t)y_1(t)+q(t)y_2(t)\\&=\left[y_1''(t)+p(t)y_1'(t)+q(t)y_1(t)\right]+\left[y_2''(t)+p(t)y_2'(t)+q(t)y_2(t)\right]\\&=L[y_1](t)+L[y_2](t).\end{aligned}$$
□

The meaning of Property 2 is that the operator L assigns to the function y_1+y_2 the sum of the functions it assigns to y_1 and y_2. For example, let

$$L[y](t)=y''(t)+2y'(t)-y(t).$$

This operator L assigns the function

$$(\cos t)'' + 2(\cos t)' - \cos t = -2\cos t - 2\sin t$$

to the function $\cos t$, and the function

$$(\sin t)'' + 2(\sin t)' - \sin t = 2\cos t - 2\sin t$$

to the function $\sin t$. Hence, L assigns the function

$$(-2\cos t - 2\sin t) + 2\cos t - 2\sin t = -4\sin t$$

to the function $\sin t + \cos t$.

Definition. An operator L which assigns functions to functions and which satisfies Properties 1 and 2 is called a linear operator. All other operators are nonlinear. An example of a nonlinear operator is

$$L[y](t) = y''(t) - 2t[y(t)]^4.$$

This operator assigns the function

$$\left(\frac{1}{t}\right)'' - 2t\left(\frac{1}{t}\right)^4 = \frac{2}{t^3} - \frac{2}{t^3} = 0$$

to the function $1/t$, and the function

$$\left(\frac{c}{t}\right)'' - 2t\left(\frac{c}{t}\right)^4 = \frac{2c}{t^3} - \frac{2c^4}{t^3} = \frac{2c(1-c^3)}{t^3}$$

to the function c/t. Hence, for $c \neq 0, 1$, and $y(t) = 1/t$, we see that $L[cy] \neq cL[y]$.

The usefulness of Properties 1 and 2 lies in the observation that the solutions $y(t)$ of the differential equation (3) are exactly those functions y for which

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t) = 0.$$

In other words, the solutions $y(t)$ of (3) are exactly those functions y to which the operator L assigns the zero function.* Hence, if $y(t)$ is a solution of (3) then so is $cy(t)$, since

$$L[cy](t) = cL[y](t) = 0.$$

If $y_1(t)$ and $y_2(t)$ are solutions of (3), then $y_1(t) + y_2(t)$ is also a solution of (3), since

$$L[y_1 + y_2](t) = L[y_1](t) + L[y_2](t) = 0 + 0 = 0.$$

Combining Properties 1 and 2, we see that all linear combinations

$$c_1y_1(t) + c_2y_2(t)$$

of solutions of (3) are again solutions of (3).

*The zero function is the function whose value at any time t is zero.

The preceding argument shows that we can use our knowledge of two solutions $y_1(t)$ and $y_2(t)$ of (3) to generate infinitely many other solutions. This statement has some very interesting implications. Consider, for example, the differential equation

$$\frac{d^2y}{dt^2}+y=0. \tag{5}$$

Two solutions of (5) are $y_1(t)=\cos t$ and $y_2(t)=\sin t$. Hence,

$$y(t)=c_1\cos t+c_2\sin t \tag{6}$$

is also a solution of (5), for every choice of constants c_1 and c_2. Now, Equation (6) contains two arbitrary constants. It is natural to suspect, therefore, that this expression represents the general solution of (5); that is, every solution $y(t)$ of (5) must be of the form (6). This is indeed the case, as we now show. Let $y(t)$ be any solution of (5). By the existence–uniqueness theorem, $y(t)$ exists for all t. Let $y(0)=y_0$, $y'(0)=y_0'$, and consider the function

$$\phi(t)=y_0\cos t+y_0'\sin t.$$

This function is a solution of (5) since it is a linear combination of solutions of (5). Moreover, $\phi(0)=y_0$ and $\phi'(0)=y_0'$. Thus, $y(t)$ and $\phi(t)$ satisfy the same second-order linear homogeneous equation and the same initial conditions. Therefore, by the uniqueness part of Theorem 1, $y(t)$ must be identically equal to $\phi(t)$, so that

$$y(t)=y_0\cos t+y_0'\sin t.$$

Thus, Equation (6) is indeed the general solution of (5).

Let us return now to the general linear equation (3). Suppose, in some manner, that we manage to find two solutions $y_1(t)$ and $y_2(t)$ of (3). Then, every function

$$y(t)=c_1y_1(t)+c_2y_2(t) \tag{7}$$

is again a solution of (3). Does the expression (7) represent the general solution of (3)? That is to say, does every solution $y(t)$ of (3) have the form (7)? The following theorem answers this question.

Theorem 2. *Let $y_1(t)$ and $y_2(t)$ be two solutions of* (3) *on the interval $\alpha<t<\beta$, with*

$$y_1(t)y_2'(t)-y_1'(t)y_2(t)$$

unequal to zero in this interval. Then,

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

is the general solution of (3).

PROOF. Let $y(t)$ be any solution of (3). We must find constants c_1 and c_2 such that $y(t)=c_1y_1(t)+c_2y_2(t)$. To this end, pick a time t_0 in the interval

(α,β) and let y_0 and y_0' denote the values of y and y' at $t=t_0$. The constants c_1 and c_2, if they exist, must satisfy the two equations

$$c_1y_1(t_0)+c_2y_2(t_0)=y_0$$
$$c_1y_1'(t_0)+c_2y_2'(t_0)=y_0'.$$

Multiplying the first equation by $y_2'(t_0)$, the second equation by $y_2(t_0)$ and subtracting gives

$$c_1\left[y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)\right]=y_0y_2'(t_0)-y_0'y_2(t_0).$$

Similarly, multiplying the first equation by $y_1'(t_0)$, the second equation by $y_1(t_0)$ and subtracting gives

$$c_2\left[y_1'(t_0)y_2(t_0)-y_1(t_0)y_2'(t_0)\right]=y_0y_1'(t_0)-y_0'y_1(t_0).$$

Hence,

$$c_1=\frac{y_0y_2'(t_0)-y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)}$$

and

$$c_2=\frac{y_0'y_1(t_0)-y_0y_1'(t_0)}{y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)}$$

if $y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)\neq 0$. Now, let

$$\phi(t)=c_1y_1(t)+c_2y_2(t)$$

for this choice of constants c_1,c_2. We know that $\phi(t)$ satisfies (3), since it is a linear combination of solutions of (3). Moreover, by construction, $\phi(t_0)=y_0$ and $\phi'(t_0)=y_0'$. Thus, $y(t)$ and $\phi(t)$ satisfy the same second-order linear homogeneous equation and the same initial conditions. Therefore, by the uniqueness part of Theorem 1, $y(t)$ must be identically equal to $\phi(t)$; that is,

$$y(t)=c_1y_1(t)+c_2y_2(t),\qquad \alpha<t<\beta. \qquad \square$$

Theorem 2 is an extremely useful theorem since it reduces the problem of finding all solutions of (3), of which there are infinitely many, to the much simpler problem of finding just two solutions $y_1(t),y_2(t)$. The only condition imposed on the solutions $y_1(t)$ and $y_2(t)$ is that the quantity $y_1(t)y_2'(t)-y_1'(t)y_2(t)$ be unequal to zero for $\alpha<t<\beta$. When this is the case, we say that $y_1(t)$ and $y_2(t)$ are a *fundamental* set of solutions of (3), since all other solutions of (3) can be obtained by taking linear combinations of $y_1(t)$ and $y_2(t)$.

Definition. The quantity $y_1(t)y_2'(t)-y_1'(t)y_2(t)$ is called the *Wronskian* of y_1 and y_2, and is denoted by $W(t)=W[y_1,y_2](t)$.

Theorem 2 requires that $W[y_1,y_2](t)$ be unequal to zero at all points in the interval (α,β). In actual fact, the Wronskian of any two solutions

$y_1(t), y_2(t)$ of (3) is either identically zero, or is never zero, as we now show.

Theorem 3. *Let $p(t)$ and $q(t)$ be continuous in the interval $\alpha < t < \beta$, and let $y_1(t)$ and $y_2(t)$ be two solutions of* (3). *Then, $W[y_1,y_2](t)$ is either identically zero, or is never zero, on the interval $\alpha < t < \beta$.*

We prove Theorem 3 with the aid of the following lemma.

Lemma 1. *Let $y_1(t)$ and $y_2(t)$ be two solutions of the linear differential equation $y'' + p(t)y' + q(t)y = 0$. Then, their Wronskian*

$$W(t) = W[y_1,y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

satisfies the first-order differential equation

$$W' + p(t)W = 0.$$

Proof. Observe that

$$\begin{aligned} W'(t) &= \frac{d}{dt}(y_1y_2' - y_1'y_2) \\ &= y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2 \\ &= y_1y_2'' - y_1''y_2. \end{aligned}$$

Since y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$, we know that

$$y_2'' = -p(t)y_2' - q(t)y_2$$

and

$$y_1'' = -p(t)y_1' - q(t)y_1.$$

Hence,

$$\begin{aligned} W'(t) &= y_1[-p(t)y_2' - q(t)y_2] - y_2[-p(t)y_1' - q(t)y_1] \\ &= -p(t)[y_1y_2' - y_1'y_2] \\ &= -p(t)W(t). \end{aligned}$$

□

We can now give a very simple proof of Theorem 3.

Proof of theorem 3. Pick any t_0 in the interval (α,β). From Lemma 1,

$$W[y_1,y_2](t) = W[y_1,y_2](t_0)\exp\left(-\int_{t_0}^{t} p(s)\,ds\right).$$

Now, $\exp\left(-\int_{t_0}^{t} p(s)\,ds\right)$ is unequal to zero for $\alpha < t < \beta$. Therefore, $W[y_1,y_2](t)$ is either identically zero, or is never zero. □

The simplest situation where the Wronskian of two functions $y_1(t),y_2(t)$ vanishes identically is when one of the functions is identically zero. More generally, the Wronskian of two functions $y_1(t),y_2(t)$ vanishes identically if one of the functions is a constant multiple of the other. If $y_2=cy_1$, say, then

$$W[y_1,y_2](t)=y_1(cy_1)'-y_1'(cy_1)=0.$$

Conversely, suppose that the Wronskian of two *solutions* $y_1(t),y_2(t)$ of (3) vanishes identically. Then, one of these solutions must be a constant multiple of the other, as we now show.

Theorem 4. *Let $y_1(t)$ and $y_2(t)$ be two solutions of* (3) *on the interval $\alpha<t<\beta$, and suppose that $W[y_1,y_2](t_0)=0$ for some t_0 in this interval. Then, one of these solutions is a constant multiple of the other.*

PROOF #1. Suppose that $W[y_1,y_2](t_0)=0$. Then, the equations

$$c_1y_1(t_0)+c_2y_2(t_0)=0$$
$$c_1y_1'(t_0)+c_2y_2'(t_0)=0$$

have a nontrivial solution c_1,c_2; that is, a solution c_1,c_2 with $|c_1|+|c_2|\neq 0$. Let $y(t)=c_1y_1(t)+c_2y_2(t)$, for this choice of constants c_1,c_2. We know that $y(t)$ is a solution of (3), since it is a linear combination of $y_1(t)$ and $y_2(t)$. Moreover, by construction, $y(t_0)=0$ and $y'(t_0)=0$. Therefore, by Theorem 1, $y(t)$ is identically zero, so that

$$c_1y_1(t)+c_2y_2(t)=0, \qquad \alpha<t<\beta,$$

If $c_1\neq 0$, then $y_1(t)=-(c_2/c_1)y_2(t)$, and if $c_2\neq 0$, then $y_2(t)=-(c_1/c_2)y_1(t)$. In either case, one of these solutions is a constant multiple of the other. □

PROOF #2. Suppose that $W[y_1,y_2](t_0)=0$. Then, by Theorem 3, $W[y_1,y_2](t)$ is identically zero. Assume that $y_1(t)y_2(t)\neq 0$ for $\alpha<t<\beta$. Then, dividing both sides of the equation

$$y_1(t)y_2'(t)-y_1'(t)y_2(t)=0$$

by $y_1(t)y_2(t)$ gives

$$\frac{y_2'(t)}{y_2(t)}-\frac{y_1'(t)}{y_1(t)}=0.$$

This equation implies that $y_1(t)=cy_2(t)$ for some constant c.

Next, suppose that $y_1(t)y_2(t)$ is zero at some point $t=t^*$ in the interval $\alpha<t<\beta$. Without loss of generality, we may assume that $y_1(t^*)=0$, since otherwise we can relabel y_1 and y_2. In this case it is simple to show (see Exercise 19) that either $y_1(t)\equiv 0$, or $y_2(t)=[y_2'(t^*)/y_1'(t^*)]y_1(t)$. This completes the proof of Theorem 4. □

Definition. The functions $y_1(t)$ and $y_2(t)$ are said to be *linearly dependent* on an interval I if one of these functions is a constant multiple of the other on I. The functions $y_1(t)$ and $y_2(t)$ are said to be *linearly independent* on an interval I if they are not linearly dependent on I.

Corollary to Theorem 4. *Two solutions $y_1(t)$ and $y_2(t)$ of* (3) *are linearly independent on the interval $\alpha<t<\beta$ if, and only if, their Wronskian is unequal to zero on this interval. Thus, two solutions $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of* (3) *on the interval $\alpha<t<\beta$ if, and only if, they are linearly independent on this interval.*

EXERCISES

1. Let $L[y](t)=y''(t)-3ty'(t)+3y(t)$. Compute
(a) $L[e^t]$, (b) $L[\cos\sqrt{3}\,t]$, (c) $L[2e^t+4\cos\sqrt{3}\,t]$,
(d) $L[t^2]$, (e) $L[5t^2]$, (f) $L[t]$, (g) $L[t^2+3t]$.

2. Let $L[y](t)=y''(t)-6y'(t)+5y(t)$. Compute
(a) $L[e^t]$, (b) $L[e^{2t}]$, (c) $L[e^{3t}]$, (d) $L[e^{rt}]$,
(e) $L[t]$, (f) $L[t^2]$, (g) $L[t^2+2t]$.

3. Show that the operator L defined by

$$L[y](t)=\int_a^t s^2 y(s)\,ds$$

is linear; that is, $L[cy]=cL[y]$ and $L[y_1+y_2]=L[y_1]+L[y_2]$.

4. Let $L[y](t)=y''(t)+p(t)y'(t)+q(t)y(t)$, and suppose that $L[t^2]=t+1$ and $L[t]=2t+2$. Show that $y(t)=t-2t^2$ is a solution of $y''+p(t)y'+q(t)y=0$.

5. (a) Show that $y_1(t)=\sqrt{t}$ and $y_2(t)=1/t$ are solutions of the differential equation

$$2t^2y''+3ty'-y=0 \tag{*}$$

on the interval $0<t<\infty$.
(b) Compute $W[y_1,y_2](t)$. What happens as t approaches zero?
(c) Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of (*) on the interval $0<t<\infty$.
(d) Solve the initial-value problem $2t^2y''+3ty'-y=0$; $y(1)=2$, $y'(1)=1$.

6. (a) Show that $y_1(t)=e^{-t^2/2}$ and $y_2(t)=e^{-t^2/2}\int_0^t e^{s^2/2}ds$ are solutions of

$$y''+ty'+y=0 \tag{*}$$

on the interval $-\infty<t<\infty$.
(b) Compute $W[y_1,y_2](t)$.
(c) Show that y_1 and y_2 form a fundamental set of solutions of (*) on the interval $-\infty<t<\infty$.
(d) Solve the initial-value problem $y''+ty'+y=0$; $y(0)=0$, $y'(0)=1$.

7. Compute the Wronskian of the following pairs of functions.

(a) $\sin at, \cos bt$ (b) $\sin^2 t, 1-\cos 2t$

(c) e^{at}, e^{bt} (d) e^{at}, te^{at}

(e) $t, t\ln t$ (f) $e^{at}\sin bt, e^{at}\cos bt$

8. Let $y_1(t)$ and $y_2(t)$ be solutions of (3) on the interval $-\infty<t<\infty$ with $y_1(0)=3, y_1'(0)=1, y_2(0)=1$, and $y_2'(0)=\frac{1}{3}$. Show that $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $-\infty<t<\infty$.

9. (a) Let $y_1(t)$ and $y_2(t)$ be solutions of (3) on the interval $\alpha<t<\beta$, with $y_1(t_0)=1$, $y_1'(t_0)=0$, $y_2(t_0)=0$, and $y_2'(t_0)=1$. Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of (3) on the interval $\alpha<t<\beta$.

(b) Show that $y(t)=y_0y_1(t)+y_0'y_2(t)$ is the solution of (3) satisfying $y(t_0)=y_0$ and $y'(t_0)=y_0'$.

10. Show that $y(t)=t^2$ can never be a solution of (3) if the functions $p(t)$ and $q(t)$ are continuous at $t=0$.

11. Let $y_1(t)=t^2$ and $y_2(t)=t|t|$.

(a) Show that y_1 and y_2 are linearly dependent on the interval $0\leqslant t\leqslant 1$.

(b) Show that y_1 and y_2 are linearly independent on the interval $-1\leqslant t\leqslant 1$.

(c) Show that $W[y_1,y_2](t)$ is identically zero.

(d) Show that y_1 and y_2 can never be two solutions of (3) on the interval $-1<t<1$ if both p and q are continuous in this interval.

12. Suppose that y_1 and y_2 are linearly independent on an interval I. Prove that $z_1=y_1+y_2$ and $z_2=y_1-y_2$ are also linearly independent on I.

13. Let y_1 and y_2 be solutions of Bessel's equation

$$t^2y''+ty'+(t^2-n^2)y=0$$

on the interval $0<t<\infty$, with $y_1(1)=1, y_1'(1)=0, y_2(1)=0$, and $y_2'(1)=1$. Compute $W[y_1,y_2](t)$.

14. Suppose that the Wronskian of any two solutions of (3) is constant in time. Prove that $p(t)=0$.

In Problems 15–18, assume that p and q are continuous, and that the functions y_1 and y_2 are solutions of the differential equation

$$y''+p(t)y'+q(t)y=0$$

on the interval $\alpha<t<\beta$.

15. Prove that if y_1 and y_2 vanish at the same point in the interval $\alpha<t<\beta$, then they cannot form a fundamental set of solutions on this interval.

16. Prove that if y_1 and y_2 achieve a maximum or minimum at the same point in the interval $\alpha<t<\beta$, then they cannot form a fundamental set of solutions on this interval.

17. Prove that if y_1 and y_2 are a fundamental set of solutions, then they cannot have a common point of inflection in $\alpha<t<\beta$ unless p and q vanish simultaneously there.

18. Suppose that y_1 and y_2 are a fundamental set of solutions on the interval $-\infty < t < \infty$. Show that there is one and only one zero of y_1 between consecutive zeros of y_2. *Hint*: Differentiate the quantity y_2/y_1 and use Rolle's Theorem.

19. Suppose that $W[y_1,y_2](t^*)=0$, and, in addition, $y_1(t^*)=0$. Prove that either $y_1(t)\equiv 0$ or $y_2(t)=[y_2'(t^*)/y_1'(t^*)]y_1(t)$. *Hint*: If $W[y_1,y_2](t^*)=0$ and $y_1(t^*)=0$, then $y_2(t^*)y_1'(t^*)=0$.

2.2 Linear equations with constant coefficients

We consider now the homogeneous linear second-order equation with constant coefficients

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0 \tag{1}$$

where a, b, and c are constants, with $a\neq 0$. Theorem 2 of Section 2.1 tells us that we need only find two independent solutions y_1 and y_2 of (1); all other solutions of (1) are then obtained by taking linear combinations of y_1 and y_2. Unfortunately, Theorem 2 doesn't tell us how to find two solutions of (1). Therefore, we will try an educated guess. To this end, observe that a function $y(t)$ is a solution of (1) if a constant times its second derivative, plus another constant times its first derivative, plus a third constant times itself is identically zero. In other words, the three terms ay'', by', and cy must cancel each other. In general, this can only occur if the three functions $y(t)$, $y'(t)$, and $y''(t)$ are of the "same type". For example, the function $y(t)=t^5$ can never be a solution of (1) since the three terms $20at^3$, $5bt^4$, and ct^5 are polynomials in t of different degree, and therefore cannot cancel each other. On the other hand, the function $y(t)=e^{rt}$, r constant, has the property that both $y'(t)$ and $y''(t)$ are multiples of $y(t)$. This suggests that we try $y(t)=e^{rt}$ as a solution of (1). Computing

$$\begin{aligned} L[e^{rt}] &= a(e^{rt})''+b(e^{rt})'+c(e^{rt}) \\ &= (ar^2+br+c)e^{rt}, \end{aligned}$$

we see that $y(t)=e^{rt}$ is a solution of (1) if, and only if

$$ar^2+br+c=0. \tag{2}$$

Equation (2) is called the *characteristic equation* of (1). It has two roots r_1,r_2 given by the quadratic formula

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

If b^2-4ac is positive, then r_1 and r_2 are real and distinct. In this case, $y_1(t)=e^{r_1t}$ and $y_2(t)=e^{r_2t}$ are two distinct solutions of (1). These solutions are clearly linearly independent (on any interval I), since e^{r_2t} is obviously not a constant multiple of e^{r_1t} for $r_2\neq r_1$. (If the reader is unconvinced of this he can compute

$$W\left[e^{r_1t},e^{r_2t}\right]=(r_2-r_1)e^{(r_1+r_2)t},$$

and observe that W is never zero. Hence, e^{r_1t} and e^{r_2t} are linearly independent on any interval I.)

Example 1. Find the general solution of the equation

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+4y=0. \tag{3}$$

Solution. The characteristic equation $r^2+5r+4=(r+4)(r+1)=0$ has two distinct roots $r_1=-4$ and $r_2=-1$. Thus, $y_1(t)=e^{-4t}$ and $y_2(t)=e^{-t}$ form a fundamental set of solutions of (3), and every solution $y(t)$ of (3) is of the form

$$y(t)=c_1e^{-4t}+c_2e^{-t}$$

for some choice of constants c_1,c_2.

Example 2. Find the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}-2y=0; \qquad y(0)=1, \quad y'(0)=2.$$

Solution. The characteristic equation $r^2+4r-2=0$ has 2 roots

$$r_1=\frac{-4+\sqrt{16+8}}{2}=-2+\sqrt{6}$$

and

$$r_2=\frac{-4-\sqrt{16+8}}{2}=-2-\sqrt{6}\ .$$

Hence, $y_1(t)=e^{r_1t}$ and $y_2(t)=e^{r_2t}$ are a fundamental set of solutions of $y''+4y'-2y=0$, so that

$$y(t)=c_1e^{(-2+\sqrt{6}\,)t}+c_2e^{(-2-\sqrt{6}\,)t}$$

for some choice of constants c_1,c_2. The constants c_1 and c_2 are determined from the initial conditions

$$c_1+c_2=1 \quad \text{and} \quad (-2+\sqrt{6}\,)c_1+(-2-\sqrt{6}\,)c_2=2.$$

From the first equation, $c_2=1-c_1$. Substituting this value of c_2 into the second equation gives

$$(-2+\sqrt{6}\,)c_1-(2+\sqrt{6}\,)(1-c_1)=2, \quad \text{or} \quad 2\sqrt{6}\,c_1=4+\sqrt{6}\ .$$

Therefore, $c_1=2/\sqrt{6}\,+\frac{1}{2}$, $c_2=1-c_1=\frac{1}{2}-2/\sqrt{6}$, and

$$y(t)=\left(\frac{1}{2}+\frac{2}{\sqrt{6}}\right)e^{(-2+\sqrt{6}\,)t}+\left(\frac{1}{2}-\frac{2}{\sqrt{6}}\right)e^{-(2+\sqrt{6}\,)t}.$$

Exercises

Find the general solution of each of the following equations.

1. $\dfrac{d^2y}{dt^2}-y=0$

2. $6\dfrac{d^2y}{dt^2}-7\dfrac{dy}{dt}+y=0$

3. $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}+y=0$

4. $3\dfrac{d^2y}{dt^2}+6\dfrac{dy}{dt}+2y=0$

Solve each of the following initial-value problems.

5. $\dfrac{d^2y}{dt^2}-3\dfrac{dy}{dt}-4y=0;\quad y(0)=1,\ y'(0)=0$

6. $2\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}-10y=0;\quad y(1)=5,\ y'(1)=2$

7. $5\dfrac{d^2y}{dt^2}+5\dfrac{dy}{dt}-y=0;\quad y(0)=0,\ y'(0)=1$

8. $\dfrac{d^2y}{dt^2}-6\dfrac{dy}{dt}+y=0;\quad y(2)=1,\ y'(2)=1$

Remark. In doing Problems 6 and 8, observe that $e^{r(t-t_0)}$ is also a solution of the differential equation $ay''+by'+cy=0$ if $ar^2+br+c=0$. Thus, to find the solution $y(t)$ of the initial-value problem $ay''+by'+cy=0$; $y(t_0)=y_0, y'(t_0)=y'_0$, we would write $y(t)=c_1e^{r_1(t-t_0)}+c_2e^{r_2(t-t_0)}$ and solve for c_1 and c_2 from the initial conditions.

9. Let $y(t)$ be the solution of the initial-value problem

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0;\qquad y(0)=1,\quad y'(0)=V.$$

For what values of V does $y(t)$ remain nonnegative for all $t\geq 0$?

10. The differential equation

$$L[y]=t^2y''+\alpha ty'+\beta y=0 \tag{*}$$

is known as Euler's equation. Observe that t^2y'', ty', and y are all multiples of t^r if $y=t^r$. This suggests that we try $y=t^r$ as a solution of (*). Show that $y=t^r$ is a solution of (*) if $r^2+(\alpha-1)r+\beta=0$.

11. Find the general solution of the equation

$$t^2y''+5ty'-5y=0,\qquad t>0$$

12. Solve the initial-value problem

$$t^2y''-ty'-2y=0;\qquad y(1)=0,\quad y'(1)=1$$

on the interval $0<t<\infty$.

2.2.1 Complex roots

If b^2-4ac is negative, then the characteristic equation $ar^2+br+c=0$ has complex roots

$$r_1=\frac{-b+i\sqrt{4ac-b^2}}{2a} \quad \text{and} \quad r_2=\frac{-b-i\sqrt{4ac-b^2}}{2a}.$$

We would like to say that e^{r_1t} and e^{r_2t} are solutions of the differential equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0. \tag{1}$$

However, this presents us with two serious difficulties. On the one hand, the function e^{rt} is not defined, as yet, for r complex. And on the other hand, even if we succeed in defining e^{r_1t} and e^{r_2t} as complex-valued solutions of (1), we are still faced with the problem of finding two *real-valued* solutions of (1).

We begin by resolving the second difficulty, since otherwise there's no sense tackling the first problem. Assume that $y(t)=u(t)+iv(t)$ is a complex-valued solution of (1). This means, of course, that

$$a\left[u''(t)+iv''(t)\right]+b\left[u'(t)+iv'(t)\right]+c\left[u(t)+iv(t)\right]=0. \tag{2}$$

This complex-valued solution of (1) gives rise to *two* real-valued solutions, as we now show.

Lemma 1. *Let $y(t)=u(t)+iv(t)$ be a complex-valued solution of* (1), *with a, b, and c real. Then, $y_1(t)=u(t)$ and $y_2(t)=v(t)$ are two real-valued solutions of* (1). *In other words, both the real and imaginary parts of a complex-valued solution of* (1) *are solutions of* (1). (*The imaginary part of the complex number $\alpha+i\beta$ is β. Similarly, the imaginary part of the function $u(t)+iv(t)$ is $v(t)$.*)

PROOF. From Equation (2),

$$\left[au''(t)+bu'(t)+cu(t)\right]+i\left[av''(t)+bv'(t)+cv(t)\right]=0. \tag{3}$$

Now, if a complex number is zero, then both its real and imaginary parts must be zero. Consequently,

$$au''(t)+bu'(t)+cu(t)=0 \quad \text{and} \quad av''(t)+bv'(t)+cv(t)=0,$$

and this proves Lemma 1. □

The problem of defining e^{rt} for r complex can also be resolved quite easily. Let $r=\alpha+i\beta$. By the law of exponents,

$$e^{rt}=e^{\alpha t}e^{i\beta t}. \tag{4}$$

Thus, we need only define the quantity $e^{i\beta t}$, for β real. To this end, recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots . \tag{5}$$

Equation (5) makes sense, formally, even for x complex. This suggests that we set

$$e^{i\beta t} = 1 + i\beta t + \frac{(i\beta t)^2}{2!} + \frac{(i\beta t)^3}{3!} + \dots .$$

Next, observe that

$$\begin{aligned} 1 + i\beta t + \frac{(i\beta t)^2}{2!} + \dots &= 1 + i\beta t - \frac{\beta^2 t^2}{2!} - \frac{i\beta^3 t^3}{3!} + \frac{\beta^4 t^4}{4!} + \frac{i\beta^5 t^5}{5!} + \dots \\ &= \left[1 - \frac{\beta^2 t^2}{2!} + \frac{\beta^4 t^4}{4!} + \dots\right] + i\left[\beta t - \frac{\beta^3 t^3}{3!} + \frac{\beta^5 t^5}{5!} + \dots\right] \\ &= \cos\beta t + i\sin\beta t. \end{aligned}$$

Hence,

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t). \tag{6}$$

Returning to the differential equation (1), we see that

$$\begin{aligned} y(t) &= e^{[-b+i\sqrt{4ac-b^2}\,]t/2a} \\ &= e^{-bt/2a}\left[\cos\sqrt{4ac-b^2}\; t/2a + i\sin\sqrt{4ac-b^2}\; t/2a\right] \end{aligned}$$

is a complex-valued solution of (1) if b^2-4ac is negative. Therefore, by Lemma 1,

$$y_1(t) = e^{-bt/2a}\cos\beta t \quad\text{and}\quad y_2(t) = e^{-bt/2a}\sin\beta t, \qquad \beta = \frac{\sqrt{4ac-b^2}}{2a}$$

are two real-valued solutions of (1). These two functions are linearly independent on any interval I, since their Wronskian (see Exercise 10) is never zero. Consequently, the general solution of (1) for $b^2-4ac<0$ is

$$y(t) = e^{-bt/2a}\left[c_1\cos\beta t + c_2\sin\beta t\right], \qquad \beta = \frac{\sqrt{4ac-b^2}}{2a}.$$

Remark 1. Strictly speaking, we must verify that the formula

$$\frac{d}{dt}e^{rt} = re^{rt}$$

is true even for r complex, before we can assert that $e^{r_1 t}$ and $e^{r_2 t}$ are complex-valued solutions of (1). To this end, we compute

$$\begin{aligned} \frac{d}{dt}e^{(\alpha+i\beta)t} &= \frac{d}{dt}e^{\alpha t}\left[\cos\beta t + i\sin\beta t\right] \\ &= e^{\alpha t}\left[(\alpha\cos\beta t - \beta\sin\beta t) + i(\alpha\sin\beta t + \beta\cos\beta t)\right] \end{aligned}$$

and this equals $(\alpha+i\beta)e^{(\alpha+i\beta)t}$, since

$$\begin{aligned}(\alpha+i\beta)e^{(\alpha+i\beta)t}&=(\alpha+i\beta)e^{\alpha t}[\cos\beta t+i\sin\beta t]\\&=e^{\alpha t}[(\alpha\cos\beta t-\beta\sin\beta t)+i(\alpha\sin\beta t+\beta\cos\beta t)].\end{aligned}$$

Thus, $(d/dt)e^{rt}=re^{rt}$, even for r complex.

Remark 2. At first glance, one might think that e^{r_2t} would give rise to two additional solutions of (1). This is not the case, though, since

$$\begin{aligned}e^{r_2t}&=e^{-(b/2a)t}e^{-i\beta t},\qquad \beta=\sqrt{4ac-b^2}\,/2a\\&=e^{-bt/2a}[\cos(-\beta t)+i\sin(-\beta t)]=e^{-bt/2a}[\cos\beta t-i\sin\beta t].\end{aligned}$$

Hence,

$$\mathrm{Re}\{e^{r_2t}\}=e^{-bt/2a}\cos\beta t=y_1(t)$$

and

$$\mathrm{Im}\{e^{r_2t}\}=-e^{-bt/2a}\sin\beta t=-y_2(t).$$

Example 1. Find two linearly independent real-valued solutions of the differential equation

$$4\frac{d^2y}{dt^2}+4\frac{dy}{dt}+5y=0. \tag{7}$$

Solution. The characteristic equation $4r^2+4r+5=0$ has complex roots $r_1=-\frac{1}{2}+i$ and $r_2=-\frac{1}{2}-i$. Consequently,

$$e^{r_1t}=e^{(-1/2+i)t}=e^{-t/2}\cos t+ie^{-t/2}\sin t$$

is a complex-valued solution of (7). Therefore, by Lemma 1,

$$\mathrm{Re}\{e^{r_1t}\}=e^{-t/2}\cos t\quad\text{and}\quad \mathrm{Im}\{e^{r_1t}\}=e^{-t/2}\sin t$$

are two linearly independent real-valued solutions of (7).

Example 2. Find the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+4y=0;\qquad y(0)=1,\quad y'(0)=1.$$

Solution. The characteristic equation $r^2+2r+4=0$ has complex roots $r_1=-1+\sqrt{3}\,i$ and $r_2=-1-\sqrt{3}\,i$. Hence,

$$e^{r_1t}=e^{(-1+\sqrt{3}\,i)t}=e^{-t}\cos\sqrt{3}\,t+ie^{-t}\sin\sqrt{3}\,t$$

is a complex-valued solution of $y''+2y'+4y=0$. Therefore, by Lemma 1, both

$$\mathrm{Re}\{e^{r_1t}\}=e^{-t}\cos\sqrt{3}\,t\quad\text{and}\quad \mathrm{Im}\{e^{r_1t}\}=e^{-t}\sin\sqrt{3}\,t$$

are real-valued solutions. Consequently,

$$y(t)=e^{-t}[c_1\cos\sqrt{3}\,t+c_2\sin\sqrt{3}\,t]$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$1 = y(0) = c_1$$

and

$$1 = y'(0) = -c_1 + \sqrt{3}\, c_2.$$

This implies that

$$c_1 = 1, c_2 = \frac{2}{\sqrt{3}} \quad \text{and} \quad y(t) = e^{-t}\left[\cos\sqrt{3}\,t + \frac{2}{\sqrt{3}}\sin\sqrt{3}\,t\right].$$

Exercises

Find the general solution of each of the following equations.

1. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$

2. $2\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 4y = 0$

3. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = 0$

4. $4\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + y = 0$

Solve each of the following initial-value problems.

5. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 2y = 0;\quad y(0) = 1,\ y'(0) = -2$

6. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = 0;\quad y(0) = 0,\ y'(0) = 2$

7. Assume that $b^2 - 4ac < 0$. Show that

$$y_1(t) = e^{(-b/2a)(t-t_0)}\cos\beta(t-t_0)$$

and

$$y_2(t) = e^{(-b/2a)(t-t_0)}\sin\beta(t-t_0), \qquad \beta = \frac{\sqrt{4ac-b^2}}{2a}$$

are solutions of (1), for any number t_0.

Solve each of the following initial-value problems.

8. $2\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + 3y = 0;\quad y(1) = 1,\ y'(1) = 1$

9. $3\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 4y = 0;\quad y(2) = 1,\ y'(2) = -1$

10. Verify that $W[e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t] = \beta e^{2\alpha t}$.

11. Show that $e^{i\omega t}$ is a complex-valued solution of the differential equation $y'' + \omega^2 y = 0$. Find two real-valued solutions.

12. Show that $(\cos t + i\sin t)^r = \cos rt + i\sin rt$. Use this result to obtain the double angle formulas $\sin 2t = 2\sin t\cos t$ and $\cos 2t = \cos^2 t - \sin^2 t$.

13. Show that
$$(\cos t_1 + i\sin t_1)(\cos t_2 + i\sin t_2) = \cos(t_1+t_2) + i\sin(t_1+t_2).$$
Use this result to obtain the trigonometric identities
$$\cos(t_1+t_2) = \cos t_1 \cos t_2 - \sin t_1 \sin t_2,$$
$$\sin(t_1+t_2) = \sin t_1 \cos t_2 + \cos t_1 \sin t_2.$$

14. Show that any complex number $a+ib$ can be written in the form $Ae^{i\theta}$, where $A = \sqrt{a^2+b^2}$ and $\tan\theta = b/a$.

15. Defining the two possible square roots of a complex number $Ae^{i\theta}$ as $\pm\sqrt{A}\, e^{i\theta/2}$, compute the square roots of i, $1+i$, $-i$, $\sqrt{i}$.

16. Use Problem 14 to find the three cube roots of i.

17. (a) Let $r_1 = \lambda + i\mu$ be a complex root of $r^2 + (\alpha-1)r + \beta = 0$. Show that
$$t^{\lambda+i\mu} = t^{\lambda}t^{i\mu} = t^{\lambda}e^{(\ln t)i\mu} = t^{\lambda}[\cos\mu\ln t + i\sin\mu\ln t]$$
is a complex-valued solution of Euler's equation
$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0. \tag{*}$$
(b) Show that $t^{\lambda}\cos\mu\ln t$ and $t^{\lambda}\sin\mu\ln t$ are real-valued solutions of (*).

Find the general solution of each of the following equations.

18. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = 0, \quad t>0$

19. $t^2\dfrac{d^2y}{dt^2} + 2t\dfrac{dy}{dt} + 2y = 0, \quad t>0$

2.2.2 Equal roots; reduction of order

If $b^2 = 4ac$, then the characteristic equation $ar^2 + br + c = 0$ has real equal roots $r_1 = r_2 = -b/2a$. In this case, we obtain only one solution
$$y_1(t) = e^{-bt/2a}$$
of the differential equation
$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{1}$$
Our problem is to find a second solution which is independent of y_1. One approach to this problem is to try some additional guesses. A second, and much more clever approach is to try and use our knowledge of $y_1(t)$ to help us find a second independent solution. More generally, suppose that we know one solution $y = y_1(t)$ of the second-order linear equation
$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{2}$$
Can we use this solution to help us find a second independent solution?

The answer to this question is yes. Once we find one solution $y=y_1(t)$ of (2), we can reduce the problem of finding all solutions of (2) to that of solving a first-order linear homogeneous equation. This is accomplished by defining a new dependent variable v through the substitution

$$y(t)=y_1(t)v(t).$$

Then

$$\frac{dy}{dt}=v\frac{dy_1}{dt}+y_1\frac{dv}{dt}$$

and

$$\frac{d^2y}{dt^2}=v\frac{d^2y_1}{dt^2}+2\frac{dv}{dt}\frac{dy_1}{dt}+y_1\frac{d^2v}{dt^2}.$$

Consequently.

$$\begin{aligned}
L[y]&=v\frac{d^2y_1}{dt^2}+2\frac{dv}{dt}\frac{dy_1}{dt}+y_1\frac{d^2v}{dt^2}+p(t)\left[v\frac{dy_1}{dt}+y_1\frac{dv}{dt}\right]+q(t)vy_1\\
&=y_1\frac{d^2v}{dt^2}+\left[2\frac{dy_1}{dt}+p(t)y_1\right]\frac{dv}{dt}+\left[\frac{d^2y_1}{dt^2}+p(t)\frac{dy_1}{dt}+q(t)y_1\right]v\\
&=y_1\frac{d^2v}{dt^2}+\left[2\frac{dy_1}{dt}+p(t)y_1\right]\frac{dv}{dt},
\end{aligned}$$

since $y_1(t)$ is a solution of $L[y]=0$. Hence, $y(t)=y_1(t)v(t)$ is a solution of (2) if v satisfies the differential equation

$$y_1\frac{d^2v}{dt^2}+\left[2\frac{dy_1}{dt}+p(t)y_1\right]\frac{dv}{dt}=0. \tag{3}$$

Now, observe that Equation (3) is really a first-order linear equation for dv/dt. Its solution is

$$\begin{aligned}
\frac{dv}{dt}&=c\exp\left[-\int\left[2\frac{y_1'(t)}{y_1(t)}+p(t)\right]dt\right]\\
&=c\exp\left(-\int p(t)\,dt\right)\exp\left[-2\int\frac{y_1'(t)}{y_1(t)}dt\right]\\
&=\frac{c\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}.
\end{aligned} \tag{4}$$

Since we only need one solution $v(t)$ of (3), we set $c=1$ in (4). Integrating this equation with respect to t, and setting the constant of integration equal to zero, we obtain that $v(t)=\int u(t)\,dt$, where

$$u(t)=\frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}. \tag{5}$$

Hence,

$$y_2(t)=v(t)y_1(t)=y_1(t)\int u(t)\,dt \tag{6}$$

is a second solution of (2). This solution is independent of y_1, for if $y_2(t)$ were a constant multiple of $y_1(t)$ then $v(t)$ would be constant, and consequently, its derivative would vanish identically. However, from (4)

$$\frac{dv}{dt}=\frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)},$$

and this quantity is never zero.

Remark 1. In writing $v(t)=\int u(t)\,dt$, we set the constant of integration equal to zero. Choosing a nonzero constant of integration would only add a constant multiple of $y_1(t)$ to $y_2(t)$. Similarly, the effect of choosing a constant c other than one in Equation (4) would be to multiply $y_2(t)$ by c.

Remark 2. The method we have just presented for solving Equation (2) is known as the method of *reduction of order*, since the substitution $y(t)=y_1(t)v(t)$ reduces the problem of solving the second-order equation (2) to that of solving a first-order equation.

Application to the case of equal roots: In the case of equal roots, we found $y_1(t)=e^{-bt/2a}$ as one solution of the equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0. \tag{7}$$

We can find a second solution from Equations (5) and (6). It is important to realize though, that Equations (5) and (6) were derived under the assumption that our differential equation was written in the form

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0;$$

that is, the coefficient of y'' was one. In our equation, the coefficient of y'' is a. Hence, we must divide Equation (7) by a to obtain the equivalent equation

$$\frac{d^2y}{dt^2}+\frac{b}{a}\frac{dy}{dt}+\frac{c}{a}y=0.$$

Now, we can insert $p(t)=b/a$ into (5) to obtain that

$$u(t)=\frac{\exp\left(-\int \frac{b}{a}\,dt\right)}{\left[e^{-bt/2a}\right]^2}=\frac{e^{-bt/a}}{e^{-bt/a}}=1.$$

Hence,

$$y_2(t)=y_1(t)\int dt=ty_1(t)$$

is a second solution of (7). The functions $y_1(t)$ and $y_2(t)$ are clearly linearly independent on the interval $-\infty<t<\infty$. Therefore, the general solution of (7) in the case of equal roots is

$$y(t)=c_1e^{-bt/2a}+c_2te^{-bt/2a}=\left[c_1+c_2t\right]e^{-bt/2a}$$

Example 1. Find the solution $y(t)$ of the initial-value problem

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=0; \qquad y(0)=1, \quad y'(0)=3.$$

Solution. The characteristic equation $r^2+4r+4=(r+2)^2=0$ has two equal roots $r_1=r_2=-2$. Hence,

$$y(t)=c_1e^{-2t}+c_2te^{-2t}$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$1=y(0)=c_1$$

and

$$3=y'(0)=-2c_1+c_2.$$

This implies that $c_1=1$ and $c_2=5$, so that $y(t)=(1+5t)e^{-2t}$.

Example 2. Find the solution $y(t)$ of the initial-value problem

$$(1-t^2)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0; \qquad y(0)=3, \quad y'(0)=-4$$

on the interval $-1<t<1$.

Solution. Clearly, $y_1(t)=t$ is one solution of the differential equation

$$(1-t^2)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0. \tag{8}$$

We will use the method of reduction of order to find a second solution $y_2(t)$ of (8). To this end, divide both sides of (8) by $1-t^2$ to obtain the equivalent equation

$$\frac{d^2y}{dt^2}+\frac{2t}{1-t^2}\frac{dy}{dt}-\frac{2}{1-t^2}y=0.$$

Then, from (5)

$$u(t)=\frac{\exp\left(-\int\frac{2t}{1-t^2}\,dt\right)}{y_1^2(t)}=\frac{e^{\ln(1-t^2)}}{t^2}=\frac{1-t^2}{t^2},$$

and

$$y_2(t)=t\int\frac{1-t^2}{t^2}\,dt=-t\left(\frac{1}{t}+t\right)=-(1+t^2)$$

is a second solution of (8). Therefore,

$$y(t)=c_1t-c_2(1+t^2)$$

for some choice of constants c_1, c_2. (Notice that all solutions of (9) are continuous at $t=\pm 1$ even though the differential equation is not defined at these points. Thus, it does not necessarily follow that the solutions of a differential equation are discontinuous at a point where the differential equation is not defined—but this is often the case.) The constants c_1 and c_2 are determined from the initial conditions

$$3=y(0)=-c_2 \quad\text{and}\quad -4=y'(0)=c_1.$$

Hence, $y(t)=-4t+3(1+t^2)$.

Exercises

Find the general solution of each of the following equations

1. $\dfrac{d^2y}{dt^2}-6\dfrac{dy}{dt}+9y=0$

2. $4\dfrac{d^2y}{dt^2}-12\dfrac{dy}{dt}+9y=0$

Solve each of the following initial-value problems.

3. $9\dfrac{d^2y}{dt^2}+6\dfrac{dy}{dt}+y=0;\quad y(0)=1,\ y'(0)=0$

4. $4\dfrac{d^2y}{dt^2}-4\dfrac{dy}{dt}+y=0;\quad y(0)=0,\ y'(0)=3$

5. Suppose $b^2=4ac$. Show that

$$y_1(t)=e^{-b(t-t_0)/2a} \quad\text{and}\quad y_2(t)=(t-t_0)e^{-b(t-t_0)/2a}$$

are solutions of (1) for every choice of t_0.

Solve the following initial-value problems.

6. $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+y=0;\quad y(2)=1,\ y'(2)=-1$

7. $9\dfrac{d^2y}{dt^2}-12\dfrac{dy}{dt}+4y=0;\quad y(\pi)=0,\ y'(\pi)=2$

8. Let a, b and c be positive numbers. Prove that every solution of the differential equation $ay''+by'+cy=0$ approaches zero as t approaches infinity.

9. Here is an alternate and very elegant way of finding a second solution $y_2(t)$ of (1).

(a) Assume that $b^2=4ac$. Show that

$$L[e^{rt}]=a(e^{rt})''+b(e^{rt})'+ce^{rt}=a(r-r_1)^2e^{rt}$$

for $r_1=-b/2a$.

(b) Show that

$$(\partial/\partial r)L[e^{rt}]=L\big[(\partial/\partial r)e^{rt}\big]=L[te^{rt}]=2a(r-r_1)e^{rt}+at(r-r_1)^2e^{rt}.$$

(c) Conclude from (a) and (b) that $L[te^{r_1t}]=0$. Hence, $y_2(t)=te^{r_1t}$ is a second solution of (1).

Use the method of reduction of order to find the general solution of the following differential equations.

10. $\dfrac{d^2y}{dt^2}-\dfrac{2(t+1)}{(t^2+2t-1)}\dfrac{dy}{dt}+\dfrac{2}{(t^2+2t-1)}y=0 \quad (y_1(t)=t+1)$

11. $\dfrac{d^2y}{dt^2}-4t\dfrac{dy}{dt}+(4t^2-2)y=0 \quad (y_1(t)=e^{t^2})$

12. $(1-t^2)\dfrac{d^2y}{dt^2}-2t\dfrac{dy}{dt}+2y=0 \quad (y_1(t)=t)$

13. $(1+t^2)\dfrac{d^2y}{dt^2}-2t\dfrac{dy}{dt}+2y=0 \quad (y_1(t)=t)$

14. $(1-t^2)\dfrac{d^2y}{dt^2}-2t\dfrac{dy}{dt}+6y=0 \quad (y_1(t)=3t^2-1)$

15. $(2t+1)\dfrac{d^2y}{dt^2}-4(t+1)\dfrac{dy}{dt}+4y=0 \quad (y_1(t)=t+1)$

16. $t^2\dfrac{d^2y}{dt^2}+t\dfrac{dy}{dt}+\left(t^2-\dfrac{1}{4}\right)y=0 \quad \left(y_1(t)=\dfrac{\sin t}{\sqrt{t}}\right)$

17. Given that the equation

$$t\frac{d^2y}{dt^2}-(1+3t)\frac{dy}{dt}+3y=0$$

has a solution of the form e^{ct}, for some constant c, find the general solution.

18. (a) Show that t^r is a solution of Euler's equation

$$t^2y''+\alpha ty'+\beta y=0, \quad t>0$$

if $r^2+(\alpha-1)r+\beta=0$.

(b) Suppose that $(\alpha-1)^2=4\beta$. Using the method of reduction of order, show that $(\ln t)t^{(1-\alpha)/2}$ is a second solution of Euler's equation.

Find the general solution of each of the following equations.

19. $t^2\dfrac{d^2y}{dt^2}+3t\dfrac{dy}{dt}+y=0$

20. $t^2\dfrac{d^2y}{dt^2}-t\dfrac{dy}{dt}+y=0$

2.3 The nonhomogeneous equation

We turn our attention now to the nonhomogeneous equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=g(t) \tag{1}$$

where the functions $p(t)$, $q(t)$ and $g(t)$ are continuous on an open interval $\alpha<t<\beta$. An important clue as to the nature of all solutions of (1) is provided by the first-order linear equation

$$\frac{dy}{dt}-2ty=-t. \tag{2}$$

The general solution of this equation is

$$y(t)=ce^{t^2}+\tfrac{1}{2}.$$

Now, observe that this solution is the sum of two terms: the first term, ce^{t^2}, is the general solution of the homogeneous equation

$$\frac{dy}{dt}-2ty=0 \tag{3}$$

while the second term, $\frac{1}{2}$, is a solution of the nonhomogeneous equation (2). In other words, every solution $y(t)$ of (2) is the sum of a particular solution, $\psi(t)=\frac{1}{2}$, with a solution ce^{t^2} of the homogeneous equation. A similar situation prevails in the case of second-order equations, as we now show.

Theorem 5. *Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the homogeneous equation*

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{4}$$

and let $\psi(t)$ be any particular solution of the nonhomogeneous equation (1). *Then, every solution $y(t)$ of* (1) *must be of the form*

$$y(t)=c_1y_1(t)+c_2y_2(t)+\psi(t)$$

for some choice of constants c_1, c_2.

The proof of Theorem 5 relies heavily on the following lemma.

Lemma 1. *The difference of any two solutions of the nonhomogeneous equation* (1) *is a solution of the homogeneous equation* (4).

PROOF. Let $\psi_1(t)$ and $\psi_2(t)$ be two solutions of (1). By the linearity of L,

$$L[\psi_1-\psi_2](t)=L[\psi_1](t)-L[\psi_2](t)=g(t)-g(t)=0.$$

Hence, $\psi_1(t)-\psi_2(t)$ is a solution of the homogeneous equation (4). □

We can now give a very simple proof of Theorem 5.

PROOF OF THEOREM 5. Let $y(t)$ be any solution of (1). By Lemma 1, the function $\phi(t)=y(t)-\psi(t)$ is a solution of the homogeneous equation (4). But every solution $\phi(t)$ of the homogeneous equation (4) is of the form $\phi(t)=c_1y_1(t)+c_2y_2(t)$, for some choice of constants c_1, c_2. Therefore,

$$y(t)=\phi(t)+\psi(t)=c_1y_1(t)+c_2y_2(t)+\psi(t). \qquad \square$$

Remark. Theorem 5 is an extremely useful theorem since it reduces the problem of finding all solutions of (1) to the much simpler problem of finding just two solutions of the homogeneous equation (4), and one solution of the nonhomogeneous equation (1).

Example 1. Find the general solution of the equation

$$\frac{d^2y}{dt^2}+y=t. \tag{5}$$

Solution. The functions $y_1(t)=\cos t$ and $y_2(t)=\sin t$ are two linearly independent solutions of the homogeneous equation $y''+y=0$. Moreover, $\psi(t)=t$ is obviously a particular solution of (5). Therefore, by Theorem 5, every solution $y(t)$ of (5) must be of the form

$$y(t)=c_1\cos t+c_2\sin t+t.$$

Example 2. Three solutions of a certain second-order nonhomogeneous linear equation are

$$\psi_1(t)=t,\qquad \psi_2(t)=t+e^t,\quad \text{and}\quad \psi_3(t)=1+t+e^t.$$

Find the general solution of this equation.
Solution. By Lemma 1, the functions

$$\psi_2(t)-\psi_1(t)=e^t\quad \text{and}\quad \psi_3(t)-\psi_2(t)=1$$

are solutions of the corresponding homogeneous equation. Moreover, these functions are obviously linearly independent. Therefore, by Theorem 5, every solution $y(t)$ of this equation must be of the form

$$y(t)=c_1e^t+c_2+t.$$

EXERCISES

1. Three solutions of a certain second-order nonhomogeneous linear equation are

$$\psi_1(t)=t^2,\ \psi_2(t)=t^2+e^{2t}$$

and

$$\psi_3(t)=1+t^2+2e^{2t}.$$

Find the general solution of this equation.

2. Three solutions of a certain second-order linear nonhomogeneous equation are

$$\psi_1(t)=1+e^{t^2},\ \psi_2(t)=1+te^{t^2}$$

and

$$\psi_3(t)=(t+1)e^{t^2}+1$$

Find the general solution of this equation.

3. Three solutions of a second-order linear equation $L[y]=g(t)$ are

$$\psi_1(t)=3e^t+e^{t^2},\ \psi_2(t)=7e^t+e^{t^2}$$

and

$$\psi_3(t)=5e^t+e^{-t^3}+e^{t^2}.$$

Find the solution of the initial-value problem

$$L[y]=g; \qquad y(0)=1, \quad y'(0)=2.$$

4. Let a, b and c be positive constants. Show that the difference of any two solutions of the equation

$$ay''+by'+cy=g(t)$$

approaches zero as t approaches infinity.

5. Let $\psi(t)$ be a solution of the nonhomogeneous equation (1), and let $\phi(t)$ be a solution of the homogeneous equation (4). Show that $\psi(t)+\phi(t)$ is again a solution of (1).

2.4 The method of variation of parameters

In this section we describe a very general method for finding a particular solution $\psi(t)$ of the nonhomogeneous equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=g(t), \tag{1}$$

once the solutions of the homogeneous equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{2}$$

are known. The basic principle of this method is to use our knowledge of the solutions of the homogeneous equation to help us find a solution of the nonhomogeneous equation.

Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of the homogeneous equation (2). We will try to find a particular solution $\psi(t)$ of the nonhomogeneous equation (1) of the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t); \tag{3}$$

that is, we will try to find functions $u_1(t)$ and $u_2(t)$ so that the linear combination $u_1(t)y_1(t) + u_2(t)y_2(t)$ is a solution of (1). At first glance, this would appear to be a dumb thing to do, since we are replacing the problem of finding one unknown function $\psi(t)$ by the seemingly harder problem of finding two unknown functions $u_1(t)$ and $u_2(t)$. However, by playing our cards right, we will be able to find $u_1(t)$ and $u_2(t)$ as the solutions of two very simple first-order equations. We accomplish this in the following manner. Observe that the differential equation (1) imposes only one condition on the two unknown functions $u_1(t)$ and $u_2(t)$. Therefore, we have a certain "freedom" in choosing $u_1(t)$ and $u_2(t)$. Our goal is to impose an additional condition on $u_1(t)$ and $u_2(t)$ which will make the expression $L[u_1y_1+u_2y_2]$ as simple as possible. Computing

$$\begin{aligned}\frac{d}{dt}\psi(t)&=\frac{d}{dt}[u_1y_1+u_2y_2]\\&=[u_1y_1'+u_2y_2']+[u_1'y_1+u_2'y_2]\end{aligned}$$

we see that $d^2\psi/dt^2$, and consequently $L[\psi]$, will contain no second-order derivatives of u_1 and u_2 if

$$y_1(t)u_1'(t)+y_2(t)u_2'(t)=0. \tag{4}$$

This suggests that we impose the condition (4) on the functions $u_1(t)$ and $u_2(t)$. In this case, then,

$$\begin{aligned}L[\psi]&=[u_1y_1'+u_2y_2']'+p(t)[u_1y_1'+u_2y_2']+q(t)[u_1y_1+u_2y_2]\\&=u_1'y_1'+u_2'y_2'+u_1[y_1''+p(t)y_1'+q(t)y_1]+u_2[y_2''+p(t)y_2'+q(t)y_2]\\&=u_1'y_1'+u_2'y_2'\end{aligned}$$

since both $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation $L[y]=0$. Consequently, $\psi(t)=u_1y_1+u_2y_2$ is a solution of the nonhomogeneous equation (1) if $u_1(t)$ and $u_2(t)$ satisfy the two equations

$$\begin{aligned}y_1(t)u_1'(t)+y_2(t)u_2'(t)&=0\\y_1'(t)u_1'(t)+y_2'(t)u_2'(t)&=g(t).\end{aligned}$$

Multiplying the first equation by $y_2'(t)$, the second equation by $y_2(t)$, and subtracting gives

$$[y_1(t)y_2'(t)-y_1'(t)y_2(t)]u_1'(t)=-g(t)y_2(t),$$

while multiplying the first equation by $y_1'(t)$, the second equation by $y_1(t)$, and subtracting gives

$$[y_1(t)y_2'(t)-y_1'(t)y_2(t)]u_2'(t)=g(t)y_1(t).$$

Hence,

$$u_1'(t)=-\frac{g(t)y_2(t)}{W[y_1,y_2](t)}\quad\text{and}\quad u_2'(t)=\frac{g(t)y_1(t)}{W[y_1,y_2](t)}. \tag{5}$$

Finally, we obtain $u_1(t)$ and $u_2(t)$ by integrating the right-hand sides of (5).

Remark. The general solution of the homogeneous equation (2) is

$$y(t)=c_1y_1(t)+c_2y_2(t).$$

By letting c_1 and c_2 vary with time, we obtain a solution of the nonhomogeneous equation. Hence, this method is known as the method of variation of parameters.

Example 1.
(a) Find a particular solution $\psi(t)$ of the equation

$$\frac{d^2y}{dt^2}+y=\tan t \tag{6}$$

on the interval $-\pi/2<t<\pi/2$.
(b) Find the solution $y(t)$ of (6) which satisfies the initial conditions $y(0)=1$, $y'(0)=1$.
Solution.
(a) The functions $y_1(t)=\cos t$ and $y_2(t)=\sin t$ are two linearly independent solutions of the homogeneous equation $y''+y=0$ with

$$W[y_1,y_2](t)=y_1y_2'-y_1'y_2=(\cos t)\cos t-(-\sin t)\sin t=1.$$

Thus, from (5),

$$u_1'(t)=-\tan t\sin t \quad\text{and}\quad u_2'(t)=\tan t\cos t. \tag{7}$$

Integrating the first equation of (7) gives

$$\begin{aligned}u_1(t)&=-\int\tan t\sin t\,dt=-\int\frac{\sin^2t}{\cos t}dt\\&=\int\frac{\cos^2t-1}{\cos t}dt=\sin t-\ln|\sec t+\tan t|.\\&=\sin t-\ln(\sec t+\tan t),\quad -\frac{\pi}{2}<t<\frac{\pi}{2}\end{aligned}$$

while integrating the second equation of (7) gives

$$u_2(t)=\int\tan t\cos t\,dt=\int\sin t\,dt=-\cos t.$$

Consequently,

$$\begin{aligned}\psi(t)&=\cos t[\sin t-\ln(\sec t+\tan t)]+\sin t(-\cos t)\\&=-\cos t\ln(\sec t+\tan t)\end{aligned}$$

is a particular solution of (6) on the interval $-\pi/2<t<\pi/2$.
(b) By Theorem 5 of Section 2.3,

$$y(t)=c_1\cos t+c_2\sin t-\cos t\ln(\sec t+\tan t)$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$1=y(0)=c_1 \quad\text{and}\quad 1=y'(0)=c_2-1.$$

Hence, $c_1 = 1$, $c_2 = 2$ and

$$y(t) = \cos t + 2\sin t - \cos t \ln(\sec t + \tan t).$$

Remark. Equation (5) determines $u_1(t)$ and $u_2(t)$ up to two constants of integration. We usually take these constants to be zero, since the effect of choosing nonzero constants is to add a solution of the homogeneous equation to $\psi(t)$.

Exercises

Find the general solution of each of the following equations.

1. $\dfrac{d^2y}{dt^2} + y = \sec t, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

2. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = te^{2t}$

3. $2\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + y = (t^2+1)e^t$

4. $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = te^{3t} + 1$

Solve each of the following initial-value problems.

5. $3y'' + 4y' + y = (\sin t)e^{-t}; \quad y(0) = 1, \; y'(0) = 0$

6. $y'' + 4y' + 4y = t^{5/2}e^{-2t}; \quad y(0) = y'(0) = 0$

7. $y'' - 3y' + 2y = \sqrt{t+1}\,; \quad y(0) = y'(0) = 0$

8. $y'' - y = f(t); \quad y(0) = y'(0) = 0$

Warning. It must be remembered, while doing Problems 3 and 5, that Equation (5) was derived under the assumption that the coefficient of y'' was one.

9. Find two linearly independent solutions of $t^2y'' - 2y = 0$ of the form $y(t) = t^r$. Using these solutions, find the general solution of $t^2y'' - 2y = t^2$.

10. One solution of the equation

$$y'' + p(t)y' + q(t)y = 0 \tag{*}$$

is $(1+t)^2$, and the Wronskian of any two solutions of (*) is constant. Find the general solution of

$$y'' + p(t)y' + q(t)y = 1 + t.$$

11. Find the general solution of $y'' + (1/4t^2)y = f\cos t$, $t > 0$, given that $y_1(t) = \sqrt{t}$ is a solution of the homogeneous equation.

12. Find the general solution of the equation

$$\frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\frac{dy}{dt} + \frac{2}{1+t^2}y = 1+t^2.$$

13. Show that $\sec t + \tan t$ is positive for $-\pi/2 < t < \pi/2$.

2.5 The method of judicious guessing

A serious disadvantage of the method of variation of parameters is that the integrations required are often quite difficult. In certain cases, it is usually much simpler to guess a particular solution. In this section we will establish a systematic method for guessing solutions of the equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t) \tag{1}$$

where a, b and c are constants, and $g(t)$ has one of several special forms.

Consider first the differential equation

$$L[y] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = a_0 + a_1t + \dots + a_nt^n. \tag{2}$$

We seek a function $\psi(t)$ such that the three functions $a\psi''$, $b\psi'$ and $c\psi$ add up to a given polynomial of degree n. The obvious choice for $\psi(t)$ is a polynomial of degree n. Thus, we set

$$\psi(t) = A_0 + A_1t + \dots + A_nt^n \tag{3}$$

and compute

$$\begin{aligned}L[\psi](t) &= a\psi''(t) + b\psi'(t) + c\psi(t)\\&= a\left[2A_2 + \dots + n(n-1)A_nt^{n-2}\right] + b\left[A_1 + \dots + nA_nt^{n-1}\right]\\&\quad + c\left[A_0 + A_1t + \dots + A_nt^n\right]\\&= cA_nt^n + (cA_{n-1} + nbA_n)t^{n-1} + \dots + (cA_0 + bA_1 + 2aA_2).\end{aligned}$$

Equating coefficients of like powers of t in the equation

$$L[\psi](t) = a_0 + a_1t + \dots + a_nt^n$$

gives

$$cA_n = a_n,\ cA_{n-1} + nbA_n = a_{n-1}, \dots, cA_0 + bA_1 + 2aA_2 = a_0. \tag{4}$$

The first equation determines $A_n = a_n/c$, for $c \neq 0$, and the remaining equations then determine $A_{n-1}, \dots, A_0$ successively. Thus, Equation (1) has a particular solution $\psi(t)$ of the form (3), for $c \neq 0$.

We run into trouble when $c = 0$, since then the first equation of (4) has no solution A_n. This difficulty is to be expected though, for if $c = 0$, then $L[\psi] = a\psi'' + b\psi'$ is a polynomial of degree $n-1$, while the right hand side of (2) is a polynomial of degree n. To guarantee that $a\psi''+b\psi'$ is a polynomial of degree n, we must take ψ as a polynomial of degree $n+1$. Thus, we set

$$\psi(t)=t[A_0+A_1t+\ldots+A_nt^n]. \tag{5}$$

We have omitted the constant term in (5) since $y=$constant is a solution of the homogeneous equation $ay''+by'=0$, and thus can be subtracted from $\psi(t)$. The coefficients $A_0, A_1,\ldots,A_n$ are determined uniquely (see Exercise 19) from the equation

$$a\psi''+b\psi'=a_0+a_1t+\ldots+a_nt^n$$

if $b\neq 0$.

Finally, the case $b=c=0$ is trivial to handle since the differential equation (2) can then be integrated immediately to yield a particular solution $\psi(t)$ of the form

$$\psi(t)=\frac{1}{a}\left[\frac{a_0t^2}{1\cdot 2}+\frac{a_1t^3}{2\cdot 3}+\ldots+\frac{a_nt^{n+2}}{(n+1)(n+2)}\right].$$

Summary. The differential equation (2) has a solution $\psi(t)$ of the form

$$\psi(t)=\begin{cases}A_0+A_1t+\ldots+A_nt^n, & c\neq 0\\ t(A_0+A_1t+\ldots+A_nt^n), & c=0,\ b\neq 0.\\ t^2(A_0+A_1t+\ldots+A_nt^n), & c=b=0\end{cases}$$

Example 1. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=t^2. \tag{6}$$

Solution. We set $\psi(t)=A_0+A_1t+A_2t^2$ and compute

$$\begin{aligned}L[\psi](t)&=\psi''(t)+\psi'(t)+\psi(t)\\ &=2A_2+(A_1+2A_2t)+A_0+A_1t+A_2t^2\\ &=(A_0+A_1+2A_2)+(A_1+2A_2)t+A_2t^2.\end{aligned}$$

Equating coefficients of like powers of t in the equation $L[\psi](t)=t^2$ gives

$$A_2=1,\qquad A_1+2A_2=0$$

and

$$A_0+A_1+2A_2=0.$$

The first equation tells us that $A_2=1$, the second equation then tells us that $A_1=-2$, and the third equation then tells us that $A_0=0$. Hence,

$$\psi(t)=-2t+t^2$$

is a particular solution of (6).

Let us now re-do this problem using the method of variation of parameters. It is easily verified that

$$y_1(t)=e^{-t/2}\cos\sqrt{3}\,t/2 \quad \text{and} \quad y_2(t)=e^{-t/2}\sin\sqrt{3}\,t/2$$

are two solutions of the homogeneous equation $L[y]=0$. Hence,

$$\psi(t)=u_1(t)e^{-t/2}\cos\sqrt{3}\,t/2+u_2(t)e^{-t/2}\sin\sqrt{3}\,t/2$$

is a particular solution of (6), where

$$u_1(t)=\int\frac{-t^2e^{-t/2}\sin\sqrt{3}\,t/2}{W[y_1,y_2](t)}dt=\frac{-2}{\sqrt{3}}\int t^2e^{t/2}\sin\sqrt{3}\,t/2\,dt$$

and

$$u_2(t)=\int\frac{t^2e^{-t/2}\cos\sqrt{3}\,t/2}{W[y_1,y_2](t)}dt=\frac{2}{\sqrt{3}}\int t^2e^{t/2}\cos\sqrt{3}\,t/2\,dt.$$

These integrations are extremely difficult to perform. Thus, the method of guessing is certainly preferrable, in this problem at least, to the method of variation of parameters.

Consider now the differential equation

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=(a_0+a_1t+\ldots+a_nt^n)e^{\alpha t}. \tag{7}$$

We would like to remove the factor $e^{\alpha t}$ from the right-hand side of (7), so as to reduce this equation to Equation (2). This is accomplished by setting $y(t)=e^{\alpha t}v(t)$. Then,

$$y'=e^{\alpha t}(v'+\alpha v) \quad \text{and} \quad y''=e^{\alpha t}(v''+2\alpha v'+\alpha^2v)$$

so that

$$L[y]=e^{\alpha t}\big[av''+(2a\alpha+b)v'+(a\alpha^2+b\alpha+c)v\big].$$

Consequently, $y(t)=e^{\alpha t}v(t)$ is a solution of (7) if, and only if,

$$a\frac{d^2v}{dt^2}+(2a\alpha+b)\frac{dv}{dt}+(a\alpha^2+b\alpha+c)v=a_0+a_1t+\ldots+a_nt^n. \tag{8}$$

In finding a particular solution $v(t)$ of (8), we must distinguish as to whether (i) $a\alpha^2+b\alpha+c\neq0$; (ii) $a\alpha^2+b\alpha+c=0$, but $2a\alpha+b\neq0$; and (iii) both $a\alpha^2+b\alpha+c$ and $2a\alpha+b=0$. The first case means that α is not a root of the characteristic equation

$$ar^2+br+c=0. \tag{9}$$

In other words, $e^{\alpha t}$ is not a solution of the homogeneous equation $L[y]=0$. The second condition means that α is a single root of the characteristic equation (9). This implies that $e^{\alpha t}$ is a solution of the homogeneous equation, but $te^{\alpha t}$ is not. Finally, the third condition means that α is a double root of the characteristic equation (9), so that both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation. Hence, Equation (7) has a particular solution $\psi(t)$ of the form (i) $\psi(t)=(A_0+\dots+A_nt^n)e^{\alpha t}$, if $e^{\alpha t}$ is not a solution of the homogeneous equation; (ii) $\psi(t)=t(A_0+\dots+A_nt^n)e^{\alpha t}$, if $e^{\alpha t}$ is a solution of the homogeneous equation but $te^{\alpha t}$ is not; and (iii) $\psi(t)=t^2(A_0+\dots+A_nt^n)e^{\alpha t}$ if both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation.

Remark. There are two ways of computing a particular solution $\psi(t)$ of (7). Either we make the substitution $y=e^{\alpha t}v$ and find $v(t)$ from (8), or we guess a solution $\psi(t)$ of the form $e^{\alpha t}$ times a suitable polynomial in t. If α is a double root of the characteristic equation (9), or if $n \geqslant 2$, then it is advisable to set $y=e^{\alpha t}v$ and then find $v(t)$ from (8). Otherwise, we guess $\psi(t)$ directly.

Example 2. Find the general solution of the equation

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=(1+t+\dots+t^{27})e^{2t}. \tag{10}$$

Solution. The characteristic equation $r^2-4r+4=0$ has equal roots $r_1=r_2=2$. Hence, $y_1(t)=e^{2t}$ and $y_2(t)=te^{2t}$ are solutions of the homogeneous equation $y''-4y'+4y=0$. To find a particular solution $\psi(t)$ of (10), we set $y=e^{2t}v$. Then, of necessity,

$$\frac{d^2v}{dt^2}=1+t+t^2+\dots+t^{27}.$$

Integrating this equation twice, and setting the constants of integration equal to zero gives

$$v(t)=\frac{t^2}{1\cdot 2}+\frac{t^3}{2\cdot 3}+\dots+\frac{t^{29}}{28\cdot 29}.$$

Hence, the general solution of (10) is

$$\begin{aligned} y(t)&=c_1e^{2t}+c_2te^{2t}+e^{2t}\left[\frac{t^2}{1\cdot 2}+\dots+\frac{t^{29}}{28\cdot 29}\right]\\ &=e^{2t}\left[c_1+c_2t+\frac{t^2}{1\cdot 2}+\dots+\frac{t^{29}}{28\cdot 29}\right]. \end{aligned}$$

It would be sheer madness (and a terrible waste of paper) to plug the expression

$$\psi(t)=t^2\left(A_0+A_1t+\dots+A_{27}t^{27}\right)e^{2t}$$

into (10) and then solve for the coefficients $A_0, A_1,\dots,A_{27}$.

Example 3. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=(1+t)e^{3t}.$$

Solution. In this case, e^{3t} is not a solution of the homogeneous equation $y''-3y'+2y=0$. Thus, we set $\psi(t)=(A_0+A_1t)e^{3t}$. Computing

$$\begin{aligned}L[\psi](t)&=\psi''-3\psi'+2\psi\\&=e^{3t}\left[(9A_0+6A_1+9A_1t)-3(3A_0+A_1+3A_1t)+2(A_0+A_1t)\right]\\&=e^{3t}\left[(2A_0+3A_1)+2A_1t\right]\end{aligned}$$

and cancelling off the factor e^{3t} from both sides of the equation

$$L[\psi](t)=(1+t)e^{3t},$$

gives

$$2A_1t+(2A_0+3A_1)=1+t.$$

This implies that $2A_1=1$ and $2A_0+3A_1=1$. Hence, $A_1=\frac{1}{2}$, $A_0=-\frac{1}{4}$ and $\psi(t)=(-\frac{1}{4}+t/2)e^{3t}$.

Finally, we consider the differential equation

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=(a_0+a_1t+\dots+a_nt^n)\times\begin{cases}\cos\omega t\\ \sin\omega t\end{cases}. \tag{11}$$

We can reduce the problem of finding a particular solution $\psi(t)$ of (11) to the simpler problem of finding a particular solution of (7) with the aid of the following simple but extremely useful lemma.

Lemma 1. *Let $y(t)=u(t)+iv(t)$ be a complex-valued solution of the equation*

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=g(t)=g_1(t)+ig_2(t) \tag{12}$$

where a, b and c are real. This means, of course, that

$$a\left[u''(t)+iv''(t)\right]+b\left[u'(t)+iv'(t)\right]+c\left[u(t)+iv(t)\right]=g_1(t)+ig_2(t). \tag{13}$$

Then, $L[u](t)=g_1(t)$ and $L[v](t)=g_2(t)$.

PROOF. Equating real and imaginary parts in (13) gives

$$au''(t)+bu'(t)+cu(t)=g_1(t)$$

and

$$av''(t)+bv'(t)+cv(t)=g_2(t). \qquad \square$$

Now, let $\phi(t)=u(t)+iv(t)$ be a particular solution of the equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=(a_0+\dots+a_nt^n)e^{i\omega t}. \tag{14}$$

The real part of the right-hand side of (14) is $(a_0+\dots+a_nt^n)\cos\omega t$, while the imaginary part is $(a_0+\dots+a_nt^n)\sin\omega t$. Hence, by Lemma 1

$$u(t)=\mathrm{Re}\{\phi(t)\}$$

is a solution of

$$ay''+by'+cy=(a_0+\dots+a_nt^n)\cos\omega t$$

while

$$v(t)=\mathrm{Im}\{\phi(t)\}$$

is a solution of

$$ay''+by'+cy=(a_0+\dots+a_nt^n)\sin\omega t.$$

Example 4. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+4y=\sin 2t. \tag{15}$$

Solution. We will find $\psi(t)$ as the imaginary part of a complex-valued solution $\phi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+4y=e^{2it}. \tag{16}$$

To this end, observe that the characteristic equation $r^2+4=0$ has complex roots $r=\pm 2i$. Therefore, Equation (16) has a particular solution $\phi(t)$ of the form $\phi(t)=A_0te^{2it}$. Computing

$$\phi'(t)=A_0(1+2it)e^{2it} \quad \text{and} \quad \phi''(t)=A_0(4i-4t)e^{2it}$$

we see that

$$L[\phi](t)=\phi''(t)+4\phi(t)=4iA_0e^{2it}.$$

Hence, $A_0=1/4i=-i/4$ and

$$\phi(t)=-\frac{it}{4}e^{2it}=-\frac{it}{4}(\cos 2t+i\sin 2t)=\frac{t}{4}\sin 2t-i\frac{t}{4}\cos 2t.$$

Therefore, $\psi(t)=\mathrm{Im}\{\phi(t)\}=-(t/4)\cos 2t$ is a particular solution of (15).

Example 5. Find a particular solution $\psi(t)$ of the equation

$$\frac{d^2y}{dt^2}+4y=\cos 2t. \tag{17}$$

Solution. From Example 4, $\phi(t)=(t/4)\sin 2t-i(t/4)\cos 2t$ is a complex-valued solution of (16). Therefore,

$$\psi(t)=\mathrm{Re}\{\phi(t)\}=\frac{t}{4}\sin 2t$$

is a particular solution of (17).

Example 6. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=te^t\cos t. \tag{18}$$

Solution. Observe that $te^t\cos t$ is the real part of $te^{(1+i)t}$. Therefore, we can find $\psi(t)$ as the real part of a complex-valued solution $\phi(t)$ of the equation

$$L[y]=\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=te^{(1+i)t}. \tag{19}$$

To this end, observe that $1+i$ is not a root of the characteristic equation $r^2+2r+1=0$. Therefore, Equation (19) has a particular solution $\phi(t)$ of the form $\phi(t)=(A_0+A_1t)e^{(1+i)t}$. Computing $L[\phi]=\phi''+2\phi'+\phi$, and using the identity

$$(1+i)^2+2(1+i)+1=\left[(1+i)+1\right]^2=(2+i)^2$$

we see that

$$\left[(2+i)^2A_1t+(2+i)^2A_0+2(2+i)A_1\right]=t.$$

Equating coefficients of like powers of t in this equation gives

$$(2+i)^2A_1=1$$

and

$$(2+i)A_0+2A_1=0.$$

This implies that $A_1=1/(2+i)^2$ and $A_0=-2/(2+i)^3$, so that

$$\phi(t)=\left[\frac{-2}{(2+i)^3}+\frac{t}{(2+i)^2}\right]e^{(1+i)t}.$$

After a little algebra, we find that

$$\begin{aligned}\phi(t)=\frac{e^t}{125}\Big\{&\left[(15t-4)\cos t+(20t-22)\sin t\right]\\&+i\left[(22-20t)\cos t+(15t-4)\sin t\right]\Big\}.\end{aligned}$$

Hence,

$$\psi(t)=\mathrm{Re}\{\phi(t)\}=\frac{e^t}{125}\left[(15t-4)\cos t+(20t-22)\sin t\right].$$

Remark. The method of judicious guessing also applies to the equation

$$L[y]=a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=\sum_{j=1}^{n}p_j(t)e^{\alpha_jt} \tag{20}$$

where the $p_j(t), j=1,\ldots,n$ are polynomials in t. Let $\psi_j(t)$ be a particular solution of the equation

$$L[y] = p_j(t)e^{\alpha_j t}, \qquad j = 1, \dots, n.$$

Then, $\psi(t) = \sum_{j=1}^{n} \psi_j(t)$ is a solution of (20) since

$$L[\psi] = L\left[\sum_{j=1}^{n} \psi_j\right] = \sum_{j=1}^{n} L[\psi_j] = \sum_{j=1}^{n} p_j(t)e^{\alpha_j t}.$$

Thus, to find a particular solution of the equation

$$y'' + y' + y = e^t + t\sin t$$

we find particular solutions $\psi_1(t)$ and $\psi_2(t)$ of the equations

$$y'' + y' + y = e^t \quad \text{and} \quad y'' + y' + y = t\sin t$$

respectively, and then add these two solutions together.

Exercises

Find a particular solution of each of the following equations.

1. $y'' + 3y = t^3 - 1$

2. $y'' + 4y' + 4y = te^{\alpha t}$

3. $y'' - y = t^2 e^t$

4. $y'' + y' + y = 1 + t + t^2$

5. $y'' + 2y' + y = e^{-t}$

6. $y'' + 5y' + 4y = t^2 e^{7t}$

7. $y'' + 4y = t\sin 2t$

8. $y'' - 6y' + 9y = (3t^7 - 5t^4)e^{3t}$

9. $y'' - 2y' + 5y = 2\cos^2 t$

10. $y'' - 2y' + 5y = 2(\cos^2 t)e^t$

11. $y'' + y' - 6y = \sin t + te^{2t}$

12. $y'' + y' + 4y = t^2 + (2t+3)(1+\cos t)$

13. $y'' - 3y' + 2y = e^t + e^{2t}$

14. $y'' + 2y' = 1 + t^2 + e^{-2t}$

15. $y'' + y = \cos t\cos 2t$

16. $y'' + y = \cos t\cos 2t\cos 3t$.

17. (a) Show that $\cos^3\omega t = \frac{1}{4}\operatorname{Re}\{e^{3i\omega t} + 3e^{i\omega t}\}$.
Hint: $\cos\omega t = (e^{i\omega t} + e^{-i\omega t})/2$.
(b) Find a particular solution of the equation

$$10y'' + 0.2y' + 1000y = 5 + 20\cos^3 10t$$

18. (a) Let $L[y] = y'' - 2r_1 y' + r_1^2 y$. Show that

$$L\left[e^{r_1 t}v(t)\right] = e^{r_1 t}v''(t).$$

(b) Find the general solution of the equation

$$y'' - 6y' + 9y = t^{3/2}e^{3t}.$$

19. Let $\psi(t) = t(A_0 + \dots + A_n t^n)$, and assume that $b \neq 0$. Show that the equation $a\psi'' + b\psi' = a_0 + \dots + a_n t^n$ determines $A_0, \dots, A_n$ uniquely.

2.6 Mechanical vibrations

Consider the case where a small object of mass m is attached to an elastic spring of length l, which is suspended from a rigid horizontal support (see Figure 1). (An elastic spring has the property that if it is stretched or compressed a distance Δl which is small compared to its natural length l, then it will exert a restoring force of magnitude $k\Delta l$. The constant k is called the spring-constant, and is a measure of the stiffness of the spring.) In addition, the mass and spring may be immersed in a medium, such as oil, which impedes the motion of an object through it. Engineers usually refer to such systems as spring-mass-dashpot systems, or as seismic instruments, since they are similar, in principle, to a seismograph which is used to detect motions of the earth's surface.

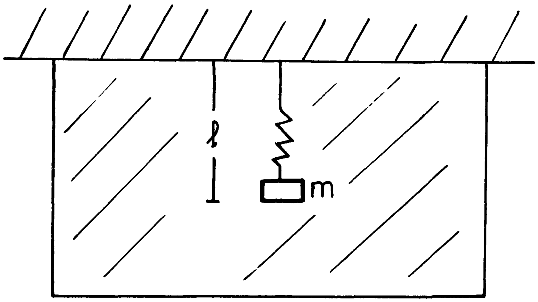

Figure 1

Spring-mass-dashpot systems have many diverse applications. For example, the shock absorbers in our automobiles are simple spring-mass-dashpot systems. Also, most heavy gun emplacements are attached to such systems so as to minimize the "recoil" effect of the gun. The usefulness of these devices will become apparent after we set up and solve the differential equation of motion of the mass m.

In calculating the motion of the mass m, it will be convenient for us to measure distances from the equilibrium position of the mass, rather than the horizontal support. The equilibrium position of the mass is that point where the mass will hang at rest if no external forces act upon it. In equilibrium, the weight mg of the mass is exactly balanced by the restoring force of the spring. Thus, in its equilibrium position, the spring has been stretched a distance Δl, where $k\Delta l = mg$. We let $y=0$ denote this equilibrium position, and we take the downward direction as positive. Let $y(t)$ denote the position of the mass at time t. To find $y(t)$, we must compute the total force acting on the mass m. This force is the sum of four separate forces W, R, D and F.

(i) The force $W = mg$ is the weight of the mass pulling it downward. This force is positive, since the downward direction is the positive y direction.

(ii) The force R is the restoring force of the spring, and it is proportional to the elongation, or compression, $\Delta l + y$ of the spring. It always acts to restore the spring to its natural length. If $\Delta l + y > 0$, then R is negative, so that $R = -k(\Delta l + y)$, and if $\Delta l + y < 0$, then R is positive, so that $R = -k(\Delta l + y)$. In either case,

$$R = -k(\Delta l + y).$$

(iii) The force D is the damping, or drag force, which the medium exerts on the mass m. (Most media, such as oil and air, tend to resist the motion of an object through it.) This force always acts in the direction opposite the direction of motion, and is usually directly proportional to the magnitude of the velocity dy/dt. If the velocity is positive; that is, the mass is moving in the downward direction, then $D = -c\,dy/dt$, and if the velocity is negative, then $D = -c\,dy/dt$. In either case,

$$D = -c\,dy/dt.$$

(iv) The force F is the external force applied to the mass. This force is directed upward or downward, depending as to whether F is positive or negative. In general, this external force will depend explicitly on time.

From Newton's second law of motion (see Section 1.7)

$$\begin{aligned} m\frac{d^2y}{dt^2} &= W + R + D + F \\ &= mg - k(\Delta l + y) - c\frac{dy}{dt} + F(t) \\ &= -ky - c\frac{dy}{dt} + F(t), \end{aligned}$$

since $mg = k\Delta l$. Hence, the position $y(t)$ of the mass satisfies the second-order linear differential equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t) \tag{1}$$

where m, c and k are nonnegative constants. We adopt here the mks system of units so that F is measured in newtons, y is measured in meters, and t is measured in seconds. In this case, the units of k are N/m, the units of c are N·s/m, and the units of m are kilograms (N·s^2/m)

(a) *Free vibrations*:

We consider first the simplest case of free undamped motion. In this case, Equation (1) reduces to

$$m\frac{d^2y}{dt^2} + ky = 0 \quad \text{or} \quad \frac{d^2y}{dt^2} + \omega_0^2 y = 0 \tag{2}$$

where $\omega_0^2 = k/m$. The general solution of (2) is

$$y(t) = a\cos\omega_0 t + b\sin\omega_0 t. \tag{3}$$

In order to analyze the solution (3), it is convenient to rewrite it as a single cosine function. This is accomplished by means of the following lemma.

Lemma 1. *Any function $y(t)$ of the form* (3) *can be written in the simpler form*

$$y(t) = R\cos(\omega_0 t - \delta) \tag{4}$$

where $R = \sqrt{a^2+b^2}$ and $\delta = \tan^{-1} b/a$.

PROOF. We will verify that the two expressions (3) and (4) are equal. To this end, compute

$$R\cos(\omega_0 t - \delta) = R\cos\omega_0 t\cos\delta + R\sin\omega_0 t\sin\delta$$

and observe from Figure 2 that $R\cos\delta = a$ and $R\sin\delta = b$. Hence,

$$R\cos(\omega_0 t - \delta) = a\cos\omega_0 t + b\sin\omega_0 t. \qquad \square$$

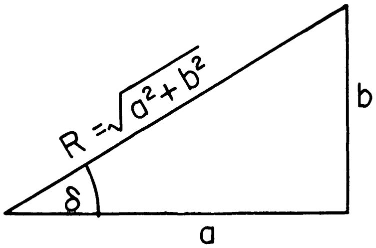

Figure 2

In Figure 3 we have graphed the function $y = R\cos(\omega_0 t - \delta)$. Notice that $y(t)$ always lies between $-R$ and $+R$, and that the motion of the mass is periodic—it repeats itself over every time interval of length $2\pi/\omega_0$. This type of motion is called simple harmonic motion; R is called the amplitude of the motion, δ the phase angle of the motion, $T_0 = 2\pi/\omega_0$ the natural period of the motion, and $\omega_0 = \sqrt{k/m}$ the natural frequency of the system.

(b) *Damped free vibrations*:

If we now include the effect of damping, then the differential equation governing the motion of the mass is

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0. \tag{5}$$

The roots of the characteristic equation $mr^2 + cr + k = 0$ are

$$r_1 = \frac{-c+\sqrt{c^2-4km}}{2m} \quad \text{and} \quad r_2 = \frac{-c-\sqrt{c^2-4km}}{2m}.$$

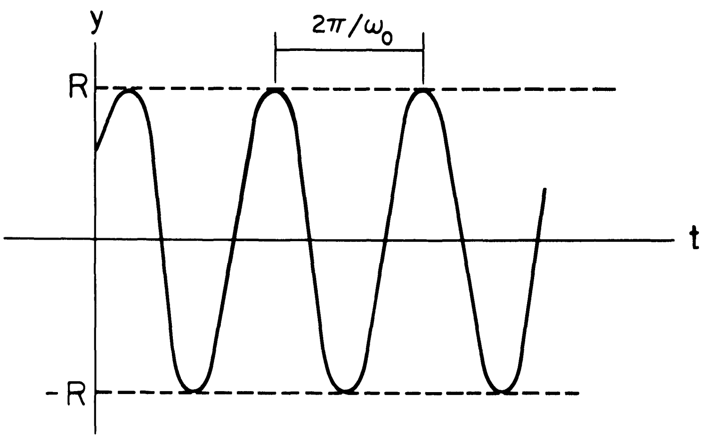

Figure 3. Graph of $y(t)=R\cos(\omega_0 t-\delta)$

Thus, there are three cases to consider, depending as to whether c^2-4km is positive, negative or zero.

(i) $c^2-4km>0$. In this case both r_1 and r_2 are negative, and every solution $y(t)$ of (5) has the form

$$y(t)=ae^{r_1 t}+be^{r_2 t}.$$

(ii) $c^2-4km=0$. In this case, every solution $y(t)$ of (5) is of the form

$$y(t)=(a+bt)e^{-ct/2m}.$$

(iii) $c^2-4km<0$. In this case, every solution $y(t)$ of (5) is of the form

$$y(t)=e^{-ct/2m}\left[a\cos\mu t+b\sin\mu t\right],\qquad \mu=\frac{\sqrt{4km-c^2}}{2m}.$$

The first two cases are referred to as overdamped and critically damped, respectively. They represent motions in which the originally displaced mass creeps back to its equilibrium position. Depending on the initial conditions, it may be possible to overshoot the equilibrium position once, but no more than once (see Exercises 2-3). The third case, which is referred to as an underdamped motion, occurs quite often in mechanical systems and represents a damped vibration. To see this, we use Lemma 1 to rewrite the function

$$y(t)=e^{-ct/2m}\left[a\cos\mu t+b\sin\mu t\right]$$

in the form

$$y(t)=Re^{-ct/2m}\cos(\mu t-\delta).$$

The displacement y oscillates between the curves $y=\pm Re^{-ct/2m}$, and thus represents a cosine curve with decreasing amplitude, as shown in Figure 4.

Now, observe that the motion of the mass always dies out eventually if there is damping in the system. In other words, any initial disturbance of

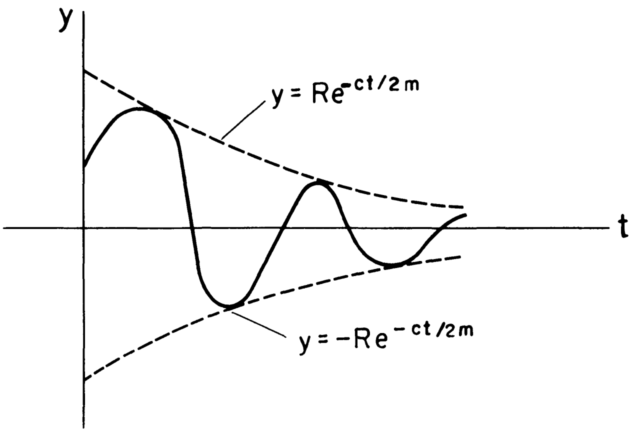

Figure 4. Graph of $\mathrm{Re}^{-ct/2m}\cos(\mu t-\delta)$

the system is dissipated by the damping present in the system. This is one reason why spring-mass-dashpot systems are so useful in mechanical systems: they can be used to damp out any undesirable disturbances. For example, the shock transmitted to an automobile by a bump in the road is dissipated by the shock absorbers in the car, and the momentum from the recoil of a gun barrel is dissipated by a spring-mass-dashpot system attached to the gun.

(c) *Damped forced vibrations*:

If we now introduce an external force $F(t)=F_0\cos\omega t$, then the differential equation governing the motion of the mass is

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=F_0\cos\omega t. \tag{6}$$

Using the method of judicious guessing, we can find a particular solution $\psi(t)$ of (6) of the form

$$\begin{aligned}\psi(t)&=\frac{F_0}{(k-m\omega^2)^2+c^2\omega^2}\left[(k-m\omega^2)\cos\omega t+c\omega\sin\omega t\right]\\ &=\frac{F_0}{(k-m\omega^2)^2+c^2\omega^2}\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}\cos(\omega t-\delta)\\ &=\frac{F_0\cos(\omega t-\delta)}{\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}}\end{aligned} \tag{7}$$

where $\tan\delta=c\omega/(k-m\omega^2)$. Hence, every solution $y(t)$ of (6) must be of the form

$$y(t)=\phi(t)+\psi(t)=\phi(t)+\frac{F_0\cos(\omega t-\delta)}{\left[(k-m\omega^2)^2+c^2\omega^2\right]^{1/2}} \tag{8}$$

where $\phi(t)$ is a solution of the homogeneous equation

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=0. \tag{9}$$

We have already seen though, that every solution $y=\phi(t)$ of (9) approaches zero as t approaches infinity. Thus, for large t, the equation $y(t)=\psi(t)$ describes very accurately the position of the mass m, regardless of its initial position and velocity. For this reason, $\psi(t)$ is called the steady state part of the solution (8), while $\phi(t)$ is called the transient part of the solution.

(d) *Forced free vibrations*:

We now remove the damping from our system and consider the case of forced free vibrations where the forcing term is periodic and has the form $F(t)=F_0\cos\omega t$. In this case, the differential equation governing the motion of the mass m is

$$\frac{d^2y}{dt^2}+\omega_0^2y=\frac{F_0}{m}\cos\omega t, \qquad \omega_0^2=k/m. \tag{10}$$

The case $\omega\neq\omega_0$ is uninteresting; every solution $y(t)$ of (10) has the form

$$y(t)=c_1\cos\omega_0t+c_2\sin\omega_0t+\frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t,$$

and thus is the sum of two periodic functions of different periods. The interesting case is when $\omega=\omega_0$; that is, when the frequency ω of the external force equals the natural frequency of the system. This case is called the *resonance* case, and the differential equation of motion for the mass m is

$$\frac{d^2y}{dt^2}+\omega_0^2y=\frac{F_0}{m}\cos\omega_0t. \tag{11}$$

We will find a particular solution $\psi(t)$ of (11) as the real part of a complex-valued solution $\phi(t)$ of the equation

$$\frac{d^2y}{dt^2}+\omega_0^2y=\frac{F_0}{m}e^{i\omega_0t}. \tag{12}$$

Since $e^{i\omega_0t}$ is a solution of the homogeneous equation $y''+\omega_0^2y=0$, we know that (12) has a particular solution $\phi(t)=Ate^{i\omega_0t}$, for some constant A. Computing

$$\phi''+\omega_0^2\phi=2i\omega_0Ae^{i\omega_0t}$$

we see that

$$A=\frac{1}{2i\omega_0}\frac{F_0}{m}=\frac{-iF_0}{2m\omega_0}.$$

Hence,

$$\begin{aligned}\phi(t)&=\frac{-iF_0t}{2m\omega_0}(\cos\omega_0t+i\sin\omega_0t)\\&=\frac{F_0t}{2m\omega_0}\sin\omega_0t-i\frac{F_0t}{2m\omega_0}\cos\omega_0t\end{aligned}$$

is a particular solution of (12), and

$$\psi(t)=\mathrm{Re}\{\phi(t)\}=\frac{F_0t}{2m\omega_0}\sin\omega_0t$$

is a particular solution of (11). Consequently, every solution $y(t)$ of (11) is of the form

$$y(t)=c_1\cos\omega_0t+c_2\sin\omega_0t+\frac{F_0t}{2m\omega_0}\sin\omega_0t \tag{13}$$

for some choice of constants c_1,c_2.

Now, the sum of the first two terms in (13) is a periodic function of time. The third term, though, represents an oscillation with increasing amplitude, as shown in Figure 5. Thus, the forcing term $F_0\cos\omega t$, if it is in resonance with the natural frequency of the system, will always cause unbounded oscillations. Such a phenomenon was responsible for the collapse of the Tacoma Bridge, (see Section 2.6.1) and many other mechanical catastrophes.

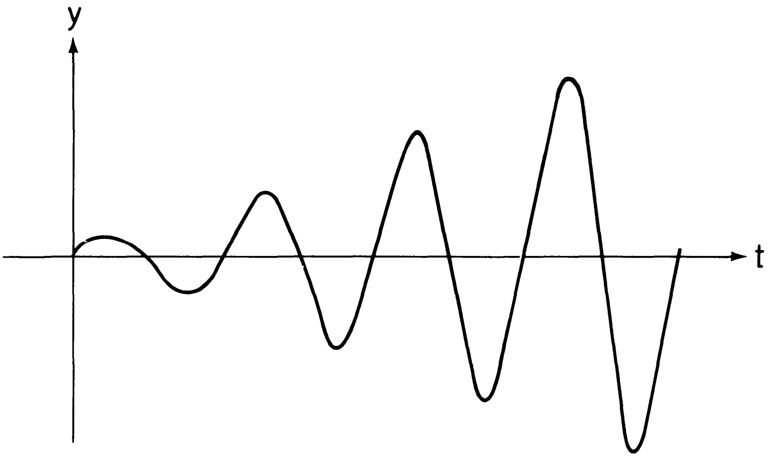

Figure 5. Graph of $f(t)=At\sin\omega_0t$

Exercises

1. It is found experimentally that a 1 kg mass stretches a spring 49/320 m. If the mass is pulled down an additional 1/4 m and released, find the amplitude, period and frequency of the resulting motion, neglecting air resistance (use $g = 9.8\ \text{m/s}^2$).

2. Let $y(t) = Ae^{r_1 t} + Be^{r_2 t}$, with $|A| + |B| \neq 0$.
(a) Show that $y(t)$ is zero at most once.
(b) Show that $y'(t)$ is zero at most once.

3. Let $y(t) = (A + Bt)e^{rt}$, with $|A| + |B| \neq 0$.
(a) Show that $y(t)$ is zero at most once.
(b) Show that $y'(t)$ is zero at most once.

4. A small object of mass 1 kg is attached to a spring with spring constant 2N/m. This spring–mass system is immersed in a viscous medium with damping constant 3 N·s/m. At time $t = 0$, the mass is lowered 1/2 m below its equilibrium position, and released. Show that the mass will creep back to its equilibrium position as t approaches infinity.

5. A small object of mass 1 kg is attached to a spring with spring-constant 1 N/m and is immersed in a viscous medium with damping constant 2 N·s/m. At time $t = 0$, the mass is lowered 1/4 m and given an initial velocity of 1 m/s in the upward direction. Show that the mass will overshoot its equilibrium position once, and then creep back to equilibrium.

6. A small object of mass 4 kg is attached to an elastic spring with spring-constant 64 N/m, and is acted upon by an external force $F(t) = A\cos^3 \omega t$. Find all values of ω at which resonance occurs.

7. The gun of a U.S. M60 tank is attached to a spring–mass–dashpot system with spring-constant $100\alpha^2$ and damping constant 200α, in their appropriate units. The mass of the gun is 100 kg. Assume that the displacement $y(t)$ of the gun from its rest position after being fired at time $t = 0$ satisfies the initial-value problem

$$100y'' + 200\alpha y' + 100\alpha^2 y = 0;\ y(0) = 0,\ y'(0) = 100\ \text{m/s}.$$

It is desired that one second later, the quantity $y^2 + (y')^2$ be less than .01. How large must α be to guarantee that this is so? (The spring–mass–dashpot mechanism in the M60 tanks supplied by the U.S. to Israel are critically damped, for this situation is preferable in desert warfare where one has to fire again as quickly as possible).

8. A spring–mass–dashpot system has the property that the spring constant k is 9 times its mass m, and the damping constant c is 6 times its mass. At time $t = 0$, the mass, which is hanging at rest, is acted upon by an external force $F(t) = (3\sin 3t)$ N. The spring will break if it is stretched an additional 5 m from its equilibrium position. Show that the spring will not break if $m \geqslant 1/5$ kg.

9. A spring–mass–dashpot system with $m = 1$, $k = 2$ and $c = 2$ (in their respective units) hangs in equilibrium. At time $t = 0$, an external force $F(t) = \pi - t$ N acts for a time interval π. Find the position of the mass at anytime $t > \pi$.

10. A 1 kg mass is attached to a spring with spring constant $k = 64$ N/m. With the mass on the spring at rest in the equilibrium position at time $t = 0$, an external force $F(t) = (\frac{1}{2}t)$ N is applied until time $t_1 = 7\pi/16$ seconds, at which time it is removed. Assuming no damping, find the frequency and amplitude of the resulting oscillation.

11. A 1 kg mass is attached to a spring with spring constant $k = 4$ N/m, and hangs in equilibrium. An external force $F(t) = (1 + t + \sin 2t)$ N is applied to the mass beginning at time $t = 0$. If the spring is stretched a length $(1/2 + \pi/4)$ m or more from its equilibrium position, then it will break. Assuming no damping present, find the time at which the spring breaks.

12. A small object of mass 1 kg is attached to a spring with spring constant $k = 1$ N/m. This spring-mass system is then immersed in a viscous medium with damping constant c. An external force $F(t) = (3 - \cos t)$ N is applied to the system. Determine the minimum positive value of c so that the magnitude of the steady state solution does not exceed 5 m.

13. Determine a particular solution $\psi(t)$ of $my'' + cy' + ky = F_0 \cos \omega t$, of the form $\psi(t) = A \cos(\omega t - \phi)$. Show that the amplitude A is a maximum when $\omega^2 = \omega_0^2 - \frac{1}{2}(c/m)^2$. This value of ω is called the *resonant frequency* of the system. What happens when $\omega_0^2 < \frac{1}{2}(c/m)^2$?

2.6.1 The Tacoma Bridge disaster

On July 1, 1940, the Tacoma Narrows Bridge at Puget Sound in the state of Washington was completed and opened to traffic. From the day of its opening the bridge began undergoing vertical oscillations, and it soon was nicknamed "Galloping Gertie." Strange as it may seem, traffic on the bridge increased tremendously as a result of its novel behavior. People came from hundreds of miles in their cars to enjoy the curious thrill of riding over a galloping, rolling bridge. For four months, the bridge did a thriving business. As each day passed, the authorities in charge became more and more confident of the safety of the bridge—so much so, in fact, that they were planning to cancel the insurance policy on the bridge.

Starting at about 7:00 on the morning of November 7, 1940, the bridge began undulating persistently for three hours. Segments of the span were heaving periodically up and down as much as three feet. At about 10:00 a.m., something seemed to snap and the bridge began oscillating wildly. At one moment, one edge of the roadway was twenty-eight feet higher than the other; the next moment it was twenty-eight feet lower than the other edge. At 10:30 a.m. the bridge began cracking, and finally, at 11:10 a.m. the entire bridge came crashing down. Fortunately, only one car was on the bridge at the time of its failure. It belonged to a newspaper reporter who had to abandon the car and its sole remaining occupant, a pet dog, when the bridge began its violent twisting motion. The reporter reached safety, torn and bleeding, by crawling on hands and knees, desperately clutching the curb of the bridge. His dog went down with the car and the span—the only life lost in the disaster.

There were many humorous and ironic incidents associated with the collapse of the Tacoma Bridge. When the bridge began heaving violently, the authorities notified Professor F. B. Farquharson of the University of Washington. Professor Farquharson had conducted numerous tests on a simulated model of the bridge and had assured everyone of its stability. The professor was the last man on the bridge. Even when the span was tilting more than twenty-eight feet up and down, he was making scientific observations with little or no anticipation of the imminent collapse of the bridge. When the motion increased in violence, he made his way to safety by scientifically following the yellow line in the middle of the roadway. The professor was one of the most surprised men when the span crashed into the water.

One of the insurance policies covering the bridge had been written by a local travel agent who had pocketed the premium and had neglected to report the policy, in the amount of $800,000, to his company. When he later received his prison sentence, he ironically pointed out that his embezzlement would never have been discovered if the bridge had only remained up for another week, at which time the bridge officials had planned to cancel all of the policies.

A large sign near the bridge approach advertised a local bank with the slogan "as safe as the Tacoma Bridge." Immediately following the collapse of the bridge, several representatives of the bank rushed out to remove the billboard.

After the collapse of the Tacoma Bridge, the governor of the state of Washington made an emotional speech, in which he declared "We are going to build the exact same bridge, exactly as before." Upon hearing this, the noted engineer Von Karman sent a telegram to the governor stating "If you build the exact same bridge exactly as before, it will fall into the exact same river exactly as before."

The collapse of the Tacoma Bridge was due to an aerodynamical phenomenon known as *stall flutter*. This can be explained very briefly in the following manner. If there is an obstacle in a stream of air, or liquid, then a "vortex street" is formed behind the obstacle, with the vortices flowing off at a definite periodicity, which depends on the shape and dimension of the structure as well as on the velocity of the stream (see Figure 1). As a result of the vortices separating alternately from either side of the obstacle, it is acted upon by a periodic force perpendicular to the direction of the stream, and of magnitude $F_0 \cos \omega t$. The coefficient F_0 depends on the shape of the structure. The poorer the streamlining of the structure; the larger the coefficient F_0, and hence the amplitude of the force. For example, flow around an airplane wing at small angles of attack is very smooth, so that the vortex street is not well defined and the coefficient F_0 is very small. The poorly streamlined structure of a suspension bridge is another

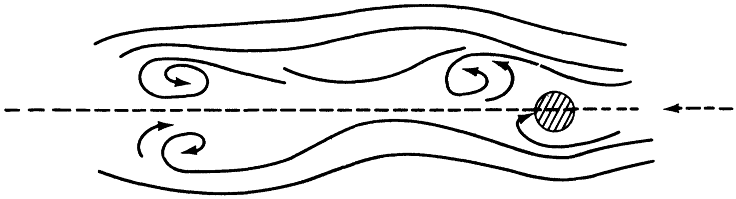

Figure 1

matter, and it is natural to expect that a force of large amplitude will be set up. Thus, a structure suspended in an air stream experiences the effect of this force and hence goes into a state of forced vibrations. The amount of danger from this type of motion depends on how close the natural frequency of the structure (remember that bridges are made of steel, a highly elastic material) is to the frequency of the driving force. If the two frequencies are the same, resonance occurs, and the oscillations will be destructive if the system does not have a sufficient amount of damping. It has now been established that oscillations of this type were responsible for the collapse of the Tacoma Bridge. In addition, resonances produced by the separation of vortices have been observed in steel factory chimneys, and in the periscopes of submarines.

The phenomenon of resonance was also responsible for the collapse of the Broughton suspension bridge near Manchester, England in 1831. This occurred when a column of soldiers marched in cadence over the bridge, thereby setting up a periodic force of rather large amplitude. The frequency of this force was equal to the natural frequency of the bridge. Thus, very large oscillations were induced, and the bridge collapsed. It is for this reason that soldiers are ordered to break cadence when crossing a bridge.

Epilog. The father of one of my students is an engineer who worked on the construction of the Bronx Whitestone Bridge in New York City. He informed me that the original plans for this bridge were very similar to those of the Tacoma Bridge. These plans were hastily redrawn following the collapse of the Tacoma Bridge.

2.6.2 Electrical networks

We now briefly study a simple series circuit, as shown in Figure 1 below. The symbol E represents a source of electromotive force. This may be a battery or a generator which produces a potential difference (or voltage), that causes a current I to flow through the circuit when the switch S is closed. The symbol R represents a resistance to the flow of current such as that produced by a lightbulb or toaster. When current flows through a coil of wire L, a magnetic field is produced which opposes any change in the current through the coil. The change in voltage produced by the coil is proportional to the rate of change of the current, and the constant of propor-

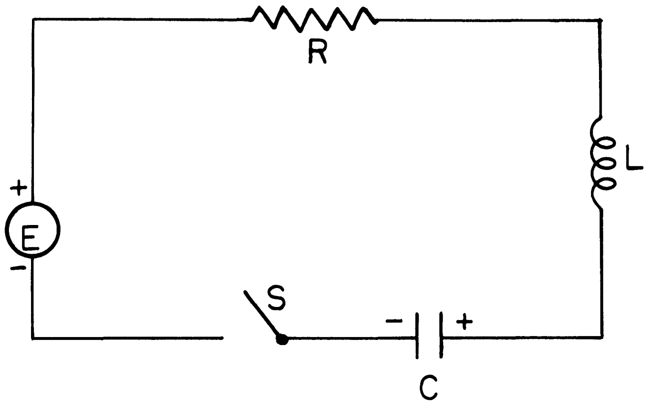

Figure 1. A simple series circuit

tionality is called the inductance L of the coil. A capacitor, or condenser, indicated by C, usually consists of two metal plates separated by a material through which very little current can flow. A capacitor has the effect of reversing the flow of current as one plate or the other becomes charged.

Let $Q(t)$ be the charge on the capacitor at time t. To derive a differential equation which is satisfied by $Q(t)$ we use the following.

Kirchoff's second law: In a closed circuit, the impressed voltage equals the sum of the voltage drops in the rest of the circuit.

Now,

(i) The voltage drop across a resistance of R ohms equals RI (Ohm's law).
(ii) The voltage drop across an inductance of L henrys equals $L(dI/dt)$.
(iii) The voltage drop across a capacitance of C farads equals Q/C.

Hence,

$$E(t) = L\frac{dI}{dt} + RI + \frac{Q}{C},$$

and since $I(t) = dQ(t)/dt$, we see that

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \tag{1}$$

Notice the resemblance of Equation (1) to the equation of a vibrating mass. Among the similarities with mechanical vibrations, electrical circuits also have the property of resonance. Unlike mechanical systems, though, resonance is put to good use in electrical systems. For example, the tuning knob of a radio is used to vary the capacitance in the tuning circuit. In this manner, the resonant frequency (see Exercise 13, Section 2.6) is changed until it agrees with the frequency of one of the incoming radio signals. The amplitude of the current produced by this signal will be much greater than that of all other signals. In this way, the tuning circuit picks out the desired station.

EXERCISES

1. Suppose that a simple series circuit has no resistance and no impressed voltage. Show that the charge Q on the capacitor is periodic in time, with frequency $\omega_0 = \sqrt{1/LC}$. The quantity $\sqrt{1/LC}$ is called the natural frequency of the circuit.

2. Suppose that a simple series circuit consisting of an inductor, a resistor and a capacitor is open, and that there is an initial charge $Q_0 = 10^{-8}$ coulombs on the capacitor. Find the charge on the capacitor and the current flowing in the circuit after the switch is closed for each of the following cases.
(a) $L=0.5$ henrys, $C=10^{-5}$ farads, $R=1000$ ohms
(b) $L=1$ henry, $C=10^{-4}$ farads, $R=200$ ohms
(c) $L=2$ henrys, $C=10^{-6}$ farads, $R=2000$ ohms

3. A simple series circuit has an inductor of 1 henry, a capacitor of 10^{-6} farads, and a resistor of 1000 ohms. The initial charge on the capacitor is zero. If a 12 volt battery is connected to the circuit, and the circuit is closed at $t=0$, find the charge on the capacitor 1 second later, and the steady state charge.

4. A capacitor of 10^{-3} farads is in series with an electromotive force of 12 volts and an inductor of 1 henry. At $t=0$, both Q and I are zero.
(a) Find the natural frequency and period of the electrical oscillations.
(b) Find the maximum charge on the capacitor, and the maximum current flowing in the circuit.

5. Show that if there is no resistance in a circuit, and the impressed voltage is of the form $E_0 \sin \omega t$, then the charge on the capacitor will become unbounded as $t \to \infty$ if $\omega = \sqrt{1/LC}$. This is the phenomenon of resonance.

6. Consider the differential equation

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = E_0 \cos \omega t. \tag{i}$$

We find a particular solution $\psi(t)$ of (i) as the real part of a particular solution $\phi(t)$ of

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = E_0 e^{i\omega t}. \tag{ii}$$

(a) Show that

$$i\omega\phi(t) = \frac{E_0}{R + i\left(\omega L - \dfrac{1}{\omega C}\right)} e^{i\omega t}.$$

(b) The quantity $Z = R + i(\omega L - 1/\omega C)$ is known as the complex impedance of the circuit. The reciprocal of Z is called the admittance, and the real and imaginary parts of $1/Z$ are called the conductance and susceptance. Determine the admittance, conductance and susceptance.

7. Consider a simple series circuit with given values of L, R and C, and an impressed voltage $E_0 \sin \omega t$. For which value of ω will the steady state current be a maximum?

2.7 A model for the detection of diabetes

Diabetes mellitus is a disease of metabolism which is characterized by too much sugar in the blood and urine. In diabetes, the body is unable to burn off all its sugars, starches, and carbohydrates because of an insufficient supply of insulin. Diabetes is usually diagnosed by means of a glucose tolerance test (GTT). In this test the patient comes to the hospital after an overnight fast and is given a large dose of glucose (sugar in the form in which it usually appears in the bloodstream). During the next three to five hours several measurements are made of the concentration of glucose in the patient's blood, and these measurements are used in the diagnosis of diabetes. A very serious difficulty associated with this method of diagnosis is that there is no universally accepted criterion for interpreting the results of a glucose tolerance test. Three physicians interpreting the results of a GTT may come up with three different diagnoses. In one case recently, in Rhode Island, one physician, after reviewing the results of a GTT, came up with a diagnosis of diabetes. A second physician declared the patient to be normal. To settle the question, the results of the GTT were sent to a specialist in Boston. After examining these results, the specialist concluded that the patient was suffering from a pituitary tumor.

In the mid 1960's Drs. Rosevear and Molnar of the Mayo Clinic and Ackerman and Gatewood of the University of Minnesota discovered a fairly reliable criterion for interpreting the results of a glucose tolerance test. Their discovery arose from a very simple model they developed for the blood glucose regulatory system. Their model is based on the following simple and fairly well known facts of elementary biology.

1. Glucose plays an important role in the metabolism of any vertebrate since it is a source of energy for all tissues and organs. For each individual there is an optimal blood glucose concentration, and any excessive deviation from this optimal concentration leads to severe pathological conditions and potentially death.

2. While blood glucose levels tend to be autoregulatory, they are also influenced and controlled by a wide variety of hormones and other metabolites. Among these are the following.

(i) *Insulin*, a hormone secreted by the β cells of the pancreas. After we eat any carbohydrates, our G.I. tract sends a signal to the pancreas to secrete more insulin. In addition, the glucose in our blood directly stimulates the β cells of the pancreas to secrete insulin. It is generally believed that insulin facilitates tissue uptake of glucose by attaching itself to the impermeable membrane walls, thus allowing glucose to pass through the membranes to the center of the cells, where most of the biological and chemical activity takes place. Without sufficient insulin, the body cannot avail itself of all the energy it needs.

(ii) *Glucagon*, a hormone secreted by the α cells of the pancreas. Any excess glucose is stored in the liver in the form of glycogen. In times of need this glycogen is converted back into glucose. The hormone glucagon increases the rate of breakdown of glycogen into glucose. Evidence collected thus far clearly indicates that hypoglycemia (low blood sugar) and fasting promote the secretion of glucagon while increased blood glucose levels suppress its secretion.

(iii) *Epinephrine* (adrenalin), a hormone secreted by the adrenal medulla. Epinephrine is part of an emergency mechanism to quickly increase the concentration of glucose in the blood in times of extreme hypoglycemia. Like glucagon, epinephrine increases the rate of breakdown of glycogen into glucose. In addition, it directly inhibits glucose uptake by muscle tissue; it acts directly on the pancreas to inhibit insulin secretion; and it aids in the conversion of lactate to glucose in the liver.

(iv) *Glucocorticoids*, hormones such as cortisol which are secreted by the adrenal cortex. Glucocorticoids play an important role in the metabolism of carbohydrates.

(v) *Thyroxin*, a hormone secreted by the thyroid gland. This hormone aids the liver in forming glucose from non-carbohydrate sources such as glycerol, lactate and amino acids.

(vi) *Growth hormone* (somatotropin), a hormone secreted by the anterior pituitary gland. Not only does growth hormone affect glucose levels in a direct manner, but it also tends to "block" insulin. It is believed that growth hormone decreases the sensitivity of muscle and adipose membrane to insulin, thereby reducing the effectiveness of insulin in promoting glucose uptake.

The aim of Ackerman et al was to construct a model which would accurately describe the blood glucose regulatory system during a glucose tolerance test, and in which one or two parameters would yield criteria for distinguishing normal individuals from mild diabetics and pre-diabetics. Their model is a very simplified one, requiring only a limited number of blood samples during a GTT. It centers attention on two concentrations, that of glucose in the blood, labelled G, and that of the net hormonal concentration, labelled H. The latter is interpreted to represent the cumulative effect of all the pertinent hormones. Those hormones such as insulin which decrease blood glucose concentrations are considered to increase H, while those hormones such as cortisol which increase blood glucose concentrations are considered to decrease H. Now there are two reasons why such a simplified model can still provide an accurate description of the blood glucose regulatory system. First, studies have shown that under normal, or close to normal conditions, the interaction of one hormone, namely insulin, with blood glucose so predominates that a simple "lumped parameter model" is quite adequate. Second, evidence indicates that normoglycemia does not depend, necessarily, on the normalcy of each kinetic mechanism of the blood glucose regulatory system. Rather, it depends on the overall performance of the blood glucose regulatory system, and this system is dominated by insulin-glucose interactions.

The basic model is described analytically by the equations

$$\frac{dG}{dt} = F_1(G,H) + J(t) \tag{1}$$

$$\frac{dH}{dt} = F_2(G,H). \tag{2}$$

The dependence of F_1 and F_2 on G and H signify that changes in G and H are determined by the values of both G and H. The function $J(t)$ is the external rate at which the blood glucose concentration is being increased. Now, we assume that G and H have achieved optimal values G_0 and H_0 by the time the fasting patient has arrived at the hospital. This implies that $F_1(G_0,H_0)=0$ and $F_2(G_0,H_0)=0$. Since we are interested here in the deviations of G and H from their optimal values, we make the substitution

$$g = G - G_0, \qquad h = H - H_0.$$

Then,

$$\frac{dg}{dt} = F_1(G_0+g, H_0+h) + J(t),$$

$$\frac{dh}{dt} = F_2(G_0+g, H_0+h).$$

Now, observe that

$$F_1(G_0+g,H_0+h) = F_1(G_0,H_0) + \frac{\partial F_1(G_0,H_0)}{\partial G}g + \frac{\partial F_1(G_0,H_0)}{\partial H}h + e_1$$

and

$$F_2(G_0+g,H_0+h) = F_2(G_0,H_0) + \frac{\partial F_2(G_0,H_0)}{\partial G}g + \frac{\partial F_2(G_0,H_0)}{\partial H}h + e_2$$

where e_1 and e_2 are very small compared to g and h. Hence, assuming that G and H deviate only slightly from G_0 and H_0, and therefore neglecting the terms e_1 and e_2, we see that

$$\frac{dg}{dt} = \frac{\partial F_1(G_0,H_0)}{\partial G}g + \frac{\partial F_1(G_0,H_0)}{\partial H}h + J(t) \tag{3}$$

$$\frac{dh}{dt} = \frac{\partial F_2(G_0,H_0)}{\partial G}g + \frac{\partial F_2(G_0,H_0)}{\partial H}h. \tag{4}$$

Now, there are no means, a priori, of determining the numbers

$$\frac{\partial F_1(G_0,H_0)}{\partial G}, \quad \frac{\partial F_1(G_0,H_0)}{\partial H}, \quad \frac{\partial F_2(G_0,H_0)}{\partial G} \quad \text{and} \quad \frac{\partial F_2(G_0,H_0)}{\partial H}.$$

However, we can determine their signs. Referring to Figure 1, we see that dg/dt is negative for $g>0$ and $h=0$, since the blood glucose concentration

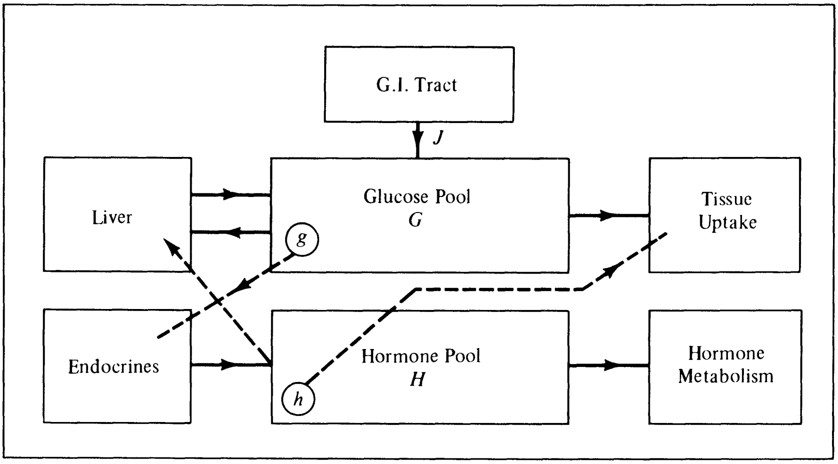

Figure 1. Simplified model of the blood glucose regulatory system

will be decreasing through tissue uptake of glucose and the storing of excess glucose in the liver in the form of glycogen. Consequently $\partial F_1(G_0,H_0)/\partial G$ must be negative. Similarly, $\partial F_1(G_0,H_0)/\partial H$ is negative since a positive value of h tends to decrease blood glucose levels by facilitating tissue uptake of glucose and by increasing the rate at which glucose is converted to glycogen. The number $\partial F_2(G_0,H_0)/\partial G$ must be positive since a positive value of g causes the endocrine glands to secrete those hormones which tend to increase H. Finally, $\partial F_2(G_0,H_0)/\partial H$ must be negative, since the concentration of hormones in the blood decreases through hormone metabolism.

Thus, we can write Equations (3) and (4) in the form

$$\frac{dg}{dt} = -m_1 g - m_2 h + J(t) \tag{5}$$

$$\frac{dh}{dt} = -m_3 h + m_4 g \tag{6}$$

where m_1, m_2, m_3, and m_4 are positive constants. Equations (5) and (6) are two first-order equations for g and h. However, since we only measure the concentration of glucose in the blood, we would like to remove the variable h. This can be accomplished as follows: Differentiating (5) with respect to t gives

$$\frac{d^2g}{dt^2} = -m_1\frac{dg}{dt} - m_2\frac{dh}{dt} + \frac{dJ}{dt}.$$

Substituting for dh/dt from (6) we obtain that

$$\frac{d^2g}{dt^2} = -m_1\frac{dg}{dt} + m_2m_3h - m_2m_4g + \frac{dJ}{dt}. \tag{7}$$

Next, observe from (5) that $m_2h = (-dg/dt) - m_1g + J(t)$. Consequently,

$g(t)$ satisfies the second-order linear differential equation

$$\frac{d^2g}{dt^2}+(m_1+m_3)\frac{dg}{dt}+(m_1m_3+m_2m_4)g=m_3J+\frac{dJ}{dt}.$$

We rewrite this equation in the form

$$\frac{d^2g}{dt^2}+2\alpha\frac{dg}{dt}+\omega_0^2g=S(t) \tag{8}$$

where $\alpha=(m_1+m_3)/2$, $\omega_0^2=m_1m_3+m_2m_4$, and $S(t)=m_3J+dJ/dt$.

Notice that the right-hand side of (8) is identically zero except for the very short time interval in which the glucose load is being ingested. We will learn to deal with such functions in Section 2.12. For our purposes here, let $t=0$ be the time at which the glucose load has been completely ingested. Then, for $t\geqslant 0$, $g(t)$ satisfies the second-order linear homogeneous equation

$$\frac{d^2g}{dt^2}+2\alpha\frac{dg}{dt}+\omega_0^2g=0. \tag{9}$$

This equation has positive coefficients. Hence, by the analysis in Section 2.6, (see also Exercise 8, Section 2.2.2) $g(t)$ approaches zero as t approaches infinity. Thus our model certainly conforms to reality in predicting that the blood glucose concentration tends to return eventually to its optimal concentration.

The solutions $g(t)$ of (9) are of three different types, depending as to whether $\alpha^2-\omega_0^2$ is positive, negative, or zero. These three types, of course, correspond to the overdamped, critically damped and underdamped cases discussed in Section 2.6. We will assume that $\alpha^2-\omega_0^2$ is negative; the other two cases are treated in a similar manner. If $\alpha^2-\omega_0^2<0$, then the characteristic equation of Equation (9) has complex roots. It is easily verified in this case (see Exercise 1) that every solution $g(t)$ of (9) is of the form

$$g(t)=Ae^{-\alpha t}\cos(\omega t-\delta),\qquad \omega^2=\omega_0^2-\alpha^2. \tag{10}$$

Consequently,

$$G(t)=G_0+Ae^{-\alpha t}\cos(\omega t-\delta). \tag{11}$$

Now there are five unknowns G_0, A, α, ω_0, and δ in (11). One way of determining them is as follows. The patient's blood glucose concentration before the glucose load is ingested is G_0. Hence, we can determine G_0 by measuring the patient's blood glucose concentration immediately upon his arrival at the hospital. Next, if we take four additional measurements G_1, G_2, G_3, and G_4 of the patient's blood glucose concentration at times t_1, t_2, t_3, and t_4, then we can determine A, α, ω_0, and δ from the four equations

$$G_j=G_0+Ae^{-\alpha t_j}\cos(\omega t_j-\delta),\qquad j=1,2,3,4.$$

A second, and better method of determining G_0, A, α, ω_0, and δ is to take n measurements $G_1,G_2,\dots,G_n$ of the patient's blood glucose concentration at times $t_1, t_2, \ldots, t_n$. Typically n is 6 or 7. We then find optimal values for G_0, A, α, ω_0, and δ such that the least square error

$$E = \sum_{j=1}^{n} \left[G_j - G_0 - Ae^{-\alpha t_j} \cos(\omega t_j - \delta) \right]^2$$

is minimized. The problem of minimizing E can be solved on a digital computer, and Ackerman et al (see reference at end of section) provide a complete Fortran program for determining optimal values for G_0, A, α, ω_0, and δ. This method is preferrable to the first method since Equation (11) is only an approximate formula for $G(t)$. Consequently, it is possible to find values G_0, A, α, ω_0, and δ so that Equation (11) is satisfied exactly at four points t_1, t_2, t_3, and t_4 but yields a poor fit to the data at other times. The second method usually offers a better fit to the data on the entire time interval since it involves more measurements.

In numerous experiments, Ackerman et al observed that a slight error in measuring G could produce a very large error in the value of α. Hence, any criterion for diagnosing diabetes that involves the parameter α is unreliable. However, the parameter ω_0, the natural frequency of the system, was relatively insensitive to experimental errors in measuring G. Thus, we may regard a value of ω_0 as the basic descriptor of the response to a glucose tolerance test. For discussion purposes, it is more convenient to use the corresponding natural period $T_0 = 2\pi/\omega_0$. The remarkable fact is that data from a variety of sources indicated that *a value of less than four hours for T_0 indicated normalcy, while appreciably more than four hours implied mild diabetes.*

Remark 1. The usual period between meals in our culture is about 4 hours. This suggests the interesting possibility that sociological factors may also play a role in the blood glucose regulatory system.

Remark 2. We wish to emphasize that the model described above can only be used to diagnose mild diabetes or pre-diabetes, since we have assumed throughout that the deviation g of G from its optimal value G_0 is small. Very large deviations of G from G_0 usually indicate severe diabetes or diabetes insipidus, which is a disorder of the posterior lobe of the pituitary gland.

A serious shortcoming of this simplified model is that it sometimes yields a poor fit to the data in the time period three to five hours after ingestion of the glucose load. This indicates, of course, that variables such as epinephrine and glucagon play an important role in this time period. Thus these variables should be included as separate variables in our model, rather than being lumped together with insulin. In fact, evidence indicates that levels of epinephrine may rise dramatically during the recovery phase of the GTT response, when glucose levels have been lowered below fasting

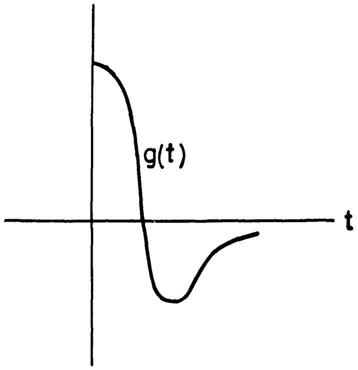

Figure 2. Graph of $g(t)$ if $\alpha^2-\omega_0^2>0$

levels. This can also be seen directly from Equation (9). If $\alpha^2-\omega_0^2>0$, then $g(t)$ may have the form described in Figure 2. Note that $g(t)$ drops very rapidly from a fairly high value to a negative one. It is quite conceivable, therefore, that the body will interpret this as an extreme emergency and thereby secrete a large amount of epinephrine.

Medical researchers have long recognized the need of including epinephrine as a separate variable in any model of the blood glucose regulatory system. However, they were stymied by the fact that there was no reliable method of measuring the concentration of epinephrine in the blood. Thus, they had to assume, for all practical purposes, the the level of epinephrine remained constant during the course of a glucose tolerance test. This author has just been informed that researchers at Rhode Island Hospital have devised an accurate method of measuring the concentration of epinephrine in the blood. Thus we will be able to develop and test more accurate models of the blood glucose regulatory system. Hopefully, this will lead to more reliable criteria for the diagnosis of diabetes.

Reference

E. Ackerman, L. Gatewood, J. Rosevear, and G. Molnar, Blood glucose regulation and diabetes, Chapter 4 in *Concepts and Models of Biomathematics*, F. Heinmets, ed., Marcel Dekker, 1969, 131–156.

Exercises

1. Derive Equation (10).

2. A patient arrives at the hospital after an overnight fast with a blood glucose concentration of 70 mg glucose/100 ml blood (mg glucose/100 ml blood = milligrams of glucose per 100 milliliters of blood). His blood glucose concentration 1 hour, 2 hours, and 3 hours after fully absorbing a large amount of glucose is 95, 65, and 75 mg glucose/100 ml blood, respectively. Show that this patient is normal. *Hint*: In the underdamped case, the time interval between two successive zeros of $G-G_0$ exceeds one half the natural period.

According to a famous diabetologist, the blood glucose concentration of a nondiabetic who has just absorbed a large amount of glucose will be at or below the fasting level in 2 hours or less. Exercises 3 and 4 compare the diagnoses of this diabetologist with those of Ackerman et al.

3. The deviation $g(t)$ of a patient's blood glucose concentration from its optimal concentration satisfies the differential equation $(d^2g/dt^2)+2\alpha(dg/dt)+\alpha^2g=0$ immediately after he fully absorbs a large amount of glucose. The time t is measured in minutes, so that the units of α are reciprocal minutes. Show that this patient is normal according to Ackerman et al, if $\alpha>\pi/120$ (min), and that this patient is normal according to the famous diabetologist if

$$g'(0)<-\left(\tfrac{1}{120}+\alpha\right)g(0).$$

4. A patient's blood glucose concentration $G(t)$ satisfies the initial-value problem

$$\frac{d^2G}{dt^2}+\frac{1}{20\,(\text{min})}\frac{dG}{dt}+\frac{1}{2500\,(\text{min})^2}G$$

$$=\frac{1}{2500\,(\text{min})^2}75\text{ mg glucose}/100\text{ ml blood};$$

$$G(0)=150\text{ mg glucose}/100\text{ ml blood},$$

$$G'(0)=-\alpha G(0)/(\text{min});\qquad \alpha\geqslant\frac{1}{200}\frac{1-4e^{18/5}}{1-e^{18/5}}$$

immediately after he fully absorbs a large amount of glucose. This patient's optimal blood glucose concentration is 75 mg glucose/100 ml blood. Show that this patient is a diabetic according to Ackerman et al, but is normal according to the famous diabetologist.

2.8 Series solutions

We return now to the general homogeneous linear second-order equation

$$L[y]=P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0 \tag{1}$$

with $P(t)$ unequal to zero in the interval $\alpha<t<\beta$. It was shown in Section 2.1 that every solution $y(t)$ of (1) can be written in the form $y(t)=c_1y_1(t)+c_2y_2(t)$, where $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of (1). Thus, the problem of finding all solutions of (1) is reduced to the simpler problem of finding just two solutions. In Section 2.2 we handled the special case where P, Q, and R are constants. The next simplest case is when $P(t)$, $Q(t)$, and $R(t)$ are polynomials in t. In this case, the form of the differential equation suggests that we guess a polynomial solution $y(t)$ of (1). If $y(t)$ is a polynomial in t, then the three functions $P(t)y''(t)$, $Q(t)y'(t)$, and $R(t)y(t)$ are again polynomials in t. Thus, in principle, we can determine a polynomial solution $y(t)$ of (1) by setting the sums of the coefficients of like powers of t in the expression $L[y](t)$ equal to zero. We illustrate this method with the following example.

Example 1. Find two linearly independent solutions of the equation

$$L[y]=\frac{d^2y}{dt^2}-2t\frac{dy}{dt}-2y=0. \tag{2}$$

Solution. We will try to find 2 polynomial solutions of (2). Now, it is not obvious, a priori, what the degree of any polynomial solution of (2) should be. Nor is it evident that we will be able to get away with a polynomial of finite degree. Therefore, we set

$$y(t)=a_0+a_1t+a_2t^2+\dots=\sum_{n=0}^{\infty}a_nt^n.$$

Computing

$$\frac{dy}{dt}=a_1+2a_2t+3a_3t^2+\dots=\sum_{n=0}^{\infty}na_nt^{n-1}$$

and

$$\frac{d^2y}{dt^2}=2a_2+6a_3t+\dots=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2},$$

we see that $y(t)$ is a solution of (2) if

$$\begin{aligned}L[y](t)&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}-2t\sum_{n=0}^{\infty}na_nt^{n-1}-2\sum_{n=0}^{\infty}a_nt^n\\&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}-2\sum_{n=0}^{\infty}na_nt^n-2\sum_{n=0}^{\infty}a_nt^n=0.\end{aligned} \tag{3}$$

Our next step is to rewrite the first summation in (3) so that the exponent of the general term is n, instead of $n-2$. This is accomplished by increasing every n underneath the summation sign by 2, and decreasing the lower limit by 2, that is,

$$\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}=\sum_{n=-2}^{\infty}(n+2)(n+1)a_{n+2}t^n.$$

(If you don't believe this, you can verify it by writing out the first few terms in both summations. If you still don't believe this and want a formal proof, set $m=n-2$. When n is zero, m is -2 and when n is infinity, m is infinity. Therefore

$$\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}=\sum_{m=-2}^{\infty}(m+2)(m+1)a_{m+2}t^m,$$

and since m is a dummy variable, we may replace it by n.) Moreover, observe that the contribution to this sum from $n=-2$ and $n=-1$ is zero since the factor $(n+2)(n+1)$ vanishes in both these instances. Hence,

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n$$

and we can rewrite (3) in the form

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n - 2\sum_{n=0}^{\infty} na_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0. \tag{4}$$

Setting the sum of the coefficients of like powers of t in (4) equal to zero gives

$$(n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0$$

so that

$$a_{n+2} = \frac{2(n+1)a_n}{(n+2)(n+1)} = \frac{2a_n}{n+2}. \tag{5}$$

Equation (5) is a recurrence formula for the coefficients $a_0, a_1, a_2, a_3, \ldots$. The coefficient a_n determines the coefficient a_{n+2}. Thus, a_0 determines a_2 through the relation $a_2 = 2a_0/2 = a_0$; a_2, in turn, determines a_4 through the relation $a_4 = 2a_2/(2+2) = a_0/2$; and so on. Similarly, a_1 determines a_3 through the relation $a_3 = 2a_1/(2+1) = 2a_1/3$; a_3, in turn, determines a_5 through the relation $a_5 = 2a_3/(3+2) = 4a_1/3\cdot 5$; and so on. Consequently, all the coefficients are determined uniquely once a_0 and a_1 are prescribed. The values of a_0 and a_1 are completely arbitrary. This is to be expected, though, for if

$$y(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$

then the values of y and y' at $t=0$ are a_0 and a_1 respectively. Thus, the coefficients a_0 and a_1 must be arbitrary until specific initial conditions are imposed on y.

To find two solutions of (2), we choose two different sets of values of a_0 and a_1. The simplest possible choices are (i) $a_0 = 1, a_1 = 0$; (ii) $a_0 = 0$, $a_1 = 1$.

(i) $$a_0 = 1, \qquad a_1 = 0.$$

In this case, all the odd coefficients $a_1, a_3, a_5, \ldots$ are zero since $a_3 = 2a_1/3 = 0$, $a_5 = 2a_3/5 = 0$, and so on. The even coefficients are determined from the relations

$$a_2 = a_0 = 1, \qquad a_4 = \frac{2a_2}{4} = \frac{1}{2}, \qquad a_6 = \frac{2a_4}{6} = \frac{1}{2\cdot 3},$$

and so on. Proceeding inductively, we find that

$$a_{2n} = \frac{1}{2\cdot 3\cdots n} = \frac{1}{n!}.$$

Hence,

$$y_1(t) = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \ldots = e^{t^2}$$

is one solution of (2).

(ii) $$a_0=0, \qquad a_1=1.$$

In this case, all the even coefficients are zero, and the odd coefficients are determined from the relations

$$a_3=\frac{2a_1}{3}=\frac{2}{3}, \qquad a_5=\frac{2a_3}{5}=\frac{2}{5}\frac{2}{3}, \qquad a_7=\frac{2a_5}{7}=\frac{2}{7}\frac{2}{5}\frac{2}{3},$$

and so on. Proceeding inductively, we find that

$$a_{2n+1}=\frac{2^n}{3\cdot 5\cdots(2n+1)}.$$

Thus,

$$y_2(t)=t+\frac{2t^3}{3}+\frac{2^2t^5}{3\cdot 5}+\ldots=\sum_{n=0}^{\infty}\frac{2^n t^{2n+1}}{3\cdot 5\cdots(2n+1)}$$

is a second solution of (2).

Now, observe that $y_1(t)$ and $y_2(t)$ are polynomials of infinite degree, even though the coefficients $P(t)=1$, $Q(t)=-2t$, and $R(t)=-2$ are polynomials of finite degree. Such polynomials are called power series. Before proceeding further, we will briefly review some of the important properties of power series.

1. An infinite series
$$y(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots=\sum_{n=0}^{\infty}a_n(t-t_0)^n \tag{6}$$
is called a power series about $t=t_0$.
2. All power series have an interval of convergence. This means that there exists a nonnegative number ρ such that the infinite series (6) converges for $|t-t_0|<\rho$, and diverges for $|t-t_0|>\rho$. The number ρ is called the radius of convergence of the power series.
3. The power series (6) can be differentiated and integrated term by term, and the resultant series have the same interval of convergence.
4. The simplest method (if it works) for determining the interval of convergence of the power series (6) is the Cauchy ratio test. Suppose that the absolute value of a_{n+1}/a_n approaches a limit λ as n approaches infinity. Then, the power series (6) converges for $|t-t_0|<1/\lambda$, and diverges for $|t-t_0|>1/\lambda$.
5. The product of two power series $\sum_{n=0}^{\infty}a_n(t-t_0)^n$ and $\sum_{n=0}^{\infty}b_n(t-t_0)^n$ is again a power series of the form $\sum_{n=0}^{\infty}c_n(t-t_0)^n$, with $c_n=a_0b_n+a_1b_{n-1}+\ldots+a_nb_0$. The quotient
$$\frac{a_0+a_1t+a_2t^2+\ldots}{b_0+b_1t+b_2t^2+\ldots}$$
of two power series is again a power series, provided that $b_0\neq 0$.

6. Many of the functions $f(t)$ that arise in applications can be expanded in power series; that is, we can find coefficients $a_0, a_1, a_2, \ldots$ so that

$$f(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots=\sum_{n=0}^{\infty} a_n(t-t_0)^n. \tag{7}$$

   Such functions are said to be *analytic* at $t=t_0$, and the series (7) is called the Taylor series of f about $t=t_0$. It can easily be shown that if f admits such an expansion, then, of necessity, $a_n=f^{(n)}(t_0)/n!$, where $f^{(n)}(t)=d^nf(t)/dt^n$.
7. The interval of convergence of the Taylor series of a function $f(t)$, about t_0, can be determined directly through the Cauchy ratio test and other similar methods, or indirectly, through the following theorem of complex analysis.

Theorem 6. *Let the variable t assume complex values, and let z_0 be the point closest to t_0 at which f or one of its derivatives fails to exist. Compute the distance ρ, in the complex plane, between t_0 and z_0. Then, the Taylor series of f about t_0 converges for $|t-t_0|<\rho$, and diverges for $|t-t_0|>\rho$.*

As an illustration of Theorem 6, consider the function $f(t)=1/(1+t^2)$. The Taylor series of f about $t=0$ is

$$\frac{1}{1+t^2}=1-t^2+t^4-t^6+\ldots,$$

and this series has radius of convergence one. Although the function $(1+t^2)^{-1}$ is perfectly well behaved for t real, it goes to infinity when $t=\pm i$, and the distance of each of these points from the origin is one.

A second application of Theorem 6 is that the radius of convergence of the Taylor series about $t=0$ of the quotient of two polynomials $a(t)$ and $b(t)$, is the magnitude of the smallest zero of $b(t)$.

At this point we make the important observation that it really wasn't necessary to assume that the functions $P(t)$, $Q(t)$, and $R(t)$ in (1) are polynomials. The method used to solve Example 1 should also be applicable to the more general differential equation

$$L[y]=P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0$$

where $P(t)$, $Q(t)$, and $R(t)$ are power series about t_0. (Of course, we would expect the algebra to be much more cumbersome in this case.) If

$$P(t)=p_0+p_1(t-t_0)+\ldots, \qquad Q(t)=q_0+q_1(t-t_0)+\ldots,$$
$$R(t)=r_0+r_1(t-t_0)+\ldots$$

and $y(t)=a_0+a_1(t-t_0)+\ldots$, then $L[y](t)$ will be the sum of three power series about $t=t_0$. Consequently, we should be able to find a recurrence formula for the coefficients a_n by setting the sum of the coefficients of like powers of t in the expression $L[y](t)$ equal to zero. This is the content of the following theorem, which we quote without proof.

Theorem 7. *Let the functions $Q(t)/P(t)$ and $R(t)/P(t)$ have convergent Taylor series expansions about $t=t_0$, for $|t-t_0|<\rho$. Then, every solution $y(t)$ of the differential equation*

$$P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0 \tag{8}$$

is analytic at $t=t_0$, and the radius of convergence of its Taylor series expansion about $t=t_0$ is at least ρ. The coefficients $a_2, a_3, \ldots$, in the Taylor series expansion

$$y(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots \tag{9}$$

are determined by plugging the series (9) into the differential equation (8) and setting the sum of the coefficients of like powers of t in this expression equal to zero.

Remark. The interval of convergence of the Taylor series expansion of any solution $y(t)$ of (8) is determined, usually, by the interval of convergence of the power series $Q(t)/P(t)$ and $R(t)/P(t)$, rather than by the interval of convergence of the power series $P(t)$, $Q(t)$, and $R(t)$. This is because the differential equation (8) must be put in the standard form

$$\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0$$

whenever we examine questions of existence and uniqueness.

Example 2.
(a) Find two linearly independent solutions of

$$L[y]=\frac{d^2y}{dt^2}+\frac{3t}{1+t^2}\frac{dy}{dt}+\frac{1}{1+t^2}y=0. \tag{10}$$

(b) Find the solution $y(t)$ of (10) which satisfies the initial conditions $y(0)=2$, $y'(0)=3$.
Solution.
(a) The *wrong* way to do this problem is to expand the functions $3t/(1+t^2)$ and $1/(1+t^2)$ in power series about $t=0$. The right way to do this problem is to multiply both sides of (10) by $1+t^2$ to obtain the equivalent equation

$$L[y]=(1+t^2)\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+y=0.$$

We do the problem this way because the algebra is much less cumbersome when the coefficients of the differential equation (8) are polynomials than when they are power series. Setting $y(t)=\sum_{n=0}^{\infty}a_nt^n$, we compute

$$\begin{aligned}L[y](t)&=(1+t^2)\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+3t\sum_{n=0}^{\infty}na_nt^{n-1}+\sum_{n=0}^{\infty}a_nt^n\\&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}\left[n(n-1)+3n+1\right]a_nt^n\\&=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)^2a_nt^n.\end{aligned}$$

Setting the sum of the coefficients of like powers of t equal to zero gives $(n+2)(n+1)a_{n+2}+(n+1)^2a_n=0$. Hence,

$$a_{n+2}=-\frac{(n+1)^2a_n}{(n+2)(n+1)}=-\frac{(n+1)a_n}{n+2}. \tag{11}$$

Equation (11) is a recurrence formula for the coefficients $a_2,a_3,\dots$ in terms of a_0 and a_1. To find two linearly independent solutions of (10), we choose the two simplest cases (i) $a_0=1$, $a_1=0$; and (ii) $a_0=0$, $a_1=1$.

(i) $$a_0=1,\qquad a_1=0.$$

In this case, all the odd coefficients are zero since $a_3=-2a_1/3=0$, $a_5=-4a_3/5=0$, and so on. The even coefficients are determined from the relations

$$a_2=-\frac{a_0}{2}=-\frac{1}{2},\qquad a_4=-\frac{3a_2}{4}=\frac{1\cdot3}{2\cdot4},\qquad a_6=-\frac{5a_4}{6}=-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}$$

and so on. Proceeding inductively, we find that

$$a_{2n}=(-1)^n\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots2n}=(-1)^n\frac{1\cdot3\cdots(2n-1)}{2^nn!}.$$

Thus,

$$y_1(t)=1-\frac{t^2}{2}+\frac{1\cdot3}{2\cdot4}t^4+\dots=\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot3\cdots(2n-1)}{2^nn!}t^{2n} \tag{12}$$

is one solution of (10). The ratio of the $(n+1)$st term to the nth term of $y_1(t)$ is

$$-\frac{1\cdot3\cdots(2n-1)(2n+1)t^{2n+2}}{2^{n+1}(n+1)!}\times\frac{2^nn!}{1\cdot3\cdots(2n-1)t^{2n}}=\frac{-(2n+1)t^2}{2(n+1)},$$

and the absolute value of this quantity approaches t^2 as n approaches infinity. Hence, by the Cauchy ratio test, the infinite series (12) converges for $|t|<1$, and diverges for $|t|>1$.

(ii) $$a_0=0,\qquad a_1=1.$$

In this case, all the even coefficients are zero, and the odd coefficients are determined from the relations

$$a_3=-\frac{2a_1}{3}=-\frac{2}{3},\qquad a_5=-\frac{4a_3}{5}=\frac{2\cdot 4}{3\cdot 5},\qquad a_7=-\frac{6a_5}{7}=-\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7},$$

and so on. Proceeding inductively, we find that

$$a_{2n+1}=(-1)^n\frac{2\cdot 4\cdots 2n}{3\cdot 5\cdots(2n+1)}=\frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)}.$$

Thus,

$$y_2(t)=t-\frac{2}{3}t^3+\frac{2\cdot 4}{3\cdot 5}t^5+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n 2^n n!}{3\cdot 5\cdots(2n+1)}t^{2n+1} \tag{13}$$

is a second solution of (10), and it is easily verified that this solution, too, converges for $|t|<1$, and diverges for $|t|>1$. This, of course, is not very surprising, since the Taylor series expansions about $t=0$ of the functions $3t/(1+t^2)$ and $1/(1+t^2)$ only converge for $|t|<1$.
(b) The solution $y_1(t)$ satisfies the initial conditions $y(0)=1, y'(0)=0$, while $y_2(t)$ satisfies the initial conditions $y(0)=0, y'(0)=1$. Hence $y(t)=2y_1(t)+3y_2(t)$.

Example 3. Solve the initial-value problem

$$L[y]=\frac{d^2y}{dt^2}+t^2\frac{dy}{dt}+2ty=0;\qquad y(0)=1,\quad y'(0)=0.$$

Solution. Setting $y(t)=\sum_{n=0}^{\infty}a_nt^n$, we compute

$$\begin{aligned}L[y](t)&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+t^2\sum_{n=0}^{\infty}na_nt^{n-1}+2t\sum_{n=0}^{\infty}a_nt^n\\&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}na_nt^{n+1}+2\sum_{n=0}^{\infty}a_nt^{n+1}\\&=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=0}^{\infty}(n+2)a_nt^{n+1}.\end{aligned}$$

Our next step is to rewrite the first summation so that the exponent of the general term is $n+1$ instead of $n-2$. This is accomplished by increasing every n underneath the summation sign by 3, and decreasing the lower limit by 3; that is,

$$\begin{aligned}\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}&=\sum_{n=-3}^{\infty}(n+3)(n+2)a_{n+3}t^{n+1}\\&=\sum_{n=-1}^{\infty}(n+3)(n+2)a_{n+3}t^{n+1}.\end{aligned}$$

Therefore,

$$L[y](t) = \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3}t^{n+1} + \sum_{n=0}^{\infty} (n+2)a_n t^{n+1}$$
$$= 2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2)a_{n+3}t^{n+1} + \sum_{n=0}^{\infty} (n+2)a_n t^{n+1}.$$

Setting the sums of the coefficients of like powers of t equal to zero gives

$$2a_2 = 0, \quad \text{and} \quad (n+3)(n+2)a_{n+3} + (n+2)a_n = 0; \qquad n = 0,1,2,\dots$$

Consequently,

$$a_2 = 0, \quad \text{and} \quad a_{n+3} = -\frac{a_n}{n+3}; \qquad n \geqslant 0. \tag{14}$$

The recurrence formula (14) determines a_3 in terms of a_0, a_4 in terms of a_1, a_5 in terms of a_2, and so on. Since $a_2 = 0$, we see that $a_5, a_8, a_{11}, \dots$ are all zero, regardless of the values of a_0 and a_1. To satify the initial conditions, we set $a_0 = 1$ and $a_1 = 0$. Then, from (14), $a_4, a_7, a_{10}, \dots$ are all zero, while

$$a_3 = -\frac{a_0}{3} = -\frac{1}{3}, \qquad a_6 = -\frac{a_3}{6} = \frac{1}{3\cdot 6}, \qquad a_9 = -\frac{a_6}{9} = -\frac{1}{3\cdot 6\cdot 9}$$

and so on. Proceeding inductively, we find that

$$a_{3n} = \frac{(-1)^n}{3\cdot 6\cdots 3n} = \frac{(-1)^n}{3^n 1\cdot 2\cdots n} = \frac{(-1)^n}{3^n n!}.$$

Hence,

$$y(t) = 1 - \frac{t^3}{3} + \frac{t^6}{3\cdot 6} - \frac{t^9}{3\cdot 6\cdot 9} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n}}{3^n n!}.$$

By Theorem 7, this series converges for all t, since the power series t^2 and $2t$ obviously converge for all t. (We could also verify this directly using the Cauchy ratio test.)

Example 4. Solve the initial-value problem

$$L[y] = (t^2 - 2t)\frac{d^2y}{dt^2} + 5(t-1)\frac{dy}{dt} + 3y = 0; \qquad y(1) = 7, \quad y'(1) = 3. \tag{15}$$

Solution. Since the initial conditions are given at $t = 1$, we will express the coefficients of the differential equation (15) as polynomials in $(t-1)$, and then we will find $y(t)$ as a power series centered about $t = 1$. To this end, observe that

$$t^2 - 2t = t(t-2) = \left[(t-1)+1\right]\left[(t-1)-1\right] = (t-1)^2 - 1.$$

Hence, the differential equation (15) can be written in the form

$$L[y] = \left[(t-1)^2 - 1\right]\frac{d^2y}{dt^2} + 5(t-1)\frac{dy}{dt} + 3y = 0.$$

Setting $y(t)=\sum_{n=0}^{\infty}a_n(t-1)^n$, we compute

$$
\begin{aligned}
L[y](t)&=\left[(t-1)^2-1\right]\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^{n-2}\\
&\quad+5(t-1)\sum_{n=0}^{\infty}na_n(t-1)^{n-1}+3\sum_{n=0}^{\infty}a_n(t-1)^n\\
&=-\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^{n-2}\\
&\quad+\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^n+\sum_{n=0}^{\infty}(5n+3)a_n(t-1)^n\\
&=-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(t-1)^n+\sum_{n=0}^{\infty}(n^2+4n+3)a_n(t-1)^n.
\end{aligned}
$$

Setting the sums of the coefficients of like powers of t equal to zero gives $-(n+2)(n+1)a_{n+2}+(n^2+4n+3)a_n=0$, so that

$$
a_{n+2}=\frac{n^2+4n+3}{(n+2)(n+1)}a_n=\frac{n+3}{n+2}a_n,\qquad n\geq 0. \tag{16}
$$

To satisfy the initial conditions, we set $a_0=7$ and $a_1=3$. Then, from (16),

$$
\begin{aligned}
a_2&=\frac{3}{2}a_0=\frac{3}{2}\cdot 7, & a_4&=\frac{5}{4}a_2=\frac{5\cdot 3}{4\cdot 2}\cdot 7, & a_6&=\frac{7}{6}a_4=\frac{7\cdot 5\cdot 3}{6\cdot 4\cdot 2}\cdot 7,\ldots\\
a_3&=\frac{4}{3}a_1=\frac{4}{3}\cdot 3, & a_5&=\frac{6}{5}a_3=\frac{6\cdot 4}{5\cdot 3}\cdot 3, & a_7&=\frac{8}{7}a_5=\frac{8\cdot 6\cdot 4}{7\cdot 5\cdot 3}\cdot 3,\ldots
\end{aligned}
$$

and so on. Proceeding inductively, we find that

$$
a_{2n}=\frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)}\cdot 7\quad\text{and}\quad a_{2n+1}=\frac{4\cdot 6\cdots(2n+2)}{3\cdot 5\cdots(2n+1)}\cdot 3\qquad(\text{for } n\geq 1).
$$

Hence,

$$
\begin{aligned}
y(t)&=7+3(t-1)+\frac{3}{2}\cdot 7(t-1)^2+\frac{4}{3}\cdot 3(t-1)^3+\ldots\\
&=7+7\sum_{n=1}^{\infty}\frac{3\cdot 5\cdots(2n+1)(t-1)^{2n}}{2^n n!}+3(t-1)+3\sum_{n=1}^{\infty}\frac{2^n(n+1)!(t-1)^{2n+1}}{3\cdot 5\cdots(2n+1)}.
\end{aligned}
$$

Example 5. Solve the initial-value problem

$$
L[y]=(1-t)\frac{d^2y}{dt^2}+\frac{dy}{dt}+(1-t)y=0;\qquad y(0)=1,\ \ y'(0)=1.
$$

Solution. Setting $y(t)=\sum_{n=0}^{\infty}a_nt^n$, we compute

$$\begin{aligned}
L[y](t) &= (1-t)\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} \\
&\quad + \sum_{n=0}^{\infty} na_n t^{n-1} + (1-t)\sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n t^{n-1} \\
&\quad + \sum_{n=0}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} n(n-2)a_n t^{n-1} \\
&\quad + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} (n+1)(n-1)a_{n+1} t^n \\
&\quad + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=1}^{\infty} a_{n-1} t^n \\
&= 2a_2 + a_1 + a_0 \\
&\quad + \sum_{n=1}^{\infty} \left\{(n+2)(n+1)a_{n+2} - (n+1)(n-1)a_{n+1} + a_n - a_{n-1}\right\} t^n.
\end{aligned}$$

Setting the coefficients of each power of t equal to zero gives

$$a_2 = -\frac{a_1 + a_0}{2} \quad \text{and} \quad a_{n+2} = \frac{(n+1)(n-1)a_{n+1} - a_n + a_{n-1}}{(n+2)(n+1)}, \qquad n \geq 1. \tag{17}$$

To satisfy the initial conditions, we set $a_0 = 1$ and $a_1 = 1$. Then, from (17),

$$a_2 = -1, \qquad a_3 = \frac{-a_1 + a_0}{6} = 0, \qquad a_4 = \frac{3a_3 - a_2 + a_1}{12} = \frac{1}{6},$$

$$a_5 = \frac{8a_4 - a_3 + a_2}{20} = \frac{1}{60}, \qquad a_6 = \frac{15a_5 - a_4 + a_3}{30} = \frac{1}{360}$$

and so on. Unfortunately, though, we cannot discern a general pattern for the coefficients a_n as we did in the previous examples. (This is because the coefficient a_{n+2} depends on the values of a_{n+1}, a_n, and a_{n-1}, while in our previous examples, the coefficient a_{n+2} depended on only one of its predecessors.) This is not a serious problem, though, for we can find the coefficients a_n quite easily with the aid of a digital computer. Sample Pascal and Fortran programs to compute the coefficients $a_2, \ldots, a_n$ in terms of a_0 and a_1, and to evaluate the "approximate" solution

$$y(t) \cong a_0 + a_1 t + \ldots + a_n t^n$$

at any point t are given below. These programs have variable values for a_0 and a_1, so they can also be used to solve the more general initial-value problem

$$(1-t)\frac{d^2y}{dt^2}+\frac{dy}{dt}+(1-t)y=0; \qquad y(0)=a_0, \quad y'(0)=a_1.$$

Pascal Program

```
Program Series (input, output);

var
  A: array[0..199] of real;
  T, sum: real;
  k, N: integer;

begin
  readln(A[0], A[1], T, N);
  page;
  A[2]:= -0.5*(A[1]+A[0]);
  sum:=A[0]+A[1]*T+A[2]*T*T;
  for k:=1 to N-2 do
    begin
      A[k+2]:=((k+1)*(k-1)*A[k+1]-A[k]+A[k-1])
                /((k+1)*(k+2));
      sum:=sum+A[k+2]*exp((k+2)*ln(T));
    end;
  writeln('For N=' ,N:3, ' and T=' ,T:6:4);
  writeln('the sum is: ',sum:11:9);
end.
```

Fortran Program

```
      DIMENSION A(200)
      READ (5, 10) A0, A(1), T, N
10    FORMAT (3F15.8, I5)
      A(2)= -0.5*(A(1)+A0)
      A(3)=(A0-A(1))/2.*3.
      SUM=A0+A(1)*T+A(2)*T**2+A(3)*T**3
      NA=N-2
      D020 K=2,NA
      A(K+2)=(A(K-1)-A(K)+(K+1.)*(K-1.)*
     1A(K+1))/(K+1.)*(K+2.)
      SUM=SUM+A(K+2)*T**(K+2)
20    CONTINUE
      WRITE (6,30) N,T,SUM
30    FORMAT (1H1, 'FOR N=', I3, ', AND T=',F10.4/1H, 'THE
     1SUM IS', F20.9)
      CALL EXIT
      END
```

See also C Program 14 in Appendix C for a sample C program. Setting $A[0]=1$, $A[1]=1$, ($A(1)=1$ for the Fortran program), $T=0.5$, and $N=20$ in these programs gives

$$y\left(\tfrac{1}{2}\right)\cong a_0+a_1\left(\tfrac{1}{2}\right)+\ldots+a_{20}\left(\tfrac{1}{2}\right)^{20}=1.26104174.$$

This result is correct to eight significant decimal places, since any larger value of N yields the same result.

Exercises

Find the general solution of each of the following equations.

1. $y''+ty'+y=0$

2. $y''-ty=0$

3. $(2+t^2)y''-ty'-3y=0$

4. $y''-t^3y=0$

Solve each of the following initial-value problems.

5. $t(2-t)y''-6(t-1)y'-4y=0;\quad y(1)=1,\ y'(1)=0$

6. $y''+t^2y=0;\quad y(0)=2,\ y'(0)=-1$

7. $y''-t^3y=0;\quad y(0)=0,\ y'(0)=-2$

8. $y''+(t^2+2t+1)y'-(4+4t)y=0;\quad y(-1)=0,\ y'(-1)=1$

9. The equation $y''-2ty'+\lambda y=0$, λ constant, is known as the Hermite differential equation, and it appears in many areas of mathematics and physics.
(a) Find 2 linearly independent solutions of the Hermite equation.
(b) Show that the Hermite equation has a polynomial solution of degree n if $\lambda=2n$. This polynomial, when properly normalized; that is, when multiplied by a suitable constant, is known as the Hermite polynomial $H_n(t)$.

10. The equation $(1-t^2)y''-2ty'+\alpha(\alpha+1)y=0$, α constant, is known as the Legendre differential equation, and it appears in many areas of mathematics and physics.
(a) Find 2 linearly independent solutions of the Legendre equation.
(b) Show that the Legendre differential equation has a polynomial solution of degree n if $\alpha=n$.
(c) The Legendre polynomial $P_n(t)$ is defined as the polynomial solution of the Legendre equation with $\alpha=n$ which satisfies the condition $P_n(1)=1$. Find $P_0(t)$, $P_1(t)$, $P_2(t)$, and $P_3(t)$.

11. The equation $(1-t^2)y''-ty'+\alpha^2y=0$, α constant, is known as the Tchebycheff differential equation, and it appears in many areas of mathematics and physics.
(a) Find 2 linearly independent solutions of the Tchebycheff equation.
(b) Show that the Tchebycheff equation has a polynomial solution of degree n if $\alpha=n$. These polynomials, when properly normalized, are called the Tchebycheff polynomials.

12. (a) Find 2 linearly independent solutions of

$$y''+t^3y'+3t^2y=0.$$

(b) Find the first 5 terms in the Taylor series expansion about $t=0$ of the solution $y(t)$ of the initial-value problem

$$y''+t^3y'+3t^2y=e^t; \qquad y(0)=0, \quad y'(0)=0.$$

In each of Problems 13–17, (a) Find the first 5 terms in the Taylor series expansion $\sum_{n=0}^{\infty}a_nt^n$ of the solution $y(t)$ of the given initial-value problem. (b) Write a computer program to find the first $N+1$ coefficients $a_0, a_1, \ldots, a_N$, and to evaluate the polynomial $a_0+a_1t+\ldots+a_Nt^N$. Then, obtain an approximation of $y(\frac{1}{2})$ by evaluating $\sum_{n=0}^{20}a_n(\frac{1}{2})^n$.

13. $(1-t)y''+ty'+y=0;\quad y(0)=1,\ y'(0)=0$

14. $y''+y'+ty=0;\quad y(0)=-1,\ y'(0)=2$

15. $y''+ty'+e^ty=0;\quad y(0)=1,\ y'(0)=0$

16. $y''+y'+e^ty=0;\quad y(0)=0,\ y'(0)=-1$

17. $y''+y'+e^{-t}y=0;\quad y(0)=3,\ y'(0)=5$

2.8.1 Singular points, Euler equations

The differential equation

$$L[y]=P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0 \tag{1}$$

is said to be singular at $t=t_0$ if $P(t_0)=0$. Solutions of (1) frequently become very large, or oscillate very rapidly, in a neighborhood of the singular point t_0. Thus, solutions of (1) may not even be continuous, let along analytic at t_0, and the method of power series solution will fail to work, in general.

Our goal is to find a class of singular equations which we can solve for t near t_0. To this end we will first study a very simple equation, known as Euler's equation, which is singular, but easily solvable. We will then use the Euler equation to motivate a more general class of singular equations which are also solvable in the vicinity of the singular point.

Definition. The differential equation

$$L[y]=t^2\frac{d^2y}{dt^2}+\alpha t\frac{dy}{dt}+\beta y=0. \tag{2}$$

where α and β are constants is known as Euler's equation.

We will assume at first, for simplicity, that $t>0$. Observe that t^2y'' and ty' are both multiples of t^r if $y=t^r$. This suggests that we try $y=t^r$ as a solution of (2). Computing

$$\frac{d}{dt}t^r = rt^{r-1} \quad \text{and} \quad \frac{d^2}{dt^2}t^r = r(r-1)t^{r-2}$$

we see that

$$\begin{aligned} L[t^r] &= r(r-1)t^r + \alpha r t^r + \beta t^r \\ &= [r(r-1)+\alpha r+\beta]t^r \\ &= F(r)t^r \end{aligned} \tag{3}$$

where

$$\begin{aligned} F(r) &= r(r-1)+\alpha r+\beta \\ &= r^2+(\alpha-1)r+\beta. \end{aligned} \tag{4}$$

Hence, $y=t^r$ is a solution of (2) if, and only if, r is a solution of the quadratic equation

$$r^2+(\alpha-1)r+\beta=0. \tag{5}$$

The solutions r_1, r_2 of (5) are

$$r_1 = -\tfrac{1}{2}\left[(\alpha-1)+\sqrt{(\alpha-1)^2-4\beta}\,\right]$$
$$r_2 = -\tfrac{1}{2}\left[(\alpha-1)-\sqrt{(\alpha-1)^2-4\beta}\,\right].$$

Just as in the case of constant coefficients, we must examine separately the cases where $(\alpha-1)^2-4\beta$ is positive, negative, and zero.

Case 1. $(\alpha-1)^2-4\beta>0$. In this case Equation (5) has two real, unequal roots, and thus (2) has two solutions of the form $y_1(t)=t^{r_1}$, $y_2(t)=t^{r_2}$. Clearly, t^{r_1} and t^{r_2} are independent if $r_1\neq r_2$. Thus the general solution of (2) is (for $t>0$)

$$y(t)=c_1t^{r_1}+c_2t^{r_2}.$$

Example 1. Find the general solution of

$$L[y]=t^2\frac{d^2y}{dt^2}+4t\frac{dy}{dt}+2y=0, \qquad t>0. \tag{6}$$

Solution. Substituting $y=t^r$ in (6) gives

$$\begin{aligned} L[t^r] &= r(r-1)t^r+4rt^r+2t^r \\ &= [r(r-1)+4r+2]t^r \\ &= (r^2+3r+2)t^r \\ &= (r+1)(r+2)t^r \end{aligned}$$

Hence $r_1 = -1$, $r_2 = -2$ and

$$y(t) = c_1 t^{-1} + c_2 t^{-2} = \frac{c_1}{t} + \frac{c_2}{t^2}$$

is the general solution of (6).

Case 2. $(\alpha-1)^2 - 4\beta = 0$. In this case

$$r_1 = r_2 = \frac{1-\alpha}{2}$$

and we have only one solution $y = t^{r_1}$ of (2). A second solution (see Exercise 11) can be found by the method of reduction of order. However, we would like to present here an alternate method of obtaining y_2 which will generalize very nicely in Section 2.8.3. Observe that $F(r) = (r - r_1)^2$ in the case of equal roots. Hence

$$L[t^r] = (r - r_1)^2 t^r. \tag{7}$$

Taking partial derivatives of both sides of (7) with respect to r gives

$$\frac{\partial}{\partial r} L[t^r] = L\left[\frac{\partial}{\partial r} t^r\right] = \frac{\partial}{\partial r}\left[(r - r_1)^2 t^r\right].$$

Since $\partial(t^r)/\partial r = t^r \ln t$, we see that

$$L[t^r \ln t] = (r - r_1)^2 t^r \ln t + 2(r - r_1)t^r. \tag{8}$$

The right hand side of (8) vanishes when $r = r_1$. Hence,

$$L[t^{r_1} \ln t] = 0$$

which implies that $y_2(t) = t^{r_1} \ln t$ is a second solution of (2). Since t^{r_1} and $t^{r_1} \ln t$ are obviously linearly independent, the general solution of (2) in the case of equal roots is

$$y(t) = (c_1 + c_2 \ln t)t^r, \qquad t > 0.$$

Example 2. Find the general solution of

$$L[y] = t^2 \frac{d^2 y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0, \qquad t > 0. \tag{9}$$

Solution. Substituting $y = t^r$ in (9) gives

$$\begin{aligned} L[t^r] &= r(r-1)t^r - 5rt^r + 9t^r \\ &= [r(r-1) - 5r + 9]t^r \\ &= (r^2 - 6r + 9)t^r \\ &= (r-3)^2 t^r. \end{aligned}$$

The equation $(r-3)^2=0$ has $r=3$ as a double root. Hence,

$$y_1(t)=t^3,\ y_2(t)=t^3\ln t$$

and the general solution of (9) is

$$y(t)=(c_1+c_2\ln t)t^3, \qquad t>0.$$

Case 3. $(\alpha-1)^2-4\beta<0$. In this case.

$$r_1=\lambda+i\mu \quad \text{and} \quad r_2=\lambda-i\mu$$

with

$$\lambda=\frac{1-\alpha}{2},\ \mu=\frac{\left[4\beta-(\alpha-1)^2\right]^{1/2}}{2} \tag{10}$$

are complex roots. Hence,

$$\begin{aligned}\phi(t)&=t^{\lambda+i\mu}=t^{\lambda}t^{i\mu}\\&=t^{\lambda}(e^{\ln t})^{i\mu}=t^{\lambda}e^{i\mu\ln t}\\&=t^{\lambda}\left[\cos(\mu\ln t)+i\sin(\mu\ln t)\right]\end{aligned}$$

is a complex-valued solution of (2). But then (see Section 2.2.1)

$$y_1(t)=\mathrm{Re}\{\phi(t)\}=t^{\lambda}\cos(\mu\ln t)$$

and

$$y_2(t)=\mathrm{Im}\{\phi(t)\}=t^{\lambda}\sin(\mu\ln t)$$

are two real-valued independent solutions of (2). Hence, the general solution of (2), in the case of complex roots, is

$$y(t)=t^{\lambda}\left[c_1\cos(\mu\ln t)+c_2\sin(\mu\ln t)\right]$$

with λ and μ given by (10).

Example 3. Find the general solution of the equation

$$L[y]=t^2y''-5ty'+25y=0, \qquad t>0. \tag{11}$$

Solution. Substituting $y=t^r$ in (11) gives

$$\begin{aligned}L[t^r]&=r(r-1)t^r-5rt^r+25t^r\\&=\left[r(r-1)-5r+25\right]t^r\\&=\left[r^2-6r+25\right]t^r\end{aligned}$$

The roots of the equation $r^2-6r+25=0$ are

$$\frac{6\pm\sqrt{36-100}}{2}=3\pm4i$$

so that

$$\begin{aligned}\phi(t) &= t^{3+4i} = t^3 t^{4i}\\ &= t^3 e^{(\ln t)4i} = t^3 e^{i(4\ln t)}\\ &= t^3[\cos(4\ln t) + i\sin(4\ln t)]\end{aligned}$$

is a complex-valued solution of (11). Consequently,

$$y_1(t) = \mathrm{Re}\{\phi(t)\} = t^3\cos(4\ln t)$$

and

$$y_2(t) = \mathrm{Im}\{\phi(t)\} = t^3\sin(4\ln t)$$

are two independent solutions of (11), and the general solution is

$$y(t) = t^3[c_1\cos(4\ln t) + c_2\sin(4\ln t)], \qquad t>0.$$

Let us now return to the case $t<0$. One difficulty is that t^r may not be defined if t is negative. For example, $(-1)^{1/2}$ equals i, which is imaginary. A second difficulty is that $\ln t$ is not defined for negative t. We overcome both of these difficulties with the following clever change of variable. Set

$$t = -x, \qquad x>0,$$

and let $y = u(x)$, $x>0$. Observe, from the chain rule, that

$$\frac{dy}{dt} = \frac{du}{dx}\frac{dx}{dt} = -\frac{du}{dx}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(-\frac{du}{dx}\right) = \frac{d}{dx}\left(-\frac{du}{dx}\right)\frac{dx}{dt} = \frac{d^2u}{dx^2}.$$

Thus, we can rewrite (2) in the form

$$(-x)^2\frac{d^2u}{dx^2} + \alpha(-x)\left(-\frac{du}{dx}\right) + \beta u = 0$$

or

$$x^2\frac{d^2u}{dx^2} + \alpha x\frac{du}{dx} + \beta u = 0, \qquad x>0 \tag{12}$$

But Equation (12) is exactly the same as (2) with t replaced by x and y replaced by u. Hence, Equation (12) has solutions of the form

$$u(x) = \begin{cases} c_1x^{r_1} + c_2x^{r_2} \\ (c_1 + c_2\ln x)x^{r_1} \\ [c_1\cos(\mu\ln x) + c_2\sin(\mu\ln x)]x^{\lambda} \end{cases} \tag{13}$$

depending on whether $(\alpha-1)^2-4\beta$ is positive, zero, or negative. Observe now that

$$x=-t=|t|$$

for negative t. Thus, for negative t, the solutions of (2) have one of the forms

$$\begin{cases} c_1|t|^{r_1}+c_2|t|^{r_2} \\ [c_1+c_2\ln|t|]|t|^{r_1} \\ [c_1\cos(\mu\ln|t|)+c_2\sin(\mu\ln|t|)]|t|^{\lambda} \end{cases}$$

Remark. The equation

$$(t-t_0)^2\frac{d^2y}{dt^2}+\alpha(t-t_0)\frac{dy}{dt}+\beta y=0 \tag{14}$$

is also an Euler equation, with a singularity at $t=t_0$ instead of $t=0$. In this case we look for solutions of the form $(t-t_0)^r$. Alternately, we can reduce (14) to (2) by the change of variable $x=t-t_0$.

EXERCISES

In Problems 1–8, find the general solution of the given equation.

1. $t^2y''+5ty'-5y=0$

2. $2t^2y''+3ty'-y=0$

3. $(t-1)^2y''-2(t-1)y'+2y=0$

4. $t^2y''+3ty'+y=0$

5. $t^2y''-ty'+y=0$

6. $(t-2)^2y''+5(t-2)y'+4y=0$

7. $t^2y''+ty'+y=0$

8. $t^2y''+3ty'+2y=0$

9. Solve the initial-value problem

$$t^2y''-ty'-2y=0; \quad y(1)=0, \quad y'(1)=1$$

on the interval $0<t<\infty$.

10. Solve the initial-value problem

$$t^2y''-3ty'+4y=0; \quad y(1)=1, \quad y'(1)=0$$

on the interval $0<t<\infty$.

11. Use the method of reduction of order to show that $y_2(t)=t^{r_1}\ln t$ in the case of equal roots.

2.8.2 Regular singular points, the method of Frobenius

Our goal now is to find a class of singular differential equations which is more general than the Euler equation

$$t^2\frac{d^2y}{dt^2}+\alpha t\frac{dy}{dt}+\beta y=0 \tag{1}$$

but which is also solvable by analytical techniques. To this end we rewrite (1) in the form

$$\frac{d^2y}{dt^2}+\frac{\alpha}{t}\frac{dy}{dt}+\frac{\beta}{t^2}y=0. \tag{2}$$

A very natural generalization of (2) is the equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{3}$$

where $p(t)$ and $q(t)$ can be expanded in series of the form

$$p(t)=\frac{p_0}{t}+p_1+p_2t+p_3t^2+\cdots$$
$$q(t)=\frac{q_0}{t^2}+\frac{q_1}{t}+q_2+q_3t+q_4t^2+\cdots \tag{4}$$

Definition. The equation (3) is said to have a *regular singular point* at $t=0$ if $p(t)$ and $q(t)$ have series expansions of the form (4). Equivalently, $t=0$ is a regular singular point of (3) if the functions $tp(t)$ and $t^2q(t)$ are analytic at $t=0$. Equation (3) is said to have a regular singular point at $t=t_0$ if the functions $(t-t_0)p(t)$ and $(t-t_0)^2q(t)$ are analytic at $t=t_0$. A singular point of (3) which is not regular is called *irregular*.

Example 1. Classify the singular points of Bessel's equation of order ν

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-\nu^2)y=0, \tag{5}$$

where ν is a constant.
Solution. Here $P(t)=t^2$ vanishes at $t=0$. Hence, $t=0$ is the only singular point of (5). Dividing both sides of (5) by t^2 gives

$$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}+\left(1-\frac{\nu^2}{t^2}\right)y=0.$$

Observe that

$$tp(t)=1 \quad\text{and}\quad t^2q(t)=t^2-\nu^2$$

are both analytic at $t=0$. Hence Bessel's equation of order ν has a regular singular point at $t=0$.

Example 2. Classify the singular points of the Legendre equation

$$(1-t^2)\frac{d^2y}{dt^2}-2t\frac{dy}{dt}+\alpha(\alpha+1)y=0 \tag{6}$$

where α is a constant.
Solution. Since $1-t^2$ vanishes when $t=1$ and -1, we see that (6) is

singular at $t=\pm 1$. Dividing both sides of (6) by $1-t^2$ gives

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\alpha\frac{(\alpha+1)}{1-t^2}y=0.$$

Observe that

$$(t-1)p(t)=-(t-1)\frac{2t}{1-t^2}=\frac{2t}{1+t}$$

and

$$(t-1)^2q(t)=\alpha(\alpha+1)\frac{(t-1)^2}{1-t^2}=\alpha(\alpha+1)\frac{1-t}{1+t}$$

are analytic at $t=1$. Similarly, both $(t+1)p(t)$ and $(t+1)^2q(t)$ are analytic at $t=-1$. Hence, $t=1$ and $t=-1$ are regular singular points of (6).

Example 3. Show that $t=0$ is an irregular singular point of the equation

$$t^2\frac{d^2y}{dt^2}+3\frac{dy}{dt}+ty=0. \tag{7}$$

Solution. Dividing through by t^2 gives

$$\frac{d^2y}{dt^2}+\frac{3}{t^2}\frac{dy}{dt}+\frac{1}{t}y=0.$$

In this case, the function

$$tp(t)=t\left(\frac{3}{t^2}\right)=\frac{3}{t}$$

is not analytic at $t=0$. Hence $t=0$ is an irregular singular point of (7).

We return now to the equation

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0 \tag{8}$$

where $t=0$ is a regular singular point. For simplicity, we will restrict ourselves to the interval $t>0$. Multiplying (8) through by t^2 gives the equivalent equation

$$L[y]=t^2\frac{d^2y}{dt^2}+t(tp(t))\frac{dy}{dt}+t^2q(t)y=0. \tag{9}$$

We can view Equation (9) as being obtained from (1) by adding higher powers of t to the coefficients α and β. This suggests that we might be able to obtain solutions of (9) by adding terms of the form t^{r+1}, t^{r+2},... to the solutions t^r of (1). Specifically, we will try to obtain solutions of (9) of the form

$$y(t)=\sum_{n=0}^{\infty}a_nt^{n+r}=t^r\sum_{n=0}^{\infty}a_nt^n.$$

Example 4. Find two linearly independent solutions of the equation

$$L[y]=2t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty=0, \qquad 0<t<\infty. \tag{10}$$

Solution. Let

$$y(t)=\sum_{n=0}^{\infty} a_n t^{n+r}, \qquad a_0\neq 0.$$

Computing

$$y'(t)=\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}$$

and

$$y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2}$$

we see that

$$\begin{aligned}L[y]&=t^r\left[2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n-1}\right.\\&\qquad\left.+\sum_{n=0}^{\infty}(n+r)a_n t^{n-1}+\sum_{n=0}^{\infty}a_n t^{n+1}\right]\\&=t^r\left[2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n-1}\right.\\&\qquad\left.+\sum_{n=0}^{\infty}(n+r)a_n t^{n-1}+\sum_{n=2}^{\infty}a_{n-2} t^{n-1}\right]\\&=[2r(r-1)a_0+ra_0]t^{r-1}+[2(1+r)ra_1+(1+r)a_1]t^r\\&\qquad+\sum_{n=2}^{\infty}[2(n+r)(n+r-1)a_n+(n+r)a_n+a_{n-2}]t^{n+r-1}\end{aligned}$$

Setting the coefficients of each power of t equal to zero gives

(i) $2r(r-1)a_0+ra_0=r(2r-1)a_0=0$,
(ii) $2(r+1)ra_1+(r+1)a_1=(r+1)(2r+1)a_1=0$,
and
(iii) $2(n+r)(n+r-1)a_n+(n+r)a_n=(n+r)[2(n+r)-1]a_n=-a_{n-2}$, $n\geqslant 2$.

The first equation determines r; it implies that $r=0$ or $r=\frac{1}{2}$. The second equation then forces a_1 to be zero, and the third equation determines a_n for $n\geqslant 2$.

(i) $r=0$. In this case, the recurrence formula (iii) is

$$a_n = \frac{-a_{n-2}}{n(2n-1)}, \qquad n \geq 2.$$

Since $a_1 = 0$, we see that all of the odd coefficients are zero. The even coefficients are determined from the relations

$$a_2 = \frac{-a_0}{2\cdot 3}, \quad a_4 = \frac{-a_2}{4\cdot 7} = \frac{a_0}{2\cdot 4\cdot 3\cdot 7}, \quad a_6 = \frac{-a_4}{6\cdot 11} = \frac{-a_0}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11}$$

and so on. Setting $a_0 = 1$, we see that

$$y_1(t) = 1 - \frac{t^2}{2\cdot 3} + \frac{t^4}{2\cdot 4\cdot 3\cdot 7} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2^n n! 3\cdot 7\cdots(4n-1)}$$

is one solution of (10). It is easily verified, using the Cauchy ratio test, that this series converges for all t.

(ii) $r=\frac{1}{2}$. In this case, the recurrence formula (iii) is

$$a_n = \frac{-a_{n-2}}{\left(n+\frac{1}{2}\right)\left[2\left(n+\frac{1}{2}\right)-1\right]} = \frac{-a_{n-2}}{n(2n+1)}, \quad n \geq 2.$$

Again, all of the odd coefficients are zero. The even coefficients are determined from the relations

$$a_2 = \frac{-a_0}{2\cdot 5}, \quad a_4 = \frac{-a_2}{4\cdot 9} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9}, \quad a_6 = \frac{-a_4}{6\cdot 13} = \frac{-a_0}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}$$

and so on. Setting $a_0 = 1$, we see that

$$\begin{aligned} y_2(t) &= t^{1/2}\left[1 - \frac{t^2}{2\cdot 5} + \frac{t^4}{2\cdot 4\cdot 5\cdot 9} + \cdots\right] \\ &= t^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2^n n! 5\cdot 9\cdots(4n+1)}\right] \end{aligned}$$

is a second solution of (10) on the interval $0 < t < \infty$.

Remark. Multiplying both sides of (10) by t gives

$$2t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + t^2 y = 0.$$

This equation can be viewed as a generalization of the Euler equation

$$2t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} = 0. \tag{11}$$

Equation (11) has solutions of the form t^r, where

$$2r(r-1) + r = 0.$$

This equation is equivalent to Equation (i) which determined r for the solutions of (10).

Let us now see whether our technique, which is known as the method of Frobenius, works in general for Equation (9). (We will assume throughout this section that $t>0$.) By assumption, this equation can be written in the form

$$L[y]=t^2\frac{d^2y}{dt^2}+t[p_0+p_1t+p_2t^2+\cdots]\frac{dy}{dt}+[q_0+q_1t+q_2t^2+\cdots]y=0.$$

Set

$$y(t)=\sum_{n=0}^{\infty}a_nt^{n+r}, \text{ with } a_0\neq 0.$$

Computing

$$y'(t)=\sum_{n=0}^{\infty}(n+r)a_nt^{n+r-1}$$

and

$$y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-2}$$

we see that

$$\begin{aligned}L[y]=t^r\Bigg\{&\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^n+\left(\sum_{m=0}^{\infty}p_mt^m\right)\left[\sum_{n=0}^{\infty}(n+r)a_nt^n\right]\\&+\left(\sum_{m=0}^{\infty}q_mt^m\right)\left(\sum_{n=0}^{\infty}a_nt^n\right)\Bigg\}.\end{aligned}$$

Multiplying through and collecting terms gives

$$\begin{aligned}L[y]=&[r(r-1)+p_0r+q_0]a_0t^r\\&+\{[(1+r)r+p_0(1+r)+q_0]a_1+(rp_1+q_1)a_0\}t^{r+1}\\&\vdots\\&+\Bigg\{[(n+r)(n+r-1)+p_0(n+r)+q_0]a_n\\&+\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]a_k\Bigg\}t^{n+r}\\&+\cdots.\end{aligned}$$

This expression can be simplified if we set

$$F(r)=r(r-1)+p_0r+q_0. \tag{12}$$

Then,

$$L[y]=a_0F(r)t^r+[a_1F(1+r)+(rp_1+q_1)a_0]t^{1+r}+\cdots$$
$$+a_nF(n+r)t^{n+r}+\left\{\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]a_k\right\}t^{n+r}$$
$$+\cdots.$$

Setting the coefficient of each power of t equal to zero gives

$$F(r)=r(r-1)+p_0r+q_0=0 \tag{13}$$

and

$$F(n+r)a_n=-\sum_{k=0}^{n-1}[(k+r)p_{n-k}+q_{n-k}]a_k,\quad n\geqslant 1. \tag{14}$$

Equation (13) is called the *indicial* equation of (9). It is a quadratic equation in r, and its roots determine the two possible values r_1 and r_2 of r for which there may be solutions of (9) of the form

$$\sum_{n=0}^{\infty} a_n t^{n+r}.$$

Note that the indicial equation (13) is exactly the equation we would obtain in looking for solutions t^r of the Euler equation

$$t^2\frac{d^2y}{dt^2}+p_0t\frac{dy}{dt}+q_0y=0.$$

Equation (14) shows that, in general, a_n depends on r and all the preceding coefficients $a_0, a_1,\dots,a_{n-1}$. We can solve it recursively for a_n provided that $F(1+r), F(2+r),\dots,F(n+r)$ are not zero. Observe though that if $F(n+r)=0$ for some positive integer n, then $n+r$ is a root of the indicial equation (13). Consequently, if (13) has two real roots r_1, r_2 with $r_1>r_2$ and r_1-r_2 not an integer, then Equation (9) has two solutions of the form

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty}a_n(r_1)t^n,\ y_2(t)=t^{r_2}\sum_{n=0}^{\infty}a_n(r_2)t^n,$$

and these solutions can be shown to converge wherever $tp(t)$ and $t^2q(t)$ both converge.

Remark. We have introduced the notation $a_n(r_1)$ and $a_n(r_2)$ to emphasize that a_n is determined after we choose $r=r_1$ or r_2.

Example 5. Find the general solution of the equation

$$L[y]=4t\frac{d^2y}{dt^2}+3\frac{dy}{dt}+3y=0. \tag{15}$$

Solution. Equation (15) has a regular singular point at $t=0$ since

$$tp(t)=\tfrac{3}{4} \quad \text{and} \quad t^2q(t)=\tfrac{3}{4}t$$

are both analytic at $t=0$. Set

$$y(t)=\sum_{n=0}^{\infty} a_n t^{n+r}, \quad a_0 \neq 0.$$

Computing

$$y'(t)=\sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1}$$

and

$$y''(t)=\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-2}$$

we see that

$$\begin{aligned} L[y]&=4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-1}\\ &\quad+3\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}+3\sum_{n=0}^{\infty}a_n t^{n+r}\\ &=\sum_{n=0}^{\infty}[4(n+r)(n+r-1)+3(n+r)]a_n t^{n+r-1}+\sum_{n=1}^{\infty}3a_{n-1}t^{n+r-1}. \end{aligned}$$

Setting the sum of coefficients of like powers of t equal to zero gives

$$4r(r-1)+3r=4r^2-r=r(4r-1)=0 \tag{16}$$

and

$$[4(n+r)(n+r-1)+3(n+r)]a_n \equiv (n+r)[4(n+r)-1]a_n=-3a_{n-1}, \quad n \geq 1. \tag{17}$$

Equation (16) is the indicial equation, and it implies that $r=0$ or $r=\frac{1}{4}$. Since these roots do not differ by an integer, we can find two solutions of (15) of the form

$$\sum_{n=0}^{\infty} a_n t^{n+r}$$

with a_n determined from (17).

$r=0$. In this case the recurrence relation (17) reduces to

$$a_n=-3\frac{a_{n-1}}{4n(n-1)+3n}=\frac{-3a_{n-1}}{n(4n-1)}.$$

Setting $a_0=1$ gives

$$a_1=-1,\quad a_2=\frac{-3a_1}{2\cdot 7}=3\frac{1}{2\cdot 7},$$

$$a_3=\frac{-3a_2}{3\cdot 11}=-3^2\frac{1}{2\cdot 3\cdot 7\cdot 11},$$

$$a_4=\frac{-3a_3}{4\cdot 15}=3^3\frac{1}{2\cdot 3\cdot 4\cdot 7\cdot 11\cdot 15},$$

and, in general,

$$a_n=\frac{(-1)^n 3^{n-1}}{n!7\cdot 11\cdot 15\cdots(4n-1)}.$$

Hence,

$$y_1(t)=\sum_{n=0}^{\infty}\frac{(-1)^n 3^{n-1}}{n!7\cdot 11\cdot 15\cdots(4n-1)}t^n \tag{18}$$

is one solution of (15). It is easily seen, using the Cauchy ratio test, that $y_1(t)$ converges for all t. Hence $y_1(t)$ is an analytic solution of (15).

$r=\frac{1}{4}$. In this case the recurrence relation (17) reduces to

$$a_n=\frac{-3a_{n-1}}{(n+\frac{1}{4})[4(n-\frac{3}{4})+3]}=\frac{-3a_{n-1}}{n(4n+1)},\quad n\geqslant 1.$$

Setting $a_0=1$ gives

$$a_1=\frac{-3}{5},\quad a_2=\frac{3^2}{2\cdot 5\cdot 9},\quad a_3=\frac{-3^3}{2\cdot 3\cdot 5\cdot 9\cdot 13},$$

$$a_4=\frac{3^4}{2\cdot 3\cdot 4\cdot 5\cdot 9\cdot 13\cdot 17},\ldots .$$

Proceeding inductively, we see that

$$a_n=\frac{(-1)^n 3^n}{n!5\cdot 9\cdot 13\cdots(4n+1)}.$$

Hence,

$$y_2(t)=t^{1/4}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{n!5\cdot 9\cdot 13\cdots(4n+1)}t^n$$

is a second solution of (15). It can easily be shown, using the Cauchy ratio test, that this solution converges for all positive t. Note, however, that $y_2(t)$ is not differentiable at $t=0$.

The method of Frobenius hits a snag in two separate instances. The first instance occurs when the indicial equation (13) has equal roots $r_1=r_2$. In this case we can only find one solution of (9) of the form

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty} a_n t^n.$$

In the next section we will prove that (9) has a second solution $y_2(t)$ of the form

$$y_2(t)=y_1(t)\ln t+t^{r_1}\sum_{n=0}^{\infty} b_n t^n$$

and show how to compute the coefficients b_n. The computation of the b_n is usually a very formidable problem. We wish to point out here, though, that in many physical applications the solution $y_2(t)$ is rejected on the grounds that it is singular. Thus, it often suffices to find $y_1(t)$ alone. It is also possible to find a second solution $y_2(t)$ by the method of reduction of order, but this too is usually very cumbersome.

The second snag in the method of Frobenius occurs when the roots r_1, r_2 of the indicial equation differ by a positive integer. Suppose that $r_1=r_2+N$, where N is a positive integer. In this case, we can find one solution of the form

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty} a_n t^n.$$

However, it may not be possible to find a second solution $y_2(t)$ of the form

$$y_2(t)=t^{r_2}\sum_{n=0}^{\infty} b_n t^n.$$

This is because $F(r_2+n)=0$ when $n=N$. Thus, the left hand side of (14) becomes

$$0\cdot a_N=-\sum_{k=0}^{N-1}\left[(k+r_2)p_{N-k}+q_{N-k}\right]a_k \tag{19}$$

when $n=N$. This equation cannot be satisfied for any choice of a_N, if

$$\sum_{k=0}^{N-1}\left[(k+r_2)p_{N-k}+q_{N-k}\right]a_k\neq 0.$$

In this case (see Section 2.8.3), Equation (9) has a second solution of the form

$$y_2(t)=y_1(t)\ln t+t^{r_2}\sum_{n=0}^{\infty} b_n t^n$$

where again, the computation of the b_n is a formidable problem.

On the other hand, if the sum on the right hand side of (19) vanishes, then a_N is arbitrary, and we can obtain a second solution of the form

$$y_2(t) = t^{r_2} \sum_{n=0}^{\infty} b_n t^n.$$

We illustrate this situation with the following example.

Example 6. Find two solutions of Bessel's equation of order $\frac{1}{2}$,

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - 1/4)y = 0, \quad 0 < t < \infty. \tag{20}$$

Solution. This equation has a regular singular point at $t=0$ since

$$tp(t) = 1 \quad \text{and} \quad t^2 q(t) = t^2 - \tfrac{1}{4}$$

are both analytic at $t=0$. Set

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}, \quad a_0 \neq 0.$$

Computing

$$y'(t) = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1}$$

and

$$y''(t) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}$$

we see that

$$\begin{aligned} L[y] &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n t^{n+r} \\ &\quad + \sum_{n=0}^{\infty} a_n t^{n+r+2} - \tfrac{1}{4} \sum_{n=0}^{\infty} a_n t^{n+r} \\ &= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + (n+r) - \tfrac{1}{4} \right] a_n t^{n+r} + \sum_{n=2}^{\infty} a_{n-2} t^{n+r}. \end{aligned}$$

Setting the sum of coefficients of like powers of t equal to zero gives

$$F(r)a_0 = \left[r(r-1) + r - \tfrac{1}{4} \right] a_0 = \left(r^2 - \tfrac{1}{4} \right) a_0 = 0 \tag{i}$$

$$F(1+r)a_1 = \left[(1+r)r + (1+r) - \tfrac{1}{4} \right] a_1 = \left[(1+r)^2 - \tfrac{1}{4} \right] a_1 = 0 \tag{ii}$$

and

$$F(n+r)a_n = \left[(n+r)^2 - \tfrac{1}{4} \right] a_n = -a_{n-2}, \quad n \geq 2 \tag{iii}$$

Equation (i) is the indicial equation, and it implies that $r_1 = \frac{1}{2}$, $r_2 = -\frac{1}{2}$.
$r_1 = \frac{1}{2}$: Set $a_0 = 1$. Equation (ii) forces a_1 to be zero, and the recurrence

relation (iii) implies that

$$a_n = \frac{-a_{n-2}}{F(n+\frac{1}{2})} = \frac{-a_{n-2}}{n(n+1)}, \quad n \geqslant 2.$$

This, in turn, implies that all the odd coefficients $a_3, a_5, \ldots,$ are zero, and the even coefficients are given by

$$a_2 = \frac{-a_0}{2\cdot 3} = \frac{-1}{2\cdot 3} = -\frac{1}{3!}$$

$$a_4 = \frac{-a_2}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4\cdot 5} = \frac{1}{5!}$$

$$a_6 = \frac{-a_4}{6\cdot 7} = \frac{-1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = -\frac{1}{7!}$$

and so on. Proceeding inductively, we see that

$$a_{2n} = \frac{(-1)^n}{(2n)!(2n+1)}$$

Hence

$$y_1(t) = t^{1/2}\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots\right)$$

is one solution of (20). This solution can be rewritten in the form

$$\begin{aligned} y_1(t) &= \frac{t^{1/2}}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \\ &= \frac{1}{\sqrt{t}} \sin t. \end{aligned}$$

$r_2 = -\frac{1}{2}$: Set $a_0 = 1$. Since $1 + r_2 = \frac{1}{2}$ is also a root of the indicial equation, we could, conceivably, run into trouble when trying to solve for a_1. However, Equation (ii) is automatically satisfied, regardless of the value of a_1. We will set $a_1 = 0$. (A nonzero value of a_1 will just reproduce a multiple of $y_1(t)$). The recurrence relation (iii) becomes

$$a_n = \frac{-a_{n-2}}{(n-\frac{1}{2})^2 - \frac{1}{4}} = \frac{-a_{n-2}}{n^2 - n} = \frac{-a_{n-2}}{n(n-1)}, \quad n \geqslant 2.$$

All the odd coefficients are again zero, and the even coefficients are

$$a_2 = \frac{-a_0}{2\cdot 1} = -\frac{1}{2!}$$

$$a_4 = \frac{-a_2}{4\cdot 3} = \frac{1}{4!}$$

$$a_6 = \frac{-a_4}{6\cdot 5} = -\frac{1}{6!}$$

and so on. Proceeding inductively, we see that

$$a_{2n} = \frac{(-1)^n}{(2n)!}.$$

Hence,

$$\begin{aligned} y_2(t) &= t^{-1/2}\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) \\ &= \frac{1}{\sqrt{t}}\cos t \end{aligned}$$

is a second solution of (20).

Remark 1. If r is a complex root of the indicial equation, then

$$y(t) = t^r \sum_{n=0}^{\infty} a_n t^n$$

is a complex-valued solution of (9). It is easily verified in this case that both the real and imaginary parts of $y(t)$ are real-valued solutions of (9).

Remark 2. We must set

$$y(t) = |t|^r \sum_{n=0}^{\infty} a_n t^n$$

if we want to solve (9) on an interval where t is negative. The proof is exactly analogous to the proof for the Euler equation in Section 2.8.1, and is left as an exercise for the reader.

We summarize the results of this section in the following theorem.

Theorem 8. *Consider the differential equation* (9) *where* $t=0$ *is a regular singular point. Then, the functions* $tp(t)$ *and* $t^2q(t)$ *are analytic at* $t=0$ *with power series expansions*

$$tp(t) = p_0 + p_1 t + p_2 t^2 + \cdots, \quad t^2 q(t) = q_0 + q_1 t + q_2 t^2 + \cdots$$

which converge for $|t| < \rho$. *Let* r_1 *and* r_2 *be the two roots of the indicial equation*

$$r(r-1) + p_0 r + q_0 = 0$$

with $r_1 \geqslant r_2$ *if they are real. Then, Equation* (9) *has two linearly independent solutions* $y_1(t)$ *and* $y_2(t)$ *on the interval* $0 < t < \rho$ *of the following form*:

(a) *If* $r_1 - r_2$ *is not a positive integer, then*

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n t^n, \quad y_2(t) = t^{r_2} \sum_{n=0}^{\infty} b_n t^n.$$

(b) *If* $r_1=r_2$, *then*

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty}a_nt^n,\quad y_2(t)=y_1(t)\ln t+t^{r_1}\sum_{n=0}^{\infty}b_nt^n.$$

(c) *If* $r_1-r_2=N$, *a positive integer, then*

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty}a_nt^n,\quad y_2(t)=ay_1(t)\ln t+t^{r_2}\sum_{n=0}^{\infty}b_nt^n$$

where the constant a may turn out to be zero.

Exercises

In each of Problems 1–6, determine whether the specified value of t is a regular singular point of the given differential equation.

1. $t(t-2)^2y''+ty'+y=0;\ t=0$ **2.** $t(t-2)^2y''+ty'+y=0;\ t=2$

3. $(\sin t)y''+(\cos t)y'+\frac{1}{t}y=0;\ t=0$ **4.** $(e^t-1)y''+e^ty'+y=0;\ t=0$

5. $(1-t^2)y''+\dfrac{1}{\sin(t+1)}y'+y=0;\ t=-1$

6. $t^3y''+(\sin t^2)y'+ty=0;\ t=0$

Find the general solution of each of the following equations.

7. $2t^2y''+3ty'-(1+t)y=0$ **8.** $2ty''+(1-2t)y'-y=0$

9. $2ty''+(1+t)y'-2y=0$ **10.** $2t^2y''-ty'+(1+t)y=0$

11. $4ty''+3y'-3y=0$ **12.** $2t^2y''+(t^2-t)y'+y=0$

In each of Problems 13–18, find two independent solutions of the given equation. In each problem, the roots of the indicial equation differ by a positive integer, but two solutions exist of the form $t^r\sum_{n=0}^{\infty}a_nt^n$.

13. $t^2y''-ty'-(t^2+\frac{5}{4})y=0$ **14.** $t^2y''+(t-t^2)y'-y=0$

15. $ty''-(t^2+2)y'+ty=0$ **16.** $t^2y''+(3t-t^2)y'-ty=0$

17. $t^2y''+t(t+1)y'-y=0$ **18.** $ty''-(4+t)y'+2y=0$

19. Consider the equation

$$t^2y''+(t^2-3t)y'+3y=0 \qquad (*)$$

(a) Show that $r=1$ and $r=3$ are the two roots of the indicial equation of $(*)$.
(b) Find a power series solution of $(*)$ of the form

$$y_1(t)=t^3\sum_{n=0}^{\infty}a_nt^n,\quad a_0=1.$$

(c) Show that $y_1(t)=t^3e^{-t}$.
(d) Show that $(*)$ has no solution of the form

$$t\sum_{n=0}^{\infty} b_n t^n.$$

(e) Find a second solution of $(*)$ using the method of reduction of order. Leave your answer in integral form.

20. Consider the equation

$$t^2y''+ty'-(1+t)y=0.$$

(a) Show that $r=-1$ and $r=1$ are the two roots of the indicial equation.
(b) Find one solution of the form

$$y_1(t)=t\sum_{n=0}^{\infty} a_n t^n.$$

(c) Find a second solution using the method of reduction of order.

21. Consider the equation

$$ty''+ty'+2y=0.$$

(a) Show that $r=0$ and $r=1$ are the two roots of the indicial equation.
(b) Find one solution of the form

$$y_1(t)=t\sum_{n=0}^{\infty} a_n t^n.$$

(c) Find a second solution using the method of reduction of order.

22. Consider the equation

$$ty''+(1-t^2)y'+4ty=0.$$

(a) Show that $r=0$ is a double root of the indicial equation.
(b) Find one solution of the form $y_1(t)=\Sigma_{n=0}^{\infty} a_n t^n$.
(c) Find a second solution using the method of reduction of order.

23. Consider the Bessel equation of order zero

$$t^2y''+ty'+t^2y=0.$$

(a) Show that $r=0$ is a double root of the indicial equation.
(b) Find one solution of the form

$$y_1(t)=1-\frac{t^2}{2^2}+\frac{t^4}{2^2\cdot 4^2}-\frac{t^6}{2^2\cdot 4^2\cdot 6^2}+\cdots.$$

This solution is known as $J_0(t)$.
(c) Find a second solution using the method of reduction of order.

24. Consider the Bessel equation of order ν

$$t^2y''+ty'+(t^2-\nu^2)y=0$$

where ν is real and positive.
(a) Find a power series solution

$$J_\nu(t)=\frac{t^\nu}{2^\nu\nu!}\sum_{n=0}^{\infty}a_nt^n,\quad a_0=1.$$

This function $J_\nu(t)$ is called the Bessel function of order ν.
(b) Find a second solution if 2ν is not an integer.

25. The differential equation

$$ty''+(1-t)y'+\lambda y=0,\quad \lambda \text{ constant},$$

is called the Laguerre differential equation.
(a) Show that the indicial equation is $r^2=0$.
(b) Find a solution $y(t)$ of the Laguerre equation of the form $\Sigma_{n=0}^{\infty}a_nt^n$.
(c) Show that this solution reduces to a polynomial if $\lambda=n$.

26. The differential equation

$$t(1-t)y''+[\gamma-(1+\alpha+\beta)t]y'-\alpha\beta y=0$$

where α, β, and γ are constants, is known as the hypergeometric equation.
(a) Show that $t=0$ is a regular singular point and that the roots of the indicial equation are 0 and $1-\gamma$.
(b) Show that $t=1$ is also a regular singular point, and that the roots of the indicial equation are now 0 and $\gamma-\alpha-\beta$.
(c) Assume that γ is not an integer. Find two solutions $y_1(t)$ and $y_2(t)$ of the hypergeometric equation of the form

$$y_1(t)=\sum_{n=0}^{\infty}a_nt^n,\quad y_2(t)=t^{1-\gamma}\sum_{n=0}^{\infty}b_nt^n.$$

27. (a) Show that the equation

$$2(\sin t)y''+(1-t)y'-2y=0$$

has two solutions of the form

$$y_1(t)=\sum_{n=0}^{\infty}a_nt^n,\quad y_2(t)=t^{1/2}\sum_{n=0}^{\infty}b_nt^n.$$

(b) Find the first 5 terms in these series expansions assuming that $a_0=b_0=1$.

28. Let $y(t)=u(t)+iv(t)$ be a complex-valued solution of (3) with $p(t)$ and $q(t)$ real. Show that both $u(t)$ and $v(t)$ are real-valued solutions of (3).

29. (a) Show that the indicial equation of

$$t^2y''+ty'+(1+t)y=0 \qquad (*)$$

has complex roots $r=\pm i$.

(b) Show that $(*)$ has 2 linearly independent solutions $y(t)$ of the form

$$y(t)=\sin(\ln t)\sum_{n=0}^{\infty}a_nt^n+\cos(\ln t)\sum_{n=0}^{\infty}b_nt^n.$$

2.8.3 Equal roots, and roots differing by an integer

Equal roots.
We run into trouble if the indicial equation has equal roots $r_1=r_2$ because then the differential equation

$$P(t)\frac{d^2y}{dt^2}+Q(t)\frac{dy}{dt}+R(t)y=0 \tag{1}$$

has only one solution of the form

$$y(t)=t^r\sum_{n=0}^{\infty}a_nt^n. \tag{2}$$

The method of finding a second solution is very similar to the method used in finding a second solution of Euler's equation, in the case of equal roots. Let us rewrite (2) in the form

$$y(t)=y(t,r)=t^r\sum_{n=0}^{\infty}a_n(r)t^n$$

to emphasize that the solution $y(t)$ depends on our choice of r. Then (see Section 2.8.2)

$$\begin{aligned}L[y](t,r)&=a_0F(r)t^r\\&\quad+\sum_{n=1}^{\infty}\left\{a_n(r)F(n+r)+\sum_{k=0}^{n-1}\left[(k+r)p_{n-k}+q_{n-k}\right]a_k\right\}t^{n+r}.\end{aligned}$$

We now think of r as a continuous variable and determine a_n as a function of r by requiring that the coefficient of t^{n+r} be zero for $n\geqslant 1$. Thus

$$a_n(r)=\frac{-\sum_{k=0}^{n-1}\left[(k+r)p_{n-k}+q_{n-k}\right]a_k}{F(n+r)}$$

With this choice of $a_n(r)$, we see that

$$L[y](t,r)=a_0F(r)t^r. \tag{3}$$

In the case of equal roots, $F(r)=(r-r_1)^2$, so that (3) can be written in the form

$$L[y](t,r)=a_0(r-r_1)^2t^r.$$

Since $L[y](t,r_1)=0$, we obtain one solution

$$y_1(t)=t^{r_1}\left[a_0+\sum_{n=1}^{\infty}a_n(r_1)t^n\right].$$

Observe now, that

$$\begin{aligned}\frac{\partial}{\partial r}L[y](t,r) &= L\left[\frac{\partial y}{\partial r}\right](t,r)\\ &= \frac{\partial}{\partial r}a_0(r-r_1)^2t^r\\ &= 2a_0(r-r_1)t^r + a_0(r-r_1)^2(\ln t)t^r\end{aligned}$$

also vanishes when $r=r_1$. Thus

$$\begin{aligned}y_2(t) &= \frac{\partial}{\partial r}y_1(t,r)|_{r=r_1}\\ &= \frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} a_n(r)t^{n+r}\right]_{r=r_1}\\ &= \sum_{n=0}^{\infty}[a_n(r_1)t^{n+r_1}]\ln t + \sum_{n=0}^{\infty} a_n'(r_1)t^{n+r_1}\\ &= y_1(t)\ln t + \sum_{n=0}^{\infty} a_n'(r_1)t^{n+r_1}\end{aligned}$$

is a second solution of (1).

Example 1. Find two solutions of Bessel's equation of order zero

$$L[y] = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + t^2y = 0, \quad t>0. \tag{4}$$

Solution. Set

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}.$$

Computing

$$y'(t) = \sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}$$

and

$$y''(t) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2}$$

we see that

$$\begin{aligned}L[y] &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n t^{n+r} + \sum_{n=0}^{\infty}a_n t^{n+r+2}\\ &= \sum_{n=0}^{\infty}(n+r)^2a_n t^{n+r} + \sum_{n=2}^{\infty}a_{n-2}t^{n+r}.\end{aligned}$$

Setting the sums of like powers of t equal to zero gives

(i) $r^2a_0 = F(r)a_0 = 0$
(ii) $(1+r)^2a_1 = F(1+r)a_1 = 0$
and
(iii) $(n+r)^2a_n = F(n+r)a_n = -a_{n-2},\ n \geqslant 2.$

Equation (i) is the indicial equation, and it has equal roots $r_1 = r_2 = 0$. Equation (ii) forces a_1 to be zero, and the recurrence relation (iii) says that

$$a_n = \frac{-a_{n-2}}{(n+r)^2}.$$

Clearly, $a_3 = a_5 = a_7 = \cdots = 0$. The even coefficients are given by

$$a_2(r) = \frac{-a_0}{(2+r)^2} = \frac{-1}{(2+r)^2}$$

$$a_4(r) = \frac{-a_2}{(4+r)^2} = \frac{1}{(2+r)^2(4+r)^2}$$

and so on. Proceeding inductively, we see that

$$a_{2n}(r) = \frac{(-1)^n}{(2+r)^2(4+r)^2\cdots(2n+r)^2}.$$

To determine $y_1(t)$, we set $r=0$. Then

$$a_2(0) = \frac{-1}{2^2}$$

$$a_4(0) = \frac{1}{2^2\cdot 4^2} = \frac{1}{2^4}\frac{1}{(2!)^2}$$

$$a_6(0) = \frac{-1}{2^2\cdot 4^2\cdot 6^2} = \frac{-1}{2^6(3!)^2}$$

and in general

$$a_{2n}(0) = \frac{(-1)^n}{2^2\cdot 4^2\cdots(2n)^2} = \frac{(-1)^n}{2^{2n}(n!)^2}.$$

Hence,

$$y_1(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^4\cdot(2!)^2} - \frac{t^6}{2^6(3!)^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$$

is one solution of (4). This solution is often referred to as the Bessel function of the first kind of order zero, and is denoted by $J_0(t)$.

To obtain a second solution of (4) we set

$$y_2(t)=y_1(t)\ln t+\sum_{n=0}^{\infty} a_{2n}'(0)t^{2n}.$$

To compute $a_{2n}'(0)$, observe that

$$\begin{aligned}\frac{a_{2n}'(r)}{a_{2n}(r)}&=\frac{d}{dr}\ln|a_{2n}(r)|=\frac{d}{dr}\ln(2+r)^{-2}\cdots(2n+r)^{-2}\\&=-2\frac{d}{dr}[\ln(2+r)+\ln(4+r)+\cdots+\ln(2n+r)]\\&=-2\left(\frac{1}{2+r}+\frac{1}{4+r}+\cdots+\frac{1}{2n+r}\right).\end{aligned}$$

Hence,

$$\begin{aligned}a_{2n}'(0)&=-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)a_{2n}(0)\\&=-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)a_{2n}(0).\end{aligned}$$

Setting

$$H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \tag{5}$$

we see that

$$a_{2n}'(0)=\frac{-H_n(-1)^n}{2^{2n}(n!)^2}=\frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}$$

and thus

$$y_2(t)=y_1(t)\ln t+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}t^{2n}$$

is a second solution of (4) with H_n given by (5).

Roots differing by a positive integer. Suppose that r_2 and $r_1=r_2+N$, N a positive integer, are the roots of the indicial equation. Then we can certainly find one solution of (1) of the form

$$y_1(t)=t^{r_1}\sum_{n=0}^{\infty}a_n(r_1)t^n.$$

As we mentioned previously, it may not be possible to find a second solution of the form

$$t^{r_2}\sum_{n=0}^{\infty}b_nt^n.$$

In this case, Equation (1) will have a second solution of the form

$$y_2(t) = \frac{\partial}{\partial r} y(t,r)\Big|_{r=r_2}$$
$$= ay_1(t)\ln t + \sum_{n=0}^{\infty} a_n'(r_2)t^{n+r_2}$$

where a is a constant, and

$$y(t,r) = t^r \sum_{n=0}^{\infty} a_n(r)t^n$$

with

$$a_0 = a_0(r) = r - r_2.$$

The proof of this result can be found in more advanced books on differential equations. In Exercise 5, we develop a simple proof, using the method of reduction of order, to show why a logarithm term will be present.

Remark. It is usually very difficult, and quite cumbersome, to obtain the second solution $y_2(t)$ when a logarithm term is present. Beginning and intermediate students are not expected, usually, to perform such calculations. We have included several exercises for the more industrious students. In these problems, and in similar problems which occur in applications, it is often more than sufficient to find just the first few terms in the series expansion of $y_2(t)$, and this can usually be accomplished using the method of reduction of order.

Exercises

In Problems 1 and 2, show that the roots of the indicial equation are equal, and find two independent solutions of the given equation.

1. $ty'' + y' - 4y = 0$

2. $t^2y'' - t(1+t)y' + y = 0$

3. (a) Show that $r = -1$ and $r = 1$ are the roots of the indicial equation for Bessel's equation of order one

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-1)y = 0.$$

(b) Find a solution:

$$J_1(t) = \tfrac{1}{2}t \sum_{n=0}^{\infty} a_n t^n, \; a_0 = 1.$$

$J_1(t)$ is called the Bessel function of order one.

(c) Find a second solution:

$$y_2(t) = -J_1(t)\ln t + \frac{1}{t}\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^n(H_n + H_{n-1})}{2^{2n}n!(n-1)!}t^{2n}\right].$$

4. Consider the equation

$$ty''+3y'-3y=0, \qquad t>0.$$

(a) Show that $r=0$ and $r=-2$ are the roots of the indicial equation.
(b) Find a solution

$$y_1(t)=\sum_{n=0}^{\infty} a_n t^n.$$

(c) Find a second solution

$$y_2(t)=y_1(t)\ln t+\frac{1}{t^2}-\frac{1}{t}+\frac{1}{4}+\frac{11}{36}t+\frac{31}{576}t^2+\cdots.$$

5. This exercise gives an alternate proof of some of the results of this section, using the method of reduction of order.

(a) Let $t=0$ be a regular singular point of the equation

$$t^2y''+tp(t)y'+q(t)y=0 \tag{i}$$

Show that the substitution $y=t^r z$ reduces (i) to the equation

$$t^2z''+[2r+p(t)]tz'+[r(r-1)+rp(t)+q(t)]z=0. \tag{ii}$$

(b) Let r be a root of the indicial equation. Show that (ii) has an analytic solution $z_1(t)=\sum_{n=0}^{\infty}a_nt^n$.

(c) Set $z_2(t)=z_1(t)v(t)$. Show that

$$v(t)=\int u(t)\,dt, \qquad \text{where } u(t)=\frac{e^{-\int[2r+p(t)]/t\,dt}}{z_1^2(t)}.$$

(d) Suppose that $r=r_0$ is a double root of the indicial equation. Show that $2r_0+p_0=1$, and conclude therefore that

$$u(t)=\frac{u_0}{t}+u_1+u_2t+\cdots.$$

(e) Use the result in (d) to show that $y_2(t)$ has an $\ln t$ term in the case of equal roots.

(f) Suppose the roots of the indicial equation are r_0 and r_0-N, N a positive integer. Show that $2r_0+p_0=1+N$, and conclude therefore, that

$$u(t)=\frac{1}{t^{1+N}}\hat{u}(t)$$

where $\hat{u}(t)$ is analytic at $t=0$.

(g) Use the result in (f) to show that $y_2(t)$ has an $\ln t$ term if the coefficient of t^N in the expansion of $\hat{u}(t)$ is nonzero. Show, in addition, that if this coefficient is zero, then

$$v(t)=\frac{v_{-N}}{t^N}+\cdots+\frac{v_{-1}}{t}+v_1t+v_2t^2+\cdots$$

and $y_2(t)$ has no $\ln t$ term.

2.9 The method of Laplace transforms

In this section we describe a very different and extremely clever way of solving the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t); \qquad y(0)=y_0, \quad y'(0)=y'_0 \tag{1}$$

where a, b and c are constants. This method, which is known as the method of Laplace transforms, is especially useful in two cases which arise quite often in applications. The first case is when $f(t)$ is a discontinuous function of time. The second case is when $f(t)$ is zero except for a very short time interval in which it is very large.

To put the method of Laplace transforms into proper perspective, we consider the following hypothetical situation. Suppose that we want to multiply the numbers 3.163 and 16.38 together, but that we have forgotten completely how to multiply. We only remember how to add. Being good mathematicians, we ask ourselves the following question.

Question: Is it possible to reduce the problem of multiplying the two numbers 3.163 and 16.38 together to the simpler problem of adding two numbers together?

The answer to this question, of course, is yes, and is obtained as follows. First, we consult our logarithm tables and find that $\ln 3.163 = 1.15152094$, and $\ln 16.38 = 2.79606108$. Then, we add these two numbers together to yield 3.94758202. Finally, we consult our anti-logarithm tables and find that $3.94758202 = \ln 51.80994$. Hence, we conclude that $3.163 \times 16.38 = 51.80994$.

The key point in this analysis is that the operation of multiplication is replaced by the simpler operation of addition when we work with the logarithms of numbers, rather than with the numbers themselves. We represent this schematically in Table 1. In the method to be discussed below, the unknown function $y(t)$ will be replaced by a new function $Y(s)$, known as the Laplace transform of $y(t)$. This association will have the property that $y'(t)$ will be replaced by $sY(s)-y(0)$. Thus, the operation of differentiation with respect to t will be replaced, essentially, by the operation of multiplication with respect to s. In this manner, we will replace the initial-value problem (1) by an algebraic equation which can be solved explicitly for $Y(s)$. Once we know $Y(s)$, we can consult our "anti-Laplace transform" tables and recover $y(t)$.

Table 1

a	$\longrightarrow$	$\ln a$
b	$\longrightarrow$	$\ln b$
$a \cdot b$	$\longrightarrow$	$\ln a + \ln b$

We begin with the definition of the Laplace transform.

Definition. Let $f(t)$ be defined for $0 \leqslant t < \infty$. The Laplace transform of $f(t)$, which is denoted by $F(s)$, or $\mathcal{L}\{f(t)\}$, is given by the formula

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \tag{2}$$

where

$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\,dt.$$

Example 1. Compute the Laplace transform of the function $f(t) = 1$.
Solution. From (2),

$$\mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st}\,dt = \lim_{A\to\infty} \frac{1-e^{-sA}}{s}$$
$$= \begin{cases} \dfrac{1}{s}, & s > 0 \\ \infty, & s \leqslant 0 \end{cases}.$$

Example 2. Compute the Laplace transform of the function e^{at}.
Solution. From (2),

$$\mathcal{L}\{e^{at}\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{at}\,dt = \lim_{A\to\infty} \frac{e^{(a-s)A} - 1}{a-s}$$
$$= \begin{cases} \dfrac{1}{s-a}, & s > a \\ \infty, & s \leqslant a \end{cases}.$$

Example 3. Compute the Laplace transform of the functions $\cos\omega t$ and $\sin\omega t$.
Solution. From (2),

$$\mathcal{L}\{\cos\omega t\} = \int_0^\infty e^{-st}\cos\omega t\,dt \quad \text{and} \quad \mathcal{L}\{\sin\omega t\} = \int_0^\infty e^{-st}\sin\omega t\,dt.$$

Now, observe that

$$\mathcal{L}\{\cos\omega t\} + i\mathcal{L}\{\sin\omega t\} = \int_0^\infty e^{-st} e^{i\omega t}\,dt = \lim_{A\to\infty} \int_0^A e^{(i\omega - s)t}\,dt$$
$$= \lim_{A\to\infty} \frac{e^{(i\omega - s)A} - 1}{i\omega - s}$$
$$= \begin{cases} \dfrac{1}{s - i\omega} = \dfrac{s + i\omega}{s^2 + \omega^2}, & s > 0 \\ \text{undefined}, & s \leqslant 0 \end{cases}.$$

Equating real and imaginary parts in this equation gives

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2} \quad \text{and} \quad \mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}, \qquad s>0.$$

Equation (2) associates with every function $f(t)$ a new function, which we call $F(s)$. As the notation $\mathcal{L}\{f(t)\}$ suggests, the Laplace transform is an operator acting on functions. It is also a linear operator, since

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]\,dt \\ &= c_1\int_0^\infty e^{-st} f_1(t)\,dt + c_2\int_0^\infty e^{-st} f_2(t)\,dt \\ &= c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.\end{aligned}$$

It is to be noted, though, that whereas $f(t)$ is defined for $0 \leqslant t < \infty$, its Laplace transform is usually defined in a different interval. For example, the Laplace transform of e^{2t} is only defined for $2 < s < \infty$, and the Laplace transform of e^{8t} is only defined for $8 < s < \infty$. This is because the integral (2) will only exist, in general, if s is sufficiently large.

One very serious difficulty with the definition (2) is that this integral may fail to exist for every value of s. This is the case, for example, if $f(t) = e^{t^2}$ (see Exercise 13). To guarantee that the Laplace transform of $f(t)$ exists at least in some interval $s > s_0$, we impose the following conditions on $f(t)$.

(i) The function $f(t)$ is piecewise continuous. This means that $f(t)$ has at most a finite number of discontinuities on any interval $0 \leqslant t \leqslant A$, and both the limit from the right and the limit from the left of f exist at every point of discontinuity. In other words, $f(t)$ has only a finite number of "jump discontinuities" in any finite interval. The graph of a typical piecewise continuous function $f(t)$ is described in Figure 1.

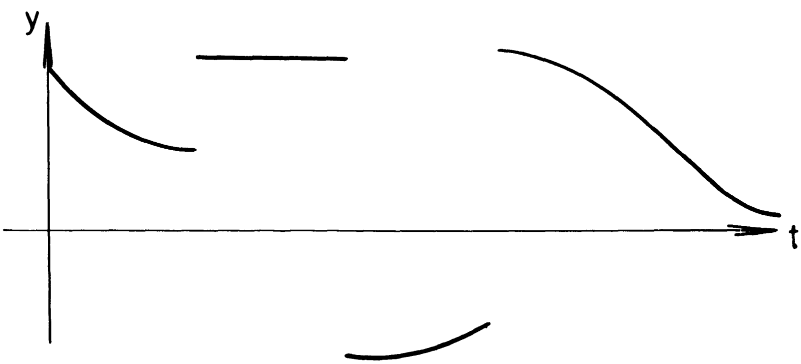

Figure 1. Graph of a typical piecewise continuous function

(ii) The function $f(t)$ is of exponential order, that is, there exist constants M and c such that

$$|f(t)| \leqslant Me^{ct}, \qquad 0 \leqslant t < \infty.$$

Lemma 1. *If $f(t)$ is piecewise continuous and of exponential order, then its Laplace transform exists for all s sufficiently large. Specifically, if $f(t)$ is piecewise continuous, and $|f(t)| \leqslant Me^{ct}$, then $F(s)$ exists for $s > c$.*

We prove Lemma 1 with the aid of the following lemma from integral calculus, which we quote without proof.

Lemma 2. *Let $g(t)$ be piecewise continuous. Then, the improper integral $\int_0^\infty g(t)\,dt$ exists if $\int_0^\infty |g(t)|\,dt$ exists. To prove that this latter integral exists, it suffices to show that there exists a constant K such that*

$$\int_0^A |g(t)|\,dt \leqslant K$$

for all A.

Remark. Notice the similarity of Lemma 2 with the theorem of infinite series (see Appendix B) which states that the infinite series $\sum a_n$ converges if $\sum |a_n|$ converges, and that $\sum |a_n|$ converges if there exists a constant K such that $|a_1| + \dots + |a_n| \leqslant K$ for all n.

We are now in a position to prove Lemma 1.

PROOF OF LEMMA 1. Since $f(t)$ is piecewise continuous, the integral $\int_0^A e^{-st} f(t)\,dt$ exists for all A. To prove that this integral has a limit for all s sufficiently large, observe that

$$\begin{aligned}\int_0^A |e^{-st} f(t)|\,dt &\leqslant M\int_0^A e^{-st}e^{ct}\,dt \\ &= \frac{M}{c-s}\left[e^{(c-s)A} - 1\right] \leqslant \frac{M}{s-c}\end{aligned}$$

for $s > c$. Consequently, by Lemma 2, the Laplace transform of $f(t)$ exists for $s > c$. Thus, from here on, we tacitly assume that $|f(t)| \leqslant Me^{ct}$, and $s > c$. □

The real usefulness of the Laplace transform in solving differential equations lies in the fact that the Laplace transform of $f'(t)$ is very closely related to the Laplace transform of $f(t)$. This is the content of the following important lemma.

Lemma 3. *Let* $F(s)=\mathcal{L}\{f(t)\}$. *Then*

$$\mathcal{L}\{f'(t)\}=s\mathcal{L}\{f(t)\}-f(0)=sF(s)-f(0).$$

Proof. The proof of Lemma 3 is very elementary; we just write down the formula for the Laplace transform of $f'(t)$ and integrate by parts. To wit,

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \lim_{A\to\infty}\int_0^A e^{-st}f'(t)\,dt\\ &= \lim_{A\to\infty} e^{-st}f(t)\Big|_0^A+\lim_{A\to\infty} s\int_0^A e^{-st}f(t)\,dt\\ &= -f(0)+s\lim_{A\to\infty}\int_0^A e^{-st}f(t),dt\\ &= -f(0)+sF(s).\end{aligned}$$ □

Our next step is to relate the Laplace transform of $f''(t)$ to the Laplace transform of $f(t)$. This is the content of Lemma 4.

Lemma 4. *Let* $F(s)=\mathcal{L}\{f(t)\}$. *Then,*

$$\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0).$$

Proof. Using Lemma 3 twice, we see that

$$\begin{aligned}\mathcal{L}\{f''(t)\}&=s\mathcal{L}\{f'(t)\}-f'(0)\\ &=s\left[sF(s)-f(0)\right]-f'(0)\\ &=s^2F(s)-sf(0)-f'(0).\end{aligned}$$ □

We have now developed all the machinery necessary to reduce the problem of solving the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t);\qquad y(0)=y_0,\quad y'(0)=y_0' \tag{3}$$

to that of solving an algebraic equation. Let $Y(s)$ and $F(s)$ be the Laplace transforms of $y(t)$ and $f(t)$ respectively. Taking Laplace transforms of both sides of the differential equation gives

$$\mathcal{L}\{ay''(t)+by'(t)+cy(t)\}=F(s).$$

By the linearity of the Laplace transform operator,

$$\mathcal{L}\{ay''(t)+by'(t)+cy(t)\}=a\mathcal{L}\{y''(t)\}+b\mathcal{L}\{y'(t)\}+c\mathcal{L}\{y(t)\},$$

and from Lemmas 3 and 4

$$\mathcal{L}\{y'(t)\}=sY(s)-y_0, \qquad \mathcal{L}\{y''(t)\}=s^2Y(s)-sy_0-y_0'.$$

Hence,

$$a\left[s^2Y(s)-sy_0-y_0'\right]+b\left[sY(s)-y_0\right]+cY(s)=F(s)$$

and this *algebraic equation* implies that

$$Y(s)=\frac{(as+b)y_0}{as^2+bs+c}+\frac{ay_0'}{as^2+bs+c}+\frac{F(s)}{as^2+bs+c}. \tag{4}$$

Equation (4) tells us the Laplace transform of the solution $y(t)$ of (3). To find $y(t)$, we must consult our anti, or inverse, Laplace transform tables. Now, just as $Y(s)$ is expressed explicitly in terms of $y(t)$; that is, $Y(s) = \int_0^\infty e^{-st}y(t)\,dt$, we can write down an explicit formula for $y(t)$. However, this formula, which is written symbolically as $y(t)=\mathcal{L}^{-1}\{Y(s)\}$, involves an integration with respect to a complex variable, and this is beyond the scope of this book. Therefore, instead of using this formula, we will derive several elegant properties of the Laplace transform operator in the next section. These properties will enable us to invert many Laplace transforms by inspection; that is, by recognizing "which functions they are the Laplace transform of".

Example 4. Solve the initial-value problem

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{3t}; \qquad y(0)=1, \quad y'(0)=0.$$

Solution. Let $Y(s)=\mathcal{L}\{y(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$s^2Y(s)-s-3\left[sY(s)-1\right]+2Y(s)=\frac{1}{s-3}$$

and this implies that

$$\begin{aligned} Y(s)&=\frac{1}{(s-3)(s^2-3s+2)}+\frac{s-3}{s^2-3s+2}\\ &=\frac{1}{(s-1)(s-2)(s-3)}+\frac{s-3}{(s-1)(s-2)}. \end{aligned} \tag{5}$$

To find $y(t)$, we expand each term on the right-hand side of (5) in partial fractions. Thus, we write

$$\frac{1}{(s-1)(s-2)(s-3)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}.$$

This implies that

$$A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)=1. \tag{6}$$

Setting $s=1$ in (6) gives $A=\frac{1}{2}$; setting $s=2$ gives $B=-1$; and setting $s=3$ gives $C=\frac{1}{2}$. Hence,

$$\frac{1}{(s-1)(s-2)(s-3)}=\frac{1}{2}\frac{1}{s-1}-\frac{1}{s-2}+\frac{1}{2}\frac{1}{s-3}.$$

Similarly, we write

$$\frac{s-3}{(s-1)(s-2)}=\frac{D}{s-1}+\frac{E}{s-2}$$

and this implies that

$$D(s-2)+E(s-1)=s-3. \tag{7}$$

Setting $s=1$ in (7) gives $D=2$, while setting $s=2$ gives $E=-1$. Hence,

$$\begin{aligned}Y(s)&=\frac{1}{2}\frac{1}{s-1}-\frac{1}{s-2}+\frac{1}{2}\frac{1}{s-3}+\frac{2}{s-1}-\frac{1}{s-2}\\&=\frac{5}{2}\frac{1}{s-1}-\frac{2}{s-2}+\frac{1}{2}\frac{1}{s-3}.\end{aligned}$$

Now, we recognize the first term as being the Laplace transform of $\frac{5}{2}e^{t}$. Similarly, we recognize the second and third terms as being the Laplace transforms of $-2e^{2t}$ and $\frac{1}{2}e^{3t}$, respectively. Therefore,

$$Y(s)=\mathcal{L}\left\{\tfrac{5}{2}e^{t}-2e^{2t}+\tfrac{1}{2}e^{3t}\right\}$$

so that

$$y(t)=\tfrac{5}{2}e^{t}-2e^{2t}+\tfrac{1}{2}e^{3t}.$$

Remark. We have cheated a little bit in this problem because there are actually infinitely many functions whose Laplace transform is a given function. For example, the Laplace transform of the function

$$z(t)=\begin{cases}\frac{5}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}, & t\neq 1, 2, \text{ and } 3\\ 0, & t=1,2,3\end{cases}$$

is also $Y(s)$, since $z(t)$ differs from $y(t)$ at only three points.* However, there is only one *continuous* function $y(t)$ whose Laplace transform is a given function $Y(s)$, and it is in this sense that we write $y(t)=\mathcal{L}^{-1}\{Y(s)\}$.

We wish to emphasize that Example 4 is just by way of illustrating the method of Laplace transforms for solving initial-value problems. The best way of solving this particular initial-value problem is by the method of

*If $f(t)=g(t)$ except at a finite number of points, then $\int_a^b f(t)\,dt=\int_a^b g(t)\,dt$.

judicious guessing. However, even though it is longer to solve this particular initial-value problem by the method of Laplace transforms, there is still something "nice and satisfying" about this method. If we had done this problem by the method of judicious guessing, we would have first computed a particular solution $\psi(t)=\frac{1}{2}e^{3t}$. Then, we would have found two independent solutions e^t and e^{2t} of the homogeneous equation, and we would have written

$$y(t)=c_1e^t+c_2e^{2t}+\tfrac{1}{2}e^{3t}$$

as the general solution of the differential equation. Finally, we would have computed $c_1=\frac{5}{2}$ and $c_2=-2$ from the initial conditions. What is unsatisfying about this method is that we first had to find *all* the solutions of the differential equation before we could find the specific solution $y(t)$ which we were interested in. The method of Laplace transforms, on the other hand, enables us to find $y(t)$ directly, without first finding all solutions of the differential equation.

Exercises

Determine the Laplace transform of each of the following functions.

1. t

2. t^n

3. $e^{at}\cos bt$

4. $e^{at}\sin bt$

5. $\cos^2 at$

6. $\sin^2 at$

7. $\sin at\cos bt$

8. $t^2\sin t$

9. Given that $\int_0^\infty e^{-x^2}dx=\sqrt{\pi}\,/2$, find $\mathcal{L}\{t^{-1/2}\}$. *Hint*: Make the change of variable $u=\sqrt{t}$ in (2).

Show that each of the following functions are of exponential order.

10. t^n

11. $\sin at$

12. $e^{\sqrt{t}}$

13. Show that e^{t^2} does not possess a Laplace transform. *Hint*: Show that $e^{t^2-st}>e^t$ for $t>s+1$.

14. Suppose that $f(t)$ is of exponential order. Show that $F(s)=\mathcal{L}\{f(t)\}$ approaches 0 as $s\to\infty$.

Solve each of the following initial-value problems.

15. $y''-5y'+4y=e^{2t}$; $y(0)=1,\ y'(0)=-1$

16. $2y''+y'-y=e^{3t}$; $y(0)=2,\ y'(0)=0$

Find the Laplace transform of the solution of each of the following initial-value problems.

17. $y''+2y'+y=e^{-t}$; $y(0)=1,\ y'(0)=3$

18. $y''+y=t^2\sin t;\quad y(0)=y'(0)=0$

19. $y''+3y'+7y=\cos t;\quad y(0)=0, y'(0)=2$

20. $y''+y'+y=t^3;\quad y(0)=2, y'(0)=0$

21. Prove that all solutions $y(t)$ of $ay''+by'+cy=f(t)$ are of exponential order if $f(t)$ is of exponential order. *Hint*: Show that all solutions of the homogeneous equation are of exponential order. Obtain a particular solution using the method of variation of parameters, and show that it, too, is of exponential order.

22. Let $F(s)=\mathcal{L}\{f(t)\}$. Prove that

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\}=s^nF(s)-s^{n-1}f(0)-\dots-\frac{df^{(n-1)}(0)}{dt^{n-1}}.$$

Hint: Try induction.

23. Solve the initial-value problem

$$y'''-6y''+11y'-6y=e^{4t};\qquad y(0)=y'(0)=y''(0)=0$$

24. Solve the initial-value problem

$$y''-3y'+2y=e^{-t};\qquad y(t_0)=1,\quad y'(t_0)=0$$

by the method of Laplace transforms. *Hint*: Let $\phi(t)=y(t+t_0)$.

2.10 Some useful properties of Laplace transforms

In this section we derive several important properties of Laplace transforms. Using these properties, we will be able to compute the Laplace transform of most functions without performing tedious integrations, and to invert many Laplace transforms by inspection.

Property 1. If $\mathcal{L}\{f(t)\}=F(s)$, then

$$\mathcal{L}\{-tf(t)\}=\frac{d}{ds}F(s).$$

PROOF. By definition, $F(s)=\int_0^\infty e^{-st}f(t)\,dt$. Differentiating both sides of this equation with respect to s gives

$$\begin{aligned}\frac{d}{ds}F(s)&=\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt\\&=\int_0^\infty\frac{\partial}{\partial s}(e^{-st})f(t)\,dt=\int_0^\infty -te^{-st}f(t)\,dt\\&=\mathcal{L}\{-tf(t)\}.\end{aligned}$$

□

Property 1 states that the Laplace transform of the function $-tf(t)$ is the derivative of the Laplace transform of $f(t)$. Thus, if we know the Laplace transform $F(s)$ of $f(t)$, then, we don't have to perform a tedious integration to find the Laplace transform of $tf(t)$; we need only differentiate $F(s)$ and multiply by -1.

Example 1. Compute the Laplace transform of te^t.
Solution. The Laplace transform of e^t is $1/(s-1)$. Hence, by Property 1, the Laplace transform of te^t is

$$\mathcal{L}\{te^t\} = -\frac{d}{ds}\frac{1}{s-1} = \frac{1}{(s-1)^2}.$$

Example 2. Compute the Laplace transform of t^{13}.
Solution. Using Property 1 thirteen times gives

$$\mathcal{L}\{t^{13}\} = (-1)^{13}\frac{d^{13}}{ds^{13}}\mathcal{L}\{1\} = (-1)^{13}\frac{d^{13}}{ds^{13}}\frac{1}{s} = \frac{(13)!}{s^{14}}.$$

The main usefulness of Property 1 is in inverting Laplace transforms, as the following examples illustrate.

Example 3. What function has Laplace transform $-1/(s-2)^2$?
Solution. Observe that

$$-\frac{1}{(s-2)^2} = \frac{d}{ds}\frac{1}{s-2} \quad\text{and}\quad \frac{1}{s-2} = \mathcal{L}\{e^{2t}\}.$$

Hence, by Property 1,

$$\mathcal{L}^{-1}\left\{-\frac{1}{(s-2)^2}\right\} = -te^{2t}.$$

Example 4. What function has Laplace transform $-4s/(s^2+4)^2$?
Solution. Observe that

$$-\frac{4s}{(s^2+4)^2} = \frac{d}{ds}\frac{2}{s^2+4} \quad\text{and}\quad \frac{2}{s^2+4} = \mathcal{L}\{\sin 2t\}.$$

Hence, by Property 1,

$$\mathcal{L}^{-1}\left\{-\frac{4s}{(s^2+4)^2}\right\} = -t\sin 2t.$$

Example 5. What function has Laplace transform $1/(s-4)^3$?

Solution. We recognize that

$$\frac{1}{(s-4)^3}=\frac{d^2}{ds^2}\frac{1}{2}\frac{1}{s-4}.$$

Hence, using Property 1 twice, we see that

$$\frac{1}{(s-4)^3}=\mathcal{L}\left\{\frac{1}{2}t^2e^{4t}\right\}.$$

Property 2. If $F(s)=\mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\left\{e^{at}f(t)\right\}=F(s-a).$$

PROOF. By definition,

$$\begin{aligned}\mathcal{L}\left\{e^{at}f(t)\right\}&=\int_0^\infty e^{-st}e^{at}f(t)\,dt=\int_0^\infty e^{(a-s)t}f(t)\,dt\\&=\int_0^\infty e^{-(s-a)t}f(t)\,dt\equiv F(s-a).\end{aligned}$$ □

Property 2 states that the Laplace transform of $e^{at}f(t)$ evaluated at the point s equals the Laplace transform of $f(t)$ evaluated at the point $(s-a)$. Thus, if we know the Laplace transform $F(s)$ of $f(t)$, then we don't have to perform an integration to find the Laplace transform of $e^{at}f(t)$; we need only replace every s in $F(s)$ by $s-a$.

Example 6. Compute the Laplace transform of $e^{3t}\sin t$.
Solution. The Laplace transform of $\sin t$ is $1/(s^2+1)$. Therefore, to compute the Laplace transform of $e^{3t}\sin t$, we need only replace every s by $s-3$; that is,

$$\mathcal{L}\{e^{3t}\sin t\}=\frac{1}{(s-3)^2+1}.$$

The real usefulness of Property 2 is in inverting Laplace transforms, as the following examples illustrate.

Example 7. What function $g(t)$ has Laplace transform

$$G(s)=\frac{s-7}{25+(s-7)^2}?$$

Solution. Observe that

$$F(s)=\frac{s}{s^2+5^2}=\mathcal{L}\{\cos 5t\}$$

and that $G(s)$ is obtained from $F(s)$ by replacing every s by $s-7$. Hence, by Property 2,

$$\frac{s-7}{(s-7)^2+25}=\mathcal{L}\{e^{7t}\cos 5t\}.$$

Example 8. What function has Laplace transform $1/(s^2-4s+9)$?
Solution. One way of solving this problem is to expand $1/(s^2-4s+9)$ in partial fractions. A much better way is to complete the square of s^2-4s+9. Thus, we write

$$\frac{1}{s^2-4s+9}=\frac{1}{s^2-4s+4+(9-4)}=\frac{1}{(s-2)^2+5}.$$

Now,

$$\frac{1}{s^2+5}=\mathcal{L}\left\{\frac{1}{\sqrt{5}}\sin\sqrt{5}\,t\right\}.$$

Hence, by Property 2,

$$\frac{1}{s^2-4s+9}=\frac{1}{(s-2)^2+5}=\mathcal{L}\left\{\frac{1}{\sqrt{5}}e^{2t}\sin\sqrt{5}\,t\right\}.$$

Example 9. What function has Laplace transform $s/(s^2-4s+9)$?
Solution. Observe that

$$\frac{s}{s^2-4s+9}=\frac{s-2}{(s-2)^2+5}+\frac{2}{(s-2)^2+5}.$$

The function $s/(s^2+5)$ is the Laplace transform of $\cos\sqrt{5}\,t$. Therefore, by Property 2,

$$\frac{s-2}{(s-2)^2+5}=\mathcal{L}\{e^{2t}\cos\sqrt{5}\,t\},$$

and

$$\frac{s}{s^2-4s+9}=\mathcal{L}\left\{e^{2t}\cos\sqrt{5}\,t+\frac{2}{\sqrt{5}}e^{2t}\sin\sqrt{5}\,t\right\}.$$

In the previous section we showed that the Laplace transform is a linear operator; that is

$$\mathcal{L}\{c_1f_1(t)+c_2f_2(t)\}=c_1\mathcal{L}\{f_1(t)\}+c_2\mathcal{L}\{f_2(t)\}.$$

Thus, if we know the Laplace transforms $F_1(s)$ and $F_2(s)$, of $f_1(t)$ and $f_2(t)$, then we don't have to perform any integrations to find the Laplace transform of a linear combination of $f_1(t)$ and $f_2(t)$; we need only take the same linear combination of $F_1(s)$ and $F_2(s)$. For example, two functions which appear quite often in the study of differential equations are the hyperbolic cosine and hyperbolic sine functions. These functions are defined by the equations

$$\cosh at=\frac{e^{at}+e^{-at}}{2},\qquad \sinh at=\frac{e^{at}-e^{-at}}{2}.$$

Therefore, by the linearity of the Laplace transform,

$$\mathcal{L}\{\cosh at\} = \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$$

and

$$\mathcal{L}\{\sinh at\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}.$$

Exercises

Use Properties 1 and 2 to find the Laplace transform of each of the following functions.

1. t^n **2.** $t^n e^{at}$ **3.** $t \sin at$ **4.** $t^2 \cos at$

5. $t^{5/2}$ (see Exercise 9, Section 2.9)

6. Let $F(s) = \mathcal{L}\{f(t)\}$, and suppose that $f(t)/t$ has a limit as t approaches zero. Prove that

$$\mathcal{L}\{f(t)/t\} = \int_s^\infty F(u)\,du. \qquad (*)$$

(The assumption that $f(t)/t$ has a limit as $t \to 0$ guarantees that the integral on the right-hand side of (*) exists.)

7. Use Equation (*) of Problem 6 to find the Laplace transform of each of the following functions:

(a) $\dfrac{\sin t}{t}$ (b) $\dfrac{\cos at - 1}{t}$ (c) $\dfrac{e^{at} - e^{bt}}{t}$

Find the inverse Laplace transform of each of the following functions. In several of these problems, it will be helpful to write the functions

$$p_1(s) = \frac{\alpha_1 s^3 + \beta_1 s^2 + \gamma_1 s + \delta_1}{(as^2+bs+c)(ds^2+es+f)} \quad \text{and} \quad p_2(s) = \frac{\alpha_1 s^2 + \beta_1 s + \gamma}{(as+b)(cs^2+ds+e)}$$

in the simpler form

$$p_1(s) = \frac{As+B}{as^2+bs+c} + \frac{Cs+D}{ds^2+es+f} \quad \text{and} \quad p_2(s) = \frac{A}{as+b} + \frac{Cs+D}{cs^2+ds+e}.$$

8. $\dfrac{s}{(s+a)^2+b^2}$

9. $\dfrac{s^2-5}{s^3+4s^2+3s}$

10. $\dfrac{1}{s(s^2+4)}$

11. $\dfrac{s}{s^2-3s-12}$

12. $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$

13. $\dfrac{3s}{(s+1)^4}$

14. $\dfrac{1}{s(s+4)^2}$

15. $\dfrac{s}{(s+1)^2(s^2+1)}$

16. $\dfrac{1}{(s^2+1)^2}$

17. Let $F(s)=\mathcal{L}\{f(t)\}$. Show that

$$f(t)=-\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

Thus, if we know how to invert $F'(s)$, then we can also invert $F(s)$.

18. Use the result of Problem 17 to invert each of the following Laplace transforms

(a) $\ln\left(\dfrac{s+a}{s-a}\right)$ (b) $\arctan\dfrac{a}{s}$ (c) $\ln\left(1-\dfrac{a^2}{s^2}\right)$

Solve each of the following initial-value problems by the method of Laplace transforms.

19. $y''+y=\sin t$; $y(0)=1, y'(0)=2$

20. $y''+y=t\sin t$; $y(0)=1, y'(0)=2$

21. $y''-2y'+y=te^t$; $y(0)=0, y'(0)=0$

22. $y''-2y'+7y=\sin t$; $y(0)=0, y'(0)=0$

23. $y''+y'+y=1+e^{-t}$; $y(0)=3, y'(0)=-5$

24. $y''+y=\begin{cases} 2, & 0\leqslant t\leqslant 3 \\ 3t-7, & 3<t<\infty \end{cases}$; $y(0)=0, y'(0)=0$

2.11 Differential equations with discontinuous right-hand sides

In many applications, the right-hand side of the differential equation $ay''+by'+cy=f(t)$ has a jump discontinuity at one or more points. For example, a particle may be moving under the influence of a force $f_1(t)$, and suddenly, at time t_1, an additional force $f_2(t)$ is applied to the particle. Such equations are often quite tedious and cumbersome to solve, using the methods developed in Sections 2.4 and 2.5. In this section we show how to handle such problems by the method of Laplace transforms. We begin by computing the Laplace transform of several simple discontinuous functions.

The simplest example of a function with a single jump discontinuity is the function

$$H_c(t)=\begin{cases} 0, & 0\leqslant t<c \\ 1, & t\geqslant c \end{cases}.$$

This function, whose graph is given in Figure 1, is often called the unit step

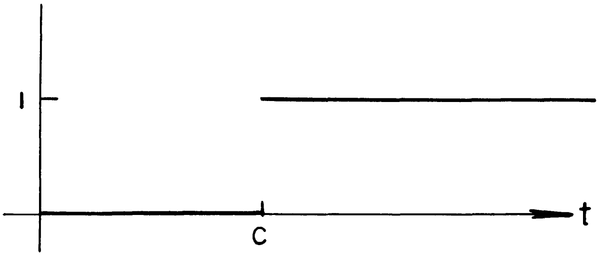

Figure 1. Graph of $H_c(t)$

function, or the Heaviside function. Its Laplace transform is

$$\begin{aligned}\mathcal{L}\{H_c(t)\} &= \int_0^\infty e^{-st}H_c(t)\,dt = \int_c^\infty e^{-st}\,dt\\ &= \lim_{A\to\infty}\int_c^A e^{-st}\,dt = \lim_{A\to\infty}\frac{e^{-cs}-e^{-sA}}{s}\\ &= \frac{e^{-cs}}{s}, \qquad s>0.\end{aligned}$$

Next, let f be any function defined on the interval $0 \leqslant t < \infty$, and let g be the function obtained from f by moving the graph of f over c units to the right, as shown in Figure 2. More precisely, $g(t)=0$ for $0 \leqslant t < c$, and $g(t)=f(t-c)$ for $t \geqslant c$. For example, if $c=2$ then the value of g at $t=7$ is the value of f at $t=5$. A convenient analytical expression for $g(t)$ is

$$g(t)=H_c(t)f(t-c).$$

The factor $H_c(t)$ makes g zero for $0 \leqslant t < c$, and replacing the argument t of f by $t-c$ moves f over c units to the right. Since $g(t)$ is obtained in a simple manner from $f(t)$, we would expect that its Laplace transform can also be obtained in a simple manner from the Laplace transform of $f(t)$. This is indeed the case, as we now show.

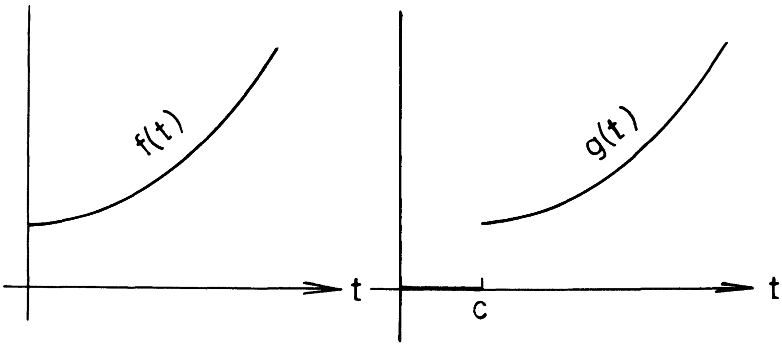

Figure 2

Property 3. Let $F(s)=\mathcal{L}\{f(t)\}$. Then,

$$\mathcal{L}\{H_c(t)f(t-c)\}=e^{-cs}F(s).$$

PROOF. By definition,

$$\begin{aligned}\mathcal{L}\{H_c(t)f(t-c)\}&=\int_0^\infty e^{-st}H_c(t)f(t-c)\,dt\\&=\int_c^\infty e^{-st}f(t-c)\,dt.\end{aligned}$$

This integral suggests the substitution

$$\xi=t-c.$$

Then,

$$\begin{aligned}\int_c^\infty e^{-st}f(t-c)\,dt&=\int_0^\infty e^{-s(\xi+c)}f(\xi)\,d\xi\\&=e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)\,d\xi\\&=e^{-cs}F(s).\end{aligned}$$

Hence, $\mathcal{L}\{H_c(t)f(t-c)\}=e^{-cs}\mathcal{L}\{f(t)\}$. □

Example 1. What function has Laplace transform e^{-s}/s^2?
Solution. We know that $1/s^2$ is the Laplace transform of the function t. Hence, by Property 3

$$\frac{e^{-s}}{s^2}=\mathcal{L}\{H_1(t)(t-1)\}.$$

The graph of $H_1(t)(t-1)$ is given in Figure 3.

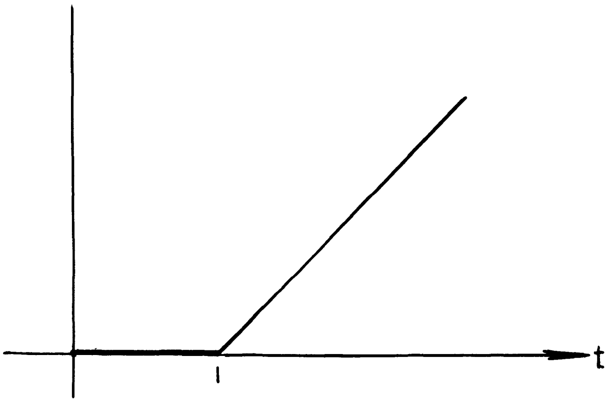

Figure 3. Graph of $H_1(t)\ (t-1)$

Example 2. What function has Laplace transform $e^{-3s}/(s^2-2s-3)$?
Solution. Observe that

$$\frac{1}{s^2-2s-3}=\frac{1}{s^2-2s+1-4}=\frac{1}{(s-1)^2-2^2}.$$

Since $1/(s^2-2^2)=\mathcal{L}\{\frac{1}{2}\sinh 2t\}$, we conclude from Property 2 that

$$\frac{1}{(s-1)^2-2^2}=\mathcal{L}\left\{\frac{1}{2}e^t\sinh 2t\right\}.$$

Consequently, from Property 3,

$$\frac{e^{-3s}}{s^2-2s-3}=\mathcal{L}\left\{\frac{1}{2}H_3(t)e^{t-3}\sinh 2(t-3)\right\}.$$

Example 3. Let $f(t)$ be the function which is t for $0\leqslant t<1$, and 0 for $t\geqslant 1$. Find the Laplace transform of f without performing any integrations.
Solution. Observe that $f(t)$ can be written in the form

$$f(t)=t[H_0(t)-H_1(t)]=t-tH_1(t).$$

Hence, from Property 1,

$$\begin{aligned}\mathcal{L}\{f(t)\}&=\mathcal{L}\{t\}-\mathcal{L}\{tH_1(t)\}\\&=\frac{1}{s^2}+\frac{d}{ds}\frac{e^{-s}}{s}=\frac{1}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}.\end{aligned}$$

Example 4. Solve the initial-value problem

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=f(t)=\begin{cases}1, & 0\leqslant t<1; \quad 0, \ 1\leqslant t<2;\\ 1, & 2\leqslant t<3; \quad 0, \ 3\leqslant t<4;\\ 1, & 4\leqslant t<5; \quad 0, \ 5\leqslant t<\infty.\end{cases}\qquad y(0)=0,\ \ y'(0)=0$$

Solution. Let $Y(s)=\mathcal{L}\{y(t)\}$ and $F(s)=\mathcal{L}\{f(t)\}$. Taking Laplace transforms of both sides of the differential equation gives $(s^2-3s+2)Y(s)=F(s)$, so that

$$Y(s)=\frac{F(s)}{s^2-3s+2}=\frac{F(s)}{(s-1)(s-2)}.$$

One way of computing $F(s)$ is to write $f(t)$ in the form

$$f(t)=[H_0(t)-H_1(t)]+[H_2(t)-H_3(t)]+[H_4(t)-H_5(t)].$$

Hence, by the linearity of the Laplace transform

$$F(s)=\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\frac{e^{-4s}}{s}-\frac{e^{-5s}}{s}.$$

A second way of computing $F(s)$ is to evaluate the integral

$$\int_0^\infty e^{-st}f(t)\,dt=\int_0^1 e^{-st}\,dt+\int_2^3 e^{-st}\,dt+\int_4^5 e^{-st}\,dt$$
$$=\frac{1-e^{-s}}{s}+\frac{e^{-2s}-e^{-3s}}{s}+\frac{e^{-4s}-e^{-5s}}{s}.$$

Consequently,

$$Y(s)=\frac{1-e^{-s}+e^{-2s}-e^{-3s}+e^{-4s}-e^{-5s}}{s(s-1)(s-2)}.$$

Our next step is to expand $1/s(s-1)(s-2)$ in partial fractions; i.e., we write

$$\frac{1}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}.$$

This implies that

$$A(s-1)(s-2)+Bs(s-2)+Cs(s-1)=1. \tag{1}$$

Setting $s=0$ in (1) gives $A=\frac{1}{2}$; setting $s=1$ gives $B=-1$; and setting $s=2$ gives $C=\frac{1}{2}$. Thus,

$$\frac{1}{s(s-1)(s-2)}=\frac{1}{2}\frac{1}{s}-\frac{1}{s-1}+\frac{1}{2}\frac{1}{s-2}$$
$$=\mathcal{L}\left\{\frac{1}{2}-e^t+\frac{1}{2}e^{2t}\right\}.$$

Consequently, from Property 3,

$$\begin{aligned}y(t)=&\left[\tfrac{1}{2}-e^t+\tfrac{1}{2}e^{2t}\right]-H_1(t)\left[\tfrac{1}{2}-e^{(t-1)}+\tfrac{1}{2}e^{2(t-1)}\right]\\&+H_2(t)\left[\tfrac{1}{2}-e^{(t-2)}+\tfrac{1}{2}e^{2(t-2)}\right]-H_3(t)\left[\tfrac{1}{2}-e^{(t-3)}+\tfrac{1}{2}e^{2(t-3)}\right]\\&+H_4(t)\left[\tfrac{1}{2}-e^{(t-4)}+\tfrac{1}{2}e^{2(t-4)}\right]-H_5(t)\left[\tfrac{1}{2}-e^{(t-5)}+\tfrac{1}{2}e^{2(t-5)}\right].\end{aligned}$$

Remark. It is easily verified that the function

$$\tfrac{1}{2}-e^{(t-n)}+\tfrac{1}{2}e^{2(t-n)}$$

and its derivative are both zero at $t=n$. Hence, both $y(t)$ and $y'(t)$ are continuous functions of time, even though $f(t)$ is discontinuous at $t=1, 2, 3, 4$, and 5. More generally, both the solution $y(t)$ of the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t); \qquad y(t_0)=y_0, \quad y'(t_0)=y_0'$$

and its derivative $y'(t)$ are always continuous functions of time, if $f(t)$ is piecewise continuous. We will indicate the proof of this result in Section 2.12.

Exercises

Find the solution of each of the following initial-value problems.

1. $y''+2y'+y=2(t-3)H_3(t);\quad y(0)=2, y'(0)=1$

2. $y''+y'+y=H_\pi(t)-H_{2\pi}(t);\quad y(0)=1, y'(0)=0$

3. $y''+4y=\begin{cases}1, & 0\leqslant t<4\\ 0, & t>4\end{cases};\quad y(0)=3, y'(0)=-2$

4. $y''+y=\begin{cases}\sin t, & 0\leqslant t<\pi\\ \cos t, & \pi\leqslant t<\infty\end{cases};\quad y(0)=1, y'(0)=0$

5. $y''+y=\begin{cases}\cos t, & 0\leqslant t<\pi/2\\ 0, & \pi/2\leqslant t<\infty\end{cases};\quad y(0)=3, y'(0)=-1$

6. $y''+2y'+y=\begin{cases}\sin 2t, & 0\leqslant t<\pi/2\\ 0, & \pi/2\leqslant t<\infty\end{cases};\quad y(0)=1, y'(0)=0$

7. $y''+y'+7y=\begin{cases}t, & 0\leqslant t<2\\ 0, & 2\leqslant t<\infty\end{cases};\quad y(0)=0, y'(0)=0$

8. $y''+y=\begin{cases}t^2, & 0\leqslant t<1\\ 0, & 1\leqslant t<\infty\end{cases};\quad y(0)=0, y'(0)=0$

9. $y''-2y'+y=\begin{cases}0, & 0\leqslant t<1\\ t, & 1\leqslant t<2\\ 0, & 2\leqslant t<\infty\end{cases};\quad y(0)=0, y'(0)=1$

10. Find the Laplace transform of $|\sin t|$. *Hint*: Observe that

$$|\sin t|=\sin t+2\sum_{n=1}^{\infty} H_{n\pi}(t)\sin(t-n\pi).$$

11. Solve the initial-value problem of Example 4 by the method of judicious guessing. *Hint*: Find the general solution of the differential equation in each of the intervals $0<t<1$, $1<t<2$, $2<t<3$, $3<t<4$, $4<t<5$, $5<t<\infty$, and choose the arbitrary constants so that $y(t)$ and $y'(t)$ are continuous at the points $t=1$, 2, 3, 4, and 5.

2.12 The Dirac delta function

In many physical and biological applications we are often confronted with an initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t);\qquad y(0)=y_0,\quad y'(0)=y_0' \tag{1}$$

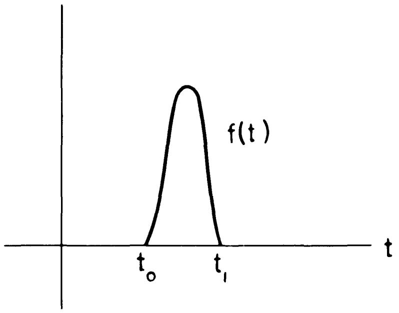

Figure 1. The graph of a typical impulsive function $f(t)$

where we do not know $f(t)$ explicitly. Such problems usually arise when we are dealing with phenomena of an impulsive nature. In these situations, the only information we have about $f(t)$ is that it is identically zero except for a very short time interval $t_0 \leqslant t \leqslant t_1$, and that its integral over this time interval is a given number $I_0 \neq 0$. If I_0 is not very small, then $f(t)$ will be quite large in the interval $t_0 \leqslant t \leqslant t_1$. Such functions are called impulsive functions, and the graph of a typical $f(t)$ is given in Figure 1.

In the early 1930's the Nobel Prize winning physicist P. A. M. Dirac developed a very controversial method for dealing with impulsive functions. His method is based on the following argument. Let t_1 get closer and closer to t_0. Then the function $f(t)/I_0$ approaches the function which is 0 for $t \neq t_0$, and ∞ for $t = t_0$, and whose integral over any interval containing t_0 is 1. We will denote this function, which is known as the Dirac delta function, by $\delta(t - t_0)$. Of course, $\delta(t - t_0)$ is not an ordinary function. However, says Dirac, let us formally operate with $\delta(t - t_0)$ as if it really were an ordinary function. Then, if we set $f(t) = I_0\delta(t - t_0)$ in (1) and impose the condition

$$\int_a^b g(t)\delta(t - t_0)\,dt = \begin{cases} g(t_0) & \text{if } a \leqslant t_0 \leqslant b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

for any continuous function $g(t)$, we will always obtain the correct solution $y(t)$.

Remark. Equation (2) is certainly a very reasonable condition to impose on $\delta(t - t_0)$. To see this, suppose that $f(t)$ is an impulsive function which is positive for $t_0 < t < t_1$, zero otherwise, and whose integral over the interval $[t_0, t_1]$ is 1. For any continuous function $g(t)$,

$$\left[\min_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t) \leqslant g(t) f(t) \leqslant \left[\max_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t).$$

Consequently,

$$\int_{t_0}^{t_1} \left[\min_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t)\,dt \leqslant \int_{t_0}^{t_1} g(t) f(t)\,dt \leqslant \int_{t_0}^{t_1} \left[\max_{t_0 \leqslant t \leqslant t_1} g(t)\right] f(t)\,dt,$$

or

$$\min_{t_0 \leqslant t \leqslant t_1} g(t) \leqslant \int_{t_0}^{t_1} g(t) f(t)\, dt \leqslant \max_{t_0 \leqslant t \leqslant t_1} g(t).$$

Thus, as $t_1 \to t_0$, $\int_{t_0}^{t_1} g(t) f(t)\, dt \to g(t_0)$.

Now, most mathematicians, of course, usually ridiculed this method. "How can you make believe that $\delta(t-t_0)$ is an ordinary function if it is obviously not," they asked. However, they never laughed too loud since Dirac and his followers always obtained the right answer. In the late 1940's, in one of the great success stories of mathematics, the French mathematician Laurent Schwartz succeeded in placing the delta function on a firm mathematical foundation. He accomplished this by enlarging the class of all functions so as to include the delta function. In this section we will first present a physical justification of the method of Dirac. Then we will illustrate how to solve the initial-value problem (1) by the method of Laplace transforms. Finally, we will indicate very briefly the "germ" of Laurent Schwartz's brilliant idea.

Physical justification of the method of Dirac. Newton's second law of motion is usually written in the form

$$\frac{d}{dt} mv(t) = f(t) \tag{3}$$

where m is the mass of the particle, v is its velocity, and $f(t)$ is the total force acting on the particle. The quantity mv is called the momentum of the particle. Integrating Equation (3) between t_0 and t_1 gives

$$mv(t_1) - mv(t_0) = \int_{t_0}^{t_1} f(t)\, dt.$$

This equation says that the change in momentum of the particle from time t_0 to time t_1 equals $\int_{t_0}^{t_1} f(t)\, dt$. Thus, the physically important quantity is the integral of the force, which is known as the impulse imparted by the force, rather than the force itself. Now, we may assume that $a > 0$ in Equation (1), for otherwise we can multiply both sides of the equation by -1 to obtain $a > 0$. In this case (see Section 2.6) we can view $y(t)$, for $t \leqslant t_0$, as the position at time t of a particle of mass a moving under the influence of the force $-b(dy/dt) - cy$. At time t_0 a force $f(t)$ is applied to the particle, and this force acts over an extremely short time interval $t_0 \leqslant t \leqslant t_1$. Since the time interval is extremely small, we may assume that the position of the particle does not change while the force $f(t)$ acts. Thus the sum result of the impulsive force $f(t)$ is that the velocity of the particle jumps by an amount I_0/a at time t_0. In other words, $y(t)$ satisfies the initial-value problem

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0; \qquad y(0) = y_0, \quad y'(0) = y_0'$$

for $0 \leqslant t < t_0$, and

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=0; \qquad y(t_0)=z_0, \quad y'(t_0)=z_0'+\frac{I_0}{a} \tag{4}$$

for $t \geqslant t_0$, where z_0 and z_0' are the position and velocity of the particle just before the impulsive force acts. It is clear, therefore, that any method which correctly takes into account the momentum I_0 transferred to the particle at time t_0 by the impulsive force $f(t)$ must yield the correct answer. It is also clear that we always keep track of the momentum I_0 transferred to the particle by $f(t)$ if we replace $f(t)$ by $I_0\delta(t-t_0)$ and obey Equation (2). Hence the method of Dirac will always yield the correct answer.

Remark. We can now understand why any solution $y(t)$ of the differential equation

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t), \qquad f(t) \text{ a piecewise continuous function,}$$

is a continuous function of time even though $f(t)$ is discontinuous. To wit, since the integral of a piecewise continuous function is continuous, we see that $y'(t)$, must vary continuously with time. Consequently, $y(t)$ must also vary continuously with time.

Solution of Equation (1) *by the method of Laplace transforms.* In order to solve the initial-value problem (1) by the method of Laplace transforms, we need only know the Laplace transform of $\delta(t-t_0)$. This is obtained directly from the definition of the Laplace transform and Equation (2), for

$$\mathcal{L}\{\delta(t-t_0)\} \equiv \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0} \qquad (\text{for } t_0 \geqslant 0).$$

Example 1. Find the solution of the initial-value problem

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=3\delta(t-1)+\delta(t-2); \qquad y(0)=1, \quad y'(0)=1.$$

Solution. Let $Y(s)=\mathcal{L}\{y(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$s^2Y-s-1-4(sY-1)+4Y=3e^{-s}+e^{-2s}$$

or

$$(s^2-4s+4)Y(s)=s-3+3e^{-s}+e^{-2s}.$$

Consequently,

$$Y(s)=\frac{s-3}{(s-2)^2}+\frac{3e^{-s}}{(s-2)^2}+\frac{e^{-2s}}{(s-2)^2}.$$

Now, $1/(s-2)^2 = \mathcal{L}\{te^{2t}\}$. Hence,

$$\frac{3e^{-s}}{(s-2)^2} + \frac{e^{-2s}}{(s-2)^2} = \mathcal{L}\{3H_1(t)(t-1)e^{2(t-1)} + H_2(t)(t-2)e^{2(t-2)}\}.$$

To invert the first term of $Y(s)$, observe that

$$\frac{s-3}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2} = \mathcal{L}\{e^{2t}\} - \mathcal{L}\{te^{2t}\}.$$

Thus, $y(t) = (1-t)e^{2t} + 3H_1(t)(t-1)e^{2(t-1)} + H_2(t)(t-2)e^{2(t-2)}$.

It is instructive to do this problem the long way, that is, to find $y(t)$ separately in each of the intervals $0 \leqslant t < 1$, $1 \leqslant t < 2$ and $2 \leqslant t < \infty$. For $0 \leqslant t < 1$, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0; \qquad y(0) = 1, \quad y'(0) = 1.$$

The characteristic equation of this differential equation is $r^2 - 4r + 4 = 0$, whose roots are $r_1 = r_2 = 2$. Hence, any solution $y(t)$ must be of the form $y(t) = (a_1 + a_2 t)e^{2t}$. The constants a_1 and a_2 are determined from the initial conditions

$$1 = y(0) = a_1 \quad \text{and} \quad 1 = y'(0) = 2a_1 + a_2.$$

Hence, $a_1 = 1$, $a_2 = -1$ and $y(t) = (1-t)e^{2t}$ for $0 \leqslant t < 1$. Now $y(1) = 0$ and $y'(1) = -e^2$. At time $t = 1$ the derivative of $y(t)$ is suddenly increased by 3. Consequently, for $1 \leqslant t < 2$, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0; \qquad y(1) = 0, \quad y'(1) = 3 - e^2.$$

Since the initial conditions are given at $t = 1$, we write this solution in the form $y(t) = [b_1 + b_2(t-1)]e^{2(t-1)}$ (see Exercise 1). The constants b_1 and b_2 are determined from the initial conditions

$$0 = y(1) = b_1 \quad \text{and} \quad 3 - e^2 = y'(1) = 2b_1 + b_2.$$

Thus, $b_1 = 0$, $b_2 = 3 - e^2$ and $y(t) = (3 - e^2)(t-1)e^{2(t-1)}$, $1 \leqslant t < 2$. Now, $y(2) = (3 - e^2)e^2$ and $y'(2) = 3(3 - e^2)e^2$. At time $t = 2$ the derivative of $y(t)$ is suddenly increased by 1. Consequently, for $2 \leqslant t < \infty$, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0; \qquad y(2) = e^2(3 - e^2), \quad y'(2) = 1 + 3e^2(3 - e^2).$$

Hence $y(t) = [c_1 + c_2(t-2)]e^{2(t-2)}$. The constants c_1 and c_2 are determined from the equations

$$e^2(3 - e^2) = c_1 \quad \text{and} \quad 1 + 3e^2(3 - e^2) = 2c_1 + c_2.$$

Thus,

$$c_1 = e^2(3-e^2), \qquad c_2 = 1+3e^2(3-e^2)-2e^2(3-e^2) = 1+e^2(3-e^2)$$

and $y(t)=[e^2(3-e^2)+(1+e^2(3-e^2))(t-2)]e^{2(t-2)}$, $t \geq 2$. The reader should verify that this expression agrees with the expression obtained for $y(t)$ by the method of Laplace transforms.

Example 2. A particle of mass 1 is attached to a spring dashpot mechanism. The stiffness constant of the spring is 1 N/ft and the drag force exerted by the dashpot mechanism on the particle is twice its velocity. At time $t=0$, when the particle is at rest, an external force e^{-t} is applied to the system. At time $t=1$, an additional force $f(t)$ of very short duration is applied to the particle. This force imparts an impulse of 3 N·s to the particle. Find the position of the particle at any time t greater than 1.
Solution. Let $y(t)$ be the distance of the particle from its equilibrium position. Then, $y(t)$ satisfies the initial-value problem

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=e^{-t}+3\delta(t-1); \qquad y(0)=0, \quad y'(0)=0.$$

Let $Y(s)=\mathcal{L}\{y(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$(s^2+2s+1)Y(s)=\frac{1}{s+1}+3e^{-s}, \quad \text{or} \quad Y(s)=\frac{1}{(s+1)^3}+\frac{3e^{-s}}{(s+1)^2}.$$

Since

$$\frac{1}{(s+1)^3}=\mathcal{L}\left\{\frac{t^2e^{-t}}{2}\right\} \quad \text{and} \quad \frac{3e^{-s}}{(s+1)^2}=3\mathcal{L}\left\{H_1(t)(t-1)e^{-(t-1)}\right\}$$

we see that

$$y(t)=\frac{t^2e^{-t}}{2}+3H_1(t)(t-1)e^{-(t-1)}.$$

Consequently, $y(t)=\frac{1}{2}t^2e^{-t}+3(t-1)e^{-(t-1)}$ for $t>1$.

We conclude this section with a very brief description of Laurent Schwartz's method for placing the delta function on a rigorous mathematical foundation. The main step in his method is to rethink our notion of "function." In Calculus, we are taught to recognize a function by its value at each time t. A much more subtle (and much more difficult) way of recognizing a function is by what it does to other functions. More precisely, let f be a piecewise continuous function defined for $-\infty<t<\infty$. To each function ϕ which is infinitely often differentiable and which vanishes for $|t|$

sufficiently large, we assign a number $K[\phi]$ according to the formula

$$K[\phi]=\int_{-\infty}^{\infty}\phi(t)f(t)\,dt. \tag{5}$$

As the notation suggests, K is an operator acting on functions. However, it differs from the operators introduced previously in that it associates a number, rather than a function, with ϕ. For this reason, we say that $K[\phi]$ is a functional, rather than a function. Now, observe that the association $\phi \rightarrow K[\phi]$ is a linear association, since

$$\begin{aligned}K[c_1\phi_1+c_2\phi_2]&=\int_{-\infty}^{\infty}(c_1\phi_1+c_2\phi_2)(t)f(t)\,dt\\&=c_1\int_{-\infty}^{\infty}\phi_1(t)f(t)\,dt+c_2\int_{-\infty}^{\infty}\phi_2(t)f(t)\,dt\\&=c_1K[\phi_1]+c_2K[\phi_2].\end{aligned}$$

Hence every piecewise continuous function defines, through (5), a linear functional on the space of all infinitely often differentiable functions which vanish for $|t|$ sufficiently large.

Now consider the functional $K[\phi]$ defined by the relation $K[\phi]=\phi(t_0)$. K is a linear functional since

$$K[c_1\phi_1+c_2\phi_2]=c_1\phi_1(t_0)+c_2\phi_2(t_0)=c_1K[\phi_1]+c_2K[\phi_2].$$

To mimic (5), we write K symbolically in the form

$$K[\phi]=\int_{-\infty}^{\infty}\phi(t)\delta(t-t_0)\,dt. \tag{6}$$

In this sense, $\delta(t-t_0)$ is a "generalized function." It is important to realize though, that we cannot speak of the value of $\delta(t-t_0)$ at any time t. The only meaningful quantity is the expression $\int_{-\infty}^{\infty}\phi(t)\delta(t-t_0)\,dt$, and we must always assign the value $\phi(t_0)$ to this expression.

Admittedly, it is very difficult to think of a function in terms of the linear functional (5) that it induces. The advantage to this way of thinking, though, is that it is now possible to assign a derivative to every piecewise continuous function and to every "generalized function." To wit, suppose that $f(t)$ is a differentiable function. Then $f'(t)$ induces the linear functional

$$K'[\phi]=\int_{-\infty}^{\infty}\phi(t)f'(t)\,dt. \tag{7}$$

Integrating by parts and using the fact that $\phi(t)$ vanishes for $|t|$ sufficiently large, we see that

$$K'[\phi]=\int_{-\infty}^{\infty}[-\phi'(t)]f(t)\,dt=K[-\phi']. \tag{8}$$

Now, notice that the formula $K'[\phi]=K[-\phi']$ makes sense even if $f(t)$ is not differentiable. This motivates the following definition.

Definition. To every linear functional $K[\phi]$ we assign the new linear functional $K'[\phi]$ by the formula $K'[\phi]=K[-\phi']$. The linear functional $K'[\phi]$ is called the derivative of $K[\phi]$ since if $K[\phi]$ is induced by a differentiable function $f(t)$ then $K'[\phi]$ is induced by $f'(t)$.

Finally, we observe from (8) that the derivative of the delta function $\delta(t-t_0)$ is the linear functional which assigns to each function ϕ the number $-\phi'(t_0)$, for if $K[\phi]=\phi(t_0)$ then $K'[\phi]=K[-\phi']=-\phi'(t_0)$. Thus,

$$\int_{-\infty}^{\infty}\phi(t)\delta'(t-t_0)\,dt=-\phi'(t_0)$$

for all differentiable functions $\phi(t)$.

Exercises

1. Let a be a fixed constant. Show that every solution of the differential equation $(d^2y/dt^2)+2\alpha(dy/dt)+\alpha^2y=0$ can be written in the form
$$y(t)=[c_1+c_2(t-a)]e^{-\alpha(t-a)}.$$

2. Solve the initial-value problem $(d^2y/dt^2)+4(dy/dt)+5y=f(t)$; $y(0)=1, y'(0)=0$, where $f(t)$ is an impulsive force which acts on the extremely short time interval $1\leqslant t\leqslant 1+\tau$, and $\int_1^{1+\tau}f(t)\,dt=2$.

3. (a) Solve the initial-value problem $(d^2y/dt^2)-3(dy/dt)+2y=f(t)$; $y(0)=1$, $y'(0)=0$, where $f(t)$ is an impulsive function which acts on the extremely short time interval $2\leqslant t\leqslant 2+\tau$, and $\int_2^{2+\tau}f(t)\,dt=-1$.

   (b) Solve the initial-value problem $(d^2y/dt^2)-3(dy/dt)+2y=0$; $y(0)=1$, $y'(0)=0$, on the interval $0\leqslant t\leqslant 2$. Compute $z_0=y(2)$ and $z_0'=y'(2)$. Then solve the initial-value problem
$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=0;\qquad y(2)=z_0,\ \ y'(2)=z_0'-1,\qquad 2\leqslant t<\infty.$$

   Compare this solution with the solution of part (a).

4. A particle of mass 1 is attached to a spring dashpot mechanism. The stiffness constant of the spring is 3 N/m and the drag force exerted on the particle by the dashpot mechanism is 4 times its velocity. At time $t=0$, the particle is stretched $\frac{1}{4}$ m from its equilibrium position. At time $t=3$ seconds, an impulsive force of very short duration is applied to the system. This force imparts an impulse of 2 N·s to the particle. Find the displacement of the particle from its equilibrium position.

In Exercises 5–7 solve the given initial-value problem.

5. $\dfrac{d^2y}{dt^2}+y=\sin t+\delta(t-\pi);\quad y(0)=0, y'(0)=0$

6. $\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+y=2\delta(t-1)-\delta(t-2);\quad y(0)=1, y'(0)=0$

7. $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+y=e^{-t}+3\delta(t-3);\quad y(0)=0, y'(0)=3$

8. (a) Solve the initial-value problem

$$\frac{d^2y}{dt^2}+y=\sum_{j=0}^{\infty}\delta(t-j\pi), \qquad y(0)=y'(0)=0,$$

and show that

$$y(t)=\begin{cases}\sin t, & n \text{ even}\\ 0, & n \text{ odd}\end{cases}$$

in the interval $n\pi<t<(n+1)\pi$.

(b) Solve the initial-value problem

$$\frac{d^2y}{dt^2}+y=\sum_{j=0}^{\infty}\delta(t-2j\pi), \qquad y(0)=y'(0)=0,$$

and show that $y(t)=(n+1)\sin t$ in the interval $2n\pi<t<2(n+1)\pi$.

This example indicates why soldiers are instructed to break cadence when marching across a bridge. To wit, if the soldiers are in step with the natural frequency of the steel in the bridge, then a resonance situation of the type (b) may be set up.

9. Let $f(t)$ be the function which is $\frac{1}{2}$ for $t>t_0$, 0 for $t=t_0$, and $-\frac{1}{2}$ for $t<t_0$. Let $K[\phi]$ be the linear functional

$$K[\phi]=\int_{-\infty}^{\infty}\phi(t)f(t)\,dt.$$

Show that $K'[\phi]\equiv K[-\phi']=\phi(t_0)$. Thus, $\delta(t-t_0)$ may be viewed as the derivative of $f(t)$.

2.13 The convolution integral

Consider the initial-value problem

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy=f(t); \qquad y(0)=y_0, \quad y'(0)=y_0'. \tag{1}$$

Let $Y(s)=\mathcal{L}\{y(t)\}$ and $F(s)=\mathcal{L}\{f(t)\}$. Taking Laplace transforms of both sides of the differential equation gives

$$a\left[s^2Y(s)-sy_0-y_0'\right]+b\left[sY(s)-y_0\right]+cY(s)=F(s)$$

and this implies that

$$Y(s)=\frac{as+b}{as^2+bs+c}y_0+\frac{a}{as^2+bs+c}y_0'+\frac{F(s)}{as^2+bs+c}.$$

Now, let

$$y_1(t)=\mathcal{L}^{-1}\left\{\frac{as+b}{as^2+bs+c}\right\}$$

and

$$y_2(t)=\mathcal{L}^{-1}\left\{\frac{a}{as^2+bs+c}\right\}.$$

Setting $f(t)=0$, $y_0=1$ and $y_0'=0$, we see that $y_1(t)$ is the solution of the homogeneous equation which satisfies the initial conditions $y_1(0)=1$, $y_1'(0)=0$. Similarly, by setting $f(t)=0$, $y_0=0$ and $y_0'=1$, we see that $y_2(t)$ is the solution of the homogeneous equation which satisfies the initial conditions $y_2(0)=0$, $y_2'(0)=1$. This implies that

$$\psi(t)=\mathcal{L}^{-1}\left\{\frac{F(s)}{as^2+bs+c}\right\}$$

is the particular solution of the nonhomogeneous equation which satisfies the initial conditions $\psi(0)=0$, $\psi'(0)=0$. Thus, the problem of finding a particular solution $\psi(t)$ of the nonhomogeneous equation is now reduced to the problem of finding the inverse Laplace transform of the function $F(s)/(as^2+bs+c)$. If we look carefully at this function, we see that it is the product of two Laplace transforms; that is

$$\frac{F(s)}{as^2+bs+c}=\mathcal{L}\{f(t)\}\times\mathcal{L}\left\{\frac{y_2(t)}{a}\right\}.$$

It is natural to ask whether there is any simple relationship between $\psi(t)$ and the functions $f(t)$ and $y_2(t)/a$. It would be nice, of course, if $\psi(t)$ were the product of $f(t)$ with $y_2(t)/a$, but this is obviously false. However, there is an extremely interesting way of combining two functions f and g together to form a new function $f*g$, which resembles multiplication, and for which

$$\mathcal{L}\{(f*g)(t)\}=\mathcal{L}\{f(t)\}\times\mathcal{L}\{g(t)\}.$$

This combination of f and g appears quite often in applications, and is known as the *convolution* of f with g.

Definition. The *convolution* $(f*g)(t)$ of f with g is defined by the equation

$$(f*g)(t)=\int_0^t f(t-u)\,g(u)\,du. \tag{2}$$

For example, if $f(t)=\sin 2t$ and $g(t)=e^{t^2}$, then

$$(f*g)(t)=\int_0^t \sin 2(t-u)e^{u^2}\,du.$$

The convolution operator $*$ clearly bears some resemblance to the multiplication operator since we multiply the value of f at the point $t-u$ by the value of g at the point u, and then integrate this product with respect to u. Therefore, it should not be too surprising to us that the convolution operator satisfies the following properties.

Property 1. The convolution operator obeys the commutative law of multiplication; that is, $(f*g)(t)=(g*f)(t)$.

PROOF. By definition,

$$(f*g)(t)=\int_0^t f(t-u)\,g(u)\,du.$$

Let us make the substitution $t-u=s$ in this integral. Then,

$$\begin{aligned}(f*g)(t)&=-\int_t^0 f(s)\,g(t-s)\,ds\\&=\int_0^t g(t-s)f(s)\,ds\equiv(g*f)(t).\end{aligned}$$ □

Property 2. The convolution operator satisfies the distributive law of multiplication; that is,

$$f*(g+h)=f*g+f*h.$$

PROOF. See Exercise 19. □

Property 3. The convolution operator satisfies the associative law of multiplication; that is, $(f*g)*h=f*(g*h)$.

PROOF. See Exercise 20. □

Property 4. The convolution of any function f with the zero function is zero.

PROOF. Obvious. □

On the other hand, the convolution operator differs from the multiplication operator in that $f*1\neq f$ and $f*f\neq f^2$. Indeed, the convolution of a function f with itself may even be negative.

Example 1. Compute the convolution of $f(t)=t^2$ with $g(t)=1$.

Solution. From Property 1,

$$(f*g)(t)=(g*f)(t)=\int_0^t 1\cdot u^2\,du=\frac{t^3}{3}.$$

Example 2. Compute the convolution of $f(t)=\cos t$ with itself, and show that it is not always positive.
Solution. By definition,

$$\begin{aligned}(f*f)(t)&=\int_0^t \cos(t-u)\cos u\,du\\&=\int_0^t(\cos t\cos^2 u+\sin t\sin u\cos u)\,du\\&=\cos t\int_0^t\frac{1+\cos 2u}{2}\,du+\sin t\int_0^t\sin u\cos u\,du\\&=\cos t\left[\frac{t}{2}+\frac{\sin 2t}{4}\right]+\frac{\sin^3 t}{2}\\&=\frac{t\cos t+\sin t\cos^2 t+\sin^3 t}{2}\\&=\frac{t\cos t+\sin t(\cos^2 t+\sin^2 t)}{2}\\&=\frac{t\cos t+\sin t}{2}.\end{aligned}$$

This function, clearly, is negative for

$$(2n+1)\pi\leqslant t\leqslant(2n+1)\pi+\tfrac{1}{2}\pi,\qquad n=0,1,2,\dots.$$

We now show that the Laplace transform of $f*g$ is the product of the Laplace transform of f with the Laplace transform of g.

Theorem 9. $\mathcal{L}\{(f*g)(t)\}=\mathcal{L}\{f(t)\}\times\mathcal{L}\{g(t)\}$.

PROOF. By definition,

$$\mathcal{L}\{(f*g)(t)\}=\int_0^\infty e^{-st}\left[\int_0^t f(t-u)g(u)\,du\right]dt.$$

This iterated integral equals the double integral

$$\iint_R e^{-st}f(t-u)g(u)\,du\,dt$$

where R is the triangular region described in Figure 1. Integrating first

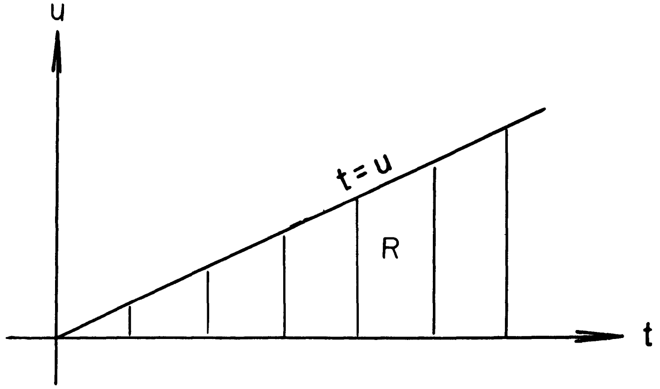

Figure 1

with respect to t, instead of u, gives

$$\mathcal{L}\{(f*g)(t)\}=\int_0^{\infty} g(u)\left[\int_u^{\infty} e^{-st} f(t-u)\,dt\right] du.$$

Setting $t-u=\xi$, we see that

$$\int_u^{\infty} e^{-st} f(t-u)\,dt=\int_0^{\infty} e^{-s(u+\xi)} f(\xi)\,d\xi.$$

Hence,

$$\begin{aligned}\mathcal{L}\{(f*g)(t)\}&=\int_0^{\infty} g(u)\left[\int_0^{\infty} e^{-su}e^{-s\xi} f(\xi)\,d\xi\right] du\\ &=\left[\int_0^{\infty} g(u)e^{-su}\,du\right]\left[\int_0^{\infty} e^{-s\xi} f(\xi)\,d\xi\right]\\ &\equiv\mathcal{L}\{f(t)\}\times\mathcal{L}\{g(t)\}.\end{aligned}$$

□

Example 3. Find the inverse Laplace transform of the function

$$\frac{a}{s^2(s^2+a^2)}.$$

Solution. Observe that

$$\frac{1}{s^2}=\mathcal{L}\{t\}\quad\text{and}\quad\frac{a}{s^2+a^2}=\mathcal{L}\{\sin at\}.$$

Hence, by Theorem 9

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\}&=\int_0^t (t-u)\sin au\,du\\ &=\frac{at-\sin at}{a^2}.\end{aligned}$$

Example 4. Find the inverse Laplace transform of the function

$$\frac{1}{s(s^2+2s+2)}.$$

Solution. Observe that

$$\frac{1}{s}=\mathcal{L}\{1\} \quad \text{and} \quad \frac{1}{s^2+2s+2}=\frac{1}{(s+1)^2+1}=\mathcal{L}\{e^{-t}\sin t\}.$$

Hence, by Theorem 9,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}=\int_0^t e^{-u}\sin u\,du$$
$$=\frac{1}{2}\left[1-e^{-t}(\cos t+\sin t)\right].$$

Remark. Let $y_2(t)$ be the solution of the homogeneous equation $ay''+by'+cy=0$ which satisfies the initial conditions $y_2(0)=0$, $y_2'(0)=1$. Then,

$$\psi(t)=f(t)*\frac{y_2(t)}{a} \tag{3}$$

is the particular solution of the nonhomogeneous equation $ay''+by'+cy=f(t)$ which satisfies the initial conditions $\psi(0)=\psi'(0)=0$. Equation (3) is often much simpler to use than the variation of parameters formula derived in Section 2.4.

Exercises

Compute the convolution of each of the following pairs of functions.

1. e^{at}, e^{bt}, $a\neq b$

2. e^{at}, e^{at}

3. $\cos at$, $\cos bt$

4. $\sin at$, $\sin bt$, $a\neq b$

5. $\sin at$, $\sin at$

6. t, $\sin t$

Use Theorem 9 to invert each of the following Laplace transforms.

7. $\dfrac{1}{s^2(s^2+1)}$

8. $\dfrac{s}{(s+1)(s^2+4)}$

9. $\dfrac{s}{(s^2+1)^2}$

10. $\dfrac{1}{s(s^2+1)}$

11. $\dfrac{1}{s^2(s+1)^2}$

12. $\dfrac{1}{(s^2+1)^2}$

Use Theorem 9 to find the solution $y(t)$ of each of the following integro-differential equations.

13. $y(t)=4t-3\int_0^t y(u)\sin(t-u)\,du$

14. $y(t)=4t-3\int_0^t y(t-u)\sin u\,du$

15. $y'(t)=\sin t+\int_0^t y(t-u)\cos u\,du,\ y(0)=0$

16. $y(t)=4t^2-\int_0^t y(u)e^{-(t-u)}\,du$

17. $y'(t)+2y+\int_0^t y(u)\,du=\sin t,\ y(0)=1$

18. $y(t)=t-e^t\int_0^t y(u)e^{-u}\,du$

19. Prove that $f*(g+h)=f*g+f*h$.

20. Prove that $(f*g)*h=f*(g*h)$.

2.14 The method of elimination for systems

The theory of second-order linear differential equations can also be used to find the solutions of two simultaneous first-order equations of the form

$$\begin{aligned} x' &= \frac{dx}{dt} = a(t)x+b(t)y+f(t) \\ y' &= \frac{dy}{dt} = c(t)x+d(t)y+g(t). \end{aligned} \tag{1}$$

The key idea is to eliminate one of the variables, say y, and then find x as the solution of a second-order linear differential equation. This technique is known as the *method of elimination*, and we illustrate it with the following two examples.

Example 1. Find all solutions of the simultaneous equations

$$\begin{aligned} x' &= 2x+y+t \\ y' &= x+3y+1. \end{aligned} \tag{2}$$

Solution. First, we solve for

$$y=x'-2x-t \tag{3}$$

from the first equation of (2). Differentiating this equation gives

$$y'=x''-2x'-1=x+3y+1.$$

Then, substituting for y from (3) gives

$$x''-2x'-1=x+3(x'-2x-t)+1$$

so that

$$x''-5x'+5x=2-3t. \tag{4}$$

Equation (4) is a second-order linear equation and its solution is

$$x(t)=e^{5t/2}\left[c_1e^{\sqrt{5}\,t/2}+c_2e^{-\sqrt{5}\,t/2}\right]-\frac{(1+3t)}{5}$$

for some constants c_1 and c_2. Finally, plugging this expression into (3) gives

$$y(t)=e^{5t/2}\left[\frac{1+\sqrt{5}}{2}c_1e^{\sqrt{5}\,t/2}+\frac{1-\sqrt{5}}{2}c_2e^{-\sqrt{5}\,t/2}\right]+\frac{t-1}{5}.$$

Example 2. Find the solution of the initial-value problem

$$\begin{aligned} x'&=3x-y, \qquad x(0)=3\\ y'&=x+y, \qquad y(0)=0. \end{aligned} \tag{5}$$

Solution. From the first equation of (5),

$$y=3x-x'. \tag{6}$$

Differentiating this equation gives

$$y'=3x'-x''=x+y.$$

Then, substituting for y from (6) gives

$$3x'-x''=x+3x-x'$$

so that

$$x''-4x'+4x=0.$$

This implies that

$$x(t)=(c_1+c_2t)e^{2t}$$

for some constants c_1, c_2, and plugging this expression into (6) gives

$$y(t)=(c_1-c_2+c_2t)e^{2t}.$$

The constants c_1 and c_2 are determined from the initial conditions

$$\begin{aligned} x(0)&=3=c_1\\ y(0)&=0=c_1-c_2. \end{aligned}$$

Hence $c_1=3$, $c_2=3$ and

$$x(t)=3(1+t)e^{2t}, y(t)=3te^{2t}$$

is the solution of (5).

Remark. The simultaneous equations (1) are usually referred to as a first-order *system* of equations. Systems of equations are treated fully in Chapters 3 and 4.

Exercises

Find all solutions of each of the following systems of equations.

1. $x'=6x-3y$
$y'=2x+y$

2. $x'=-2x+y+t$
$y'=-4x+3y-1$

3. $x' = -3x + 2y$
$y' = -x - y$

4. $x' = x + y + e^t$
$y' = x - y - e^t$

Find the solution of each of the following initial-value problems.

5. $x' = x + y, \quad x(0) = 2$
$y' = 4x + y, \quad y(0) = 3$

6. $x' = x - 3y, \quad x(0) = 0$
$y' = -2x + 2y, \quad y(0) = 5$

7. $x' = x - y, \quad x(0) = 1$
$y' = 5x - 3y, \quad y(0) = 2$

8. $x' = 3x - 2y, \quad x(0) = 1$
$y' = 4x - y, \quad y(0) = 5$

9. $x' = 4x + 5y + 4e^t \cos t, \quad x(0) = 0$
$y' = -2x - 2y, \quad y(0) = 0$

10. $x' = 3x - 4y + e^t, \quad x(0) = 1$
$y' = x - y + e^t, \quad y(0) = 1$

11. $x' = 2x - 5y + \sin t, \quad x(0) = 0$
$y' = x - 2y + \tan t, \quad y(0) = 0$

12. $x' = y + f_1(t), \quad x(0) = 0$
$y' = -x + f_2(t), \quad y(0) = 0$

2.15 Higher-order equations

In this section we briefly discuss higher-order linear differential equations.

Definition. The equation

$$L[y] = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0(t)y = 0, \quad a_n(t) \neq 0 \tag{1}$$

is called the general nth order homogeneous linear equation. The differential equation (1) together with the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)} \tag{1'}$$

is called an initial-value problem. The theory for Equation (1) is completely analogous to the theory for the second-order linear homogeneous equation which we studied in Sections 2.1 and 2.2. Therefore, we will state the relevant theorems without proof. Complete proofs can be obtained by generalizing the methods used in Sections 2.1 and 2.2, or by using the methods to be developed in Chapter 3.

Theorem 10. *Let $y_1(t), \ldots, y_n(t)$ be n independent solutions of (1); that is, no solution $y_j(t)$ is a linear combination of the other solutions. Then, every solution $y(t)$ of (1) is of the form*

$$y(t) = c_1 y_1(t) + \ldots + c_n y_n(t) \tag{2}$$

for some choice of constants $c_1, \ldots, c_n$. For this reason, we say that (2) is the general solution of (1).

To find n independent solutions of (1) when the coefficients $a_0, a_1, \ldots, a_n$ do not depend on t, we compute

$$L[e^{rt}] = (a_n r^n + a_{n-1}r^{n-1} + \ldots + a_0)e^{rt}. \tag{3}$$

This implies that e^{rt} is a solution of (1) if, and only if, r is a root of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_0 = 0. \tag{4}$$

Thus, if Equation (4) has n distinct roots $r_1, \ldots, r_n$, then the general solution of (1) is $y(t) = c_1 e^{r_1 t} + \ldots + c_n e^{r_n t}$. If $r_j = \alpha_j + i\beta_j$ is a complex root of (4), then

$$u(t) = \operatorname{Re}\{e^{r_j t}\} = e^{\alpha_j t}\cos\beta_j t$$

and

$$v(t) = \operatorname{Im}\{e^{r_j t}\} = e^{\alpha_j t}\sin\beta_j t$$

are two real-valued solutions of (1). Finally, if r_1 is a root of multiplicity k; that is, if

$$a_n r^n + \ldots + a_0 = (r - r_1)^k q(r)$$

where $q(r_1) \neq 0$, then $e^{r_1 t}, te^{r_1 t}, \ldots, t^{k-1}e^{r_1 t}$ are k independent solutions of (1). We prove this last assertion in the following manner. Observe from (3) that

$$L[e^{rt}] = (r - r_1)^k q(r) e^{rt}$$

if r_1 is a root of multiplicity k. Therefore,

$$\begin{aligned} L[t^j e^{r_1 t}] &= L\left[\frac{\partial^j}{\partial r^j} e^{rt}\right]\Bigg|_{r=r_1} \\ &= \frac{\partial^j}{\partial r^j} L[e^{rt}]\Bigg|_{r=r_1} \\ &= \frac{\partial^j}{\partial r^j}(r-r_1)^k q(r) e^{rt}\Bigg|_{r=r_1} \\ &= 0, \text{ for } 1 \leqslant j < k. \end{aligned}$$

Example 1. Find the general solution of the equation

$$\frac{d^4 y}{dt^4} + y = 0. \tag{5}$$

Solution. The characteristic equation of (5) is $r^4 + 1 = 0$. We find the roots of this equation by noting that

$$-1 = e^{i\pi} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i}.$$

Hence,

$$\begin{aligned} r_1 &= e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i), \\ r_2 &= e^{3\pi i/4} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}(1-i), \\ r_3 &= e^{5\pi i/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}(1+i), \end{aligned}$$

and

$$r_4 = e^{7\pi i/4} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}}(1-i)$$

are 4 roots of the equation $r^4+1=0$. The roots r_3 and r_4 are the complex conjugates of r_2 and r_1, respectively. Thus,

$$e^{r_1 t} = e^{t/\sqrt{2}}\left[\cos\frac{t}{\sqrt{2}} + i\sin\frac{t}{\sqrt{2}}\right]$$

and

$$e^{r_2 t} = e^{-t/\sqrt{2}}\left[\cos\frac{t}{\sqrt{2}} + i\sin\frac{t}{\sqrt{2}}\right]$$

are 2 complex-valued solutions of (5), and this implies that

$$y_1(t) = e^{t/\sqrt{2}}\cos\frac{t}{\sqrt{2}}, \qquad y_2(t) = e^{t/\sqrt{2}}\sin\frac{t}{\sqrt{2}},$$
$$y_3(t) = e^{-t/\sqrt{2}}\cos\frac{t}{\sqrt{2}}, \quad \text{and} \quad y_4(t) = e^{-t/\sqrt{2}}\sin\frac{t}{\sqrt{2}}$$

are 4 real-valued solutions of (5). These solutions are clearly independent. Hence, the general solution of (5) is

$$y(t) = e^{t/\sqrt{2}}\left[a_1\cos\frac{t}{\sqrt{2}} + b_1\sin\frac{t}{\sqrt{2}}\right]$$
$$+ e^{-t/\sqrt{2}}\left[a_2\cos\frac{t}{\sqrt{2}} + b_2\sin\frac{t}{\sqrt{2}}\right].$$

Example 2. Find the general solution of the equation

$$\frac{d^4y}{dt^4} - 3\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} - \frac{dy}{dt} = 0. \tag{6}$$

Solution. The characteristic equation of (6) is

$$0 = r^4 - 3r^3 + 3r^2 - r = r(r^3 - 3r^2 + 3r - 1)$$
$$= r(r-1)^3.$$

Its roots are $r_1 = 0$ and $r_2 = 1$, with $r_2 = 1$ a root of multiplicity three. Hence, the general solution of (6) is

$$y(t) = c_1 + (c_2 + c_3 t + c_4 t^2)e^t.$$

The theory for the nonhomogeneous equation

$$L[y] = a_n(t)\frac{d^ny}{dt^n} + \ldots + a_0(t)y = f(t), \quad a_n(t) \neq 0 \tag{7}$$

is also completely analogous to the theory for the second-order nonhomogeneous equation. The following results are the analogs of Lemma 1 and Theorem 5 of Section 2.3.

Lemma 1. *The difference of any two solutions of the nonhomogeneous equation* (7) *is a solution of the homogeneous equation* (1).

Theorem 11. *Let $\psi(t)$ be a particular solution of the nonhomogeneous equation* (7), *and let $y_1(t),\ldots,y_n(t)$ be n independent solutions of the homogeneous equation* (1). *Then, every solution $y(t)$ of* (7) *is of the form*

$$y(t)=\psi(t)+c_1y_1(t)+\ldots+c_ny_n(t)$$

for some choice of constants $c_1, c_2,\ldots,c_n$.

The method of judicious guessing also applies to the nth-order equation

$$a_n\frac{d^ny}{dt^n}+\ldots+a_0y=\left[b_0+b_1t+\ldots+b_kt^k\right]e^{\alpha t}. \tag{8}$$

It is easily verified that Equation (8) has a particular solution $\psi(t)$ of the form

$$\psi(t)=\left[c_0+c_1t+\ldots+c_kt^k\right]e^{\alpha t}$$

if $e^{\alpha t}$ is not a solution of the homogeneous equation, and

$$\psi(t)=t^j\left[c_0+c_1t+\ldots+c_kt^k\right]e^{\alpha t}$$

if $t^{j-1}e^{\alpha t}$ is a solution of the homogeneous equation, but $t^je^{\alpha t}$ is not.

Example 3. Find a particular solution $\psi(t)$ of the equation

$$L[y]=\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+3\frac{dy}{dt}+y=e^t. \tag{9}$$

Solution. The characteristic equation

$$r^3+3r^2+3r+1=(r+1)^3$$

has $r=-1$ as a triple root. Hence, e^t is not a solution of the homogeneous equation, and Equation (9) has a particular solution $\psi(t)$ of the form

$$\psi(t)=Ae^t.$$

Computing $L[\psi](t)=8Ae^t$, we see that $A=\frac{1}{8}$. Consequently, $\psi(t)=\frac{1}{8}e^t$ is a particular solution of (9).

There is also a variation of parameters formula for the nonhomogeneous equation (7). Let $v(t)$ be the solution of the homogeneous equation

(1) which satisfies the initial conditions $v(t_0)=0$, $v'(t_0)=0,\ldots,v^{(n-2)}(t_0)=0$, $v^{(n-1)}(t_0)=1$. Then,

$$\psi(t)=\int_{t_0}^{t}\frac{v(t-s)}{a_n(s)}f(s)\,ds$$

is a particular solution of the nonhomogeneous equation (7). We will prove this assertion in Section 3.12. (This can also be proven using the method of Laplace transforms; see Section 2.13.)

Exercises

Find the general solution of each of the following equations.

1. $y'''-2y''-y'+2y=0$

2. $y'''-6y''+5y'+12y=0$

3. $y^{(\mathrm{iv})}-5y'''+6y''+4y'-8y=0$

4. $y'''-y''+y'-y=0$

Solve each of the following initial-value problems.

5. $y^{(\mathrm{iv})}+4y'''+14y''-20y'+25y=0;\quad y(0)=y'(0)=y''(0)=0,\ y'''(0)=0$

6. $y^{(\mathrm{iv})}-y=0;\quad y(0)=1,\ y'(0)=y''(0)=0,\ y'''(0)=-1$

7. $y^{(\mathrm{v})}-2y^{(\mathrm{iv})}+y'''=0;\quad y(0)=y'(0)=y''(0)=y'''(0)=0,\ y^{(\mathrm{iv})}(0)=-1$

8. Given that $y_1(t)=e^t\cos t$ is a solution of

$$y^{(\mathrm{iv})}-2y'''+y''+2y'-2y=0, \tag{*}$$

find the general solution of (*). *Hint*: Use this information to find the roots of the characteristic equation of (*).

Find a particular solution of each of the following equations.

9. $y'''+y'=\tan t$

10. $y^{(\mathrm{iv})}-y=g(t)$

11. $y^{(\mathrm{iv})}+y=g(t)$

12. $y'''+y'=2t^2+4\sin t$

13. $y'''-4y'=t+\cos t+2e^{-2t}$

14. $y^{(\mathrm{iv})}-y=t+\sin t$

15. $y^{(\mathrm{iv})}+2y''+y=t^2\sin t$

16. $y^{(\mathrm{vi})}+y''=t^2$

17. $y'''+y''+y'+y=t+e^{-t}$

18. $y^{(\mathrm{iv})}+4y'''+6y''+4y'+y=t^3e^{-t}$

Hint for (18): Make the substitution $y=e^{-t}v$ and solve for v. Otherwise, it will take an awfully long time to do this problem.

3 Systems of differential equations

3.1 Algebraic properties of solutions of linear systems

In this chapter we will consider simultaneous first-order differential equations in several variables, that is, equations of the form

$$
\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, \ldots, x_n), \\
\frac{dx_2}{dt} &= f_2(t, x_1, \ldots, x_n), \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, \ldots, x_n).
\end{aligned}
\tag{1}
$$

A solution of (1) is n functions $x_1(t), \ldots, x_n(t)$ such that $dx_j(t)/dt = f_j(t, x_1(t), \ldots, x_n(t))$, $j = 1, 2, \ldots, n$. For example, $x_1(t) = t$ and $x_2(t) = t^2$ is a solution of the simultaneous first-order differential equations

$$\frac{dx_1}{dt} = 1 \quad \text{and} \quad \frac{dx_2}{dt} = 2x_1$$

since $dx_1(t)/dt = 1$ and $dx_2(t)/dt = 2t = 2x_1(t)$.

In addition to Equation (1), we will often impose initial conditions on the functions $x_1(t), \ldots, x_n(t)$. These will be of the form

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \ldots, x_n(t_0) = x_n^0. \tag{1'}$$

Equation (1), together with the initial conditions (1)$'$, is referred to as an initial-value problem. A solution of this initial-value problem is n functions $x_1(t), \ldots, x_n(t)$ which satisfy (1) and the initial conditions

$$x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0.$$

For example, $x_1(t) = e^t$ and $x_2(t) = 1 + e^{2t}/2$ is a solution of the initial-value problem

$$\frac{dx_1}{dt}=x_1, \qquad x_1(0)=1,$$

$$\frac{dx_2}{dt}=x_1^2, \qquad x_2(0)=\frac{3}{2},$$

since $dx_1(t)/dt=e^t=x_1(t)$, $dx_2(t)/dt=e^{2t}=x_1^2(t)$, $x_1(0)=1$ and $x_2(0)=\frac{3}{2}$.

Equation (1) is usually referred to as a system of n first-order differential equations. Equations of this type arise quite often in biological and physical applications and frequently describe very complicated systems since the rate of change of the variable x_j depends not only on t and x_j, but on the value of all the other variables as well. One particular example is the blood glucose model we studied in Section 2.7. In this model, the rates of change of g and h (respectively, the deviations of the blood glucose and net hormonal concentrations from their optimal values) are given by the equations

$$\frac{dg}{dt}=-m_1g-m_2h+J(t), \qquad \frac{dh}{dt}=-m_3h+m_4g.$$

This is a system of two first-order equations for the functions $g(t)$ and $h(t)$.

First-order systems of differential equations also arise from higher-order equations for a single variable $y(t)$. Every nth-order differential equation for the single variable y can be converted into a system of n first-order equations for the variables

$$x_1(t)=y, \quad x_2(t)=\frac{dy}{dt},\dots,x_n(t)=\frac{d^{n-1}y}{dt^{n-1}}.$$

Examples 1 and 2 illustrate how this works.

Example 1. Convert the differential equation

$$a_n(t)\frac{d^ny}{dt^n}+a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}}+\dots+a_0y=0$$

into a system of n first-order equations.

Solution. Let $x_1(t)=y$, $x_2(t)=dy/dt,\dots,$ and $x_n(t)=d^{n-1}y/dt^{n-1}$. Then,

$$\frac{dx_1}{dt}=x_2, \quad \frac{dx_2}{dt}=x_3,\dots, \frac{dx_{n-1}}{dt}=x_n,$$

and

$$\frac{dx_n}{dt}=-\frac{a_{n-1}(t)x_n+a_{n-2}(t)x_{n-1}+\dots+a_0x_1}{a_n(t)}.$$

Example 2. Convert the initial-value problem

$$\frac{d^3y}{dt^3}+\left(\frac{dy}{dt}\right)^2+3y=e^t; \qquad y(0)=1, \quad y'(0)=0, \quad y''(0)=0$$

into an initial-value problem for the variables y, dy/dt, and d^2y/dt^2.

Solution. Set $x_1(t)=y$, $x_2(t)=dy/dt$, and $x_3(t)=d^2y/dt^2$. Then,

$$\frac{dx_1}{dt}=x_2, \qquad \frac{dx_2}{dt}=x_3, \qquad \frac{dx_3}{dt}=e^t-x_2^2-3x_1.$$

Moreover, the functions x_1, x_2, and x_3 satisfy the initial conditions $x_1(0)=1$, $x_2(0)=0$, and $x_3(0)=0$.

If each of the functions $f_1,\ldots,f_n$ in (1) is a linear function of the dependent variables $x_1,\ldots,x_n$, then the system of equations is said to be linear. The most general system of n first-order linear equations has the form

$$\begin{aligned}\frac{dx_1}{dt}&=a_{11}(t)x_1+\ldots+a_{1n}(t)x_n+g_1(t)\\ &\ \vdots\\ \frac{dx_n}{dt}&=a_{n1}(t)x_1+\ldots+a_{nn}(t)x_n+g_n(t).\end{aligned} \tag{2}$$

If each of the functions $g_1,\ldots,g_n$ is identically zero, then the system (2) is said to be homogeneous; otherwise it is nonhomogeneous. In this chapter, we only consider the case where the coefficients a_{ij} do not depend on t.

Now, even the homogeneous linear system with constant coefficients

$$\begin{aligned}\frac{dx_1}{dt}&=a_{11}x_1+\ldots+a_{1n}x_n\\ &\ \vdots\\ \frac{dx_n}{dt}&=a_{n1}x_1+\ldots+a_{nn}x_n\end{aligned} \tag{3}$$

is quite cumbersome to handle. This is especially true if n is large. Therefore, we seek to write these equations in as concise a manner as possible. To this end we introduce the concepts of *vectors* and *matrices*.

Definition. A *vector*

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\ \vdots\\ x_n\end{bmatrix}$$

is a shorthand notation for the sequence of numbers $x_1,\ldots,x_n$. The numbers $x_1,\ldots,x_n$ are called the *components* of $\mathbf{x}$. If $x_1=x_1(t),\ldots,$ and $x_n=x_n(t)$, then

$$\mathbf{x}(t)=\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}$$

is called a vector-valued function. Its derivative $d\mathbf{x}(t)/dt$ is the vector-valued function

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \vdots \\ \dfrac{dx_n(t)}{dt} \end{bmatrix}.$$

Definition. A *matrix*

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is a shorthand notation for the array of numbers a_{ij} arranged in m rows and n columns. The element lying in the ith row and jth column is denoted by a_{ij}, the first subscript identifying its row and the second subscript identifying its column. $\mathbf{A}$ is said to be a square matrix if $m=n$.

Next, we define the product of a matrix $\mathbf{A}$ with a vector $\mathbf{x}$.

Definition. Let $\mathbf{A}$ be an $n\times n$ matrix with elements a_{ij} and let $\mathbf{x}$ be a vector with components $x_1,\dots,x_n$. We define the product of $\mathbf{A}$ with $\mathbf{x}$, denoted by $\mathbf{Ax}$, as the vector whose ith component is

$$a_{i1}x_1+a_{i2}x_2+\dots+a_{in}x_n, \qquad i=1,2,\dots,n.$$

In other words, the ith component of $\mathbf{Ax}$ is the sum of the product of corresponding terms of the ith row of $\mathbf{A}$ with the vector $\mathbf{x}$. Thus,

$$\mathbf{Ax}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n \\ \vdots \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 6 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 3+4+4 \\ -3+0+6 \\ 3+2+1 \end{bmatrix}=\begin{bmatrix} 11 \\ 3 \\ 6 \end{bmatrix}.$$

Finally, we observe that the left-hand sides of (3) are the components of the vector $d\mathbf{x}/dt$, while the right-hand sides of (3) are the components of the vector $\mathbf{Ax}$. Hence, we can write (3) in the concise form

$$\dot{\mathbf{x}}=\frac{d\mathbf{x}}{dt}=\mathbf{Ax},\quad \text{where}\quad \mathbf{x}=\begin{bmatrix} x_1\\ \vdots\\ x_n\end{bmatrix}\quad\text{and}\quad \mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{bmatrix}. \tag{4}$$

Moreover, if $x_1(t),\dots,x_n(t)$ satisfy the initial conditions

$$x_1(t_0)=x_1^0,\dots,x_n(t_0)=x_n^0,$$

then $\mathbf{x}(t)$ satisfies the initial-value problem

$$\dot{\mathbf{x}}=\mathbf{Ax},\qquad \mathbf{x}(t_0)=\mathbf{x}^0,\quad\text{where}\quad \mathbf{x}^0=\begin{bmatrix} x_1^0\\ \vdots\\ x_n^0\end{bmatrix}. \tag{5}$$

For example, the system of equations

$$\frac{dx_1}{dt}=3x_1-7x_2+9x_3$$
$$\frac{dx_2}{dt}=15x_1+x_2-x_3$$
$$\frac{dx_3}{dt}=7x_1+6x_3$$

can be written in the concise form

$$\dot{\mathbf{x}}=\begin{bmatrix} 3 & -7 & 9\\ 15 & 1 & -1\\ 7 & 0 & 6\end{bmatrix}\mathbf{x},\qquad \mathbf{x}=\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix},$$

and the initial-value problem

$$\frac{dx_1}{dt}=x_1-x_2+x_3,\qquad x_1(0)=1$$
$$\frac{dx_2}{dt}=3x_2-x_3,\qquad x_2(0)=0$$
$$\frac{dx_3}{dt}=x_1+7x_3,\qquad x_3(0)=-1$$

can be written in the concise form

$$\dot{\mathbf{x}}=\begin{bmatrix} 1 & -1 & 1\\ 0 & 3 & -1\\ 1 & 0 & 7\end{bmatrix}\mathbf{x},\qquad \mathbf{x}(0)=\begin{bmatrix} 1\\ 0\\ -1\end{bmatrix}.$$

Now that we have succeeded in writing (3) in the more manageable form (4), we can tackle the problem of finding all of its solutions. Since these equations are linear, we will try and play the same game that we played, with so much success, with the second-order linear homogeneous equation. To wit, we will show that a constant times a solution and the sum of two solutions are again solutions of (4). Then, we will try and show that we can find every solution of (4) by taking all linear combinations of a finite number of solutions. Of course, we must first define what we mean by a constant times $\mathbf{x}$ and the sum of $\mathbf{x}$ and $\mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are vectors with n components.

Definition. Let c be a number and $\mathbf{x}$ a vector with n components $x_1,\dots,x_n$. We define $c\mathbf{x}$ to be the vector whose components are $cx_1,\dots,cx_n$, that is

$$c\mathbf{x}=c\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix} cx_1\\ cx_2\\ \vdots\\ cx_n\end{bmatrix}.$$

For example, if

$$c=2 \quad\text{and}\quad \mathbf{x}=\begin{bmatrix}3\\1\\7\end{bmatrix},\quad\text{then}\quad 2\mathbf{x}=2\begin{bmatrix}3\\1\\7\end{bmatrix}=\begin{bmatrix}6\\2\\14\end{bmatrix}.$$

This process of multiplying a vector $\mathbf{x}$ by a number c is called scalar multiplication.

Definition. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors with components $x_1,\dots,x_n$ and $y_1,\dots,y_n$ respectively. We define $\mathbf{x}+\mathbf{y}$ to be the vector whose components are $x_1+y_1,\dots,x_n+y_n$, that is

$$\mathbf{x}+\mathbf{y}=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix}=\begin{bmatrix} x_1+y_1\\ x_2+y_2\\ \vdots\\ x_n+y_n\end{bmatrix}.$$

For example, if

$$\mathbf{x}=\begin{bmatrix}1\\6\\3\\2\end{bmatrix}\quad\text{and}\quad \mathbf{y}=\begin{bmatrix}-1\\-6\\7\\9\end{bmatrix},$$

then

$$\mathbf{x}+\mathbf{y}=\begin{bmatrix}1\\6\\3\\2\end{bmatrix}+\begin{bmatrix}-1\\-6\\7\\9\end{bmatrix}=\begin{bmatrix}0\\0\\10\\11\end{bmatrix}.$$

This process of adding two vectors together is called vector addition.

Having defined the processes of scalar multiplication and vector addition, we can now state the following theorem.

Theorem 1. *Let* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *be two solutions of* (4). *Then* (*a*) $c\mathbf{x}(t)$ *is a solution, for any constant* c, *and* (*b*) $\mathbf{x}(t)+\mathbf{y}(t)$ *is again a solution.*

Theorem 1 can be proven quite easily with the aid of the following lemma.

Lemma. *Let* $\mathbf{A}$ *be an* $n\times n$ *matrix. For any vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *and constant* c, (*a*) $\mathbf{A}(c\mathbf{x})=c\mathbf{A}\mathbf{x}$ *and* (*b*) $\mathbf{A}(\mathbf{x}+\mathbf{y})=\mathbf{A}\mathbf{x}+\mathbf{A}\mathbf{y}$.

PROOF OF LEMMA.
(a) We prove that two vectors are equal by showing that they have the same components. To this end, observe that the ith component of the vector $c\mathbf{A}\mathbf{x}$ is

$$ca_{i1}x_1+ca_{i2}x_2+\ldots+ca_{in}x_n=c(a_{i1}x_1+\ldots+a_{in}x_n),$$

and the ith component of the vector $\mathbf{A}(c\mathbf{x})$ is

$$a_{i1}(cx_1)+a_{i2}(cx_2)+\ldots+a_{in}(cx_n)=c(a_{i1}x_1+\ldots+a_{in}x_n).$$

Hence $\mathbf{A}(c\mathbf{x})=c\mathbf{A}\mathbf{x}$.
(b) The ith component of the vector $\mathbf{A}(\mathbf{x}+\mathbf{y})$ is

$$a_{i1}(x_1+y_1)+\ldots+a_{in}(x_n+y_n)=(a_{i1}x_1+\ldots+a_{in}x_n)+(a_{i1}y_1+\ldots+a_{in}y_n).$$

But this is also the ith component of the vector $\mathbf{A}\mathbf{x}+\mathbf{A}\mathbf{y}$ since the ith component of $\mathbf{A}\mathbf{x}$ is $a_{i1}x_1+\ldots+a_{in}x_n$ and the ith component of $\mathbf{A}\mathbf{y}$ is $a_{i1}y_1+\ldots+a_{in}y_n$. Hence $\mathbf{A}(\mathbf{x}+\mathbf{y})=\mathbf{A}\mathbf{x}+\mathbf{A}\mathbf{y}$. □

PROOF OF THEOREM 1.
(a). If $\mathbf{x}(t)$ is a solution of (4), then

$$\frac{d}{dt}c\mathbf{x}(t)=c\frac{d\mathbf{x}(t)}{dt}=c\mathbf{A}\mathbf{x}(t)=\mathbf{A}(c\mathbf{x}(t)).$$

Hence, $c\mathbf{x}(t)$ is also a solution of (4).
(b). If $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions of (4) then

$$\frac{d}{dt}(\mathbf{x}(t)+\mathbf{y}(t))=\frac{d\mathbf{x}(t)}{dt}+\frac{d\mathbf{y}(t)}{dt}=\mathbf{A}\mathbf{x}(t)+\mathbf{A}\mathbf{y}(t)=\mathbf{A}(\mathbf{x}(t)+\mathbf{y}(t)).$$

Hence, $\mathbf{x}(t)+\mathbf{y}(t)$ is also a solution of (4). □

An immediate corollary of Theorem 1 is that any linear combination of solutions of (4) is again a solution of (4). That is to say, if $\mathbf{x}^1(t),\ldots,\mathbf{x}^j(t)$ are j solutions of (4), then $c_1\mathbf{x}^1(t)+\ldots+c_j\mathbf{x}^j(t)$ is again a solution for any choice of constants $c_1, c_2, \ldots, c_j$. For example, consider the system of equations

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -4x_1, \quad \text{or} \quad \frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6}$$

This system of equations was derived from the second-order scalar equation $(d^2y/dt^2) + 4y = 0$ by setting $x_1 = y$ and $x_2 = dy/dt$. Since $y_1(t) = \cos 2t$ and $y_2(t) = \sin 2t$ are two solutions of the scalar equation, we know that

$$\begin{aligned} \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= c_1 \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix} \\ &= \begin{pmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix} \end{aligned}$$

is a solution of (6) for any choice of constants c_1 and c_2.

The next step in our game plan is to show that every solution of (4) can be expressed as a linear combination of finitely many solutions. Equivalently, we seek to determine how many solutions we must find before we can generate all the solutions of (4). There is a branch of mathematics known as linear algebra, which addresses itself to exactly this question, and it is to this area that we now turn our attention.

Exercises

In each of Exercises 1–3 convert the given differential equation for the single variable y into a system of first-order equations.

1. $\dfrac{d^3y}{dt^3} + \left(\dfrac{dy}{dt}\right)^2 = 0$ **2.** $\dfrac{d^3y}{dt^3} + \cos y = e^t$ **3.** $\dfrac{d^4y}{dt^4} + \dfrac{d^2y}{dt^2} = 1$

4. Convert the pair of second-order equations

$$\frac{d^2y}{dt^2} + 3\frac{dz}{dt} + 2y = 0, \qquad \frac{d^2z}{dt^2} + 3\frac{dy}{dt} + 2z = 0$$

into a system of 4 first-order equations for the variables

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = z, \quad \text{and} \quad x_4 = z'.$$

5. (a) Let $y(t)$ be a solution of the equation $y'' + y' + y = 0$. Show that

$$\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of the system of equations

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{x}.$$

(b) Let

$$\mathbf{x}(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

be a solution of the system of equations

$$\dot{\mathbf{x}}=\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{x}.$$

Show that $y=x_1(t)$ is a solution of the equation $y''+y'+y=0$.

In each of Exercises 6–9, write the given system of differential equations and initial values in the form $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}, \mathbf{x}(t_0)=\mathbf{x}^0$.

6. $\dot{x}_1=3x_1-7x_2, \quad x_1(0)=1$
$\dot{x}_2=4x_1, \quad x_2(0)=1$

7. $\dot{x}_1=5x_1+5x_2, \quad x_1(3)=0$
$\dot{x}_2=-x_1+7x_2, \quad x_2(3)=6$

8. $\dot{x}_1=x_1+x_2-x_3, \quad x_1(0)=0$
$\dot{x}_2=3x_1-x_2+4x_3, \quad x_2(0)=1$
$\dot{x}_3=-x_1-x_2, \quad x_3(0)=-1$

9. $\dot{x}_1=-x_3, \quad x_1(-1)=2$
$\dot{x}_2=x_1, \quad x_2(-1)=3$
$\dot{x}_3=-x_2, \quad x_3(-1)=4$

10. Let

$$\mathbf{x}=\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{y}=\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

Compute $\mathbf{x}+\mathbf{y}$ and $3\mathbf{x}-2\mathbf{y}$.

11. Let

$$\mathbf{A}=\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ -1 & -1 & 2 \end{pmatrix}.$$

Compute $\mathbf{A}\mathbf{x}$ if

$$\text{(a) } \mathbf{x}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(b) } \mathbf{x}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{(c) } \mathbf{x}=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

12. Let $\mathbf{A}$ be any $n\times n$ matrix and let $\mathbf{e}^j$ be the vector whose jth component is 1 and whose remaining components are zero. Verify that the vector $\mathbf{A}\mathbf{e}^j$ is the jth column of $\mathbf{A}$.

13. Let

$$\mathbf{A}=\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}.$$

Compute $\mathbf{A}\mathbf{x}$ if

$$\text{(a) } \mathbf{x}=\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \text{(b) } \mathbf{x}=\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \text{(c) } \mathbf{x}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{(d) } \mathbf{x}=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

14. Let $\mathbf{A}$ be a 3×3 matrix with the property that

$$\mathbf{A}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Compute

$$\mathbf{A}\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}.$$

Hint: Write

$$\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

as a linear combination of

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

15. Let **A** be a 2×2 matrix with the property that

$$\mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}.$$

Find **A**. *Hint*: The easy way is to use Exercise 12.

3.2 Vector spaces

In the previous section we defined, in a natural manner, a process of adding two vectors $\mathbf{x}$ and $\mathbf{y}$ together to form a new vector $\mathbf{z}=\mathbf{x}+\mathbf{y}$, and a process of multiplying a vector $\mathbf{x}$ by a scalar c to form a new vector $\mathbf{u}=c\mathbf{x}$. The former process was called vector addition and the latter process was called scalar multiplication. Our study of linear algebra begins with the more general premise that we have a set **V** of elements $\mathbf{x},\mathbf{y},\mathbf{z},\ldots$ and that we have one process that combines two elements $\mathbf{x}$ and $\mathbf{y}$ of **V** to form a third element $\mathbf{z}$ in **V** and a second process that combines a number c and an element $\mathbf{x}$ in **V** to form a new element $\mathbf{u}$ in **V**. We will denote the first process by addition; that is, we will write $\mathbf{z}=\mathbf{x}+\mathbf{y}$, and the second process by scalar multiplication; that is, we will write $\mathbf{u}=c\mathbf{x}$, if they satisfy the usual axioms of addition and multiplication. These axioms are:

(i) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ (commutative law)
(ii) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$ (associative law)
(iii) There is a unique element in **V**, called the zero element, and denoted by $\mathbf{0}$, having the property that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for all $\mathbf{x}$ in **V**.
(iv) For each element $\mathbf{x}$ in **V** there is a unique element, denoted by $-\mathbf{x}$ and called minus $\mathbf{x}$, such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.
(v) $1\cdot\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}$ in **V**.
(vi) $(ab)\mathbf{x}=a(b\mathbf{x})$ for any numbers a,b and any element $\mathbf{x}$ in **V**.
(vii) $a(\mathbf{x}+\mathbf{y})=a\mathbf{x}+a\mathbf{y}$
(viii) $(a+b)\mathbf{x}=a\mathbf{x}+b\mathbf{x}$.

A set **V**, together with processes addition and multiplication satisfying (i)–(viii) is said to be a *vector space* and its elements are called *vectors*. The numbers a, b will usually be real numbers, except in certain special cases where they will be complex numbers.

Remark 1. Implicit in axioms (i)–(viii) is the fact that if $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbf{V}$, then the linear combination $a\mathbf{x}+b\mathbf{y}$ is again in $\mathbf{V}$ for any choice of constants a and b.

Remark 2. In the previous section we defined a vector $\mathbf{x}$ as a sequence of n numbers. In the more general context of this section, a quantity $\mathbf{x}$ is a vector by dint of its being in a vector space. That is to say, a quantity $\mathbf{x}$ is a vector if it belongs to a set of elements $\mathbf{V}$ which is equipped with two processes (addition and scalar multiplication) which satisfy (i)–(viii). As we shall see in Example 3 below, the set of all sequences

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

of n real numbers is a vector space (with the usual operations of vector addition and scalar multiplication defined in Section 3.1). Thus, our two definitions are consistent.

Example 1. Let $\mathbf{V}$ be the set of all functions $x(t)$ which satisfy the differential equation

$$\frac{d^2x}{dt^2}-x=0 \tag{1}$$

with the sum of two functions and the product of a function by a number being defined in the usual manner. That is to say,

$$(f_1+f_2)(t)=f_1(t)+f_2(t)$$

and

$$(cf)(t)=cf(t).$$

It is trivial to verify that $\mathbf{V}$ is a vector space. Observe first that if x^1 and x^2 are in $\mathbf{V}$, then every linear combination $c_1x^1+c_2x^2$ is in $\mathbf{V}$, since the differential equation (1) is linear. Moreover, axioms (i), (ii), and (v)–(viii) are automatically satisfied since all we are doing at any time t in function addition and multiplication of a function by a number is adding or multiplying two numbers together. The zero vector in $\mathbf{V}$ is the function whose value at any time t is zero; this function is in $\mathbf{V}$ since $x(t)\equiv 0$ is a solution of (1). Finally, the negative of any function in $\mathbf{V}$ is again in $\mathbf{V}$, since the negative of any solution of (1) is again a solution of (1).

Example 2. Let **V** be the set of all solutions $x(t)$ of the differential equation $(d^2x/dt^2) - 6x^2 = 0$, with the sum of two functions and the product of a function by a number being defined in the usual manner. **V** is not a vector space since the sum of any two solutions, while being defined, is not necessarily in **V**. Similarly, the product of a solution by a constant is not necessarily in **V**. For example, the function $x(t) = 1/t^2$ is in **V** since it satisfies the differential equation, but the function $2x(t) = 2/t^2$ is not in **V** since it does not satisfy the differential equation.

Example 3. Let **V** be the set of all sequences

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

of n real numbers. Define $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ as the vector addition and scalar multiplication defined in Section 3.1. It is trivial to verify that **V** is a vector space under these operations. The zero vector is the sequence

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and the vector $-\mathbf{x}$ is the vector

$$\begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix}.$$

This space is usually called n dimensional Euclidean space and is denoted by R^n.

Example 4. Let **V** be the set of all sequences

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

of n complex numbers $x_1, \ldots, x_n$. Define $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$, for any complex number c, as the vector addition and scalar multiplication defined in Section 3.1. Again, it is trivial to verify that **V** is a vector space under these operations. This space is usually called complex n dimensional space and is denoted by C^n.

Example 5. Let **V** be the set of all $n\times n$ matrices **A**. Define the sum of two matrices **A** and **B** to be the matrix obtained by adding together corresponding elements of **A** and **B**, and define the matrix $c\mathbf{A}$ to be the matrix obtained by multiplying every element of **A** by c. In other words,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

and

$$c\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nn} \end{bmatrix}.$$

Axioms (i), (ii), and (v)–(viii) are automatically satisfied since all we are doing in adding two matrices together or multiplying a matrix by a number is adding or multiplying two numbers together. The zero vector, or the matrix **0**, is the matrix whose every element is the number zero, and the negative of any matrix **A** is the matrix

$$\begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & & \vdots \\ -a_{n1} & \cdots & -a_{nn} \end{bmatrix}.$$

Hence **V** is a vector space under these operations of matrix addition and scalar multiplication.

Example 6. We now present an example of a set of elements which comes close to being a vector space, but which doesn't quite make it. The purpose of this example is to show that the elements of **V** can be just about anything, and the operation of addition can be a rather strange process. Let **V** be the set consisting of three animals, a cat, a dog, and a mouse. Whenever any two of these animals meet, one eats up the other and changes into a different animal. The rules of eating are as follows.

(1) If a dog meets a cat, then the dog eats up the cat and changes into a mouse.
(2) If a dog meets another dog, then one dog eats up the other and changes into a cat.
(3) If a dog meets a mouse, then the dog eats up the mouse and remains unchanged.
(4) If a cat meets another cat, then one cat eats up the other and changes into a dog.
(5) If a cat meets a mouse, then the cat eats up the mouse and remains unchanged.
(6) If a mouse meets another mouse, then one mouse eats up the other and remains unchanged.

Clearly, "eating" is a process which combines two elements of **V** to form a third element in **V**. If we call this eating process addition, and denote it by $+$, then rules 1–6 can be written concisely in the form

1. $D+C=M$	2. $D+D=C$	3. $D+M=D$
4. $C+C=D$	5. $C+M=C$	6. $M+M=M$.

This operation of eating satisfies all the axioms of addition. To see this, note that axiom (i) is satisfied since the eating formulae do not depend on the order of the two animals involved. This is to say, $D+C=C+D$, etc. Moreover, the result of any addition is again an animal in **V**. This would not be the case, for example, if a dog ate up a cat and changed into a hippopotamus. The associative law (ii) is also satisfied, but it has to be verified explicitly. For example, suppose that we have an encounter between two cats and a dog. It is not obvious, a priori, that it does not make a difference whether the two cats meet first and their resultant meets the dog or whether one cat meets the dog and their resultant meets the other cat. To check that this is so we compute

$$(C+C)+D=D+D=C$$

and

$$(C+D)+C=M+C=C.$$

In a similar manner we can show that the result of any encounter between three animals is independent of the order in which they meet. Next, observe that the zero element in **V** is the mouse, since every animal is unchanged after eating a mouse. Finally, "minus a dog" is a cat (since $D+C=M$), "minus a cat" is a dog and "minus a mouse" is a mouse. However, **V** is *not* a vector space since there is no operation of scalar multiplication defined. Moreover, it is clearly impossible to define the quantities aC and aD, for all real numbers a, so as to satisfy axioms (v)–(viii).

Example 7. Let **V** be the set of all vector-valued solutions

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t)\\ \vdots \\ x_n(t)\end{bmatrix}$$

of the vector differential equation

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}, \qquad \mathbf{A}=\begin{bmatrix} a_{11} & \dots & a_{1n}\\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn}\end{bmatrix}. \tag{2}$$

V is a vector space under the usual operations of vector addition and scalar multiplication. To wit, observe that axioms (i), (ii), and (v)–(viii) are automatically satisfied. Hence, we need only verify that

(a) The sum of any two solutions of (2) is again a solution.
(b) A constant times a solution of (2) is again a solution.
(c) The vector-valued function

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t)\\ \vdots \\ x_n(t)\end{bmatrix}=\begin{bmatrix} 0\\ \vdots \\ 0\end{bmatrix}$$

is a solution of (2) (axiom (iii)).
(d) The negative of any solution of (2) is again a solution (axiom (iv)).

Now (a) and (b) are exactly Theorem 1 of the previous section, while (d) is a special case of (b). To verify (c) we observe that

$$\frac{d}{dt}\begin{bmatrix} 0\\ \vdots \\ 0\end{bmatrix}=\begin{bmatrix} 0\\ \vdots \\ 0\end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix} 0\\ \vdots \\ 0\end{bmatrix}=\begin{bmatrix} 0\\ \vdots \\ 0\end{bmatrix}.$$

Hence the vector-valued function $\mathbf{x}(t)\equiv\mathbf{0}$ is always a solution of the differential equation (2).

Exercises

In each of Problems 1–6, determine whether the given set of elements

$$\mathbf{x}=\begin{bmatrix} x_1\\ x_2 \\ x_3\end{bmatrix}$$

form a vector space under the properties of vector addition and scalar multiplication defined in Section 3.1.

1. The set of all elements $\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $3x_1-2x_2=0$

2. The set of all elements $\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $x_1+x_2+x_3=0$

3. The set of all elements $\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $x_1^2+x_2^2+x_3^2=1$

4. The set of all elements $\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $x_1+x_2+x_3=1$

5. The set of elements $\mathbf{x}=\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$ for all real numbers a and b

6. The set of all elements $\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where

$$x_1+x_2+x_3=0, \quad x_1-x_2+2x_3=0, \quad 3x_1-x_2+5x_3=0$$

In each of Problems 7–11 determine whether the given set of functions form a vector space under the usual operations of function addition and multiplication of a function by a constant.

7. The set of all polynomials of degree $\leqslant 4$

8. The set of all differentiable functions

9. The set of all differentiable functions whose derivative at $t=1$ is three

10. The set of all solutions of the differential equation $y''+y=\cos t$

11. The set of all functions $y(t)$ which have period 2π, that is $y(t+2\pi)=y(t)$

12. Show that the set of all vector-valued solutions

$$\mathbf{x}(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

of the system of differential equations

$$\frac{dx_1}{dt}=x_2+1, \qquad \frac{dx_2}{dt}=x_1+1$$

is not a vector space.

3.3 Dimension of a vector space

Let $\mathbf{V}$ be the set of all solutions $y(t)$ of the second-order linear homogeneous equation $(d^2y/dt^2)+p(t)(dy/dt)+q(t)y=0$. Recall that every solution $y(t)$ can be expressed as a linear combination of any two linearly independent solutions. Thus, if we knew two "independent" functions $y^1(t)$ and

$y^2(t)$ in **V**, then we could find every function in **V** by taking all linear combinations $c_1y^1(t)+c_2y^2(t)$ of y^1 and y^2. We would like to derive a similar property for solutions of the equation $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}$. To this end, we define the notion of a finite set of vectors generating the whole space, and the notion of independence of vectors in an arbitrary vector space **V**.

Definition. A set of vectors $\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^n$ is said to *span* **V** if the set of all linear combinations $c_1\mathbf{x}^1+c_2\mathbf{x}^2+\ldots+c_n\mathbf{x}^n$ exhausts **V**. That is to say, the vectors $\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^n$ span **V** if every element of **V** can be expressed as a linear combination of $\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^n$.

Example 1. Let **V** be the set of all solutions of the differential equation $(d^2x/dt^2)-x=0$. Let x^1 be the function whose value at any time t is e^t and let x^2 be the function whose value at any time t is e^{-t}. The functions x^1 and x^2 are in **V** since they satisfy the differential equation. Moreover, these functions also span **V** since every solution $x(t)$ of the differential equation can be written in the form

$$x(t)=c_1e^t+c_2e^{-t}$$

so that

$$x=c_1x^1+c_2x^2.$$

Example 2. Let $\mathbf{V}=\mathsf{R}^n$ and let $\mathbf{e}^j$ denote the vector with a 1 in the jth place and zeros everywhere else, that is,

$$\mathbf{e}^1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\qquad \mathbf{e}^2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\ldots,\mathbf{e}^n=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}.$$

The set of vectors $\mathbf{e}^1,\mathbf{e}^2,\ldots,\mathbf{e}^n$ span R^n since any vector

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$$

can be written in the form

$$\mathbf{x}=\begin{bmatrix}x_1\\0\\\vdots\\0\end{bmatrix}+\begin{bmatrix}0\\x_2\\\vdots\\0\end{bmatrix}+\ldots+\begin{bmatrix}0\\0\\\vdots\\x_n\end{bmatrix}=x_1\mathbf{e}^1+x_2\mathbf{e}^2+\ldots+x_n\mathbf{e}^n.$$

Definition. A set of vectors $\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n$ in $\mathbf{V}$ is said to be *linearly dependent* if one of these vectors is a linear combination of the others. A very precise mathematical way of saying this is as follows. A set of vectors $\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n$ is said to be linearly dependent if there exist constants $c_1,c_2,\dots,c_n$, *not all zero* such that

$$c_1\mathbf{x}^1+c_2\mathbf{x}^2+\dots+c_n\mathbf{x}^n=\mathbf{0}.$$

These two definitions are equivalent, for if $\mathbf{x}^j$ is a linear combination of $\mathbf{x}^1,\dots,\mathbf{x}^{j-1},\mathbf{x}^{j+1},\dots,\mathbf{x}^n$, that is

$$\mathbf{x}^j=c_1\mathbf{x}^1+\dots+c_{j-1}\mathbf{x}^{j-1}+c_{j+1}\mathbf{x}^{j+1}+\dots+c_n\mathbf{x}^n,$$

then the linear combination

$$c_1\mathbf{x}^1+\dots+c_{j-1}\mathbf{x}^{j-1}-\mathbf{x}^j+c_{j+1}\mathbf{x}^{j+1}+\dots+c_n\mathbf{x}^n$$

equals zero and not all the constants are zero. Conversely, if $c_1\mathbf{x}^1+c_2\mathbf{x}^2+\dots+c_n\mathbf{x}^n=\mathbf{0}$ and $c_j\neq 0$ for some j, then we can divide by c_j and solve for $\mathbf{x}^j$ as a linear combination of $\mathbf{x}^1,\dots,\mathbf{x}^{j-1},\mathbf{x}^{j+1},\dots,\mathbf{x}^n$. For example, if $c_1\neq 0$ then we can divide by c_1 to obtain that

$$\mathbf{x}^1=-\frac{c_2}{c_1}\mathbf{x}^2-\frac{c_3}{c_1}\mathbf{x}^3-\dots-\frac{c_n}{c_1}\mathbf{x}^n.$$

Definition. If the vectors $\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n$ are not linearly dependent, that is, none of these vectors can be expressed as a linear combination of the others, then they are said to be *linearly independent*. The precise mathematical way of saying this is that the vectors $\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n$ are linearly independent if the equation

$$c_1\mathbf{x}^1+c_2\mathbf{x}^2+\dots+c_n\mathbf{x}^n=\mathbf{0}$$

implies, of necessity, that all the constants $c_1,c_2,\dots,c_n$ are zero.

In order to determine whether a set of vectors $\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n$ is linearly dependent or linearly independent, we write down the equation $c_1\mathbf{x}^1+c_2\mathbf{x}^2+\dots+c_n\mathbf{x}^n=\mathbf{0}$ and see what this implies about the constants $c_1,c_2,\dots,c_n$. If all these constants must be zero, then $\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n$ are linearly independent. On the other hand, if not all the constants $c_1,c_2,\dots,c_n$ must be zero, then $\mathbf{x}^1,\mathbf{x}^2,\dots,\mathbf{x}^n$ are linearly dependent.

Example 3. Let $\mathbf{V}=\mathsf{R}^3$ and let $\mathbf{x}^1$, $\mathbf{x}^2$, and $\mathbf{x}^3$ be the vectors

$$\mathbf{x}^1=\begin{bmatrix}1\\-1\\1\end{bmatrix},\qquad \mathbf{x}^2=\begin{bmatrix}1\\2\\3\end{bmatrix}\quad\text{and}\quad \mathbf{x}^3=\begin{bmatrix}3\\0\\5\end{bmatrix}.$$

To determine whether these vectors are linearly dependent or linearly independent, we write down the equation $c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + c_3\mathbf{x}^3 = \mathbf{0}$, that is

$$c_1\begin{bmatrix}1\\-1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\2\\3\end{bmatrix}+c_3\begin{bmatrix}3\\0\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The left-hand side of this equation is the vector

$$\begin{bmatrix}c_1+c_2+3c_3\\-c_1+2c_2\\c_1+3c_2+5c_3\end{bmatrix}.$$

Hence the constants c_1, c_2, and c_3 must satisfy the equations

$$c_1+c_2+3c_3=0, \tag{i}$$

$$-c_1+2c_2=0, \tag{ii}$$

$$c_1+3c_2+5c_3=0. \tag{iii}$$

Equation (ii) says that $c_1=2c_2$. Substituting this into Equations (i) and (iii) gives

$$3c_2+3c_3=0 \quad \text{and} \quad 5c_2+5c_3=0.$$

These equations have infinitely many solutions c_2, c_3 since they both reduce to the single equation $c_2+c_3=0$. One solution, in particular, is $c_2=-1$, $c_3=1$. Then, from Equation (ii), $c_1=-2$. Hence,

$$-2\begin{bmatrix}1\\-1\\1\end{bmatrix}-\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}3\\0\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

and $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$ are linearly dependent vectors in R^3.

Example 4. Let $\mathbf{V}=\mathsf{R}^n$ and let $\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n$ be the vectors

$$\mathbf{e}^1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, \quad \mathbf{e}^2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \ldots, \mathbf{e}^n=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}.$$

To determine whether $\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n$ are linearly dependent or linearly independent, we write down the equation $c_1\mathbf{e}^1 + \ldots + c_n\mathbf{e}^n = \mathbf{0}$, that is

$$c_1\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}+\ldots+c_n\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\\vdots\\0\end{bmatrix}.$$