A Homological Characterization and Unique Factorization Property of Regular Local Rings

A Thesis submitted for the completion of requirements for the degree of

Master of Science

(Research)

by

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Integrated BS-MS Programme (2018-2023)

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To my dearest Posto and Papu

Declaration

I, Trishita Patra, Roll No. 18MS188, a student of Department of Mathematics and Statistics, IISER Kolkata, hereby declare that this report titled **"A Homological Characterization and Unique Factorization of Property Regular Local Rings"**, is my own work, and to the best of my knowledge, contains neither material previously published or written by any other entity, nor substantial proportions of any material, that has been accepted for the award of any other degree or diploma at IISER Kolkata or any other educational institution, except where due acknowledgement is made within this report.

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Certificate

This is to certify that the work entitled **"A Homological Characterization and Unique Factorization Property of Regular Local Rings"** embodies the study done by **Trishita Patra** under my guidance and supervision for the partial fulfilment of the degree of *Master of Science (MS), Indian Institute of Science Education and Research Kolkata, India.* It has not previously formed the basis for the award of any Degree, Diploma, Associate-ship or Fellowship to her.

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Abstract

The two main theorems of this thesis are Serre's characterization of regular local rings, which says that a Noetherian local ring is regular if and only if its global dimension is finite and in that case global dimension and Krull dimension are equal; the second theorem says that any regular local ring is a unique factorization domain. We study notion of dimension from both homological algebraic and ring theoretic perspective, and properties of a regular local ring. We finally prove the theorems using results from commutative algebra and homological algebra.

Acknowledgement

My academic journey at IISER Kolkata, the ultimate product of which is this thesis, has instilled in me a deep appreciation for the systematic, logical, and rigorous approach to mathematics. I am grateful to the BS-MS program at IISER Kolkata for providing an excellent environment. Alongside honing my mathematical maturity and skills, it deepened my curiosity in science and research, playing a significant role in shaping me as a person. My whole-hearted gratitude goes to my supervisor, Dr. Md. Ali Zinna, for the insightful meetings, guidance, and support throughout the entire period of my thesis.

Acknowledging only those directly related to the thesis would underestimate the significance of those who have positively influenced both my academic and non-academic life until the present date. Without them, this document wouldn't exist. I express my gratitude to my family and friends for making my life easier and supporting me throughout my stay at IISER K, turning this (1.75+1.75+1.5)-year journey, including the pandemic-riddled online era, into a memorable experience.

I extend my thanks to all my teachers, especially Dibyendu Sir, Mr. Dibyendu Pan, for making me write IAT; Dhiren Sir, Mr. Dhiren Majhi, for instilling the perspective that 'Problem is a gift' and for his casual motivational speeches during English classes in school; and Dida, Mrs. Ashalata Patra, for being my first teacher and trying to convince me to join IISER. I would like to thank Papu, Mr. Susanta Patra, for always prioritizing us, for always being a strong backup, and Didi, Ms. Triparna Patra, for existing and taking care of us. I would like to thank Dadu - Mr. Bibhutibhusan Patra, my dearest Neem - Aloo Posto, Sinu - Lola, Babu - Snappy Patra, Mani - Wendy Patra, Sadu and all my fur-siblings for loving so much.

Beyond academic, intellectual and social development, life at the Mohanpur campus has heightened my sensitivity to emotional aspects. I have learnt a lot from my experience here. Last but not least, I earnestly thank all the dogs and cats of the campus for directly or indirectly taking care of my well-being, and being a constant source of endearment and love.

Contents

Chapter 1

Introduction

The primary objective of this thesis is to present a proof of the following two theorems of commutative ring theory. The first one is Serre's characterization of a regular local ring,

Theorem 1.0.1. *Let A be a Noetherian local ring. Then A is regular iff gl. dim$A < \infty$. Furthermore, if gl. dim$A < \infty$, then gl. dim$A$ =dim A.*

This provides us with a tool to determine when a local ring is regular and is an important point of intersection of ideal theory[1] and homological algebra. The second one is about unique factorization property of regular local rings.

Theorem 1.0.2. *Any regular local ring is unique factorization daomain.*

In the second chapter, we are going to study results from commutative algebra and see important results like Nakayama's Lemma, the Hilbert Basis Theorem and the Artin-Rees Theorem [5]. The results of this chapter are going to be required mainly in the third chapter when we develop the notion of dimension from the perspective of ring theory.

In the third chapter, we discuss the three different formulations of dimension for rings and mosules [2] : the Krull dimension, the Chevalley dimension and dimension developped using the Hilbert-Samuel Polynomial[2]. We see Samuel's Theorem and a very important result, the Dimension Theorem 3.2.1 which says that for a finitely generated module over a Noetherian local ring, the three dimensions are equivalent [5]. We also discuss the Principal Ideal Theorem with which the theorey of Noetherian rings gained mathematical profundity [3].

[1]Not talking politics here!

[2]This part is going to be a bit technical. If at this point, you have lost interest, here's a nice article on the longest running study on human happiness, The Harvard Grant Study [7]

In the fourth chapter, we enter into homological algebra and learn about homological tools which are very useful in extracting information from algebraic objects [6], [4].

In the fifth chapter, we see the notion of dimension from the perspective of homological algebra [8]. We discuss projective and injective dimension of rings and modules. We, of course, see what global dimension of a ring means in terms of projective and injective dimension. One of the significant results of this chapter is Lemma 5.3.3 and Corollary 5.3.7.

Finally, in the sixth chapter, we discuss regular local rings and its properties, using which (along with other important results from the previous chapters) the two main theorems are proved. Note that the first theorem is required to prove the second one.

Now here I must mention that it was not possible to prove each and every result used in this thesis, in most of those cases, appropriate references has been cited for details. For ideas of the proofs presented in this thesis, different references were consulted. It is regrettable if, by any chance, any such reference has been overlooked and not mentioned in the Bibliography section. It is clarified that no originality is being claimed, and the relevant sources are acknowledged.

Chapter 2

Commutative Algebra

We start this report by discussing results from commutative algebra which will either, like Nakayama's lemma, be directly used to prove the main theorems, or be used to understand subsequent chapters. Throughout this chapter, by a ring we mean commutative ring with 1 and any module is assumed to be a unitary module.

2.1 Ring of Fractions and Localization

The formation of ring of fractions and the process of localization are important technical tools used in commutative algebra. Localization serves as a formal mechanism for introducing "denominators" into a given ring or module and this process is often used to obtain local rings. Two very important properties of localization are that it preserves exactness and Noetherian property.

Definition 2.1.1 (**Local ring**)**.** A ring is called a *local ring* if it is non-zero and has a unique maximal ideal.

The process of formation of the rational field $\mathbb{Q}$ out of the ring of the integers $\mathbb{Z}$ can be easily extended to obtain ring of fractions (in this case, the quotient field) out of an integral domain[1], but to generalize the process to get ring of fractions out of an arbitrary ring, say A, we need a multiplicatively closed subset S of A.

Definition 2.1.2 (**Multiplicatively Closed Set**)**.** A set S is called *multiplicatively closed* if for any $a, b \in S$, we have $ab \in S$.

Example 2.1.1. $S = A - \{0\}$ for any integral domain A.

Example 2.1.2. $S = A - \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of ring A.

[1] For details, please refer to Chapter 3 of [1].

Construction of ring of fractions and module of fractions: Let M be an A-module and S be a multiplicative subset of ring A. We define a relation $M \times S$ in the following way:

$$(m, s) \sim (m', s')$$

$$\text{if } \exists\, s'' \in S \ni s''(s'm - sm') = 0$$

It can be easily checked that this $\sim$ relation is reflexive, symmetric and transitive and hence an equivalence relation. we use the following notations:

$$S^{-1}M = \text{ The set of eqivalence classes on } M \times S$$

$$\frac{m}{s} = \text{ The class containing the element } (m, s) \in M \times S$$

Now for A-module $M = A$, we define in $S^{-1}A$, addition and multiplication in the following way:

$$+ \,:\, a/s + a'/s' = (s'a + sa')\,/ss'$$
$$\cdot \,:\, a/s \cdot a'/s' = aa'/ss'.$$

The well-definedness of the above operations can be easily checked. Under $+$ and $\cdot$, $S^{-1}A$ is aring with the zero element $\frac{0}{1}$ and multiplicative identity $\frac{1}{1}$. $S^{-1}A$ is called the ring of fractions of A with respect to S. It is noted that

$$S^{-1}A = 0 \Longleftrightarrow 0 \in S$$

Consider the A-module M, then under the following (well-defined) operations:

$$m/s + m'/s' = (s'm + sm')\,/ss'$$
$$a/s \cdot m/s' = am/ss'.$$

The set $S^{-1}M$ is easily verified to be an $S^{-1}A$- module and is called the module of fractions of M with respect to S. We use the following notation, for a prime ideal $\mathfrak{p}$ of A:

$$A_{\mathfrak{p}} = S^{-1}A \text{ and } M_{\mathfrak{p}} = S^{-1}M, \text{ where } S = A - \{\mathfrak{p}\}.$$

$A_{\mathfrak{p}}$ is called the **localization of A at $\mathfrak{p}$**.

Module Homomorphism: For an$A$-module M and N, $f \in \mathrm{Hom}_A(M, N)$ gives rise to $S^{-1}f \in \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ (well-)defined as

$$S^{-1}f \,:\, S^{-1}M \to S^{-1}N$$

$$\left(S^{-1}f\right)(m/s) = f(m)/s$$

The following relations are easily verified.

- $S^{-1}f = \mathrm{Id}\,|_{S^{-1}M}$, when $f = \mathrm{Id}|_M \in \mathrm{Hom}_A(M, M)$ for some A-module M.
- $S^{-1}f \circ S^{-1}g = S^{-1}(f \circ g)$, for $f \in \mathrm{Hom}_A(M, N)$ and $g \in \mathrm{Hom}_A(P, M)$ where M, N, P are A-modules.

Proposition 2.1.1. *The assignments $M \mapsto S^{-1}M$, $f \mapsto S^{-1}f$ define an exact functor from A-modules to $S^{-1}A$-modules.*

Proof. The only non-trivial thing, we need to proof is that, given an exact sequence of A-modules,

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

the following sequence of $S^{-1}A$-modules

$$0 \to S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \to 0$$

is exact.

Exactness at $S^{-1}M''$. Any element of $S^{-1}M''$ is of the form m''/s, with $m'' \in M'', s \in S$.Since g is surjective, $\exists\, m \in M$ be such that $g(m) = m''$. We have $S^{-1}g(m/s) = g(m)/s = m''/s$ implying $S^{-1}g$ is surjective.

Exactness at $S^{-1}M$. We have $\mathrm{Im}\, S^{-1}f \subset \ker S^{-1}g$, since

$$g \circ f = 0 \implies S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = 0$$

Now consider $m/s \in \ker S^{-1}g \subset S^{-1}M$. So

$$S^{-1}g(m/s) = g(m)/s = 0$$

So there exists $t \in S$ such that $g(tm) = tg(m) = 0$. Sincer $\mathrm{Im} f = \ker g$, there exists $m' \in M'$ such that $tm = f(m')$. Now $S^{-1}f(m'/ts) = f(m')/ts = tm/ts = m/s$, i.e. $m/s \in \mathrm{Im}\, S^{-1}f$. Hence $\mathrm{Im}\, S^{-1}f = \ker S^{-1}g$.

Exactness at $S^{-1}M'$. Let $m'/s \in S^{-1}M'$ be such that $f(m')/s = S^{-1}f(m'/s) = 0$. Then there exists $t \in S$ such that $f(tm') = tf(m') = 0$ which, f being injective, implies $tm' = 0$ proving injectivity of $S^{-1}f$. Hence $m'/s = 0$. The proposition is hence proved. □

Some important properties of ring of fractions:

1. We define i_M as followed

$$i_M : M \to S^{-1}M \text{ by } m \mapsto \frac{m}{1}$$

Since $i_A : A \to S^{-1}A$ is a ring homomorphism, any $S^{-1}A$-module can be regarded as an A-module. Hence $S^{-1}M$ is an A-module and it is easily verified that i_M is an A-homomorphism, functorial in M.

2. The A-bilinear map $S^{-1}A \times M \to S^{-1}M$ (well-)defined as

$$(a/s, m) \mapsto am/s$$

by definition of tensor product, induces an unique A-homomorphism

$$\varphi : S^{-1}A \otimes_A M \to S^{-1}M \text{ given by } \varphi(a/s \otimes m) = am/s$$

The map φ is an $S^{-1}A$-isomorphism and is functorial in M.

3. Let I be an ideal of A. $IS^{-1}A$ denotes the ideal of $S^{-1}A$ generated by $i_A(I)$. If we regard $S^{-1}I$ as a subset of $S^{-1}A$, then $IS^{-1}A = S^{-1}I$.

4. If $I \cap S \neq \emptyset$, then $IS^{-1}A = S^{-1}A$.

5. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$, then $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$. For two such prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$, we have

$$\mathfrak{p}_1 \subset \mathfrak{p}_2 \Longleftrightarrow S^{-1}\mathfrak{p}_1 \subset S^{-1}\mathfrak{p}_2$$

6. The map $\varphi : \mathfrak{p} \mapsto S^{-1}\mathfrak{p}(= \mathfrak{p}S^{-1}A)$, from the set of prime ideals $\mathfrak{p}$ of A with $\mathfrak{p} \cap S = \emptyset$ onto the set of all prime ideals of $S^{-1}A$, has an inverse.

From the last two properties, we conclude the following Proposition.

Proposition 2.1.2. *The map $\varphi : \mathfrak{p} \mapsto S^{-1}\mathfrak{p}(= \mathfrak{p}S^{-1}A)$ is an inclusion preserving bijection of the set of prime ideals $\mathfrak{p}$ of A with $\mathfrak{p} \cap S = \emptyset$ onto the set of all prime ideals of $S^{-1}A$.*

Corollary 2.1.3. *Let A be a commutative ring and let $\mathfrak{p}$ be a prime ideal of A. Then $A_\mathfrak{p}$ is a local ring with $\mathfrak{p}A_\mathfrak{p}$ as its unique maximal ideal.*

Proof. Clearly $\mathfrak{p}A_\mathfrak{p}$ is a prime ideal. Let now $\wp$ be any prime ideal of $A_\mathfrak{p}$. Then $i_A^{-1}(\wp)$ is a prime ideal of A such that $i_A^{-1}(\wp) \subset \mathfrak{p}$. Hence $\wp = i_A^{-1}(\wp)A_\mathfrak{p} \subset \mathfrak{p}A_\mathfrak{p}$ concluding that it is the only maximal ideal. Hence the corollary is proved. □

2.2 Some Lemmas

Definition 2.2.1 (Jacobson Radical). The Jacobson radical of a ring is the intersection of all its maximal ideals.

Note. Jacobson radical of a ring A, is denoted by $\underline{r}(A)$ or $\underline{r}$.

Lemma 2.2.1 (Nakayama's Lemma). *Let M be a finitely generated A-module. If $\underline{r}M = M$, then $M = 0$.*

Proof. Let us assume $M \neq 0$ and let $x_1, \dots, x_n$ be a minimal set of generators for M. Since $M = \underline{r}M$, we have $x_1 = \sum_{1 \leq i \leq n} a_i x_i$ with $a_i \in \underline{r}$. We know that that if $a \in \underline{r}(A)$, then $1 - a$ is invertible. So we have,

$$(1 - a_1)\, x_1 = \sum_{2 \leq i \leq n} a_i x_i \implies x_1 = \sum_{2 \leq i \leq n} (1 - a_1)^{-1} a_i x_i$$

which implies $x_2, \dots, x_n$ generate M leading to a contradiction. Hence $M = 0$. □

Lemma 2.2.2 (Prime Avoidance Lemma). *Let I be an ideal of a ring A and $I \subset \cup_{1 \leq i \leq n} J_i$, where J_i's are prime ideals of A. Then $I \subset J_i$ for some $i \in \{1, 2, \dots, n\}$.*

Lemma 2.2.3. *Let $I, J_0, J_1, \dots, J_n$ be ideals of a ring A with J_0 prime and $I \subset \bigcup_{0 \leq i \leq n} J_i$. Then there exists a proper subset L of $\{0, 1, 2, \dots, n\}$ such that $I \subset \bigcup_{j \in L} J_j$.*

Proof. Assume the above mentioned result is false. Then for every i with $0 \leq i \leq n$, there exists $a_i \in I - \bigcup_{j \neq i} J_j \implies a_i \in J_i$. We consider $a = a_0 + a_1 a_2 \cdots a_n$. We have $a \in I$ and hence $a \in J_i$ for some i.
Case I: $a \in J_0$ then $a_1 a_2 \cdots a_n \in J_0$. Since J_0 is prime, this implies that $a_i \in J_0$ for some $i \geq 1$, which is not possible by choice of a_i.
Case II:$a \in J_i$ for some $i \geq 1$. Then $a_0 \in J_i$ which is again a contradiction. □

2.3 Noetherian Modules

Definition 2.3.1 (Noetherian Modules). We say, an A-module N is Noetherian, if it satisfies any one of the following equivalent conditions:

(i) Every submodule of N is finitely generated:

(ii) N satisfies the ascending chain condition for submodules i.e. every sequence

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq N_3 \dots$$

of submodules of N is finite.

(iii) Every nonempty set of submodules of M has a maximal element.

Definition 2.3.2 (**Noetherian Rings**)**.** A ring A is called Noetherian if it is a Noetherian A-module.

Example 2.3.1. Any finite ring is Noetherian, so is any field.

Some important results related to Noetherian modules:

1. If A is a Noetherian ring and $A \to B$ is a surjective ring homomorphism then B is Noetherian.
2. For any exact sequence of A-modules,

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

   we have, N is Noetherian if and only if N', N'' are Noetherian.

3. Any a finitely generated module of a Noetherian ring is Noetherian.
4. The localization of a Noetherian ring at a prime ideal is Noetherian .

2.3.1 Hilbert Basis Theorem

Theorem 2.3.1 (**Hilbert Basis Theorem**)**.** *Let A be a Noetherian ring. Then the polynomial ring $A[X_1, \dots, X_n]$ in n variables over A is Noetherian.*

Proof. Since we have the polynomial ring in n variables over A, $A[X_1, \dots, X_n] = A'[X_n]$, the polynomial ring in one variable over A', where $A' = A[X_1, \dots, X_{n-1}]$, by induction on n, it is sufficient to prove the theorem for $n = 1$, i.e. we shall prove that the polynomial ring $B = A[X]$ in one variable is Noetherian.

Let J be an ideal of B, we want to show that J is finitely generated. Let $I = \{0\} \bigcup \{$ leading coefficients of elements of $J\}$. Clearly, I is an ideal of A.
Case I : $I = 0$, then $J = 0$, trivially an ideal of B.
Case II : $I \neq 0$. A being Noetherian, I is finitely generated. Hence we have non-zero elements $\alpha_1, \dots, \alpha_r \in A$ such that $I = (\alpha_1, \dots, \alpha_r)$.
Let $f_i \in J$ with leading coefficient α_i for $1 \leq i \leq r$. Let $N = \max_{1 \leq i \leq r} (\deg f_i)$.

Claim : $J = (f_1, \ldots, f_r) + J'$ where $J' = J \cap \left(A + AX + \cdots + AX^{N-1}\right)$.
To prove the claim, it is sufficient to show that any $f = a_m X^m + \cdots + a_0$ in J belongs to $(f_1, \ldots, f_r) + J'$. If $m \leq N - 1$, it is clear. So we assume that $m \geq N$ and let $a_m = \sum_{1 \leq i \leq r} \beta_i \alpha_i, \beta_i \in A$. Now we have

$$\deg\left(f - \sum_{1 \leq i \leq r} \beta_i X^{m - \deg f_i} f_i\right) \leq m - 1$$

By induction on m,

$$f - \sum_{1 \leq i \leq r} \beta_i X^{m - \deg f_i} f_i \in (f_1, \ldots, f_r) + J'$$

$$\implies f \in (f_1, \ldots, f_r) + J'$$

Since J' is an A-submodule of $A + AX + \cdots + AX^{n-1}$, by Property (3), it has a finite set of generators $g_1, \ldots, g_s$ over A. Clearly $f_1, \ldots, f_r, g_1, \ldots, g_s$ generate the ideal J □

Corollary 2.3.2. *Let A be a Noetherian ring and B a finitely generated A-algebra. Then B is Noetherian.*

Proof. We have any finitely generated A-algebra is a quotient of a polynomial ring $A[X_1, \ldots, X_n]$. Using Property (1) and (2) along with Hilbert Basis theorem, the result follows. □

2.4 Primary Decomposition

Definition 2.4.1 (Homothety). For any $a \in A$ and A-module M, the A-homomorphism $a_M : M \to M$, defined by $a_M(x) = ax$ for $x \in M$, is called *homothety by a.*

Definition 2.4.2 (Primary Submodule). A submodule N of A-module M is said to be primary in M (or a primary submodule of M), if

1. $N \neq M$.
2. For any $a \in A$, the homothety $a_{M/N} : M/N \to M/N$ is either injective or nilpotent.

For $M = A$, primary submodules are called primary ideals.

Definition 2.4.3 ($\mathfrak{p}$-primary). For a primary submodule N of an A-module M and $\mathfrak{p} = \left\{a \in A \mid a_{M/N} \text{ is not injective }\right\}$, $\mathfrak{p}$ is a prime ideal, which is called prime ideal belonging to N in M and N is said to be $\mathfrak{p}$-primary (in M).

Example 2.4.1. Consider $A = M = \mathbb{Z}$ and $N = (p^n)$ for some prime element $p \in \mathbb{Z}$ and $n \in \mathbb{N}$. Let $\mathfrak{p} = (p)$. Then N is $\mathfrak{p}$-primary in $\mathbb{Z}$.

Definition 2.4.4 (Primary Decomposition). For a submodule N of M, a decomposition of the form $N = N_1 \cap N_2 \cap \cdots N_r$, where each N_i is a primary submodule of M, is called a *primary decomposition of N (in M).*

Definition 2.4.5 (Reduced Primary Decomposition). A primary decomposition is said to be reduced, if

1. $N \neq \cap_{i \in J} N_i$, for any proper subset J of $\{1, 2, \ldots, r\}$.
2. The prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ belonging respectively to $N_1, \ldots, N_r$ in M are distinct.

Example 2.4.2. Any proper submodule of a Noetherian module admits a reduced primary decomposition[2].

Definition 2.4.6 (Associated Prime). A prime ideal $\mathfrak{p}$ of A is said to be *associated prime of A-module M*, or *associated to M*, if there exists $x \neq 0$ in M such that $\mathfrak{p}$ is the annihilator of x, that is $\mathfrak{p} = \{a \in A \mid ax = 0\}$.

Remark. $\mathfrak{p} \in \operatorname{Ass}(M) =$ set of all associated primes of M, $\iff$ if there exists an A-monomorphism[3] $A/\mathfrak{p} \to M$.

Proposition 2.4.1. *Let M be a finitely generated A-module, where A is a Noetherian ring. Let a reduced primary decomposition of 0 in M be*

$$0 = N_1 \cap \cdots \cap N_r$$

with each N_i being $\mathfrak{p}_i$-primary. Then

$$\operatorname{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$$

Particularly, $\operatorname{Ass}(M)$ *is finite. Also,* $M = 0 \iff \operatorname{Ass}(M) = \emptyset$.

Proof. Please refer to the proof of Theorem 3.2.15, Chapter 3 of [2] for details. □

Corollary 2.4.2. *Let A be a Noetherian ring and N be a submodule of a finitely generated A-module M with reduced primary decomposition $N = N_1 \cap \cdots \cap N_r$. Then*

[2] Please refer to the proof of Proposition 1.15, Chapter 1 of [5] for details.
[3] Injective A-homomorphism

$\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ *where* $\mathfrak{p}_i$ *are the prime ideals belonging to* N_i *in* M*, for* $1 \leq i \leq r$.

Remark. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of prime ideals corresponding to a reduced primary decomposition of N, as above, is independent of the decomposition.

Definition 2.4.7 (Zero-divisor of a Module). We say, an element $a \in A$ is a *zero divisor of an* A*-module* M, if there exists $m \in M, m \neq 0$. such that $am = 0$.

Proposition 2.4.3. *Let* A *be a Noetherian ring and* M*, a finitely generated* A*-module. Then the set of zero-divisors of* M *is* $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.

Proof. $(\Leftarrow)$
Let $a \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$. Let $\mathfrak{p}$ be the annihilator of $x \in M, x \neq 0$. Then $ax = 0$.
$(\Rightarrow)$
Let a be a zero-divisor of M and let $x \in M, x \neq 0$ be such that $ax = 0$. Let $0 = N_1 \cap \cdots N_r$ be a reduced primary decomposition of 0 in M. Then $x \notin N_i$ for some N_i.
Let $\mathfrak{p}_i$ be the prime ideal belonging to N_i in M. By Proposition 2.4.1, $\mathfrak{p}_i \in \mathrm{Ass}(M)$. Now we have $ax = 0$, i.e. the homothety a_{M/N_i} is not injective, which implies $a \in \mathfrak{p}_i$. □

Definition 2.4.8 (Relevant Definitions). Here are few relevant definitions and notations :

1. For a ring A, *Spec* A denotes the set of all prime ideals of A.
2. The *nilradical,* $\mathfrak{n}(A)$ of a ring A is defined to be the subset $\{a \in A \mid a^n = 0$ for some $n \in \mathbb{N}\}$ of A.
3. The *annihilator of* A*-module* (M) is defined as $\mathrm{ann}(M) = \{a \in A \mid aM = 0\}$. It is clear that $\mathrm{ann}M$ is an ideal of A.
4. The *support of* M, $\mathrm{Supp}(M)$ is the set $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$.

Proposition 2.4.4. *For a ring* A*,*

$$\mathfrak{n}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}$$

Proof. It is clear that $\mathfrak{n}(A)$ is an ideal of A.
$(\Leftarrow)$
Let $a \in \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$ and let S be the multiplicative subset $\{1, a, a^2, \ldots\}$

of A. Since $S \cap \mathfrak{p} \neq \emptyset$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$, by Proposition 2.1.2, we have that $\mathrm{Spec}\,(S^{-1}A) = \emptyset$. Hence $S^{-1}A = 0 \iff 0 \in S$, which implies $a^n = 0$ for some $n \in \mathbb{N}$, i.e. $a \in \mathfrak{n}(A)$.
$(\Rightarrow)$
Let $a \in \mathfrak{n}(A)$. Then

$$\begin{aligned} a^n = 0 \text{ for some } n \in \mathbf{N} &\Rightarrow a^n \in \mathfrak{p} \text{ for every } \mathfrak{p} \in \mathrm{Spec}(A) \\ &\Rightarrow a \in \mathfrak{p} \text{ for every } \mathfrak{p} \in \mathrm{Spec}(A). \end{aligned}$$

□

Proposition 2.4.5. *For an A-module M, the A-homomorphism, $M \xrightarrow{\varphi} \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} M_{\mathfrak{p}}$, induced by the canonical homomorphisms $M \to M_{\mathfrak{p}}$, is injective. In particular,*

$$M = 0 \iff \mathrm{Supp}(M) = \emptyset$$

Proof. Let $x \in M$ be such that $\varphi(x) = 0$. This means that, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, there exists $s_{\mathfrak{p}} \in A - \mathfrak{p}$, which annhilates x, i.e. $s_{\mathfrak{p}} x = 0$. Thus $\mathrm{ann}(Ax) \not\subset \mathfrak{p}$, for every $\mathfrak{p} \in \mathrm{Spec}(A)$, implying $\mathrm{ann}(Ax) = A$ and $x = 0$. □

2.5 Artinian Modules

Definition 2.5.1 (Artinian Modules). We say, an A-module N is Artinian, if N satisfies the descending chain condition for submodules i.e. every sequence

$$N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq N_3 \dots$$

of submodules of N is finite.

Definition 2.5.2 (Artinian Rings). A ring A is called Artinian if it is a Artinian A-module.

Example 2.5.1. Any finite ring is Artinian, so is any field.

Example 2.5.2. The ring of intgers $\mathbb{Z}$ is Noetherian, but not Artinian, since for any $n \in \mathbb{Z}$ with $n > 1$, we have the descending chain,

$$(n) \supset (n^2) \supset (n^3) \dots$$

Definition 2.5.3 (Length of a Module). An A-module M is of *finite length* n if it possesses a Jordan-Hölder series of length n. That is there exists a sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

of submodules of M such that M_i/M_{i+1} is a simple[4] A-module for $i \in \{0, 1, 2, \ldots, n-1\}$. We denote, the length of A-module M as $\ell_A(M) = n$.

Some important results related to Artinian modules:

1. For any exact sequence of A-modules,

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

   we have,

   (i) N is Artinian $\iff N', N''$ are Artinian.

   (ii) N is of finite length $\iff N'$ and N'' are of finite length. And in this case,

$$\ell_A(N) = \ell_A(N') + \ell_A(N'')$$

2. An A-module N is Artinian $\iff$ every non-empty family of submodules of N contains a minimal element.

3. Every Artinian ring is of finite length and thus Noetherian.

4. Any finitely generated module over an artinian ring is of finite length.

Proposition 2.5.1. *Let A be a Noetherian ring and N a finitely generated A-module. Then, N is of finite length $\iff$ every $\mathfrak{p} \in \mathrm{Supp}(N)$ is a maximal ideal.*

Proof. For details of the proof, please refer to Proposition 1.32, Chapter 1 of [5]. □

2.6 Graded and Filtered Modules

A graded ring is an abstraction of polynomial rings in several variables. It can be decomposed into a direct sum of subgroups, each corresponding to a specific degree.

Definition 2.6.1 (Graded Ring). A ring A is called *graded*, if it has a *gradation*, i.e. it can be decomposed as $A = \bigoplus_{n \geq 0} A_n$, as a direct sum of subgroups A_n of A, where n runs over the set $\mathbb{Z}^+$, such that $A_m A_n \subset A_{m+n}$ for all $m, n \in \mathbb{Z}^+$.

[4] A module is called simple, if does not have any proper submodule.

A_n is called the n^{th} *homogenous component of* A and the non-zero elements of A_n are called homogeneous elements of A of degree n.

Example 2.6.1. For any ring A, we have the trivial gradation with $A_0 = A$ and $A_i = 0 \; \forall \, i \geq 1$.

Example 2.6.2. $A = k[X_1, \ldots, X_r]$, where A is a polynomial ring in n variables over field k and A_n is the set of polynomials over k in $X_1, \ldots, X_r$ of degree n.

Remark. A_0 is a subring and $1 \in A_0$. Each A_n is an A_0-module and A is an A_0-algebra. If we have a ring homomorphism $k \to A_0$, A can be regarded as k-algebra.

Definition 2.6.2 (Graded A-module). Let M be an A-module, where $A = \bigoplus_{n>0} A_n$ be a graded ring. M is called a graded A-module, if we ahve A-gradation on M, i.e. M can decomposed as $M = \bigoplus_{n\geq 0} M_n$ of M as a direct sum of subgroups M_n of M such that $A_m M_n \subset M_{m+n}$ for all $m, n \in \mathbb{Z}^+$.

Definition 2.6.3 (Homomorphism of Graded Rings). Let $A = \bigoplus_n A_n$ and $B = \bigoplus_n B_n$ be two graded rings. A ring homomorphism $f : A \to B$ is called a homomorphism of graded rings if $f\left(A_n\right) \subset B_n$, for every $n \in \mathbb{Z}^+$.

Definition 2.6.4 (Homomorphism of Graded Modules). Let $M = \bigoplus_{n\geq 0} M_n$ and $N = \bigoplus_{n\geq 0} N_n$ be two graded A-modules. A homomorphism $f : M \to N$ of graded A-modules of degree r is an A homomorphism such that $f\left(M_n\right) \subset M_{n+r}$, for every $n \in \mathbb{Z}^+$.

Definition 2.6.5 (Graded Submodule). For A and M, as above, a submodule N of M is called a *graded submodule* if $N = \bigoplus_n \left(N \cap M_n\right)$. An ideal which is a graded submodule of A is called a *homogeneous ideal of* A.

Example 2.6.3. For any homomorphism of graded A modules M, M', $f : M \to M'$, $\ker f$ and $\mathrm{Im} f$ are graded submodules of M and M' respectively.

Some important results related to Graded modules:

1. Let $A = \bigoplus_n A_n$ be a graded ring and $M = \bigoplus_n M_n$ a graded A-module. If N is a graded submodule of M, then M/N has an A-gradation induced from that of M,
$$M/N = \bigoplus_n \left(M_n + N\right)/N$$
2. For A and M as above, if M is noetherian, then each M_n is a finitely generated A_0-module.
3. If A is generated by A_1 as an A_0-algebra, then

$$A \text{ is noetherian} \iff A_0 \text{ is noetherian and } A-i \text{ is a finitely generated } A_0\text{-module.}$$

Definition 2.6.6 (Filtered Ring). A ring A is called *filtered*, if it has a *filtration*, i.e. there is a sequence of ideals of A,

$$A = A_0 \supset A_1 \supset A_1 \supset \cdots \supset A_n \supset \cdots$$

such that $A_m A_n \subset A_{m+n}$ for all $m, n \in \mathbb{Z}^+$.

Definition 2.6.7 (Filtered A-module). An A-module M is called filtered, if we have a filtration on M, i.e. a sequence of submodules of M,

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$$

such that $A_m M_n \subset M_{m+n}$ for all $m, n \in \mathbb{Z}^+$.

Definition 2.6.8 ($\mathfrak{a}$-good Filtration). Let $\mathfrak{a}$ be an ideal of filtered ring A and M be a filtered A-module with filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$$

The filtration is said to be $\mathfrak{a}$-good, if

1. It is compatible with $\mathfrak{a}$, i.e. $\mathfrak{a}M_n \subset M_{n+1}$ for every $n \geq 0$.
2. For $n \gg 1$, i.e. for sufficiently large n,$\mathfrak{a}M_n = M_{n+1}$.

Definition 2.6.9 ($\mathfrak{a}$-adic Filtration). Let $\mathfrak{a}$ be an ideal of filtered ring A and M be an A-module. Then $\mathfrak{a}$-adic filtration are the filtrations defined by $\mathfrak{a}$, on A

$$A = \mathfrak{a}^0 \supset \mathfrak{a} \supset \mathfrak{a}^2 \supset \cdots$$

and on M,

$$M = \mathfrak{a}^0 M \supset \mathfrak{a}M \supset \mathfrak{a}^2 M \supset \cdots$$

It is clear that an $\mathfrak{a}$-adic filtration is $\mathfrak{a}$-good.

Theorem 2.6.1 (Artin-Rees Theorem). *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module and N submodule of M. Then, for any $\mathfrak{a}$-good filtration on M, the induced filtration on N is $\mathfrak{a}$-good.*

Proof. For details of the proof, please refer to Theorem 1.39, Chapter 1 of [5]. □

Definition 2.6.10 (Graded Ring and Module associated to $\mathfrak{a}$-adic Filtration).

Let A be a filtered ring with a filtration

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots$$

and let M be a filtered A-module with a filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots$$

Consider the direct sum $G(A) = \bigoplus_{n\geq 0} A_n/A_{n+1}$ of abelian groups A_n/A_{n+1}. To make $G(A)$ a graded ring, we define a multiplication in the following manner,
Let $\bar{a} \in A_n/A_{n+1}, \bar{b} \in A_m/A_{m+1}$ be homogeneous elements of degrees n and m respectively and let $a \in A_n, b \in A_m$ be such that $\bar{a}, \bar{b}$ are the images of a, b under the natural maps $A_n \to A_n/A_{n+1}, A_m \to A_m/A_{m+1}$ respectively. Now $ab \in A_{n+m}$. We define $\bar{a}\bar{b}$ to be the image of ab under the natural map $A_{n+m} \to A_{n+m}/A_{n+m+1}$ and thus $\bar{a}\bar{b}$ is a homogenous element of degree $n+m$.
Clearly, the multiplication operation on $G(A)$ is well-defined and hence $G(A)$ is a graded ring.
Similary, the filtration on M, induces a graded $G(A)$-module $G(M) = \bigoplus_{n\geq 0} M_n/M_{n+1}$. $G(A)$ is said to be the *graded ring associated to the filtration* $A = A_0 \supset A_1 \supset \cdots$ and $G(M)$, the *graded module associated the filtration* $M = M_0 \supset M_1 \supset \cdots$.

Lemma 2.6.2. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A contained in $\underline{r}(A)$. If $G_{\mathfrak{a}}(A)$ is an integral domain, then so is A.*

Proof. Let $a, b \in A, a \neq 0, b \neq 0$. Now according to a Corollary[5] of Artin-Rees Theorem, $\bigcap_{n\geq 0} \underline{r}^n = 0$, where $\underline{r}$ is the Jacobson radical of a Noetherian ring.
Since $\mathfrak{a} \subset \underline{r}(A)$, we have $\bigcap_{n\geq 0} \mathfrak{a}^n = 0$. Hence there exists $m, n \in \mathbb{Z}^+$such that $a \in \mathfrak{a}^m, a \notin \mathfrak{a}^{m+1}$ and $b \in \mathfrak{a}^n, b \notin \mathfrak{a}^{n+1}$.
Let $\bar{a}, \bar{b}$ be the images of a, b in $\mathfrak{a}^m/\mathfrak{a}^{m+1}, \mathfrak{a}^n/\mathfrak{a}^{n+1}$ respectively. Then we have $\bar{a} \neq 0, \bar{b} \neq 0$. Since $G_{\mathfrak{a}}(A)$ is an integral domain, we have $\bar{a}\bar{b} \neq 0 \implies, ab \neq 0$. Hence A is an integral domain. $\square$

[5]The proof can be done using Theorem 2.6.1 and Nakayama's Lemma. For details, please refer to proof of Corollary 1.41, Chapter 1 of [5].

Chapter 3

Dimension Theory

Dimension allows us to measure the size, complexity, and intrinsic structure of the mathematical objects. There are three formulations of the algebraic analogue of dimension for rings and modules. The Krull dimension of a ring captures the maximum length of chains of prime ideals within the ring, we also have Chevalley dimension and another one is developped using Hilbert-Samuel polynomial. We are going to briefly discuss these three dimensions and see that for a Noetherian local ring, these three dimensions are equivalent.

3.1 The Hilbert-Samuel Polynomial

Definition 3.1.1 (Polynomial function). A map $f : \mathbb{N} \to \mathbb{Q}$ is said to be a polynomial function if there exists $g(X) \in \mathbb{Q}[X]$ such that $f(n) = g(n)$ for $n \gg 1$.

Uniqueness : The polynomial $g(X)$, mentioned above, is uniquely determined for given polynomial function f. If $g(X), h(X) \in \mathbb{Q}[X]$ be such that $g(n) = h(n)$ for $n \gg 1$. Then $(g-h)(X) \in \mathbb{Q}[X]$ is a polynomial of degree, say, $m \leq \max(\deg g, \deg h)$, then $(g-h)(X) \in \mathbb{Q}[X]$ can have more than m distinct roots in $\mathbb{C}$, but $g(n) = h(n)$ for $n \gg 1 \implies (g-h)(X)$ has infinitely many roots, implying that it is identically zero, i.e. $g(X) = h(X)$.

The degree of g is called the *degree of* f and the leading coefficient of g is called the *leading coefficient of* f.

Remark. If a polynomial function $f \neq 0$, and $f(n) \geq 0$ for $n \gg 1$, i.e. $\forall n > n_0$ for some $n_0 \in \mathbb{N}$, then the leading coefficient of f is positive.

Lemma 3.1.1. *Let for a map $f : \mathbb{N} \longrightarrow \mathbb{Q}$, $\Delta f : \mathbb{N} \to \mathbb{Q}$ is defined by*

$$(\Delta f)(n) = f(n+1) - f(n)$$

Let $d \in \mathbb{N}$. Then f is a polynomial function of degree d if and only if Δf is a polynomial function of degree $d-1$.

Proof. The ($\Rightarrow$) part is obvious and ($\Leftarrow$) part can be easily proven by induction on s. For details, please refer to Chapter 8 of [2]. $\square$

We now consider a graded ring, $A = \bigoplus_{n \geq 0} A_n$ such that A_0 is Artinian and A is generated as an A_0-algebra by s elements $x_1, \ldots, x_s$ of A_1. Let $M = \bigoplus_{n \geq 0} M_n$ be a finitely generated graded A-module. Then, **For each n, M_n is finitely generated and $\ell_{A_0}(M_n) \leq \infty$.**
A being a finitely generated A_0-algebra, by Corollary 2.3.2, is Noetherian. Hence as an A_0-module, by Result 3 of graded modules in Section 2.6, each M_n is finitely generated and hence, by Result 4 of Artinian modules in Section 2.5, by of finite length. Now we define $\chi(M, \,) : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ by

$$\chi(M, n) = \ell_{A_0}(M_n)$$

Proposition 3.1.2 (Hilbert). *The map $\chi(M, .)$ is a polynomial function of degree $\leq r - 1$ where r is as above.*

Proof. For details, please refer to Chapter 8 of [2]. $\square$

Remark. We assign the degree -1 to the zero polynomial. Then clearly, Δf is a polynomial function of degree $-1 \Leftrightarrow f$ is a polynomial function of degree ≤ 0.

Definition 3.1.2 (Hilbert Polynomial). The polynomial defing the polynomial function $\chi(M, n)$ is called the *Hilbert polynomial of M* and is also denoted by the same notation.

Now for the rest of this chapter A is assumed to be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and any A-module is assumed to be finitely generated.

Definition 3.1.3 (Ideal of Definition). If for an ideal $\mathfrak{a}$ of A, $\mathfrak{m}^n \subset \mathfrak{a} \subset \mathfrak{m}$ for some $n \in \mathbb{N}$, then $\mathfrak{a}$ is said to be an *ideal of definition of A*

Definition 3.1.4. Let A be a Noetherian local ring, M a finitely generated A-module and $\mathfrak{a}$ an ideal of definition of A. As we did in Section 2.6, the graded ring associated to the $\mathfrak{a}$-adic filtration on A is

$$G_{\mathfrak{a}}(A) = \oplus_{n\geq 0}\mathfrak{a}^n/\mathfrak{a}^{n+1}$$

Similarly, for an A-module M, $G_{\mathfrak{a}}(M) = \bigoplus_{n\geq 0} \mathfrak{a}^n M/\mathfrak{a}^{n+1}M$ is the graded $G_{\mathfrak{a}}(A)$-module corresponding to the $\mathfrak{a}$-adic filtration on M.
Since $\mathfrak{a}$ is an ideal of definition of A, $\mathrm{Supp}(A/\mathfrak{a}) = \{\mathfrak{m}\}$. Hence, by Proposition 2.5.1, $A/\mathfrak{a}$ is of finite length and thus Artinian.
A being a Noetherian ring, $\mathfrak{a}$ is finitely generated, let $\mathfrak{a}$ be generated by r elements. Using Hilbert's Prroposition 3.1.2 for $G_{\mathfrak{a}}(A)$-module $G_{\mathfrak{a}}(M)$, we have that

$$\chi\left(G_{\mathfrak{a}}(M), n\right) = \ell_{A/\mathfrak{a}}\left(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M\right)$$

is a polynomial of degree less than or equal to $r-1$.
Also, we have $\mathrm{Supp}\left(M/\mathfrak{a}^n M\right) = \{\mathfrak{m}\}$. Hence, by Proposition 2.5.1, the A-module $M/\mathfrak{a}^n M$ is of finite length . Now we define

$$P_{\mathfrak{a}}(M, n) := \ell_A\left(M/\mathfrak{a}^n M\right)$$

Theorem 3.1.3 **(Samuel).** *For a Noetherian local ring A, a finitely generated A-module M and an ideal of definition of A, $\mathfrak{a}$, generated by r elements, $P_{\mathfrak{a}}(M,n)$ is a polynomial function $\leq r$.*

Proof. From the exact sequence

$$0 \to \mathfrak{a}^n M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^n M \to 0$$

we obtain,

$$\ell_{A/\mathfrak{a}}\left(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M\right) + \ell_{A/\mathfrak{a}}\left(M/\mathfrak{a}^n M\right) = \ell_{A/\mathfrak{a}}\left(M/\mathfrak{a}^{n+1}M\right)$$

It is clear that $\ell_A\left(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M\right) = \ell_{A/\mathfrak{a}}\left(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M\right)$. Hence

$$\chi\left(G_{\mathfrak{a}}(M), n\right) = P_{\mathfrak{a}}(M, n+1) - P_{\mathfrak{a}}(M,n) = \Delta P_{\mathfrak{a}}(M,n)$$

The assertion follows from the Lemma 3.1.1 and the remark following it. □

Lemma 3.1.4. *Let M be an A-module and $\mathfrak{a}, \mathfrak{a}'$ ideals of definition of A. Then $P_{\mathfrak{a}}(M,n)$ and $P_{\mathfrak{a}'}(M,n)$ have the same degree.*

Proof. It suffices to prove that $\deg(P_{\mathfrak{a}}(M,n)) = \deg(P_{\mathfrak{m}}(M,n))$, where $\mathfrak{m}$ is the maximal ideal. Now $\mathfrak{a}$ being an ideal of definition of A,

$$\exists\, q \in \mathbb{N} \ni \mathfrak{m}^q \subset \mathfrak{a} \subset \mathfrak{m}$$

Thus $\forall n \in \mathbb{N}$, we have $\mathfrak{m}^{qn} \subset \mathfrak{a}^n \subset \mathfrak{m}^n$, which implies $P_{\mathfrak{m}}(M,qn) \geq P_{\mathfrak{a}}(M,n) \geq P_{\mathfrak{m}}(M,n)$ proving the lemma. □

Definition 3.1.5 (Dimension developped from Hilbert-Samuel Polynomial). For A, M as above and an ideal of definition $\mathfrak{a}$ of A such that $\mathfrak{m}^n \subset \mathfrak{a} \subset \mathfrak{m}$ for some $n \in \mathbb{N}$. By Samuel's Theorem 3.1.3, we have $\deg(P_{\mathfrak{a}}(M,n))$ is finite. By Lemma 3.1.4, we have that $\deg(\ P_{\mathfrak{a}}(M,n))$ is independent of the choice of ideal of definition. We define

$$\mathrm{d}(M) = \deg(P_{\mathfrak{a}}(M,n))$$

Proposition 3.1.5. *Let A be a Noetherian local ring and $\mathfrak{a}$ be an ideal of definition of A. Consider the an exact sequence of A-modules*

$$0 \to M' \to M \to M'' \to 0$$

Then

$$P_{\mathfrak{a}}(M'',n) + P_{\mathfrak{a}}(M',n) = P_{\mathfrak{a}}(M,n) + R(n),$$

where $R(n)$ is a polynomial function and $\deg R(n) < \mathrm{d}(M) = \deg P_{\mathfrak{a}}(M,n)$ and the leading coefficient of $R(n)$ is non-negative.

Proof. The given exact sequence, for each $n \in \mathbb{N}$, induces an exact sequence of A-modules,

$$0 \to M'/M' \cap \mathfrak{a}^n M \to M/\mathfrak{a}^n M \to M''/\mathfrak{a}^n M'' \to 0$$

which gives

$$\ell_A\left(M''/\mathfrak{a}^n M''\right) + \ell_A\left(M'/M' \cap \mathfrak{a}^n M\right) = \ell_A\left(M/\mathfrak{a}^n M\right)$$

Let $M'_n = M' \cap \mathfrak{a}^n M$. Now we have

$$\ell_A\left(M'/M'_n\right) = P_{\mathfrak{a}}(M,n) - P_{\mathfrak{a}}(M'',n)$$

So $\ell_A\left(M'/M'_n\right)$ is a polynomial function. By Artin-Rees Theorem 2.6.1, there exists $p \in \mathbb{N}$ such that $\mathfrak{a}M'_n = M'_{n+1}$ for $n \geq p$.

now for any $n \in \mathbb{N}$, $\mathfrak{a}^{p+n}M' \subset M'_{n+p} = \mathfrak{a}^n M'_p \subset \mathfrak{a}^n M'$, implying

$$\ell_A\left(M'/\mathfrak{a}^{p+n}M'\right) \geq \ell_A\left(M'/M'_{n+p}\right) \geq \ell_A\left(M'/\mathfrak{a}^n M'\right)$$

$$\implies P_{\mathfrak{a}}\left(M', n+p\right) \geq \ell_A\left(M'/M'_{n+p}\right) \geq P_{\mathfrak{a}}\left(M', n\right)$$

Hence the polynomial functions, $P_{\mathfrak{a}}\left(M', n\right)$ and $\ell_A\left(M'/M'_n\right)$ have the same degree and the same leading coefficient, and $R(n) = P_{\mathfrak{a}}\left(M', n\right) - \ell_A\left(M'/M'_n\right)$ is a polynomial function. Since $\deg P_{\mathfrak{a}}\left(M'', n\right) \leq \deg P_{\mathfrak{a}}(M, n)$, we have

$$\deg(R(n)) < \deg(\ell_A\left(M'/M'_n\right)) \leq \deg P_{\mathfrak{a}}(M, n)$$

Since, by the above inequality related to the length of A-modules, we have $R(n) \geq 0$ for $n \gg 1$, the leading coefficient of $R(n)$ is non-negative. Thus the proof is complete. $\square$

Corollary 3.1.6. *For any submodule M' of M, $d(M') \leq d(M)$.*

Proposition 3.1.7. *let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, generated by $x_1, \ldots, x_r$ and residue field $k = \frac{A}{\mathfrak{m}}$. Let $G(A) = G_{\mathfrak{m}}(A)$, as defined in Section 2.6. We define the graded k-algebra homomorphism,*

$$\varphi : k\left[X_1, \ldots, X_r\right] \to G(A)$$

$$\text{by } \varphi(X_i) = \bar{x}_i = x_i + \mathfrak{m}^2 \text{ for } 1 \leq i \leq r$$

Then $\deg\chi(G(A), n) = r - 1 \iff \varphi$ is an isomorphism.

Proof. For details of the proof, please refer to Proposition 3.8, Chapter 3 of [5]. $\square$

Corollary 3.1.8. *We have $\deg P_{\mathfrak{m}}(A, n) = r$ if and only if*

$$\varphi : k\left[X_1, \ldots, X_r\right] \to G(A)$$

is an isomorphism.

Proof. In the proof of Samuel's Theorem 3.1.3, we have seen that $\Delta P_{\mathfrak{m}}(A, n) = \chi(G(A), n)$. The result directly follows from the previous Proposition. $\square$

3.2 Dimension Theorem

Definition 3.2.1 (Chain of Prime Ideals). A *chain of prime ideals of length s* in a ring A, is a strictly increasing sequence $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_s$ of prime ideals $\mathfrak{p}_j$ of A.

Definition 3.2.2 (Height of a Prime Ideals). The *height* of a prime ideal $\mathfrak{p}$, ht$\mathfrak{p}$, is defined by

$$\mathrm{ht}\mathfrak{p} = \sup\left\{r \mid \text{ there exists in } A \text{ a chain } \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_r = \mathfrak{p}\right\}$$

Definition 3.2.3 (Coheight of a Prime Ideals). The *coheight* of a prime ideal $\mathfrak{p}$, coht $\mathfrak{p}$, is defined by

$$\mathrm{coht}\,\mathfrak{p} = \sup\left\{r \mid \text{ there exists in } A \text{ a chain } \mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r\right\}$$

Definition 3.2.4 (Krull Dimension). The *Krull dimension of a non-zero A-module* M, $\dim_A M$ (or, $\dim M$) is the supremum of lengths of chains of prime ideals belonging to Supp(M), i.e.

$$\dim M = \sup_{\mathfrak{p} \in \mathrm{Supp}(M)} \mathrm{coht}\,\mathfrak{p}$$

If $M = 0$, then Supp$(M) = \emptyset$. we define, dim$M = -1$.
The *Krull dimension of a ring* A, $\dim A$ is the supremum of lengths of all chains of prime ideals in A.

Some important results :

1. Let S be a multiplicative subset of A with $\mathfrak{p} \cap S = \emptyset$, then by Proposition 2.1.2, we have that ht$\mathfrak{p} = \mathrm{ht}\ S^{-1}\mathfrak{p}$.
2. For a prime ideal $\mathfrak{p}$ of $A, \dim A/\mathfrak{p} = \mathrm{coht}\ \mathfrak{p}$ and $\dim A\mathfrak{p} = \mathrm{ht}\ \mathfrak{p}$.

Definition 3.2.5 (Chevalley Dimension). For a non-zero A-module M, the *Chevalley dimension*, denoted as s(M), is defined to be the least integer n for which there exist n elements

$$a_1, \ldots, a_n \in \mathfrak{m} \ni \ell_A\left(\frac{M}{(a_1, \ldots, a_r)\,M}\right) < \infty$$

Since $M/\mathfrak{m}M$ is of finite length, such an integer n exists. If $M = 0$, we define $s(M) = -1$.
The *Chevalley dimension of a ring A* is defined to be its Chevalley dimension as an A-module.

Theorem 3.2.1 (Dimension Theorem). *For finitely generated module M over a Noetherian local ring A,* $\dim M = d(M) = s(M)$.

Proof. **Motivation for the proof :** "There's a way of doing it!" Hermione said crossly, "There just has to be!"[1]
For the mathematical motivation and details of the proof, please refer to Theorem 3.10, Chapter 3 of [5]. The theorem is proved by showing that $\dim M \leq d(M) \leq s(M) \leq \dim M$. □

Corollary 3.2.2. *Let M be a finitely generated module over a Noetherian local ring A. Then* $\dim_A M < \infty$.

Proof. Since clearly $\mathrm{s}(M) < \infty$, the result follows. □

Now onwards, for a finitely generated module M over a Noetherian local ring A, we refer to the common value, $\dim M = d(M) = s(M)$ as *dimension of M* and denote it as $\dim M$ or $\dim_A M$.

Corollary 3.2.3. *Let $\mathfrak{a}$ an ideal of a Noetherian ring A and let $\mathfrak{a}$ be generated by r elements (A being Noetherian, the existence of $r \in \mathbb{N}$ is guaranteed). Then, for any minimal prime ideal $\mathfrak{p}$ of A with $\mathfrak{a} \subset \mathfrak{p}$, we have $ht\,\mathfrak{p} \leq r$.*

Proof. By Corollary 2.1.3, we have that $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$. $\mathfrak{p}$ being a minimal prime ideal with $\mathfrak{a} \subset \mathfrak{p}$ and using Proposition 2.5.1, we have

$$\mathrm{Supp}\left(\frac{A_\mathfrak{p}}{\mathfrak{a}A_\mathfrak{p}}\right) = \{\mathfrak{p}A_\mathfrak{p}\} \iff \ell_{A_\mathfrak{p}}\left(\frac{A_\mathfrak{p}}{\mathfrak{a}A_\mathfrak{p}}\right) < \infty \implies \mathrm{s}(A_\mathfrak{p}) \leq r$$

Hence, by Dimension Theorem 3.2.1, $\mathrm{ht}\mathfrak{p} = \dim A_\mathfrak{p} = \mathrm{s}A_\mathfrak{p} \leq r$. □

Corollary 3.2.4 (Principal Ideal Theorem). *Let (a) be a principal ideal of a Noetherian ring A. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the minimal prime ideals of A containing $(a) = Aa$. Then $ht\,\mathfrak{p}_i \leq 1$ for $1 \leq i \leq k$.*
Also, if a is not a zero-divisor of A, then $ht\,\mathfrak{p}_i = 1$ for $1 \leq i \leq k$.

Proof. The first result directly follows from the previous Corollary. For the second assertion, suppose a is not a zero divisor of A.

[1] Since I do not want a copyright infringement lawsuit from Rowling, I must mention, this is taken from Harry Potter and the Goblet of Fire.

Claim : a cannot belong to any minimal prime ideal of A.
To prove the claim, we show that

$$\cup_{\substack{\mathfrak{p} \text{ minimal} \\ \text{prime ideal}}} \mathfrak{p} \subset \text{The set of zero-divisors of } A$$

Let $\mathfrak{p}$ be a minimal prime ideal of A. By Corollary 2.1.3, we have that $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. Also clearly $\mathfrak{p}A_{\mathfrak{p}}$ is the only prime ideal of $A_{\mathfrak{p}}$. So, using Proposition 2.4.4, we have

$$\mathfrak{n}(A_{\mathfrak{p}}) = \cap_{\mathfrak{p}\in \mathrm{Spec}(A_{\mathfrak{p}})}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$$

where $\mathfrak{n}(A_{\mathfrak{p}})$ is the nilradical of $A_{\mathfrak{p}}$. Now take $x \in \mathfrak{p}$. Then $\frac{x}{q} \in A_{\mathfrak{p}}$ for some $\frac{1}{q} \in A_{\mathfrak{p}}$,

$$\begin{aligned}
&\Longrightarrow \left(\frac{x}{q}\right)^n \in A_{\mathfrak{p}} = 0 \text{ for some } n \in \mathbb{N} \\
&\Longrightarrow \frac{x^n}{q^n} = 0 \text{ in } S^{-1}A, \text{ where } S = A - \mathfrak{p} \\
&\Longrightarrow \exists \text{ non-zero } y \in S \ni yx^n = 0 \\
&\Longrightarrow x \text{ is a zero divisor of } A.
\end{aligned}$$

The claim implies, for every $\mathfrak{p} \in \mathrm{Ass}(A/Aa)$, we have that $\mathrm{ht}\mathfrak{p} \geq 1$. Thus, for $1 \leq i \leq k$, $\mathrm{ht}\mathfrak{p}_i = 1$. $\square$

Chapter 4

A Mathematician's Kit of Homological Tools

As the title suggests, in this chapter we will study Homological Algebra as a tool to extract information from algebraic objects. We will be discussing it only as much as we will recquire for application in this report. Throughout this chapter, A denotes a commutative ring with 1 and any module is assumed to be a unitary module.

4.1 Complexes and Homology

Definition 4.1.1 (Complex). A sequence of A-modules X_n and A-homomorphisms d_n with $d_n \circ d_{n+1} = 0$ for every $n \in \mathbb{Z}$,

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

is said to be a *complex* $\underline{X}$ *of A-modules.*

We have,

$$d_n \circ d_{n+1} = 0 \ \Rightarrow \ \text{Im } d_{n+1} \subset \ker d_n$$

Definition 4.1.2 (Homology Module). The n^{th} *homology module of* $\underline{X}$, denoted by $H_n(\underline{X})$, is defined to be $\frac{\ker d_n}{\text{Im } d_{n+1}}$.

The homology measures how much the sequence differs from being exact, that is, how underexact it is. $\underline{X}$ is called a **left** (resp. **right**) **complex** if $X_n = 0$ for $n < 0$ (resp. $n > 0$). We write X^n for X_{-n} and H^n for H_{-n}. If $\underline{X}$ is a right complex, we denote it by

$$0 \to X^0 \to X^1 \to \cdots$$

The complex $\underline{X}$ with $X_n = 0\,\forall n \in \mathbb{Z}$ will be denoted by 0.

4.1.1 Morphism of Complexes

Definition 4.1.3 (Morphism of Complexes). Let $\underline{X}$, $\underline{Y}$ be complexes of A-modules. A *morphism* $f : \underline{X} \to \underline{Y}$ *of complexes* is a family $\{f_n : X_n \to Y_n\}_{n\in\mathbb{Z}}$ of A-homomorphisms such that, for every $n \in \mathbb{Z}$, the following diamgram is commutative.

$$\begin{tikzcd} X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d_{n+1}"] & Y_{n+1} \arrow[d, "d'_{n+1}"] \\ X_n \arrow[r, "f_n"] & Y_n \end{tikzcd}$$

Definition 4.1.4. Let $f : \underline{X} \to \underline{Y}$ be a morphism of complexes. Then we have the following commutative diagram :

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "d_n"] & Y_n \arrow[d, "d'_n"] \\ X_{n-1} \arrow[r, "f_{n-1}"] & Y_{n-1} \end{tikzcd}$$

Clearly, $f_n(\ker\, d_n) \subset \ker\, d'_n$ and $f_n(\mathrm{Im}\, d_{n+1}) \subset \mathrm{Im}\, d'_{n+1}$. Thus, f_n **induces an** $A-$**homomorphism**

$$H_n(f) : H_n(\underline{X}) \to H_n(\underline{Y})$$

For $\bar{\alpha} \in H_n(\underline{X}) = \frac{\ker\, d_n}{\mathrm{Im}\, d_{n+1}}$,

$$\begin{aligned} H_n(f)(\bar{\alpha}) &= H_n(f)(\alpha + \mathrm{Im}\, d_{n+1}) \\ &= f_n(\alpha) + \mathrm{Im}\, d'_{n+1} \end{aligned}$$

Properties. The following properties of $A-$homomorphism $H_n(f)$ can be seen easily.

- Let $g : \underline{Y} \to \underline{Z}$ be a morphism of complexes. We have the composition of homomorphism of complexes, $gf : \underline{X} \to \underline{Z}$ defined in obvious way. Then

$$H_n(gf) = H_n(g)H_n(f)\,, \text{for every } n \in \mathbb{Z}$$

- Evidently,

$$H_n(1_{\underline{X}}) = 1_{H_n(\underline{X})}.$$

Definition 4.1.5 (Exactness of Sequence of Complexes). A sequence of complexes

$$0 \to \underline{X} \xrightarrow{f} \underline{Y} \xrightarrow{g} \underline{Z} \to 0$$

is said to be exact if, the sequence

$$0 \to X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \to 0$$

is exact for every $n \in \mathbb{Z}$.

4.1.2 Connecting Homomorphism

Let a sequence of complexes of $A-$modules, $0 \to \underline{X} \xrightarrow{f} \underline{Y} \xrightarrow{g} \underline{Z} \to 0$ be exact. For $n \in \mathbb{Z}$, we have the following commutative diagram.

$$\begin{tikzcd}
 & & Y_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d'_{n+1}"] & Z_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "d_n"] & Y_n \arrow[r, "g_n"] \arrow[d, "d'_n"] & Z_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d_{n-1}"] & Y_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d'_{n-1}"] & Z_{n-1} \arrow[r] & 0 \\
0 \arrow[r] & X_{n-2} \arrow[r, "f_{n-2}"] & Y_{n-2} & &
\end{tikzcd}$$

Figure 4.1: Exact sequence of complexes

Let $\bar{z} \in H_n(\underline{Z}) = \frac{\ker\, d''_n}{\text{Im}\, d''_{n+1}}$ and let $z \in \ker\, d''_n \subset Z_n$ represents $\bar{z} \cdots (*)$
Since g_n is surjective, we have,

$$y \in Y_n \ \ni\ g_n(y) = z \quad \cdots (**)$$

$$\text{Let } y' = d'_n(y) \ \in Y_{n-1}$$

$$\begin{aligned}
\text{Then, } g_{n-1}(y') &= g_{n-1} \circ d'_n(y) \\
&= d''_n \circ g_n(y) \\
&= d''_n(z) \\
&= 0
\end{aligned}$$

Since Im(f_{n-1}) =kerg_{n-1}, $\exists\, x \in X_{n-1} \ni y' = f_{n-1}(x) \implies d'_n(y) = f_{n-1}(x)$. Now,

$$\begin{aligned} f_{n-2} \circ d_{n-1}(x) &= d'_{n-1} \circ f_{n-1}(x) \\ &= d'_{n-1} \circ d'_n(y) \\ &= 0 \end{aligned}$$

Since, f_{n-2} is injective, $\ker(f_{n-2}) = 0$. Hence, we obtain,

$$d_{n-1}(x) \in \ker\,(f_{n-2}) = 0 \implies x \in \ker\,(d_{n-1})$$

Let $\bar{x}$ be the canonical image of x in $H_{n-1}(\underline{X})$.

$$\bar{x} = x + \text{Im}\ d_n$$

Note that $\bar{x}$ is independent of the choice of z and y.

$\bar{x}$ independent of choice of y.

Let $y_1, y_2 \in Y_n$ in step (**) above, such that

$$g_n(y_1) = g_n(y_2) = z \implies g_n(y_1 - y_2) = 0$$

Assume we obtain x_1 , $x_2 \in \ker\, d_{n-1}$ respectively from y_1 , y_2. Now,

$$y_1 - y_2 \in \ker\,(g_n) = \text{Im}\,(f_n)$$

$$\implies \exists\, x' \in X_n \ni f_n(x') = (y_1 - y_2)$$

$$\begin{aligned} \text{Now, } f_{n-1} \circ d_n(x') &= d'_n \circ f_n(x') \\ &= d'_n(y_1 - y_2) \\ &= f_{n-1}(x_1 - x_2) \end{aligned}$$

Since, $\ker\,(f_{n-1}) = 0$, we obtain,

$$d_n(x') = (x_1 - x_2) \implies (x_1 - x_2) \in \text{Im}(d_n)$$

$$\implies \bar{x_1} = \bar{x_2}.$$

$\bar{x}$ indepent of choice of z.

Let $z_1, z_2 \in \ker\, d''_n \subset Z_n$ in step (*) above, such that

$$\bar{z_1} = \bar{z_2} \implies z_1 - z_2 \in \text{Im}\ d''_{n+1}$$

$$\therefore\ \exists\, p \in Z_{n+1} \ni\ d''_{n+1}(p) = z_1 - z_2$$

Let $g_n(y_1) = z_1$ and $g_n(y_2) = z_2$. Since g_{n+1} is surjective,

$$\exists\, q \in Y_{n+1} \ni\ g_{n+1}(q) = p$$

Now we have

$$\begin{aligned} g_n \circ d'_{n+1}(q) &= d''_{n+1} \circ g_{n+1}(q) \\ &= z_1 - z_2 \\ &= g_n(y_1 - y_2) \end{aligned}$$

Let $s = d'_{n+1}(q)$ and $r = (y_1 - y_2)$.

$$\begin{aligned} &\implies g_n(s-r) = 0 \\ &\implies s - r \in \ker\,(g_n) = \operatorname{Im}\,(f_n) \\ &\implies \exists\, w \in X_n \ni\ f_n(w) = s - r \\ &\implies d'_n \circ f_n(w) = d'_n(s-r) = f_{n-1}(x_2 - x_1) \\ &\implies f_{n-1}[d_n(w) - (x_2 - x_1)] = 0 \end{aligned}$$

Since f_{n-1} is injective, $\ker(f_{n-1}) = 0$, which implies

$$d_n(w) = (x_2 - x_1) \implies x_2 - x_1 \in \operatorname{Im}\, d_n$$

$$\implies \bar{x_2} = \bar{x_1}.$$

Definition 4.1.6 (Connecting Homomorphism). We define,

$$\partial_n : H_n(\underline{Z}) \to H_{n-1}(\underline{X})$$

$$\text{by } \partial_n(\bar{z}) = \bar{x}$$

∂_n is an $A-$homomorphism, since all the maps involved in connecting $\bar{z}$ to $\bar{x}$ are $A-$ homomorphic. The homomorphisms $\{\partial_n\}_{n\in\mathbb{Z}}$ are called *connecting homomorphisms.*

Proposition 4.1.1. *Let* $0 \to \underline{X} \xrightarrow{f} \underline{Y} \xrightarrow{g} \underline{Z} \to 0$ *be an exact sequence of complexes of* $A-$ *modules. Then the sequence*

$$\cdots \to H_n\underline{X} \xrightarrow{H_n(f)} H_n\underline{Y} \xrightarrow{H_n(g)} H_n\underline{Z} \xrightarrow{\partial_n} H_{n-1}\underline{X} \xrightarrow{H_{n-1}(f)} H_{n-1}\underline{Y} \to \cdots$$

is exact.

Proof. **<u>STEP I :</u> Exactness at $H_n(\bar{Z})$.** We want to prove, Im $H_n(g) = \ker \partial_n$.
($\Rightarrow$)
If $\bar{z} \in$ Im $H_n(g)$, we can choos,e

$$y \in \ker d'_n \ni y' = d'_n(y) = 0\,, \text{ where } H_n(g)(\bar{y}) = \bar{z}.$$

Hence, $f_{n-1}(x) = 0 \implies x = 0$, which implies,

$$\partial_n(\bar{z}) = \bar{x} = 0 \implies \bar{z} \in \ker\,(\partial_n)$$

Hence Im $H_n(g) \supset \ker \partial_n$.
($\Leftarrow$)
Suppose we have,$\bar{z} \in \ker \partial_n$, i.e. $\bar{x} = \partial_n(\bar{z}) = 0 \in H_{n-1}\underline{X}$. Then

$$\exists x' \in X_n \ni d_n(x') = x$$

Let $y'' = y - f_n(x') \in Y_n$. Then we have,

$$d'_n(y'') = d'_n(y) - d'_n \circ f_n(x') = d'_n(y) - f_{n-1} \circ d_n(x') = 0$$

So $y'' \in \ker d'_n$. Now

$$g_n(y'') = g_n(y - f_n(x')) = g_n(y) - g_n \circ f_n(x') = z - 0 = z$$

$$\implies \bar{z} \in \text{Im } H_n(g)$$

Hence Im $H_n(g) \subset \ker \partial_n$.

<u>STEP II:</u> Exactness at $H_{n-1}(\bar{X})$. We want to prove, Im $\partial_n = \ker H_{n-1}(f)$.
($\Rightarrow$)
Let $\bar{x} \in$ Im ∂_n. With the notation same as in the proof above the definition of connecting homomorphism, we have $H_{n-1}(f)\partial_n(\bar{z}) = H_{n-1}(f)(\bar{x})$, which is the canonical image of $y' = f_{n-1}(x)$ in $H_{n-1}(Y)$.
Since $y' \in$ Im d'_n,

$$H_{n-1}(f)(\bar{x}) = 0 \implies \bar{x} \in \ker\,(H_{n-1}(f))$$

Hence, Im $\partial_n \subset \ker H_{n-1}(f)$.
($\Leftarrow$)
Let $x_{n-1} \in H_{n-1}(\underline{X})$ be such that $H_{n-1}(f)\,(x_{n-1}) = 0$.

$$\implies f_{n-1}\,(x_{n-1}) \in \text{Im } d'_n$$

Let $y_{n-1} \in Y_n$ be such that $d'_n(y_{n-1}) = f_{n-1}(x_{n-1})$.
Let $\bar{z}_{n-1}$ be the class of $g_n(y_{n-1})$ in $H_n(\underline{Z})$. So $\partial_n(\bar{z}_{n-1}) = \bar{x}_{n-1}$.
Hence, Im $\partial_n \supset$ ker $H_{n-1}(f)$.

<u>STEP III :</u> Exactness at $H_n(\bar{Y})$. We want to prove, Im $()H_n(f) =$ ker $(H_n(g))$.
$(\Rightarrow)$
We have

$$H_n(g) \circ H_n(f) = H_n(g \circ f) = 0$$

$$\implies \text{Im } (H_n(f)) \subset \text{ker } (H_n(g))$$

$(\Leftarrow)$
Let $\bar{y} \in H_n(\underline{Y})$ be such that $H_n(g)(\bar{y}) = 0$.
Let $y \in \text{ker} d'_n$ be a representative of $\bar{y}$. Then

$$\exists z \in Z_{n+1} \ni g_n(y) = d''_{n+1}(z)$$

Now, since g_{n+1} is surjective, we choose

$$y' \in Y_{n+1} \ni z = g_{n+1}(y')$$

We obtain, from the following commutative diagram:

$$\begin{tikzcd}
Y_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d'_{n+1}"] & Z_{n+1} \arrow[d, "d''_{n+1}"] \\
Y_n \arrow[r, "g_n"] & Z_n
\end{tikzcd}$$

$$g_n\left(y - d'_{n+1}(y')\right) = 0$$

Since ker $g_n =$ Im f_n,

$$\exists x \in X_n \ni y - d'_{n+1}(y') = f_n(x)$$

Since $d'_{n+1}(y') \in$ Im d'_{n+1}, the class of $f_n(x)$ in $H_n(\underline{Y})$ is the same as that of y. To complete the proof,i.e. to show $\bar{y} \in$ Im $(H_n(f))$, it is enough to prove that $x \in \ker d_n$.
We have

$$d'_n(f_n(x)) = d'_n\left(y - d'_{n+1}(y)\right) = 0$$

which implies,

$$f_{n-1}(d_n(x)) = 0 \Rightarrow d_n(x) = 0$$

Hence, Im $(H_n(f)) \supset$ ker $(H_n(g))$. It completes the proof. $\square$

Proposition 4.1.2. *Let the following diagram of complexes with rows exact*

$$\begin{tikzcd}
0 \arrow[r] & \underline{X} \arrow[r, "f"] \arrow[d, "\alpha"] & \underline{Y} \arrow[r, "g"] \arrow[d, "\beta"] & \underline{Z} \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \underline{X}' \arrow[r, "f'"] & \underline{Y}' \arrow[r, "g'"] & \underline{Z}' \arrow[r] & 0
\end{tikzcd}$$

be commutative. Then the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(\underline{X}) \arrow[r] \arrow[d, "H_n(\alpha)"] & H_n(\underline{Y}) \arrow[r] \arrow[d, "H_n(\beta)"] & H_n(\underline{Z}) \arrow[r, "\partial_n"] \arrow[d, "H_n(\gamma)"] & H_{n-1}(\underline{X}) \arrow[r] \arrow[d, "H_{n-1}(\alpha)"] & H_{n-1}(\underline{Y}) \arrow[r] \arrow[d, "H_{n-1}(\beta)"] & \cdots \\
\cdots \arrow[r] & H_n(\underline{X}') \arrow[r] & H_n(\underline{Y}') \arrow[r] & H_n(\underline{Z}') \arrow[r, "\partial'_n"] & H_{n-1}(\underline{X}') \arrow[r] & H_{n-1}(\underline{Y}') \arrow[r] & \cdots
\end{tikzcd}$$

of homology modules is commutative.

Proof. Clearly,

$$H_n(g') \circ H_n(\beta) = H_n(g' \circ \beta) = H_n(\gamma \circ g) = H_n(\gamma) \circ H_n(g).$$

Similarly commutativity can be proven for the parts not involving connecting homomorphisms. For the ones involving connecting homomorphisms, we need to prove that

$$H_{n-1}(\alpha) \circ \partial_n = \partial'_n \circ H_n(\gamma)\, \forall n \in \mathbb{Z}$$

Let $\bar{z} \in H_n(\underline{Z})$ with z as a representative. We have g_n surjective. Let $y \in Y_n$ be such that $g_n(y) = z$. By assumption, the following diagram is commutative.

$$\begin{tikzcd}
Y_n \arrow[r, "g_n"] \arrow[d, "\beta_n"] & Z_n \arrow[d, "\gamma_n"] \\
Y'_n \arrow[r, "g'_n"] & Z'_n
\end{tikzcd}$$

Therefore $\gamma_n(z)$ is a representative of $H_n(\gamma)(\bar{z})$ and $\beta_n(y)$ is a lift of $\gamma_n(z)$ in Y'_n. We use the lift y to compute $\partial_n(\bar{z})$ and the lift $\beta_n(y)$ to compute $\partial'_n \circ (H_n(\gamma)(\bar{z}))$. Clearly,

$$\left(H_{n-1}(\alpha) \circ \partial_n\right)(\bar{z}) = \left(\partial'_n \circ H_n(\gamma)\right)(\bar{z})$$

Since $\bar{z} \in H_n(\underline{Z})$ is arbitrary, the proof is complete. □

4.1.3 Homotopic Morphism

We have seen how homomorphism of homology arises. We are interested to know when different A-homomorphisms between two chain complexes give rise to same A-homomorphism between the corresponding complexes of homology modules. One reason for this being the fact that when we write the complexes later used to define Ext and Tor, some choices will be made, and we don't want our answers to depend on the choices.

Definition 4.1.7 (Homotopy). Let $\underline{X}, \underline{Y}$ be complexes of A-modules and $f, g : \underline{X} \to \underline{Y}$ be two morphisms. A family $h = \{h_n\}_{n \in \mathbf{Z}}$ of A-homomorphisms is said to be a *homotopy* between f and g when $h_n : X_n \to Y_{n+1}$ satisfy the relation

$$h_{n-1}d_n + d'_{n+1}h_n = f_n - g_n \text{ for every } n \in \mathbb{Z}.$$

$$\begin{tikzcd}
X_{n+1} \arrow[r, "f_{n+1}", "g_{n+1}"'] \arrow[d, "d_{n+1}"] & Y_{n+1} \arrow[d, "d'_{n+1}"] \\
X_n \arrow[r, "f_n", "g_n"'] \arrow[d, "d_n"] \arrow[ur, "h_n"] & Y_n \arrow[d, "d'_n"] \\
X_{n-1} \arrow[r, "f_{n-1}", "g_{n-1}"'] \arrow[ur, "h_{n-1}"'] & Y_{n-1}
\end{tikzcd}$$

f and g are said to be homotopic, if there is a homotopy between between them. It can be easily seen that, **homotopy is an equivalence relation.**

Proposition 4.1.3. *Let $f, g : \underline{X} \to \underline{Y}$ be morphisms of complexes of A-modules such that f and g are homotopic. Then, for every $n \in \mathbb{Z}$, $H_n(f) = H_n(g)$.*

Proof. Take $\bar{x} \in H_n(X)$ and let $x \in \ker d_n$ be a representative of $\bar{x}$. By the definition of homotopy, we have a family $h = \{h_n\}_{n \in \mathbf{Z}}$ of A-homomorphisms such that

$$\begin{aligned} f_n(x) - g_n(x) &= h_{n-1} \circ d_n(x) + d'_{n+1} \circ h_n(x) \\ &= d'_{n+1} \circ h_n(x) \in \operatorname{Im} d'_{n+1} \end{aligned}$$

$$\therefore H_n(f)(\bar{x}) = H_n(g)(\bar{x})$$

This completes the proof. □

4.2 Projective Modules

Definition 4.2.1 (Free A-module). An $A-$module, M is said to be free (or A-free) if it has a basis, i.e. a set $S = \{e_i\}_{i \in I}$of linearly independent elements which generates M. For an free $A-$ module M,

$$M = \sum_{i \in I} Ae_i \text{ and } Ann(e_i) = \{0\} \forall\, i \in I$$

Example 4.2.1. A itself is an A-free module with basis $\{1\}$.

Example 4.2.2. The $A-$modules A^n $\forall\, n \in \mathbb{N}$ are $A-$free with bases $\{e_i\}_{i \in \{1,2,\cdots,n\}}$, where $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0) \cdots e_n = (0, 0, \cdots, 0, 1)$.

Example 4.2.3. Any finite cyclic group is not a $\mathbb{Z}$-free module. Since any finite cyclic group is isomorphic to $\mathbb{Z}_n$ for some $n \in \mathbb{N}$, and for every $a \in \mathbb{Z}_n$, $na = 0$, i.e. $Ann(a) \neq \{0\}$, so there is no basis for $\mathbb{Z}_n$ as $\mathbb{Z}$-module.

Projective modules enlarges the class of free modules.

Proposition 4.2.1. *Let P be an A-module. The following conditions are equivalent:*

(i) P is a direct summand of a free A-module.

(ii) For any diagram of A-homomorphisms,

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & M'' \arrow[r] & 0 \end{tikzcd}$$

with exact row, $\exists$ an A-homomorphism $\bar{f} : P \to M \ni \varphi \circ \bar{f} = f$;

(iii) If $0 \to M' \to M \xrightarrow{\varphi} P \to 0$ is an exact sequence A-modules, it splits.

Proof. $\underline{\textbf{(i)} \implies \textbf{(ii)}}$:
Let F be a free A-module with basis $(e_i)_{i \in I}$ and Q be an A-module such that $P \oplus Q = F$. We define,

$$g : F \to M'' \text{ by } g\big|_P = f \text{ and } g\big|_Q = 0$$

Since φ is surjective, we can choose $x_i \in M \ni \varphi(x_i) = g(e_i)$. We define,

$$\bar{g} : F \to M \text{ by } \bar{g}(e_i) = x_i \forall i \in I$$

We obtain $\varphi \circ \bar{g}(e_i) = \varphi(x_i) = g(e_i)$, which implies

$$\varphi \circ \bar{g} = g$$

Clearly, $\bar{f} = \bar{g}\big|_P$ satisfies $\varphi \circ \bar{f} = g\big|_P = f$.
(ii) $\implies$ (iii):
Considering the diagram

$$\begin{tikzcd} & P \arrow[d, "1_P"] & \\ M \arrow[r, "\varphi"] & M'' \arrow[r] & 0 \end{tikzcd}$$

assuming the assertion of (ii) being true, we get an A-homomorphism $f : P \to M$ such that $\varphi \circ f = 1_P$, i.e. every exact sequence $0 \to M' \to M \to P \to 0$ splits.
(iii) $\implies$ (i):
Let $F \to P \to 0$ be an exact sequence with F being an free A-module.[1] Let $K = \ker(F \to P)$ then, assuming (iii) is true,we have the exact sequence

$$0 \to K \to F \to P \to 0$$

which splits implying that $F = K \oplus P$, i.e. P is a direct summand of F. □

Definition 4.2.2 (Projective A-module). An A-module P is said to be a *projective A-module*, if it satisfies any of the equivalent conditions of Proposition 4.1.2.

Example 4.2.4. Any free module is an projective module (In the first part of the proof of Proposition 4.1.2, it has been proved).

Remark. The converse of Example 4.2.4 is not true.

Example 4.2.5. Consider the non-trivial commutative rings R_1 and R_2 with multiplicative identity 1. Let $R = R_1 \oplus R_2$. Here, R_1 and R_2 are projective $A-$modules, but not free.
R_1 and R_2 are projective : Clearly R is a free $R-$module with basis $\{(1,0),(0,1)\}$.As direct summand of free $A-$modules, R_1 and R_2 are projective.
R_1 and R_2 are not free : Suppose, R_1 is free. Then we have $\{e_i\}_{i\in I}$ as bases of R_1. Note, we have the projection maps, $\pi_i : R \to R_i \forall i \in \{1,2\}$. R_1 and R_2 are $R-$modules over R, via

$$\cdot : R \times R_i \to R_i$$

$$(x,y)\cdot r = \pi_i(x,y)r$$

[1]For instance, F can be chosen as the free A-module on a set of generators of P.

Now fix $j \in I$. Both R_1 and R_2 being non-trivial, $1 \neq 0$.

$$(1,1) \cdot e_j = \pi_1(1,1)e_j = e_j = \pi_1(1,0)e_j = (1,0) \cdot e_j$$

$$\Rightarrow\Leftarrow$$

This is because $(1,1) \neq (1,0)$ in R and $\{e_i\}_{i\in I}$ being bases, $\alpha \cdot e_j = 0 \implies \alpha = 0$.

Corollary 4.2.2. *Direct summands and direct sums of projective modules are projective.*

Corollary 4.2.3. *For projective A-module P and an exact sequence of A-modules, $0 \to M' \to M \to M'' \to 0$, the following sequence is exact.*

$$0 \to P \otimes_A M' \to P \otimes_A M \to P \otimes_A M'' \to 0$$

Proof. Let F be a free A-module and Q be an A-modules such that $F = P \oplus Q$. We have the following exact sequence,

$$0 \to F \otimes_A M' \to F \otimes_A M \to F \otimes_A M'' \to 0$$

Now we have that direct sum commutes with tensor products, so we get the exact sequence

$$0 \to (P \otimes_A M')\oplus(Q \otimes_A M') \to (P \otimes_A M)\oplus(Q \otimes_A M) \to (P \otimes_A M'')\oplus(Q \otimes_A M'') \to 0$$

which implies

$$0 \to P \otimes_A M' \to P \otimes_A M \to P \otimes_A M'' \to 0$$

is exact. □

Corollary 4.2.4. *Let P be a projective A-module and $A \to B$ be a ring homomorphism. Then $B \otimes_A P$ is a projective B-module.*

Proof. Let F be a free A-module of which P be a direct summand, i.e. for some A-module Q,

$$F = P \oplus Q$$

Let $\{e_i\}_{i\in I}$ be a basis of $A-$free module F .Now $A \to B$ being a ring homomorphism, $B \otimes_A F$ is a free $B-$module with $\{1 \otimes e_i\}_{i\in I}$ as bases. Since direct sum commutes with tensor products, we have

$$B \otimes_A F = B \otimes_A (P \oplus Q) = (B \otimes_A P) \oplus (B \otimes_A Q)$$

$\therefore B \otimes_A P$ is a direct summand of the free B-module $B \otimes_A F$ and hence, a projective B-module. □

4.3 Projective Resolutions

Definition 4.3.1 (Projective Resolution). A projective resolution of an A-module M is a pair of a left complex of A-modules and A-homomorphism $(\underline{P}, \varepsilon)$, where P is a left complex with each P_i being projective and $\varepsilon : P_0 \to M$ is surjective so that the sequence

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact.

Proposition 4.3.1. *Every A-module M has a projective resolution.*

Proof. Given any A-module M, we coose a free A-module P_0 such that $P_0 \to M \to 0$ is an exact sequence. Now we inductively define P_i and d_i. Suppose we have an exact sequence

$$P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with each P_i being A-free. We set $K_n = \ker d_n$. We choose a free A-module P_{n+1} such that the sequence $P_{n+1} \xrightarrow{\varphi_{n+1}} K_n \to 0$ is exact.
We define $d_{n+1} = \alpha \circ \varphi_{n+1}$, where $\alpha : K_n \hookrightarrow P_n$ is the canonical inclusion. It is can be seen that the following sequence is exact.

$$P_{n+1} \xrightarrow{d_{n+1}} P_n \to \cdots \xrightarrow{d_0} P_0 \to M \to 0$$

Thus the proposition is proved. □

Corollary 4.3.2. *Every A-module has a projective resolution $(\underline{P}, \varepsilon)$ with P_i being A-free.*

Corollary 4.3.3. *Let M be a finitely generated module over a Noetherian ring A. Then M has a projective resolution $(\underline{P}, \epsilon)$ with each P_i being a finitely generated free A-module.*

Proof. We use the notation as used in Proposition 4.3.1. Now M being finitely generated, P_0 can be chosen to be a finitely generated free A-module. Then, by Property 3 of Noetherian modules in Section 2.3, $K_0 = \ker \varepsilon$ is finitely generated. Now, in the line of the proof of Proposition 4.3.1, each P_n can be chosen inductively to be finitely generated. This proves the Corollary. □

Definition 4.3.2. Let $(\underline{P}, \varepsilon)$, $(\underline{P}', \varepsilon')$ be projective resolutions of two A -modules M, M' respectively and $f : M \to M'$ be a A-homomorphism. A *morphism* $F : \underline{P} \to \underline{P}'$ *of complexes* is said to be *over* f if $f \circ \varepsilon = \varepsilon' \circ F_0$.

$$
\begin{tikzcd}
P_1 \arrow[r, "F_1"] \arrow[d, "d_0"] & P_1' \arrow[d, "d_0'"] \\
P_0 \arrow[r, "F_0"] \arrow[d, "\varepsilon"] & P_0' \arrow[d, "\varepsilon'"] \\
M \arrow[r, "f"] \arrow[d] & M' \arrow[d] \\
0 & 0
\end{tikzcd}
$$

Proposition 4.3.4. *Let $(\underline{P}, \varepsilon)$, $(\underline{P}', \varepsilon')$ be projective resolutions of two A-modules M, M' respectively and $f : M \to M'$ be a A-homomorphism. Then there exists a morphism $F : \underline{P} \to \underline{P}'$ over f.*
Also, if $F, G : \underline{P} \to \underline{P}'$ are morphisms over f, then F and G are homotopic.

Proof. **To prove the existence of** F. Consider the diagram with exact row,

$$
\begin{tikzcd}
 & P_0 \arrow[d, "f\varepsilon"] & \\
P_0' \arrow[r, "\varepsilon'"] & M' \arrow[r] & 0
\end{tikzcd}
$$

we have that P_0 is projective. By Proposition 4.2.1 there exists an A homomorphism $F_0 : P_0 \to P_0'$ such that $f \circ \varepsilon = \varepsilon' \circ F_0$.
Now F_n is defined by induction on n, assuming F_p to be defined for $p < n$. We set $d_0 = \varepsilon$, $d_0' = \varepsilon'$ and $F_{-1} = f$. So we have

$$d'_{n-1} \circ F_{n-1} \circ d_n = F_{n-2} \circ d_{n-1} \circ d_n = 0$$

which implies $\operatorname{Im}(F_{n-1} \circ d_n) \subset \ker d'_{n-1} = \operatorname{Im} d'_n$. Hence we obtain the diagram with exact row,

$$
\begin{tikzcd}
 & P_n \arrow[d, "F_{n-1}d_n"] & \\
P_n' \arrow[r, "d_n'"] & \operatorname{Im} d_n' \arrow[r] & 0
\end{tikzcd}
$$

Now P_n being projective,

$$\exists\, F_n : P_n \to P'_n \ni d'_n F_n = F_{n-1} d_n$$

Thus the existence of F is proved.

To prove F and G are homotopic. Since F, G are morphisms over f, we have that $\varepsilon' \circ F_0 = f \circ \varepsilon = \varepsilon' \circ G_0$, which implies

$$\varepsilon' \circ (F_0 - G_0) = 0 \implies \operatorname{Im}(F_0 - G_0) \subset \ker \varepsilon' = \operatorname{Im} d'_1$$

from which we obtain a diagram,

$$\begin{tikzcd} & P_0 \arrow[d, "F_0 - G_0"] & \\ P'_1 \arrow[r, "d'_1"] & \operatorname{Im} d'_1 \arrow[r] & 0 \end{tikzcd}$$

Now P_0 being projective, $\exists$ an A-homomorphism $h_0 : P_0 \to P'_1$ such that $d'_1 \circ h_0 = F_0 - G_0$. We assume that inductively, for $n_0 < n, h_{n_0} : P_{n_0} \to P'_{n_0+1}$ has been defined so that

$$d'_{n_0+1} \circ h_{n_0} + h_{n_0-1} \circ d_{n_0} = F_{n_0} - G_{n_0}$$

Now we define $h_n : P_n \to P'_{n+1}$ In the following way. Since

$$d'_n \circ (F_n - G_n) = (F_{n-1} - G_{n-1}) \circ d_n = (d'_n \circ h_{n-1} + h_{n-2} \circ d_{n-1}) \circ d_n = d'_n \circ h_{n-1} \circ d_n$$

we get

$$d'_n \circ (F_n - G_n - h_{n-1} \circ d_n) = 0 \implies \operatorname{Im}(F_n - G_n - h_{n-1} \circ d_n) \subset \ker d'_n = \operatorname{Im} d'_{n+1}$$

Thus we obtain a diagram with exact row,

$$\begin{tikzcd} & P_n \arrow[d, "F_n - G_n - h_{n-1} d_n"] & \\ P'_{n+1} \arrow[r, "d'_{n+1}"] & \operatorname{Im} d'_{n+1} \arrow[r] & 0 \end{tikzcd}$$

Now P_n being projective, $\exists$ an A-homomorphism $h_n : P_n \to P'_{n+1}$ with

$$d'_{n+1} \circ h_n = F_n - G_n - h_{n-1} \circ d_n \implies d'_{n+1} \circ h_n + h_{n-1} \circ d_n = F_n - G_n$$

Thus the family of A-homomorphisms $h = \{h_n\}$ defines a homotopy between F and G. $\square$

Definition 4.3.3 (Projective Resolution of an Exact Sequence). A projective resolution of an exact sequence of A-modules, $0 \to M' \to M \to M'' \to 0$, is an exact sequence $0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0$ where $(\underline{P}', \varepsilon'), (\underline{P}, \varepsilon), (\underline{P}'', \varepsilon'')$ are projective resolutions of M', M, M'' respectively, so that the following diagram commutative.

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r] \arrow[d, "\varepsilon'"] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & P''_0 \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

Proposition 4.3.5. *Every exact sequence has a projective resolution.*

Proof. Let $(\underline{P}', \varepsilon'), (\underline{P}'', \varepsilon'')$ be projective resolutions of two A-modules M', M'' respectively and $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ be an exact sequence with M also being an A-module. Then we want to prove that M has a projective resolution $(\underline{P}, \varepsilon)$ so that the sequence of complexes,

$$0 \to \underline{P}' \xrightarrow{f} \underline{P} \xrightarrow{g} \underline{P}'' \to 0$$

is exact and the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r, "f_0"] \arrow[d, "\varepsilon'"] & P_0 \arrow[r, "g_0"] \arrow[d, "\varepsilon"] & P''_0 \arrow[r] \arrow[d, "\varepsilon''"] & 0 & (*) \\
0 \arrow[r] & M' \arrow[r, "i"] & M \arrow[r, "j"] & M'' \arrow[r] & 0
\end{tikzcd}$$

We prove this by defining P_n such that the exactness of complexes is preserved and and verify that the corresponfing $d_n$and ε along with P_n's for a projective resolution of M.

For $n \in \mathbb{Z}$, we define projective A-modules $P_n = P'_n \oplus P''_n$, A-homomorphism f_n :

$P'_n \to P_n$ by $f_n(x') = (x', 0)$ and $g_n : P_n \to P''_n$ by $g_n(x', x'') = x''$.
Let us assume that there exist A-homomorphisms $l : P''_0 \to M$ and $k_n : P''_n \to P'_{n-1}$, for $n \in \mathbb{N}$ such that the following conditions, marked by (**), are satisfied,

$$\varepsilon'' = j \circ l$$
$$i \circ \varepsilon' \circ k_1 + l \circ d''_1 = 0$$
$$d'_{n-1} \circ k_n + k_{n-1} \circ d''_n = 0, \text{ for } n > 1$$

Now $d_n : P_n \to P_{n-1}$, for $n \in \mathbb{N}$ and $\varepsilon : P_0 \to M$ is defined as follows (Theses definitions are marked as for future reference (***)):

$$d_n(x', x'') = (d'_n(x') + k_n(x''), d''_n(x''))$$
$$\varepsilon(x', x'') = i \circ \varepsilon'(x') + l(x'')$$

It can be easily verified that $(\underline{P}, \varepsilon)$ is a projective resolution of M and that the diagram $(*)$ is commutative. Now if we can prove the following, we are done.

To prove the existence of l and k_n. We have that P''_0 is projective and j is a surjective homomorphism. So existence of l follows from Proposition 4.2.1.
Now k_n is constructed by induction on n. We have

$$j \circ (-l \circ d''_1) = -\varepsilon'' \circ d''_1 = 0 \implies \operatorname{Im}(-l \circ d''_1) \subset \ker j = \operatorname{Im}(i \circ \varepsilon')$$

We have the diagram,

$$\begin{tikzcd} & P''_1 \arrow[d, "-ld''_1"] & & \\ P'_0 \arrow[r, "i\varepsilon'"] & M \arrow[r, "j"] & M'' \arrow[r] & 0 \end{tikzcd}$$

P''_1 being projective, from the above relation, the existence of $k_1 : P''_1 \to P'_0$ is guranteed. Now assume that $k_{n-1} : P''_{n-1} \to P'_{n-2}$ has been constructed, by induction. We set $d'_0 = i \circ \varepsilon'$ and $P'_{-1} = M$. Consider the diagram
With $k_0 = l$, we have

$$d'_{n-2} \circ (-k_{n-1} \circ d''_n) = k_{n-2} \circ d''_{n-1} \circ d''_n = 0$$
$$\implies \operatorname{Im}(-k_{n-1} \circ d''_n) \subset \ker d'_{n-2} = \operatorname{Im} d'_{n-1}$$

Now P''_n being projective, the existence of $k_n : P''_n \to P'_{n-1}$ as well as the Proposition

$$\begin{tikzcd} & P''_n \arrow[d, "-k_{n-1}d''_n"] & \\ P'_{n-1} \arrow[r, "d'_{n-1}"] & P'_{n-2} \arrow[r, "d'_{n-2}"] & P'_{n-3} \end{tikzcd}$$

is proved. □

Proposition 4.3.6. *Consider the following commutative diagram of A-modules with exact rows,*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "i_1"] \arrow[d, "f'"] & M \arrow[r, "p_1"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "i_2"] & N \arrow[r, "p_2"] & N'' \arrow[r] & 0 \end{tikzcd}$$

Let the following two projective resolution of exact sequences

$$0 \to \underline{P'} \to \underline{P} \to \underline{P''} \to 0$$
$$0 \to \underline{Q'} \to \underline{Q} \to \underline{Q''} \to 0$$

correspond to

$$0 \to M' \to M \to M'' \to 0$$
$$0 \to N' \to N \to N'' \to 0$$

respectively. Let $F' : \underline{P'} \to \underline{Q'}$ *be be a morphism over* f' *and* $F'' : \underline{P''} \to \underline{Q''}$ *be over* f''*. Then* $\exists$ *a morphism* $F : \underline{P} \to \underline{Q}$ *over* f *such that the following diagram is commutative.*

$$\begin{tikzcd} 0 \arrow[r] & \underline{P'} \arrow[r] \arrow[d, "F'"] & \underline{P} \arrow[r] \arrow[d, "F"] & \underline{P''} \arrow[r] \arrow[d, "F''"] & 0 \\ 0 \arrow[r] & \underline{Q'} \arrow[r] & \underline{Q} \arrow[r] & \underline{Q''} \arrow[r] & 0 \end{tikzcd}$$

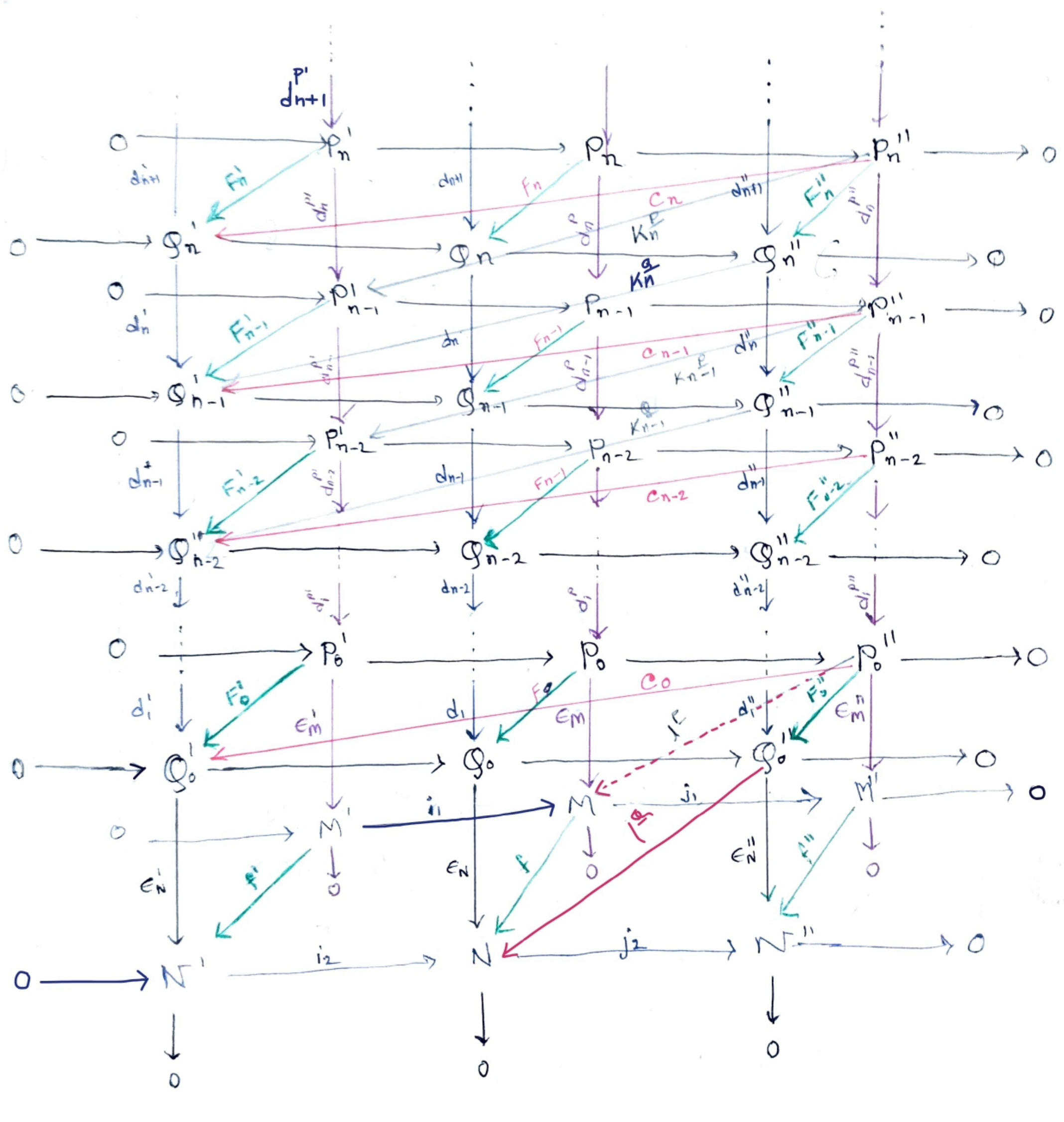
Proof. We have that P''_n is projective, so we can assume, for $n \in \mathbb{Z}, P_n = P'_n \oplus P''_n$. We can also assume that $P'_n \hookrightarrow P_n$ is the natural inclusion and $P_n \to P''_n$ is the natural epimorphism. Similar assumptions are made for the Q 's.

Now it can be is easily seen that, as done in the the proof of Proposition 4.3.5, $d_n : P_n \to P_{n-1}$ and $\varepsilon : P_0 \to M$ are given by formulæ $(***)$ of with conditions $(**)$ being fulfilled. Same process is followed for the Q 's.

Claim : For $n \in \mathbb{N}$, there exists A-homomorphisms $c_n : P''_n \to Q'_n$ which satisfies the following conditions:

$$i_2 \circ \varepsilon' \circ c_0 + l^{\underline{Q}} \circ F''_0 = f \circ l^{\underline{P}}$$
$$d'_n \circ c_n - c_{n-1} \circ d''_n = F'_{n-1} \circ k^P_n - k^Q_n \circ F''_n.$$

Now, the claim[2] can be proved by induction on n. Here we have many[3] A-homomorphisms. The notations are same as was in the proof of Proposition 4.3.5, except for that we have two sets of morphisms, one for $\underline{P}$'s and one for $\underline{Q}$'s, which are denoted as superscript. Here I am attaching the, apparently complicated, diagram[4] using which the existence of c_n's are proved.



[2]Since I shall not lie, I am avoiding to write 'easily' here.
[3]I repeat MANY.
[4]It's good enough for understanding, get a pen/chalk and draw your own diagram.

Now if someone starts writing down the proof, it can be seen that the existence of c_n's follows from the given relations and commutativity of the diagrams we have. We now define $F : \underline{P} \to \underline{Q}$ by

$$F_n\left(x', x''\right) = \left(F_n'\left(x'\right) + c_n\left(x''\right), F_n''\left(x''\right)\right)$$

It can be easily[5] verified that F is an A-homomorphism between the complexes $\underline{P}$ and $\underline{Q}$, F is over f and the diagram at the statement of the Proposition is commutative for this F. This completes the proof[6]. □

4.4 The Functors Tor

Definition 4.4.1 (Homology Modules). Let $\underline{P} = (\underline{P}, \varepsilon)$ be a projective resolution of an A-module M. Then, for any A-module N, the following left complex[7]

$$\cdots \to P_N \otimes_A N \xrightarrow{d_n \otimes 1_N} P_{n-1} \otimes_A N \to \cdots \to P_0 \otimes_A N \to 0$$

is denoted as $\underline{P} \otimes_A N$ and the homology modules $H_n\left(\underline{P} \otimes_A N\right) = \frac{\ker(d_n \otimes 1_N)}{\mathrm{Im}(d_{n+1} \otimes 1_N)}$ of this complex is denoted by $H_n(M, N; P)$.

Definition 4.4.2 (Morphism of Homology Modules). Let $\underline{P}, \underline{P}'$ be projective resolutions of A-modules M, M', respectively and $f : M \to M'$ be an A-homomorphism. Let $F : \underline{P} \to \underline{P}'$ be a morphism over f. Note that the existence of F is guranteed by Proposition 4.3.4. For every $n \in \mathbb{N}$, this morphism F defines an A-homomorphism

$$H_n\left(f, N; \underline{P}, \underline{P}'\right) : H_n(M, N; \underline{P}) \to H_n\left(M', N; \underline{P}'\right).$$

By Proposition 4.3.6, we have that the homomorphism $H_n\left(f, N; \underline{P}, \underline{P}'\right)$ is independent of the choice of the morphism (i.e. here F) over f, since, by Proposition 4.3.4, two morphisms, $\underline{P} \to \underline{P}'$ over f, are homotopic.

Now we see connecting homomorphisms[8]. Given an exact sequence of A modules, $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ and projective resolution of the exact sequence,

$$0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0$$

[5] Yes! I mean it.

[6] Yay!

[7] Note that this complex not necessarily be exact, when this sequence is exact, $H_n\left(\underline{P} \otimes_A N\right) = 0 \,\forall\, n \geq 0$.

[8] We discussed it explicitly it in Subsection 4.1.2, remember?

For $n \geq 1$, let

$$\partial_n(N, (*)) : H_n \left(M'', N; \underline{P''}\right) \to H_{n-1} \left(M', N; \underline{P'}\right)$$

be connecting homomorphisms defined by $(*)$.

Lemma 4.4.1. *Let $\underline{P}, \underline{P'}, \underline{P''}$ be projective resolutions of A-modules M, M', M'' respectively and $f : M \to M', g : M' \to M''$ be two A-homomorphisms. For every $n \in \mathbb{Z}^+$, we then have*

$$H_n \left(g \circ f, N; \underline{P}, \underline{P''}\right) = H_n \left(g, N; \underline{P'}, \underline{P''}\right) \circ H_n \left(f, N; \underline{P}, \underline{P'}\right)$$

In particular,

$$H_n \left(1_M, N; \underline{P}, \underline{P}\right) = 1_{H_n}(M, N; \underline{P}) \ .$$

Proof. Let $F : \underline{P} \to \underline{P'}$ be a morphism over f and $G : \underline{P'} \to \underline{P''}$ be a morphism over g. It is clear that $G \circ F : \underline{P} \to \underline{P''}$ is a morphism over $g \circ f : M \to M''$, proving the first part. Now 1_P is evidently over 1_M, which proves the second part. □

Lemma 4.4.2. *Let $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ an exact sequenceof A-modules with the projective resolution, denoted as (*),*

$$0 \to \underline{P'} \to \underline{P} \to \underline{P''} \to 0$$

Then the following sequence is exact.

$$\cdots \to H_n \left(M'', N; \underline{P''}\right) \xrightarrow{\partial_n} H_{n-1} \left(M', N; \underline{P'}\right) \xrightarrow{H_{n-1}(i)} H_{n-1}(M, N; \underline{P}) \to$$
$$\xrightarrow{H_{n-1}(j)} H_{n-1} \left(M'', N; \underline{P''}\right) \to \cdots \to H_0 \left(M'', N; \underline{P''}\right) \to 0$$

where ∂_n has been used to denote the connecting homomorphism $\partial_n(N, (*))$ and $H_{n-1}(i)$ has been used instead of $H_{n-1} \left(i, N; \underline{P'}, \underline{P}\right)$ and similar.

Proof. The proof follows from Proposition 4.1.1. □

Lemma 4.4.3. *Let*

$$0 \to M' \to M \to M'' \to 0$$
$$0 \to L' \to L \to L'' \to 0$$

be two exact sequence of A-modules with projective resolutions (respectively),

$$0 \to \underline{P'} \to \underline{P} \to \underline{P''} \to 0$$
$$0 \to \underline{Q'} \to \underline{Q} \to \underline{Q''} \to 0$$

Let $F' : \underline{P}' \to \underline{Q}'\big($ resp. $F : \underline{P} \to \underline{Q}$, $F'' : \underline{P}'' \to \underline{Q}''\big)$ be a morphism over $f' : M' \to L'\big($ resp. $f : M \to \bar{L}$, $f'' : M'' \to L''\big)$. Now if the following diagram with exact rows is commutative,

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & L' \arrow[r] & L \arrow[r] & L'' \arrow[r] & 0 \end{tikzcd}$$

and if diagram of complexes is also commutative,

$$\begin{tikzcd} 0 \arrow[r] & \underline{P}' \arrow[r] \arrow[d, "F'"] & \underline{P} \arrow[r] \arrow[d, "F"] & \underline{P}'' \arrow[r] \arrow[d, "F''"] & 0 & \text{(*)} \\ 0 \arrow[r] & \underline{Q}' \arrow[r] & \underline{Q} \arrow[r] & \underline{Q}'' \arrow[r] & 0 & \text{(**)} \end{tikzcd}$$

Then, for every $n \geq 1$, the diagram

$$\begin{tikzcd} H_n(M'', N; \underline{P}'') \arrow[r, "{H_n(f'', N; \underline{P}'', \underline{Q}'')}"] \arrow[d, "{\partial_n(N, (*))}"'] & H_n(L'', N; \underline{Q}'') \arrow[d, "{\partial_n(N, (**))}"] \\ H_{n-1}(M', N; \underline{P}') \arrow[r, "{H_{n-1}(f', N; \underline{P}', \underline{Q}')}"] & H_{n-1}(L', N; \underline{Q}') \end{tikzcd}$$

is commutative.

Proof. The proof follows from Proposition 4.1.2. □

Proposition 4.4.4. *Let $\underline{P}, \underline{Q}$ be two projective resolutions of A-module M. Then, for every $n \in \mathbb{Z}^+$,*

$$H_n\left(1_M, N; \underline{P}, \underline{Q}\right) : H_n(M, N; \underline{P}) \to H_n(M, N; \underline{Q})$$

is an isomorphism.

Let $\underline{P}', \underline{Q}'$ be projective resolutions of another A-module M' and $f : M \to M'$ be an A-homomorphism. Then the following diagram is commutative.

$$\begin{tikzcd}
H_n(M'', N; \underline{P}) \arrow[r, "H_n(1_M, N; \underline{P}, \underline{Q})"] \arrow[d, "H_n(f, N, \underline{P}, \underline{P}')"'] & H_n(M, N; \underline{Q}) \arrow[d, "H_n(f, N; \underline{Q}, \underline{Q}')"] \\
H_n(M', N; \underline{P}') \arrow[r, "H_n(1_{M'}, N; \underline{P}', \underline{Q}')"] & H_n(M', N; \underline{Q}')
\end{tikzcd}$$

Moreover, let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules with two projective resolutions,

$$0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0$$

$$0 \to \underline{Q}' \to \underline{Q} \to \underline{Q}' \to 0$$

denoted as (*) and (**) respectively. Then, for $n \geq 1$ the following diagram is commutative.

$$\begin{tikzcd}
H_n(M'', N; \underline{P}'') \arrow[r, "H_n(1_{M''}, N; \underline{P}'', \underline{Q}'')"] \arrow[d, "\partial_n(N, (*))"'] & H_n(M'', N; \underline{Q}'') \arrow[d, "\partial_n(N, (**))"] \\
H_{n-1}(M', N; \underline{P}') \arrow[r, "H_{n-1}(1_{M'}, N; \underline{P}', \underline{Q}')"] & H_{n-1}(M', N; \underline{Q}')
\end{tikzcd}$$

Proof. For the details of the proof, please refer to Proposition 2.15, Chapter 2 of [5]. □

Definition 4.4.3 (Tor Functor and Connecting Homomorphism). $\mathrm{Tor}_n^A(M, N)$, for $n \in \mathbb{Z}^+$, is defined to be

$$\mathrm{Tor}_n^A(M, N) := H_n(M, N; \underline{Q})$$

where Q is a projective resolution of M. As a consequence of Proposition 4.4.4, for a fixed A-module N, a sequence $\{\mathrm{Tor}_n^A(M, N)\}_{n \in \mathbf{Z}^+}$of functors from A modules to A-modules, does not depend on the choice of the projective resolution $\underline{Q}$ of M. Furthermore, for an exact sequence of A-modules, $0 \to M' \to M \to M'' \to 0$, we have A-homomorphisms $\{\partial_n : \mathrm{Tor}_n^A(M'', N) \to \mathrm{Tor}_{n-1}^A(M', N)\}_{n \geq 1}$ called the *connecting homomorphisms.*

Let $\underline{Q}$ be a projective resolution of an A-module M. Let N, N' be two A-modules and $f : N \to N'$ be a A-homomorphism. Then the morphism of complexes, $1_{\underline{Q}} \otimes f$:

$\underline{Q} \otimes N \to \underline{Q} \otimes N'$ induces an A-homomorphism, for every $n \in \mathbb{Z}$,

$$\mathrm{Tor}_n^A(M, f) : \mathrm{Tor}_n^A(M, N) \to \mathrm{Tor}_n^A(M, N')$$

Now for an exact sequence A-modules, $0 \to N' \to N \to N'' \to 0$, by Corollary 4.2.3, the sequence of complexes

$$0 \to \underline{Q} \otimes_A N' \to \underline{Q} \otimes_A N \to \underline{Q} \otimes_A N'' \to 0$$

is exact. Hence, this defines connecting homomorphisms

$$\partial_n : \mathrm{Tor}_n^A(M, N'') \to \mathrm{Tor}_{n-1}^A(M, N').$$

Theorem 4.4.5. *(i) Let N be an A-module. With N being fixed, the assignments $\left\{M \mapsto \mathrm{Tor}_n^A(M, N)\right\}_{n \in \mathbb{Z}^+}$ and $\left\{M \mapsto \mathrm{Tor}_n^A(N, M)\right\}_{n \in \mathbb{Z}^+}$ are sequences of functors from A-modules to A-modules.*

(ii) For an exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$, the sequences

$$\cdots \to \mathrm{Tor}_n^A(M'', N) \xrightarrow{\partial_n} \mathrm{Tor}_{n-1}^A(M', N) \to \mathrm{Tor}_{n-1}^A(M, N) \to$$
$$\to \mathrm{Tor}_{n-1}^A(M'', N) \to \cdots \to \mathrm{Tor}_0^A(M'', N) \to 0$$

and

$$\cdots \to \mathrm{Tor}_n^A(N, M'') \xrightarrow{\partial_n} \mathrm{Tor}_{n-1}^A(N, M') \to \mathrm{Tor}_{n-1}^A(N, M) \to$$
$$\mathrm{Tor}_{n-1}^A(N, M'') \to \cdots \to \mathrm{Tor}_0^A(N, M'') \to 0$$

are exact.

(iii) For a commutative diagram of A-modules with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{K'} \arrow[r] & \underline{K} \arrow[r] & \underline{K''} \arrow[r] & 0
\end{tikzcd}$$

the induced diagrams, for every $n \geq 1$,

$$\begin{tikzcd}
\mathrm{Tor}_n^A(M'', N) \arrow[r, "\partial_n"] \arrow[d] & \mathrm{Tor}_{n-1}^A(M', N) \arrow[d] \\
\mathrm{Tor}_n^A(K'', N) \arrow[r, "\partial_n"] & \mathrm{Tor}_{n-1}^A(K', N)
\end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Tor}_n^A(N, M'') \arrow[r, "\partial_n"] \arrow[d] & \mathrm{Tor}_{n-1}^A(N, M') \arrow[d] \\ \mathrm{Tor}_n^A(N, K'') \arrow[r, "\partial_n"] & \mathrm{Tor}_{n-1}^A(N, K') \end{tikzcd}$$

are commutative.

(iv)The functor $\mathrm{Tor}_n^A(M, N)$*, for every* $n \in \mathbb{Z}^+$*, is A-linear in both* M *and* N*.*

(v) There exists an A-isomorphism $M \otimes_A N \simeq \mathrm{Tor}_0^A(M, N)$ *which is functorial in both* M *and* N*.*

Proof. For the details of the proof, please refer to Theorem 2.16, Chapter 2 of [5]. □

Lemma 4.4.6. *For a projective A-module* P*, we have*

$$\mathrm{Tor}_n^A(P, M) = 0 \text{ and } \mathrm{Tor}_n^A(M, P) = 0$$

for every A-module M *and every* $n \geq 1$*.*

Proof. P ibeing projective, we have the following projective resolution of P,

$$0 \to P \stackrel{1_P}{\to} P \to 0$$

Computing Tor, using the above resolution, we get, for every $n \geq 1$, $\mathrm{Tor}_n^A(P, M) = 0$.

Now, let $(\underline{Q}, \varepsilon)$ be a projective resolution of M, then by Corollary 4.2.3, the sequence

$$\ldots \to Q_n \otimes_A P \to Q_{n-1} \otimes_A P \to \ldots \to Q_0 \otimes_A P \stackrel{\varepsilon \otimes 1}{\to} M \otimes_A P \to 0$$

is exact . It follows that $\mathrm{Tor}_n^A(M, P) = 0$ for $n \geq 1$. □

Proposition 4.4.7. *Let* M, N *be A-modules. Then, for every* $n \in \mathbf{Z}^+$ *there exists an isomorphism* $\mathrm{Tor}_n^A(M, N) \simeq \mathrm{Tor}_n^A(N, M)$*, which is functorial in* M *and* N*.*

Proof. Using Theorem 4.4.5 and Lemma 4.4.6, it can be proved. For further details of the proof, please refer to Proposition 2.18, Chapter 2 of [5]. □

4.5 The Functors Ext

Let $(\underline{P}, \varepsilon)$ be a projective resolution of A-module M. Let N be another A-module. The complex

$$0 \to \mathrm{Hom}_A(P_0, N) \to \mathrm{Hom}_A(P_1, N) \to \ldots \to \mathrm{Hom}_A(P_n, N) \to \ldots$$

is denoted by $\mathrm{Hom}_A(\underline{P}, N)$ and the homology module $H^n(\mathrm{Hom}_A(\underline{P}, N))$ is denoted by $\mathrm{Ext}^n_A(M, N)$. For A-homomorphisms $f : M \to M'$ and $g : N \to N'$,

$$\mathrm{Ext}^n_A(f, N) : \mathrm{Ext}^n_A(M', N) \to \mathrm{Ext}^n_A(M, N)$$

and

$$\mathrm{Ext}^n_A(M, g) : \mathrm{Ext}^n_A(M, N) \to \mathrm{Ext}^n_A(M, N')$$

are defined in the same way as they were in the case of Tor. For an exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$ and any A-module N, connecting homomorphisms

$$\partial^{n-1} : \mathrm{Ext}^{n-1}_A(M', N) \to \mathrm{Ext}^n_A(M'', N)$$

are defined as it was in the previous section, and similarly for exact sequences in the second variable.

Theorem 4.5.1. *i) Let N be an A-module. When N bis fixed, the assignment $\{M \mapsto \mathrm{Ext}^n_A(M, N)\}_{n\in\mathbb{Z}^+}$is a sequence of contravariant functors from A modules to A-modules and the assignment $\{M \mapsto \mathrm{Ext}^n_A(N, M)\}_{n\in\mathbf{Z}^+}$is a sequence of functors from A-modules to A-modules.*

(ii) Let $0 \to M' \to M \to M'' \to 0$be an exact sequence of A-modules. Then the following sequences

$$0 \to \mathrm{Ext}^0_A(M'', N) \to \ldots \to \mathrm{Ext}^{n-1}_A(M'', N) \to \mathrm{Ext}^{n-1}_A(M, N) \to$$
$$\to \mathrm{Ext}^{n-1}_A(M', N) \xrightarrow{\partial^{n-1}} \mathrm{Ext}^n_A(M'', N) \to \ldots$$

and

$$0 \to \mathrm{Ext}^0_A(N, M') \to \ldots \to \mathrm{Ext}^{n-1}_A(N, M') \to \mathrm{Ext}^{n-1}_A(N, M) \to$$
$$\mathrm{Ext}^{n-1}_A(N, M'') \overset{\partial^{n-1}}{\to} \mathrm{Ext}^n_A(N, M') \to \ldots$$

are exact.

(iii) For a commutative diagram of A-modules with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \end{array}$$

the induced diagrams, for every $n \geq 1$

$$\begin{array}{ccc} \mathrm{Ext}_A^{n-1}(M', N) & \xrightarrow{\partial^{n-1}} & \mathrm{Ext}_A^n(M'', N) \\ \uparrow & & \uparrow \\ \mathrm{Ext}_A^{n-1}(K', N) & \xrightarrow{\partial^{n-1}} & \mathrm{Ext}_A^n(K'', N) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Ext}_A^{n-1}(N, M'') & \xrightarrow{\partial^{n-1}} & \mathrm{Ext}_A^n(N, M') \\ \downarrow & & \downarrow \\ \mathrm{Ext}_A^{n-1}(N, K'') & \xrightarrow{\partial^{n-1}} & \mathrm{Ext}_A^n(N, K') \end{array}$$

are commutative.

(iv) The functor $\mathrm{Ext}_A^n(M, N)$*, for every* $n \in \mathbb{Z}^+$ *is* A*-linear in both* M *and* N*.*

(v) There exists an A*-isomorphism*

$$\mathrm{Ext}_A^0(M, N) \simeq \mathrm{Hom}_A(M, N)$$

which is functorial in M *and* N*.*

(vi) For a projective A*-module* P*,* $\mathrm{Ext}_A^n(P, N) = 0$*, for every* A*-module* N *and for every* $n \geq 1$*.*

Proof. The proofs are similar to that of the functors Tor in the previous section. □

Chapter 5

Homological Dimensions

In this chapter, we will study homological dimensions using tools discussed in the previous chapter. Throughout this chapter, A is a commutative ring with 1 and any module is supposed to be a unitary module.

5.1 Projective Dimension

Definition 5.1.1 (Length of a Resolution). Consider a non-zero A-module M with the following projective resolution,

$$\ldots \to P_n \xrightarrow{d} P_{n-1} \to \ldots \to P_0 \xrightarrow{\varepsilon} M \to 0$$

We say that this resolution is of length n, if $P_n \neq 0$ and $P_i = 0$, for $i > n$.

Definition 5.1.2 (Homological Dimension). The homological dimension AKA projective resolution of a A-module $M \neq 0$, denoted by $\operatorname{hd}_A M$, is the least integer n, if it exists, such that M has a projective resolution of length n.
If no such integer exists, we set $\operatorname{hd}_A M = \infty$. If $M = 0$, we set $\operatorname{hd}_A M = -1$.

Remark. An A-module M is projective $\Longleftrightarrow$ $\operatorname{hd}_A M \leq 0$.

Definition 5.1.3 (Global Dimension). The global dimension of a ring A, denoted by $gl.\dim A$, is defined by

$$\operatorname{gl} \cdot \dim A = \sup_M \operatorname{hd}_A M \ ,$$

where the supremum is taken over all A-modules M.

Proposition 5.1.1. *Let M ba an A-module. Then the following conditions are equiv-*

alent:

(i) M is projective;

(ii) For all A-modules N and all $j \geq 1$, $\operatorname{Ext}_A^j(M, N) = 0$.

(iii) For all A-modules N, $\operatorname{Ext}_A^1(M, N) = 0$.

Proof. The Proposition can be easily proved by Theorem 4.5.1, by showing $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. □

Proposition 5.1.2. *Let M ba an A-module. For $n \in \mathbb{Z}^+$, the following conditions are equivalent:*

(i) $\operatorname{hd}_A M \leq n$.

(ii) For all A-modules N and all $j \geq n+1$, $\operatorname{Ext}_A^j(M, N) = 0$.

(iii) For all A-modules N, $\operatorname{Ext}_A^{n+1}(M, N) = 0$.

(iv) Given any exact sequence $0 \to K_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$, where each P_j is A-projective, we have that K_n is A-projective.

Proof. $(i) \Rightarrow$**(ii)**. By hypothesis, M has a projective resolution of length $\leq n$. Using such resolution to compute Ext, we obtain that the assertion of (ii) is true.

(ii)⇒(iii). It is trivial.

(iii)⇒(iv). If $n = 0$, by Proposition 5.1.1, M is projective and the assertion is obvious. Let then $n \geq 1$. Now the given exact sequence induces short exact sequences, for $0 \leq j \leq n-1$,

$$0 \to K_{j+1} \to P_j \to K_j \to 0$$

where $K_{j+1} = \operatorname{Im}(P_{j+1} \to P_j), 0 \leq j \leq n-2$ and $K_0 = M$. For any A module N, these sequences yield the following exact sequences,

$$\operatorname{Ext}_A^{n-j}(P_j, N) \to \operatorname{Ext}_A^{n-j})(K_{j+1}, N) \to \operatorname{Ext}_A^{n-j+1}(K_j, N) \to \operatorname{Ext}_A^{n-j+1}(P_j, N), \; 0 \leq j \leq n-1$$

Since P_j is projective, we have $\operatorname{Ext}_A^{n-j}(P_j, N) = 0$, and $\operatorname{Ext}_A^{n-j+1}(P_j, N) = 0$, for $0 \leq j \leq n-1$, by Proposition 5.1.1. Thus

$$\operatorname{Ext}_A^1(K_n, N) \simeq \operatorname{Ext}_A^2(K_{n-1}, N) \simeq \ldots \simeq \operatorname{Ext}_A^{n+1}(K_0, N)$$

Since $K_0 = M$, we get that $\operatorname{Ext}_A^{n+1}(K_0, N) = 0 \implies \operatorname{Ext}_A^1(K_n, N) = 0$. Since N is arbitrary, by Proposition refprop4.1, we have that K_n is projective.

(iv)⇒(i). This follows from the proof of the Proposition 4.3.1. □

Corollary 5.1.3. *Let M be a non-zero A-module. Then*

$$\operatorname{hd}_A M = \sup_n \{n \mid \exists \text{ an } A\text{-module } N \text{ with } \operatorname{Ext}_A^n(M, N) \neq 0\} \ .$$

Corollary 5.1.4. *Let M' be a direct summand of M. Then $\operatorname{hd}_A M' \leq \operatorname{hd}_A M$.*

Proof. For $n \in \mathbb{Z}^+$, and any A-module N, we have, by Theorem 4.5.1(iv), $\operatorname{Ext}_A^n(M', N)$ is a direct summand of $\operatorname{Ext}_A^n(M, N)$. Now using the the previous corollary, the result follows. □

Proposition 5.1.5. *Let P be a projective A-module and the sequence of A-modules $0 \to M' \to P \to M'' \to 0$ be exact. Then we have,*

(i) M'' being projective implies that M' is also projective.

(ii) $\operatorname{hd}_A M'' \geq 1$ implies that $\operatorname{hd}_A M'' = \operatorname{hd}_A M' + 1$ where both sides may be infinite.

Proof. (i) Let M'' be projective. Then the sequence $0 \to M' \to P \to M'' \to 0$ splits and thus M' is a direct summand of P and hence projective.

(ii) Let N be an A-module. For $n \in \mathbb{N}$, we have an exact sequence

$$0 \to \operatorname{Ext}_A^n(M', N) \to \operatorname{Ext}_A^{n+1}(M'', N) \to 0$$

Now, since P is projective, we have that $\operatorname{Ext}_A^n(P, N) = 0 = \operatorname{Ext}_A^{n+1}(P, N)$. By Corollary 5.1.3, the assertion follows. □

Lemma 5.1.6. *For an A-module M with $\operatorname{hd}_A M < \infty$, if $a \in A$ is a non-zero divisor of both A and M, then $hd_{\frac{A}{Aa}} \frac{M}{aM} < \infty$.*

Proof. The lemma is proved by induction on $\operatorname{hd}_A M$. We may obviously assume $M \neq 0$. If $\operatorname{hd}_A M = 0$, then M is A-projective and in that case, by Corollary 4.2.4, $\frac{M}{aM} = M \otimes \frac{A}{Aa}$ is $\frac{A}{Aa}$-projective. So $\operatorname{hd}_{\frac{A}{Aa}} \frac{M}{aM} \leq 0 < \infty$.

Let $\operatorname{hd}_A M > 0$ and let the following sequence be exact,

$$0 \to N \to P \to M \to 0 \quad (*)$$

where P is A-projective. Using Proposition 5.1.5, we have that $\operatorname{hd}_A N = \operatorname{hd}_A M - 1$. Now the exact sequence $(*)$ induces the exact sequence of A/Aa-modules,

$$\operatorname{Tor}_1^A\left(M, \frac{A}{Aa}\right) \to \frac{N}{aN} \to \frac{P}{aP} \to \frac{M}{aM} \to 0$$

Since a is not a zero divisor of A, we have an exact sequence

$$0 \to A \xrightarrow{\varphi} A \to A \to A/Aa \to 0$$

where φ is the homothety a_A. Now this induces the exact sequence

$$0 \to \operatorname{Tor}_1^A\left(M, \frac{A}{Aa}\right) \to M \xrightarrow{a} M$$

Since a is not a zero divisor of M, it follows that $\operatorname{Tor}_1^A\left(M, \frac{A}{Aa}\right) = 0$. Therefore the following sequence is exact,

$$0 \to \frac{N}{aN} \to \frac{P}{aP} \to \frac{M}{aM} \to 0$$

By induction hypothesis, we have that $\operatorname{hd}_{\frac{A}{Aa}} \frac{N}{aN} < \infty$. Now $\frac{P}{aP}$ being $\frac{A}{aA}$-projective, by Proposition 5.1.5, it follows that $\operatorname{hd}_{\frac{A}{Aa}} \frac{M}{aM} < \infty$. $\square$

5.2 Injective Dimension

Definition 5.2.1 (Injective Module). An A-module Q is said to be injective, if, given any diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i"] \arrow[dr, "f"'] & M \arrow[d] \\
 & & Q
\end{tikzcd}$$

of A-homomorphisms with exact row, there exists an A-homomorphism $\bar{f} : M \to Q$ such that $\bar{f} \circ i = f$.

Definition 5.2.2 (Divisible Integral Domain). A module M over an integral domain A is divisible, if for any $x \in M$ and $0 \neq a \in A$, there exists $y \in M$ such that $x = ay$.

Note the following results,

1. An A-module N is injective $\iff$ any A homomorphism from any ideal of A into N can be extended to an A homomorphism of A into N. (It can be proved using Zorn's Lemma.)
2. Any injective module over an integral domain, is divisible.
3. Any divisible module over a principal ideal domain, is injective.
4. Any module is isomorphic to a submodule of an injective module.

Remark. The $\mathbb{Z}$-modules $\mathbb{Q}$ and $\frac{\mathbb{Q}}{\mathbb{Z}}$ are $\mathbb{Z}$ injective.

Proposition 5.2.1. *Let N be an A-module. Then the following conditions are equivalent:*

(i) N is injective.

(ii) For all A-modules M,$\mathrm{Ext}^1_A(M, N) = 0$.

(iii) For all integers $i \geq 1$ and for all A-modules M, $\mathrm{Ext}^i_A(M, N) = 0$.

(iv) For all finitely generated A-modules M, $\mathrm{Ext}^1_A(M, N) = 0$.

(v) For all ideals $\mathfrak{a}$ of A, $\mathrm{Ext}^1_A(\frac{A}{\mathfrak{a}}, N) = 0$.

Proof. The Proposition can be easily proved by Properties of Ext functor and result 1,mentioned above, by showing $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$. $\square$

Definition 5.2.3 (Injective Dimension). The injective dimension of a non-zero A-module N, denoted as $\operatorname{inj\,dim}_A N$, is defined by

$$\operatorname{inj\,dim}_A N = \sup\{n \mid \exists \text{ an A-module M with } \mathrm{Ext}^n_A(M, N) \neq 0\}$$

if it exists, otherwise it is set to be ∞. If $N = 0$, we set $\operatorname{inj\,dim}_A N = -1$.

Remark. By Proposition 5.2.1, an A-module N is injective $\iff$ $\operatorname{inj\,dim}_A N \leq 0$.

Using the result 4, mentioned above and Proposition 5.2.1, it can be shown that for any A-module N,

$$\operatorname{inj\,dim}_A N = \sup\{n \mid \exists \text{ a finitely generated } A\text{-module } M \text{ such that } \mathrm{Ext}^n_A(M, N) \neq 0\}.$$

5.3 Global Dimension

We have already seen the definition of global dimension of a ring, in the first section of this chapter, here we shall see its relation with injective and homological dimension of modules over the ring. One important result of this section is that for a Noetherian local ring A, the global dimension of the ring is equal to the homological dimension of its residue field as an A-module.

Proposition 5.3.1. *For any ring A,*

$$gl\ \dim A = \sup_N \operatorname{inj}\dim_A N.$$

Proof.

$$\begin{aligned}
\mathrm{gl}\cdot\dim A &= \sup_M \mathrm{hd}_A M \\
&= \sup_M \sup \{n \mid \exists \,\text{an}\, A\text{-module } N \text{ such that } \operatorname{Ext}^n_A(M,N) \neq 0\} \\
&= \sup \{n \mid \exists A\text{-modules } M \text{ and } N \text{ such that } \operatorname{Ext}^n_A(M,N) \neq 0\} \\
&= \sup_N \sup \{n \mid \exists \,\text{an}\, A\text{-module } M \text{ such that } \operatorname{Ext}^n_A(M,N) \neq 0\} \\
&= \sup_N \operatorname{inj}\dim_A N.
\end{aligned}$$

□

Theorem 5.3.2. *For any ring A*

$$gl. \dim A = \sup_M \{\mathrm{hd}_A M \mid M \textit{ finitely generated }\}.$$

Proof.

$$\begin{aligned}
\mathrm{gl}\cdot\dim A =& \sup_N \operatorname{inj}\dim_A N \text{ by Proposition 5.3.1} \\
&= \sup_N \sup\{n \mid \exists \text{ a finitely generated } A\text{-module } M \\
&= \sup_{\substack{M \text{ finitely} \\ generated}} \mathrm{wup}\,\{n \mid \exists \operatorname{Ext}^n_A(M,N) \neq 0\}, \\
&= \sup_{\substack{M \text{ finitely} \\ generated}} \mathrm{hd}_A M \text{ by Corollary 5.1.3}
\end{aligned}$$

□

For the rest of this chapter, A is assumed to be a local ring and modules are to be considered as finitely generated. $\mathfrak{m}$ and $k = \frac{A}{\mathfrak{m}}$ denotes the maximal ideal and residue field respectively. Now we discuss a very useful result, which we will be using quite often in the remaining portion of this thesis.

Lemma 5.3.3. *Let M be an A-module. A set of elements $x_1, \ldots, x_n$ of M is a minimal set of generators of $M \iff$ their canonical images $\bar{x}_1, \ldots, \bar{x}_n$ in $\frac{M}{\mathfrak{m}M}$ form a basis of the k-vector space $\frac{M}{\mathfrak{m}M}$.*
In particular, the cardinality of any minimal set of generators of $M =$ the rank of the k-vector space $\frac{M}{\mathfrak{m}M}$.

Proof. It is sufficient to prove that $x_1, \ldots, x_n \in M$ **generate** M **over** $A \iff \bar{x}_1, \ldots, \bar{x}_n$ **generate** $\frac{M}{\mathfrak{m}M}$ **over** k.
$(\Rightarrow)$
If $x_1, \ldots, x_n \in M$ generate M, then obviously $\bar{x}_1, \ldots, \bar{x}_n$ generate $\frac{M}{\mathfrak{m}M}$ over k.
$(\Leftarrow)$
Now suppose $x_1, x_2, \ldots, x_n$ are such that $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ generate $\frac{M}{\mathfrak{m}M}$ over k. Let M' be the submodule of M generated by $x_1, \ldots, x_n$. If $M'' = \frac{M}{M'}$, we have an exact sequence

$$0 \to M' \xrightarrow{i} M \to M'' \to 0$$

which induces an exact sequence

$$\frac{M'}{\mathfrak{m}M'} \xrightarrow{\bar{i}} \frac{M}{\mathfrak{m}M} \to \frac{M''}{\mathfrak{m}M''} \to 0$$

Since $x_1, \ldots, x_n$ are in M' and since $\bar{x}_1, \ldots, \bar{x}_n$ generate $\frac{M}{\mathfrak{m}M}, \bar{i}$ is a surjective homomorphism implying that $\frac{M''}{\mathfrak{m}M''} = 0$. Since M'' is finitely generated, by Nakayama's lemma, we have that $M'' = 0$. Thus $M' = M$. This completes the proof. □

In Section 4.2, we have seen that a projective module not necessarily be free. Now we will see a criterion which enables a projective module to be free.

Proposition 5.3.4. *Let M be a finitely generated module over a local ring A. Then the following conditions are equivalent:*

(i) M is free.

(ii) M is projective.

Moreover, if A is Noetherian, then (i) and (ii) are also equivalent to

(iii) For all A-modules N and all $j \geq 1$, $\mathrm{Tor}_j^A(M, N) = 0$.

(iv) $\mathrm{Tor}_{n+1}^A(M, k) = 0$.

Proof. **(i)⇒ (ii)**. We have seen this in Section 4.2.

(ii) ⇒(i). Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of M and let $\varphi : F \to M$ be a surjective A-homomorphism, where F is an A-free module with a basis of n elements. If $K = \ker \varphi$, we have the exact sequence

$$0 \to K \to F \xrightarrow{\varphi} M \to 0$$

Now M being projective, this sequence splits, so that we have an exact sequence,

$$0 \to \frac{K}{\mathfrak{m}K} \to \frac{F}{\mathfrak{m}F} \xrightarrow{\bar{\varphi}} \frac{M}{\mathfrak{m}M} \to 0$$

Using Lemma 5.3.3, we have that $\bar{\varphi}$ is an isomorphism, and $\frac{K}{\mathfrak{m}K} = 0$. Since the first sequence (in this proof) splits, we have that K is finitely generated. Now Nakayama's lemma implies that $K = 0$, thus φ is an isomorphism and M is free.

(ii) ⇒(iii).M being a A-projective, it has a projective resolution

$$0 \to M \xrightarrow{1_M} M \to 0$$

Using this resolution to compute Tor, assertion (iii) is proved.

(iii) ⇒(iv).This is trivial.

(iv) ⇒(i). The proof is similar to that of *(ii)* ⇒ *(i)* part. It should be noted that in this case, the exactness of the second sequence (in *(ii)* ⇒ *(i)* part) follows from the hypothesis $\mathrm{Tor}_1^A(M, k) = 0$, and A being Noetherian, K is finitely generated. □

Proposition 5.3.5. *Let M be a finitely generated module over a Noetherian local ring and $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $\mathrm{hd}_A M \leq n$.

(ii) For all A-modules N and all $j \geq n+1$, $\mathrm{Tor}_j^A(M, N) = 0$.

(iii) $\mathrm{Tor}_1^A(M, k) = 0$.

Proof. It can be easily shown that *(i)* ⇒ *(ii)* ⇒ *(iii)* ⇒ *(i)* using Proposition 5.3.4 and Proposition 5.1.5. □

Using Proposition 4.4.7 and Theorem 5.3.2, we obtain the following result.

Proposition 5.3.6. *For a Noetherian local ring A and any $n \in \mathbb{Z}^+$, the following statements are equivalent:*

(i) $\mathrm{gl} \cdot \dim A \leq n$.

(ii) For all A-modules M, N and $j \geq n+1$, $\mathrm{Tor}_j^A(M, N) = 0$.

(iii) $\mathrm{Tor}_{n+1}^A(k, k) = 0$.

Proof. **(i) ⇒ (ii)**.This directly follows from Proposition 5.3.5.

(ii) ⇒ (iii). This is trivial.

(iii) ⇒ (i). We assume that (iii) holds. Using Proposition 5.3.5, we have that, for all A-modules M, $\mathrm{Tor}_{n+1}^A(k, M) = 0$. Now, by Proposition 4.4.7, we have that

$$\mathrm{Tor}_{n+1}^A(M, k) \simeq \mathrm{Tor}_{n+1}^A(k, M)$$

Using Proposition 5.3.5 again, we get that if M is finitely generated, then $\mathrm{hd}_A M \leq n$. Now Theorem 5.3.2 implies,

$$\mathrm{gl} \cdot \dim A \leq n$$

Thus the proof is complete. □

From this Proposition we obtain a very useful result stated below.

Corollary 5.3.7. *For a Noetherian local ring A, we have* $\mathrm{gl.}\dim_A = \mathrm{hd}_A k$.

Chapter 6

Regular Local Rings

In this chapter, we are going to study regular local rings applying the results obtained in the previous chapters using homological tools and dimension theory. In the first part, regular local rings will be characterised as Noetherian local rings of finite global dimension. The second part will centre around the UFD propery of regular local rings. Throughout this chapter, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $k = A/\mathfrak{m}$ is the residue field.

6.1 Regular Local Rings

Let the Krull dimension of A, dim$A$ be r. By Samuel's Theorem 3.1.3, we know that the number of generating elements of $\mathfrak{m} \geq r$.

Definition 6.1.1 (Regular). A Noetherian local ring is called *regular* if its maximal ideal has exactly r generating elements.

Before we proceed to see an interesting theorem, note that, if the k-algebra $G(A) = \bigoplus_{j\geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}$ is isomorphic as a graded k-algebra to a polynomial algebra $k[X_1, \ldots, X_r]$, then dim$A = r$.

> Let $\varphi : k[X_1, \ldots, X_s] \to G(A)$ be an isomorphism of graded k-algebras and let $x_1, \ldots, x_r \in \mathfrak{m}$ be such that $\varphi(X_j) = x_j$ modulo $\mathfrak{m}^2$, for $1 \leq j \leq r$. By Lemma 5.3.3, the elements $x_1, \ldots, x_s$ generate $\mathfrak{m}$ and we have, by Corollary 3.1.8, deg $P_{\mathfrak{m}}(A, n) = r$, which is same as the dim(A), by Dimension Theorem 3.2.1.

Theorem 6.1.1. *Let A be a Noetherian local ring and $\mathfrak{m}$ be its maximal ideal with the residue field be denoted as $k = A/\mathfrak{m}$. Then the following conditions are equivalent:*

(i) A is regular.

(ii) *The rank of the k-vector space* $\mathfrak{m}/\mathfrak{m}^2 = \dim A$.

(iii) $G(A) = \bigoplus_{j\geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}$*, the graded module associated to the* $\mathfrak{m}$*-adic filtrations on* A*, is isomorphic as a graded* k*-algebra to the polynomial algebra in* r *variables over* k*,* $k[X_1, \ldots, X_r]$ *with* $r = \dim A$.

Proof. **(i) $\Rightarrow$ (iii)**. Let $r = \dim A$ and $\{x_1, \ldots, x_r\}$ be a set of generators of $\mathfrak{m}$. Let $\varphi : k[X_1, \ldots, X_r] \to G(A)$ be the graded k-algebra homomorphism defined by

$$\varphi(X_j) = \bar{x_j} = x_j + \mathfrak{m}^2, 1 \leq j \leq r$$

Since, using Dimension Theorem, $\deg P_{\mathfrak{m}}(A, n) = \dim A = r$, using Corollary 3.1.8, we hvae that φ is an isomorphism.

(iii) $\Rightarrow$ (ii). Because of the isomorphism $\varphi : k[X_1, \ldots, X_s] \to G(A)$, we have $\mathfrak{m}/\mathfrak{m}^2 = G(A)_1 \simeq k[X_1, \ldots, X_r]_1$, which implies, $\operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 = r$.

(ii) $\Rightarrow$ (i). $\dim(A) = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 =$ number of generators of $\mathfrak{m}$ (using Lemma 5.3.3). Hence A is regular.

This completes the proof. □

Corollary 6.1.2. *A regular local ring is an integral domain.*

Proof. Since k is a field, $G(A) \simeq k[X_1, \ldots, X_r]$ is an integral domain. Using Lemma 2.6.2, we conclude A is an integral domain. □

Definition 6.1.2 (Regular System of Parameters). Let us consider a regular local ring A with $\dim A = r$. Any generating set of A of cardinality r is said to be a *regular system of parameters of* A.

Proposition 6.1.3. *Let* A *be a regular local ring with* $\dim A = r$ *and let* $a_1, \ldots, a_j$ *be any* j *elements of* $\mathfrak{m}$, $0 \leq j \leq r$*. Then the following statements are equivalent:*

(i) $\{a_1, \ldots, a_j\}$ *is a part of a regular system of parameters of* A.

(ii) *The images* $\bar{a}_1, \ldots, \bar{a}_j$ *of* $a_1, \ldots, a_j$ *under the canonical map* $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ *are linearly independent over.*

(iii) $A/(a_1, \ldots, a_j)$ *is a regular local ring of dimension* $r - j$.

Proof. **(i) $\Leftrightarrow$ (ii)**. It follows from Lemma 5.3.3.

(i) $\Rightarrow$ (iii). Let consider the Noetherian local ring,

$$\bar{A} = \frac{A}{(a_1, \ldots, a_j)}$$

with the maximal ideal

$$\overline{\mathfrak{m}} = \frac{\mathfrak{m}}{(a_1, \ldots, a_j)}$$

Now let $a_1, \ldots, a_j, a_{j+1}, \ldots, a_r$ be a regular system of parameters of A, obviously the canonical images of $a_{j+1}, \ldots, a_r$ in $\overline{\mathfrak{m}}$ generate $\overline{\mathfrak{m}}$. So by Theorem 3.1.3, we get

$$\dim \bar{A} \leq r - j$$

Let $s = \dim \bar{A} = s(\bar{A})$, Chavelly dimension of $\bar{A}$ and let $b_1, \ldots, b_s \in \mathfrak{m}$ be such that if $\bar{b}_1, \ldots, \bar{b}_s$ are their canonical images in $\overline{\mathfrak{m}}$, then $\bar{A}/\left(\bar{b}_1, \ldots, \bar{b}_s\right)$ is of finite length. Since

$$\frac{A}{(a_1, \ldots, a_j, b_1, \ldots, b_s)} \simeq \frac{\bar{A}}{\left(\bar{b}_1, \ldots, \bar{b}_s\right)}$$

we have $s + j \geq \dim A = r$. Thus $\dim \bar{A} = r - j$. Since $\overline{\mathfrak{m}}$ is generated by $r - j$ elements, $\bar{A}$ is regular, proving the assertion (iii).

(iii) $\Rightarrow$ (i). Let $a_{j+1}, \ldots, a_r \in \mathfrak{m}$ be such that $\bar{a_{j+1}}, \ldots, \bar{a_r}$, the canonical images in $\overline{\mathfrak{m}} = \mathfrak{m}/\left(a_1, \ldots, a_j\right)$ generate $\overline{\mathfrak{m}}$. Then $a_1, \ldots, a_j, a_{j+1}, \ldots, a_r$ generate $\mathfrak{m}$, proving (i). □

Corollary 6.1.4. *Let $\{a_1, \ldots, a_j\}$ be a part of a regular system of parameters of a regular local ring A. Then $\mathfrak{p} = (a_1, \ldots, a_j)$ is a prime ideal of A of height j.*

Proof. **To prove : $\mathfrak{p}$ is a prime ideal.** By Proposition 6.1.3, we have that $A/\mathfrak{p}$ is a regular local ring and now by Corollary 6.1.2, we know that it is an integral domain implying $\mathfrak{p}$ is a prime ideal.
To prove : ht $\mathfrak{p} = j$. We do this by induction on j. If $j = 0$, then $\mathfrak{p} = 0 \implies$ ht $\mathfrak{p} = 0$.
Let $j > 0$. By induction hypothesis, we have that $(a_1, \ldots, a_{j-1})$ is a prime ideal of height $j - 1$. $\{a_1, \ldots, a_j\}$, being part of a regular system of parameters of A, is a minimal system of generators of $\mathfrak{p}$, so $(a_1, \ldots, a_{j-1}) \subsetneq \mathfrak{p}$ and hence ht $\mathfrak{p} \geq j$. Now, by Corollary 5.1.4, ht $\mathfrak{p} \leq j$. Thus ht $\mathfrak{p} = j$. □

Definition 6.1.3 (M**-sequence**)**.** For a non-zero A-module, M, a sequence $a_1, \dots, a_r$ of elements of $\mathfrak{m}$ is called an *M-sequence* if a_i is not a zero-divisor of $\frac{M}{(a_1,\dots,a_{i-1})M}$ for $1 \le i \le r$.
That is, for $i = 1$, the condition means that a_1 is not a zero divisor of M and similar for higher i values.

Proposition 6.1.5. *For an A-module $M \neq 0$ and an M-sequence $a_1, \dots, a_r$, we have* $\dim M \geq r$*.*

Proof. We shall prove this theorem by induction on r. For $r = 0$, $0 \geq \dim M$. So we assume $r > 0$.
$M'' = M/a_1M$. Since a_1 is not a zero-divisor of M, we have that homothety by a_1, $\varphi : M \to M$ defined by $\varphi(m) = a_1m$ is injective. Hence we get an exact sequence,

$$0 \to M \xrightarrow{\varphi} M \to M'' \to 0$$

Now applying Proposition 3.1.5 to this exact sequence, we obtain

$$P_{\mathfrak{m}}(M'', n) = R(n)$$

where $R(n)$ is a polynomial function with $\deg R(n) < \deg P_{\mathfrak{m}}(M, n)$. Hence using Dimension Theorem,

$$\dim M'' = \deg P_{\mathfrak{m}}(M'', n) < \deg P_{\mathfrak{m}}(M, n) = \dim M$$

Now observe that,

- a_2 is not a zero divisor of $M'' = M/a_1M$.
- a_3 is not a zero divisor of $M''/a_2M'' \cong M/(a_1, a_2)M$, using the Second Isomorphism Theorem for modules, $\frac{I+J}{I} \cong \frac{J}{I\cap J}$, where I, J are submodules of M.
- Continuing this process, we can see that $a_2, \dots, a_r$ is an M''-sequence.

Now we have by induction hypothesis,

$$r - 1 \leq \dim M'' \leq \dim M - 1$$

which implies $r \leq \dim M$. □

The following Corollary of the Proposition 6.1.5 is going to provide us with a criterion when a Noetherian local ring is regular. We shall also be using this corollary to prove the main of the next section.

Corollary 6.1.6. *A Noetherian local ring A is regular $\Longleftrightarrow$ its maximal ideal is generated by an A-sequence.*

Proof. $(\Rightarrow)$
Let A be a regular Noetherian local ring with $\dim A = r$ and let $\{a_1, \ldots, a_r\}$ be a regular system of parameters of A.
Now by Corollary 6.1.2, A is an integral domain. Hence a_1 is not a zero-divisor in A, by (i)$\Leftrightarrow$(iii) relation of Proposition 6.1.3 $\frac{A}{(a_1)}$ is a regular local ring of $\dim\frac{A}{(a_1)} = r-1$. Again by Corollary 6.1.2, $\frac{A}{(a_1)}$ being an integral domain, a_2 is not a zero-divisor in $\frac{A}{(a_1)}$. Continuing this process, we obtain $a_1, \ldots, a_r$ is an $A-$sequence.
$(\Leftarrow)$
Suppose $a_1, \ldots, a_r$ be an $A-$sequence generating $\mathfrak{m}$. By Proposition 6.1.5, we have $r \leq \dim A$. On the other hand, by Samuel's Theorem 3.1.3, we have $\dim A \leq r$, which implies, $r = \dim A$, completing the proof. □

6.2 Homological Characterisation of Regular Local Rings

In this section, we are going to prove, a Noetherian local ring A is regular $\Leftrightarrow$ gl.dim$A < \infty$. Before going to the proof, we shall see some Lemmas, which will be required subsequently.

Lemma 6.2.1. *For a Noetherian local ring A with maximal ideal $\mathfrak{m}$, if $a \in \mathfrak{m} - \mathfrak{m}^2$ is a non-zero element, then the exact sequence*

$$0 \to \frac{Aa}{\mathfrak{m}a} \to \frac{\mathfrak{m}}{\mathfrak{m}a} \to \frac{\mathfrak{m}}{Aa} \to 0$$

of A/Aa-modules splits.

Proof. Let $r = \mathrm{rank}_k \frac{\mathfrak{m}}{\mathfrak{m}^2}$. Since $a \notin \mathfrak{m}^2$, $\exists\, a_1, \ldots, a_{r-1} \in \mathfrak{m}$ with canonical images $\bar{a_1}, \ldots, \bar{a_{r-1}} \in \frac{\mathfrak{m}}{\mathfrak{m}^2} \ni \{\bar{a}, \bar{a_1}, \ldots, \bar{a_{r-1}}\}$ form a basis of k-vector space $\frac{\mathfrak{m}}{\mathfrak{m}^2}$, where $k = \frac{A}{\mathfrak{m}}$. So by Lemma 5.3.3, $\{a, a_1, \ldots, a_{r-1}\}$ is a minimal set of generators of $\mathfrak{m}$. Let $\mathfrak{a} = (a_1, \ldots, a_{r-1})$. Let $b \in A$ be such that $ba \in$ a. Since $\{a, a_1, \ldots, a_{r-1}\}$ is a minimal set of generators $\mathfrak{m}$, b cannot be a unit and thus $b \in \mathfrak{m}$, which implies $\mathfrak{a} \cap Aa \subset \mathfrak{a} \cap \mathfrak{m}a$. Since, obviously $\mathfrak{a} \cap Aa \supset \mathfrak{a} \cap \mathfrak{m}a$, we have that $\mathfrak{a} \cap Aa = \mathfrak{a} \cap \mathfrak{m}a$. Now using the Second Isomorphism Theorem for rings, we see

$$\frac{\mathfrak{a} + \mathfrak{m}a}{\mathfrak{m}a} \simeq \frac{\mathfrak{a}}{\mathfrak{a} \cap \mathfrak{m}a} = \frac{\mathfrak{a}}{\mathfrak{a} \cap Aa} \simeq \frac{\mathfrak{a} + Aa}{Aa} = \frac{\mathfrak{m}}{Aa}$$

which implies that the canonical homomorphism $\frac{\mathfrak{m}}{\mathfrak{m}a} \to \frac{\mathfrak{m}}{Aa}$ maps $\frac{\mathfrak{a}+\mathfrak{m}a}{\mathfrak{m}a}$ isomorphically onto $\frac{\mathfrak{m}}{Aa}$. Thus it is proved that the exact sequence splits. $\square$

Corollary 6.2.2. *For a Noetherian local ring A with* $\operatorname{gl.dim} A < \infty$*, if $a \in \mathfrak{m} - \mathfrak{m}^2$ is not a zero divisor of A, then*

$$\operatorname{gl.dim} A/Aa < \infty.$$

Proof. We have, by the Third Isomorphism Theorem for rings, an A/Aa-isomorphism

$$\frac{\frac{A}{Aa}}{\frac{\mathfrak{m}}{Aa}} \simeq \frac{A}{\mathfrak{m}} = k$$

which induces an exact sequence of A/Aa-modules,

$$0 \to \frac{\mathfrak{m}}{Aa} \to \frac{A}{Aa} \to k \to 0$$

Now, by Corollary 5.3.7, $gl.\dim \frac{A}{Aa} = \operatorname{hd}_{\frac{A}{Aa}} k$. In view of Proposition 5.1.5, it suffices to show that $\operatorname{hd}_{\frac{A}{Aa}} \frac{\mathfrak{m}}{Aa} < \infty$. By hypothesis, we have

$$gl.\dim A < \infty \implies \operatorname{hd}_A \mathfrak{m} < \infty$$

Now we have, by Lemma 5.1.6, $\operatorname{hd}_{\frac{A}{Aa}} \frac{\mathfrak{m}}{\mathfrak{m}a} < \infty$ and by Lemma 6.2.1, $\frac{\mathfrak{m}}{Aa}$ is a direct summand of $\frac{\mathfrak{m}}{\mathfrak{m}a}$. Using Corollary 5.1.4, we obatain that $\operatorname{hd}_{A/Aa} \mathfrak{m}/Aa < \infty$ and we are done! $\square$

Lemma 6.2.3. *Let M be a non-zero A-module and let $a \in \mathfrak{m}$ be not a zero-divisor of M. Then $\operatorname{hd}_A M/aM = \operatorname{hd}_A M + 1$, where both sides may be infinite.*

Proof. Since a is not a zero-divisor, we have the exact sequence,

$$0 \to M \xrightarrow{a_M} M \longrightarrow M/aM \to 0$$

whch induces the following exact sequence,

$$\operatorname{Tor}_{n+1}^A(M,k) \to \operatorname{Tor}_{n+1}^A(M/aM,k) \to \operatorname{Tor}_n^A(M,k) \xrightarrow{\operatorname{Tor}_n^A(a_M,k)} \operatorname{Tor}_n^A(M,k)$$

for every $n \in \mathbb{N}$. Now, since a being in $\mathfrak{m}, a_k$ is zero, we get

$$\operatorname{Tor}_n^A(a_M,k) = a \operatorname{Tor}_n^A(1_M,k) = \operatorname{Tor}_n^A(1_M,a_k) = 0$$

Thus the following exact sequence,

$$\mathrm{Tor}_{n+1}^A(M,k) \to \mathrm{Tor}_{n+1}^A(M/aM,k) \to \mathrm{Tor}_n^A(M,k) \to 0$$

Let $\mathrm{hd}_A M = s$, then by Proposition 5.3.5, $\mathrm{Tor}_{s+1}^A(M,k) = 0$, in which case,

$$\mathrm{Tor}_{n+1}^A(M/aM,k) \cong \mathrm{Tor}_n^A(M,k)$$

Hence, $\mathrm{hd}_A\, M/aM = \mathrm{hd}_A\, M + 1$. $\square$

Lemma 6.2.4. *Foe a Noetherian local ring A with the maximal ideal, $\mathfrak{m} \neq \mathfrak{m}^2$ and each element of $\mathfrak{m} - \mathfrak{m}^2$, a zero-divisor, any A-module of finite homological dimension is free.*

Proof. According to Proposition 2.4.3, the set of zero-divisors of $A = \mathrm{Ass}(A)\mathfrak{p}$. Hence,

$$\mathfrak{m} - \mathfrak{m}^2 \subset \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p} \implies \mathfrak{m} \subset \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p} \cup \mathfrak{m}^2$$

Now, since $\mathfrak{m} \neq \mathfrak{m}^2$, we have, by Lemma 2.2.3 and Prime Avoidance Lemma, $\mathfrak{m} \in \mathrm{Ass}(A)$. We know that,

$$\mathfrak{m} \in \mathrm{Ass}(A) \iff k = \frac{A}{\mathfrak{m}} \hookrightarrow A \text{ is an } A - \text{monomorphism}$$

Assume M to be a A-module with finite homological dimension, i.e. $\mathrm{hd}_A\, M = n < \infty$. If $n = -1$, then $M = 0$ and there is nothing to prove. Let $n \geq 0$. We have the exact sequence,

$$0 \to k \hookrightarrow A \to \frac{A}{k} \to 0$$

which induces the following exact sequence,

$$\mathrm{Tor}_{n+1}^A\left(M, \frac{A}{k}\right) \to \mathrm{Tor}_n^A(M,k) \to \mathrm{Tor}_n^A(M,A)$$

By Proposition 5.3.5, we have $\mathrm{Tor}_{n+1}^A(M, \frac{A}{k}) = 0$ and $\mathrm{Tor}_n^A(M,k) \neq 0$, implying $\mathrm{Tor}_n^A(M,A) \neq 0$. Now, since A is free as A-module, $\mathrm{Tor}_n^A(M,A) = 0\, \forall\, n \geq 1$, which implies that, $n = 0$. So

$$\mathrm{Tor}_1^A(M,k) = 0 \implies \mathrm{hd}_A M \leq 0 \iff M \text{ is projective.}$$

We have used Proposition 5.3.5, for the first implication. Now by Proposition 5.3.4, M is free. $\square$

Theorem 6.2.5 (Serre's Characterization of Regular Local Rings). *A Noetherian local ring A is regular iff gl.dim$A$ $< \infty$. Moreover, if gl.dim$A$ $< \infty$, then gl.dim$A$ $=$dim A.*

Proof. In view of the Corollary 6.1.6, it is sufficient to prove that,
The maximal ideal $\mathfrak{m}$ (of A) is generated by an A-sequence. $\Longleftrightarrow$ gl.dim$A \leq \infty \implies$ gl.dim$A =$ dim$A$.
$(\Rightarrow)$
Let $\mathfrak{m}$ be generated by an A-sequence $a_1, \ldots, a_r$. Let $I = (a_1)$ and $J = (a_2, a_3, \ldots, a_r)$. Now using the Second and Third Isomorphism Theorem, we have

$$\frac{A}{\mathfrak{m}} = \frac{A}{I + J} \cong \frac{A_1}{a_1 A_1}$$

where $A_1 = \frac{A}{J}$. By applying Lemma 6.2.3 repeatedly, we obtain

$$\begin{aligned} \text{hd}_A \frac{A}{\mathfrak{m}} &= \text{hd}_A \frac{A_1}{a_1 A_1} \\ &= \text{hd}_A \frac{A}{J} + 1 \\ &\ \ \vdots \\ &= \text{hd}_A A + r \\ &= r \end{aligned}$$

Now using Corollary 5.3.7, we have

$$\text{gl} \cdot \dim A = \text{hd}_A \frac{A}{\mathfrak{m}} = r < \infty$$

Moreover, by Proposition 6.1.5 , we have $r \leq \dim A$ and Theorem 3.1.3 implies that $\dim A \leq r$. Hence

$$gl \cdot \dim A = r = \dim A$$

$(\Leftarrow)$
Let $\text{gl} \cdot \dim A < \infty$. We wa t to prove that $\mathfrak{m}$ is generated by an A-sequence. We use induction on $r = \text{rank}_k \mathfrak{m}/\mathfrak{m}^2$ to complete the proof.
If $r = 0$, then $\mathfrak{m} = \mathfrak{m}^2$ and by Nakayama's lemma, $\mathfrak{m} = 0$ implying that $\mathfrak{m}$ is generated by the empty sequence. Now, let $r > 0$.
Claim : There exists $a \in \mathfrak{m} - \mathfrak{m}^2$ which is not a zero divisor.
Proof of the claim : Suppose every element of $\mathfrak{m} - \mathfrak{m}^2$ is a zero-divisor. By Corollary 5.3.7, $\text{hd}_A \frac{A}{\mathfrak{m}} < \infty$. Now using Lemma 6.2.4, we have that $\frac{A}{\mathfrak{m}}$ is free $\implies \mathfrak{m} = 0$, leading to a contradicting, since we assumed that $r > 0$. Hence the claim is true.

Now by Corollary 6.2.2, we have $\operatorname{gl.dim} \frac{A}{Aa} < \infty$. Let $\overline{\mathfrak{m}} = \frac{\mathfrak{m}}{Aa}$. We have $r = \operatorname{rank}_k \frac{\mathfrak{m}}{\mathfrak{m}^2}$. By Lemma 5.3.3, the cardinality of the minimal set of generators of $\mathfrak{m}$ is r. Since a is not a zero-divisor, $\bar{\mathfrak{m}}$ is generated by a minimal set of $r-1$ elements. Using Lemma 5.3.3 again, we have that $\frac{\overline{\mathfrak{m}}}{\overline{\mathfrak{m}}^2}$ is a k-vector space with $\operatorname{rank}_k \frac{\overline{\mathfrak{m}}}{\overline{\mathfrak{m}}^2} = r-1$.
By induction hypothesis, $\frac{\mathfrak{m}}{Aa}$ is generated by an $\frac{A}{Aa}$-sequence $\bar{a}_1, \ldots, \bar{a}_{r-1}$ where $a_i \in \mathfrak{m}$ and $\bar{a}_i = a_i + Aa$, for $1 \leq i \leq r-1$. This implies $a, a_1, \ldots, a_{r-1}$ is an A-sequence which generates $\mathfrak{m}$. This completes the proof. □

Corollary 6.2.6. *Let $\mathfrak{p}$ be a prime ideal of a regular local ring A. Then the localization of A at $\mathfrak{p}$, $A_\mathfrak{p}$ is a regular local ring.*

Proof. In view of the Theorem 6.2.5, it is sufficient to prove that $\operatorname{gl\cdot dim} A_\mathfrak{p} \leq \operatorname{gl\cdot dim} A$. We consider an A-free resolution of the A-module $\frac{A}{\mathfrak{p}}$ with $n \leq \operatorname{gl.dim} A$.

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to \frac{A}{\mathfrak{p}} \to 0$$

Now we use Proposition 2.1.1, which says that localization is an exact functor and property (2) of localization in Section 2.1, from which we obtain that for an A-module M, $A_\mathfrak{p} \otimes_A M \cong M_\mathfrak{p}$. By tensoring the previous resolution with $A_\mathfrak{p}$, we obtain an $A_\mathfrak{p}$-free resolution of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$,

$$0 \to F_n \otimes_A A_\mathfrak{p} \to F_{n-1} \otimes_A A_\mathfrak{p} \to \cdots \to F_0 \otimes_A A_\mathfrak{p} \to \frac{A_\mathfrak{p}}{\mathfrak{p}A_\mathfrak{p}} \to 0$$

Note that, we have a ring homomorphism $A \to A_\mathfrak{p}$. So for any free A-module, F with basis $\{e_i\}_{i\in I}$, $F \otimes_A A_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module with $\{e_i \otimes 1\}_{i\in I}$ as a basis.
Hence $\operatorname{hd}_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \leq n$. Now Corollary 5.3.7 implies that,

$$\operatorname{gl}\cdot\dim A_\mathfrak{p} = \operatorname{hd}_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \leq n \leq \operatorname{gl}\cdot\dim A$$

□

6.3 Unique Factorization in Regular Local Rings

In this section we shall discuss the unique factorization property of regular local rings. Initially we will discuss some Propositions which will be required and then proceed to the main proof.

Lemma 6.3.1. *Let S be a multiplicative subset of a Noetherian ring A and M be a* *finitely generated A-module. Then for any A-module N, the canonical map*

$$\varphi_M : \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right)$$

given by $f \mapsto S^{-1}f$

induces an $S^{-1}A$*-isomorphism*

$$\bar{\varphi}_M : S^{-1}\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{S^{-1}A}\left(S^{-1}M, S^{-1}N\right),$$

which is functorial both in M *and* N.

Proof. **Case I** : If $M = A$, and if $\operatorname{Hom}_A(A, N)$ and $\operatorname{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N)$ are identified respectively with N and $S^{-1}N$, φ_A is the canonical map i_N[1] and hence $\bar{\varphi}_A = 1_{S^{-1}N}$. The assertion is true.
Case II : Let M be any finitely generated free A-module. We have $M = \oplus_{i=1}^{n} Ae_i$ for some $n \in \mathbb{N}$ with $\operatorname{ann}(a_i) = 0$. Now we have that Hom and S^{-1} are additive functors, from the previous case it follows that $\bar{\varphi}_M$ is an isomorphism M.
Case III : Let M be any finitely generated A-module. A being Noetherian, we have an exact sequence

$$F_1 \to F_0 \to M \to 0$$

where F_0 and F_1 are finitely generated free A-modules, which induces the following two exact sequences

$$0 \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(F_0, N) \to \operatorname{Hom}_A(F_1, N)$$

and

$$S^{-1}F_1 \to S^{-1}F_0 \to S^{-1}M \to 0$$

these results in the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \to S^{-1}\operatorname{Hom}_A(M,N) & \longrightarrow & S^{-1}\operatorname{Hom}_A(F_0,N) & \longrightarrow & S^{-1}\operatorname{Hom}_A(F_1,N) \\
\Big\downarrow{\scriptstyle \bar{\varphi}_M} & & \Big\downarrow{\scriptstyle \bar{\varphi}_{F_0}} & & \Big\downarrow{\scriptstyle \bar{\varphi}_{F_1}} \\
0 \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) & \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}F_0, S^{-1}N) & \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}F_1, S^{-1}N)
\end{array}$$

Now, by Case II, $\bar{\varphi}_{F_0}$ and $\bar{\varphi}_{F_1}$ are isomorphisms, using the above diagram, we easily obtain that $\bar{\varphi}_M$ is an isomorphism. This completes the proof. □

[1] As it is defined in Section 2.1

Lemma 6.3.2. *A finitely generated module P over a Noetherian ring A is projective $\iff$ $P_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free for every $\mathfrak{p} \in \operatorname{Spec}(A)$.*

Proof. $(\Rightarrow)$
Let P be a projective A-module. From Corollary 4.2.4 and Proposition 5.3.4, it follows that $P_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free for every $\mathfrak{p} \in \operatorname{Spec}(A)$.
$(\Leftarrow)$
Let P be a finitely generated A-module such that for every $\mathfrak{p} \in \operatorname{Spec}(A)$, $P_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free. We choose a finitely generated free A-module F such that the sequence $F \to P \to 0$ is exact. Now to prove that P is projective, we need to show that the following sequence is exact.

$$\operatorname{Hom}_A(P,F) \xrightarrow{\psi} \operatorname{Hom}_A(P,P) \to 0$$

Let $C = \operatorname{coker} \psi = \frac{\operatorname{Hom}_A(P,P)}{\operatorname{Im}(\psi)}$. Now P being finitely generated, so is $\operatorname{Hom}_A(P,P)$ and hence C. Now, by Proposition 2.4.5, we have

$$\psi \text{ is surjective } \iff C = 0 \iff \operatorname{Supp}(C) = \emptyset$$

Hence to prove the exactness of the above sequence, we have to show that $C_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$.
Let $\mathfrak{p} \in \operatorname{Spec}(A)$. By Proposition 2.1.1, we have $C_{\mathfrak{p}} = \operatorname{coker} \psi_{\mathfrak{p}}$ and by Lemma 6.3.1, we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(P,F)_{\mathfrak{p}} \arrow[r, "\psi_{\mathfrak{p}}"] \arrow[d, "\bar{\varphi}_P"', "\simeq"] & \operatorname{Hom}_A(P,P)_{\mathfrak{p}} \arrow[d, "\simeq"', "\bar{\varphi}_P"] \\
\operatorname{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}},F_{\mathfrak{p}}) \arrow[r, "\theta"] & \operatorname{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}},P_{\mathfrak{p}})
\end{tikzcd}$$

where the vertical maps are isomorphisms. Now, by hypothesis, $P_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free. Hence the sequence $F_{\mathfrak{p}} \to P_{\mathfrak{p}} \to 0$ splits implying the surjectivity of θ, from which it follows that $\psi_{\mathfrak{p}}$ is surjective and thus $C_{\mathfrak{p}} = 0$. Thus the lemma is proved. □

Now we will state a corollary of Cancellation Lemma[2], which will be required to prove the main theorem.

Corollary 6.3.3. *Let $\mathfrak{a}$ be a non-zero projective ideal of a ring A such that $\mathfrak{a}$ has a finite free resolution. Then $\mathfrak{a} \cong A$.*

[2] It will not discussed here, please refer to Chapter 5 of [5] for further details.

Proof. For the proof, please refer to Section 5.1 of [5]. □

Now onwards we assume that A to be an integral domain. An element $p \in A$ is said to be prime if Ap is a prime ideal. Unique Factorisation is the property of rings analogous to the well-known Fundamental Theorem of Arithmetic.

Definition 6.3.1 (Unique Factorisation Domain). An integral domain A with 1 is called a unique factorization domain if

1. A is a factorization domain[3]
2. The factorisation into irreducible elements is unique upto order and associates, i.e., if $x \in A - 0$ is factored as $x = ux_1x_2 \dots x_n = vy_1y_2 \dots y_m$ where u, v are units and x_i, y_j are irreducible elements, then $n = m$ and each x_i is an associate of some y_j.

Which implies that an UFD is a FD where every irreducible element is a prime element, i.e. **every element can be uniquely written as $u\prod_{1\leq i\leq n} p_i$, where u is a unit in A, p_i are prime elements and $n \in \mathbb{N}$.**

Lemma 6.3.4. *Let A be a Noetherian domain. Then A is a unique factorization domain $\iff$ every prime ideal of height 1 of A is principal.*

Proof. ($\Rightarrow$)
Let A be a UFD and $\mathfrak{p}$ be a prime ideal of height 1. We want to show that $\mathfrak{p}$ is a principal ideal.
Let $a \in \mathfrak{p}$, and $a \neq 0$. Then $a = u\prod_{1\leq i\leq n} p_i$, where u is a unit in A, p_i are prime elements and $n \in \mathbb{N}$. $\mathfrak{p}$ being a prime ideal, $p_j \in \mathfrak{p}$ for some $j \in \{1, 2, \dots, n\}$. So $Ap_j \subset \mathfrak{p}$ is a non-zero prime ideal, which implies $\mathrm{ht}Ap_j \leq \mathrm{ht}_{\mathfrak{p}} = 1$. On the other hand, A being a Noetherian domain, using Principal Ideal Theorem (Corollary 3.2.4), we have $\mathrm{ht}Ap_j = 1$.

$$\therefore \mathfrak{p} = Ap_j$$

($\Leftarrow$)
Suppose, every prime ideal of height 1 is principal. Since A is Noetherian, every element can be written as $u\prod_{i\leq i\leq n} p_i$, with each p_i being irreducible[4]. Therefore we only need to prove that any irreducible element of A is prime.
Let $a \in A$ be an irreducible element and let $\mathfrak{p}$ be a minimal prime ideal with $Aa \subset \mathfrak{p}$. By Principal Ideal Theorem (Corollary 3.2.4), we have ht $\mathfrak{p} = 1$. Therefore, by

[3]FD is an integral domain with 1 such that if every non-unit element can be expressed as a product of irreducible elements.

[4]An element $a \in A$ is irreducible if it is not a unit and its only divisors are units of A.

hypothesis, $\mathfrak{p} = Ap$, for some $p \in A$.

$$Aa \subset Ap \implies p|a$$

$\mathfrak{p}$ being a prime ideal, p is prime and a being an irreducible element, p is associated to a, i.e. $p = ua$ for some unit of u of A Hence $\mathfrak{p} = Ap = Aa$ implying a is a prime. This completes the proof. □

Theorem 6.3.5. *Any regular local ring is unique factorization daomain.*

Proof. Let A be regular local ring of dimension r. We prove the theorem by **induction on** r.
If $r = 0$, then $\mathfrak{m} = 0$, i.e. $\frac{A}{\mathfrak{m}} = A$ is a field. Hence A is a UFD.
Let $r \geq 1$. Let $\mathfrak{p}$ be a prime ideal of height 1. We need to prove that $\mathfrak{p}$ is principal.Then by Lemma 6.3.4, the desired result follows.
Claim : There is a prime element in $\mathfrak{m} - \mathfrak{m}^2$.
Since $r \geq 1$, $m \neq 0$, so by Nakayama's Lemma 2.2.1, we have $\mathfrak{m} \neq \mathfrak{m}^2$. Let $a \in \mathfrak{m} - \mathfrak{m}^2$. Then $\bar{a} \in \frac{\mathfrak{m}}{\mathfrak{m}^2}$ is a basis element and hence by Lemma 5.3.3, $a \in A$ is part of regular system of parameters of A and Corollary 6.1.4 implies, (a) is a prime ideal of height 1, i.e. a is a prime element.
Case I : $a \in \mathfrak{p}$
$(a) \subset \mathfrak{p}$ and (a) is a minimal prime ideal containg a. By Principal Ideal Theorem (Corollary 3.2.4), $\mathrm{ht}(a) = 1$. Now since $\mathrm{ht}\,\mathfrak{p} = 1$, $(a) = \mathfrak{p}$. We are done!
Case II : $a \notin \mathfrak{p}$
Consider the multiplicative set $S = \{1, a, a^2, \ldots\}$ and let $B = S^{-1}A$. So $\mathfrak{p} \cap S = \emptyset$, using Proposition 2.1.2, we have $\mathfrak{p}B$ is a prime ideal of B of height 1, since $\mathfrak{p}$ is a prime ideal of height 1.
Let $\mathfrak{q}B$ be a prime ideal of B, where $\mathfrak{q}$ is a prime ideal of A. Clearly $a \notin \mathfrak{q}$, since otherwise, $\frac{a}{a} = \frac{1}{1} \in \mathfrak{q}B$ leading to a contradiction. Now $\mathfrak{q} \neq \mathfrak{m}$. Clearly, $B_{\mathfrak{q}B} = A_{\mathfrak{q}}$. Since $\mathfrak{q} \neq \mathfrak{m}$,

$$\begin{aligned}\dim B_{\mathfrak{q}B} &= \dim A_{\mathfrak{q}} \\ &= \mathrm{ht}\mathfrak{q} \\ &< \mathrm{ht}\mathfrak{m} \\ &= r\end{aligned}$$

$B_{\mathfrak{q}B}$ is a local ring of dimension less than r and by Corollary 6.2.6, $B_{\mathfrak{q}B}$ is regular. By induction hypothesis, $B_{\mathfrak{q}B}$ is a unique factorization domain.
Now, if $\mathfrak{p}B_{\mathfrak{q}B} \neq B_{\mathfrak{q}B}$, then $\mathfrak{p}B_{\mathfrak{q}B}$, being a prime ideal of $B_{\mathfrak{q}B}$ of height 1, is principal,

by Lemma 6.3.4. Therefore, by Lemma 6.3.2,

$$\mathfrak{p}B_{\mathfrak{q}B} \text{ is } B_{\mathfrak{q}B}\text{-free} \iff \mathfrak{p}B \text{ is } B\text{-projective}$$

Claim : $\mathfrak{p}$ admits of a finite free resolution as an A-module.
Since A is regular, its global dimension is finite by Theorem 6.2.5. So there is a $n \in \mathbb{N}$ such that $\mathrm{hd}_A\mathfrak{p} = n$. Now by Corollary 4.3.3, $\mathfrak{p}$ admits of a free resolution as an A-module. Let it be

$$\ldots \to F_{n+1} \to F_n \to \ldots \to F_0 \to \mathfrak{p} \to 0$$

Let $K_n = \mathrm{Im}(F_n \to F_{n-1})$. Consider the following resolution

$$0 \to K_n \to F_{n-1} \to F_{n-2} \to \ldots \to F_0 \to \mathfrak{p} \to 0$$

By Proposition 5.1.2, K_n is A-projective and by Proposition 5.3.4, K_n is A-free. Hence $\mathfrak{p}$ admits of a finite free resolution as an A-module.
Claim : $\mathfrak{p}B$ is principal as B-module.
Now by Proposition 2.1.1 we have that localization is an exact functor, implying that $\mathfrak{p}B$ has a finite free resolution as a B-module. Therefore, by Corollary 6.3.3 $\mathfrak{p}B$ is principal. Let $p \in \mathfrak{p}$ be such that $\mathfrak{p}B = Bp$.
Claim : $\mathfrak{p}$ is principal as A-module.
We claim that $\mathfrak{p} = Ap$. We can assume that $a \nmid p$. Since if there exists $m \in \mathbb{N}$ be such that $a^m|p$, but $a^{m+1} \nmid p$. Let

$$p = a^m q$$

Since $\mathfrak{p}$ is a prime ideal and $a \notin \mathfrak{p}$, $a^m \notin \mathfrak{p}$ and we have $q \in \mathfrak{p}$ implying $\mathfrak{p}B = Bq$ with $a \nmid q$. Hence the assumption is valid.
Now we prove the claim. By Proposition 2.1.2, we have $\mathfrak{p}B \cap A = \mathfrak{p}$, if we can show $Bp \cap A = Ap$, we will be done. It is clear that $Ap \subset Bp \cap A$. Let $\frac{xp}{a^n} \in Bp \cap A$ with $n \in \mathbb{N}$ and $x \in A$. Since a is a prime element $a \nmid p$, we have $a^n|x$ implying $\frac{x}{a^n} \in A$, i.e. $\frac{xp}{a^n} \in Ap$, so we get

$$\mathfrak{p}B = Bp \Rightarrow \mathfrak{p}B \cap A = Bp \cap A \Rightarrow \mathfrak{p} = Ap$$

which completes the proof[5].

□

[5] Finally!

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