Homological Characterization and Unique Factorization Property

of Regular Local Rings

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Aim : To Prove

Theorem 1 (Serre's Characterization of Regular Local Rings)

Let A be a noetherian local ring. Then A is regular iff gl.dim$A < \infty$ and moreover, if gl.dim$A < \infty$, then gl.dim$A =$dim A.

Aim : To Prove

Theorem 1 (Serre's Characterization of Regular Local Rings)

Let A be a noetherian local ring. Then A is regular iff gl.dim$A$ $< \infty$ and moreover, if gl.dim$A$ $< \infty$, then gl.dim$A$ $=$dim A.

Theorem 2

Any regular local ring is unique factorization daomain.

Topics studied

- Presented required results from **Commutative Algebra** last semester.
- Homological Algebra

    Complexes and Homology
    Projective Modules
    Projective Resolution
    The Functors Tor
    The Functors Ext
    Projective Dimension
    Injective Dimension
    Global Dimension

Cont...

- Dimension Theory
  - The Hilbert-Samuel Polynomial
  - Dimension Theorem
- Regular Local Rings
  - Homological Characterisation
  - UFD Property

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Local ring and Localisation

- A is a commutative ring with identity and M is a unitary A-module. A is called a **local ring**, if $A \neq 0$ and has a unique maximal ideal.

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- The intersection of all maximal ideals of A is called **Jacobson radical**, $\underline{r}(A)$.

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- The intersection of all maximal ideals of A is called **Jacobson radical**, $\underline{r}$(A).

Nakayama's Lemma

Let M be a finitely generated A-module. If $\underline{r}M = M$, then $M = 0$.

Samuel Theorem

Let A be noetherian local ring with maximal ideal $\mathfrak{m}$.

Definition

$\mathfrak{a}$ is called a **ideal of definitiion of** A if, $\mathfrak{m}^n \subset \mathfrak{a} \subset \mathfrak{m}$, for some integer $n \in \mathbb{N}$.

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Define, $P_{\mathfrak{a}}(M, n) = l_A(M/\mathfrak{a}^n M)$.

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Theorem (Samuel)

For a local ring A, a finitely generated A-module, M and an ideal of definition of A, $\mathfrak{a}$ generated by r elements. Then $P_{\mathfrak{a}}(M, n)$ is a polynomial function of degree less than or equal to r.

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Define, $\mathrm{d}(M) = \deg(P_{\mathfrak{a}}(M, n))$.

Dimension Theory

– **Height** of a prime ideal $\mathfrak{p}$,
$\mathrm{ht}\mathfrak{p} = \sup \{r \mid \text{there exists in } A \text{ a chain } \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}\}$.

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- **Krull Dimension of** M,

$$\dim M = \sup_{\mathfrak{p} \in \mathrm{Supp}(M)} \mathrm{coht}\,\mathfrak{p}$$

where, $\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$.

Dimension Theorem

– $\dim A = \sup\{$ lengths of chains of prime ideals in $A\}$.

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Dimension Theorem

Let M be a finitely generated module over a noetherian local ring A. Then $\dim M = d(M) = s(M)$.

Global Dimension

Homological Dimension

The least integer n, if exixts, such that there exists a projective resolution of M, of length n.
If no such n exists, $\mathrm{hd}_A(M) = \infty$.
If $M = 0$, $\mathrm{hd}_A(M) = -1$.

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Global Dimension

$$\text{gl.dim}A = sup_M(\text{hd}_A M)$$

Result

For a noetherian local ring *A*, we have

$$\text{gl.}\dim_A = \text{hd}_A\, k$$

Regular Local Ring

Definition

A noetherian local ring, A with $\dim A = r$, is said to be **regular**, if $\mathfrak{m}$ can be generated by r elements.

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A noetherian local ring, A with $\dim A = r$, is said to be **regular**, if $\mathfrak{m}$ can be generated by r elements.

Set of generators of $\mathfrak{m}$ for regular local rings is called **regular system of parameters** of A

Difinition

For a non-zero A-module, M, a sequence $a_1, \ldots, a_r$ of elements of $\mathfrak{m}$ is called an M**-sequence** if a_i is not a zero-divisor of $M/(a_1, \ldots, a_{i-1})M$ for $1 \leq i \leq r$.
That is, for $i = 1$, the condition means that a_1 is not a zero divisor of M ...

Cont...

Proposition

Let M be a non-zero A-module and $a_1, \ldots, a_r$ an M-sequence. Then $r \leq \dim M$.

Corollary*

A noetherian local ring A is regular if and only if its maximal ideal is generated by an A-sequence.

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Proof

Step 1: $(\Rightarrow)$ Let $\{a_1, \ldots, a_r\}$ generate $\mathfrak{m}$. A is I.D so is $A/(a_1)$. Cont.
Step 2: $(\Leftarrow)$ Let $\{a_1, \ldots, a_r\}$ be an A-sequence. Use previous Proposiition and Samuel's theorem.

Required Results

Corollary

A regular local ring is an integral domain.

Proposition

Let A be a regular local ring of dimension r and let $a_1, \ldots, a_j$ be any j elements of $\mathfrak{m}$, $0 \leq j \leq r$. Then the following statements are equivalent:

- $\{a_1, \ldots, a_j\}$ is a part of a regular system of parameters of A.
- $A/\left(a_1, \ldots, a_j\right)$ is a part of a regular local ring of dimension $r - j$.

Cont...

Lemma*

Let M be a non-zero A-module and let $a \in \mathfrak{m}$ be not a zero divisor of M. Then $\mathrm{hd}_A M/aM = \mathrm{hd}_A M + 1$, where both sides may be infinite.

Proof

Step 1: $0 \to M \xrightarrow{a_M} M \longrightarrow M/aM \to 0$
Step 2: Exact sequence:

$$\mathrm{Tor}_{n+1}^A(M,k) \to \mathrm{Tor}_{n+1}^A(M/aM,k) \to \mathrm{Tor}_n^A(M,k) \xrightarrow{\mathrm{Tor}_n^A(a_M,k)} \mathrm{Tor}_n^A(M,k)$$

Step 3: For every $n \in \mathbb{N}$. Now, since a being in $\mathfrak{m}$, a_k is zero, we get $\mathrm{Tor}_n^A(a_M,k) = a\,\mathrm{Tor}_n^A(1_M,k) = \mathrm{Tor}_n^A(1_M,a_k) = 0$
Step 4: For A be a noetherian local ring, M a finitely generated A-module, $\mathrm{hd}_A M \leq n \Leftrightarrow \mathrm{Tor}_{n+1}^A(M,k) = 0$.

Cont...

Lemma*

Let A be a noetherian local ring such that $\mathfrak{m} \neq \mathfrak{m}^2$ and such that every element of $\mathfrak{m} - \mathfrak{m}^2$ is a zero-divisor. Then any A-module of finite homological dimension is free.

Proof

Step 1: $\mathfrak{m} \in \mathrm{Ass}(M) \iff k = A/\mathfrak{m} \hookrightarrow A$ is an $A -$ monomorphism
Step 2: Exact sequence: $0 \to k \to A \to A/k \to 0$
Step 3:$\mathrm{Tor}_{n+1}^A(M, A/k) \to \mathrm{Tor}_n^A(M, k) \to \mathrm{Tor}_n^A(M, A)$
Step 4: We have $\mathrm{hd}_A M = n$, $\mathrm{Tor}_{n+1}^A(M, A/k) = 0$ and $\mathrm{Tor}_n^A(M, k) \neq 0$, implying $\mathrm{Tor}_n^A(M, A) \neq 0$. Now, since A is free as A-module, $\mathrm{Tor}_n^A(M, A) = 0 \, \forall \, n \geq 1$, which implies that, $n = 0$.
Step 5:$\mathrm{Tor}_1^A(M, k) = 0 \implies \mathrm{hd}_A M \leq 0 \iff M$ is projective.. Hence M is free.

Required Results

Lemma

Let $\mathfrak{a}, \mathfrak{b}_0, \mathfrak{b}_1, \ldots, \mathfrak{b}_n$ be ideals of a ring A with $\mathfrak{b}_0$ prime and $\mathfrak{a} \subset \bigcup_{0 \leq i \leq n} \mathfrak{b}_i$. Then there exists a proper subset J of $\{0, 1, 2, \ldots, n\}$ such that $\mathfrak{a} \subset \bigcup_{j \in J} \mathfrak{b}_j$.

Proposition

Let A be a local ring and M a finitely generated A module. Then the following conditions are equivalent:
(i) M is free.
(ii) M is projective.

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Serre's Characterization of Regular Local Rings

Theorem

Let A be a noetherian local ring. Then A is regular iff gl.dim$A < \infty$ and moreover, if gl.dim$A < \infty$, then gl.dim$A =$dim A.

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Theorem

Let A be a noetherian local ring. Then A is regular iff gl.dim$A < \infty$ and moreover, if gl.dim$A < \infty$, then gl.dim$A$ =dim A.

Skech of Proof

Step 1: The maximal ideal $\mathfrak{m}$ of A is generated by an A-sequence. $\iff$ gl.dim$A \leq \infty \implies$ gl.dim$A =$ dim$A$.
Step 2:($\Rightarrow$) Take A-sequence, $\mathrm{hd}_A \frac{A}{\mathfrak{m}} = r =$ gl.dim$A$. Also by Proposition and Samuel's Theorem, dim$A$ = r.
Step 3:($\Leftarrow$) Induction on $r = \mathrm{rank}_k\, \mathfrak{m}/\mathfrak{m}^2$. $r = 0$, Nakayama's Lemma.
step 4:$r > 0$. There exists $a \in \mathfrak{m} - \mathfrak{m}^2$ which is not a zero divisor.Since otherwise, $A/\mathfrak{m}$ is free, i.e. $\mathfrak{m} = 0$.
Step 5: $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is a k-vector space with $\mathrm{rank}_k\, \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = r - 1$. use induction hypothesis.

Required Results

Corollary*

A noetherian local ring A is regular if and only if its maximal ideal is generated by an A-sequence.

Lemma*

Let M be a non-zero A-module and let $a \in \mathfrak{m}$ be not a zero divisor of M. Then $\operatorname{hd}_A M/aM = \operatorname{hd}_A M + 1$, where both sides may be infinite.

Lemma*

Let A be a noetherian local ring such that $\mathfrak{m} \neq \mathfrak{m}^2$ and such that every element of $\mathfrak{m} - \mathfrak{m}^2$ is a zero-divisor. Then any A-module of finite homological dimension is free.

Corollary*

Let A be a noetherian local ring with $\operatorname{gl.dim} A < \infty$. If $a \in \mathfrak{m} - \mathfrak{m}^2$ is not a zero divisor of A, then $\operatorname{gl.dim} A/Aa < \infty$.

Corollary

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Let A be a regular local ring and let $\mathfrak{p}$ be a prime ideal of A. Then $A_\mathfrak{p}$ is a regular local ring.

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Skech of Proof

Step 1: To prove : $\mathrm{gl} \cdot \dim A_{\mathfrak{p}} \leq \mathrm{gl} \cdot \dim A$
Step 2: An A-free resolution of the A-module $A/\mathfrak{p}$ with $n \leq \mathrm{gl.} \dim A$.
$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to A/\mathfrak{p} \to 0$
Step 3: $A_{\mathfrak{p}}$, we obtain an $A_{\mathfrak{p}}$-free resolution of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$,
$0 \to F_n \otimes_A A_{\mathfrak{p}} \to F_{n-1} \otimes_A A_{\mathfrak{p}} \to \cdots \to F_0 \otimes_A A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to 0$
Step 4: $\mathrm{gl} \cdot \dim A_{\mathfrak{p}} = \mathrm{hd}_{A_{\mathfrak{p}}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \leq n \leq \mathrm{gl} \cdot \dim A$

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UFD property of Regular Local Rings

Let A be an ID.

Definition(Prime element)

$p \in A$ is said to be prime if Ap is aprime ideal.

Definition(UFD)

An ID, A is called an UFD if every element can be written as $u \prod_{1 \leq i \leq n} p_i$, where u is a unit in A, p_i are prime elements and $n \in \mathbb{N}$.

UFD property of Regular Local Rings

Lemma

Let A be a noetherian domain. Then A is a unique factorization domain $\iff$ every prime ideal of height 1 of A is principal.

UFD property of Regular Local Rings

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Let A be a noetherian domain. Then A is a unique factorization domain $\iff$ every prime ideal of height 1 of A is principal.

Skech of Proof

Step 1:($\Rightarrow$) To show $\mathfrak{p}$ with height 1 is principal. Let $a \in \mathfrak{p}$. then $\exists p | a \ni \ Ap \subset \mathfrak{p} \implies Ap = \mathfrak{p}$
Step 2: ($\Leftarrow$) Any irreducible element, a is prime. Let $\mathfrak{p}$ be a minimal prime ideal with containg Aa.
By Principal Ideal Theorem, $\mathfrak{p} = Ap$ with p prime $p|a$. Hence $\mathfrak{p} = Ap = Aa$ implying a is a prime.

UFD property of Regular Local Rings

Theorem

Any regular local ring is a UFD.

UFD property of Regular Local Rings

Theorem

Any regular local ring is a UFD.

Skech of Proof

Step 1: Induction on dim$A = r$.
Step 2: $r = 0$, A is field. $r \geq 1$, Use previous Lemma.By Nakayama's Lemma $\mathfrak{m} \neq \mathfrak{m}^2$. Show $a \in \mathfrak{m} - \mathfrak{m}^2$ is prime.
Step 3: Case I : $a \in \mathfrak{p}$.
step 4: Case II : $a \notin \mathfrak{p}$.
Step 5: Let $S = \{1, a, a^2, \dots\}$ and let $B = S^{-1}A$. Show $\mathfrak{p}B$ is B-projective.
i) $B_{qB}$is UFD for qB prime ideal, by IH.
ii) $\mathfrak{p}B_{qB}$ is principal, hence B_{qB}-free.
Step 6:$\mathfrak{p}B$ has a finite resolution.
Step 7: $\mathfrak{p}B = Bp$. Claim :$\mathfrak{p} = Ap$.

Required Results

Step II : Proposition*

Let M be an A-module. A set of elements $x_1, \ldots, x_n$ of M is a minimal set of generators of M if and only if their canonical images $\bar{x}_1, \ldots, \bar{x}_n$ in $M/\mathfrak{m}M$ form a basis of the k-vector space $M/\mathfrak{m}M$. In particular, the cardinality of any minimal set of generators of M is equal to the rank of the k-vector space $M/\mathfrak{m}M$.

Step II : Corollary

Let $\{a_1, \ldots, a_j\}$ be a part of a regular system of parameters of a regular local ring A. Then $\mathfrak{p} = (a_1, \ldots, a_j)$ is a prime ideal of A of height j.

Step 5 : Lemma*

Let A be a noetherian ring and P a finitely generated A module. Then P is projective if and only if, $P_\mathfrak{p}$ is $A_\mathfrak{p}$-free for every $\mathfrak{p} \in \mathrm{Spec}(A)$.

Required Results

Proposition

For an A-module M and $n \in \mathbb{Z}^+$, TFCE:
(i) $\mathrm{hd}_A M \leq n$.
(ii) If $0 \to K_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ is exact with P_j being A-projective for $0 \leq j \leq n-1$, then K_n is A-projective.

Corollary*

Let $\mathfrak{a}$ be a non-zero projective ideal of a ring A such that $\mathfrak{a}$ has a finite free resolution. Then $\mathfrak{a} \cong A$.

Thank You!