A → commutative ring with 1

Ring of fractions and localisation

S multiplicative subset of A. $\begin{cases} 1 \in S \subset A \\ s, s' \in S \Rightarrow ss' \in S \end{cases}$

Example: p → prime ideal of A, $S = A - p$.

Equivalence relation on $M \times S$: $(m, s) \sim (m', s')$ if $\exists\, s'' \in S \ni s''(s'm - sm') = 0$
| M → A-module, $m, m' \in M$, $s, s' \in S$.

$S^{-1}M$ → the set of equivalence class. | $\frac{m}{s}$ → class containing (m, s)

- $M = A$, Define addition: $\frac{a}{s} + \frac{a'}{s'} = (s'a + sa')/ss'$

  Multiplication: $\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$

$S^{-1}A$ → ring under $(+, \cdot)$; $\left.\begin{array}{l} \frac{0}{1} \to \text{zero element} \\ \frac{1}{1} \to \text{unit element} \end{array}\right\}$

↓

- Ring of fractions of A w.r.t. S. $\boxed{S^{-1}A = 0 \Leftrightarrow 0 \in S}$

For M an A-module, $+ : \frac{m}{s} + \frac{m'}{s'} = \frac{s'm + sm'}{ss'}$

$\cdot : \frac{a}{s} \cdot \frac{m}{s'} = \frac{am}{ss'}$

$S^{-1}A$-module $S^{-1}M$.
↳ module of fractions of M w.r.t. S.

- P → prime ideal

  $S = A - P$

  $S^{-1}A = A_P$ & $S^{-1}M = M_P$

Example: A → integral domain.
(0) → prime ideal.

$A_{(0)} = S^{-1}A$ is the quotient field of A.

M, N · A-modules.

$f \in \mathrm{Hom}_A(M, N)$

Define, $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$, by $(S^{-1}f)\left(\frac{m}{s}\right) = \frac{f(m)}{s}$.

→ well-defined

→ $\in \mathrm{Hom}_{S^{-1}(A)}(S^{-1}M, S^{-1}N)$.

Proposition 1: The assignments

$$M \mapsto S^{-1}M$$
$$f \mapsto S^{-1}f$$

define an exact functor from A-modules to $S^{-1}A$-modules.

Proof: Non-trivial part.

Given exact sequence, $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ (A-modules)

the sequence, $0 \xrightarrow{\cancel{S^{-1}f}} S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \longrightarrow 0$

is exact. ∎

$i_M : M \longrightarrow S^{-1}M$

$m \longmapsto \frac{m}{1}$

$i_A : A \longrightarrow S^{-1}A$

→ ring homomorphism.

- $S^{-1}A$-module can be regarded as A-module through i_A.
- $S^{-1}M$ is an A-module. i_M → A-homomorphism, which is functorial in M.
- $f: S^{-1}A \times M \longrightarrow S^{-1}M$, $\left(\frac{a}{s}, m\right) \longmapsto \frac{am}{s}$ → well-defined → A-bilinear

It induces an A-homomorphism

$$\varphi : S^{-1}A \otimes_A M \longrightarrow S^{-1}M$$

$$\left(\frac{a}{s} \otimes m\right) \longmapsto \frac{am}{s}$$

$$\begin{tikzcd}
S^{-1}A \times M \arrow[d, "\Phi"'] \arrow[dr, "f"] & \\
S^{-1}A \otimes_A M \arrow[r, "\varphi"'] & S^{-1}M
\end{tikzcd}$$

$$f = \Phi \circ \varphi$$

Proposition 2: φ is an $S^{-1}A$ isomorphism and is functorial in M.

- I is an ideal of A.
  $IS^{-1}A \Rightarrow$ ideal of $S^{-1}A$ generated by $i_A(I)$

  $IS^{-1}A = S^{-1}I$; regarding $S^{-1}I$ as a subset of $S^{-1}A$.

- If $I \cap S \neq \Phi$, then. $IS^{-1}A = S^{-1}A$.

  (if $s \in I \cap S$. $\frac{1}{1} = \frac{s}{1} \cdot \frac{1}{s} \in IS^{-1}A$.)

Proposition 3: The map $\varphi : \mathfrak{p} \longmapsto S^{-1}\mathfrak{p} \; (= \mathfrak{p}S^{-1}A)$ is an inclusion preserving bijection of the set of prime ideals $\mathfrak{p}$ of A with $\mathfrak{p} \cap S = \Phi$ onto the set of all prime ideals of $S^{-1}A$.

Proof ideas:

- $S^{-1}\mathfrak{p} \neq S^{-1}A$
- $S^{-1}\mathfrak{p}$ * is a prime ideal
- φ has an inverse. $\theta : \mathfrak{p} \longmapsto i_A^{-1}(\mathfrak{p})$
  ↳ prime ideal of $S^{-1}A$.

Local ring: $\begin{cases} \text{If } A \neq 0. \\ A \text{ has a unique maximal ideal.} \end{cases}$

Example: Field.

Corollary 4: $A \rightarrow$ commutative ring

$\mathfrak{p} \rightarrow$ prime ideal of A.

Then $A_{\mathfrak{p}}$ is a local ring with $\mathfrak{p}A_{\mathfrak{p}}$ as its unique maximal ideal.

- $A_{\mathfrak{p}} \rightarrow$ The localization of A at $\mathfrak{p}$.

Noetherian modules:

Prop^n 5: Let A be a ring. For an A-module M, TFAE:

i) Every submodule of M is finitely generated.

ii) M satisfies the ascending chain condition for submodules, i.e.,

Every sequence of submodules of M,

$$M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \quad \text{is finite.}$$

iii) Every non-empty set of submodules of M has a maximal element.

Noetherian Module: If an A-module M satisfies any of the three equivalent conditions of prop^n 5.

A ring is noetherian, if its is a noetherian A-module.

Prop^n 6:

Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of A-modules.

Then M is noetherian

$\Updownarrow$

M', M'' are noetherian.

Prop^n 7:

Let A $\rightarrow$ noetherian ring.

M $\rightarrow$ finitely generated A-module.

Then M is noetherian.

Prop^n 1·8:

Let S be a multiplicative subset of a noetherian ring A.

Then $S^{-1}A$ is noetherian.

In particular, the localization of a noetherian ring at a prime ideal is noetherian.

Theorem 9 (Hilbert Basis Theorem):

let A be a noetherian ring. Then the polynomial ring $A[X_1, \ldots, X_n]$ in n variables over A is noetherian.

Proof:

We'll prove it by induction on n.

$n = 1$:

To show:
$B = A[X]$ is noetherian.

Let J be an ~~ino~~ ideal of B.

We'll show J is finitely generated.

Let $I = \{0\} \cup \{$ highest degree coefficients of elements of $J \}$

• I is an ideal of A.

If $I = 0$, then $J = 0$. Nothing to prove.

Let $I \neq 0$.

Since A is noetherian, I is finitely generated,

$I = (c_1, \ldots, c_r)$.

Let $f_i \in J \ni c_i \to$ highest degree coefficient of f_i.

$N = \max\limits_{1 \leq i \leq r} (\deg f_i)$

Claim: $J = (f_1, \ldots, f_r) + J'$

where $J' = J \cap (A + AX + \cdots + AX^{N-1})$.

Let $f = a_m x^m + \cdots + a_0 \in J$.

Case - I: $m \leq N - 1$.

$f \in \cancel{\notin} J'$. Done!

Case - II: $m \geq N-1$. ($m \geq N$)

Let $a_m = \sum_{1 \leq i \leq r} d_i c_i$, $d_i \in A$.

Then $g = \deg \Big(\underbrace{f - \sum_{1 \leq i \leq r} d_i X^{m - \deg f_i} f_i}_{g_1} \Big) \leq m-1$.

By induction on m, (continuing this process, we would get,

$\deg(g_K) \leq N-1 = m-K$

$\Rightarrow g_K \in J'$

$\Rightarrow g_1 \in (f_1, \ldots, f_r) + J'$

$\Rightarrow f \in (f_1, \ldots, f_r) + J'$

$\Rightarrow J \subset (f_1, \ldots, f_r) + J'$

Clearly $(f_1, \ldots, f_r) + J' \subset J$

$\therefore J = (f_1, \ldots, f_r) + J'$.

$M = (A + \cdots + AX^{N-1})$ is a finitely generated A-module & A is noetherian.

~~By prop^n $\exists$ M is noetherian~~.

finitely generated over A,

$\Rightarrow M \cap J$ is finitely generated over A, & thus over $A[X]$.

$\Rightarrow J$ is finitely generated over $A[X]$.

For $n > 1$:

$$0 \longrightarrow A[X_1] \xhookrightarrow{j} A[X_1, \dots, X_n] \xrightarrow{\pi} \frac{A[X_1, \dots, X_n]}{A[X_1]} \longrightarrow 0$$

is exact.

Noetherian

Noetherian (by induction hypothesis)

$\therefore$ $A[X_1, \dots, X_n]$ is noetherian. $\square$

Corollary 10:

$A \to$ Noetherian ring.

$B \to$ finitely generated A-algebra.

Then B is noetherian.

Proof:

$$B \cong \frac{A[X_1, \dots, X_n]}{I}$$

Use Hilbert basis theorem. & proof similar to $n > 1$ case (above) $\square$

- **Jacobson radical $r(A)$:**

The intersection of ~~of~~ all maximal ideal of A.

- $a \in r(A)$, then $1-a$ is unit.

Lemma 11 (Nakayam):

Let M be a finitely generated A-module.

if $rM = M$, then $M = 0$

proof: Suppose $M \neq 0$.

$x_1, \ldots, x_n$ be a minimal set of generators of M.

Since $\underline{r} M = M$.

$$x_1 = \sum_{1 \le i \le n} a_i x_i \, , \quad a_i \in \underline{r} .$$

$$\Rightarrow (1 - a_1) x_1 = \sum_{2 \le i \le n} a_i x_i$$

$$\Rightarrow x_1 = \sum_{2 \le i \le n} (1 - a_1)^{-1} a_i x_i$$

$\Rightarrow x_2, \ldots, x_n$ generate M,

$\Rightarrow \Leftarrow$.

Lemma 12:

$I, J_0, \ldots, J_n \rightarrow$ ideals of A

$J_0 \rightarrow$ primary ideal.

$$I \subset \bigcup_{i=0}^{n} J_i$$

$\exists$ a proper subset K of $\{0, 1, \ldots, n\}$

$$\ni I \subset \bigcup_{k \in K} J_k .$$

Lemma 13:

$M, N \rightarrow$ non-zero finitely generated modules over a local ring.

Then $M \otimes_A N \neq 0$.

Primary decomposition

$A \to$ commutative ring with 1.

$M \to$ A-module.

Homothety by a : $a_M : M \longrightarrow M$, $x \longmapsto ax$ | $a \in A$

$\downarrow$

A-homomorphism.

• N is primary in M : $N \to$ submodule of M.

(i) $N \neq M$

(ii) For any $a \in M$, $a_{\frac{M}{N}}$ is either injective or, nilpotent.

propⁿ 14:

$N \to$ primary submodule of an A-module M.

$P = \{ a \in A \mid a_{\frac{M}{N}}$ is ~~either~~ not injective$\}$.

Then P is a prime ideal of A.

▪ P is called prime ideal belonging to N in M.

N is P-primary (in M).

▪ Primary decomposition of N in M:

$N = N_1 \cap \dots \cap N_r$.

$N_i \to$ primary submodules of M. $1 \leq i \leq r$.

Reduced decomposition :

If, (i) N cannot be expressed as the intersection of a proper subset of $\{N_1, ---, N_r\}$.

(ii) The prime ideals $P_1, ----, P_r$ belonging to $N_1, -----, N_r$ in M are distinct.

Propn 15 :

$M \rightarrow$ noetherian A-module. Then any proper submodule of M admits a reduced primary decomposition.

Lemma ~~1~~16 :

Let $N_1, ---, N_r$ be P-primary submodule of M. Then $N = N_1 \cap ---- \cap N_r$ is P-primary.

Lemma 17 :

Any irreducible submodule N of noetherian module M is primary.

Lemma 18 :

$M \rightarrow$ noetherian A-module.

Then any proper submodule of M is finite intersection of irreducible submodules.

$\mathfrak{p} \to$ prime ideal of A.

* A prime ideal $\mathfrak{p}$ ~~is~~ of A is said to be **associated to M**, if $\exists$ $x \neq 0$ in M $\ni$ $\mathfrak{p} = \operatorname{ann}(x) = \{a \in A \mid ax = 0\}$.

Ass (M) = The set of all prime ideals of A associated to M.

- $\mathfrak{p} \in \operatorname{Ass}(M) \Leftrightarrow \exists$ A-monomorphism $\frac{A}{\mathfrak{p}} \to M$.

propⁿ 19:

$A \to$ noetherian ring.

$M \to$ finitely generated A-module.

$0 = N_1 \cap \cdots \cap N_r$; reduced primary decomposition of 0 in M. with $N_i \to \mathfrak{p}_i$ primary; $1 \leq i \leq r$.

Then $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$. moreover, $\operatorname{Ass}(M)$ is finite.

In particular,

$M = 0 \Leftrightarrow \operatorname{Ass}(M) = \phi$.

proof:

Step - I: $\operatorname{Ass}(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

part - I:

Let $\mathfrak{p} \in \operatorname{Ass}(M)$.

$\Rightarrow \exists$ $x \neq 0$ in M $\ni$ $\mathfrak{p} = \operatorname{Ann}(x)$.

Since $x \neq 0$, we may assume,

$x \notin N_1 \cup N_2 \cdots \cup N_j$.

$x \in N_{j+1} \cap \cdots \cap N_r$; $1 \leq j \leq r$.

(Rearranging indices if needed)

For any $a \in P_i$,

The homothety a_{M/N_i} is nilpotent.

Since A is noetherian P_i is finitely generated.,

$\exists\ n_i \in \mathbb{N} \ni P_i^{n_i} M \subset N_i$.

Clearly. $\prod_{1 \leq i \leq j} P_i\, x \subset (N_1 \cap \cdots \cap N_j) \cap (N_{j+1} \cap \cdots N_r)$
$= 0$

$\Rightarrow \prod_{1 \leq i \leq j} P_i \subset P$, since $P = \mathrm{Ann}(x)$
↳ prime ideal.

$\Rightarrow P_k \subset P$ for <u>some</u> $1 \leq k \leq j$,

<u>part - II</u>:

$Px = 0 \Rightarrow a_{M/N_k}$ is not injective $\forall\ a \in P$

$\Rightarrow a \in P_k \quad \forall\ 1 \leq k \leq r$

$\Rightarrow P \subset P_k \quad \underline{\forall} \quad k \mid 1 \leq k \leq r$

$\therefore \boxed{P = P_k}$.

<u>Step - II</u>: $\{P_1, \ldots, P_r\} \subset \mathrm{Ass}(M)$

<u>Subpart - I</u>: <u>To show:</u>
$P_i \in \mathrm{Ass}(M)$ for $1 \leq i \leq r$

WLOG, we'll show it for $i = 1$.

Since the given primary decomposition is reduced, $\exists\ x \in N_2 \cap \cdots \cap N_r$,
$\ni x \notin N_1$.

As N_1 is P_1-primary,

$\exists\ n \in \mathbb{N} \ni \quad P_1^{n_1} x \subset N_1.$

$\& \ P_1^{n-1} x \not\subset N.$

Let $\quad y \in P_1^{n-1} x, \quad y \notin N_1.$

$\Rightarrow P_1 y \in P_1^{n} x \subset N_1.$

$\Rightarrow P_1 y \in \bigcap_{i=1}^{n} N_i = 0$

$\Rightarrow \boxed{P_1 \in Ann(y)}$

Subpart - II:

Let $\ a \in Ann(y)$, i.e., $ay = 0$

$a_{\frac{M}{N_1}}$ is not injective, & since $\bar{a}\bar{y} = \bar{0}$.

$\Rightarrow \ a \in P_1$

$\Rightarrow \boxed{ann(y) \subset P_1}$

$\therefore \ ann(y) = P_1$

$\Rightarrow \ P_1 \in Ass(M).$

$\therefore \ Ass(M) = \{P_1, \ldots, P_n\}$ (Q.E.D.)

Corollary 20:

A → noetherian ring.

N → submodule of M

M → finitely generated A-module.

$N = N_1 \cap \ldots \cap N_n$; reduced primary decomposition.

Then $Ass\left(\frac{M}{N}\right) = \{P_1, \ldots, P_r\}$.

where P_i are the prime ideals belonging to N_i in M, for $1 \le i \le r$.

In particular, the set of prime ideals corresponding to a reduced primary decomposition, ~~is~~ of N is independent of the decomposition.

Zero-divisor: $a \in A$, if $\exists\, x \in M, x \neq 0$ $\ni$ $ax = 0$.

Prop^n 21:

$A \to$ noetherian ring

$M \to$ finitely generated A-module.

Then the set of zero-divisors of M is $I = \bigcup_{P \in Ass(M)} P$.

proof:

$(\Rightarrow)$ Let $a \in P$ for some $P \in Ass(M)$.

$\exists\, x \neq 0 \ni Px = 0 \Rightarrow ax = 0$.

$(\Leftarrow)$ Let a be a zero-divisor of M.

$\exists\, x \neq 0 \ni \boxed{ax = 0}$.

Let $0 = N_1 \cap \ldots \cap N_r$, reduced primary decomposition of 0 in M.

Then $x \notin N_i$, for some N_i, $N_i \to P_i$-primary,

$P_i \in Ass(M)$ (by prop^n 19)

a_{M/N_i} is not injective, $a \in P_i$. ■

Graded and filtered modules & Artin - Rees Theorem.

- Let A be a ring.
- **Gradation** on A is a decomposition of A ~~as~~, - $A = \bigoplus_{n \geq 0} A_n$ of as a direct sum of subgroups of A_n of A; $n \in \mathbb{Z}^+$, such that.

$$A_m A_n \subset A_{m+n} \quad \forall\ m, n \in \mathbb{Z}^+.$$

- **Graded ring**: A ring with gradation.
- The non-zero elements of A_n are called homogenous elements of degree n.

$A_n \to n^{th}$ homogenous component of A.

Prop^n 1.36:

A_0 is a subring and $1 \in A_0$.

Moreover, each A_n is an A_0-module and A is an A_0-algebra.

proof:

Let $1 = e_0 + e_1 + \cdots + e_n$, $e_i \in A_i$

For any $a \in A_j$,

$$a = a \cdot 1 = ae_0 + ae_1 + \cdots + ae_n$$

with $ae_j \in A_{j+1}$.

It follows. $a = ae_0$.

Consequently, $\forall\ b \in A$

$$b = be_0$$

$\Rightarrow\ 1 = e_0 \in A_0$.

- Now, since $A_0 A_0 \subset A_0$, we have closure of multiplication.

$\Rightarrow\ A_0$ is a subring.

Since $A_0 A_n \subset A_n\ \forall\ n \in \mathbb{Z}^+$, other result follows. ∎

- Let A be as above. $K \to A_0$ be a ring homomorphism.

Then A is a K-algebra.

$A \to$ graded K-algebra.

- Let $A \to$ graded ring. $A = \bigoplus_{n \geq 0} A_n$.

A-gradation on M is a decomposition,

$$M = \bigoplus_{n \geq 0} M_n$$

$M_n \to$ subgroups of M. $\ni$

$$A_n M_m \subset M_{m+n} \quad \forall\ m, n \in \mathbb{Z}^+.$$

$M \to$ graded A-module.

Note: Each M_n is an A_0-module.

$$M = \bigoplus_{n \geq 0} M_n$$
$$N = \bigoplus_{n \geq 0} N_n$$
} Two graded ~~A-homomorphism~~ A-modules.

~~A-~~ Homomorphisms:

A-homomorphism $f : M \to N$ of graded A-modules of degree r is an A-homomorphism. $\ni$

$$f(M_n) \subset M_{n+r} \quad \forall\, n \in \mathbb{Z}^+.$$

- If $r = 0$, $f \to$ homomorphism of graded modules.

Homomorphism of graded rings:

Homomorphism $f : A \to B$

$$f(A_n) \subset B_n \quad \forall\, n \in \mathbb{Z}^+$$

$A = \bigoplus_{n \geq 0} A_n$

$B = \bigoplus_{n \geq 0} B_n$

- A submodule N of M is called a graded submodule if $N = \bigoplus_n (N \cap M_n)$

- Homogenous ideal of A: An ideal which is a graded submodule of M.

- ~~Not~~ For gradaded submodule N, we can induce an A-gradation of $\frac{M}{N}$, from M,

$$\frac{M}{N} = \bigoplus_n \left(\frac{M_n + N}{N} \right).$$

Prop^n 1.1:

(i) $A = \bigoplus_n A_n$ graded ring

$M = \bigoplus_n M_n$ graded module.

If M is noethethrian, then each M_n is a finitely generated A_0-module.

(ii) Assume that A is generated by A_1 as an A_0-algebra. Then

A is noetherian

$$\Updownarrow$$

A_0 is noetherian and A_1 is a finitely generated A_0-module.

proof:

(i) $n \in \mathbb{Z}^+$. $N = \bigoplus_{m \geq n} M_m$

$N \rightarrow$ submodule of M.

Since M is noetherian, N is finitely generated.

$\{x_1, \ldots, x_r\} \longrightarrow N$, generators.

$x_i = y_i + z_i$; $y_i \in M_n$
$z_i \in \bigoplus_{m \geq n+1} M_m$

Claim: M_n is generated over A_0 by $y_1, \ldots, y_r$.

~~For~~ Let $t \in M_n$.

$$t = \sum_{i=1}^{r} a_i x_i \quad ; a_i \in A.$$

$$a_i = b_i + c_i \quad ; \begin{cases} b_i \in A_0 \\ c_i \in \bigoplus_{m \geq 1} A_m \end{cases} \quad 1 \leq i \leq r$$

$$\Rightarrow a_i x_i = \underbrace{b_i x_i}_{\in M_n} + \underbrace{c_i x_i}_{\notin M_n}, \quad A_i M_j \subset M_{i+j}$$

$$\Rightarrow t = \sum_{i=1}^{n} b_i y_i. \quad \blacksquare$$

(ii) $(\Rightarrow)$

Let $A_+ = \bigoplus_{m \geq 1} A_m$.

Then A_+ is an ideal of A.

$$A_0 \simeq \frac{A}{A_+} \quad : \quad \left. \begin{array}{l} f : A \longrightarrow A_0 \\ a = \sum_i a_i \longmapsto a_0 \\ \operatorname{Ker} f = A_+. \end{array} \right\} \checkmark$$

So if A is noetherian, A_0 is noetherian.

By taking $M = A$, by (i),

~~(I) H~~ it follows each A_n (in particular, A_1) is finitely generated over A_0.

$(\Leftarrow)$.

Suppose $A_0 \to$ noetherian
$A_1 \to$ finitely generated A_0-module.

Let $x_1, \ldots, x_n$ generate A_1 over A_0.

Since A_1 generates A as an A_0-algebra,

we have $A = A_0[x_1, \ldots, x_n]$.

By theorem 10, A is noetherian. ∎

Filtration: Sequence.

$$A = A_0 \supset A_1 \supset \cdots \supset A_n \supset \cdots$$

ideals of A $\ni$ $A_m A_n \subset A_{m+n}$.

$\forall\, m, n \in \mathbb{Z}^+$.

$A \rightarrow$ filtered ring.

Filtration on A-module M:

$$M = M_0 \supset M_1 \supset \cdots \supset M_n \supset \cdots$$

$A_m M_n \subset M_{m+n}$ $\forall\, m, n \in \mathbb{Z}^+$

$\downarrow$

filtered module.

- Let $I \rightarrow$ ideal of A.

  $M \rightarrow$ filtered A-module

Filtration is compatible with I if

$$I M_n \subset M_{n+1} \quad \forall\, n \geq 0.$$

I-good: (i) It is compatible with I,

(ii) For $n >> 1$, $I M_n = M_{n+1}$

I-adic filtrations:

$$A = I^0 \supset I \supset I^2 \supset \cdots \quad \text{on } A$$

$$M = I^0 M \supset IM \supset \cdots \quad \text{on } M.$$

- I-adic filtration is I-good.
- Let M be an A-module with filtration compatible with I.

Consider direct sum $\bar{A} = A \oplus I \oplus \cdots$

We make $\bar{A}$ into a graded ring under the multiplication induced by that in A.

~~Let~~ $\bar{M} = M_0 \oplus M_1 \oplus \cdots$

↳ induced structure of graded $\bar{A}$-module on $\bar{M}$.

Lemma 1.2:

$I \to$ ideal of A.

$M \to$ Noetherian A-module

$M = M_0 \supset M_1 \supset \cdots$

↳ filtration compatible with I.

Then $\bar{M}$ is finitely generated as an $\bar{A}$-module

$\Updownarrow$

The filtration is I-good.

Proof:

$(\Rightarrow)$ $\bar{M}$ is finitely generated over $\bar{A}$.

Then $\exists\ n \in \mathbb{Z}^+ \ni$

$M_0 \oplus \cdots \oplus M_n$ generates $\bar{M}$ over $\bar{A}$.

Claim : $I M_m = M_{m+1}$, for $m \geq n$.

Let $x \in M_{m+1}$.

Then $x = \sum_i a_i x_i$, $x_i \in M$ are homogenous elements of degree $d_i \leq n$.

$a_i \in I^{m+1-d_i}$

now, $a_i = \sum_j b_{ij} c_{ij}$ with $b_{ij} \in I$

$c_{ij} \in I^{m-d_i}$

$\Rightarrow x \sum_{i,j} b_{i,j} (c_{i,j}\, x_i) \in I M_m$

$\Rightarrow \boxed{M_{m+1} \subset I M_m}$

Since I-compatibility were given,

we are done!

The filtration is I-good,

$(\Leftarrow)$

Let the filtration be I-good.

Then $\exists\ n \in \mathbb{Z}^+ \ni$ for $m \geq n$.

$I M_m = M_{m+1}$.

Since M is noetherian, the A-module.

the A-module $M_0 \oplus M_1 \oplus \cdots \oplus M_n$ is finitely generated.

Since $I M_m = M_{m+1}$.

for $m \geq n$,

$M_0 \oplus M_1 \oplus \cdots \oplus M_n$ generates $\bar{M}$ as an $\bar{A}$-module.

$\Rightarrow \bar{M}$ is finitely generated over $\bar{A}$. ∎

· Given filtration on M, we can induced a filtration on submodule N of M, by setting $N_n = N \cap M_n$, $(n \geq 0)$

$M = M_0 \supset M_1 \supset \cdots\cdots$

Theorem 1·3 (Artin-Rees Theorem):

Let A be a noetherian ring,

$I \to$ ideal of A

$M \to$ finitely generated module

$N \to$ submodule of M.

Then for any I-good filtration on M, the induced filtration on N is I-good.

Proof:

Let $M = M_0 \supset M_1 \supset \cdots\cdots$

be an I-good filtration on M.

Let, $\bar{A} = A \oplus I \oplus I^2 \oplus \cdots\cdots$

$\bar{M} = M_0 \oplus M_1 \oplus M_2 \oplus \cdots\cdots$

$\bar{N} = N \oplus (M_1 \cap N) \oplus (M_2 \cap N) \oplus \cdots$

Since the filtration on M is I-good,

it follows from Lemma 1·2,

$\bar{M}$ is finitely generated $\bar{A}$-module. ✓

Since A is noetherian,

I is finitely generated,

hence by propn. 1·1.,

$\bar{A}$ is noetherian, ✓

Therefore by propn 1·7, $\bar{M}$ is a noetherian $\bar{A}$ module.

$\Rightarrow$ $\bar{N}$ is finitely generated. over $\bar{A}$.

By Lemma 1·2; the induced filtration on

N is I-good. (Q.E.D.)

Corollary 1·4:

$A \to$ noetherian ring

$I \to$ ideal of A.

$M \to$ finitely generated A-module.

$N \to$ submodule of M,

Then $\exists$ $n_0 \in \mathbb{Z}^+ \ni$ $\forall$ $n \geq n_0$,

we have. $I(I^n M \cap N) = I^{n+1} M \cap N$

$\Rightarrow$ Proof!. $M = I^0 M \supset IM \supset \cdots$

consider I-adic filtration on M.

Apply theorem 1·3.

commutative ring

Corollary 1·5 :

$A \to$ Noetherian ring

$\underline{r} \to$ Jacobson radical.

Then $\bigcap_{n \geq 0} \underline{r}^n = 0$

proof : $N = \bigcap_{n \geq 0} \underline{r}^n$.

Applying corollary ~~1·5~~ 1·4 to $M = A$.

and $I = \underline{r}$, we get.

$$r N = N.$$

By Nakayam's lemma, $N = 0$. ∎

For filtered ring A.

$$A = A_0 \supset A_1 \supset \cdots \longrightarrow (1)$$

~~module~~ $M = M_0 \supset M_1 \supset \cdots$

Consider, $G(A) = \bigoplus_{n \geq 0} \dfrac{A_n}{A_{n+1}}$

↳ graded ring associated to the filtration (1)

by defining multiplication :

$$\left.\begin{array}{l} \bar{a} \in \dfrac{A_m}{A_{n+1}} \\ \bar{b} \in \dfrac{A_m}{A_{m+1}} \end{array}\right\} \text{homogenous elements of degree } n \,\&\, m.$$

$\left.\begin{array}{l}\exists\, a \in A_n, \\ b \in A_m\end{array}\right\} \ni$ $A_n \longrightarrow \dfrac{A_n}{A_{n+1}}$, $a \longmapsto \bar{a}$ | $A_m \longrightarrow \dfrac{A_m}{A_{m+1}}$, $b \longrightarrow \bar{b}$

$a b \in A_{n+m}$

$$A_{n+m} \longrightarrow \frac{A_{n+m}}{A_{n+m+1}}$$

Define $\bar{a} \cdot \bar{b} = \overline{a \cdot b}$

$$a \cdot b \longmapsto \overline{ab}$$

Similarly $G(M) = \bigoplus_{n \geq 0} \frac{M_n}{M_{n+1}}$

$\hookrightarrow$ graded $G(A)$-module.

~~$G(A$~~

$\left.\begin{array}{l} G_I(A) \\ G_I(M) \end{array}\right\} \rightarrow$ graded ring associated to I adic filtrations on A & M.

Lemma 6:

$A \rightarrow$ noetherian ring

$I \rightarrow$ ideal of A

$I \subset \mathfrak{r}(A)$

If $G_I(A)$ is an integral domain, then so is A.

Proof:

Let $a, b \in A$, $a \neq 0$, $b \neq 0$.

By cor 1.5, $\bigcap_{n \geq 0} I^n = 0$.

$\exists\ m, n \in \mathbb{Z}^+ \ni a \in I^m,\ a \notin I^{m+1}$

$b \in I^n,\ b \notin I^{n+1}$

$\bar{a}, \bar{b}$ images of a, b in $\frac{I^m}{I^{m+1}}, \frac{I^n}{I^{n+1}}$ respectively,

$\Rightarrow \left.\begin{array}{l} \bar{a} \neq 0 \\ \bar{b} \neq 0 \end{array}\right\} \Rightarrow \overline{a \cdot b} \neq 0 \Rightarrow ab \neq 0$ (Q.E.D.)