PHYS379: Group Project – Chaos Modelling Topic

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Introduction

In PHYS379, you should undertake an open-ended group project and submit a group report describing the project. The group project is worth twenty credits. Please see the document "PHYS379-group-project-notes.pdf" for details about organisation and assessment.

Open-ended project

Your task in the project is to model chaos. Precisely what you do is your choice - the project is open-ended. You will need to formulate a research question (or questions) that is more specific than simply "model chaos". What are you trying to discover? You should do some research in the library or on the internet to find out what the interesting questions are. It may be that you will not finalise the research question until the latter stages of the project, once you have found what is (or is not) interesting. Remember that this is group work - it will make sense to plan for parallel tasks in order to use your personnel effectively. For example, you might want parallel numerical and analytic investigations, investigations looking at different aspects of the same problem, or you might deliberately duplicate effort in order to check important results (in the "real" world, results should always be checked before publication, getting different people to work independently is the best way to do this). There are some ideas below, you can follow some or all of them, or you can follow your own ideas.

Getting started

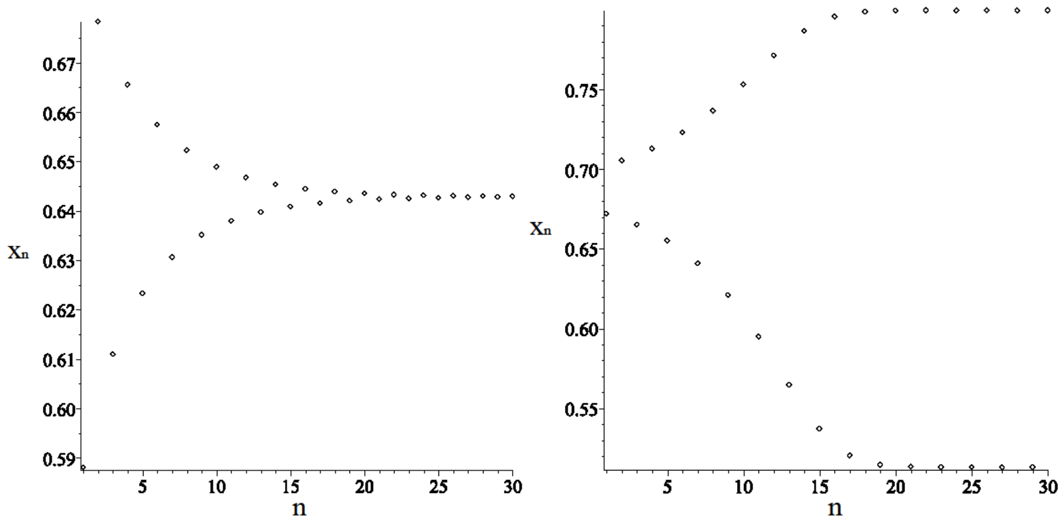

Figure 1 Trajectory x_n versus number of iterations n for the logistic map Eq.1 with (left) $x_0 = 0.3$ and $r = 2.8$ and (right) $x_0 = 0.3$ and $r = 3.2$.

The logistic map (see [1], Sections 6.1 and 6.2 of Ref.[2], Chapter 9 of Ref.[3]) relates the value of a population x_{n+1} at a given time step to its value at a previous time step x_n by a mapping:

$$x_{n+1} = r\, x_n(1 - x_n) \qquad (1)$$

defined for $0 \leq x_n \leq 1$ for all n and $0 < r \leq 4$, where r is a fixed parameter. This mapping is deterministic, i.e. for each given x_n, the value of x_{n+1} is defined precisely, there's no random element present at all. Nevertheless, there's a lot of rich behaviour for different values of r. As an example, Fig.1 shows iterated values of x_n for two different values of r. In the left panel ($r = 2.8$), there is a stable fixed point at about $x_n = 0.643$ (also called a stable attractor). By contrast, in the right panel ($r = 3.2$), x_n oscillates between two values (at about 0.513 and 0.799): we say that these points form a stable attractor of period 2. At some point in r (it's actually $r = 3$), the single stable fixed point bifurcated into two fixed points.

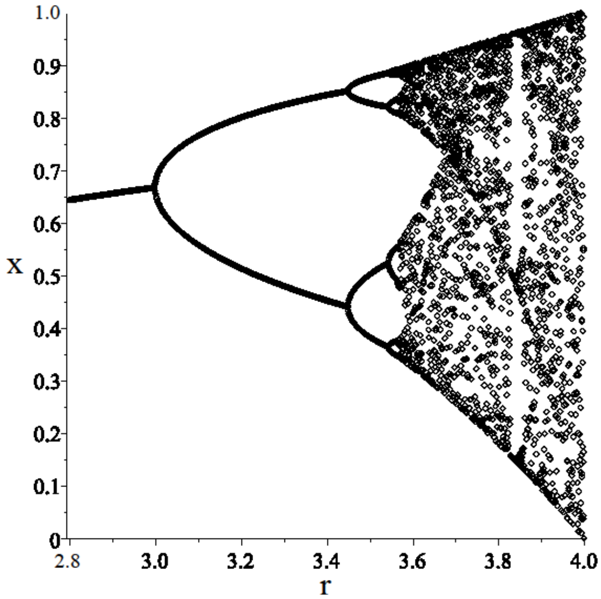

Figure 2 Bifurcation diagram of the logistic map Eq.1 showing the value of x after discarding the first 1000 iterations.

Figure 2 shows the bifurcation diagram of the logistic map for interesting values of r: $2.8 \leq r \leq 4$, in which the value of x_n is plotted after discarding the first 1000 iterations. The bifurcation at $r = 3$ is clearly visible. Further bifurcations occur at larger values of r, until, at about $r = 3.57$, there is a chaotic region: here the value of x_n can change dramatically depending on the values of x_0, n and r: there is sensitivity to initial conditions [4]. Write some code for the logistic map Eq.1 to reproduce Fig.1. Look at the behaviour for other values of r and x_0. Try to reproduce Fig.2 (see if you can do a better job than me, I only included a relatively small number of points). Work through Sections 6.1 to 6.6 of Ref.[2] to understand the logistic map better and to familiarise yourself with concepts and terminology related to chaos (see also Chapter 9 of Ref.[3]). Try not to simply copy the code in Ref.[2], you should be able to write your own.

Open-ended part

Please note:

1. **The aim is for your group to do its own, original calculations.** A literature review and, perhaps, reproduction of other people's results may be important steps along the way, but these are not the final aim.
2. This topic has run for many years with mixed success. One main difficulty is to get sufficient numerical accuracy to be able to differentiate effects of chaos from those of numerical error.

Ideas include:

- More aspects of the logistics map. See Projects 6.22, 6.23, 6.24 of Ref.[2].
- Higher-dimensional models such as the Hénon map (see Problem 6.15 of Ref.[2], Chapter 13 of Ref.[3] and Ref.[5]) or the Lorenz model (see Problem 6.16 of Ref.[2], Chapter 12 of Ref.[3] and Ref.[6]).
- Driven, damped simple pendulum. See Problems 6.17 and 6.18 of Ref.[2].
- Kicked systems. See Section 6.9 of Ref.[2] and Chapter 12 of Ref.[3].
- Double pendulum. See Section 6.9 of Ref.[2] and Chapter 5 of Ref.[3].
- Two-dimensional billiards. See Project 6.26 of Ref.[2] and Chapter 3 of Ref.[3].
- Chaotic scattering. See Project 6.28 of Ref.[2] and Chapter 6 of Ref.[3].

Bibliography

[1] “Logistic map”, https://en.wikipedia.org/wiki/Logistic_map

[2] “An Introduction to Computer Simulation Methods” by H. Gould, J. Tobochnik, and W. Christian (Pearson, San Francisco, 2007).

[3] “Chaos” by H. J. Korsch, H.-J. Jodl, and T. Hartmann (Springer, Berlin, 2008).

[4] “Chaos theory”, https://en.wikipedia.org/wiki/Chaos_theory

[5] “Hénon map”, https://en.wikipedia.org/wiki/H\%C3\%A9non_map

[6] “Lorenz system”, https://en.wikipedia.org/wiki/Lorenz_system