Definition 36.1 (Linear transformation)**.** Let $T : U \to V$ be a function from a subspace U of $\mathbb{F}^n$ to a subspace V of $\mathbb{F}^m$. We then say that T is linear transformation from U to V if

1. $\forall \mathbf{u}_1, \mathbf{u}_2 \in U$, $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ and
2. $\forall \mathbf{u}_1 \in U, c \in \mathbb{F}$, $T(c\mathbf{u}_1) = cT(\mathbf{u}_1)$

Definition 36.2 (Matrix representation of a Linear transformation)**.** Let $T : U \to V$ be a linear transformation from U to V. Let $B = \{\mathbf{u}_1, ..., \mathbf{u}_p\}$ be a basis for U and let $C = \{\mathbf{v}_1, ..., \mathbf{v}_q\}$ be a basis for V. The matrix representation of the linear transformation, with respect to basis B of U and basis C of V, is the $(q \times p)$ matrix

$${}_C[T]_B = ([T(\mathbf{u}_1)]_C, [T(\mathbf{u}_2)]_C, ..., [T(\mathbf{u}_p)]_C).$$

Special Case:
Use the standard basis S_1 in the domain and S_2 in the codomain. We then have

$${}_{S_2}[T]_{S_1} = ([T(\mathbf{e}_1)]_{S_2}, [T(\mathbf{e}_2)]_{S_2}, ..., [T(\mathbf{e}_p)]_{S_2})$$

which is the same as

$${}_{S_2}[T]_{S_1} = ([T(\mathbf{e}_1)], [T(\mathbf{e}_2)], ..., [T(\mathbf{e}_p)]) = [T]_S.$$

The simplest example of a linear transformation happens when we have a matrix $A \in M_{p\times q}(\mathbb{F})$. Then $T_A : \mathbb{F}^q \to \mathbb{F}^p$ is defined by $\forall \mathbf{x} \in \mathbb{F}^q$, $T_A(\mathbf{x}) = A\mathbf{x}$, and $[T_A]_S = A$.

Lemma 36.3. Let $T : U \to V$ be a linear transformation from U to V, with basis B for U and basis C for V. Let ${}_C[T]_B$ be the matrix representation of the linear transformation. Then $\forall \mathbf{x} \in U, [T(\mathbf{x})]_C ={}_C [T]_B\, [\mathbf{x}]_B$.

Example 36.4. Let T be the linear transformation given by $T : C^3 \to C^2$ such that $T\left(\begin{pmatrix} z \\ w \\ v \end{pmatrix}\right) = \begin{pmatrix} z+w+v \\ w-v \end{pmatrix}$. Find $[T]_S$ and use it to evaluate $T\left(\begin{pmatrix} 2i \\ -3 \\ 1+i \end{pmatrix}\right)$.

$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,so $[T]_S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

$$\text{Thus } T\left(\begin{pmatrix} 2i \\ -3 \\ 1+i \end{pmatrix}\right) = [T]_S \left[\begin{pmatrix} 2i \\ -3 \\ 1+i \end{pmatrix}\right]_S$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2i \\ -3 \\ 1+i \end{pmatrix} = \begin{pmatrix} -2+3i \\ -4-i \end{pmatrix}$$

Lemma 36.5. Let $T : U \to V$ be a linear transformation from U to V.
Let $B = \{\mathbf{u}_1, ..., \mathbf{u}_p\}$ and $D = \{\mathbf{w}_1, ..., \mathbf{w}_p\}$ be bases for U
and let $C = \{\mathbf{v}_1, ..., \mathbf{v}_q\}$ and $F = \{\mathbf{y}_1, ..., \mathbf{y}_q\}$ be bases for V. Then

$$_F[T]_D =_F [I]_C\, {}_C[T]_B\, {}_B[I]_D$$

Thus
$$\begin{aligned}[T(\mathbf{x})]_F &=_F [T]_D\, [\mathbf{x}]_D \\ &=_F [I]_C\, {}_C[T]_B\, {}_B[I]_D\, [\mathbf{x}]_D \\ &=_F [I]_C\, {}_C[T]_B\, [\mathbf{x}]_B \\ &=_F [I]_C\, [T(\mathbf{x})]_C\end{aligned}$$

Example 36.6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $_{S_2}[T]_{S_1} = \begin{pmatrix} 1 & 3 & 0 \\ 4 & -1 & 5 \end{pmatrix}$. If $D = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$ and $F = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$, evaluate $_F[T]_D$.

We calculate $_{S_1}[I]_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ $_{S_2}[I]_F = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$,so that

$_F[I]_{S_2} = (_{S_2}[I]_F)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}$

Now $_F[T]_D =_F [I]_{S_2}\, {}_{S_2}[T]_{S_1}\, {}_{S_1}[I]_D$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 4 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 4 & 4 & -6 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 & 3 \\ -2 & -2 & 3 \end{pmatrix}$

Note: The following is Topic 23 Eg 3, narrated a little differently.

Example 36.7. Find the matrix representation for the linear transformation T, projection onto the plane $P : 2x - 5y + 8z = 0$.

Then T is a function from $\mathbb{R}^3$ to P. One convenient basis for the plane P is $B = \left\{ \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} \right\}$; these vectors are linearly independent and each lies in the plane, so $P = \text{Span}\,(B)$. We will use an intermediate basis B_1 for $\mathbb{R}^3$, where $B_1 = \left\{ \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix} \right\}$. Note that we are augmenting B to get a basis for $\mathbb{R}^3$, by inserting the normal vector to P.
Recall: if $\mathbf{v} \in P$, then its projection in P is $\mathbf{v}$. Thus,

$T\left(\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$, so $\left[T\left(\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}\right)\right]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$T\left(\begin{pmatrix}4\\0\\-1\end{pmatrix}\right) = \begin{pmatrix}4\\0\\-1\end{pmatrix} = 0\begin{pmatrix}5\\2\\0\end{pmatrix} + 1\begin{pmatrix}4\\0\\-1\end{pmatrix}$, so $\left[T\left(\begin{pmatrix}4\\0\\-1\end{pmatrix}\right)\right]_B = \begin{pmatrix}0\\1\end{pmatrix}$.

Recall that $\begin{pmatrix}2\\-5\\8\end{pmatrix}$ is perpendicular to P at the origin, so its projection onto P is $\mathbf{0}$.

$T\left(\begin{pmatrix}2\\-5\\8\end{pmatrix}\right) = \begin{pmatrix}0\\0\\0\end{pmatrix} = 0\begin{pmatrix}5\\2\\0\end{pmatrix} + 0\begin{pmatrix}4\\0\\-1\end{pmatrix}$, so $\left[T\left(\begin{pmatrix}2\\-5\\8\end{pmatrix}\right)\right]_B = \begin{pmatrix}0\\0\end{pmatrix}$.

We have ${}_B[T]_{B_1} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}$.

To get ${}_B[T]_S$, we need ${}_{B_1}[I]_S = ({}_S[I]_{B_1})^{-1} = \begin{pmatrix}5 & 4 & 2\\2 & 0 & -5\\0 & -1 & 8\end{pmatrix}^{-1} = \frac{1}{93}\begin{pmatrix}5 & 34 & 20\\16 & -40 & -29\\2 & -5 & 8\end{pmatrix}$.

Now ${}_B[T]_S =_B [T]_{B_1}\,{}_{B_1}[I]_S$

$$= \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}\frac{1}{93}\begin{pmatrix}5 & 34 & 20\\16 & -40 & -29\\2 & -5 & 8\end{pmatrix}$$

$$= \frac{1}{93}\begin{pmatrix}5 & 34 & 20\\16 & -40 & -29\end{pmatrix}$$

We can compute $\left[T\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right)\right]_B = \frac{1}{93}\begin{pmatrix}5 & 34 & 20\\16 & -40 & -29\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{93}\begin{pmatrix}133\\-151\end{pmatrix}$.

This means that $T\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = \frac{133}{93}\begin{pmatrix}5\\2\\0\end{pmatrix} - \frac{151}{93}\begin{pmatrix}4\\0\\-1\end{pmatrix} = \frac{1}{93}\begin{pmatrix}61\\26651\end{pmatrix}$.