Some of my white board writing from week 11

1.) I focused on the relationships between moments and cumulants.

The moment generating function (MGF) $M_X(t) = E(\exp(tX))$ has the obvious property $M_X^{(r)}(t)\big|_{t=0} = E(X^r) = \mu_r'$ with μ_r' being a short-hand notation for the raw moment $E(X^r)$ (as opposed to $\mu_r = E(X-\mu_1')^r$ used to denote the central moments). then We have by simple Taylor expansion

$$M_X(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \dots + \mu_r' \frac{t^r}{r!} + O(t^{r+1})$$

as $t \to 0$. The cumulant generating function is $K_X(t) = \log M_X(t)$ (i.e. $e^{K_X(t)} = M_X(t)$ holds).

Since $K_X(0) = 0$ we get the Taylor expansion

$$K_X(t) = \kappa_1 t + \kappa_2 \frac{t^2}{2!} + \dots + \kappa_r \frac{t^r}{r!} + O(t^{r+1}).$$

Substituting in the relation we get:

$$e^{\kappa_1 t}\, e^{\kappa_2 \frac{t^2}{2!}}\, e^{\kappa_3 \frac{t^3}{3!}} \dots = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \dots \quad (1)$$

and expanding the exponents on the LHS we get.

$$\left(1 + \kappa_1 t + \kappa_1^2 \frac{t^2}{2!} + \dots\right)\left(1 + \kappa_2 \frac{t^2}{2!} + \frac{1}{2!}\left(\frac{\kappa_2 t^2}{2!}\right)^2 + \dots\right)\left(1 + \kappa_3 \frac{t^3}{3!} + \frac{1}{2!}\left(\frac{\kappa_3 t^3}{3!}\right)^2 + \dots\right)$$

Equating the coefficients in front of the powers of t in the LHS and RHS of (1) we get:

$\kappa_1 = \mu_1'$

$\kappa_2 + \kappa_1^2 = \mu_2' \to \kappa_2 = \mu_2' - (\mu_1')^2 = E(X)^2 - (EX)^2 = Var X = \sigma^2$

$\kappa_3 = 2\mu_1'^3 - 3\mu_1'\mu_2' + \mu_3'$

$\kappa_4 = -6(\mu_1')^4 + 12(\mu_1')^2\mu_2' - 3(\mu_2')^2 - 4\mu_1'\mu_3' + \mu_4'$

In particular for $N(0,1)$ we get $\kappa_4 = 0$ since $\mu_4' = 3(\mu_2')^2$ holds for $N(0,1)$

To summarize: - the first cumulant is the first moment, the second cumulant is the variance, the third cumulant is the skewness, the fourth cumulant is the kurtosis.

We also introduce:

- STANDARDIZED SKEWNESS: $\rho_3 = \kappa_3/(\kappa_2^{3/2})$
- Standardized kurtosis $\rho_4 = \kappa_4/(\kappa_2^2)$

These are useful in the forthcoming Edgeworth expansions.

2) Then I spoke about Cramer's condition and its importance and discussed the details in the formulation of Theorem 8.1. I specifically stressed that the constants $C_1(F), C_2(F), C_3(F)$ can be expressed by using ρ_3 and ρ_4. Namely: $C_1(F) = \frac{1}{6}\frac{\kappa_3}{\sigma^3} = \frac{1}{6}\rho_3$; $C_2(F) = \frac{1}{24}\rho_4$

$C_3(F) = \frac{C_1^2(F)}{2} = \frac{\rho_3^2}{72}$.

In section 8.3.2, when discussing formula (19), I also wrote all the first 6 Hermite polynomials explicitly.

$H_1(z) = z$; $H_2(z) = z^2 - 1$; $H_3(z) = z^3 - 3z$; $H_4(z) = z^4 - 6z^2 + 3$,

$H_5(z) = z^5 - 10z^3 + 15z$; $H_6(z) = z^6 - 15z^4 + 45z^2 - 15$

3) Next, I discussed the Cornish-Fisher expansion. I gave heuristic justification to the formula given in Theorem 8.2. It goes as follows:

Since $Z_n = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \approx$ standard normal, the quantile z_α, which is theoretically defined as the solution of $F_{Z_n}(z_\alpha) = 1-\alpha$, should be in a vincinity of the u_α quantile defined as the solution of $\Phi(u_\alpha) = 1-\alpha$.

I also discussed some "famous" u_α quantiles such as

$u_{0.01} = 2.326$, $u_{0.025} = 1.96$, $u_{0.05} = 1.645$, $u_{0.1} = 1.28$

Then the argument, by using Taylor expansion, goes as follows:

$$1-\alpha = F_{Z_n}(z_\alpha) \overset{\text{using Theorem 8.1}}{\approx} \Phi(z_\alpha) - \frac{C_1(F)p_1(z_\alpha)\varphi(z_\alpha)}{\sqrt{n}} - \frac{C_2(F)p_2(z_\alpha)+C_3(F)p_3(z_\alpha)}{n}$$

$$\underset{\substack{\uparrow \\ \text{apply Taylor by expanding "everywhere" around } u_\alpha}}{=} \Phi(u_\alpha) + \varphi(u_\alpha)(z_\alpha - u_\alpha) + \varphi(u_\alpha)[\text{polynomials containing } z_\alpha, u_\alpha]$$

Since $\Phi(u_\alpha) = 1-\alpha$, we cancel with $(1-\alpha)$ on the LHS and get

$$\varphi(u_\alpha)\underbrace{[\text{some polynomials of } z_\alpha, u_\alpha]}_{H_0} = 0$$

Set this $= 0$ and express z_α by using u_α from the resulting relation. In this way we finally obtain the expression given in Theorem 8.2

$$z_\alpha = u_\alpha + \frac{(u_\alpha^2-1)\rho_3}{6\sqrt{n}} + \frac{(u_\alpha^3-3u_\alpha)\rho_4}{24n} - \frac{(2u_\alpha^3-5u_\alpha)\rho_3^2}{36n} + o(n^{-1})$$

4) I then applied this Theorem to illustrate the power and accuracy of the Cornish-Fisher expansion on an example given on p. 69:

Example from p. 69 in detail:

$\frac{W_n}{n}$ is an average of n i.i.d. squared standard normals.

Hence the CLT will give us: (Note: for χ^2_1 r.v. $E\chi_1^2 = 1$, $Var\chi_1^2 = 2$)

$$\frac{\sqrt{n}\left(\frac{W_n}{n}-1\right)}{\sqrt{2}} = \frac{W_n - n}{\sqrt{2n}}$$ is about standard normal (this is the first order asymptotics). Then since

$$P(W_n < z_\alpha) = P\left(\frac{W_n - n}{\sqrt{2n}} < \frac{z_\alpha - n}{\sqrt{2n}}\right)$$ we know that

$\frac{z_\alpha - n}{\sqrt{2n}}$ should be "close" to u_α so then $\underline{z_\alpha = n + \sqrt{2n}\, u_\alpha}$

is the first order approximation of the $(1-\alpha)100\%$ quantile of W_n.

To improve it by using higher order Cornish-Fisher expansion we proceed as follows:

The mgf of the χ^2_n random variable (denoted generically as X here) is known to be $M_X(t) = (1-2t)^{-\frac{n}{2}}$

Hence $K_X(t) = -\frac{n}{2}\log(1-2t)$ and we get:

$$K_X'(t) = \frac{n}{1-2t}, \quad K_X''(t) = \frac{2n}{(1-2t)^2}, \quad K_X'''(t) = \frac{8n}{(1-2t)^3}, \quad K_X^{IV}(t) = \frac{48n}{(1-2t)^4}$$

Hence $K_X'(0) = n$, $K_X''(0) = 2n$, $K_X'''(0) = 8n$, $K_X^{IV}(0) = 48n$

In our case we only need to specialise this for χ^2_1 random variable so we get $K_X'(0) = 1$, $K_X''(0) = 2$, $K_X'''(0) = 8$, $K_X^{IV}(0) = 48$

Then $\rho_3 = \frac{8}{2^{3/2}} = 2\sqrt{2}$, $\rho_4 = \frac{48}{2^2} = 12$

This leads to $$\eta \approx n + \sqrt{2n}\left[u_\alpha + \frac{(u_\alpha^2 - 1)2\sqrt{2}}{6\sqrt{n}} + \frac{(u_\alpha^3 - 3u_\alpha)12}{24n} - \frac{(2u_\alpha^3 - 5u_\alpha)8}{36n}\right]$$

We apply these approximations e.g., for $n = 5$ to get:

$$5 + \sqrt{2*5} * 2.326 = 12.36$$

$$5 + \sqrt{2*5}\left(2.326 + \frac{2.326^2-1}{6\sqrt{5}} \cdot 2\sqrt{2}\right) = 15.296$$

$$5 + \sqrt{2*5}\left(2.326 + \frac{2.326^2-1}{6\sqrt{5}} 2\sqrt{2} + \frac{(2.326^3 - 3 * 2.326)\,12}{24*5} - \frac{(2*2.326^3 - 5*2.326)\,8}{36*5}\right)$$

$$= 15.16.$$

The true value of the quantile is 15.09 and we see the <u>increasing precision</u> popping up when we increase the order of the approximation.

5.) I then started discussing the idea of the saddlepoint method. I believe that the derivations, as presented in the lecture notes, are detailed enough and I did not write anything specific on the white board. I still need to go through pages 72-73 of this lecture at the beginning of lecture in week 12.