Some of my white board writing in week 8 (1)

1) I discussed the exponential distribution example on p. 46-47 of the notes. Looking at the amount of material provided in the notes, I believe that you should be able to reconstruct the details of this example by yourself.

2) Proof of the Neyman-Pearson Lemma. Again, I did complete derivation on the white board but looking at the content that is put on p. 50-51 of the notes, I believe that you should be able to reconstruct the details of the proof yourself.

3.) Example about uniformly most powerful (UMP) α-test for the normal distribution:

$\mathbb{X} = (X_1, X_2, \dots, X_n)$ i.i.d. $N(\theta, 1)$. Consider $H_0: \theta = \theta_0 \in R^1$ versus a composite $H_1: \theta > \theta_0$

all these are alternative

θ_0

Looking for UMP α-test φ^* which means: if we take any competitor $\varphi \in \Phi_\alpha = \{$set of all tests φ such that $E_{\theta_0}\varphi \le \alpha\}$ then we claim that $E_\theta \varphi^* \ge E_\theta \varphi$ for all $\theta > \theta_0$

We first simplify the problem by considering testing simple $H_0: \theta = \theta_0$ versus simple $H_1: \theta = \theta_1$ for a fixed $\theta_1 > \theta_0$. Because this is a Neyman-Pearson lemma-type problem, for it we have the most powerful α test and it is given by

$$\varphi^* = \begin{cases} 1 & \text{if } L(\mathbb{X}, \theta_1)/L(\mathbb{X}, \theta_0) > C \\ 0 & \text{if } L(\mathbb{X}, \theta_1)/L(\mathbb{X}, \theta_0) \le C \end{cases}$$

Notice that

$$\frac{L(X, \theta_1)}{L(X, \theta_0)} = \exp\left((\theta_1-\theta_0)\sum_{i=1}^{n} X_i + \frac{n}{2}(\theta_0^2-\theta_1^2)\right)$$

Since $\theta_1-\theta_0>0$, $\frac{L(X,\theta_1)}{L(X,\theta_0)}$ is monotonically increasing in $T=\sum_{i=1}^{n} X_i$ and $\frac{L(X,\theta_1)}{L(X,\theta_0)} > C$ is equivalent to $\sum_{i=1}^{n} X_i > C_1$ or, by renaming constants, to $\bar{X} > \bar{\bar{C}}$.

To find $\bar{\bar{C}}$, we must "exhaust the given level α" which means $E_{\theta_0}\varphi^* = 1 * P_{\theta_0}(\bar{X} > \bar{\bar{C}}) = \alpha$ must hold (see the statement of the NP Lemma).

But $E_{\theta_0}\varphi^* = P_{\theta_0}(\bar{X} > \bar{\bar{C}}) = P_{\theta_0}\left(\frac{\sqrt{n}(\bar{X}-\theta_0)}{1} > \frac{\sqrt{n}(\bar{\bar{C}}-\theta_0)}{1}\right)$

$= P\left(Z > \frac{\sqrt{n}(\bar{\bar{C}}-\theta_0)}{1}\right) = \alpha$ where $Z \sim N(0,1)$.

This implies that $\frac{\sqrt{n}(\bar{\bar{C}}-\theta_0)}{1} = Z_\alpha$ must hold where Z_α is the upper $\alpha * 100\%$ point of the $N(0,1)$.

Then $\bar{\bar{C}} = \theta_0 + \frac{Z_\alpha}{\sqrt{n}}$ and φ^* becomes:

$$\varphi^*(X) = \begin{cases} 1 & \text{if } \bar{X} > \theta_0 + \frac{Z_\alpha}{\sqrt{n}} \\ 0 & \text{if } \bar{X} \le \theta_0 + \frac{Z_\alpha}{\sqrt{n}} \end{cases}$$

NOW WE NOTICE that the resulting $\varphi^*(X)$ above, although having been constructed for a particular $H_1: \theta = \theta_1$, <u>DOES NOT</u> involve this $\theta_1 > \theta_0$ in its shape. Hence the <u>SAME</u> test φ^* will be the most powerful α-test for any chosen $\theta_1 > \theta_0$! Therefore, $\varphi^*(X)$ will be the <u>UNIFORMLY</u> most powerful for testing also $H_0: \theta = \theta_0$ versus $H_1: \theta > \theta_0$.

Notice that we used the <u>monotonicity</u> of the Likelihood ratio in our argument.

This example was <u>generalized</u> in the Blackwell-Girshick (BG) Theorem (p.52 of the notes).

I also gave an <u>example</u> of applying the BG theorem to derive UMP α tests.

<u>Example:</u> Assume that $\mathbb{X} = (X_1, X_2, \dots, X_n)$ are i.i.d. from $f(x,\theta) = \begin{cases} 2x/\theta^2, & 0 < x < \theta \\ 0 & \text{else.} \end{cases}$ Construct a <u>UMP α test</u>

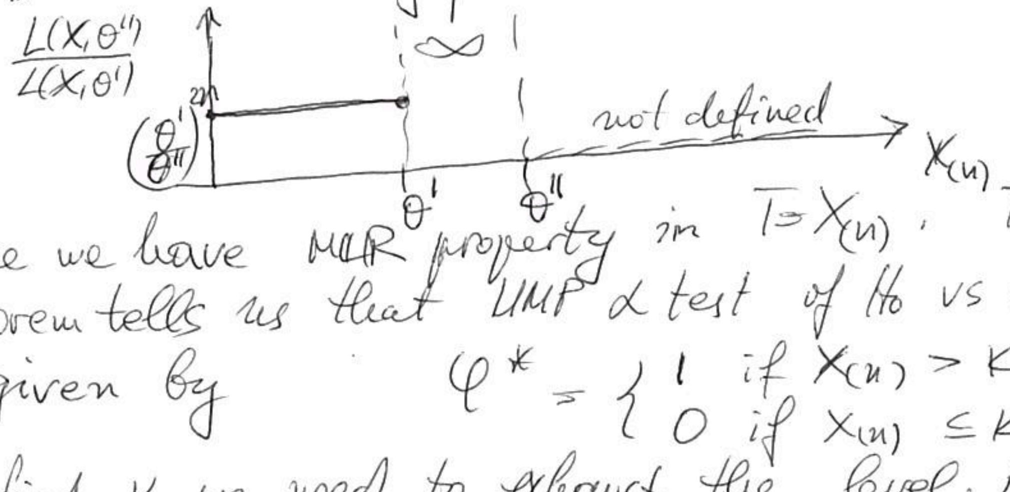
of $H_0: \theta \le \theta_0$ versus $H_1: \theta > \theta_0$.

<u>Solution:</u> First we want to show that the family $L(X,\theta)$ in this case is a MLR family in the statistic $T = X_{(n)}$.

Indeed $L(\mathbb{X},\theta) = \frac{2^n \prod_{i=1}^n x_i}{\theta^{2n}} I_{(X_{(n)},\infty)}(\theta)$. Now take two values $0 < \theta' < \theta''$ and consider

$$\frac{L(\mathbb{X},\theta'')}{L(X,\theta')} = \left(\frac{\theta'}{\theta''}\right)^{2n} \frac{I_{(X_{(n)},\infty)}(\theta'')}{I_{(X_{(n)},\infty)}(\theta')}$$

Putting $X_{(n)}$ on the OX axis we have the graph:



Hence we have MLR property in $T = X_{(n)}$. Then BG theorem tells us that UMP α test of H_0 vs H_1 exists <u>and</u> is given by $\varphi^* = \begin{cases} 1 & \text{if } X_{(n)} > K \\ 0 & \text{if } X_{(n)} \le K \end{cases}$.

To find K we need to exhaust the level, i.e. must satisfy $E_{\theta_0}\varphi^* = \alpha$. However $E_{\theta_0}\varphi^* = P_{\theta_0}(X_{(n)} > K) = 1 - P_{\theta_0}(X_{(n)} \le K)$ $= 1 - [P_{\theta_0}(X_1 \le K)]^n = 1 - \left(\frac{K}{\theta_0}\right)^{2n} = \alpha \Rightarrow \boxed{K = \theta_0 (1-\alpha)^{\frac{1}{2n}}}$ and <u>φ^* is completely</u> determined.