Some of my white board writing in week 6

Completeness, Lehmann-Scheffe:

- $X = (X_1, X_2, \dots, X_n)$ i.i.d. $N(0, \theta)$. Show, $T = \bar{X}$ is not complete for θ. It suffices to find a counterexample. Here it is: take $g(t) = t \neq 0$. We have $E_\theta g(T) = E_\theta(\bar{X}) = 0 \ \forall \theta > 0$ but $g(t) \not\equiv 0$

- $X = (X_1, X_2, \dots, X_n)$ i.i.d. Bernoulli with parameter θ. The statistic $T = \sum_{i=1}^n X_i$ is complete for $\theta \in (0,1)$:

We know $T \sim Bin(n, \theta) \longrightarrow P_\theta(T=t) = \binom{n}{t}\theta^t(1-\theta)^{n-t}$

Take $E_\theta g(T) = 0 \ \forall \theta \in (0,1) \Rightarrow \sum_{t=0}^n g(t)\binom{n}{t}\theta^t(1-\theta)^{n-t} = 0$

$\Rightarrow \underbrace{(1-\theta)^n}_{\neq 0} \cdot \sum_{t=0}^n g(t)\binom{n}{t}\eta^t = 0 \quad \forall \ \eta = \frac{\theta}{1-\theta} \in (0, \infty)$.

Then all coefficients $g(t)\binom{n}{t} = 0$ must hold and

Since $\binom{n}{t} \neq 0 \Rightarrow g(t) = 0, t = 0, 1, 2, \dots, n \Rightarrow P_\theta(g(T) = 0) = 1$

and T is <u>complete</u>.

We also know: T is sufficient. Hence if we start with an unbiased estimator W of $\tau(\theta) = \theta(1-\theta)$ and calculate $E(W|T)$ in a second step, we will get the UMVUE of $\tau(\theta) = \theta(1-\theta)$.

Suggestions for W: $W = X_1(1-X_2)$, (or $\tilde{W} = I_{(X_1=1)}(X) I_{(X_2=0)}(X)$).

We see: $E_\theta W = E_\theta X_1 - E_\theta(X_1X_2) = \theta - E_\theta X_1 E_\theta X_2 = \theta - \theta^2 = \theta(1-\theta)$

(Similarly $E_\theta \tilde{W} = EI_{(X_1=1)}(X) \cdot EI_{(X_2=0)}(X) =$
$= P(X_1 = 1)P(X_2 = 0) = \theta(1-\theta)$.

Now we get:

$$E_\theta(W|T=t) = 1 * P_\theta(W=1|T=t) + 0 =$$

$$= \frac{P(W=1 \cap T=t)}{P(T=t)} = \frac{P(X_1=1 \cap X_2=0 \cap \sum_{i=3}^{n} X_i = t-1)}{P(\sum_{i=1}^{n} X_i = t)} =$$

made sure to have intersection of independent events

$$= \frac{P(X_1=1)\,P(X_2=0)\,P(Bin(n-2,\theta)=t-1)}{P(Bin(n,\theta)=t)} = \frac{\theta(1-\theta)\binom{n-2}{t-1}\theta^{t-1}(1-\theta)^{n-t-1}}{\binom{n}{t}\theta^t(1-\theta)^{n-t}} =$$

$$= \frac{\binom{n-2}{t-1}}{\binom{n}{t}} = \boxed{\bar{X}(1-\bar{X})\frac{n}{n-1}}$$ being the UMVUE of $\tau(\theta)=\theta(1-\theta)$

- For uniform in $[0,\theta)$ distribution, $Y=\frac{n+1}{n}X_{(n)}$ is the UMVUE of the parameter $\tau(\theta)=\theta$. Justification:

From the previous lectures we know that Y is unbiased for θ and that $X_{(n)}$ is a sufficient statistic for θ. Now we show that $X_{(n)}$ is also complete. Recall that

$$f_{X_{(n)}}(t,\theta) = \begin{cases} \frac{nt^{n-1}}{\theta^n}, & 0<t<\theta \\ 0 & \text{else} \end{cases}$$ (derived last lecture)

Take $E_\theta g(X_{(n)}) = 0 \ \forall\ \theta \in (0,\infty) \Rightarrow \cancel{\frac{n}{\theta^n}}\int_0^\theta g(t)t^{n-1}dt = 0$

$0 = \frac{d}{d\theta}\int_0^\theta g(t)t^{n-1}dt = g(\theta)\theta^{n-1}$. But since $\theta>0$, this implies $g(\theta)=0$ for all $\theta>0$. Equivalently: $P_\theta(g(X_{(n)})=0)=1$ and $X_{(n)}$ is complete.

Now: $Y=\frac{n+1}{n}X_{(n)}$ is unbiased for θ and is a function of complete and sufficient statistic. Conditioning Y on $X_{(n)}$ does not change it $E(Y/X_{(n)}) = \frac{n+1}{n}X_{(n)}$

Hence, by Lehmann-Scheffe: $Y=\frac{n+1}{n}X_{(n)}$ is UMVUE of $\tau(\theta)=\theta$.

- For $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ i.i.d. Poisson(θ)

(i.e. $f(x_i, \theta) = \dfrac{e^{-\theta}\theta^{x_i}}{x_i!}$, $x_i = 0, 1, 2, \ldots$)

we have calculated the score $V(\mathbf{X}, \theta) = -n + \dfrac{\sum_{i=1}^n X_i}{\theta}$

If $\tau(\theta) = \theta$ and since we can factorize

$V(\mathbf{X}, \theta) = \frac{n}{\theta}(\bar{X} - \theta)$, we can immediately claim that $\bar{X}$ is the UMVUE of $\tau(\theta) = \theta$ and it attains the CR Bound. I also showed that this can be checked directly as follows:

$E_\theta \bar{X} = \theta$ (unbiased for θ), $Var_\theta \bar{X} = \frac{1}{n} Var_\theta X_1 = \boxed{\dfrac{\theta}{n}}$ holds

Now: $L(\mathbf{X}, \theta) = \dfrac{e^{-n\theta}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \to \ln L = -n\theta + \left(\sum_{i=1}^n x_i\right)\ln\theta - \ln\left(\prod_{i=1}^n x_i!\right)$

$\to \dfrac{\partial}{\partial\theta}\ln L = -n + \dfrac{\sum_{i=1}^n x_i}{\theta} \Rightarrow \dfrac{\partial^2}{\partial\theta^2}\ln L = -\dfrac{\sum_{i=1}^n x_i}{\theta^2}$

$\to E\left(-\dfrac{\partial^2}{\partial\theta^2}\ln L\right) = \dfrac{E\left(\sum_{i=1}^n X_i\right)}{\theta^2} = \dfrac{n\theta}{\theta^2} = \dfrac{n}{\theta} = I_{\mathbf{X}}(\theta)$

and CR Bound $= \dfrac{(\tau'(\theta))^2}{I_{\mathbf{X}}(\theta)} = \dfrac{1}{n/\theta} = \boxed{\dfrac{\theta}{n}}$ and we see that it coincides with $Var_\theta \bar{X}$.

However, take now $\tau(\theta) = e^{-\theta} = P_\theta(X_1 = 0)$

The score gives now: $V(X, \theta) = -n + \dfrac{n\bar{X}}{\theta} = ne^{\theta}\left(\frac{1}{\theta}e^{-\theta}\bar{X} - e^{-\theta}\right)$

Hence (since $\frac{1}{\theta}e^{-\theta}\bar{X}$ is <u>not</u> a statistic), the CR Bound is <u>not</u> attainable for any unbiased estimator of $e^{-\theta}$. However UMVUE (with a variance slightly bigger than the bound) can be constructed.

It was advertised as being $\left(1-\frac{1}{n}\right)^{n\bar{X}}$ and below I justify this:

First we note that $T=\sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$ (this is a known property of the Poisson distribution. $T=\sum_{i=1}^{n} X_i$ is known to be sufficient for θ from our previous lectures. Now we will show that it is also complete:

Take $E_\theta g(T)=0 \ \forall \theta>0 \Rightarrow \sum_{t=0}^{\infty} g(t) \frac{e^{-n\theta}(n\theta)^t}{t!}=0$

for all $\theta>0$. This means $\underbrace{e^{-n\theta}}_{\neq 0} \cdot \underbrace{\sum_{t=0}^{\infty} g(t) \frac{(n\theta)^t}{t!}}_{\text{This polynomial of } \theta} = 0 \ \forall \theta>0$

must be then $\equiv 0 \ \forall \theta>0 \to$ the coefficients

$\frac{g(t)n^t}{t!}$ must be all $=0$ which implies $g(t)=0, t=0,1,\dots$

i.e. $P_\theta(g(T)=0)=1$ and $T=\sum_{i=1}^{n} X_i$ is complete.

To find an unbiased starting estimator for $\tau(\theta)=e^{-\theta}$ we use the interpretation $e^{-\theta}=P_\theta(X_1=0)$ of $\tau(\theta)$.

Hence $W=I_{(X_1=0)}(\mathbf{X})$ would be unbiased for $\tau(\theta)$:

$E_\theta W = 1 * P_\theta(X_1=0)=e^{-\theta}$. If we now condition on the complete & sufficient $T=\sum_{i=1}^{n} X_i$, we will get the UMVUE:

$$E_\theta(W|T=t)=1*P(W=1|T=t)=\frac{P(W=1\cap T=t)}{P(T=t)}=$$

$$=\frac{P(X_1=0\cap \sum_{i=1}^{n} X_i=t)}{P(\sum_{i=1}^{n} X_i=t)}=\frac{P(X_1=0\cap \sum_{i=2}^{n} X_i=t)}{P(\sum_{i=1}^{n} X_i=t)}=$$

$$=\frac{e^{-\theta}\cdot \cancel{e^{-(n-1)\theta}}((n-1)\theta)^t}{\cancel{t!}\,\cancel{e^{-n\theta}}(n\theta)^t/\cancel{t!}}=\left(\frac{n-1}{n}\right)^t=\left(1-\frac{1}{n}\right)^{n\bar{X}} \quad \text{qed}$$

- I also justified why for the uniform distribution in $[0, \theta]$, the maximal observation $X_{(n)}$ is the MLE, i.e. $\hat{\theta}_{MLE} = X_{(n)}$.

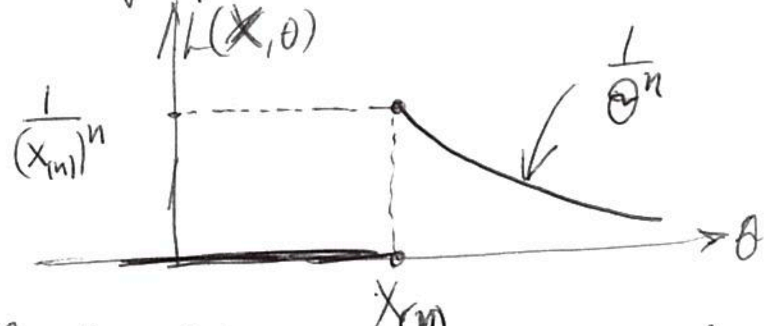
I noticed that $L(\underline{X}, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$ is not differentiable for all θ, hence instead of trying to solve the equation $V(\underline{X}, \hat{\theta}_{MLE}) = 0$ to find the MLE (which is what we would do in "regular" cases), we look <u>directly</u> into the shape of $L(\underline{X}, \theta)$ to see which is the argument that maximizes it.

Since $f(x, \theta) = \frac{1}{\theta} I_{(x, \infty)}(\theta)$ then

$$L(\mathbb{X}, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} I_{(x_i, \infty)}(\theta) = \boxed{\frac{1}{\theta^n} I_{(X_{(n)}, \infty)}(\theta)}$$

using properties of indicators

If we now <u>graph</u> $L(\mathbb{X}, \theta)$ we get after plugging the sample:



and clearly $L(\mathbb{X}, \theta)$ is maximized when $\theta = X_{(n)}$, i.e. $\hat{\theta}_{MLE} = X_{(n)}$ by direct inspection