Some of my white board writing in week 10 (1)

1.) Regarding the multinomial distribution:
I explained the formula

$$P(X_1=x_1, X_2=x_2, \dots, X_k=x_k) = \frac{n!}{x_1!\,x_2!\cdots x_k!}\, w_1^{x_1} w_2^{x_2} \cdots w_k^{x_k},$$

$0 < w_i < 1$, $\sum_{i=1}^{k} w_i = 1$ for calculating the probability of a particular outcome $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$ with $x_1+x_2+\cdots+x_k = n$, n being the number of independent trials.

I also discussed two examples:

i) If a die is tossed 6 times, what is the probability that each number (1, 2, 3, 4, 5, 6) turns up once.
Applying the above formula with $k=6$, $x_i = 1$, $i=1,2,\dots,6$, and $w_i = \frac{1}{6}$, $i=1,2,\dots,6$ we get $6!\left(\frac{1}{6}\right)^6 = \frac{5}{324}$

ii) Out of 7 tosses, what is the probability that each number (1, 2, 3, 4, 5, 6) turns up at least once.
Answer: $6 \cdot \frac{7!}{2!\,(1!)^6} \cdot \left(\frac{1}{6}\right)^7 = \frac{35}{648}$

2.) I discussed in detail the proof of Theorem 7.2. (p. 60) but I see that all details are presented in the lecture note so I will abstain from reproducing them again here.

3.) I discussed a simple method to derive the density of the r-th order statistic as stated in Theorem 7.3, p. 61

Details: we introduce a <u>discrete</u> random variable
$Y = \{$ number of realisations $X_1, X_2, \ldots, X_n$ that happen to be $\leq x \}$
Then $Y \sim \mathrm{Bin}(n, F_X(x))$.
Now we first derive $F_{X_{(n)}}(x)$ (the cdf of $X_{(n)}$) and then differentiate it to find the density. The <u>main</u> observation we make is that

$$F_{X_{(r)}}(x) = P(X_{(r)} \leq x) = P(Y \geq r)$$

Hence we can state that

$$F_{X_{(r)}}(x) = \sum_{k=r}^{n} \binom{n}{k} (F_X(x))^k (1-F_X(x))^{n-k}$$

Now to get the density we need to differentiate each of the summands in $\sum_{k=r}^{n}$ by applying the $(uv)' = u'v + v'u$ formula each time.
We get:

$$\begin{aligned} f_{X_{(r)}}(x) = {} & \binom{n}{r} r f_X(x) F_X(x)^{r-1} (1-F_X(x))^{n-r} - \cancel{\binom{n}{r}(n-r) F_X(x)^r (1-F_X(x))^{n-r-1} f_X(x)} \\ & + \cancel{\binom{n}{r+1}(r+1) f_X(x) F_X(x)^r (1-F_X(x))^{n-r-1}} - \cancel{(\quad)} \\ & + \cancel{(\quad)} - \cancel{(\quad)} \\ & + \cancel{(\quad)} - - - - \cancel{(\quad)} \\ & + \cancel{(\quad)} \qquad + \cancel{(n-n) * (\quad)} = 0 \end{aligned}$$

<u>Huge</u> cancellation happens and, because of the equality $\binom{n}{r}(n-r) = \binom{n}{r+1}(r+1)$ each of the summands after the first one disappears. Hence

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) F_X(x)^{r-1} (1-F_X(x))^{n-r}$$ holds

4) I also discussed the idea of the proof of Theorem 7.4 on p. 62. Again, we first get the cdf and then find the mixed partial derivative

$\frac{\partial^2}{\partial u \partial v} F_{X_{(i)}, X_{(j)}}(u,v)$ to calculate the density $f_{X_{(i)}, X_{(j)}}(u,v)$.

With the <u>discrete</u> variables U and V as introduced on p. 62 we see that

$$(U, V, n-U-V) \sim \text{Multinomial}\left(n; F_X(u), F_X(v)-F_X(u), 1-F_X(v)\right)$$

Then we observe that

$$F_{X_{(i)}, X_{(j)}}(u,v) = P(U \geq i \cap U+V \geq j) =$$

$$= \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} P(U=k, V=m) + P(U \geq j)$$

Since the second summand does <u>not</u> involve v, its mixed partial derivative w.r. <u>u and v</u> will be <u>zero</u> and hence

$$f_{X_{(i)}, X_{(j)}}(u,v) = \frac{\partial^2}{\partial u \partial v} \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \frac{n!}{k!\, m!\, (n-k-m)!} (F_X(u))^k (F_X(v)-F_X(u))^m (1-F_X(v))^{n-k-m}$$

Again, a huge cancellation happens when we calculate the partial derivatives by using the product rule for differentiation and we end up with

$$f_{X_{(i)}, X_{(j)}}(u,v) = \frac{n!}{(i-1)!\,(j-1-i)!\,(n-j)!} f_X(u) f_X(v) (F_X(u))^{i-1} (F_X(v)-F_X(u))^{j-1-i} (1-F_X(v))^{n-j}$$

for $n \geq j > i \geq 1$, $\quad -\infty < u < v < \infty$ (and $= 0$ else)

5. I also discussed in detail the example stating that for the range $R = X_{(n)} - X_{(1)}$ for order statistic $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ from the uniform $(0,1)$ distribution, it holds $f_R(u) = \begin{cases} n(n-1)u^{n-2}(1-u), & 0<u<1 \\ 0 & \text{else} \end{cases}$

Proof:

To this end, first we note that by using the formula from Theorem 7.4 we have (with $i=1$ and $j=n$):

$$f_{X_{(1)},X_{(n)}}(x,y) = n(n-1)(F_X(y) - F_X(x))^{n-2} f_X(x) f_X(y)$$

We introduce the variable of interest

$U = X_{(n)} - X_{(1)}$ and one auxiliary variable

$V = X_{(n)}$ so that we could apply

the density transformation formula

$$f_{\binom{U}{V}}(u,v) = f_{\binom{X_{(1)}}{X_{(n)}}}(x(u,v), y(u,v)) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

Note: $\begin{vmatrix} X_{(1)} = V - U =: x \\ X_{(n)} = V =: y \end{vmatrix}$ and $\left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \right| = 1$

Hence $f_{\binom{U}{V}}(u,v) = n(n-1)(F_X(v) - F_X(v-u))^{n-2} f_X(v-u) f_X(v) * 1$

To get $f_U(u)$ (which we are interested in) we need to integrate out the unwanted variable V from the joint density $f_{\binom{U}{V}}(u,v)$. We need to be careful with the integration range when doing this:

Since $0 < X_{(1)} < X_{(n)} < 1$ we get

$$\Updownarrow$$

$$0 < v-u < v < 1$$

$$\Updownarrow$$

$$\underline{0 < u < v < 1}$$. This means that for

a <u>fixed</u> u, v ranges in the interval $(u, 1)$.

Therefore:

$$f_R(u) = \int_u^1 f_{(U,V)}(u,v)\,dv = \int_u^1 n(n-1)\,(v-(v-u))^{n-2} * 1\,dv$$

$$= \begin{cases} n(n-1)u^{n-2}(1-u) & \text{if } 0<u<1 \\ 0 & \text{else} \end{cases}$$

I also advised you to repeat this exercise by using $X_{(1)} = V$ as an auxiliary variable. The intermediate calculations will be slightly different but at the end after integrating out v again (BUT THIS TIME in the range $(0, 1-u)$ (!)) you will get the same final result for the density $f_R(u)$.

6.) I also discussed one more problem in class ((7/d) from tutorial sheet 4) but because it is completely solved in the solutions to tutorial set 4, I abstain from reproducing the derivation here.