Grau

HM 2 Physik, Schriftl. Prüfung 316. 2.06 - 1 - WS05/06

1. Aufg. $Q(-1) = -1 + 2 + 5 - 6 = 0$,

$Q(2) = 8 + 8 - 10 - 6 = 0$, $Q(-3) = -27 + 18 + 15 - 6 = 0$ (1)

Grad $Q = 3 \Rightarrow Q$ hat höchstens 3 verschiedene N.S.

$\Rightarrow -1, 2, -3$ sind genau die N.S. von Q (1)

2. $$R(z) = \frac{A}{z+1} + \frac{B}{z-2} + \frac{C}{z+3}$$ (3)

$$\Leftrightarrow A(z-2)(z+3) + B(z+1)(z+3) + C(z+1)(z-2) = 6z^2 + 6z - 6$$

$z = -1$: $A \cdot (-6) = 6 - 6 - 6 \Rightarrow A = 1$

$z = 2$: $B \cdot 15 = 24 + 12 - 6 = 30 \Rightarrow B = 2$

$z = -3$: $C \cdot 10 = 54 - 18 - 6 = 30 \Rightarrow C = 3$ (2)

$$\Rightarrow R(z) = \frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{z+3}$$ (1)

3. $$\int_0^1 \frac{6x^2 + 6x - 6}{x^3 + 2x^2 - 5x - 6}\,dx = \int_0^1 R(x)\,dx \overset{2.}{=}$$

$$= \int_0^1 \frac{dx}{x+1} + 2\int_0^1 \frac{dx}{x-2} + 3\int_0^1 \frac{dx}{x+3} =$$ (1)

$$= \ln(x+1)\Big|_0^1 + 2\ln|x-2|\Big|_0^1 + 3\ln(x+3)\Big|_0^1$$ (1)

$$= \ln 2 - \ln 1 + 2(\ln 1 - \ln 2) + 3(\ln 4 - \ln 3)$$

$$= \ln 2 - 2\ln 2 + 6\ln 2 - 3\ln 3 = 5\ln 2 - 3\ln 3$$ (1)

$a = 5$, $b = -3$

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2. Aufg. $z = x + iy$

$$z^2 = x^2 - y^2 + 2xyi = -5 + 12i$$

$\Leftrightarrow$ $x^2 - y^2 = -5$ und $2xy = 12$ (1)

$\Downarrow$ $\qquad$ $\Updownarrow$

$y = \frac{6}{x}$

$$x^2 - \frac{36}{x^2} = -5 \quad (1)$$

$\Leftrightarrow$ $x^4 + 5x^2 - 36 = 0 \Leftrightarrow x^2 = \frac{1}{2}(-5 \pm \sqrt{\underbrace{25+144}_{=169}})$

$\Leftrightarrow$ $x^2 = \frac{1}{2}(-5 \pm 13)$ (1)

$\Rightarrow$ $x^2 = \frac{1}{2}(13-5) = 4 \Rightarrow x = \pm 2$

$\Rightarrow$ $y = \pm 3$ (1)

$\Rightarrow$ $z = \pm(2 + 3i)$ (1)

sind genau die beiden Lösungen, da eine Gl. 2-ten Grades zwei Lösungen besitzt.
(Evtl. Probe)

$|z| = \sqrt{4+9} = \sqrt{13}$ (1)

$\tan(\text{Polarwinkel}(z)) = \frac{3}{2}$ (1)

Variante für $|z|$ und Polarwinkel von z | (1) $\boxed{\|7\|}$

$|z|^2 = |-5 + 12i| = \sqrt{25+144} = \sqrt{169} = 13 \Rightarrow |z| = 13$ falls nicht schon vergeben

Sei $\alpha :=$ Polarwinkel von z. Dann ist 2α der Polarwinkel von $-5+12i$. $-\frac{12}{5} = \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$\Rightarrow$ $\tan^2\alpha - \frac{5}{6}\tan\alpha - 1 = 0 \Rightarrow \tan\alpha = \frac{1}{2}\left(\frac{5}{6} \pm \sqrt{\frac{25}{36}+4}\right)$

$\Rightarrow$ $\tan\alpha = \frac{1}{12}(5+13) = \frac{18}{12} = \frac{3}{2} > 0$

geom. ?ites.

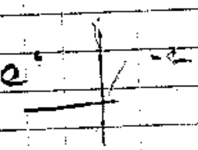
(2), falls z nicht explizit berechnet.

Variante A2

$a = -5 + 12i$

Gesucht $z \in \mathbb{C}$, $z = x + iy$

mit $z^2 = a$



$\alpha :=$ Polarwinkel von z, $\beta :=$ Polarwinkel von a

$|z|^2 = |a| = \sqrt{25 + 144} = \sqrt{169} = 13 \Rightarrow \underline{|z| = \sqrt{13}}$

$\tan \beta = -\frac{12}{5}$

$z^2 = a \Rightarrow 2\alpha = \beta$, $0 < \alpha < \frac{\pi}{2}$, $\frac{\pi}{2} < \beta < \pi$

$-\frac{12}{5} = \tan \beta = \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

$\Rightarrow \tan^2 \alpha - \frac{5}{6} \tan \alpha - 1 = 0$

$\Rightarrow \tan \alpha = \frac{1}{2}\left(\frac{5}{6} \pm \sqrt{\frac{25}{36} + 4}\right) = \frac{1}{12}(5 \pm 13)$

$\Rightarrow \tan \alpha = \frac{18}{12} = \frac{3}{2}$, da $0 < \alpha < \frac{\pi}{2}$

$\tan \alpha = \frac{y}{\sqrt{|z|^2 - y^2}} \Rightarrow \left(\frac{3}{2}\right)^2 = \frac{y^2}{13 - y^2}$

$\Rightarrow 13 - y^2 = \frac{4}{9} y^2 \Rightarrow 13 = \frac{13}{9} y^2$

$\Rightarrow y^2 = 9 \Rightarrow y = 3.$

$x = \sqrt{13 - 9} = 2$

$z = 2 + 3i$ ist Lösung

$z = \pm(2 + 3i)$ sind die Lösungen

$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

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3. Aufg.

1. $g'(x) = 2x + \frac{1}{x^2}$, $g''(x) = 2 - \frac{2}{x^3}$, $g'''(x) = \frac{6}{x^4}$ (1)

2. $g'(x) > 0$ für $x > 0$ $\Rightarrow$ g streng monoton wachsend (1)

3. $\lim\limits_{x\to 0} g(x) = -\infty$, $\lim\limits_{x\to\infty} g(x) = \infty$ (1)

4. g streng monoton wachsend
$\Rightarrow$ g besitzt Umkehrfunktion f (1)

(a) $D = g(]0,\infty[) =]-\infty,\infty[= \mathbb{R}$
nach Nr. 3 und ZWS für stetige Fkt. (1)

$f(D) =]0,\infty[$ = Def.-Ber. g (1)

(b) $f(0) = 1$, da $g(1) = 0$. (1)

(c) $f'(x) = \frac{1}{g'(f(x))} \Rightarrow \underline{f'(0)} = \frac{1}{g'(f(0))} = \frac{1}{g'(1)} = \underline{\frac{1}{3}}$ (1)

$f''(x) = -\frac{g''(f(x))f'(x)}{(g'(f(x)))^2} = -g''(f(x))(f'(x))^3$ (1)

$\Rightarrow \underline{f''(0)} = -g''(f(0))(f'(0))^3 = -\underbrace{g''(1)}_{=0}(f'(0))^3 = \underline{0}$ (1)

$f'''(0) = -g'''(f(0))(f'(0))^4 - 3\underbrace{g''(f(0))}_{=0}(f'(0))^2 \underbrace{f''(0)}_{=0}$ oder

$\Rightarrow f'''(0) = -g'''(1)\cdot\left(\frac{1}{3}\right)^4 = 6\cdot\frac{1}{3^4} = \frac{2}{27}$ (1)

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4. Aufg.

1. $|f(x) - f(y)| = \left|\frac{3+x}{7+x} - \frac{3+y}{7+y}\right| =$

$$= \frac{|21 + 7x + 3y + xy - 21 - 3x - 7y - xy|}{\underbrace{(7+x)}_{\geq 3}\underbrace{(7+y)}_{\geq 3}} \leq \frac{|4x - 4y|}{9}$$

$$= \frac{4}{9}|x-y| \quad \forall\, x, y \in [-4, 4]$$

$$\Rightarrow L = \frac{4}{9}$$

(3)

2. Für $|x| \leq 4$ gilt

$$\left|\frac{x+3}{x+7}\right| = \frac{|x+3|}{\underbrace{x+7}_{\geq 3}} \leq \frac{1}{3}|x+3| \leq \frac{1}{3}(|x|+3) \leq \frac{7}{3} < 4$$

$$\Rightarrow |f(x)| < 4 \Rightarrow f(x) \in [-4, 4]$$

(2)

3. $f(x) = x \iff x = \frac{3+x}{7+x} \iff x^2 + 7x = 3 + x$

$$\iff x^2 + 6x - 3 = 0 \iff x = \frac{1}{2}(-6 \pm \sqrt{\underbrace{36+12}_{=48=3\cdot 16}})$$

$$\iff x = -3 \pm 2\sqrt{3}$$

(1)

$-3 - 2\sqrt{3} < -3 - 2 < -4 \Rightarrow -3 - 2\sqrt{3} \notin [-4, 4]$ (1)

$-3 < -3 + 2\sqrt{3} < -3 + 4 = 1 \Rightarrow 2\sqrt{3} - 3 \in [-4, 4]$ (1)

$\Rightarrow \tilde{x} = 2\sqrt{3} - 3$ eindeutig best.

4. Folgt aus 2. (1)

5. $|x_n - \tilde{x}| = |f(x_{n-1}) - f(\tilde{x})| \le L\,|x_{n-1} - \tilde{x}|$ (2)

$= \frac{4}{9}\,|x_{n-1} - \tilde{x}| \le \dots \underset{n\text{-mal}}{\le} \left(\frac{4}{5}\right)^n |x_0 - \tilde{x}|$ (2)

$\le \left(\frac{4}{5}\right)^n \cdot 8$

6. Aus 5. folgt:

$|x_n - \tilde{x}| \le \left(\frac{4}{5}\right)^n \cdot 8 \xrightarrow[n\to\infty]{} 0$ (1)

$\Rightarrow \lim\limits_{n\to\infty} x_n = \tilde{x}$ (1)

Variante zu 1. und 2. ||15||

$f'(x) = \frac{7+x-3-x}{(7+x)^2} = \frac{4}{(7+x)^2} > 0$ (1)

$\Rightarrow$ f streng monoton wachsend

$f(-4) = \frac{-1}{3} > -4$, $f(4) = \frac{7}{11} < 4$

$\Rightarrow f([-4,4]) \subset [-4,4]$ (1)

f mon. wachsend.

$\left|\frac{f(x)-f(y)}{x-y}\right| \underset{\text{M.W.S.}}{=} |f'(\xi)| = \frac{4}{(7+\xi)^2} \le \frac{4}{9} \quad \forall x,y \in [-4,4]$ (2)

$-4 < \xi < 4$, $(7+\xi)^2 \ge 3^2$

$\Rightarrow |f(x) - f(y)| \le \frac{4}{9}|x-y|$, also $L = \frac{4}{5}$ (1)