Lösungsvorschlag zur Elektrodynamik-Klausur 16.02.17

Aufgabe 1

a.) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ $\qquad$ $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$

$\vec{\nabla}\cdot\vec{B} = 0$ $\qquad$ $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

b.) $$\int_V d^3x\,\frac{\partial\rho}{\partial t} \underset{V\text{ zeitunabhängig}}{=} \frac{d}{dt}\int_V d^3x\,\rho = -\int_{\partial V} d\vec{a}\cdot\vec{j} \underset{\text{Gauß}}{=} -\int_V d^3x\,\vec{\nabla}\cdot\vec{j}$$

V beliebig $\longrightarrow$ $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$

c.) $$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda^T g \Lambda = g$$

d.) $$x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}, \quad x_\mu = \begin{pmatrix} ct \\ -\vec{x} \end{pmatrix} \text{ oder auch } x_\mu = (ct, -\vec{x}^T)$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\vec{\nabla} \end{pmatrix}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \vec{\nabla} \end{pmatrix}$$

e.) $$j^\mu = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}, \quad \partial_\mu j^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}^T\right)\begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} = \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

f.) Ampère: $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$

$$0 \overset{!}{=} \dot{\rho} + \vec{\nabla}\cdot\vec{j} = \dot{\rho} + \frac{c}{4\pi}\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{\equiv 0} \Longrightarrow \vec{\nabla}\cdot\vec{j} = 0$$

Mit Maxwell-Verschiebungsstrom:

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

$$0 \overset{!}{=} \dot{\rho} + \vec{\nabla}\cdot\vec{j} = \dot{\rho} + \frac{c}{4\pi}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) - \frac{1}{4\pi}\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t}$$

mit $-\frac{1}{4\pi}\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = -\dot{\rho}$ nach Gauß'schem Gesetz $\longrightarrow$ konsistent

g.) $$p = p_1 + p_2 = \left\{\begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right\} * 125\,\frac{\text{GeV}}{c^2}$$

$$M^2c^2 = p^2 = \left(4 - \frac{1}{2} - 2 - \frac{1}{2}\right)\left(125\,\frac{GeV}{c^2}\right)^2 \Longrightarrow M = 125\,\frac{GeV}{c^2}$$

h.) $\int\limits_D d^3x\, \rho(\vec{x}) = A\sigma = \frac{1}{4\pi} \int\limits_{\partial D} d\vec{a}\cdot\vec{E} = \frac{A}{4\pi}\left(\vec{n}\cdot\vec{E_2} - \vec{n}\cdot\vec{E_1}\right)$

$D \to$ Kästchen um Flächenladung

$$\Longrightarrow \vec{n}\cdot\left(\vec{E_2} - \vec{E_1}\right) = 4\pi\,\sigma$$

Aufgabe 2

a.) $\rho = \lambda\,\delta(x)\,\delta(y)$, $\quad r = \sqrt{x^2+y^2}$, $\vec{E} \equiv \vec{E}(r) = E(r)\left(\frac{x}{r}, \frac{y}{r}, 0\right)$

$$\int\limits_{\partial V} d\vec{a}\cdot\vec{E} = 4\pi \int\limits_V d^3x\,\rho \Longrightarrow \int dz\; 2\pi r\; E(r) = 4\pi \int dz\,\lambda \Longrightarrow E(r) = \frac{2\lambda}{r}$$

b.) Nun ist $\quad r = \sqrt{(x-x_0)^2+y^2}$, $\quad r' = \sqrt{(x+x_0)^2+y^2}$

Spiegeldraht entlang $x = -x_0 \longrightarrow$ Potential auf Platte verschwindet

Feld plus Spiegelfeld: $\quad \vec{E} = \frac{2\lambda}{r}\begin{pmatrix} \frac{x-x_0}{r} \\ \frac{y}{r} \\ 0 \end{pmatrix} - \frac{2\lambda}{r'}\begin{pmatrix} \frac{x+x_0}{r'} \\ \frac{y}{r'} \\ 0 \end{pmatrix}$

c.) $x = 0$: $\quad \vec{E} = \frac{4\lambda}{\sqrt{x_0^2+y^2}}\begin{pmatrix} \frac{-x_0}{\sqrt{x_0^2+y^2}} \\ 0 \\ 0 \end{pmatrix}$, $\left|\vec{E}\right| \underset{\substack{\uparrow \\ 1.h)}}{=} 4\pi\sigma \Longrightarrow \sigma = -\frac{1}{\pi}\frac{\lambda x_0}{x_0^2+y^2}$

d.) $\int\limits_{-\infty}^{\infty} dy\, \frac{1}{\pi}\frac{x_0}{x_0^2+y^2} = \int\limits_{-\infty}^{\infty} dy\, \frac{1}{\pi x_0}\frac{1}{1+\frac{y^2}{x_0^2}} \underset{\substack{\uparrow \\ \xi = \frac{y}{x_0}}}{=} \frac{1}{\pi}\int\limits_{-\infty}^{\infty} d\xi\, \frac{1}{1+\xi^2} = \frac{1}{\pi}\left[\arctan\xi\right]_{-\infty}^{\infty} =$

$$\Longrightarrow \int\limits_{-\infty}^{\infty} dy\; \sigma = -\lambda$$

Aufgabe 3

a.) $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \Longrightarrow \int d^3x\, \vec{j} = -\int d^3x\, \vec{x}\,\vec{\nabla}\cdot\vec{j} = \int d^3x\, \vec{x}\,\frac{\partial\rho}{\partial t} = \frac{\partial\vec{P}}{\partial t}$

b.) $\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \int dt'\, \frac{\vec{j}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\,\delta\!\left(t - t' - \frac{|\vec{x}-\vec{x}'|}{c}\right) = \frac{1}{c}\int d^3x'\, \frac{\vec{j}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}$

$$= \frac{1}{cx}\int d^3x'\, \vec{j}\left(\vec{x}', t - \frac{x}{c}\right) = \frac{1}{cx}\,\dot{\vec{P}}(\tilde{t})\Big|_{\tilde{t} = t - \frac{x}{c}} = \frac{1}{cx\tau}\,\vec{P_0}\, f'\!\left(\frac{1}{\tau}\left(t - \frac{x}{c}\right)\right)$$

τ punktförmiger Dipol

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{c\tau}\,\vec{P_0}\times\left(\frac{\vec{x}}{x^3}f'\!\left(\frac{1}{\tau}\left(t-\frac{x}{c}\right)\right) + \frac{\vec{x}}{c\tau x^2}f''\!\left(\frac{1}{\tau}\left(t-\frac{x}{c}\right)\right)\right) \underset{\uparrow}{\approx} \frac{1}{c^2\tau^2}\, f''\!\left(\frac{1}{\tau}\left(t-\frac{x}{c}\right)\right)\vec{P_0}\times$$

c.) $\phi(\vec x,t)=\int d^3x'\int dt'\,\frac{\varrho(\vec x',t')}{|\vec x-\vec x'|}\,\delta\!\left(t-t'-\frac{|\vec x-\vec x'|}{c}\right)=\frac{1}{c}\int d^3x'\,\frac{\varrho\!\left(\vec x',t-\frac{|\vec x-\vec x'|}{c}\right)}{|\vec x-\vec x'|}$

$|\vec x-\vec x'|\approx \vec x+\hat x\cdot\vec x'+\dots$

$$\hat= \int d^3x'\left(\frac{1}{x}+\frac{\vec x\cdot\vec x'}{x^3}+\dots\right)\left(\varrho(\vec x',t-\tfrac{x}{c})+\dot\varrho(\vec x',t-\tfrac{x}{c})\,\frac{1}{c}\,\frac{\vec x\cdot\vec x'}{x}+\dots\right)$$

$$\underset{x\gg c\tau}{\approx}\ \frac{1}{cx^2}\,\vec x\cdot\int d^3x'\,\vec x'\,\dot\varrho(\vec x',t-\tfrac{x}{c})=\frac{1}{cx^2}\,\vec x\cdot\dot{\vec p}(\tilde t)\Big|_{\tilde t=\frac{x}{c}}=\frac{1}{c\tau x^2}\,\vec x\cdot\vec p_0\,f'\!\left(\tfrac{1}{\tau}(t-\right.$$

d.) $\vec E=-\vec\nabla\phi-\frac{1}{c}\frac{\partial\vec A}{\partial t}$

$$-\vec\nabla\phi\underset{x\gg c\tau}{\approx}\frac{1}{c^2\tau^2x^2}\,\vec x\cdot\vec p_0\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\frac{\vec x}{x}\ ,\quad -\frac{1}{c}\frac{\partial\vec A}{\partial t}\approx-\frac{\vec p_0}{c^2\tau^2x}\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)$$

$$\longrightarrow \vec E=\frac{1}{c^2\tau^2x}\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\left[\hat x(\hat x\cdot\vec p_0)-\vec p_0\right]=\frac{1}{c^2\tau^2x}\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\hat x\times(\hat x\times\vec p_0)=\vec B\times\hat x$$

e.) $\vec S=\frac{c}{4\pi}\vec E\times\vec B=-\frac{c}{4\pi}\vec B\times(\vec B\times\hat x)=-\frac{c}{4\pi}\left(\vec B\,\hat x\cdot\vec B-\hat x\,\vec B^2\right)$

$$=\frac{c}{4\pi}\left[\frac{1}{c^2\tau^2x}\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\right]^2\left(\hat x\,(\vec p_0\times\hat x)^2-(\vec p_0\times\hat x)\,\overbrace{\hat x\cdot(\vec p_0\times\hat x)}^{\equiv 0}\right)$$

$$=\frac{c}{4\pi}\left[\frac{1}{c^2\tau^2x}\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\right]^2\hat x\,(\vec p_0\times\hat x)^2$$

f.) $\frac{dP}{d\Omega}=S\cdot\hat x\,x^2=\frac{c}{4\pi}\left[\frac{1}{c^2\tau^2}\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\right]^2(\vec p_0\times\hat x)^2$

Mit $\int_{-1}^{1}d\cos\vartheta\,(\vec p_0\times\hat x)^2=\int_{-1}^{1}d\cos\vartheta\,\vec p_0^{\,2}(1-\cos^2\vartheta)=\frac{4}{3}$

$$\longrightarrow\ P=2\pi\int_{-1}^{1}\frac{dP}{d\Omega}\,d\cos\vartheta=\frac{2}{3}c\,\vec p_0^{\,2}\left[\frac{1}{c^2\tau^2}\,f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\right]^2$$

g.) $W=\int_{-\infty}^{\infty}dt\,P=\frac{2}{3}\frac{\vec p_0^{\,2}}{c^3\tau^4}\int_{-\infty}^{\infty}dt\left[f''\!\left(\tfrac{1}{\tau}(t-\tfrac{x}{c})\right)\right]^2=\frac{2}{3}\frac{\vec p_0^{\,2}}{c^3\tau^3}\int_{-\infty}^{\infty}dx\,\left(f''(x)\right)^2$

$$=\frac{2}{3}\frac{\vec p_0^{\,2}}{c^3\tau^3}\sqrt{\frac{\pi}{2}}\left(4-\frac{16}{4}+\frac{3\cdot16}{16}\right)=\sqrt{2\pi}\,\frac{\vec p_0^{\,2}}{c^3\tau^3}$$

wobei

$f(x)=e^{-x^2}$

$\int_{-\infty}^{\infty}dx\,x^{2n}e^{-\alpha x^2}=\sqrt{\frac{\pi}{\alpha}}\,\frac{(2n-1)!!}{(2\alpha)^n}$

$f'(x)=-2x\,e^{-x^2}$

$f''(x)=4x^2e^{-x^2}-2\,e^{-x^2}$

$(f''(x))^2=4\,e^{-2x^2}-16x^2e^{-2x^2}+16x^4\,e^{-2x^2}$