d) $t=0$

$|\psi(t=0)\rangle = \psi_{u_1}(\vec{r}_1)\,\psi_{u_2}(\vec{r}_2) = |1u_1\rangle|2u_2\rangle = \frac{1}{\sqrt{2}}|u_1u_2\rangle_+ + \frac{1}{\sqrt{2}}|u_1u_2\rangle_- = 2\frac{1}{\sqrt{2}}[\psi_1(1)\psi_2(2)]$

$E^{\pm}_{St} = E_0 + K \pm A$ ① ②

$|\psi,t\rangle = \frac{1}{\sqrt{2}} e^{-i/\hbar E^+_{St}\cdot t}|u_1u_2\rangle_+ + \frac{1}{\sqrt{2}} e^{-i/\hbar E^-_{St}\cdot t}|u_1u_2\rangle_-$

$= \frac{1}{\sqrt{2}}\Big[e^{-i/\hbar(E_0+K+A)t}(\psi_1(1)\psi_2(2)+\psi_2(1)\psi_1(2))$

$+ e^{-i/\hbar(E_0+K-A)\cdot t}(\psi_1(1)\psi_2(2)-\psi_2(1)\psi_1(2))\Big]$

$= \frac{1}{\sqrt{2}} e^{-i/\hbar(E_0+K)\cdot t}\Big[e^{-i/\hbar At}\psi_1(1)\psi_2(2) + e^{-i/\hbar At}\psi_2(1)\psi_1(2)$

$+ e^{i/\hbar At}\psi_1(1)\psi_2(2) + e^{i/\hbar At}\psi_2(1)\psi_1(2)\Big]$

$= \frac{1}{\sqrt{2}} e^{-i/\hbar(E_0+K)t}\Big[2\cos\left(\frac{At}{\hbar}\right)\psi_1(1)\psi_2(2) - 2i\sin\left(\frac{At}{\hbar}\right)\psi_2(1)\psi_1(2)\Big]$

soll übrig bleiben

$t=0$: $\frac{2}{\sqrt{2}}[\psi_1(1)\psi_2(2)]$

$\downarrow$ nach t

$\psi_1(2)\psi_2(1)$

$\Rightarrow \cos\left(\frac{At}{\hbar}\right) \overset{!}{=} 0$

$\psi \sim \psi_2(1)\psi_1(2)$ $\quad \sin\left(\frac{At}{\hbar}\right) \overset{!}{=} 1$

$\Longrightarrow \boxed{t = \frac{\pi\hbar}{2A}}$

Verbesserung Semestralklausur

16.7.08

Nr. 1:

Basis: $\{|1\rangle, |2\rangle\}$

a) $\hat{H} = \varepsilon\{|1\rangle\langle 1| - |2\rangle\langle 2| + \sqrt{3}[|1\rangle\langle 2| + |2\rangle\langle 1|]\}$

$(\hat{H})_{ij} = \langle i|\hat{H}|j\rangle$

Bsp.: $\langle 1|\hat{H}|1\rangle = \varepsilon(\overbrace{\langle 1|1\rangle\langle 1|1\rangle}^{1} - \overbrace{\langle 1|2\rangle\langle 2|1\rangle}^{0} + \sqrt{3}[\overbrace{\langle 1|1\rangle\langle 2|1\rangle}^{0}$

$+ \underbrace{\langle 1|2\rangle\langle 1|1\rangle}_{0}) = \varepsilon = \hat{H}_{11}$

$\hat{H}_{22} = -\varepsilon$

$\hat{H}_{12} = \sqrt{3}\varepsilon \qquad \Longrightarrow \quad \hat{H} = \varepsilon\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

$\hat{H}_{21} = \sqrt{3}\varepsilon$

b) $P(\lambda) = \det \begin{vmatrix} \varepsilon-\lambda & \sqrt{3}\varepsilon \\ \sqrt{3}\varepsilon & -\varepsilon-\lambda \end{vmatrix} = \lambda^2 - \varepsilon^2 - 3\varepsilon^2$

$= \lambda^2 - 4\varepsilon^2 = (\lambda-2\varepsilon)(\lambda+2\varepsilon) \overset{!}{=} 0$

$\Rightarrow \lambda_{1/2} = \pm 2\varepsilon$

$\begin{pmatrix} \varepsilon-\lambda & \sqrt{3}\varepsilon \\ \sqrt{3}\varepsilon & -\varepsilon-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

- $\lambda = 2\varepsilon$:

$\begin{pmatrix} -\varepsilon & \sqrt{3}\varepsilon \\ \sqrt{3}\varepsilon & -3\varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sigma \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

Normierung $\rightarrow \sigma = \frac{1}{\sqrt{\sqrt{3}^2 + 1^2}} = \frac{1}{2}$

$\Rightarrow \underset{\text{zu } \lambda=2\varepsilon}{EZ} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \frac{\sqrt{3}}{2}|1\rangle + \frac{1}{2}|2\rangle$

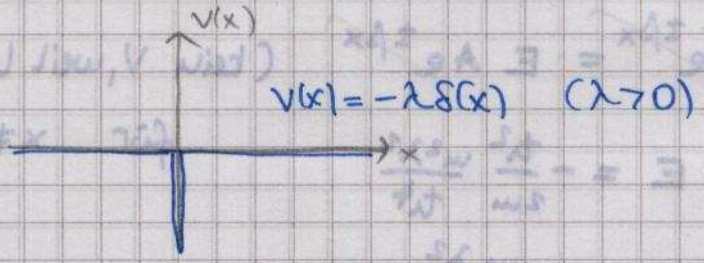
- $\lambda = -2\varepsilon$

$\rightarrow \begin{pmatrix} 3\varepsilon & \sqrt{3}\varepsilon \\ \sqrt{3}\varepsilon & \varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sigma \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

$\Rightarrow$ normiert: EZ zu $\lambda = -2\varepsilon$:

$\frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = \frac{1}{2}|1\rangle - \frac{\sqrt{3}}{2}|2\rangle$

Nr. 2:



a) Wellenfkt. stetig! $\rightarrow \psi(0^+) = \psi(0^-)$ (I)

(hier: Ableitung nicht stetig)

$-\frac{\hbar^2}{2m}\psi''(x) - \lambda\delta(x)\psi(x) = E\psi(x)$

$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\psi''(x)\,dx - \lambda\psi(0) = E\int_{-\varepsilon}^{\varepsilon}\psi(x)\,dx$

$-\frac{\hbar^2}{2m}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] - \lambda\psi(0) = E\int_{-\varepsilon}^{\varepsilon}\psi(x)\,dx$

$\varepsilon \to 0$:

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left[\psi'(0^+)-\psi'(0^-)\right]-\lambda\psi(0) = 0$$

weil Integral von 0 bis 0 $\Rightarrow$ immer 0

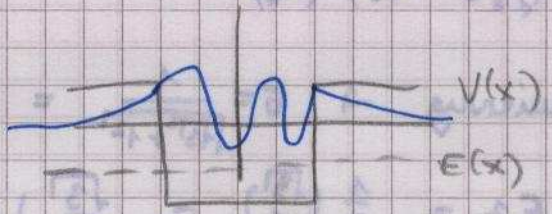
$$(\mathrm{II}) \Rightarrow \quad \psi'(0^+)-\psi'(0^-) = -\frac{2m\lambda}{\hbar^2}\psi(0) \neq 0$$

$\Rightarrow$ $\psi'(x)$ unstetig bei $x=0$!

$\Rightarrow$ 2 Anschlussbedingungen: (I) & (II)

b)

$E<V$: $\rightarrow$ ψ exp. abfallend!



für δ-Fkt.: verkleinern!

$\Rightarrow$ für δ-Fkt.: in beide Richtungen von 0 weg exponentiell abfallend!

$$\Rightarrow \text{Ansatz:} \quad \psi(x) = \begin{cases} Ae^{-\beta x} & x>0 \\ Be^{\beta x} & x<0 \end{cases}$$

(I) $\quad \psi(0^+) = \psi(0^-) \quad \Rightarrow \quad A = B$

(II) $\quad \psi'(0^+)-\psi'(0^-) = -\frac{2m\lambda}{\hbar^2}A$

$$-\beta A - \beta A = -\frac{2m\lambda}{\hbar^2}A \quad \Rightarrow \quad \beta = \frac{m\lambda}{\hbar^2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}Ae^{\pm\beta x} = E\,Ae^{\pm\beta x}$$ (kein V, weil hier Betrachtung für $x \neq 0$)

$$-\frac{\hbar^2}{2m}\beta^2 = E = -\frac{\hbar^2}{2m}\frac{m^2\lambda^2}{\hbar^4}$$

$$\Rightarrow \quad E = -\frac{m\lambda^2}{2\hbar^2}$$

Norm.: $$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 \stackrel{!}{=} 1 = A^2\left[\int_{-\infty}^{0} e^{2\beta x}dx + \int_0^{\infty} e^{-2\beta x}dx\right]$$

$$= 2A^2\int_0^{\infty} e^{-2\beta x}dx = \frac{2A^2}{(-2\beta)}e^{-2\beta x}\Big|_0^{\infty} = \frac{A^2}{\beta} \quad \Rightarrow A = \sqrt{\beta}$$

$$\Rightarrow \quad \psi(x) = \begin{cases} \sqrt{m\lambda/\hbar^2}\; e^{-\frac{m\lambda x}{\hbar^2}} & x>0 \\ \sqrt{\frac{m\lambda}{\hbar^2}}\; e^{\frac{m\lambda x}{\hbar^2}} & x<0 \end{cases}$$

c) $E>0 \;(\Rightarrow E>V)$

$$\psi(x) = \begin{cases} e^{ikx} + R(E)\, e^{-ikx} & x<0 \\ T(E)\, e^{ikx} & x>0 \end{cases}$$

(I) $\psi(0^+) = \psi(0^-) \Rightarrow 1 + R(E) = T(E)$

(II) $\psi'(0^+) - \psi'(0^-) = -2\frac{m\lambda}{\hbar^2}\psi(0)$

$\rightarrow \; ik \underbrace{T(E)}_{=1+R(E)} - (ik - ikR(E)) = -\frac{2m\lambda}{\hbar^2}(1+R(E))$

$\rightarrow \; (\cancel{1}+R-\cancel{1}+R)\, ik = -\frac{2m\lambda}{\hbar^2}(1+R)$

$2Rik = -\frac{2m\lambda}{\hbar^2}(1+R)$

$\Rightarrow \; R(E) = \frac{-2m\lambda/\hbar^2}{2ik + \frac{2m\lambda}{\hbar^2}} = \frac{-1}{1+\frac{ik\hbar^2}{m\lambda}}$

$r(E) = |R(E)|^2 = \frac{1}{1+\frac{k^2\hbar^2}{m\lambda^2}} = \frac{1}{1+\frac{2\hbar^2 E}{m\lambda^2}}$

$T(E) = 1 + R(E) = \frac{\frac{ik\hbar^2}{m\lambda}}{1+\frac{ik\hbar^2}{m\lambda}}$

$t(E) = |T(E)|^2 = \frac{k^2\hbar^4/m^2\lambda^2}{1+\frac{k^2\hbar^4}{m^2\lambda^2}}$

$= \frac{\frac{2\hbar^2 E}{m\lambda^2}}{1+\frac{2\hbar^2 E}{m\lambda^2}}$

Nr. 3: $\quad V(x) = -\lambda\delta(x) \quad ; \quad \psi^{(var)}(x,\beta) = e^{-\frac{\beta^2 x^2}{2}}$

$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \lambda\delta(x)$

$E^{(var)}(\beta) = \frac{\langle \psi^{(var)} | \hat{H} | \psi^{(var)} \rangle}{\langle \psi^{(var)} | \psi^{(var)} \rangle}$

i) $\langle \psi^{(var)} | \psi^{(var)} \rangle = \int_{-\infty}^{\infty} dx\, e^{-\beta^2 x^2} = \sqrt{\frac{\pi}{\beta^2}} = \frac{\sqrt{\pi}}{\beta}$

(ii) $\langle \hat{H} \rangle_{var} = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx\, e^{-\frac{\beta^2 x^2}{2}} \frac{\partial^2}{\partial x^2} e^{-\frac{\beta^2 x^2}{2}} - \lambda \int_{-\infty}^{\infty} dx\, \delta(x) e^{-\beta^2 x^2}$

$= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx\, e^{-\frac{\beta^2 x^2}{2}} (-\beta^2 + \beta^4 x^2) e^{-\frac{\beta^2 x^2}{2}} - \lambda$

$= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx\, (-\beta^2) e^{-\beta^2 x^2} - \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx\, \beta^4 x^2 e^{-\beta^2 x^2} - \lambda$

$= +\frac{\hbar^2}{2m} \beta \sqrt{\pi} - \frac{\hbar^2}{2m} \beta^4 \cdot \frac{1}{2} \frac{\sqrt{\pi}}{\beta^3} - \lambda$

$= \frac{\hbar^2 \beta}{4m} \sqrt{\pi} - \lambda$

$\Rightarrow E^{(var)} = \frac{\hbar^2 \beta^2}{4m} - \frac{\lambda \beta}{\sqrt{\pi}}$

$\frac{dE}{d\beta} \overset{!}{=} 0 = \frac{\hbar^2 \beta}{2m} - \frac{\lambda}{\sqrt{\pi}} \qquad \Rightarrow \beta = \frac{2m\lambda}{\hbar^2 \sqrt{\pi}}$

$\Rightarrow E_{min} = -\frac{m\lambda^2}{\pi \hbar^2}$

Nr. 4:

3-dim. - Osz.

$\hat{H} = \frac{\vec{p}^2}{2m} + \frac{1}{2} m\omega^2 r^2$

a) $\vec{p}^2 = p_x^2 + p_y^2 + p_z^2 \quad ; \quad r^2 = x^2 + y^2 + z^2$

$\Rightarrow \hat{H} = \frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{p_y^2}{2m} \frac{m\omega^2 y^2}{2} + \frac{p_z^2}{2m} \frac{m\omega^2 z^2}{2}$

$=: \hat{H}_x + \hat{H}_y + \hat{H}_z$

$\psi_{n_1 n_2 n_3}(x,y,z) \Rightarrow \hat{H} \psi_{n_1 n_2 n_3} = E \psi_{n_1 n_2 n_3}$

$\psi_{n_1 n_2 n_3}(x,y,z) = \psi_{n_1}(x) \cdot \psi_{n_2}(y) \cdot \psi_{n_3}(z)$

$(\hat{H}_x + \hat{H}_y + \hat{H}_z) \psi_{n_1}(x) \psi_{n_2}(y) \psi_{n_3}(z) = (E_x + E_y + E_z) \psi_{n_1} \psi_{n_2} \psi_{n_3}$

$\hookrightarrow$ $\hat{H}_i$ wirkt nur auf ψ_{n_i}!

wobei $E_i = \hbar\omega(n_i + \frac{1}{2})$

$\Rightarrow E = \hbar\omega(n_1+n_2+n_3+\frac{3}{2}) = \hbar\omega(\tilde{n}+\frac{3}{2})$

$\tilde{n} = 0,1,2,\dots$

(i) Grundzustand: $\tilde{n}=0$

$\rightarrow E = \frac{3}{2}\hbar\omega$

einzige Möglichkeit für $\tilde{n}=0$

keine Entartung! ($\rightarrow n_1=n_2=n_3=0$)

(ii) $\tilde{n}=1$

$E = \frac{5}{2}\hbar\omega$

3 Möglichkeiten: $(n_1,n_2,n_3) = (1,0,0),(0,1,0),(0,0,1)$

$\Rightarrow$ dreifache Entartung!

(iii) $\tilde{n}=2$

$E = \frac{7}{2}\hbar\omega$

$(n_1,n_2,n_3) = (2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)$

$\Rightarrow$ sechsfache Entartung!

(iv) $\tilde{n}=3$

$E = \frac{9}{2}\hbar\omega$

$(n_1,n_2,n_3) = (3,0,0),(0,3,0),(0,0,3),(2,1,0),(2,0,1),(1,2,0),(0,2,1),$
$(1,0,2),(0,1,2),(1,1,1)$

$\Rightarrow$ zehnfache Entartung!

b) $\xi = \sqrt{\frac{m\omega_0}{\hbar}}\, r \quad , \quad e_n = \frac{E_n}{\hbar\omega}$

$\Rightarrow$ DGL

$\psi_n(\xi,\theta,\varphi) = f_n(\xi)\,\xi^l e^{-\xi^2/2}\, Y_{lm}(\theta,\varphi)$

$\rightarrow f_n'' + \frac{1}{\xi}2(l+1)f_n' - 2\xi f_n' + (2e_n(l)-3-2l)f_n = 0$

Potenzreihenansatz: $f_n(\xi) = \sum_{k=0}^{\infty} a_k \xi^{2k}$

$\cdot\; f_n''(\xi) = \sum_{k=1}^{\infty} a_k\, 2k\cdot(2k-1)\,\xi^{2(k-1)}$

$= \sum_{k=0}^{\infty} a_{k+1}\, 2\cdot(k+1)(2k+1)\,\xi^{2k}$

• $\frac{1}{\xi} 2(L+1) f_n' = \sum_{k=1}^{\infty} 2k \cdot 2(L+1) a_k \xi^{2(k-1)}$

$= \sum_{k=0}^{\infty} 2(k+1) 2(L+1) a_{k+1} \xi^{2k}$

• $2\xi f_n' = \sum_{k=0}^{\infty} 4k a_k \xi^{2k}$

$\Rightarrow \sum_{k=0}^{\infty} \xi^{2k} \left[2(k+1)(2k+1) a_{k+1} + 4(k+1)(L+1) a_{k+1} - 4k a_k \right.$

$\left. + (2e_n(L) - 3 - 2L) a_k \right] = 0$

$\Rightarrow [\ldots] \overset{!}{=} 0$

$\Leftrightarrow a_{k+1} = - \frac{e_n(L) - 3/2 - L - 2k}{(k+1)(2k+1+2(L+1))} a_k$

$e_n(L) \overset{!}{=} 2n + L + 3/2$

$\Rightarrow a_{k+1} \sim (n-k) a_k$ ⇒ Reihe bricht bei $k=n$ ab

$\Rightarrow f_n(\xi) = \sum_{k=0}^{\infty} a_k \xi^{2k}$

⇒ Polynom $2n$-ten Grades !

• [$e_n(L)$ ⇒ n ist hier nicht Quantenzahl, ... nur Grad des Pol.]

$e_n(L) = 2n + L + \frac{3}{2}$

$E_n(L) = \hbar\omega\left((\underbrace{2n+L}_{\hat{n}}) + \frac{3}{2} \right) = \hbar\omega\left(\hat{n} + \frac{3}{2}\right)$

(i) $\hat{n} = 1$: $\hat{n} = 2n + L \Rightarrow n = 0, L = 1$

m-Entartung: $-L \le m \le L$ ⇒ Entartung: $2L+1$

⇒ $L = 1$ ⇒ dreifache Entartung

(ii) $\hat{n} = 2$: $n_1 = 0, L_1 = 2$ oder $n_2 = 1, L_2 = 0$

$g(\hat{n}) = g(2) = 2L_1 + 1 + 2L_2 + 1 = 6$ ⇒ sechsfache Entartung!

(iii) $\hat{n} = 3$: $n_1 = 0, L_1 = 3$ oder $n_2 = 1, L_2 = 1$

⇒ $g(3) = 10$ ⇒ zehnfache Entartung!

⇒ Entartungen wie in a) !