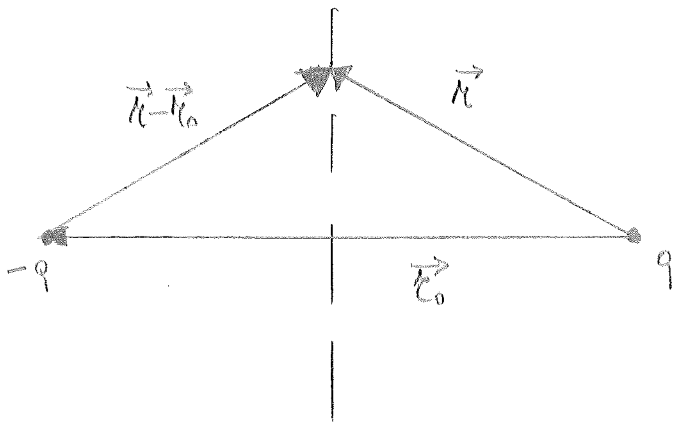
1. Aufgabe

a) (i) Near q : $\phi(\vec{r}) = \dfrac{q}{|\vec{r}|}$ in cgs units

(ii) Boundary condition on the conducting surface:

$$\phi(\vec{r}) = 0$$

b)



(i) The potential on the conducting surface is

$$\phi(\vec{r}) = \frac{q}{|\vec{r}|} + \frac{-q}{|\vec{r}-\vec{r}_0|} = 0 \quad \text{since } |\vec{r}| = |\vec{r}-\vec{r}_0|$$

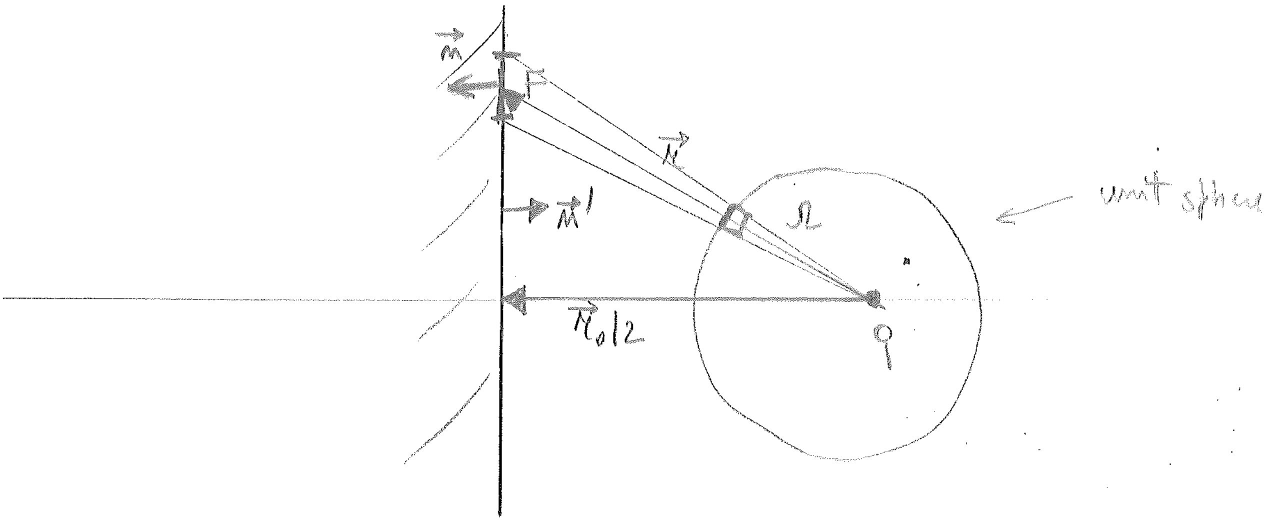
which is the boundary condition in a).

(ii) The electric field on the conducting surface is:

$$\vec{E}(\vec{r}) = -\vec{\nabla}\frac{q}{|\vec{r}|} - \vec{\nabla}\frac{-q}{|\vec{r}-\vec{r}_0|} = q\left(\frac{\vec{r}}{|\vec{r}|^3} - \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\right)$$

$$= q\,\frac{\vec{r}_0}{|\vec{r}|^3}$$ because $|\vec{r}| = |\vec{r}-\vec{r}_0|$; $\vec{E}$ is orthogonal to the conducting surface.

c)



$$\int_F d\vec f\cdot\vec E(\vec r) = \int_F df\,\vec n'\cdot\vec E(\vec r) = q\int_F df\,\frac{\vec n'\cdot\vec r_0}{|\vec r|^3} = -2q\underbrace{\int_F df\,\frac{\vec n\cdot\vec r}{|\vec r|^3}}_{\substack{=\Omega \\ \text{solid angle}}} = -2q\Omega$$

<u>obs</u> $\vec r = \dfrac{\vec r_0}{2} + \vec r_\perp$ with $\vec r_\perp\cdot\vec n' = 0$

hence $\vec n'\cdot\vec r_0 = 2\,\vec n'\cdot\vec r$

and $\vec n = -\vec n'$

<u>obs</u> We have not specified the convention on the direction of $d\vec f$. If someone defines $d\vec f = df\,\vec n$, this is also fine and the result is

$$\int_F d\vec f\cdot\vec E(\vec r) = 2q\Omega$$

2. Aufgabe

a) Maxwell equations in vacuum:

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0 \quad , \quad \vec{\nabla}\cdot\vec{B} = 0$$

From them it follows

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2\vec{E} = -\frac{1}{c}\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

i.e. $\quad \vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$

and analogously $\quad \vec{\nabla}^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$

We assume now:

$$\vec{E} = \vec{E}_0^{(+)} e^{ikz - i\omega t} + \vec{E}_0^{(-)} e^{-ikz - i\omega t}$$

(according to the picture the wave propagates along the z direction)

from $\vec{\nabla}\cdot\vec{E} = 0 = i\vec{k}\cdot\left[\vec{E}_0^{(+)} e^{ikz - i\omega t} - \vec{E}_0^{(-)} e^{-ikz - i\omega t}\right]$

with $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$ it follows $E_{0\,z}^{(+)} = E_{0\,z}^{(-)} = 0$, i.e. $E_z = 0$.

Analogously $B_z = 0$.

b) Region (I): $\vec{E}_I = \vec{E}^{(+)}_{0,I}\, e^{ikz-i\omega t} + \vec{E}^{(-)}_{0,I}\, e^{-ikz-i\omega t}$

Region (III): $\vec{E}_{III} = \vec{E}^{(+)}_{0,III}\, e^{ikz-i\omega t}$

The wave is linearly polarized in the x direction if $\vec{E}_0^{(\pm)} = \begin{pmatrix} E_0^{(\pm)} \\ 0 \\ 0 \end{pmatrix}$

The corresponding expression for the $\vec{B}$ field has: $\vec{B}_0^{(\pm)} = \begin{pmatrix} 0 \\ \pm E_0^{(\pm)} \\ 0 \end{pmatrix}$

This is because:

(i) $\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$ implies $-k^2 + \frac{\omega^2}{c^2} = 0$. i.e. $\omega = kc$

(ii) $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ implies $\pm \vec{k}\times\vec{E}_0^{(\pm)} = \frac{\omega}{c}\vec{B}_0^{(\pm)}$

c) The Maxwell equations in the medium are

$\vec{D} = \varepsilon\vec{E}$, $\vec{B} = \mu\vec{H} = \vec{H}$

$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} = \frac{4\pi}{c}\sigma\vec{E}$

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$

d) $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \frac{1}{c}\vec{\nabla}\times\frac{\partial \vec{H}}{\partial t} = 0$ implies

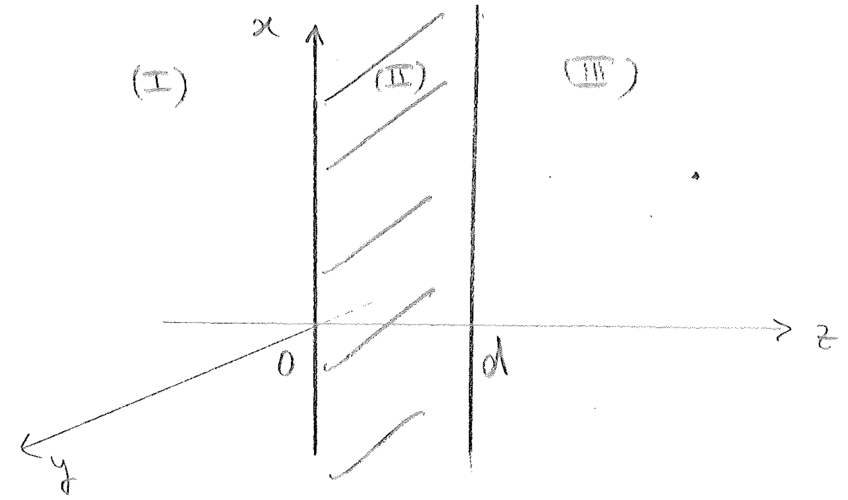
$-\vec{\nabla}^2\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\sigma\vec{E}\right) = 0$

$\left(\vec{\nabla}^2 - \frac{\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{4\pi\sigma}{c^2}\frac{\partial \vec{E}}{\partial t}$

We assume $\vec{E}_{II} = \vec{E}^{(+)}_{0,II}\, e^{ik_{II}z-i\omega t} + \vec{E}^{(-)}_{0,II}\, e^{-ik_{II}z-i\omega t}$. From this

$-k_{II}^2 + \frac{\varepsilon}{c^2}\omega^2 = -i\frac{4\pi\sigma}{c^2}\omega \implies n^2 = \frac{k_{II}^2c^2}{\omega^2} = \varepsilon + i\frac{4\pi\sigma}{\omega}$

e) We choose



The boundary conditions require $\vec{E}, \vec{B}$ and their derivatives to be continuous.

(i) <u>Conditions in $z=0$</u>

Continuity: $E_{0,I}^{(+)} + E_{0,I}^{(-)} = E_{0,II}^{(+)} + E_{0,II}^{(-)}$

Continuity of the derivative: $E_{0,I}^{(+)} - E_{0,I}^{(-)} = n\left(E_{0,II}^{(+)} - E_{0,II}^{(-)}\right)$; $n = \frac{k_{II}}{k} = \frac{k_{II}c}{\omega}$

(ii) <u>Conditions in $z=d$</u>

Continuity: $E_{0,II}^{(+)} e^{ik_{II}d} + E_{0,II}^{(-)} e^{-ik_{II}d} = E_{0,III}^{(+)} e^{ikd}$

Continuity of the derivative: $n\left(E_{0,II}^{(+)} e^{ik_{II}d} - E_{0,II}^{(-)} e^{-ik_{II}d}\right) = E_{0,III}^{(+)} e^{ikd}$

f)
$$R = \left| \frac{E_{0,I}^{(-)}}{E_{0,I}^{(+)}} \right|^2$$

$$T = \left| \frac{E_{0,III}^{(+)}}{E_{0,I}^{(+)}} \right|^2$$

g) We define $\rho \equiv \frac{E_{0,I}^{(-)}}{E_{0,I}^{(+)}}$; $\tau \equiv \frac{E_{0,III}^{(+)}}{E_{0,I}^{(+)}}$; $\alpha \equiv \frac{E_{0,II}^{(+)}}{E_{0,I}^{(+)}}$; $\beta \equiv \frac{E_{0,II}^{(-)}}{E_{0,I}^{(+)}}$

From the continuity conditions it follows:

$$\begin{cases} 1+\rho = \alpha+\beta \\ 1-\rho = n(\alpha-\beta) \\ \alpha e^{ik_{II}d} + \beta e^{-ik_{II}d} = \tau e^{ikd} \\ n(\alpha e^{ik_{II}d} - \beta e^{-ik_{II}d}) = \tau e^{ikd} \end{cases}$$

from the first two $\beta = \frac{n-1+(n+1)\rho}{2n}$; $\alpha = \frac{n+1+(n-1)\rho}{2n}$

from the third $\tau e^{ikd} = \frac{n+1+(n-1)\rho}{2n} e^{ik_{II}d} + \frac{n-1+(n+1)\rho}{2n} e^{-ik_{II}d}$

substituting everything in the last one:

$$n\left\{\frac{n+1+(n-1)\rho}{2n} e^{ik_{II}d} - \frac{n-1+(n+1)\rho}{2n} e^{-ik_{II}d}\right\} = \frac{n+1+(n-1)\rho}{2n} e^{ik_{II}d} + \frac{n-1+(n+1)\rho}{2n} e^{-ik_{II}d}$$

it follows

$$\left((n-1)^2 e^{ik_{II}d} - (n+1)^2 e^{-ik_{II}d}\right)\rho = (n^2-1)e^{-ik_{II}d} - (n^2-1)e^{ik_{II}d}$$

and

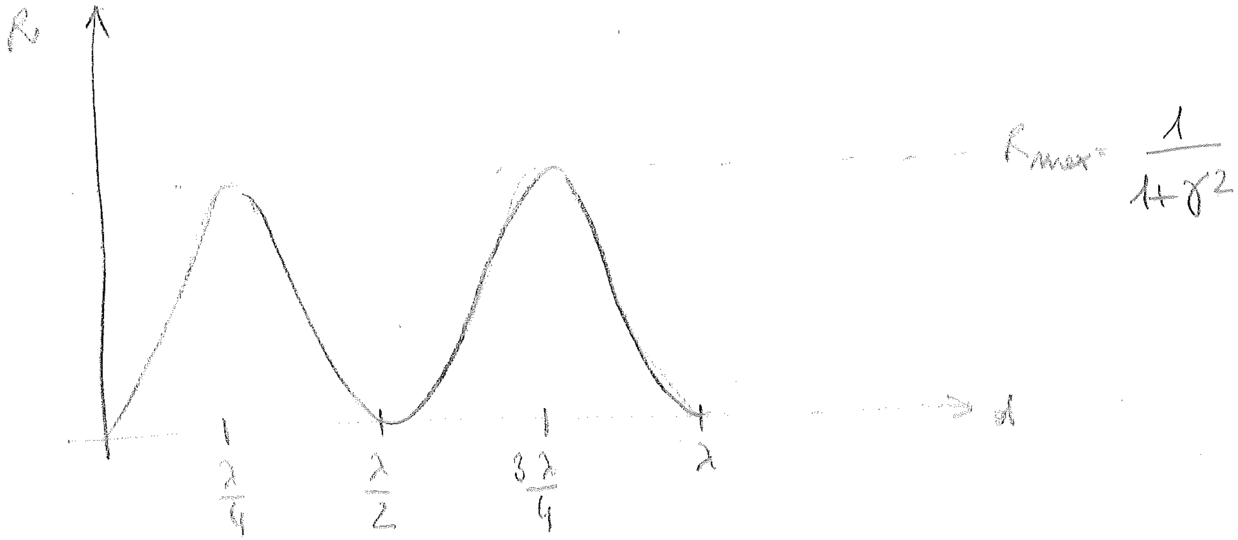
$$\rho = \frac{(n^2-1)e^{-ik_{II}d} - (n^2-1)e^{ik_{II}d}}{(n-1)^2 e^{ik_{II}d} - (n+1)^2 e^{-ik_{II}d}} \quad ; \quad R = |\rho|^2$$

$$= (n^2-1)\frac{-2i\sin k_{II}d}{-4n\cos k_{II}d + 2i(n^2+1)\sin k_{II}d}$$

For $\sigma = 0$, $n = \sqrt{\varepsilon}$, $k_{II} = nk = \sqrt{\varepsilon}\, k$

$$R = |\rho|^2 = \frac{(n^2-1)^2 \; 4 \sin^2 k_{II} d}{16 n^2 \underbrace{\cos^2 k_{II} d}_{1 - \sin^2 k_{II} d} + 4 (n^2+1)^2 \sin^2 k_{II} d} = \frac{\sin^2 nkd}{\sin^2 nkd + \gamma^2}$$

with $\gamma \equiv \dfrac{2n}{1-n^2}$



$$\lambda \equiv \frac{2\pi}{nk}$$

Hence $R = R_{max}$ for $d = \frac{\lambda}{4}(2j+1)$; i.e. $\lambda = \dfrac{4d}{2j+1}$ $j \in \mathbb{N}$

$R = 0$ (no reflection) for $d = \frac{\lambda}{2} j$; i.e. $\lambda = \dfrac{2d}{j}$ $j \in \mathbb{N}$