K1

Klausur

$\boxed{A1}$

$$V(x) = \begin{cases} 0 & \forall x \in (-a, a) \\ \infty & \forall x \notin (-a, a) \end{cases}$$

$$\left(-\frac{\hbar^2}{2m}\Delta + V(x)\right)\Psi(x) = E\,\Psi(x)$$

$$\boxed{\Psi(x) = A e^{ikx} + B e^{-ikx}} \quad (1) \qquad (2)$$

RB: $\Psi(a) = A e^{ika} + B e^{-ika} \stackrel{!}{=} 0$ $\quad (2) \qquad (2)$

$\Psi(-a) = A e^{-ika} + B e^{ika} \stackrel{!}{=} 0$

$$\det = e^{2ika} - e^{-2ika} = 2i \sin 2ka \stackrel{!}{=} 0 \qquad (2)$$

$\Rightarrow \quad 2ka = n\pi$

$$\boxed{k = \frac{n\pi}{2a}} \quad (3) \qquad (2)$$

(2): $\quad B = -A e^{2ika} \stackrel{(3)}{=} -A e^{in\pi} = A e^{i(n+1)\pi} \quad \curvearrowright (1) \Rightarrow$

$$\boxed{\Psi(x) = A\left(e^{ikx} + e^{-ikx + i(n+1)\pi}\right)} \quad (4) \qquad (2)$$

Fallunterscheidung:

$n = 2\ell$

$\curvearrowright (4) \Rightarrow \quad \Psi(x) = A\left(e^{ikx} + e^{i(2\ell+1)\pi} e^{-ikx}\right)$

$$= 2iA \sin kx \stackrel{(3)}{=} 2iA \sin(\pi\ell x/a) = \Psi_{2\ell}(x) \quad (5) \qquad (2)$$

$k = \pi\ell/a$, Normierung: $\quad A = 1/\sqrt{a} \quad \Rightarrow$

K1

$$\boxed{\Psi_{2\ell}(x) = \frac{1}{\sqrt{a}} \sin(\pi \ell x / a)}$$ (2)

Parität: (−1)

$n = 2\ell + 1$ ~ (4) ⇒

$\Psi(x) = A\left(e^{ikx} + e^{i(2\ell+2)\pi} e^{-ikx}\right)$

$= 2A \cos kx$ (2)

$k = \frac{(2\ell+1)\pi}{2a}$

$\Psi_{2\ell+1}(x) = 2A \cos\left(\frac{(2\ell+1)\pi}{2a} x\right)$ (6) (2)

normiert:

$\Psi_{2\ell+1}(x) = \frac{1}{\sqrt{a}} \cos \frac{(2\ell+1)\pi x}{2a}$ (2) Parität (+1)

$E = \frac{\hbar^2 k^2}{2m}$

$E_{2\ell} = \hbar^2 \ell^2 \pi^2 / 2m a^2$, (2)

$E_{2\ell+1} = \hbar^2 (2\ell+1)^2 \pi^2 / 8 m a^2$ (2)

$$\boxed{E_0 = \hbar^2 \pi^2 / 8 m a^2}$$ (2)

26 Punkte

$$\left(-\frac{\hbar^2}{2m}\Delta + V(x)\right)\Psi = E\Psi$$

$$-\frac{\hbar^2}{2m}\Psi'' + \lambda\delta(x)\Psi(0) = E\Psi(x)$$

$$\Psi''(x) + \frac{2mE}{\hbar^2}\Psi(x) = \frac{2m\lambda}{\hbar^2}\delta(x)\Psi(0)$$

$$\int_{-\varepsilon}^{\varepsilon}\dots\,dx \Rightarrow \text{AB}$$

$$\boxed{\Psi'(\varepsilon) - \Psi'(-\varepsilon) = \frac{2m\lambda}{\hbar^2}\Psi(0)} \qquad (4)$$

1 $\Psi_1(x) = Ae^{ikx} + Be^{-ikx}$

RB $\Psi_1(-a) = 0 \Rightarrow Ae^{-ika} + Be^{ika} = 0$

$$\Rightarrow \boxed{B = -Ae^{-2ika}} \Rightarrow$$

$$\boxed{\Psi_1(x)} = A\left(e^{ikx} - e^{-2ika}e^{-ikx}\right)$$

$$= Ae^{-ika}\left(e^{ikx+ika} - e^{-ika}e^{-ikx}\right)$$

$$= \boxed{2iAe^{-ika}\sin k(x+a)} \to \boxed{A\sin k(x+a)} \qquad (4)$$

$$\Psi_2(x) = Ce^{ikx} + De^{-ikx}$$

$$\Psi_2(a) = 0 \Rightarrow Ce^{ika} + De^{-ika} = 0 \Rightarrow \boxed{D = -Ce^{2ika}}$$

$$\Rightarrow \boxed{\Psi_2(x)} = C\left(e^{ikx} - e^{2ika}e^{-ikx}\right) = Ce^{ika}\left(e^{ikx-ika} - e^{ika-ikx}\right)$$

$$= \boxed{2iCe^{ika}\sin k(x-a)} \to \boxed{C\sin k(x-a)} \qquad (4)$$

AB bei $x=0$:

$\Psi_1(0) = \Psi_2(0)$ oder

$\boxed{A \sin ka = -C \sin ka}$ (*) (2)

$\Psi'(\varepsilon) - \Psi'(-\varepsilon) = \frac{2m\lambda}{\hbar^2}\Psi(0)$

$\Psi_2'(\varepsilon) - \Psi_1'(-\varepsilon) = \frac{2m\lambda}{\hbar^2}\Psi(0)$

$\boxed{kC\cos ka - kA\cos ka = \frac{2m\lambda}{\hbar^2} A \sin ka}$ (* *)

(2)

Falls $ka = n\pi$, dann folgt aus (* *): $\boxed{A = C}$ (*) ist leer!

$\Psi_1(x) = A \sin k(x+a) = A \sin(kx + n\pi) = A(-1)^n \sin kx$ (2)

$\Psi_2(x) = C \sin k(x-a) = A(-1)^n \sin kx$ (2)

$\Psi(x) = A(-1)^n \begin{cases} \sin kx & x<0 \\ \sin kx & x>0 \end{cases}$ (4)

$\equiv \Psi_{k=n\pi/a}(x)$ ist antisymmetrische Lösung, $A \to A^{a.s.}$

Falls $ka \neq n\pi$, dann folgt aus (*):

$\boxed{A = -C}$ $\curvearrowright$ (* *) $\Rightarrow$ $-2kA\cos ka = \frac{2m\lambda}{\hbar^2} A \sin ka$

oder $\boxed{\frac{k}{\kappa}\operatorname{ctg} ka = -1 \quad \text{mit} \quad \kappa \equiv m\lambda/\hbar^2}$ $\Rightarrow$ (2)

$\Psi(x) = \begin{cases} A \sin k(x+a) \\ -A \sin k(x-a) \end{cases}$ für $x \lessgtr 0 \to A_s \begin{cases} \sin k(x+a) \\ -\sin k(x-a) \end{cases}$

auch so: $\Psi(x) = A_s\{\Theta(-x)\sin k(x+a) - \Theta(x)\sin k(x-a)\}$

$\Psi_{k\neq n\pi/a}(x)$ ist symmetrisch! (4)

K2

Für symmetrische Lösung

$$\frac{k}{\varkappa} = \frac{k}{m\lambda/\hbar} = -\operatorname{tg} ka \qquad (4)$$

$$\lambda \to \infty \Rightarrow \operatorname{tg} ka = 0 \Rightarrow ka = n\pi \qquad (2)$$

$$\lambda \to 0 \Rightarrow \operatorname{tg} ka = \infty \Rightarrow ka = (2\ell+1)\pi/2 \qquad (2)$$

Normierung:

$$\int \psi^* \psi \, dx = 1 \Rightarrow$$

$$\int \psi_s^*(x)\, \psi_s(x) = 1 = A_s^2 \left[\int_{-a}^{0} \sin^2 k(x+a)\, dx + \int_{0}^{a} \sin^2 k(x-a)\, dx \right] =$$

$$= \frac{A_s^2}{k}\left(ka - \frac{1}{2}\sin 2ka\right)$$

$$\Rightarrow \quad A_s = \frac{1}{\sqrt{a - \frac{\sin 2ka}{2k}}} \qquad (2)$$

$$A_{as} = \frac{1}{\sqrt{a}} \qquad (2)$$

K3

Aufgabe 3

(a)
$$\Psi_0(x) = \frac{1}{\sqrt{\sqrt{\pi}\, x_0}}\, e^{-\frac{1}{2}(x/x_0)^2} \qquad (2)$$

mit
$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

(b)
$$\Delta A \, \Delta B \geq \frac{1}{2} \left| (\Psi, [A,B] \Psi) \right| \qquad (2)$$

(c)
$$\left(\frac{p^2}{2m} + \frac{m}{2} \omega^2 (x^2 + bx) \right) \Psi = E \Psi$$

$$\left(\frac{p^2}{2m} + \frac{m}{2} \omega^2 \left(x + \frac{b}{2}\right)^2 - \frac{m\omega^2 b^2}{8} \right) \Psi = E \Psi$$

$$x + \frac{b}{2} = x' \Rightarrow p' = p$$

$$\left(\frac{p'^2}{2m} + \frac{m}{2} \omega^2 x'^2 \right) \Psi(x') = E_n \Psi(x')$$

$$\Psi(x') \to \Psi_n(x') = \Psi_n(x + b/2)$$

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right) \qquad (8)$$

$$\Rightarrow E = \hbar\omega \left(n + \frac{1}{2}\right) - m\omega^2 b^2/8$$