1 a) $\frac{d\langle A\rangle}{dt} = \frac{d}{dt}\langle\Psi|A|\Psi\rangle =$

$$= \left(\frac{\partial\Psi}{\partial t}, A\Psi\right) + \underbrace{\left(\Psi, \frac{\partial A}{\partial t}\Psi\right)}_{=0} + \left(\Psi, A\frac{\partial\Psi}{\partial t}\right) =$$

$$\overset{i\hbar\frac{\partial\Psi}{\partial t} = H\Psi}{=} \left(H\Psi\cdot\frac{1}{i\hbar}, A\Psi\right) + \left(\Psi, A\cdot\frac{1}{i\hbar}H\Psi\right) \overset{H \text{ hermitesch}}{=}$$

$$= \frac{i}{\hbar}(\Psi, HA\Psi) - \frac{i}{\hbar}(\Psi, AH\Psi) =$$

$$= \frac{i}{\hbar}(\Psi, \underbrace{[H,A]}_{=0}\Psi) = 0$$

b) $(A\Psi, \Psi) = (a\Psi, \Psi) = a^*(\Psi, \Psi)$

$\parallel$

$(\Psi, A\Psi) = (\Psi, a\Psi) = a(\Psi, \Psi)$

$\Rightarrow a = a^*$, d.h. $a \in \mathbb{R}$

c) $\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\vec{S}_1\vec{S}_2 =$

$= \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2S_{1x}S_{2x} + 2S_{1y}S_{2y} + 2S_{1z}S_{2z} = *$

NR: $S_{1+}S_{2-} + S_{2+}S_{1-} =$

$= (S_{1x} + iS_{1y})(S_{2x} - iS_{2y}) + (S_{2x} + iS_{2y})(S_{1x} - iS_{1y})$

$= S_{1x}S_{2x} + S_{1y}S_{2y} + iS_{1y}S_{2x} - iS_{2y}S_{1x}$

$+ S_{1x}S_{2x} + S_{1y}S_{2y} + iS_{1x}S_{2y} - iS_{1y}S_{2x} =$

$= 2(S_{1x}S_{2x} + S_{1y}S_{2y})$

$* = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{2+}S_{1-}$

Vorfaktor egal.

$$\vec{S}^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) =$$

$$= (\vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{2+}S_{1-})(\dots) =$$

$$= \frac{3}{4}\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + \frac{3}{4}\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) +$$

$$+ 2\cdot\hbar^2\cdot\frac{1}{2}\cdot(-\frac{1}{2})(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) +$$

$$+ 0 + \hbar^2|\uparrow\downarrow\rangle + \hbar^2|\downarrow\uparrow\rangle + 0 =$$

$$= 2\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$\Rightarrow$ Gesamtspin 1

d) $|\psi\rangle_H = e^{iHt/\hbar}|\psi\rangle_S$

$$A_H = e^{iHt/\hbar}A_S e^{-iHt/\hbar}$$

$$A_S|\psi\rangle_S = a|\psi\rangle_S$$

$$A_H|\psi\rangle_H = e^{iHt/\hbar}A_S \cancel{e^{-iHt/\hbar}}\,\cancel{e^{iHt/\hbar}}|\psi\rangle_S =$$

$$= e^{iHt/\hbar}A_S|\psi\rangle_S = a e^{iHt/\hbar}|\psi_S\rangle =$$

$$= a|\psi\rangle_H$$

2a) $S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

b) $S_-\chi_{s,m_s} = \hbar\sqrt{\underbrace{s(s+1)}_{=2} - \underbrace{m_s(m_s-1)}_{=0}}\;\chi_{s,m_s-1}$

$$= \sqrt{2}\hbar\chi_{s,m_s-1}$$

$$\Rightarrow S_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

3a) $L_+|1,0\rangle \rightarrow |1,1\rangle$

b) $L_+|1,1\rangle \rightarrow 0$

$\Rightarrow L_-L_+|1,1\rangle \rightarrow 0$

c) $L^2L_+|2,1\rangle \rightarrow L^2|2,2\rangle \rightarrow |2,2\rangle$

d) $|1,2\rangle$ gibt es nicht! $\rightarrow 0$

e) $L_+L^2L_+|7,5\rangle \rightarrow L_+L^2|7,6\rangle$

$\Rightarrow L_+|7,6\rangle \rightarrow |7,7\rangle$

f) $L_-L^2|3,-3\rangle \rightarrow L_-|3,-3\rangle \rightarrow 0$

g) $L^2L_+|1,-1\rangle \rightarrow L^2|1,0\rangle \rightarrow |1,0\rangle$

h) $L_+|0,0\rangle \rightarrow 0$

4a) $H = +\frac{p^2}{2m} + V(r) =$

$$= -\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{2mr^2} + V(r)$$

b) ~~$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$~~ $H\Psi = E\Psi$

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)R(r) + \left(\frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right)R(r) = ER(r)$$

$$\left[-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r) - E\right]R(r) = 0$$

c) $\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) - E\right]u(r) = 0$

d) Normierbarkeit von Ψ:

$$\int_0^\infty r^2 dr\, |R(r)|^2 < \infty$$

$$\Rightarrow \int_0^\infty dr\, |u(r)|^2 < \infty$$

$\Rightarrow$ $u(r)$ muss schneller gegen 0 gehen als $r^{-1/2}$ (für $r \to \infty$).

außerdem: $u(0) = 0$, damit Ψ in 0 regulär ist

5a) $\Psi_{I}(x) \equiv 0$

$$\Psi_{II}(x) = A e^{ikx} + B e^{-ikx}, \quad k = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$\Psi_{III}(x) = C e^{qx} + D e^{-qx}, \quad q = \frac{\sqrt{-2mE}}{\hbar}$$

b) $\Psi_{II}(0) = 0$ (Stetigkeit in 0)

$\Psi_{II}(x_0) = \Psi_{III}(x_0)$ (" in x_0)

$\Psi_{II}'(x_0) = \Psi_{III}'(x_0)$ (Diff.barkeit in x_0)

$\Psi_{III}(x \to \infty) = 0$ (Normierbarkeit)

$\Rightarrow \Psi_{II}(x) = a \sin kx, \quad C = 0$

$\Rightarrow a \sin kx_0 = D e^{-qx_0}$

$a k \cos kx_0 = -D q e^{-qx_0}$

$$\Rightarrow \underbrace{\frac{1}{k} \tan kx_0 = -\frac{1}{q}}_{\text{Bed. f. Energie}}, \quad D = a \sin kx_0\, e^{qx_0}$$

c) $\int_0^\infty |\Psi(x)|^2 \, dx \stackrel{!}{=} 1$

$$\int_0^{x_0} dx\, a^2 \sin^2 kx + \int_{x_0}^\infty dx\, a^2 \sin^2 kx_0\, e^{2qx_0} e^{-2qx} =$$

$$= a^2 \frac{1}{2}\int_0^{x_0} dx\, (1 - \cos 2kx) +$$

$$+ a^2 \sin^2 kx_0\, e^{2qx_0} \int_{x_0}^\infty dx\, e^{-2qx} =$$

$$= a^2 \frac{1}{2}\left[x - \frac{1}{2k}\sin 2kx\right]_0^{x_0} +$$

$$+ a^2 \sin^2 kx_0\, e^{2qx_0} \left[-\frac{1}{2q} e^{-2qx}\right]_{x_0}^\infty$$

$$= \frac{1}{2} a^2 \left(x_0 - \frac{1}{2k}\sin 2kx_0\right) +$$

$$+ a^2 \sin^2 kx_0\, \cancel{e^{2qx_0}} \cdot \frac{1}{2q}\, \cancel{e^{-2qx_0}}$$

$$= a^2\left(\frac{x_0}{2} - \frac{1}{4k}\sin 2kx_0 + \frac{1}{2q}\sin^2 kx_0\right)$$

$$\Rightarrow a = \left(\frac{x_0}{2} - \frac{1}{4k}\sin 2kx_0 + \frac{1}{2q}\sin^2 kx_0\right)^{-1/2}$$

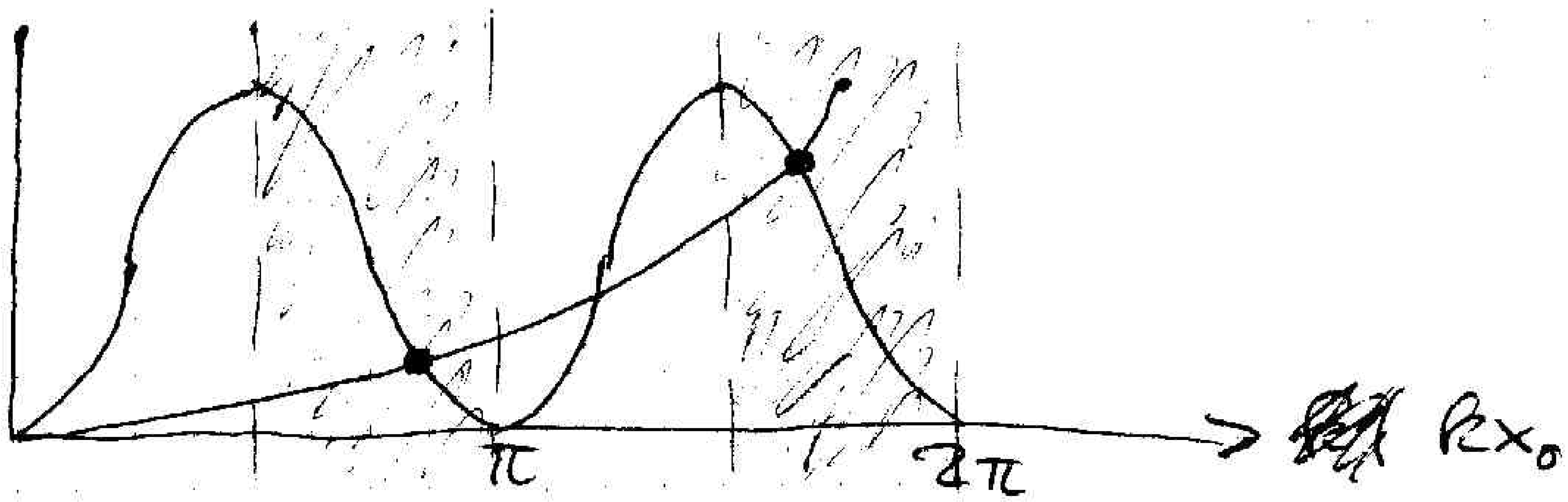
d) Im Bereich $x > 0$ entsprechen die Lösungen denen des Potenzialtopfs endlicher Tiefe, die an 0 verschwinden, d.h. ungerade sind.

e) $\tan kx_0 = -\frac{k}{q}$

$$\Rightarrow \frac{1}{\sin^2 kx_0} = \left(\frac{q}{k}\right)^2 + 1 = \frac{V_0}{E+V_0}$$

$$\Rightarrow \sin^2 kx_0 = (kx_0)^2 \cdot \frac{1}{V_0 x_0^2 \cdot \frac{2m}{\hbar^2}} =$$

$$= \left[\frac{kx_0}{\pi(n+3/4)}\right]^2$$



Wegen "–" in der 1. Zeile zählen nur die Schnittpunkte in den Bereichen $(\frac{\pi}{2}, \pi)$, $(\frac{3}{2}\pi, 2\pi)$, ... (schraffiert)

Man erhält also $n+1$ gebundene Zustände.

6a) $a^+ a = \frac{m\omega}{2\hbar}\left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right) =$

$= \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2} + \frac{i}{m\omega}\underbrace{(xp - px)}_{i\hbar}\right) =$

$= \frac{m\omega}{2\hbar}x^2 + \frac{p^2}{2\hbar m\omega} \cancel{+\, i} - \frac{1}{2} =$

$= \frac{1}{\hbar\omega} H_0 - \frac{1}{2} \quad \Rightarrow H_0 = \hbar\omega\left(a^+ a + \frac{1}{2}\right)$

$\Rightarrow H = \hbar\omega\left(a^+ a + \frac{1}{2}\right) + \alpha \frac{1}{2} m\omega^2 x^2$ siehe c!

b) $H a^+ |n\rangle^\circ = \hbar\omega\left(a^+ a a^+ + \frac{1}{2}a^+\right)|n\rangle^\circ =$

$= a^+\left[\hbar\omega\left(a a^+ + \frac{1}{2}\right)\right]|n^\circ\rangle =$

$= a^+\left[\hbar\omega\left(a^+ a + \frac{3}{2}\right)\right]|n^\circ\rangle = a^+ \hbar\omega\left(n + \frac{3}{2}\right)|n\rangle^\circ$

$\Rightarrow |n\rangle^\circ \sim |n+1\rangle^\circ$

$\langle n^\circ | a a^+ | n^\circ\rangle = \langle n^\circ | 1 + a^+ a | n^\circ\rangle =$

$= \langle n^\circ | 1 + n | n^\circ\rangle = (n+1)\underbrace{\langle n^\circ | n^\circ\rangle}_{=1}$

$\Rightarrow a^+ |n^\circ\rangle = \sqrt{n+1}\,|n+1^\circ\rangle$

$\underset{\displaystyle a\sqrt{n+1}\,|n+1^\circ\rangle}{\underset{\shortparallel}{a a^+}}|n^\circ\rangle = (1 + a^+ a)|n^\circ\rangle = n+1\,|n^\circ\rangle$

$\Rightarrow a|n+1\rangle^\circ = \sqrt{n+1}\,|n^\circ\rangle$

$\Rightarrow a|n^\circ\rangle = \sqrt{n}\,|n-1^\circ\rangle$

c) $H_1 = \frac{1}{2} m\omega^2 x^2$

$$a + a^+ = \sqrt{\frac{2m\omega}{\hbar}}\, x$$

$$\Rightarrow \frac{1}{2} m\omega^2 x^2 = \frac{\hbar\omega}{4}(a+a^+)^2$$

$$\Rightarrow H = \hbar\omega(a^+a + \tfrac{1}{2}) + \alpha \frac{\hbar\omega}{4}(a+a^+)^2$$

(gehört noch zu a!)

i) $$E_n^1 = \langle n^0 | H_1 | n^0 \rangle = \frac{\hbar\omega}{4}\langle n^0 | a^2 + a^{+2} + a^+a + aa^+ | n^0\rangle = \frac{\hbar\omega}{4}\langle n^0 | a^+a + aa^+ | n^0 \rangle = \frac{\hbar\omega}{4}\langle n^0 | 2a^+a + 1 | n^0\rangle = \frac{\hbar\omega}{4}(2n+1)$$

ii) $$|n^1\rangle = \sum_{m\neq n} |m^0\rangle \frac{\langle m^0 | H_1 | n^0\rangle}{E_n^0 - E_m^0} = \sum_{m\neq n} |m^0\rangle \frac{\langle m^0 | a^2 + a^{+2} | n^0 \rangle}{E_n^0 - E_m^0} = |n-2^0\rangle \frac{\sqrt{n(n-1)}}{+2\hbar\omega} + |n+2^0\rangle \frac{\sqrt{(n+1)(n+2)}}{-2\hbar\omega}$$

d) $$E_n = \hbar\omega'(n+\tfrac{1}{2}) = \hbar\omega\sqrt{1+\alpha}\,(n+\tfrac{1}{2}) = \hbar\omega(n+\tfrac{1}{2})(1+\tfrac{1}{2}\alpha + \ldots) = \underbrace{\hbar\omega(n+\tfrac{1}{2})}_{E_n^0} + \underbrace{\alpha\cdot\tfrac{1}{2}\hbar\omega(n+\tfrac{1}{2})}_{E_n^1} + \ldots$$