Differential Geometry
My study notes

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1 Chapter 1 Curves

1.1 Intro

1.2 Parametrized Curves

Our goal is to characterize certain subset in $\mathbb{R}^3$ that are, in a certain sense, one-dimensional and to which the methods of differential calculus can be applied. A natural way of defining such subsets is through differentiable functions.

Definition 1 (parametrized differentiable curve). *is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ to $\mathbb{R}^3$. The image set $\alpha(I)$ is called the trace of the curve.*

Definition 2 (tangent vector). *The vector $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the tangent vector.*

1.3 Regular curves; Arc length

Let $\alpha : I \to \mathbb{R}^3$, for each $t \in I$, where $\alpha'(t) \neq 0$, there's a well-defined straight line which contains the point $\alpha(t)$ and the vector $\alpha'(t)$. And the line is called the tangent line.

Definition 3 (singular point). *A point $t \in I$ is said to be a singular point of the curve $\alpha : I \to \mathbb{R}^3$, if $\alpha'(t) = 0$.*

Our focus is the curves without singular points.

Definition 4 (regular curve). *A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be regular if $\alpha'(t) \neq 0, \forall t \in I$.*

From now on, we shall automatically assume the functions are differentiable.

Definition 5 (arc length). *Given $t_0 \in I$, we say that the arc length of a parametirized curve $\alpha : I \to \mathbb{R}^3$ from the point t_0 is given by*

$$s(t) = \int_{t_0}^{t} dt$$

where

$$|\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

is the length of the tangent vector.

The arc length function is differentiable and $ds/dt = |\alpha'(t)|$

curves parametrized by arc length When the parameter t is the arc length s measured from some point, $ds/ds = 1 = |\alpha'(t)|$. And conversely, if $|\alpha'(t)| = 1$, we have

$$s(t) = t - t_0$$

i.e t is the arc length f α measure from some point t_0.

We shall pause to discuss a bit of the regular condition for curves. If $\alpha'(t_0) = 0$, then $s'(t_0) = 0$, so then $s(t)$ is not necessarily increasing, and solving for s in terms of t comes into doubt, i.e we no longer have the parametrization w.r.t the arc length. But $\alpha'(t) \neq 0$, we have:

Corollary 6. *Given a regular parametrized curve $\alpha : I \to \mathbb{R}^3$, it's possible to obtain a curve $\beta : J \to \mathbb{R}^3$ parametrized by arc length that has the same trace as α.*

Proof. Let $s = s(t) = \int_{t_0}^{t} |\alpha'(t)|dt$. Since $ds/dt = |\alpha'(t)| \neq 0$. The function $s = s(t)$ has a differentiable inverse $t = t(s), s \in J$. So now let $\beta = \alpha \circ t : J \to \mathbb{R}^3$. Then $\beta(J) = \alpha(I)$. And $|\beta'(s)| = |\alpha'(t)(dt/ds)| = 1$. This shows that β has the same trace as α, and β is parametrized by arc length. □

Let us now restrict to the curves parametrized by arc length for convention.

change of orientation Given the curve α parametrized by arc length $s \in (a, b)$, we may consider curve defined in $(-b, -a)$ by $\beta(-s) = \alpha(s)$ which has the same trace but is described in the opposite direction.

1.4 The vector product in $\mathbb{R}^3$

Definition 7 (vector product). *Let $u, v \in \mathbb{R}^3$ be two vectors, the vector product of u, v is the unique vector $u \wedge v$ characterized by*

$$(u \wedge v) \cdot w = \det(u, v, w) \;\; \forall w \in \mathbb{R}^3$$

Observe that $(u \wedge v) \cdot (u \wedge v) = |u \wedge v|^2 > 0$ This means that the determinant of $u, v, u \wedge v$ is positive, that is $\{u, vu \wedge v\}$ is a positive basis.

Fact 8. *$(u \wedge v) \cdot (x \wedge y) = \begin{vmatrix} u \cdot x & u \cdot y \\ v \cdot x & v \cdot y \end{vmatrix}$* And it follows that

$$|u \wedge v|^2 = u^2 v^2 (1 - \cos^2 \theta) = u^2 v^2 \sin^2 \theta$$

The vector product is not associative

Fact 9. *$(u \wedge v) \wedge w = (u \cdot w)v - (v \cdot w)u$* And if we let $u(t), v(t)$ be two differentiable functions, then $u(t) \wedge v(t)$ is also differentiable. And

Fact 10.

$$\frac{d}{dt}(u(t) \wedge v(t)) = \frac{du}{dt} \wedge v(t) + \frac{dv}{dt} \wedge u(t) \tag{1}$$

1.5 The local theory of Curves parametrized by arc length

Let $\alpha : I = (a, b) \to \mathbb{R}$ be a curve parametrized by arc length, then then tangent vector $\alpha'(s)$ has unit length. And the norm $|\alpha''(s)|$ measures the rate of change of the angle which neighboring tangents make with the tangent at s.

Definition 11 (curvature). *Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length s. The number $|\alpha''(s)| = k(s)$ is called the curvature of α at s.*

curvature under orientation Notice that by a change of orientation, the tangent vector changes its direction. i.e given $\beta(-s) = \alpha(s)$, we have that

$$\frac{d\beta}{d(-s)}(-s) = -\frac{d\alpha}{ds}(s)$$

Therefore, the curvature and $\alpha''(s)$ remain invariant under orientation.

signed curvature By requiring the basis $\{t(s), n(s)\}$ to have the same orientation as the basis $\{e_1, e_2\}$, The curvature k is then defined by

$$\frac{dt}{ds} = kn$$

and might be positive or negative.

Definition 12 (normal vector). *At points where $k(s) \neq 0$, a unit vector $n(s)$ in the direction $\alpha''(s)$ is well defined by the equation $\alpha''(s) = k(s)n(s)$, and $\alpha''(s)$ is normal to $\alpha'(s)$. Since $\langle \alpha'(s), \alpha'(s) \rangle = 1$ and by differentiate we obtain $\langle \alpha'(s), \alpha''(s) \rangle = 0$*

Definition 13 (osculating plane). *The plane determined by $\alpha'(s)$ and $n(s)$ is called the osculating plane at s.*

At points where $k(s) = 0$, the normal vector is not defined.

Definition 14 (singular point of k order). *A point $s \in I$ is a singular point of 1 order if $\alpha''(s) = 0$, and singular point of 0 order if $\alpha'(s) = 0$*

From now on we shall restrict to the curves that have no singular points of order 1. And denote the unit vector $\alpha'(s)$ to be $t(s)$, i.e $t(s) = \alpha'(s)$. Thus $t'(s) = k(s)n(s)$.

Definition 15 (binormal vector). *The unit vector $b(s) \equiv t(s) \wedge n(s)$ is normal to osculating plane.*

And since $b(s)$ is unit vector, $b(s)$ is normal to $b'(s)$. And observe that

$$b'(s) = t(s) \wedge n'(s) + t'(s) \wedge n(s) = t(s) \wedge n'(s)$$

therefore, $b'(s)$ is normal to $t(s)$. It follows that $b'(s)$ is parallel to $n(s)$. And we write

$$b'(s) = \tau(s)n(s)$$

Definition 16 (torsion). *For the $\tau(s)$ given above, we shall call it the torsion of α at s, when $\alpha''(s) \neq 0$.*

Corollary 17 ($\tau \equiv 0$). *If α is a plane curve, then the plane of the curve agrees with the osculating plane, hence $\tau \equiv 0$. Conversely, if $\tau \equiv 0$, and that $k \neq 0$ we have that $b(s) = b_0$. And $(\alpha(s)b_0)' = \alpha'(s)b_0 = 0$. Therefore $\alpha(s) \cdot b_0$ is some constant, that means the curve is normal to vector b_0.*

Note that the condition that $k \neq 0$ is essential here. The following exercise will give an example where τ can be defined to be identically zero and yet not a plane curve.

Exercise 10 Consider the map

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}) & t > 0 \\ (t, e^{-1/t^2}, 0) & t < 0 \\ (0, 0, 0) & t = 0 \end{cases}$$

1. The curve is differentiable. [check when t = 0]
2. α is regular for all t.

3. The curvature $k(t) \neq 0$ for $t \neq 0,\ t \neq \pm\sqrt{2/3}$
4. The limit of the osculating plane when $t \to 0, t > 0$ is the plane $y = 0$, and the limit of the osculating plane when $t \to 0, t < 0$ is the plane $z = 0$. So the binormal vector is discontinuous at $t = 0$. And this is why we excluded points where $k = 0$.
5. So then τ can be defined s.t $\tau \equiv 0$ but α is not a plane curve.

In contrast to the curvature, torsion might be positive or negative.

torsion under orientation Note that $b = t \wedge n$. It follows that $b'(s)$, the torsion, remain invariant under a change of orientation.

Definition 18 (Frenet trihedron). *To each value of parameter s, we have associated with 3 orthogonal unit vectors $t(s), n(s), b(s)$. The trihedron thus formed is referred to as the Frenet trihedron.*

Note that $t'(s) = kn,\ b'(s) = \tau n$. And

$$n'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s) = -\tau b - kt$$

Definition 19 (Frenet formula).

$$\begin{aligned} t' &= kn \\ n' &= -\tau b - kt \\ b' &= \tau n \end{aligned}$$

Recall in Corollary 6, we have that we can obtain an arc-length-parametrized curve via any other parametrization curve. And so we are allowed to extend all local concepts previously defined to regular curves with an arbitrary parameter. For example, we say that the curvature $k(t) : I \to \mathbb{R}^3$, at $t \in I$ is the curvature of a reparametrization $\beta : J \to \mathbb{R}^3$ by arc length at the corresponding point $s = s(t)$. And this is independent of the choice of β.

In application, it's convenient to have the explicit formulas for the geometry entities in terms of arbitrary parameter.[Will present in application 2]

Application 1: [torsion formula] The torsion τ of α is given by

$$\tau = \frac{-\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}$$

Proof. $\alpha'(s) = t, \alpha''(s) = kn, \alpha'''(s) = k'n + kn' = k'n + k(-kt - \tau b)$. [Please note that both $k(s)$ and $k'(s)$ are scalars!]

$$\begin{aligned} \alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s) &= t \wedge kn \cdot (k'n + k(-kt - \tau b)) \\ &= kb \cdot (k'n - k^2 t - k\tau b) \\ &= -k^2 \tau \end{aligned}$$

□

Application 2 (Exercise 12): geometry entities in terms of arbitrary parameter Let $\alpha : I \to \mathbb{R}^3$ be a regular curve (arbitrary parameter). And let $\beta : J \to \mathbb{R}^3$ be a reparametrization of $\alpha(I)$ by the acr length $s = s(t)$. Let $t = t(s)$ be the inverse function of s, and set $\alpha' = d\alpha/dt,\ \alpha'' = d^2\alpha/dt^2$. Then

1. $dt/ds = 1/|\alpha'(t)|$.
2. $d^2t/ds^2 = -\frac{\alpha'(t)}{|\alpha'(t)|^4} \cdot \alpha''$
3. The curvature of α at $t \in I$ is

$$k(t) = k(t(s)) = |\beta''(s)| = |(\alpha \circ t)''(s)| = \frac{|(\alpha'' \wedge \alpha')|}{|\alpha'|^3}$$

We may first calculate $(\alpha \circ t)'(s) = \alpha' \cdot dt/ds = \frac{\alpha'}{|\alpha'|}$, therefore

$$|(\alpha \circ t)''(s)| = |\alpha''(\alpha' \cdot \alpha') - \alpha'(\alpha'\alpha'')|/|\alpha'|^4$$

Recall that $a \wedge (b \wedge c) = (ac)b - (ab)c$. Therefore

$$\begin{aligned} &= |\alpha' \wedge (\alpha'' \wedge \alpha')|/|\alpha'|^4 \\ &= |\alpha'| \cdot |(\alpha'' \wedge \alpha')|/|\alpha'|^4 \\ &= \frac{|(\alpha'' \wedge \alpha')|}{|\alpha'|^3} \end{aligned}$$

4. The torsion of α at $t \in I$ is

$$\tau(t) = \tau(t(s)) =$$

Proof. Recall the torsion formula (1.5) given above,

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}$$

And here we shall replace $\alpha(s)$ with $\beta(s)$, and proceed with taking derivative w.r.t t.

$$\begin{aligned} \tau(t) &= \tau(t(s)) \\ &= -\frac{\beta'(s) \wedge \beta''(s) \cdot \beta'''(s)}{|k(s)|^2} \\ &= -\frac{\alpha' \frac{dt}{ds} \wedge (\alpha''(\frac{dt}{ds})^2 + \alpha' \frac{d^2t}{ds^2}) \cdot \beta'''(s)}{|k(s)|^2} \\ &= -\frac{\alpha' \frac{dt}{ds} \wedge (\alpha''(\frac{dt}{ds})^2) \cdot (\alpha'''(\frac{dt}{ds})^3 + \alpha'' \frac{d^2t}{ds^2} + \alpha'' \frac{dt}{ds}\frac{d^2t}{ds^2} + \alpha' \frac{d^3t}{ds^3})}{|k(s)|^2} \\ &= -\frac{\alpha' \wedge \alpha'' \cdot \alpha'''}{|k(s)|^2}(\frac{dt}{ds})^6 \\ &= -\frac{\alpha' \wedge \alpha'' \cdot \alpha'''}{|(\alpha'' \wedge \alpha')|^2} \end{aligned}$$

□

5. Recall the signed curvature defined at 1.5, and suppose that $\alpha(t) = (x(t), y(t))$. then

$$k(t) = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}$$

Proof. It's helpful to have some discussion about the signed curvature with arc length parameter first. Let's first take a look at $\beta(s) = (x(s), y(s))$. Then $\beta'(s) = (x'(s), y'(s))$, so then $n'(s) = (-y'(s), x'(s))$ in order to have $\{t(s), n(s)\}$ to have the same orientation as $\{e_1, e_2\}$. Therefore

$$t'(s) = kn(s),$$

□

Physically, we can think of a curve in $\mathbb{R}^3$ as being obtained from a straight line by bending(curvature) and twisting(torsion).

Conversely, the following theorem tells us that k and τ completely describes the local behavior of the curve.

Theorem 20 (Fundamental theorem of the local theory of curves). *Given differentiable function $k(s) > 0$ and $\tau(s)$, $s \in I$. There exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$, s.t s is the arc length, $k(s)$ is the curvature and $\tau(s)$ is the torsion of α. Moreover, any other curve $\bar{\alpha}$ satisfying the same condition, differs from a rigid motion; i.e there exists an orthogonal linear map ρ of $\mathbb{R}^3$ with positive determinant and a vector c, s.t $\bar{\alpha} = \rho\alpha + c$*

Before getting into the proof, it shall be useful to know some basic facts about the property of rigid motion.

Exercise 6 A translation by a vector $v \in \mathbb{R}^3$ is the map $A : \mathbb{R}^3 \to \mathbb{R}^3$ given by $A(p) = p + v$. A linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation if $\langle \rho u, \rho v\rangle = \langle u, v\rangle, \ \forall u, v \in \mathbb{R}^3$. A rigid motion is the result of composing a translation with an orthogonal transformation with positive determinant. (Because we expect the rigid motion to preserve orientation)

1. The norm of the vector and the angle between two vectors are invariant under a positive orthogonal transformation: Because $\langle \rho u, \rho u\rangle = \langle u, u\rangle$ similar argument works for angle.

2. The positive orthogonal transformation and the vector product communicate. i.e $\rho(u \wedge v) = \rho u \wedge \rho v$

   Proof. It suffice to show that the two vector has the same norm and that their angle is 0. □

3. The arc length, curvature, torsion of a parametrized curve are invariant under rigid motion.

   Proof. Let $\beta(t) = A \circ \rho\alpha(t)$. Then $\beta'(t) = \rho\alpha'(t), \ \ \beta''(t) = \rho\alpha''(t)$.

$$s_\beta(t) = \int_{t_0}^{t} |\beta'(t)|dt = \int_{t_0}^{t} |\rho\alpha'(t)|dt = \int_{t_0}^{t} |\alpha'(t)|dt = s_\alpha(t)$$

□

1.6 The local canonical form

One of the most effective methods of solving problems in geometry consists of finding a coordinate system which is adapted to the problem. In the study of local property of a curve, in the neighborhood of s, we have a natural coordinate system, namely the Frenet trihedron at s, it is therefore convenient to refer the curve to this trihedron.

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by acr length without singular point of order 1. We shall write the equations of the curve, in a neighborhood of s_0, using the trihedron $t(s_0), n(s_0), b(s_0)$ as a basis for $\mathbb{R}^3$. We may assume, WLOG, $s_0 = 0$ and consider Taylor expansion

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \alpha''(0)\frac{s^2}{2} + \alpha'''(0)\frac{s^3}{6} + o(s^3) \tag{2}$$

$$= \alpha(0) + st + k\frac{s^2}{2}n + (k'n - k^2 t - \tau k b) \tag{3}$$

$$= \alpha(0) + (s - \frac{k^2 s^3}{6})t + (\frac{k}{2}s^2 + \frac{s^3}{6}k')n - \tau k\frac{s^3}{6}b \tag{4}$$

Corollary 21 (local $y(s)$). *We see that $y(s) \geq 0$ for sufficiently small s, and $y(s) = 0$ iff $s = 0$.*

Corollary 22 (osculating plane). *at s is the limit position of the plane determined by the tangent line at s and the point $\alpha(s+h)$ when $h \to 0$*

Proof. Let's assume $s = 0$, thus the tangent line is constant $\cdot e_1$, every plane containing the tangent line is of the form $z = cy$ or $y = 0$. The plane $y = 0$ contains no points near $\alpha(0)$ so the plane must be of the form of $z = cy$. And for $z = cy$ to pass the point $\alpha(s+h)$ is

$$c = \frac{z}{y} =$$

□

2 Chapter 2 Regular Surface

2.1 Some useful Prerequisites

2.1.1 Continuity in $\mathbf{R}^n$

A (open) ball in $\mathbb{R}^n$ with center $p = (x_1^0, x_2^0, \ldots, x_n^0)$ and radius ϵ is the set in $\mathbb{R}^n$

$$B_\epsilon(p) = \{(x_1, x_2, \ldots, x_n) : (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + \ldots + (x_n - x_n^0)^2 < \epsilon^2\}$$

It is convenient to say that an open set in $\mathbb{R}^n$ containing a point p is a neighborhood of p.
A map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be continuous at p

$$F(B_\delta(p)) \subset B_\epsilon(p)$$

Definition 23. *Homeomorphism: A function f: $\mathcal{X} \to \mathcal{Y}$ between two topological spaces is homeomorphism if it satisifies the following properties:*

1. *f is a bijection*
2. *f is continuous*
3. *the inverse function f^{-1} is continuous*

Note: The third requirement f^{-1} is essential, for instance take $f : (0, 2\pi] \to \mathcal{S}^1$(the unit sphere in $\mathbb{R}^2$). then f is bijection and f is continuous, but f^{-1} is not continuous: since $f^{-1}((0,1)) = 0$, and at this point, it's not continous!

Definition 24. *Isomorphism: A linear map A is isomorphism if the matrix of A is invertible. So usually isomorphism is $\mathbb{R}^n$ to $\mathbb{R}^n$. (same dimension)*

Definition 25. *Diffeomorphism: F: $V \subset \mathbb{R}^n \to W \subset \mathbb{R}^n$, where V and W are open sets, s.t F has a differentiable inverse.*

2.1.2 Differentiability in $\mathbf{R}^n$

Recall the definition of the derivative of function $f : \mathbb{R} \to \mathbb{R}$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \quad x_o + h \in U$$

when f has derivatives at all points of a neighborhood of x_0, we can consider the second derivative $f''(x_0)$ of f at x_0. We say that f is differentiable at x_0 if f has continuous derivatives of all orders.

Now we extend the definition to function $f : \mathbb{R}^n \to \mathbb{R}$: if function f has continuous partial derivatives of all orders, then we say that f is differentiable.

Now we extend the definition to function $f : \mathbb{R}^n \to \mathbb{R}^m : f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots f_m(x_1, \ldots, x_n))$ if function f's components, i.e $f_i(x_1, \ldots, x_n),\ i \in [m]$ have continuous partial derivatives of all orders.

For the case $n = 1$, we obtain the notion of a differentiable curve in $\mathbb{R}^m$, further, we define the **tangent vector** to a map $\alpha : \mathbb{R} \to \mathbb{R}^m$ at $t_0 \in U$ is the vector in $\mathrm{R}^m : \alpha'(t_0) = (\alpha_1(t_0), \alpha_2(t_0), \ldots, \alpha_m(t_0))$

Example Given a vector $w \in \mathbb{R}^m$ and a point $x_0 \in U \subset \mathbb{R}^m$, we can always find a differentiable curve $\alpha : (-\epsilon, \epsilon) \to U$ with $\alpha(0) = p, \alpha'(0) = w$, simply by define $\alpha(t) = p_0 + wt,\ t \in (-\epsilon, \epsilon)$.

Definition 26 (The differential of differentiable map). *: Let F: $U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map, for each $p \in U$, we associate a linear map $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ which is called the differential of F at p, defined as follows: Let $w \in \mathbb{R}^n$, and let $\alpha : (-\epsilon, \epsilon) \to U$ be a differentiable curve s.t $\alpha(0) = p, \alpha'(0) = w$, then by the chain rule $\beta = F \circ \alpha$ is also differentiable, and we have:*

$$dF_p(w) = \beta'(0).$$

Proposition 27. *The above definition of dF_p doesn't depend on the choice of curve which passes through p with tangent vector w. And in fact, the map $dF_p(w)$: is linear.*

The proof is based on choosing the vector basis to deal with "vector".

Proof. Let us choose the canonical basis in $\mathbb{R}^n : e_1, \ldots, e_n$ and $\mathbb{R}^m : f_1, \ldots, f_m$,

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)) \cdot (e_1, e_2, \ldots, e_n)^\mathsf{T},\ t \in (-\epsilon, \epsilon)$$

$$\alpha'(0) = (\alpha_1'(t), \alpha_2'(t), \ldots, \alpha_n'(t)) \cdot (e_1, e_2, \ldots, e_n)^\mathsf{T},\ t \in (-\epsilon, \epsilon) = w$$

Since we've chose the vectoe basis, w's coordinate $(\alpha_1'(t), \alpha_2'(t), \ldots, \alpha_n'(t))$ is unique, and so doesn't depend on the choice of the curves. And

$$\beta(t) = F \circ \alpha(t) = (F_1(\alpha(t)), F_2(\alpha(t)), \ldots, F_m(\alpha(t))) \cdot (f_1, f_2, \ldots, f_m)^\mathsf{T}$$

$$\beta'(0) = \begin{pmatrix} \nabla F_1 \\ \nabla F_2 \\ \cdots \\ \nabla F_m \end{pmatrix} \cdot (\nabla \alpha) \cdot (f_1, f_2, \ldots, f_m)^\mathsf{T} \tag{5}$$

$$= \beta'(0) = \begin{pmatrix} \nabla F_1 \\ \nabla F_2 \\ \cdots \\ \nabla F_m \end{pmatrix} \cdot (w) \cdot (f_1, f_2, \ldots, f_m)^\mathsf{T} \tag{6}$$

$$= dF_p(w) \tag{7}$$

Therefore, it's easy to see that dF_p doesn't depend □

Example Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be (u, v) $\to$ (x(u, v), y(u, v), z(u, v)) Then the matrix of the linear map dF_p is given by :

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Proposition 28 (The chain rule for maps). *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $G : V \subset \mathbb{R}^m \to \mathbb{R}^k$ be two differentiable maps, where U and V are open sets such that $F(U) \subset V$ Then $G \circ F$ is also differentiable and*

$$d(G \circ F)_p = d(G)_{F(p)} \circ d(F)_p, \;\; p \in U$$

Proof. Let $w_1 \in \mathbb{R}^n$, $\alpha : (-\epsilon, \epsilon) \to U$, $\alpha(0) = p, \alpha'(0) = w_1$, $dF_p(w_1) = w_2 \in V$, easily observe that

$$dG_{F(p)}(w_2) = dG_{F(p)} \circ dF_p(w_1) = d(G \circ F)_p$$

□

This is also saying the matrix of the composite linear map is given by the product of the two Jacobian matrices.

An important property of a function defined on an open interval $f : (a, b) \to \mathbb{R}$ is that if $f'(0) \equiv 0$ indicates that $f \equiv c$. The following is the generalization to the high-dimension.

Definition 29 (connectness). *We say that an open set $U \subset \mathbb{R}^n$ is connect if given any two point $p, q \in U$, there exists a continuous map $\alpha : [a, b] \to U$, s.t $\alpha(a) = p$, $\alpha(b) = q$. This means that any two points of U can be jointed by a continuous curve in U.*

Proposition 30. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function defined on a connected open set U, and if $df_p : \mathbb{R}^n \to \mathbb{R} \equiv 0$ every point at $p \in U$, then f is constant on U.*

Proof. Hint: We need to use the result of one dimension function above, so we need to firstly construct function $\mathbb{R} \to \mathbb{R}$. And it's natural to think of constructing a function from an open interval to $\mathbb{R}^n$

And we start by proving the constantness of f locally and then use the connectness and to prove globally.

Let $p \in U$, and $B_\delta(p) \subset U$ be an open ball around p. Any point $q \in B_\delta(p)$ can be joined to p by the "radical" segment $\beta : [0, 1] \to U$, $\beta(t) = tq + (1 - t)p$. Since U is open, we can extend β to $(o - \epsilon, 1 + \epsilon)$. So $g = f \circ \beta : (o - \epsilon, 1 + \epsilon) \to \mathbb{R}$ satisfies $dg_t = (df \circ d\beta)_t = 0$, hence $g \equiv c$, i.e $f(\beta(0)) = f(\beta(1))$, or $f(p) = f(q)$. So f is constant on $B_\delta(p)$. Now we use connectness to prove globally.

Let r be any other arbitrary point in U, and by the connectness, there exists a continuous curve $\alpha : [a, b] \to U$, with $\alpha(a) = p, \alpha(b) = r$. By the similar proof above, for each $t \in [a, b]$, there exists an interval I_t, such that f is constant on I_t. And $[a, b] \subset \cup I_t$, and since $[a, b]$ is compact, we can find finite of I_t to cover $[a, b]$. such that f is constant on theses intervals. So f is constant on $[a, b]$, and since r is arbitrary, we conclude that f is constant in U. □

Theorem 31. *Inverse function theorem: Let F: $U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping and suppose that at $p \in U$, the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^n$ is isomorphism, then there exists a neighborhood V of p in U, and a neighborhood W of F(p) in $\mathbb{R}^n$, s.t F: $V \to$ W has an differentiable inverse F^{-1}: $W \to$ V.*

Note: Inverse function theorem is also saying that there's a neighborhood V of p and a neighborhood W of $F(p)$ s.t $F : V \to W$ is a diffeomorphism.

3 Regular Surface

3.1 prerequisites

3.2 Regular Surfaces; Inverse Images of Regular Values

In this section, we describe some criteria that are helpful in deciding whether a given subset in $\mathbb{R}^3$ is a regular surface or not.

Definition 32. *A subset $S \in \mathbb{R}^3$ is a regular surface: if for each $p \in S, \exists$ an open neighborhood V of $\mathbb{R}^3$ and a map $\boldsymbol{x} : U \subset \mathbb{R}^2 \to V \cap S$ of an open set U. s.t:*

1. *$\boldsymbol{x}$ is differentiable: if we write $\boldsymbol{x} = \boldsymbol{x}(x(u,v), y(u,v), z(u,v)), (u,v) \in U$ then x(u,v), y(u,v), z(u,v) have partial dirivatives of all orders in U.*
2. *$\boldsymbol{x}$ is homeomorphism: but since from 1. we know $\boldsymbol{x}$ is continuous already, so we will only need the continuity of $\boldsymbol{x}^{-1}$*
3. *(the regular condition):For each $p \in U$, the differential $dF_p : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one: i.e the matrix of the linear map $d\boldsymbol{x}_p$ is linearly independent.*

The mapping $\boldsymbol{x}$ is called a parameterization or a system of coordinates in p. The neighborhood $V \cap S$ of p in S is called a coordinate neighborhood.

Remark 33.

- *The regular surface we defined is some subset. NOT a map. This is different from the definition of curves in Chap1. And This is achieved by covering S with the traces of parameterization which satisfy condition 1,2,3.*
- *Condition 1 is natural. The one-to-oneness of conditional 2 is to prevent self-intersections. This is important if we want to define the tangent plane of point p. The continuity of $\boldsymbol{x}^{-1}$ is essential for proving some certain objects defined in terms of a parameterization does not depend on the parameterization but the set S itself.*
- *Condition 3 will guarantee the existence of the tangent plane of all points of S.*

Exercise 2,3

1. the set $\{x^2 + y^2 \leq 1\}$ is not a regular surface: Since you can't find a local diffeomorphism for the boundary point $(1,0)$.
2. the set $\{x^2 + y^2 < 1\}$ is a regular surface.

The following two propositions will be useful for deciding whether a subset is a regular surface or not, especially for the case of $z = f(x,y)$ and $f(x,y,z)$ =constant.

Proposition 34. *If $f : U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$. Then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x,y))$ for $(x,y) \in U$ is a regular surface.*

Proof. Just check the definition. □

Definition 35. *Critical point: Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function, a point p is said to be a critical point if dF_p is not a surjective mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called critical value of F. A point of $\mathbb{R}^m$ which is not a critical value of F is called a regular value of F.*

Example 1 If $f : \mathbb{R} \to \mathbb{R}$, then the critical value: $f'(0) = 0$.

Example 2 If $f : \mathbb{R}^3 \to \mathbb{R}$, then df_p is given by (f_x, f_y, f_z). So the critical points p are those with $f_x = f_y = f_z = 0$ at p. And $a \in f(U)$ is a regular value of $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ iff f_x, f_y, f_z doesn't vanish simultaneously at any point of the inverse image:

$$f^{-1}(a) = \{(x, y, z) \in U : f(x, y, z) = a\}$$

Observation 36. *What does it mean by dF_p is not a surjective mapping? Compute the Jacobian matrix of dF_p at point p, I claim that if $n < m$, then it's always non surjective, and when $n \geq m$, as long as the matrix is full rank, then it's surjective.*

But actually this observation may not be important, since we will only use the case for $f : \mathbb{R}^3 \to \mathbb{R}$.

Proposition 37. *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, and a is a regular value of f. Then $f^{-1}(a) = \{f(x, y, z) = a\}$ is a regular surface in $\mathbb{R}^3$.*

Proof. Take $p \in f^{-1}(a)$, since a is a regular value, we know that at least one of f_x, f_y, f_z is not zero at p, WLOG, assume that $f_z \neq 0$, since $f(x, y, z) = a$ at p and f is differentiable, we know from the implicit function theorem that $z = g(x, y)$ in a neighborhood of p, So locally around p, (x, y, z) can be written as a graph $(x, y, g(x, y))$ and by the Proposition(34) we know that it's a regular surface. $\square$

Example(Regular Surface f(x, y, z) = a) The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface.

However, it's worth noticing that the Proposition 37 is only a sufficient condition for a regular surface, that means, the inverse of a non regular value c can still be regular surface. The counterexample is given below.

Exercise 4 Let $f(x, y, z) = z^2$ then 0 is not a regular value, but $f^{-1}(0)$ is a regular surface.

Fact 38. *Notice that a graph can be always written in form of the inverse of regular value by letting $z = f(x, y)$ and define $F = f(x, y) - z$ then the graph is equivalent to $F^{-1}(0)$, and 0 is a regular value of F, since F_z is always non-zero.*

The regular surface is not necessarily connected!

Definition 39 (connected regular surface). *A regular surface S is said to be connected if any two of its points can be joined by a continuous curve in S.*

Example (unconnected regular surface) The hyperboloid of two sheets

$$-x^2 - y^2 + z^2 = 1$$

By Proposition 37, we know that the subset is a regular surface, but it is not connected. That is, we cannot find a continuous curve to connect every two points of the surface. Take $p = (x_1, y_1, z_1), q = (x_2, y_2, z_2) \in S$ with $z_1 > 0, z_2 < 0$, and suppose there exists a continuous curve $\alpha(t) = (x(t), y(t), z(t)),\ t \in [a, b]\ \alpha(a) = p, \alpha(b) = q$, and $z(a) = z_1 > 0, z(b) = z_2 < 0$, so by the continuity of $z(t)$, apply intermediate value theorem, we know that there exists $c \in (a, b)$ s.t $z(c) = 0$ and that $\alpha(c) \in S$, which is impossible.

We shall generalize the above result:

Corollary 40 (function on connected regular surface). *If $f : S \subset \mathbb{R}^3 \to \mathbb{R}$ is a non zero continuous function defined on a connected surface S, then f doesn't change the sign on S.*

Proposition 34 tells us that a graph is a regular surface, conversely the following proposition states that a regular surface is locally the graph of a differentiable function.

Proposition 41. *Let $S \subset \mathbb{R}^3$ be a regular surface, $p \in S$, then there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following form: $z = f(x, y), x = g(y, z), y = h(x, z)$*

Proof. Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$ be a parameterization of S in p, and $\boldsymbol{x} = (x(u, v), y(u, v), z(u, v))$, by the 3rd condition of the definition of regular surface, WLOG, assume that

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0$$

[So then we can use the inverse theorem function to recover (u, v) from (x, y). But it's better to be careful to write this idea down.]
Let us define $\pi(x, y, z) = (x, y)$ the projection function that maps points in $\mathbb{R}^3$ to the xy plane. So the function $\pi \circ \boldsymbol{x} : (u, v) \to (x, y)$ satisfies that the Jacobian matrix is not zero at p. And now we can apply the inverse function theorem: there exists neighborhood of $p : V_1$ and neighborhood of $\pi \circ \boldsymbol{x}(p) : V_2$, s.t $(\pi \circ \boldsymbol{x})^{-1} : V_2 \to V_1,\ \ (\pi \circ \boldsymbol{x})^{-1}(x, y) = (u(x, y), v(x, y))$ is also differentiable. Since $\boldsymbol{x}$ is homeomorphism, we know that π restricted on $\boldsymbol{x}(V_1) = V$ is one-to-one. And so we compose with $z = z(u, v) = z(u(x, y), v(x, y))$ to conclude that V is the graph of the differentiable function $z = z(u(x, y), v(x, y))$. □

The next position tells us that if we know S is a regular surface, and with a candidate paramterization $\boldsymbol{x}$ such that the 1st and 3rd condition holds, then we don't need to check the continuity of $\boldsymbol{x}^{-1}$

Proposition 42. *Let $p \in S$ where S is the regular surface, and $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ s.t the 1st, 3rd condition holds, and assume and $\boldsymbol{x}$ is one-to-one, then $\boldsymbol{x}^{-1}$ is continuous*

Proof. We use the result in Proposition 41 that $\pi \circ \boldsymbol{x}$ is diffeomorphism, so that restricted to $\boldsymbol{x}(V_1)$, $\boldsymbol{x}^{-1} = (\pi \circ \boldsymbol{x})^{-1} \circ \pi$ is continuous. □

Example The one-side cone sheet

$$Z = +\sqrt{x^2 + y^2}$$

is not a regular surface! Note that we've been given lots of conditions to verify regular surface, but it's still not straightforward to prove that a surface isn't regular. But we can invoke Proposition 41 to do this. Justify this yourself.

Exercise 3 the two-sheeted cone $\{x^2 + y^2 = z^2\}$ is not a regular surface, same argument.

3.3 Change of Parameters; Differentiable functions on surface

Before we talk about the differentiability of function defined on regular surface, it is necessary to show that when p belongs two coordinate neighborhoods, with parameter (u, v), (ξ, η). it is possible to pass from one of the these pairs of coordinates to the other by means of differentiable transformation.
The following proposition shows this is true.

Proposition 43 (change of parameters). *Let p be a point of a regular surface S, and let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$, $\boldsymbol{y} : V \subset \mathbb{R}^2 \to S$ be two parameterization of S s.t $p \in \boldsymbol{x}(U) \cap \boldsymbol{y}(V) = W$. Then "the change of coordinates": $h = \boldsymbol{x}^{-1} \circ \boldsymbol{y}$: $\boldsymbol{y}^{-1}(W) \to \boldsymbol{x}^{-1}(W)$ is a diffeomorphism. that is, h is differentiable and h^{-1} is also differentiable.*

In other words, if $\boldsymbol{x}, \boldsymbol{y}$ are given by:

$$\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \text{ for}(u,v) \in U$$

$$\boldsymbol{y}(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta)), \text{ for}(\xi,\eta) \in V$$

Then the change of coordinate h, is given by:

$$u = u(\xi,\eta), v = v(\xi,\eta), \text{ for}(\xi,\eta) \in \boldsymbol{y}^{-1}(W)$$

has the property that u and v have continuous partial derivatives of all orders. and the map h can be inverted, by:

$$\xi = \xi(u,v), \eta = \eta(u,v), \text{ for}(u,v) \in \boldsymbol{x}^{-1}(W)$$

where the function ξ, η also have continuous partial derivatives of all orders. Since

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} \cdot \frac{\partial(\xi,\eta)}{\partial(u,v)} = 1$$

this implies that the Jacobian determinants of both h and h^{-1} are nonzero everywhere.

https://math.stackexchange.com/questions/54669/proving-that-this-function-is-a-diffeomorphism

Proof. **Important note: When you try to prove a function is a diffeomorphism, you need to firstly prove that it's homeomorphism(bijective): three steps: homeomorphism, differentiability, differentiability of inverse.** So firstly, $\boldsymbol{x}^{-1} \circ \boldsymbol{y}$ is homeomorphism since both $\boldsymbol{x}^{-1}$ and $\boldsymbol{y}$ are homeomorphism.
The hard part is to deal with the differentiability of $\boldsymbol{x}^{-1}$. Since $\boldsymbol{x}^{-1}$ is a function from a regular surface, but we don't know what does it mean by differential of a regular surface yet, so we cannot bring in the results for functions defined on $\mathbb{R}^3$. Instead, we need to fatten the map $\boldsymbol{x}$. In a fancy language, you change perspective from looking at the surface S to looking at a tubular neighborhood of S.

The idea is to augment $\boldsymbol{x}^{-1}$ defined on S to be on the open set in $\mathbb{R}^3$ by using inverse function theorem: Let $r \in \boldsymbol{y}^{-1}(W)$ and let $q = h(r)$. WLOG assume:

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

We extend the map $\boldsymbol{x}$ to a map $F : U \times \mathbb{R} \to \mathbb{R}^3$ defined by :

$$F(u,v,t) = (x(u,v), y(u,v), z(u,v) + t), \ (u,v) \in U, \ t \in \mathbb{R}$$

Clearly, F is differential in i, v, t of all orders. And $F : U \times 0 = \boldsymbol{x}$. And the jacobian matrix of dF_q is given by:

$$\begin{vmatrix} \frac{\partial x}{u} & \frac{\partial x}{v} & 0 \\ \frac{\partial y}{u} & \frac{\partial y}{v} & 0 \\ \frac{\partial z}{u} & \frac{\partial z}{v} & 1 \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$$

So by the inverse function theorem, there exists a neighborhood M of $\boldsymbol{x}(q)$ in $\mathbb{R}^3$ s.t.F^{-1} exists, and is differentiable in M. By the continuity of $\boldsymbol{y}$, there exists a neighborhood N of r in V, s.t $\boldsymbol{y}(N) \subset M$. Notice that, restricted to N, $h|N = \pi_{uv} \circ F^{-1} \circ \boldsymbol{y}$ is differentiable by the chain rule of three differentiable function π_{uv}, F^{-1} and $\boldsymbol{y}$. So h is differentiable at r, since r is arbitrary, we conclude that h is differentiable on $\boldsymbol{y}^{-1}(W)$. By symmetry, we know that h^{-1} is also differentiable, so h is a diffeomorphism. $\square$

Observation 44. *We make use of two facts: 1. regular condition of $\boldsymbol{x}$ which guarantees the differentiability of $\boldsymbol{x}^{-1}$ 2. continuity of $\boldsymbol{x}^{-1}$.*

Now we are ready to discuss the differential of functions defined on regular surface.

Definition 45. *Let $f : V \subset S \to \mathbb{R}$ be a function defined on an open set of a regular surface. for $p \in V$, we say that f is differentiable at p if: for some parameterization $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$ with $p \in \boldsymbol{x}(U) \subset V$. the composition $f \circ \boldsymbol{x} : U \to \mathbb{R}$ is differentiable at $\boldsymbol{x}^{-1}(p)$. f is differentiable if it's differentiable for all $p \in V$.*

The definition is natural, but we need to show that it is well-defined (doesn't depend on the choice of the parameterization), which was showed by the proposition above.

Notation: We denote $f(u, v)$ to represent $f \circ \boldsymbol{x}$, and said that $f(u, v)$ is the expression of f in the system of coordinates of $\boldsymbol{x}$.

Example 1 Let a differentiable function $f : V \to \mathbb{R}$ where V is an open set in $\mathbb{R}^3$ which contains the regular surface S. Then the restriction of f to S is differentiable function on S, since for any $p \in S$, any parameterization $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$ satisfies $f \circ \boldsymbol{x}$ is differentiable.

- height function: for any unit vector v, define $h : S \to \mathbb{R}$, s.t $h(p) = v \cdot p$
- square distance: for a fixed point $p_0 \in \mathbb{R}^3$, define $d(p) : S \to \mathbb{R}$, s.t $d(p) = |p - p_0|^2$

Example 2 Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$, Then $\boldsymbol{x}^{-1} : \boldsymbol{x}(U) \to \mathbb{R}^2$ is differentiable. This allows that U and $\boldsymbol{x}(U)$ are diffeomorphic. (every regular surface is locally diffeomorphic to a plane)

Definition 46 (differentiability mapping between regular surfaces). *Let $f : V_1 \subset S_1 \to S_2$ of an open set V_1 from regular surface S_1 to another regular surface S_2. f is said to be differentiable at p if: for some parameterization*

$$\boldsymbol{x}_1 : U_1 \to S_1,\ \boldsymbol{x}_2 : U_2 \to S_2$$

with $p \in \boldsymbol{x}_1(U_1)$, $f(\boldsymbol{x}_1(U_1)) \subset \boldsymbol{x}_2(U_2)$ s.t the composition $\boldsymbol{x}_2^{-1} \circ f \circ \boldsymbol{x}_1$ is differentiable at $x_1^{-1}(p)$

The natural notion of equivalence associated with differentiability is the notion of diffeomorphism. Two regular surfaces S_1 and S_2 are diffeomorphic iff there exists a differentiable map $\phi : S_1 \to S_2$ and a differentiable inverse $\phi^{-1} : S_2 \to S_1$. In other words, from the point of view of differentiability, two diffeomorphic regular surfaces are indistinguishable.

Example Let $\boldsymbol{x} : U \to S$ be a parameterization of $p \in S$, then $\boldsymbol{x}^{-1} : \boldsymbol{x}(U) \to U$ is differentiable. Since any parameterization $\boldsymbol{y} : V \to S \supset \boldsymbol{x}(q)$ we have that $\boldsymbol{x}^{-1} \circ \boldsymbol{y}$ is differentiable.

Example Let a differentiable function $f : V \subset \mathbb{R}^3 \to \mathbb{R}^3$ where $S_1 \subset$ open set V and $f(S_1) \subset S_2$. Then the restriction of $f|S_1 : S_1 \to S_2$ is differentiable.

- Let S be symmetric relative to the xy plane, i.e if $(x, y, z) \in S$ then $(x, y, -z) \in S$ too. Then the map which takes point in S to its symmetric point is differentiable. Since it's the restriction of $\sigma(x, y, z) = (x, y, -z)$ on S.
- Let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$: be the rotation of angle $theta$ about the z axis. And let $S \subset \mathbb{R}^3$ be a regular surface invariant under such rotation. i.e if $p \in S$, then $R_{z,\theta}(p) \in S$. Then $R_{z,\theta}$ is differentiable.

Definition 47 (regular curve). *A regular curve is a subset $C \subset \mathbb{R}^3$ s.t for $\forall p \in C$, there exists a neighborhood V of p in $\mathbb{R}^3$ and a differentibale homeomorphism $\alpha : I \subset \mathbb{R} \to V \cap C$, s.t the differential $d\alpha_t$ is one-to-one for $t \in I$.*

This kind of definition is very useful in proving some property of the curve itself doesn't depend on the choice of the parameterization, like the arc length [Exercise 15].

Recall that we defined curve by paramterization in chapter 1, actually we can define surface in this way too.

Definition 48 (parameterized surface x). *$U \subset \mathbb{R}^2 \to \mathbb{R}$ is a differentiale map from an open set $V \subset \mathbb{R}^2$ to $\mathbb{R}^3$ The set $\boldsymbol{x}(U)$ is called the trace of $\boldsymbol{x}$, $\boldsymbol{x}$ is said to be regular if $d\boldsymbol{x}_p$ is one-to-one mapping for $\forall p \in U$. A point $p \in U$ where $d\boldsymbol{x}_p$ is not one-to-one is called a singular point.*

Fact 49. *A regular parameterized surface possibly has self-intersection in its trace.*

Proposition 50. *Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parameterized surface and let $q \in U$, there exists a neighborhood V of q in $\mathbb{R}^2$, s.t $\boldsymbol{x}(V) \subset \mathbb{R}^3$ is a regular surface.*

The proposition shows that we can extend the local concepts and property of differential geometry to a regular parameterized surface.

3.4 The tangent plane; The differential of a Map

In this section we shall show that the condition 3 for a regular surface guarantees that for every $p \in S$, the set of tangent vectors to the parameterized curves of S passing through p, constitutes a plane. (By a tangent vector to S at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a paramterized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.)

Proposition 51. *Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$ be a parameterization of a regular surface, and Let $q \in U$, the vector subspace of dimension 2*

$$d\boldsymbol{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vector to S at $\boldsymbol{x}(q)$.

Proof. Let $w \in d\boldsymbol{x}_q(\mathbb{R}^2) : w = d\boldsymbol{x}_q(v)$. we take a curve $\alpha : \alpha(t) = q + t \cdot v$. So $\alpha(0) = q, \alpha'(0) = v$. So the curve $\beta = \boldsymbol{x} \circ \alpha : \beta(0) = \boldsymbol{x}(q)$. And by the definition of $d\boldsymbol{x}_q(v)$: $w = \beta'(0)$, so w is a tangent vector to S.
Conversely, Let w be a tangent vector to S at $\boldsymbol{x}(q)$, i.e exist a curve $\alpha : (-\epsilon, \epsilon) \to S$ s.t. $\alpha(0) = \boldsymbol{x}(q), w = \alpha'(0)$. Then $\beta = \boldsymbol{x}^{-1} \circ \alpha$ satisfies: $\beta(0) = q$ and $(\boldsymbol{x} \circ \beta)'(0) = w$. So $w \in d\boldsymbol{x}_q$ ☐

By the proposition, the plane $d\boldsymbol{x}_q$ which passes through $\boldsymbol{x}(q) = p$ doesn't depend on the parameterization of $\boldsymbol{x}$. This plane is called the **tangent plane to** S **at** p, denoted by $T_p(S)$.
Now fix a parameterization of $\boldsymbol{x} : (u, v) \to S \subset \mathbb{R}^3$. For $\forall w \in T_p(S)$ there exists a curve α, s.t $\beta = \boldsymbol{x} \circ \alpha$, $\beta'(0) = w$. suppose $\alpha(t) = (u(t), v(t))$, $\boldsymbol{x} \circ \alpha = \boldsymbol{x}((u(t), v(t)))$. So

$$\frac{\partial \beta}{\partial t} = \begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}$$

So

$$\beta'(0) = \boldsymbol{x}_u \cdot u'(0) + \boldsymbol{x}_v \cdot v'(0) = w$$

So the choice of the parameterization $\boldsymbol{x}$ determines a basis $\boldsymbol{x}_u(q), \boldsymbol{x}_v(q)$ of $T_p(S)$, called the basis associated to $\boldsymbol{x}$. And w has coordinate $(u'(0), v'(0))$.

Let us get some exercises on the tangent plane of regular surface.

Exercise 1 If the regular surface is given by $S = \{f(x, y, z) = 0\}$ where 0 is the regular value of the function f, then the tangent plane of $p = (x_0, y_0, z_0) \in S$ is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{8}$$

Proof. Let $\alpha : (-\epsilon, \epsilon) \to S$, s.t $\alpha(0) = p$, $\alpha(t) = (x(t), y(t), z(t)) \in S$. then $f(x(t), y(t), z(t)) = 0$.

$$f_x(x_0, y_0, z_0)x'(0) + f_y(x_0, y_0, z_0)y'(0) + f_z(x_0, y_0, z_0)z'(0) = 0$$

And so we know that the normal vector of the tangent plane is $(f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0))$ which passes through point (x_0, y_0, z_0), so we have the desired equation. □

Continue Exercise 1 Determine the equation of the tangent plane of regular surface given by $x^2+y^2-z^2 = 1$ at the points $(x, y, 0)$, and show that they are parallel to the z axis.

Solution. $f_x = 2x, f_y = 2y, f_z = -2z$. So the tangent plane is given by

$$2x_0(x - x_0) + 2y_0(y - y_0) = 0$$

which is of course parallel to the z axis. ■

Exercise 2 If the regular surface is given by a graph of a differential function $S = \{(x, y, f(x, y))\}$, then the tangent plane at $p = (x_0, y_0, f(x_0, y_0)) \in S$ is given by

$$z = f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f(x_0, y_0) \tag{9}$$

With the notion of tangent plane, we can talk about the differential of the differentiable map between two surfaces:

Definition 52 (differentiable map between two surfaces). *Let $\phi : V \subset S_1 \to S_2$ be a differentiable map from an opens set V of regular surface S_1 to regular surface S_2. If $p \in V$, $\forall w \in T_p(S_1)$, exists curve $\alpha : (\epsilon, \epsilon) \to S_1$ s.t $\alpha'(0) = w, \alpha(0) = p$. And so $\beta = \phi \circ \alpha : (\epsilon, \epsilon) \to S_2$ satisfy $\beta(0) = \phi(p)$, $\beta'(0) \in T_{\phi_p}(S_2)$*

Proposition 53. *In the discussion above, for the given w, $\beta'(0)$ doesn't depend on the choice of α. And the map $d\phi_p : T_p(S_1) \to T_{\phi(p)}(S_2)$ defined by $d\phi_p(w) = \beta'(0)$ is linear.*

Proof. Let $\boldsymbol{x}_1 : U \subset \mathbb{R}^2 \to S_1 \subset \mathbb{R}^3 : (u_1, v_1) \to (x_1(u_1, v_1), y_1(u_1, v_1), z_1(u_1, v_1))$. $\boldsymbol{x}_2 : U \subset \mathbb{R}^2 \to S_2 \subset \mathbb{R}^3 : (u_2, v_2) \to (x_2(u_2, v_2), y_2(u_2, v_2), z_2(u_2, v_2))$. curve $\alpha : \alpha(t) = (u_1(t), v_1(t))$ s.t $(\boldsymbol{x}_1 \circ \alpha)(0) = p$, $(\boldsymbol{x}_1 \circ \alpha)'(0) = w \in T_p(S_1)$. And actually from the above discussion: $\forall w \in T_p(S_1)$

$$w = \boldsymbol{x}_{1u} \cdot u'(0) + \boldsymbol{x}_{1v} \cdot v'(0) = \begin{pmatrix} \boldsymbol{x}_{1u} & \boldsymbol{x}_{1v} \end{pmatrix} \cdot \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

And $\beta = \phi \circ \boldsymbol{x}_1 \circ \alpha = \boldsymbol{x}_2 \circ (\boldsymbol{x}_2^{-1} \circ \phi \circ \boldsymbol{x}_1 \circ \alpha)$ with $\beta(0) = \phi(p)$.

$$\frac{\partial \beta}{\partial t}|_{t=0} = \begin{pmatrix} \frac{\partial \boldsymbol{x}_2}{\partial u_2} & \frac{\partial \boldsymbol{x}_2}{\partial v_2} \end{pmatrix} \cdot \frac{\partial(\boldsymbol{x}_2^{-1} \circ \phi \circ \boldsymbol{x}_1 \circ \alpha)}{\partial t} = \begin{pmatrix} \boldsymbol{x}_{2u} & \boldsymbol{x}_{2v} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial(\boldsymbol{x}_2^{-1} \circ \phi \circ \boldsymbol{x}_1)}{\partial u_1} & \frac{\partial(\boldsymbol{x}_2^{-1} \circ \phi \circ \boldsymbol{x}_1)}{\partial v_1} \end{pmatrix} \cdot \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$$

So we can see that $\beta'(0)$ is linear in w. And of course doesn't depend on the choice of α. □

The map $d\phi_p$ is called the differential of ϕ at $p \in S_1$. In a similar way we can define the differential of function $f : U \subset S \to \mathbb{R}$ at $p \in U$ as a linear map $df_p : T_p(S) \to \mathbb{R}$

Example 1. Let $v \in \mathbb{R}^3$ be a unit vector and the height function $h : S \to \mathbb{R}$, $h(p) = v \cdot p$. Then Let $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$, $\alpha'(0) = w$ $dh_p(w) = v \cdot \alpha'(0) = w \cdot v$

Fact 54. *If V and W are two finite-dimensional vector spaces and a basis is defined for both spaces. Then every linear map from V to W can be represented by a matrix. i.e $f : V \to W$, $f(x) = Ax$.*

Corollary 55. *If the above linear map f is defined in $\mathbb{R}^n \to \mathbb{R}^m$, and restricted to regular surface S. then*

$$df_p(w) = A \cdot w = f(w)$$

Example 2. Let

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

Let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle θ about the z axis. Then $R_{z,\theta}$ restricted to S^2 is differentiable. And is linear.

$$(dR_{z,\theta})_p(w) = R_{z,\theta}(w)$$

We have seen that a regular surface is locally a plane, and extended the notion of calculus on regular surface, including differential. Now, what about inverse theorem? Recall that inverse theorem says that if a function whose differential is an isomorphism at p, then it's locally diffeomorphism. This happens for regular surface, too.

Definition 56 (local diffeomorphism). *A mapping $\phi : U \subset S_1 \to S_2$ is a local diffeomorphism at $p \in U$, if there exists a neighborhood in $V \subset U$ of p s.t ϕ restricted on V is a diffeomorphism onto an open set $\phi(V) \subset S_2$.*

Proposition 57. *If S_1 and S_2 are regular surfaces and $\phi : S_1 \to S_2$ is a differentiable mapping of an open set $U \subset S_1$ s.t the differential $d\phi_p$ of ϕ at $p \in U$ is an isomorphism, then ϕ is local diffeomorphism at p.*

Definition 58 (unit normal vector of Tangent Plane). *Given a point p in regular surface, there are two unit vectors that are orthogonal to the tangent plane at p. By fixing a parametrization $\boldsymbol{x} : U \subset S$, we can define the position unit normal vector*

$$N(q) = \frac{\boldsymbol{x}_u \wedge \boldsymbol{x}_v}{|\boldsymbol{x}_u \wedge \boldsymbol{x}_v|}$$

3.5 The First Fundamental Form; Area

Definition 59 (The First Fundamental Form). *The natural inner product of $\mathbb{R}^3 \supset S$ induced on each tangent plane: $T_p(S)$ of a regular surface S an inner product, to be denoted by $\langle , \rangle_p$: If $w_1, w_2 \in T_p(S)$, then $\langle w_1, w_2 \rangle_p$ is equal to the inner product of w_1 and w_2 as vectors in $\mathbb{R}^3$. So at each point $p \in S$, there corresponds a quadratic form:*

$$I_p(w) = T_p(S) \to \mathbb{R},\ I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0$$

And this is called the first fundamental form of the regular surface S at $p \in S$.

Geometrically, the first fundamental form allows us to make measurements on the surface. (lengths of curves, angles of tangent vector, areas of regions)

First fundamental form in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ associated to a parameterization $\boldsymbol{x}(u, v)$ at p. Since a tangent vector $w \in T_p(S)$ can be written as

$$w = \boldsymbol{x}_u u' + \boldsymbol{x}_v v'$$

we have

$$\begin{aligned} I_p(w) &= \langle \boldsymbol{x}_u, \boldsymbol{x}_u\rangle_p (u')^2 + 2\langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle u'v' + \langle \boldsymbol{x}_v, \boldsymbol{x}_v\rangle_p (v')^2 \\ &= E(u')^2 + 2Fu'v' + G(v')^2 \end{aligned}$$

where

$$\begin{aligned} E(u_0, v_0) &= \langle \boldsymbol{x}_u, \boldsymbol{x}_u\rangle_p \\ F(u_0, v_0) &= \langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle_p \\ G(u_0, v_0) &= \langle \boldsymbol{x}_v, \boldsymbol{x}_v\rangle_p \end{aligned}$$

are the coefficients of the first fundamental form of the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ of $T_p(S)$

Example 1. A coordinate system for a plane $P \subset \mathbb{R}^3$ passing through $p_0 = (x_0, y_0, z_0)$ and containing the orthonormal vectors $w_1 = (a_1, a_2, a_3)$ and $w_2 = (b_1, b_2, b_3)$ is given as follows:

$$\boldsymbol{x}(u, v) = p_0 + uw_1 + vw_2,\ (u, v) \in \mathbb{R}^2$$

So $\boldsymbol{x}_u = w_1,\ \boldsymbol{x}_v = w_2,$ E = 1, F = 0, G = 1.

Example 2. The right cylinder over the circle $x^2 + y^2 = 1$ admits the parameterization $\boldsymbol{x} : U \to \mathbb{R}^3$, where $\boldsymbol{x}(u, v) = (\cos u, \sin u, v),\ U = \{(u, v) \in \mathbb{R}^2;\ 0 < u < 2\pi, v \in \mathbb{R}\}.\ \boldsymbol{x}_u = (-\sin u, \cos u, 0),\ \boldsymbol{x}_v = (0, 0, 1).$ E = G = 1. F = 0.

As mentioned before, the importance of the first fundamental form I comes from the fact that knowing I we can treat metric questions on a regular surface without references to the ambient space $\mathbb{R}^3$.

Arc length Thus the arc length s of a parametrized curve $\alpha : I \to S$ is given by

$$s(t) = \int_0^t |\alpha'(t)|dt = \int_0^t \sqrt{I(\alpha'(t)}dt.$$

So if we put the curve in a regular surface and also with a local parameterization $\boldsymbol{x} : U \to S$ s.t $\boldsymbol{x}(U) \supset \alpha$. Then $\alpha(t) : t \to S$, s.t $\alpha = \boldsymbol{x} \circ \beta = \boldsymbol{x}(u(t), v(t))$ then $\alpha'(t) = \boldsymbol{x}_u u' + \boldsymbol{x}_v v'$. And

$$\begin{aligned} s(t) &= \int_0^t \sqrt{I(\alpha'(t))}dt \\ &= \int_0^t \sqrt{I(\boldsymbol{x}_u u' + \boldsymbol{x}_v v')}dt \\ &= \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}dt \\ &= \int_0^t \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2} \end{aligned}$$

So the "element" of arc length, ds of S (actually Riemannian metric)

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

Angle of two regular curves Two parameterized regular curves $\alpha : I \to S,\ \beta : I \to S$ interesct at $t = t_0$ is given by

$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(t_0)\rangle}{|\alpha'(t_0)||\beta'(t_0)|}$$

In particular, the angle ϕ of the coordinate curves of a parameterization $\boldsymbol{x}(u, v)$ is :

$$\cos\phi = \frac{\langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle}{|\boldsymbol{x}_u||\boldsymbol{x}_v|} = \frac{F}{\sqrt{EG}}$$

It follows that the coordinate curves are orthogonal iff F(u, v) = 0. for all (u, v). such a parameterization is called an orthogonal parameterization.

The area of regular surface Another metric problem can be treated by the first fundamental form is the computation of the area of a bounded region of a regular surface.

Definition 60 ((regular) domain). *A domain of S is an open and connected subset of S s.t its boundary is the image in S of a differentiable homeomorphism from a circle, and the homeomorphism is regular(i.e the differential is non-zero) except at a finite number of points.*

Definition 61 (region). *A region of S is the union of the domain and its boundary.*

A region is bounded if it's contained in a ball.

Let $bx : U \to S$ be a parametriczation and Q be a compact region in $\mathbb{R}^2$, and $Q \subset U$. Then $\boldsymbol{x}(Q) \subset S$ is a bounded region in S.
The function $|\boldsymbol{x}_u \wedge \boldsymbol{x}_v|$ defined in U, measures the area of the parallelogram generated by $\boldsymbol{x}_u, \boldsymbol{x}_v$. And now we are interested in the integral

$$I = \int\int_Q |\boldsymbol{x}_u \wedge \boldsymbol{x}_v| dudv \tag{10}$$

We shall first show that I doesn't depend on the choice of the parametrization. Assume $\bar{\boldsymbol{x}}$ is another parametrization

$$\begin{aligned} I &= \int\int_{\bar{Q}} |\bar{\boldsymbol{x}}_u \wedge \bar{\boldsymbol{x}}_v| d\bar{u}d\bar{v} \\ &= \int\int_{\bar{Q}} |\boldsymbol{x}_u \wedge \boldsymbol{x}_v| \Big| \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} \Big| d\bar{u}d\bar{v} \\ &= \int\int_{\bar{Q}} |\bar{\boldsymbol{x}}_u \wedge \bar{\boldsymbol{x}}_v| dudv \end{aligned}$$

Definition 62 (area of R). *Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parametrization $\boldsymbol{x} : U \to S$. The positive number*

$$\int\int_Q |\boldsymbol{x}_u \wedge \boldsymbol{x}_v| dudv = A(R), Q = \boldsymbol{x}^{-1}(R) \tag{11}$$

is called the area of R.

Fact 63.

$$|\boldsymbol{x}_u \wedge \boldsymbol{x}_v|^2 + \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle^2 = |\boldsymbol{x}_u|^2 |\boldsymbol{x}_v|^2 \tag{12}$$

Therefore $|\boldsymbol{x}_u \wedge \boldsymbol{x}_v| = \sqrt{EG - F^2}$

3.6 Orientation of surface

Since every point $p \in S$ has a tangent plane $T_p(S)$, the choice of an orientation of $T_p(S)$ induces an orientation in a neighborhood of p .
By fixing a parametrization $\boldsymbol{x}(u,v)$ of a neighborhood of p in S, we determine an orientation of the tangent plane $T_p(S)$, namely the orientation of the associated ordered basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$.
If p belong to coordinate neighborhood of another parametrization $\bar{\boldsymbol{x}}(\bar{u}, \bar{v})$, the new basis is expressed in terms of the first one by

$$\begin{pmatrix} \bar{\boldsymbol{x}}_{\bar{u}} \\ \bar{\boldsymbol{x}}_{\bar{v}} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{u}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_u \\ \boldsymbol{x}_v \end{pmatrix} \tag{13}$$

The basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ and $\{\tilde{\boldsymbol{x}}_{\tilde{u}}, \tilde{\boldsymbol{x}}_{\tilde{v}}\}$ determines the same orientation of $T_p(S)$ iff the Jacobian matrix is positive.

Definition 64 (Orientation). *The orientation of $T_P(S)$ is an* ***ordered*** *basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$. Suppose another parameterization of p, $\tilde{\boldsymbol{x}} : V \to S$. We say that the ordered basis $\{\tilde{\boldsymbol{x}_u}, \tilde{\boldsymbol{x}_v}\}$ determines the same orientation of $T_p(S)$ iff $\dfrac{\partial(u,v)}{\partial(\tilde{u},\tilde{v})} > 0$*

Definition 65 (orientable of a regular surface). *A regular surface S is said to be orientable if it is possible to cover it with a family of coordinate neighborhood in such a way that any point in S which belongs to two neighborhood of the familty, then the change of the coordinate has positive Jacobian.*

Example: Graph A surface which is the graph of a differentiable function is an orientable surface. In fact, any surface which can be covered by one coordinate neighborhood is orientable.

Corollary 66. *If a regular surface can be covered by two coordinate neighborhoods whose intersection is connected, then the surface is orientable.*

Proof. It suffice to show that every point p in the intersection, satisfies the positive Jacobian, but by the connectness we know that every Jacobian sign of the points are the same. □

Example: sphere The sphere is an orientable surface, since we can cover it with two coordinate neighborhoods whose intersection is connected.

Recall that given a system of coordinate $\boldsymbol{x}(u,v)$ at $p \in S$, we have a definite choice of unit normal vector N at p by the rule

$$N(p) = \frac{\boldsymbol{x}_u \wedge \boldsymbol{x}_v}{|\boldsymbol{x}_u \wedge \boldsymbol{x}_v|}(p)$$

Taking another system $\bar{\boldsymbol{x}}(\bar{u},\bar{v})$ we see that

$$\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}} = \boldsymbol{x}_u \wedge \boldsymbol{x}_v \frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}$$

Remember that the Jacobian matrix is a scalar, Therefore vector $\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}$ and vector $\boldsymbol{x}_u \wedge \boldsymbol{x}_v$ are parallel. And their normalized unit vector is equal up to a sign, which depends on the Jacobian matrix.

$$N(q) = \frac{\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}}{|\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}|} = \frac{\boldsymbol{x}_u \wedge \boldsymbol{x}_v}{|\boldsymbol{x}_u \wedge \boldsymbol{x}_v|} sgn(\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}) \tag{14}$$

Definition 67 (field of unit normal vectors). *which is a differentiable map $N : U \subset S \to \mathbb{R}^3$ defined on regular surface S, s.t associateing to each $p \in U$ a unit normal vector $N(q) \in \mathbb{R}^3$ to S at q.*

Proposition 68. *A regular surface $S \subset \mathbb{R}^3$ is orientable iff there exists a field of unit normal vectors $N : S \to \mathbb{R}^3$ on S.*

Proof. If S is orientable, then we can cover S with a family of coordinate neighborhood s.t the change of any two coordinate has a positive Jacobian. We define the unit normal vector by 14, such definition is well-defined. Because if we change the parametrization, by 14, along with the Jacobian is positive we know that the normal vector $N(u,v) = N(\bar{u},\bar{v})$. And we know that such given map is differentiable and the given vector is unit vector, so we complete the proof.

On the other hand, let $N : S \to \mathbb{R}^3$ be a field of unit normal vectors, and consider a family of connected coordinate neighborhood covering S, we know that for each point $p \in S$ with parametrization $\boldsymbol{x}(u,v)$ we can find unit normal vector $\boldsymbol{x}_u \wedge \boldsymbol{x}_v$, and such vector must be parallel to $N(p)$. And

$$< N(p), \frac{\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}}{|\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}|} >= f(p)$$

f(p) is either 1 or -1. And by the connectness we know that f(p) must be constant. Let's just suppose it to be 1. So then $N(p) = \frac{\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}}{|\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}|}$ exactly. Thus if the Jacobian matrix is negative then we have

$$N(1) = \frac{\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}}{|\bar{\boldsymbol{x}}_{\bar{u}} \wedge \bar{\boldsymbol{x}}_{\bar{v}}|} = -\frac{\boldsymbol{x}_u \wedge \boldsymbol{x}_v}{|\boldsymbol{x}_u \wedge \boldsymbol{x}_v|} = -N(q)$$

which is impossible, hence we are done. □

Proposition 69. *If a regular surface is given by the inverse of a differentiable function's regular value, i.e* $S = \{f(x, y, z) = a\}$ *where* a *is a regular value, then* S *is orientable.*

Proof. It's very easy to find such a field unit normal vector of this surface by

$$N(p) = (f_x, f_y, f_z)/\sqrt{f_x^2 + f_y^2 + f_z^2}$$

where f_x, f_y, f_z are all valued at p □

4 The Geometry of the Gauss Map

As a remark, we shall mention that the orientation is a global property for a regular surface, and actually for any surface, the surface is locally diffeomorphic to a plane, which is orientable.

An important counter example of non orientable surface—Mobius strip The parametrization is give by