第八章:多元函数微分学

性质

$$\begin{gathered}\text{连续可偏导} \Rightarrow \text{可微}\\ \text{可微} \Rightarrow \text{函数连续}\\ \text{可微} \Rightarrow \text{偏导存在}\\ \text{其余关系均不正确}\end{gathered} \tag{1}$$

复合函数二阶偏导数

$$\begin{aligned} & z=f(x,y), u=\varphi(x,y), v=\psi(x,y), \text{求} z=f(u,v) \text{对} x \text{的} 1 \text{和} 2 \text{阶偏导}\\ & \text{一阶：} \quad \frac{\partial z}{\partial x}=f_1^{'}\frac{\partial\varphi}{\partial x}+f_2^{'}\frac{\partial\psi}{\partial x}\\ & \text{二阶：} \quad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(f_1^{'})\frac{\partial\varphi}{\partial x}+f_1^{'}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial}{\partial x}(f_2^{'})\frac{\partial\psi}{\partial x}+f_2^{'}\frac{\partial^2\psi}{\partial x^2}\\ & \text{其中} \begin{cases} \frac{\partial}{\partial x}(f_1^{'}) & =(f_{11}^{''}\frac{\partial\varphi}{\partial x}+f_{12}^{''}\frac{\partial\psi}{\partial x})\\ \frac{\partial}{\partial x}(f_2^{'}) & =(f_{21}^{''}\frac{\partial\varphi}{\partial x}+f_{22}^{''}\frac{\partial\psi}{\partial x}) \end{cases} \end{aligned} \tag{2}$$

隐函数求导法

如果二元函数$F(x,y)$=0满足

- 函数$F(x,y)$在点(x_0,y_0)某邻域内有连续的偏导数
- $F(x_0,y_0)=0$
- $F_y'(x_0,y_0)\neq 0$

则方程$F(x,y)=0$在点(x_0,y_0)某邻域能唯一确定一个连续函数$y=y(x)$,它满足$y_0=y(x_0)$,并有连续的导数,且

$$y'=\frac{dy}{dx}=-\frac{F_x'}{F_y'} \tag{3}$$

推广:如果三元方程$F(x,y,z)=0$满足如下三个条件

- 函数$F(x,y,z)$在点(x_0,y_0,z_0)某邻域有连续偏导数
- $F(x_0,y_0,z_0)=0$
- $F_z'(x_0,y_0,z_0)\neq 0$

则方程$F(x,y,z)=0$在点(x_0,y_0,z_0)某邻域能唯一确定一个连续函数$z=z(x,y)$,它满足$z_0=z(x_0,y_0)$,并有连续的导数,且

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}, \qquad \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'} \tag{4}$$

$$n\text{个方程}, m\text{个变量组成方程组}(m>n)\text{能确定} n \text{个} m-n \text{元函数} \tag{5}$$

拉普拉斯方程

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\\ \text{在极坐标变换} x=r\cos\theta, y=r\sin\theta \text{下, 转化为}\\ \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0\end{gathered} \tag{6}$$

空间曲线定义& 性质

参数方程

$$
\text{设}l:\begin{cases}x=x(t)\\y=y(t)\\z=z(t)\end{cases}\text{为一空间曲线},M_0\text{为曲线上}t=t_0\text{的点}
$$

$$
\text{则}l\text{上}M_0\text{点的切线：}\quad\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)} \tag{7}
$$

$$
\vec{l}=(x'(t_0),y'(t_0),z'(t_0))\text{为}M_0\text{点处的切向量},\text{方向指向参数增加的方向}
$$

$$
\text{称过}M_0\text{点与切线垂直的平面为其法平面},\text{方程为}
$$

$$
x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)=0
$$

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> 可以把x看做x,y,z的参数,则切向量变为$(1,y'_x,z'_x)$

交面式方程

$$
S:\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases},M_0(x_0,y_0,z_0)\text{是}S\text{上一点}
$$

$$
\text{则两个曲面在}M_0\text{的法向量分别为}
$$

$$
\vec{n_1}=\{F'x,F'y,F'z\}|_{(x_0,y_0,z_0)},\vec{n_2}=\{G'x,G'y,G'z\}|_{(x_0,y_0,z_0)}
$$

$$
\text{切线的方向向量为}\vec{l}=\vec{n_1}\times\vec{n_2}=\begin{vmatrix}i&j&k\\F'x(M_0)&F'y(M_0)&F'z(M_0)\\G'x(M_0)&G'y(M_0)&G'z(M_0)\end{vmatrix}=\{a,b,c\} \tag{8}
$$

$$
\text{则切线方程}\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}
$$

$$
\text{法平面方程}a(x-x_0)+b(y-y_0)+c(z-z_0)=0
$$

曲面的切平面和法线

$$
\begin{gathered}
\text{设}S:F(x,y,z)=0\text{为一曲面},M_0(x_0,y_0,z_0)\text{为曲面上一点}\\
M_0\text{点的法向量}\vec{n}=(F'_x,F'_y,F'_z)\\
\text{切平面方程为}\quad(x-x_0)F'_x+(y-y_0)F'_y+(z-z_0)F'_z=0\\
\text{法线方程为}\quad\frac{x-x_0}{F'_x}=\frac{y-y_0}{F'_y}=\frac{z-z_0}{F'_z}
\end{gathered} \tag{9}
$$

多元函数泰勒公式

$$
\begin{aligned}
&f(x,y)\text{在}(x_0,y_0)\text{点处的}n\text{阶}Taylor\text{公式：}\\
f(x_0+\Delta x,y_0+\Delta y)=f(x_0,y_0)&+\frac{1}{1!}(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})f(x_0,y_0)\\
&+\frac{1}{2!}(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^2f(x_0,y_0)+\dots\\
&+\frac{1}{n!}(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^nf(x_0,y_0)\\
&+\frac{1}{(n+1)!}(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y})^{n+1}f(x_0+\theta\Delta x,y_0+\theta\Delta y)\\
&\theta\in(0,1)
\end{aligned} \tag{10}
$$

极值问题

驻点:

$$\text{同时使} f_x'(x,y)=0 \text{和} f_y'(x,y)=0 \text{成立的点称为驻点} \tag{11}$$

极值点:

$$\begin{gathered}
\text{必要条件：驻点(邻域连续二阶偏导)} \\
\text{充分条件：} f_x'(x_0,y_0)=0, f_y'(x_0,y_0)=0 \\
\& \quad f_{xx}''(x_0,y_0)=A, f_{xy}''(x_0,y_0)=B, f_{yy}''(x_0,y_0)=C \\
1)B^2-AC<0\text{时}, f(x,y)\text{在}(x_0,y_0)\text{取极值, 且当}A>0\text{时取极小值}, A<0\text{时取极大值} \\
2)B^2-AC>0\text{时}, (x_0,y_0)\text{不是}f(x,y)\text{的极值点} \\
3)B^2-AC=0\text{时}, f(x_0,y_0)\text{不一定取极值, 需另外讨论(一般用极值定义)}
\end{gathered} \tag{12}$$

条件极值:拉格朗日乘数法

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>
> 求函数 $z=f(x,y)$ 在条件 $\varphi(x,y)=0$ 下的最大值或最小值

$$\begin{gathered}
\text{构造辅助函数} F(x,y,\lambda)=f(x,y)+\lambda\varphi(x,y) \\
\text{然后求解方程组} \begin{cases} \frac{\partial F}{\partial x}=\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0 \\ \frac{\partial F}{\partial y}=\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0 \\ \frac{\partial F}{\partial \lambda}=\varphi(x,y)=0 \end{cases} \\
\text{所有满足此方程的解}(x,y,\lambda)\text{中}(x,y)\text{是}f(x,y)\text{在条件}\varphi(x,y)=0\text{下的可能的极值点} \\
\text{最后比较这些可能极值点(可能还有端点)的函数值求出最大最小值}
\end{gathered} \tag{13}$$

方向导数

方向余弦

$$\begin{gathered}
\vec{l}=(a,b,c) \\
\text{方向余弦为} \cos\alpha=\frac{a}{\sqrt{a^2+b^2+c^2}}, \cos\beta=\frac{b}{\sqrt{a^2+b^2+c^2}}, \cos\gamma=\frac{c}{\sqrt{a^2+b^2+c^2}}
\end{gathered} \tag{14}$$

方向导数

$$\begin{gathered}
\text{数量场} u=f(x,y,z) \text{在} P_0(x_0,y_0,z_0) \text{上可微, 则其沿任何方向} \vec{l}=(\cos\alpha,\cos\beta,\cos\gamma) \text{的方向导数均存在, 且} \\
\frac{\partial u}{\partial \vec{l}}|_{P_0}=f_x'(x_0,y_0,z_0)\cos\alpha+f_y'(x_0,y_0,z_0)\cos\beta+f_z'(x_0,y_0,z_0)\cos\gamma
\end{gathered} \tag{15}$$

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>
> 可微⇒方向导数存在, 其余均不成立(可见P_{209})

梯度

$$\begin{gathered}
\text{梯度是一个向量, 它的方向是函数在该点的方向导数最大的方向, 它的模是最大方向导数的值} \\
\nabla u|_{(x_0,y_0,z_0)}=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})|_{(x_0,y_0,z_0)} \\
\text{梯度} \nabla u|_{P_0} \text{垂直于} P_0 \text{点的等值面}
\end{gathered} \tag{16}$$

第九章:多元函数积分

二重积分

极坐标

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一般使用情况

1. 积分区域为圆或圆环或扇形
2. 被积函数为$f(x^2+y^2)$, $f(\frac{x}{y})$,可以消去其中一个变量

$$\begin{gathered}\iint_D f(x,y)dxdy, \qquad dxdy = rd\theta dr \\ \text{设}D\text{是}OXY\text{平面上有界闭区域}, f(x,y)\text{是}D\text{上连续函数} \\ \text{可积区域}\begin{cases} a \le \ \theta \le \beta \\ \varphi(\theta) \le r \le \psi(\theta) \end{cases} \\ \text{则}\iint_D f(x,y)dxdy = \int_\alpha^\beta d\theta \int_{\varphi(\theta)}^{\psi(\theta)} f(r\cos\theta, r\sin\theta)rdr \end{gathered} \tag{17}$$

对称性

$$\text{若}D\text{关于}y=x\text{对称, 则}\iint_D f(x,y)dxdy = \iint_D f(y,x)dxdy \tag{18}$$

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例:求$\int_0^{+\infty} e^{-x^2}dx$

$$\begin{aligned}\text{原式} &= (\int_0^{+\infty} e^{-x^2}dx \int_0^{+\infty} e^{-x^2}dx)^{\frac{1}{2}} \\ &= (\int_0^{+\infty} e^{-x^2}dx \int_0^{+\infty} e^{-y^2}dy)^{\frac{1}{2}} \\ &= (\iint_{0\le x\le +\infty} e^{-(x^2+y^2)}dxdy)^{\frac{1}{2}} \\ &= (\int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} e^{-r^2}\cdot rdr)^{\frac{1}{2}} \\ &= \frac{\sqrt{\pi}}{2}\end{aligned} \tag{19}$$

二重积分的换元法

$$\begin{gathered} x = x(u,v), y = y(u,v)\text{均有一阶偏导} \\ J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \ne 0 \\ (\text{雅可比行列式在几个点或一条线上为}0\text{, 下式仍成立}) \\ \text{区域}D\text{到}D'\text{为双射} \\ \text{则}\iint f(x,y)dxdy = \iint f(x(u,v), y(u,v))|J(u,v)|dudv \end{gathered} \tag{20}$$

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例题1:

$$\begin{aligned}&\text{求}x+y=c,x+y=d,y=ax,y=bx\quad(0<c<d,0<a<b)\text{围成的面积}\\&\text{令}u=x+y,v=\frac{y}{x},\text{则}c\le u\le d,a\le v\le b\\&S=\int_a^b dv\int_c^d J(u,v)du=\int_a^b dv\int_c^d\frac{u}{(1+v)^2}du\\&=\int_a^b\frac{1}{(1+v)^2}dv\int_c^d udu\end{aligned}\tag{21}$$

例题2:

$$\begin{aligned}&D:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\text{求}\iint_D\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dxdy\\&\text{令}x=a\rho\cos\theta,y=b\rho\sin\theta,\text{则}D':\rho=1\\&\text{原式}=\int_0^{2\pi}d\theta\int_0^1\sqrt{1-\rho^2}J(\rho,\theta)d\rho=\int_0^{2\pi}d\theta\int_0^1\sqrt{1-\rho^2}ab\rho d\rho=\frac{2}{3}\pi ab\end{aligned}\tag{22}$$

三重积分

柱形域

$$\begin{aligned}&\begin{cases}\varphi(x,y)\le z\le\psi(x,y)\\(x,y)\in D_{xy}\end{cases}\\&\iiint_\Omega f(x,y,z)dxdydz=\iint_{D_{xy}}dxdy\int_{\varphi(x,y)}^{\psi(x,y)}f(x,y,z)dz\end{aligned}\tag{23}$$

片形域

$$\begin{aligned}&\begin{cases}a\le z\le b\\(x,y)\in D_z\end{cases}\\&\iiint_\Omega f(x,y,z)dxdydz=\int_a^b dz\iint_{D_z}f(x,y,z)dxdy\end{aligned}\tag{24}$$

轮换对称性

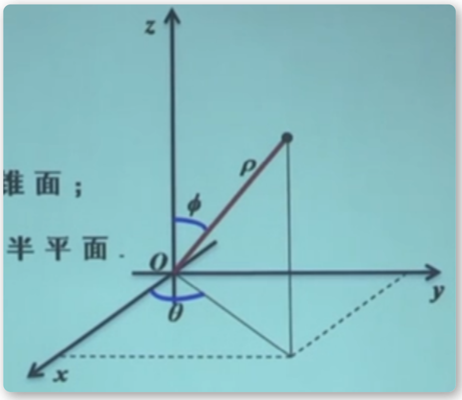
$$\begin{aligned}&\text{若}\Omega\text{关于}x,y,z\text{具有轮换对称性,则}\\&\iiint_\Omega f(x,y,z)d\Omega=\iiint_\Omega f(y,z,x)d\Omega=\iiint_\Omega f(z,x,y)d\Omega\end{aligned}\tag{25}$$

球坐标

$$\begin{aligned}&(x,y,z)\longrightarrow(\rho,\theta,\phi)\\&\rho=c:\text{表示以}c\text{为半径的球面}\\&\phi=c:\text{表示与}oz\text{轴正向夹角位}\phi\text{的锥面}\\&\theta=c:\text{表示与}ox\text{轴正向夹角位}\theta\text{的半平面}\\&\text{转换公式}:\begin{cases}x=\rho\sin\phi\cos\theta\\y=\rho\sin\phi\sin\theta\\z=\rho\cos\phi\end{cases}\qquad\text{其中}:\begin{cases}\rho\ge0\\0\le\theta\le2\pi\\0\le\phi\le\pi\end{cases}\\&\text{则}x^2+y^2+z^2=\rho^2\\&dxdydz=\rho^2\sin\phi\cdot d\theta d\phi d\rho\\&\text{若}\begin{cases}a\le\theta\le b\\c\le\phi\le d\\g(\theta,\phi)\le\rho\le h(\theta,\phi)\end{cases}\end{aligned}\tag{26}$$

$$g(\theta,\phi) \le \rho \le h(\theta,\phi)$$

$$\text{则} \iiint_\Omega f(x,y,z)dxdydz = \int_a^b d\theta \int_c^d d\phi \int_{g(\theta,\phi)}^{h(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot\rho^2\sin\phi d\rho$$



第一型积分

第一型曲线积分

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>
> 类似于弧微分/弧长部分

$$\begin{gathered}
1.\text{若曲线}C:\begin{cases} x = x(t) \\ y = y(t) \quad a \le t \le b \\ z = z(t) \end{cases} \\
\text{则} \int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt \\
2.\text{若}C: y = y(x), a \le x \le b, \text{则} \\
\int_L f(x,y)ds = \int_a^b f(x,y(x))\sqrt{1+y'^2(x)}dx \\
3.\text{若}C: \rho = \rho(\theta), \alpha \le \theta \le \beta, \text{则} \\
\int_L f(x,y)ds = \int_\alpha^\beta f(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)\sqrt{\rho^2+\rho'^2}d\theta
\end{gathered} \tag{27}$$

第一型曲面积分

$$\begin{gathered}
S\text{为单值连续可微曲面}: z = g(x,y) \in D \\
dS = \frac{dxdy}{|\cos\gamma|} \\
\text{曲面}z = g(x,y)\text{上任一点法向量为}\{g'_x, g'_y, -1\} \\
|\cos\gamma| = \frac{1}{\sqrt{g_x'^2+g_y'^2+1}} \\
\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{g_x'^2+g_y'^2+1}dxdy(D\text{为}S\text{在}XOY\text{的投影})
\end{gathered} \tag{28}$$

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>
> 例题:
>
> $$\text{计算}I = \oiint_\Sigma (x^2+y^2)dS, \text{其中}\Sigma: x^2+y^2+z^2 = 2(x+y+z)$$
>
> $$\text{解：显然球心为}(1,1,1)\text{、半径为}\sqrt{3}$$

$$\text{根据轮换对称性, 原式} = \frac{2}{3}\oiint_{\Sigma}(x^2+y^2+z^2)dS = \frac{3}{4}\oiint_{\Sigma}(x+y+z)dS \tag{29}$$
$$= 4\oiint_{\Sigma}xdS = 4\bar{x}\oiint_{\Sigma}dS = 4\cdot 1\cdot 4\pi(\sqrt{3})^2 = 48\pi$$
$$\text{形心公式}: \bar{x} = \frac{\oiint_{\Sigma}xdS}{\oiint_{\Sigma}dS}$$

第二型积分

第二型曲线积分

$$\text{若曲线}C: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t\text{取值为从}a\text{到}b(\text{起点到终点})$$
$$\text{则} \int_C P(x,y)dx + Q(x,y)dy = \int_a^b [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]dt \tag{30}$$

> 由曲线积分求功时,首先求出变力F的表达式,若已知F的大小$|F|$,它的方向与向量$\vec{l}$同向
>
> 则$F = |F|\frac{l}{|l|}$

第二型曲面积分

$$\iint_{\Sigma} Pdydz + Qdzdx + Rdxdy$$
$$= \pm\iint_{D_{yz}} P(x(y,z),y,z)dydz \pm \iint_{D_{zx}} Q(x,y(z,x),z)dzdx \pm \iint_{D_{xy}} R(x,y,z(x,y))dxdy \tag{31}$$
$$\text{正负号取决于投影的方向, 其中上侧/前侧/右侧为正号}$$
$$= \pm\iint_{D_{xy}} [P(x,y,z(x,y))(-\frac{\partial z}{\partial x}) + Q(x,y,z(x,y))(-\frac{\partial z}{\partial y}) + R(x,y,z(x,y))]dxdy$$

注意:第二型曲线/曲面积分的奇偶性与其他不同

第二型曲线积分的奇偶性

$$\text{设}L\text{为平面上分段光滑的定向曲线}, P(x,y), Q(x,y)\text{连续}$$
$$L\text{关于}y\text{轴对称}(x=0), \text{则}$$
$$\int_L Pdx = \begin{cases} 0, & \text{若}P\text{关于}x\text{为奇函数} \\ 2\int_{L_1} Pdx, & \text{若}P\text{关于}x\text{为偶函数} \end{cases} \tag{32}$$
$$\int_L Qdy = \begin{cases} 0, & \text{若}Q\text{关于}x\text{为偶函数} \\ 2\int_{L_1} Qdy, & \text{若}Q\text{关于}x\text{为奇函数} \end{cases}$$
$$\text{其中}L_1\text{是}L\text{在右半平面的部分}$$

第二型曲面积分的奇偶性

$$\text{若}\Sigma\text{关于}z=0(XoY\text{面})\text{对称, 则}$$
$$\begin{cases} \iint_{\Sigma} f(x,y,z)dxdy = 0, & f(x,y,z)\text{关于}z\text{为偶函数} \\ \iint_{\Sigma} f(x,y,z)dxdy = 2\iint_{\Sigma'} f(x,y,z)dxdy, & f(x,y,z)\text{关于}z\text{为奇函数} \end{cases} \tag{33}$$
$$\Sigma\text{为}XoY\text{平面上方的部分}$$

一/二型积分的联系

第一/二型曲线积分的联系

$$\begin{gathered}\text{设}L_{\overset{\frown}{AB}}\text{为光滑分段曲线}\\ \int_{L_{\overset{\frown}{AB}}}Pdx+Qdy=\int_{L_{\overset{\frown}{AB}}}(P\cos\alpha+Q\cos\beta)ds\\ \text{其中}\cos\alpha,\cos\beta\text{为曲线弧}L_{\overset{\frown}{AB}}\text{沿着}A\text{到}B\text{方向的切线的方向余弦}\\ \text{推广到空间, 可得：}\\ \int_{L_{\overset{\frown}{AB}}}Pdx+Qdy+Rdz=\int_{L_{\overset{\frown}{AB}}}(P\cos\alpha+Q\cos\beta+R\cos\gamma)ds\\ \text{其中}(\cos\alpha,\cos\beta,\cos\gamma)\text{是}L_{\overset{\frown}{AB}}\text{上任意点}(x,y,z)\text{处指向曲线方向的单位切向量}\end{gathered}\tag{34}$$

第一/二型曲面积分的联系

$$\begin{gathered}\cos\alpha dS=dydz,\cos\beta dS=dzdx,\cos\gamma dS=dxdy\\ \iint_{\Sigma}Pdydz+Qdzdx+Rdxdy=\iint_{\Sigma}(P\cos\alpha+Q\cos\beta+R\cos\gamma)dS\\ \text{其中}\cos\gamma=\pm\frac{1}{\sqrt{1+z_x'^2+z_y'^2}},\text{上侧取}'+'\text{号}\\ \cos\alpha=\pm\frac{-z_x'}{\sqrt{1+z_x'^2+z_y'^2}},\text{符号与}\cos\gamma\text{相同}\\ \cos\beta=\pm\frac{-z_y'}{\sqrt{1+z_x'^2+z_y'^2}},\text{符号与}\cos\gamma\text{相同}\end{gathered}\tag{35}$$

第十章:多元函数积分学公式

格林/高斯/斯托克斯公式

格林公式

$$\begin{gathered}\text{设平面上有界闭区域}D\text{由分段光滑的曲线}L\text{围成, 函数}P(x,y),Q(x,y)\text{在}D\text{有}\\ \text{连续一阶偏导数, 则有：}\\ \iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=\oint_L Pdx+Qdy\\ \text{其中}L\text{是}D\text{取正向的边界曲线}\\ \text{若取}P=-y,Q=x,\text{则有}S_D=\frac{1}{2}\oint_{L^+}-ydx+xdy,\text{其中}L^+\text{取正向}\\ \\ \text{对于复连通区域}D,\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=\sum_{i=1}^{n}\int_{C_i}Pdx+Qdy\\ C_i\text{全部取正方向}\end{gathered}\tag{36}$$

> 🌈
>
> 例题1:
>
> 求 $\oint \frac{xdy-ydx}{\ldots}$ L：无重复点，分段光滑且不过原点的逆时针闭曲线

$$
\begin{aligned}
&求\oint_L \frac{\quad}{x^2+y^2},L:无重复点、分段光滑且不过原点的逆时针闭曲线\\
&解:P=\frac{-y}{x^2+y^2},Q=\frac{x}{x^2+y^2}\\
&\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}\\
&1.若原点(0,0)不在L围成的区域内,则原式=\iint_D 0dxdy=0\\
&2.若原点(0,0)在L围成的区域内,在原点周围取一个适当大小的圆l,半径为r\\
&则\iint_{D-D_l}(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=\int_{L^+}\frac{xdy-ydx}{x^2+y^2}+\int_{l^+}\frac{xdy-ydx}{x^2+y^2}=0\\
&则\int_{L^+}\frac{xdy-ydx}{x^2+y^2}=-\int_{l^+}\frac{xdy-ydx}{x^2+y^2}=\int_{l^-}\frac{xdy-ydx}{x^2+y^2}\\
&=\int_0^{2\pi}\frac{r\cos\theta r\cos\theta+r\sin\theta r\sin\theta}{r^2}d\theta\\
&=2\pi
\end{aligned}
\tag{37}
$$

例题2:

$$
\begin{aligned}
&D:x=a\cos\theta,y=b\sin\theta,求D的面积\\
&S_D=\frac{1}{2}\oint_{L^+}-ydx+xdy\\
&=\frac{1}{2}\int_0^{2\pi}((-b\sin\theta)\cdot(-a\cos\theta)+a\cos\theta b\cos\theta)d\theta=\pi ab
\end{aligned}
\tag{38}
$$

高斯公式

$$
\begin{aligned}
&设\Omega是空间中的有界闭区域,由分块光滑的曲面S所围成.\\
&函数P(x,y,z),Q(x,y,z),R(x,y,z)在\Omega有连续的一阶偏导数,则:\\
&\iiint_\Omega(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})dV=\iint_S Pdydz+Qdzdx+Rdxdy\\
&=\iint_S(P\cos\alpha+Q\cos\beta+R\cos\gamma)dS\\
&其中S是\Omega的整个边界的外侧,\cos\alpha,\cos\beta,\cos\gamma是S上点(x,y,z)处的外法向量的方向余弦
\end{aligned}
\tag{39}
$$

斯托克斯公式

$$
\begin{aligned}
&设\Gamma为分段光滑的有向闭曲线,S是以\Gamma为边界的分块光滑有向曲面\\
&\Gamma的正向与S法向量指向符合右手螺旋\\
&函数P(x,y,z),Q(x,y,z),R(x,y,z)是S+\Gamma上有连续偏导的函数,则:\\
&\oint_C Pdx+Qdy+Rdz\\
&=\iint_S(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})dydz+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})dzdx+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy\\
&=\iint_S\begin{vmatrix}dydz&dzdx&dxdy\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\P&Q&R\end{vmatrix}=\iint_S\begin{vmatrix}\cos\alpha&\cos\beta&\cos\gamma\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\P&Q&R\end{vmatrix}dS
\end{aligned}
\tag{40}
$$

与路径无关

平面曲线积分与路径无关

$$\begin{aligned}&\text{设}D\text{为}**\text{单连通}**\text{区域}, P, Q\text{为}D\text{上具有连续一阶偏导的函数}\\&\text{在}D\text{上以下四个结论等价：}\\&1.D\text{上任意闭曲线}C, \text{有}\oint_C Pdx+Qdy=0\\&2.D\text{上第二型曲线积分}\int_{\widehat{AB}} Pdx+Qdy\text{与路径无关}\\&3.Pdx+Qdy\text{是全微分}\\&4.\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\\&\text{若为复连通区域, 结论123仍然等价, 4不再等价}\end{aligned} \tag{41}$$

平面曲线积分与路径无关

$$\begin{aligned}&\text{设}\Omega\text{为空间二维单连通区域}, P, Q, R\text{为}\Omega\text{上具有连续一阶偏导的函数}\\&\text{在}\Omega\text{上以下四个结论等价：}\\&1.\Omega\text{上任意闭曲线}C, \text{有}\oint_C Pdx+Qdy+Rdz=0\\&2.\Omega\text{上第二型曲线积分}\int_{\widehat{AB}} Pdx+Qdy+Rdz\text{与路径无关}\\&\int_{\widehat{AB}} Pdx+Qdy+Rdz=\\&\int_{x_1}^{x_2} P(x,y_1,z_1)dx+\int_{y_1}^{y_2} Q(x_2,y,z_1)dy+\int_{z_1}^{z_2} R(x_2,y_2,z)dz\\&3.Pdx+Qdy+Rdz\text{是全微分}\\&4.\frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x},\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\end{aligned} \tag{42}$$

🌈

例题:

$$\begin{aligned}&\text{证明}(2x+yz)dx+(4y+xz)dy+xydz\text{是全微分, 并求原函数}\\&1.\text{特殊路径法, 解：}P=2x+yz, Q=4y+xz, R=xy\\&\text{验证}\frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x},\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\text{成立}\\&\text{所以原式是全微分, 其原函数为：}\\u(x,y,z)&=\int_{(0,0,0)}^{(x,y,z)}(2x+yz)dx+(4y+xz)dy+xydz+C\\&=\int_0^x 2xdx+\int_0^y 4ydx+\int_0^z xydz+C\\&=x^2+2y^2+xyz+c\\&\text{方法2：不定积分法, }\frac{\partial u}{\partial z}=xy, \text{则}u=xyz+P(x,y)\\&\text{所以}\frac{\partial u}{\partial x}=yz+\frac{\partial P}{\partial x}=2x+yz\Rightarrow P=x^2+Q(y)\\&u=xyz+x^2+Q(y),\frac{\partial u}{\partial y}=xz+\frac{\partial Q}{\partial y}=xz+4y\Rightarrow Q=2y^2+C\\&\text{综上, }u=xyz+x^2+2y^2+C\end{aligned} \tag{43}$$

多元函数积分学物理应用

质心与转动惯量

质心/形心

$$(\bar{x},\bar{y},\bar{z}) = (\iiint_\Omega x\rho(x,y,z)dV, \iiint_\Omega y\rho(x,y,z)dV, \iiint_\Omega z\rho(x,y,z)dV) / \iiint_\Omega \rho dV \tag{44}$$

转动惯量

$$\begin{gathered} J = mr^2 \\ \text{关于}x\text{轴及原点的转动惯量分别为} \\ I_x = \iiint_\Omega (y^2+z^2)\rho(x,y,z)dV \\ I_o = \iiint_\Omega (x^2+y^2+z^2)\rho(x,y,z)dV \end{gathered} \tag{45}$$

通量和散度

通量

散度

$$divF = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot F \tag{46}$$

环流量与旋度

> 🌈
>
> 设有向量场$F(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$,则

环流量

$$\begin{gathered} F\text{沿分段光滑的闭曲线}\Gamma\text{的环流量为} \\ \oint_\Gamma F\cdot\tau ds = \oint (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds \\ = \oint_\Gamma Pdx + Qdy + Rdz = \oint_\Gamma F\cdot ds \\ \text{其中}\tau = (\cos\alpha,\cos\beta,\cos\gamma)\text{是}\Gamma\text{上任意点}(x,y,z)\text{处的单位切向量} \end{gathered} \tag{47}$$

旋度

$$\begin{gathered} rot\ F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k} \\ = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ \text{若向量场}A\text{的旋度}rot\ A\text{处处为零, 则称向量场}A\text{为无旋场} \end{gathered} \tag{48}$$

第十一章:无穷级数

常数项级数

收敛级数的基本性质

$$
\begin{gathered}
1.\text{级数收敛的必要条件}: \lim_{n\to\infty} u_n = 0; \text{若不趋于零, 则必发散} \\
2.\text{若两个级数} \sum_{n=1}^{\infty} u_n \text{和} \sum_{n=1}^{\infty} v_n \text{同时收敛, 则} \\
\sum_{n=1}^{\infty}(\lambda u_n + \mu v_n) = \lambda \sum_{n=1}^{\infty} u_n + \mu \sum_{n=1}^{\infty} v_n, \text{其中} \lambda, \mu \text{为常数} \\
\text{若两个级数一个收敛一个发散, 则} \sum_{n=1}^{\infty}(u_n \pm v_n) \text{发散} \\
\text{若两个级数都发散, 则} \sum_{n=1}^{\infty}(u_n \pm v_n) \text{敛散性需讨论} \\
3.\text{级数的敛散性与其有限项无关, 即添加去掉改变有限项不改变敛散性} \\
4.\text{若级数收敛, 其任意加括号(不改变顺序)得到的新级数仍然收敛, 且和不变} \\
\text{若加括号后收敛, 原级数未必收敛; 若加括号后发散, 原级数一定发散}
\end{gathered} \tag{49}
$$

重要级数

> 几何级数

$$
\text{对于} a \neq 0, |q| < 1 \text{时}, a\sum_{n=0}^{\infty} q^n = \frac{a}{1-q} \text{收敛}, |q| \geq 1 \text{时发散} \tag{50}
$$

> 调和级数

$$
\begin{gathered}
1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \text{发散} \\
\text{证明}: S_{2n} - S_n = \frac{1}{n+1} + .. + \frac{1}{2n} > \frac{1}{2n} * n = \frac{1}{2} \neq \lim_{n\to\infty}(S_{2n} - S_n) = 0
\end{gathered} \tag{51}
$$

> p级数

$$
\begin{gathered}
1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots \\
p \leq 1 \text{发散}, p > 1 \text{收敛} \\
\text{证明}: p > 1 \text{时}, \frac{1}{k^p} = \int_{k-1}^{k} \frac{1}{k^p} dx \leq \int_{k-1}^{k} \frac{1}{x^p} dx \\
S_n = 1 + \sum_{k=2}^{n} \frac{1}{k^p} \leq 1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{x^p} dx = 1 + \int_{1}^{n} \frac{1}{x^p} dx \\
= 1 + \frac{1}{p-1}(1 - \frac{1}{n^{p-1}}) < 1 + \frac{1}{p-1} \text{有界, 则收敛} \\
p \leq 1 \text{时}, n^p \leq n, \frac{1}{n^p} \geq \frac{1}{n}, \text{而} \sum \frac{1}{n} \text{为调和级数发散, 则} p \text{级数发散}
\end{gathered} \tag{52}
$$

$p \le 1$时, $n^p \le n$, $\frac{1}{n^p} \ge \frac{1}{n}$, 而$\sum \frac{1}{n}$为调和级数发散, 则$p$级数发散

正项级数$u_n \ge 0$

$$\text{正项级数}S_n\text{单调非减, 收敛} \Leftrightarrow S_n\text{有界} \tag{53}$$

比较审敛法

$$\text{若}\sum u_n, \sum v_n\text{都是正项级数, 且}u_n \le v_n\text{, 则}\begin{cases}\sum v_n\text{收敛} \Rightarrow \sum u_n\text{收敛}\\ \sum u_n\text{发散} \Rightarrow \sum v_n\text{发散}\end{cases} \tag{54}$$

极限形式

$$\begin{gathered}\text{设}u_n, v_n > 0, \lim_{n\to\infty}\frac{v_n}{u_n} = A\text{, 则有}\\ \begin{cases}0 < A < +\infty, \sum u_n, \sum v_n\text{同时收敛或发散}\\ A = 0, \sum u_n\text{收敛} \Rightarrow \sum v_n\text{收敛}\\ A = +\infty, \sum u_n\text{发散} \Rightarrow \sum v_n\text{发散}\end{cases}\end{gathered} \tag{55}$$

比值和根值判别法

比值判别法(达朗贝尔判别法)

$$\text{若}\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \rho\begin{cases}\rho < 1, \sum u_n\text{收敛}\\ \rho > 1, \sum u_n\text{发散, 且}\lim_{n\to\infty} u_n = +\infty\\ \rho = 1\text{, 无法判别}\end{cases} \tag{56}$$

根值判别法(柯西判别法)

$$\text{若}\lim_{n\to\infty}\sqrt[n]{u_n} = \rho\begin{cases}\rho < 1, \sum u_n\text{收敛}\\ \rho > 1, \sum u_n\text{发散, 且}\lim_{n\to\infty} u_n = +\infty\\ \rho = 1\text{, 无法判别}\end{cases} \tag{57}$$

与p级数比较

$$\begin{gathered}\text{设}u_n > 0, \lim_{n\to\infty}\frac{u_n}{\frac{1}{n^p}} = A\\ \text{若}0 \le A < +\infty, p > 1\text{, 则}\sum_{n=1}^{\infty} u_n\text{收敛}\\ \text{若}0 < A \le +\infty, p \le 1\text{, 则}\sum_{n=1}^{\infty}\text{发散}\end{gathered} \tag{58}$$

积分判别法

$$\begin{gathered}\text{设}\sum_{n=1}^{\infty} u_n\text{为正项级数, 若}\exists\text{单调下降的正值函数}f(x)(x \ge 1)\text{使得}u_n = f(n)\\ n = 1, 2\ldots\text{, 则级数}\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} f(n)\text{收敛的充要条件是无穷积分}\int_1^{+\infty} f(x)dx\text{收敛}\end{gathered} \tag{59}$$

交错级数

> 🌈
> 定义:$u_n > 0, \sum_{n=1}^{\infty}(-1)^{n-1}u_n$

莱布尼兹判别法

$$
\begin{gathered}
\text{若} u_n \geq u_{n+1} \text{且} \lim_{n\to\infty} u_n = 0 \\
\text{则交错级数收敛且其和} 0 < S < u_1, \text{余项} |r_n| < u_{r+1} \\
\text{比较} u_n \text{和} u_{n+1} \text{可以使用求导令} \lim_{n\to\infty} \text{的方式}
\end{gathered} \tag{60}
$$

任意项级数

> 🌈
> 定义:u_n为任意实数,则$\sum_{n=1}^{\infty} u_n$为任意项级数

$$
\begin{gathered}
\text{若级数} \sum_{n=1}^{\infty} |u_n| \text{收敛, 则称级数} \sum_{n=1}^{\infty} u_n \text{绝对收敛} \\
\text{若} \sum u_n \text{收敛}, \sum |u_n| \text{发散, 则称级数} \sum_{n=1}^{\infty} u_n \text{条件收敛} \\
\text{绝对收敛级数与条件收敛级数之和为条件收敛级数} \\
\text{若用比值或根值法判断} \sum_{n=1}^{\infty} |u_n| \text{发散, 则} \sum_{n=1}^{\infty} u_n \text{发散, 否则仍需判断是否条件收敛} \\
\text{条件收敛级数的全部正项或负项构成的级数一定发散}
\end{gathered} \tag{61}
$$

幂级数

> 🌈
> 阿贝尔定理

$$
\begin{gathered}
\text{如果幂级数} \sum_{n=0}^{\infty} a_n x^n \text{当} x = x_0 \neq 0 \text{时收敛, 该幂级数在满足} |x| < |x_0| \text{的所有} x \text{处绝对收敛} \\
\text{如果幂级数} \sum_{n=0}^{\infty} a_n x^n \text{当} x = x_1 \text{时发散, 该幂级数在满足} |x| > |x_1| \text{的所有} x \text{处发散}
\end{gathered} \tag{62}
$$

> 🌈
> 可能的收敛情况

$$
\begin{gathered}
1.\text{仅在} x = 0 \text{时收敛} \\
2.\text{在} x \in (-\infty, +\infty) \text{收敛} \\
3.\text{在} |x| < R \text{绝对收敛, 两个端点} x = \pm R \text{需要单独判断} \\
(-R, R) \text{称为收敛区间, 加上收敛的端点为收敛域,} R \text{称为收敛半径}
\end{gathered} \tag{63}
$$

收敛半径和收敛域求法

$$
\begin{gathered}
1.\text{求收敛半径} \\
\text{使用比值法或根值法, 如果} \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = l \text{或} \lim_{n\to\infty} \sqrt[n]{|a_n|} = l, \text{则}
\end{gathered}
$$

$$
\begin{cases} 1/l, & 0 < l < +\infty \end{cases} \tag{64}
$$

$$R=\begin{cases}0, & l=+\infty\\ +\infty, & l=0\end{cases}$$

如果只有偶数项等特殊情况，直接比较相邻两项(带x)根据比值/根值判别法进行求解

幂级数的运算和函数的性质

逐项积分

$$\begin{aligned}&\sum_{n=0}^{+\infty}a_nx^n\text{的和函数}S(x)\text{在}I\text{上可积，则}\\ \int_0^x S(t)dt&=\int_0^x(\sum_{n=0}^{\infty}a_nt^n)dt=\sum_{n=0}^{\infty}\int_0^x a_nt^ndt\\ &=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}(x\in I)\end{aligned}\tag{65}$$

逐项积分后得到的新幂级数与原幂级数收敛半径相同，但端点处(收敛域)需要重新检验

逐项求导

$$\begin{aligned}S(x)\text{在}(-R,R)\text{内可导，}S'(x)&=(\sum_{n=0}^{\infty}a_nx^n)'=\sum_{n=0}^{\infty}(a_nx^n)'\\ &=\sum_{n=0}^{\infty}a_nx^{n-1}\end{aligned}\tag{66}$$

逐项求导后得到的新幂级数与原幂级数收敛半径相同，但端点处(收敛域)需要重新检验

> 泰勒级数和麦克劳林展开

$$\begin{gathered}\text{泰勒展开：}f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,x\in U(x_0)\\ \text{充要条件：}\lim_{n\to\infty}R_n(x)=0\\ \text{若}x_0=0\text{，则麦克劳林展开：}\\ f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n,x\in(-r,r)\end{gathered}\tag{67}$$

常用麦克劳林展开

> 核心公式

$$\begin{gathered}1.e^x=1+x+\frac{x^2}{2}+\ldots+\frac{x^n}{n!},x\in(-\infty,+\infty)\\ 2.\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots,x\in(-\infty,+\infty)\\ 3.\frac{1}{1-x}=1+x+x^2+x^3+\ldots+x^n,x\in(-1,1)\end{gathered}\tag{68}$$

> 推出公式

$$
\begin{aligned}
&4.[3.x=-x]\frac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^n x^n+\ldots,x\in(-1,1)\\
&5.[\text{对4积分}]\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+\frac{(-1)^{n-1}}{n}x^n+..,x\in(-1,1]\\
&6.[\text{对2求导}]\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n!)}+..,x\in(-\infty,+\infty)\\
&7.[1.x=x\ln a]a^x=1+x\ln a+\frac{x^2(\ln a)^2}{2!}+..+\frac{x^n(\ln a)^n}{n!}+..,x\in(-\infty,+\infty)\\
&8.[4.x=x^2]\frac{1}{1+x^2}=1-x^2+x^4-\ldots+(-1)^n x^{2n}+..,x\in(-1,1)\\
&9.[\text{对8积分}]\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^n\frac{x^{2n+1}}{2n+1}+,,.x\in[-1,1]
\end{aligned}
\tag{69}
$$

傅里叶级数

> 🌈
>
> 三角函数系

$$
\begin{gathered}
\{1,\cos\frac{\pi}{l}x,\sin\frac{\pi}{l}x,\cos\frac{2\pi}{l}x,\sin\frac{2\pi}{l}x,\ldots,\cos\frac{n\pi}{l}x,\sin\frac{n\pi}{l}x\ldots\}\\
\text{在区间}[-l,l]\text{上正交,即任在其中取两个不同的函数}f(x),g(x)\\
\text{都有}\int_l^l f(x)g(x)dx=0
\end{gathered}
\tag{70}
$$

> 🌈
>
> 傅里叶级数

$$
\begin{gathered}
\text{若}f(x)\text{以}2l\text{为周期,且下述积分存在}\\
\begin{cases}
a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx & \\
a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi}{l}xdx & n=1,2,..\\
b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi}{l}xdx &
\end{cases}\\
\text{则称}\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x)\text{为}f(x)\text{的傅里叶级数}
\end{gathered}
\tag{71}
$$

> 🌈
>
> 性质

$$
\begin{gathered}
\text{若}f(x)\text{为奇函数,则}a_0=a_n=0,b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}xdx,n=1,2,..\\
\text{此时}f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{l}x\\
\text{若}f(x)\text{为偶函数,则}b_n=0,a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}xdx,n=0,1,\ldots\\
\text{此时}f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{l}x
\end{gathered}
\tag{72}
$$

> 🌈

收敛性(狄利克雷收敛定理)

$$\begin{gathered}\text{函数}f(x)\text{在区间}[-l.l]\text{上满足：}\\ 1.\text{连续或只有有限个第一类间断点}\\ 2.\text{只有有限个极值点}\\ \text{则}f(x)\text{在}[-l,l]\text{上的傅里叶级数收敛，并且}\\ \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x)\\ =\begin{cases}f(x),\text{若}x\in(-l,l)\text{为}f(x)\text{的连续点}\\ \frac{1}{2}[f(x+0)+f(x-0)],\text{若}x\in(-l,l)\text{为}f(x)\text{的第一类间断点}\\ \frac{1}{2}[f(-l+0)+f(l-0)],\text{若}x=\pm l\end{cases}\end{gathered}\tag{73}$$

有限区间上函数的傅里叶级数

🌈

$[0,l]$上的正弦型级数

$$\begin{gathered}\text{将}f(x)\text{奇延拓成}[-l,l]\text{上的奇函数}\\ g(x)=\begin{cases}f(x),x\in[0,l]\\ -f(x),x\in[-l,0)\end{cases}\\ \text{则}g(x)\text{在}[-l,l]\text{上的}F-\text{级数就是}f(x)\text{在}[0,l]\text{上的正弦型级数}\\ \text{则}a_0=a_n=0,b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}xdx,n=1,2,..\\ \sum_{i=1}^{\infty}b_n\sin\frac{n\pi}{l}x=\begin{cases}\frac{1}{2}[f(x^+)+f(x^-)],x\in(0,l)\\ 0,x=0\ or\ x=l\end{cases}\end{gathered}\tag{74}$$

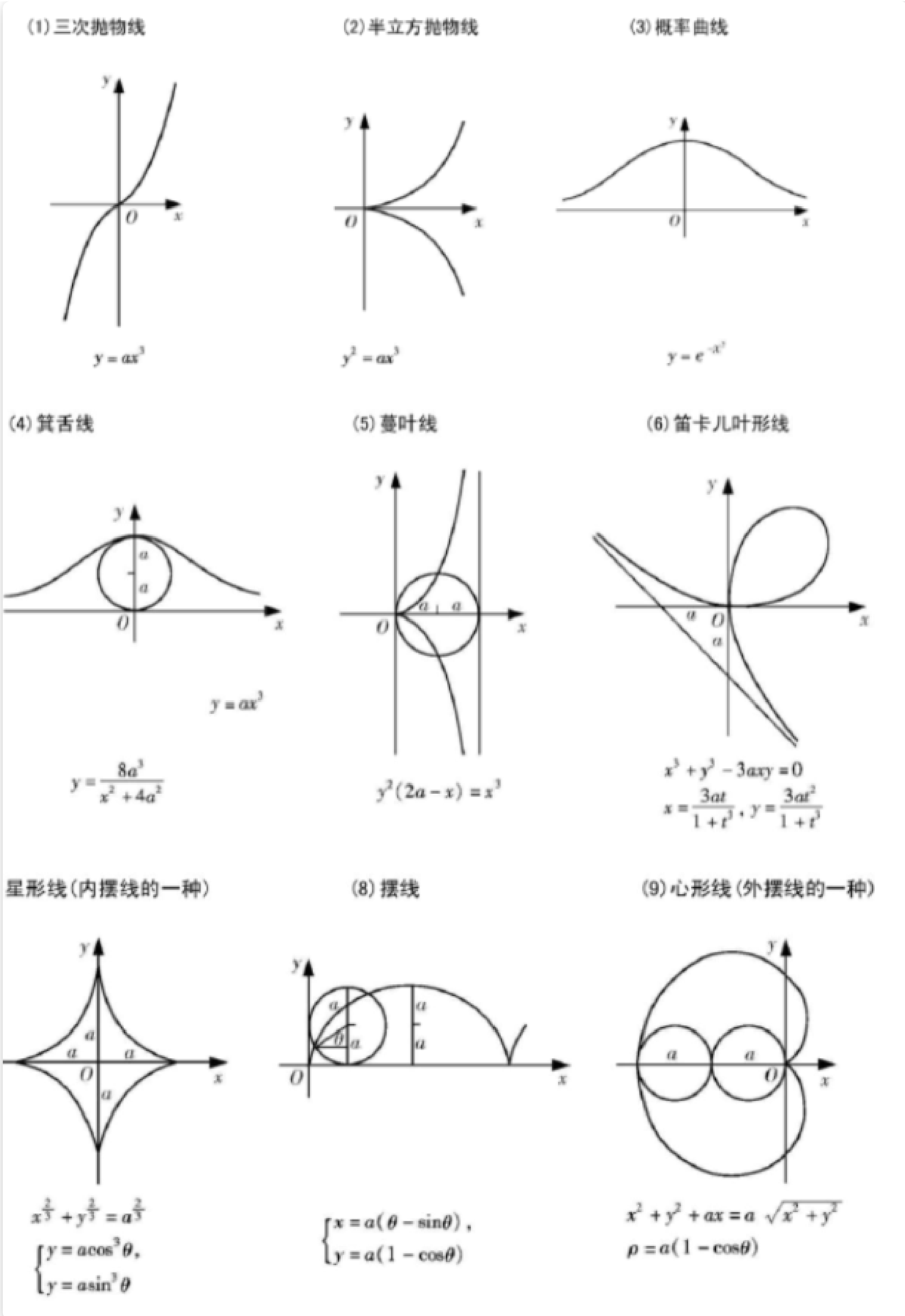
🌈

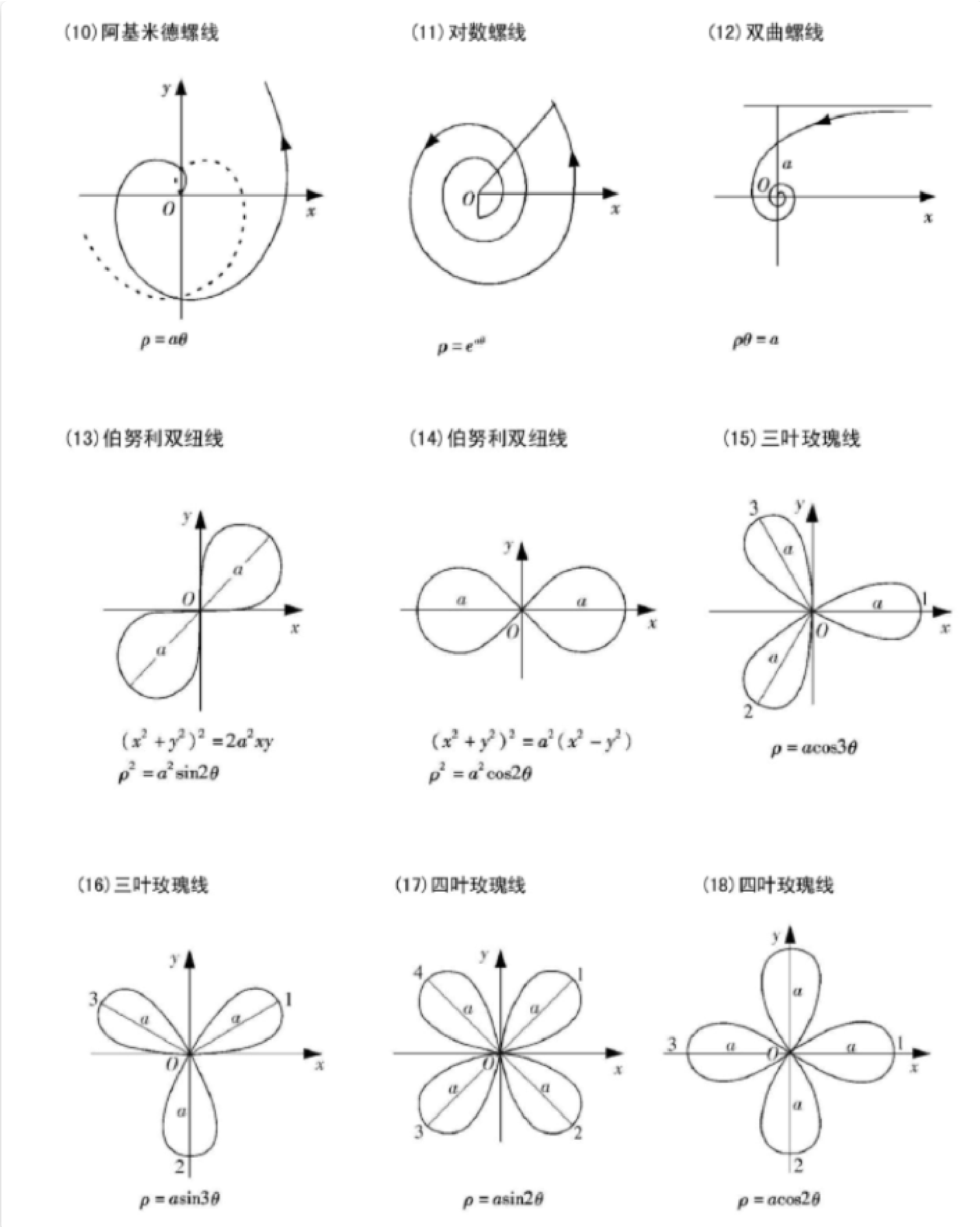
$[0,l]$上的余弦型级数

$$\begin{gathered}\text{将}f(x)\text{偶延拓成}[-l,l]\text{上的偶函数}\\ g(x)=\begin{cases}f(x),x\in[0,l]\\ f(-x),x\in[-l,0)\end{cases}\\ \text{则}g(x)\text{在}[-l,l]\text{上的}F-\text{级数就是}f(x)\text{在}[0,l]\text{上的余弦型级数}\\ \text{则}b_n=0,a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi}{l}xdx,n=0,1,...\\ \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{l}x=\frac{1}{2}[f(x^+)+f(x^-)],x\in[0,l]\end{gathered}\tag{75}$$

附录

常见曲线


(1) 三次抛物线
$y = ax^3$
(2) 半立方抛物线
$y^2 = ax^3$
(3) 概率曲线
$y = e^{-x^2}$
(4) 箕舌线
$y = ax^3$
$y = \frac{8a^3}{x^2 + 4a^2}$
(5) 蔓叶线
$y^2(2a - x) = x^3$
(6) 笛卡儿叶形线
$x^3 + y^3 - 3axy = 0$
$x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$
星形线（内摆线的一种）
$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$
$\begin{cases} y = a\cos^3\theta, \\ y = a\sin^3\theta \end{cases}$
(8) 摆线
$\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta) \end{cases}$
(9) 心形线（外摆线的一种）
$x^2 + y^2 + ax = a\sqrt{x^2 + y^2}$
$\rho = a(1 - \cos\theta)$

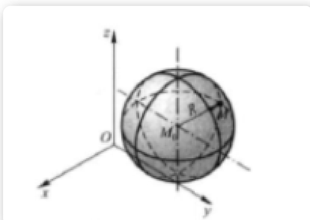


常见曲面

二次曲面

球面

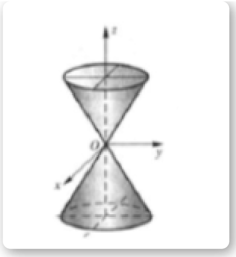
$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2, \text{球心为} M_0(x_0,y_0,z_0) \tag{76}$$



椭圆锥面

$$\text{定义：}\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2 \tag{77}$$

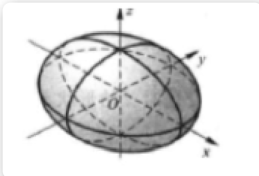
$$\text{常见：}$$



椭球面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{78}$$

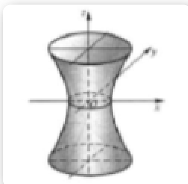
其中$\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{c^2}=1$表示$XOZ$平面上的椭圆绕$z$轴旋转而成的椭球面



单叶双曲面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \tag{79}$$

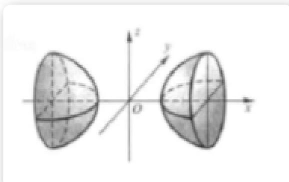
其中$\dfrac{x^2+y^2}{a^2}-\dfrac{z^2}{c^2}=1$表示$XOZ$平面上的双曲线绕$z$轴旋转而成的单叶双曲面



双叶双曲面

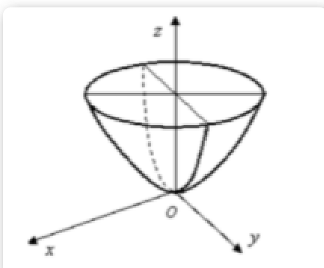
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \tag{80}$$

其中$\dfrac{x^2}{a^2}-\dfrac{y^2+z^2}{c^2}=1$表示$XOZ$平面上的双曲线绕$x$轴旋转而成的双叶双曲面



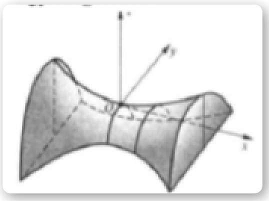
椭圆抛物面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z \tag{81}$$



双曲抛物面(马鞍面)

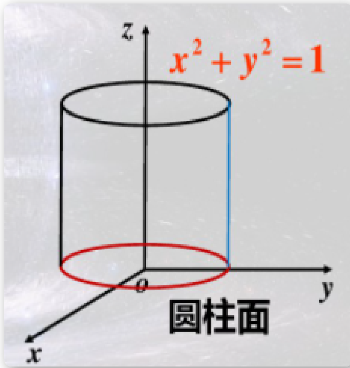
$$
\begin{gathered}
\frac{x^2}{a^2}-\frac{y^2}{b^2}=z \\
\text{常见：} z=xy\text{也为马鞍面, 证明：} \\
\text{令} x=\frac{u+v}{\sqrt{2}}, y=\frac{u-v}{\sqrt{2}}, z=z \\
\text{方程变为} \frac{u^2-v^2}{2}=z, \text{为} a^2=b^2=2\text{的马鞍面}
\end{gathered}
\tag{82}
$$



柱面

圆柱面

$$
x^2+y^2=R^2 \tag{83}
$$



椭圆柱面

$$
\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{84}
$$

双曲柱面

TODO

其他

X+Y+Z=0图像