Tips:

- 先看定义域!!!!!再看周期和奇偶性!!!!!
- 反对幂三指
- 定积分先看奇偶性
- 根号下多项式用配方
- 三角函数可以尝试万能公式
- 上凸下凹

克莱姆法则

基础知识

$$
\begin{aligned}
\text{三角形面积}: S &= \frac{absin\theta}{2}(\theta\text{为}a,b\text{两边夹角}) \\
\text{圆锥侧面积}: S &= \pi rl \\
\text{圆台侧面积}: S &= \pi l(R+r) \\
\text{椭圆面积}: S &= \pi ab \\
\text{液体压强}: P &= \rho gh \\
\text{两点间引力大小}: F &= k\frac{m_1m_2}{r^2} \\
\text{拐点}: f''(x_0) &= 0,\text{且}x_0\text{左右}f''(x)\text{异号} \\
\text{球面方程}: (x-x_0)^2 &+ (y-y_0)^2 + (z-z_0)^2 = R^2 \\
\text{等比数列求和}: S_n &= \frac{a_1(1-q^n)}{1-q} \\
\sqrt[n]{n!} &= +\infty
\end{aligned} \tag{1}
$$

放缩关系

- 四种平均数大小关系

> 🌈
> 平方平均数≥算数平均数≥几何平均数≥调和平均数

$$
\begin{aligned}
\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}} \\
\sqrt{\frac{\sum_{i=1}^n x_i^2}{n}} \geq \frac{\sum_{i=1}^n x_i}{n} \geq \sum_{i=1}^n {}^{f_i}\sqrt{\prod_{i=1}^n x_i^{f_i}} \geq \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}
\end{aligned} \tag{2}
$$

- 三角函数单位圆结论

$$
\sin x < x < \tan x \tag{3}
$$

正弦定理和余弦定理

- 正弦定理:
  $\forall \triangle ABC$,a,b,c分别为$\angle A,\angle B,\angle C$的对边,$R$为$\triangle ABC$的外接圆半径,则有

$$
\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R \tag{4}
$$

- 余弦定理:
  $\forall \triangle ABC$,三条边分别为a,b,c,则有

$$
\begin{gathered}
a^2 = b^2 + c^2 - 2bc\cos\alpha \\
\text{其中}\alpha\text{为}b,c\text{的夹角}
\end{gathered} \tag{5}
$$

三角函数

二倍角公式:

$$
\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
&= 1 - 2\sin^2\alpha \\
&= 2\cos^2\alpha - 1
\end{aligned} \tag{6}
$$

辅助角公式:

$$\begin{aligned} a\sin x+b\cos x&=\sqrt{a^2+b^2}\sin(x+\phi) \\ \text{其中}\sin\phi=\frac{b}{\sqrt{a^2+b^2}},\cos\phi&=\frac{a}{\sqrt{a^2+b^2}},\tan\phi=\frac{b}{a} \end{aligned} \tag{7}$$

两角和差的正余弦公式

$$\begin{aligned} \sin(\alpha\pm\beta)&=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta \\ cos(\alpha\pm\beta)&=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta \end{aligned} \tag{8}$$

和差化积/积化和差

$$\begin{aligned} &\text{和差化积：} \\ \sin\alpha+\sin\beta&=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \sin\alpha-\sin\beta&=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\ \cos\alpha+\cos\beta&=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\ \cos\alpha-\cos\beta&=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\ &\text{积化和差：} \\ \sin\alpha\cos\beta&=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \\ \cos\alpha\sin\beta&=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \\ \cos\alpha\cos\beta&=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \\ \sin\alpha\sin\beta&=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)] \end{aligned} \tag{9}$$

其他结论

$$\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2} \tag{10}$$

三阶行列式计算

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1b_2c_3+b_1c_2a_3+c_1a_2b_3-a_3b_2c_1-b_3c_3a_1-c_3a_2b_1 \tag{11}$$

$$\text{左对角线}-\text{右对角线} \tag{12}$$

二项式定理

$$(a+b)^n=C_n^0a^n+C_n^1a^{n-1}b+\ldots+C_n^nb^n \tag{13}$$

欧拉公式

$$e^{i\theta}=\cos\theta\pm i\sin\theta \tag{14}$$

高数

第一章:极限/连续

可积/可导/可微/连续/有界/极限存在

$$
\begin{aligned}
&\text{定义：}\\
&x_0\text{极限存在：}\\
&\quad\text{数列：}\lim_{n\to\infty}x_n=A\Leftrightarrow\forall\varepsilon>0,\exists\text{正整数}N,\text{使得当}n>N\text{时就有}|x_n-A|<\varepsilon\\
&\quad\text{函数：}\lim_{n\to\infty}f(x)=A\Leftrightarrow\forall\varepsilon>0,\exists\text{正整数}X,\text{使得当}|x|>X\text{时就有}|f(x)-A|<\varepsilon\\
&x_0\text{可导：}\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\qquad\text{极限存在}\\
&x_0\text{连续：}\lim_{x\to x0}f(x)=f(x_0)\qquad\lim_{x\to x0}[f(x_0+\Delta x)-f(x_0)]=0\\
&\text{可积的充分条件：函数有界,在区间上连续或有有限个间断点}\\
&[a,b]\text{可积的必要条件：}[a,b]\text{有界}\\
&\text{性质：}\\
&x_0\text{极限存在：}x_0\text{左右极限均存在且相等}\\
&x_0\text{可导：}x_0\text{左右导数均存在且相等}\\
&x_0\text{连续：}x_0\text{左右均连续}\qquad\lim_{x\to x0}f(x)=f(x_0)\\
&\text{可积函数的积分一定连续}\\
+&\text{数列极限存在}\Rightarrow\text{数列收敛,不存在}\Rightarrow\text{数列发散}\\
+&x_0\text{可微}\Leftrightarrow x_0\text{可导}\Rightarrow x_0\text{连续}\\
+&\text{连续}\Rightarrow\text{可积(黎曼可积)/存在原函数,第一类间断点一定没有原函数,第二类间断点可能存在原函数}\\
+&\text{有界闭区间连续函数}\Rightarrow\text{有界}\\
+&\text{可积}\Rightarrow\text{有界(黎曼积分)}\\
+&\text{极限存在}\Rightarrow\text{数列有界}
\end{aligned}\tag{15}
$$

等价无穷小

$$
\begin{gathered}
x\sim\sin x\sim\tan x\sim\arcsin x\sim\arctan x\sim e^x-1\sim\ln(1+x)\\
1-\cos x\sim\frac{1}{2}x^2\\
a^x-1\sim x\ln a\\
\sqrt[n]{1+x}-1\sim\frac{1}{n}x\\
(1+\beta x)^\alpha-1\sim\alpha\beta x\\
\log_a(1+x)\sim\frac{1}{\ln a}x
\end{gathered}\tag{16}
$$

间断点

第一类间断点:$f(x_0+0)$和$f(x_0-0)$均存在

- 可去间断点:$f(x_0+0)=f(x_0-0)\neq f(x_0)$或在$x=x_0$处无定义
- 跳跃间断点:$f(x_0+0)\neq f(x_0-0)$

第二类间断点:$f(x_0+0)$和$f(x_0-0)$至少有一个不存在

- 无穷间断点:$f(x_0+0)$和$f(x_0-0)$至少有一个为∞
- 振荡间断点

第二章:导数/微分

函数导数

$$\begin{aligned}(a^x)' &= a^x \ln x \\ (\tan x)' &= \frac{1}{\cos^2 x} \\ (\cot x)' &= -\frac{1}{\sin^2 x} \\ (\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\ (\arccos x)' &= -\frac{1}{\sqrt{1-x^2}} \\ (\arctan x)' &= \frac{1}{1+x^2} \\ (\text{arccot}\, x)' &= -\frac{1}{1+x^2} \\ (\int_a^{f(x)} g(t)dt)' &= g(f(x))f'(x)\end{aligned} \tag{17}$$

$$[\ln(ax+b)]^{(n)} = (-1)^{n-1}(n-1)!\frac{a^n}{(ax+b)^n} \tag{18}$$

洛必达的条件:

$$f(x), g(x)\text{在}x=a\text{的空心邻域可导}, g'(x) \neq 0 \tag{19}$$

第三章:积分

常见不可积函数

$$\begin{gathered}\int \frac{\sin x}{x}dx, \int \frac{\cos x}{x}dx, \int \sin x^2 dx, \int \cos x^2 dx \\ \int e^{ax^2}, \int \frac{x^n}{\ln x}dx(n \neq 1), \int \frac{\ln x}{x+a}dx(a \neq 0), \\ \int \frac{e^x}{x}dx, \int (\sin x)^m dx(m \notin Z), \int \frac{1}{\sqrt{x^4+a}}dx(a \neq 0)\end{gathered} \tag{20}$$

一元函数积分学几何应用:

平面图形的面积:

$$\begin{gathered}\text{极坐标方程：设}r_1(\theta), r_2(\theta)\text{在}[\alpha,\beta]\text{连续}, r_1(\theta) \leq r_2(\theta) \qquad (\forall\theta \in [\alpha,\beta]) \\ S = \frac{1}{2}\int_\alpha^\beta [r_2^2(\theta) - r_1^2(\theta)]d\theta\end{gathered} \tag{21}$$

旋转体体积

$$\begin{gathered}\text{由曲线}f(x), x=a, x=b\text{及}x\text{轴所围图形绕}x\text{轴旋转一周形成的旋转体体积} \\ V = \pi\int_a^b f^2(x)dx\end{gathered} \tag{22}$$

旋转体侧面积

$$\begin{aligned}S &= 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}dx \\ &= 2\pi\int_\alpha^\beta y(t)\sqrt{x'^2(t)+y'^2(t)}dt\end{aligned} \tag{23}$$

$$= 2\pi \int_{\alpha}^{\beta} r(\theta)\sin\theta\sqrt{r^2(\theta)+r'^2(\theta)}d\theta$$

弧微分与弧长:

$$\begin{aligned}
\text{直角坐标方程}: y &= f(x) \qquad (a \le x \le b) \\
ds &= \sqrt{1+f'^2(x)}dx \\
s &= \int_a^b \sqrt{1+f'^2(x)}dx \qquad \text{其中} f(x)\text{在}[a,b]\text{有连续的导数} \\
\\
\text{参数方程}: x &= x(t), y = y(t) \qquad (\alpha \le t \le \beta) \\
ds &= \sqrt{x'^2(t)+y'^2(t)}dt \\
s &= \int_{\alpha}^{\beta} \sqrt{x'^2(t)+y'^2(t)}dt \qquad \text{其中} x(t), y(t)\text{在}[\alpha,\beta]\text{有连续的导数}, x'^2(t)+y'^2(t) \ne 0 \\
\\
\text{极坐标方程}: r &= r(\theta) \qquad (\alpha \le \theta \le \beta) \\
\text{则} x &= r(\theta)\cos\theta,\ y = r(\theta)\sin\theta \qquad (\alpha \le \theta \le \beta) \\
ds &= \sqrt{r^2(\theta)+r'^2(\theta)}d\theta \\
s &= \int_{\alpha}^{\beta} \sqrt{r^2(\theta)+r'^2(\theta)}d\theta \qquad \text{其中} r = r(\theta)\text{在}[\alpha,\beta]\text{有连续的导数}
\end{aligned} \tag{24}$$

平面曲线的曲率和曲率半径:

$$\begin{aligned}
\text{直角坐标方程}: y &= y(x),\ \text{二阶可导} \\
K &= \left|\frac{d\alpha}{ds}\right| = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}} \\
\\
\text{参数方程}: x &= x(t), y = y(t) \qquad (t \in [\alpha,\beta]) \\
K &= \left|\frac{d\alpha}{ds}\right| = \frac{|x'(t)y''(t)-x''(t)y'(t)|}{[x'^2(t)+y'^2(t)]^{\frac{3}{2}}} \\
\text{曲率半径}: \rho &= \frac{1}{K}
\end{aligned} \tag{25}$$

柱面被曲面所截面积公式

$$\begin{gathered}
\text{设有} OXY \text{平面上的光滑曲线} L, \text{以} L \text{为准线, 母线平行于} z \text{轴作柱面} \\
\text{此柱面在} xy \text{平面与连续曲面} z = f(x,y)(\ge 0) \text{之间部分的面积为} \\
S = \int_L f(x,y)ds
\end{gathered} \tag{26}$$

一元函数积分学物理应用:

平面曲线或平面的质心(形心):

$$\begin{gathered}
\text{平面曲线}: \\
\bar{x} = \frac{M_y}{M} = \frac{1}{l}\int_0^l x(s)ds, \bar{y} = \frac{M_x}{M} = \frac{1}{l}\int_0^l y(s)ds \\
\text{参数方程}: \bar{x} = \frac{\int_{\alpha}^{\beta} \varphi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}dt}{\int_{\alpha}^{\beta} \sqrt{\varphi'^2(t)+\psi'^2(t)}dt} \\
\bar{y} = \frac{\int_{\alpha}^{\beta} \psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}dt}{\int_{\alpha}^{\beta} \sqrt{\varphi'^2(t)+\psi'^2(t)}dt} \\
\text{平面}:
\end{gathered} \tag{27}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x[f(x) - g(x)]dx}{\int_a^b [f(x) - g(x)]dx}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{2}\int_a^b [f^2(x) - g^2(x)]dx}{\int_a^b [f(x) - g(x)]dx}$$

函数积分:先看定义域,反对幂三指

$$\begin{aligned} &\int_0^x f(t)dt = [\int_0^x (x-t)f(t)dt]' \\ &\int f(x)dx = \int_a^x f(t)dt + C \qquad (f(x)\text{为连续函数}) \end{aligned} \tag{28}$$

三角函数万能代换:

$$\begin{aligned} t = \tan\frac{x}{2}\ ,\ x = 2\arctan t\ ,\ dx = \frac{2dt}{1+t^2} \\ \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2t}{1+t^2}, \\ \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-t^2}{1+t^2} \end{aligned} \tag{29}$$

三角函数代换结论:

> $$(1)\int\frac{dx}{\sqrt{x^2 \pm a^2}} \quad = \ln|\,x + \sqrt{x^2 \pm a^2}\,| + C \tag{30}$$

$$\begin{aligned} (1)\text{证明:令}x = \frac{a}{\cos t},\text{则}\int\frac{dx}{\sqrt{x^2 - a^2}} &= \int\frac{1}{\cos t}dt \\ &= \ln|\sec t + \tan t| + C_1 && (\text{见}\int\sec x\text{项}) \\ &= \frac{1}{a}\ln|\,x + \sqrt{x^2 - a^2}\,| + C_1 \\ &= \ln|\,x + \sqrt{x^2 \pm a^2}\,| + C && (C = C_1 + \ln a) \end{aligned} \tag{31}$$

> $$(2)\int\sqrt{a^2 - x^2}dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C \tag{32}$$

$$\begin{aligned} (2)\text{证明:令}x = a\sin t,\text{则}\int\sqrt{a^2 - x^2}dx = a^2\int\cos^2 t dt &= a^2\int\frac{\cos 2t + 1}{2}dt \\ &= \frac{a^2}{2}(\frac{\sin 2t}{2} + t) + C \\ &= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C \qquad (\text{代回}) \end{aligned} \tag{33}$$

$$(3)\int\sqrt{x^2\pm a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}\pm\frac{a^2}{2}\ln|\,x+\sqrt{x^2\pm a^2}\,|+C \tag{34}$$

$$\text{证明：} \tag{35}$$

$$\int\sec x\;dx=\ln|\sec x+\tan x|+C \tag{36}$$

$$\begin{aligned}
\int\sec x\;dx&=\int\frac{1}{\cos x}dx=\int\frac{1}{\cos^2 x}d(\sin x)=\int\frac{1}{1-\sin^2 x}d(\sin x)\;\text{，令 } t=\sin x\\
&=\int\frac{1}{1-t^2}dt=\frac{1}{2}\int(\frac{1}{1-t}+\frac{1}{1+t})dt=\frac{1}{2}|\ln{(1+t)}-\ln{(1-t)}|+C\,\text{，带回}t=\sin x\\
&=\frac{1}{2}\ln|\frac{1+\sin x}{1-\sin x}|+C\\
&=\frac{1}{2}\ln|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}|+C\\
&=\ln|\frac{1+\sin x}{\cos x}|+C\\
&=\ln|\sec x+\tan x|+C
\end{aligned}$$

$$\begin{aligned}
\int\arcsin x\;dx&=x\arcsin x-\int\frac{x}{\sqrt{1-x^2}}\;dx\\
&=x\arcsin x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}d(1-x^2)\\
&=x\arcsin x+\sqrt{1-x^2}+C\\
\int\arccos x\;dx&=x\arccos x+\int\frac{x}{\sqrt{1-x^2}}\;dx\\
&=x\arccos x-\sqrt{1-x^2}+C\\
\int\arctan x\;dx&=x\arctan x-\int\frac{x}{1+x^2}dx\\
&=x\arctan x-\frac{1}{2}\ln\left(1+x^2\right)+C
\end{aligned} \tag{38}$$

$$\arctan A+\arctan\frac{1}{A}=\frac{\pi}{2} \tag{39}$$

$$\begin{aligned}
&\text{证明：设}\arctan A=a\text{，则}A=\tan a\\
&\quad\text{设}\arctan\frac{1}{A}=b\text{，则}\frac{1}{A}=\tan b\\
&\quad\text{则}\tan a*\tan b=1\Rightarrow\cos a\cos b-\sin a\sin b=0\\
&\quad\text{则}\cos{(a+b)}=0\Rightarrow a+b=\frac{\pi}{2}
\end{aligned} \tag{40}$$

反常积分()广义积分

> 🌈
>
> 常见的反常积分

$$
\begin{aligned}
\int_a^{+\infty}\frac{dx}{x^p}=\begin{cases}\text{收敛}, p>1\\ \text{发散}, p\le 1\end{cases}(a>0)\\
\int_a^{+\infty}\frac{dx}{x\ln^p x}=\begin{cases}\text{收敛}, p>1\\ \text{发散}, p\le 1\end{cases}(a>1)\\
\int_a^{+\infty}x^k e^{-\lambda x}dx=\begin{cases}\text{收敛}, \lambda>0\\ \text{发散}, \lambda\le 0\end{cases}(k\ge 0)\\
\int_a^{b}\frac{dx}{(x-a)^p}=\begin{cases}\text{收敛}, p<1\\ \text{发散}, p\ge 1\end{cases}
\end{aligned}
\tag{41}
$$

> 🌈
>
> 性质

$$
\begin{gathered}
\text{若}\int_{-\infty}^{+\infty}f(x)dx\text{收敛}, \text{则}\int_{-\infty}^{+\infty}=\begin{cases}2\int_0^{+\infty}f(x)dx, f(x)\text{为偶函数}\\ 0, f(x)\text{为奇函数}\end{cases}\\
\text{收敛}+\text{收敛}=\text{收敛}, \text{收敛}+\text{发散}=\text{发散}, \text{发散}+\text{发散}=\text{不确定}\\
\text{若}\int_{-\infty}^{0}f(x)dx\text{和}\int_{0}^{+\infty}f(x)dx\text{均收敛}, \text{则}\int_{-\infty}^{+\infty}f(x)dx\text{收敛}, \text{其他情况}\int_{-\infty}^{+\infty}f(x)dx\text{均发散}
\end{gathered}
\tag{42}
$$

华里士Wallis公式:三角函数幂次的积分

$(\cos x)^n$ 和$(\sin x)^n$的周期相同,n为奇数时周期为2π,n为偶数时周期为π`

$$
\int_0^{\frac{\pi}{2}}\sin^n x dx=\int_0^{\frac{\pi}{2}}\cos^n x dx=\begin{cases}\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, n\text{为正偶数}\\ \frac{n-1}{n}\frac{n-1}{n-2}\cdots\frac{2}{3}, n\text{为大于1的奇数}\end{cases}
$$

$$
\begin{gathered}
\int_0^{\pi}sin^n x dx=2\int_0^{\frac{\pi}{2}}\sin^n x dx\\
\int_0^{\pi}\cos^n x dx=\begin{cases}2\int_0^{\frac{\pi}{2}}\cos^n x dx, n\text{为偶数}\\ 0, n\text{为奇数}\end{cases}\\
\int_0^{2\pi}\sin^n x dx=\int_0^{2\pi}\cos^n x dx=\begin{cases}4\int_0^{\frac{\pi}{2}}\sin^n x dx, n\text{为偶数}\\ 0, n\text{为奇数}\end{cases}\\
\int_0^{\frac{\pi}{2}}f(\sin x)dx=\int_0^{\frac{\pi}{2}}f(\cos x)dx\\
\int_0^{\pi}f(\sin x)dx\ne\int_0^{\pi}f(\cos x)dx\\
\int_0^{\pi}xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi}f(\sin x)dx=\pi\int_0^{\frac{\pi}{2}}f(\sin x)dx
\end{gathered}
\tag{43}
$$

第四章:微分中值定理

罗尔(中值)定理

> 🌈
> [https://baike.baidu.com/item/%E7%BD%97%E5%B0%94%E4%B8%AD%E5%80%BC%E5%AE%9A%E7%90%86/1876399?fromtitle=%E7%BD%97%E5%B0%94%E5%AE%9A%E7%90%86&fromid=7253372&fr=aladdin]

如果 R 上的函数$f(x)$ 满足以下条件:

1. 在闭区间 $[a,b]$上连续,
2. 在开区间(a,b) 内可导,
3. $f(a)=f(b)$

则至少存在一个 $\xi \in (a,b)$,使得$f'(\xi)=0$。

拉格朗日中值定理

> 🌈
> 设$f(x)$在$[a,b]$上连续,在(a,b)内可导,则$\exists\xi \in (a,b)$使得

$$f(b)-f(a)=f'(\xi)(b-a) \tag{44}$$

柯西中值定理

> 🌈
> 设$f(x),g(x)$在$[a,b]$上连续,在(a,b)内可导,且$g'(x)\neq 0$,则$\exists\xi \in (a,b)$使得

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)} \tag{45}$$

驻点/极值点和拐点

- 驻点:函数的一阶导数为0的点
- 极值点:
  - 必要条件:$f'(x)=0$或$f'(x)$不存在
  -
- 拐点:函数二阶导数为0且三阶导数不为0的点
  - 必要条件:$f''(x)=0$
  - 求法:
    - 求$f''(x)$
    - 令$f''(x)=0$,求出此方程在区间内的实根,并且找出区间内$f''(x)$不存在的点
    - 对于上述每一个点x_0,检查$f''(x_0)$左右两侧的符号,若相反,则x_0是拐点

凹凸性

- 定义: 设$f(x)$在$[a,b]$上连续,在(a,b)内可导,若对$\forall x,x_0\in(a,b)$且$x\neq x_0$恒有

$$f(x_0)+f'(x_0)(x-x_0)>(<)f(x) \tag{46}$$

  则称$f(x)$在$[a,b]$上是**凸(凹)**的
- 几何意义:若$y=f(x)$在任意点处的切线处该点外总在曲线上方.则该曲线是凸的
- 充要条件: $f(x)$是凸函数的充要条件为$f'(x)$是单调减函数
- 求凹凸性区间:
  - 求出$f''(x)=0$和$f''(x)$不存在的所有点
  - 按顺序将区间分为若干个不相交的子区间,讨论$f''(x)$在每个子区间上的符号

- 一般用列表法

渐近线

求$y=f(x)$的渐近线的方法

$$\begin{gathered}1.x=a\text{是垂直渐近线}\Leftrightarrow\lim_{x\to a^+}f(x)=\infty\text{或}\lim_{x\to a^-}f(x)=\infty\\ 2.\text{当}x\to+\infty\text{时}y=b\text{是水平渐近线}\Leftrightarrow\lim_{x\to+\infty}=b\\ 3.\text{当}x\to+\infty\text{时}y=kx+b(k\neq0)\text{是斜渐近线}\Leftrightarrow\lim_{x\to+\infty}\frac{f(x)}{x}=k\neq0\text{且}\lim_{x\to+\infty}[f(x)-kx]=b\\ \text{把}2,3\text{中的}x\to+\infty\text{换为}x\to-\infty\text{可得曲线}f(x)\text{当}x\to-\infty\text{时的水平或斜渐近线}\end{gathered}\tag{47}$$

> 总结
>
> $$\begin{gathered}1.\text{求间断点}x_0,\text{验证极限是否为}\infty,\text{是则}x=x_0\text{为竖直渐近线}\\ 2.\text{求}\lim_{x\to\pm\infty}f(x)\text{得到水平渐近线}\\ 3.\text{求}\lim_{x\to\infty}\frac{f(x)}{x},\text{求出}k,\text{带回求}b\\ \text{其中}2.\text{和}3.\text{中的极限最多存在一个}\end{gathered}\tag{48}$$

第五章:泰勒公式

一元皮亚诺/拉格朗日定义:

带皮亚诺余项的n阶泰勒公式

$$\begin{aligned}&\text{设}f(x)\text{在}x=x_0\text{处具有}n\text{阶导数,则}\\ &\qquad f(x)=T_n(x)+R_n(x)\\ \text{其中}\quad &x=x_0\text{的}n\text{次泰勒多项式}\quad T_n(x)=f(x_0)+f'(x_0)(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\\ &x=x_0\text{处的}n\text{阶皮亚诺余项}\quad R_n(x)=o((x-x_0^n))\quad(x\to x_0),\text{即}\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n}=0\end{aligned}\tag{49}$$

带拉格朗日余项的n阶泰勒公式

$$\begin{aligned}&\text{设}f(x)\text{在包含}x_0\text{的区间}(a,b)\text{内有}n+1\text{阶导数,在区间}[a,b]\text{上有}n\text{阶连续导数,则}\forall x\in[a,b],\text{有}\\ &\qquad f(x)=T_n(x)+R_n(x)\\ \text{其中}\quad&[a,b]\text{上拉格朗日余项的}n\text{阶泰勒公式}\quad T_n(x)\text{同皮亚诺余项}\\ &x=x_0\text{处的}n\text{阶拉格朗日余项}\quad R_n(x)=\frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}\\ &\xi\text{在}x\text{和}x_0\text{中间,也可以表示为}\xi=x_0+\theta(x-x_0)\quad 0<\theta<1\\ &***\quad n=0\text{时},f(x)\text{即为拉格朗日中值定理}\end{aligned}\tag{50}$$

> 当$n=0$时上面两定理又称之为带皮亚诺余项与带拉格朗日余项的**麦克劳林公式**

第六章:常微分方程

变量可分离的方程

$$\begin{gathered}y'=f(x)g(y)\\ \text{同除}g(y),\text{分离变量,则}\int\frac{dy}{g(y)}=\int f(x)dx+C\end{gathered}\tag{51}$$

一阶线性方程

$$\begin{gathered}
\text{形如 } y' = p(x)y + q(x) \\
q(x) = 0\text{时, 方程}y' = p(x)y\text{为一阶线性齐次方程} \\
\text{通解}: y = Ce^{\int p(x)dx} \\
>>> **\text{注释}: \text{若通解中出现}\ln|g(x)|, \text{可以去掉绝对值} \\
\\
q(x) \neq 0\text{时, 方程为一阶线性非齐次方程} \\
\text{通解}: y = e^{\int p(x)dx}[C + \int q(x)e^{-\int p(x)dx}dx]
\end{gathered} \tag{52}$$

齐次方程

$$\begin{gathered}
\text{形如}y' = f(\frac{y}{x}) \\
\text{令}u = \frac{y}{x}, \text{则}y = ux \Rightarrow y' = u + xu' \\
\text{原方程化为} \quad xu' = f(u) - u
\end{gathered} \tag{53}$$

伯努利方程

$$\begin{aligned}
&\text{形如}y' = p(x)y + y^{\lambda}q(x) \qquad (\lambda \neq 0, 1) \\
\Rightarrow \quad & y^{-\lambda}y' = p(x)y^{1-\lambda} + q(x) \\
\Rightarrow \quad & \frac{1}{1-\lambda}(y^{1-\lambda})' = p(x)y^{1-\lambda} + q(x) \\
\Rightarrow \quad & (y^{1-\lambda})' = (1-\lambda)p(x)y^{1-\lambda} + (1-\lambda)q(x) \\
\Rightarrow \quad & \text{通解}: y^{1-\lambda} = e^{(1-\lambda)\int p(x)dx}[C + (1-\lambda)\int q(x)e^{-(1-\lambda)\int p(x)dx}dx]
\end{aligned} \tag{54}$$

可降阶的高阶方程:

可降阶的高阶方程

$$\begin{gathered}
\text{形如}F(x, y^{(n)}, y^{(n+1)}) = 0 \\
\text{设}y^{(n)} = u, \text{方程转化为一阶方程}F(x, u, u') = 0
\end{gathered} \tag{55}$$

$$\begin{gathered}
\text{形如}F(y, y', y'') = 0 \\
\text{设}y' = u = \frac{dy}{dx}, \\
\text{则}y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = u\frac{du}{dy} \\
\text{方程化为一阶方程}F(y, u, u\frac{du}{dy}) = 0
\end{gathered} \tag{56}$$

常系数微分方程

常系数齐次线性微分方程

$$\begin{gathered}
y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0 \\
\text{设其解为}y = e^{\lambda x}, \text{带入并消去}e^{\lambda x}\text{可得} \\
\text{特征方程}: \\
\Rightarrow \quad \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \\
\text{此方程的根称为原方程的特征根}
\end{gathered}$$

$$\begin{aligned}
&1)\text{如果}\lambda\text{是单实特征根, 则}y = e^{\lambda x} \\
&2)\text{如果}\lambda\text{是}k\text{重实特征根, 则方程基础解系中对应的}k\text{个解}
\end{aligned} \tag{57}$$

$$
\begin{aligned}
&y_1 = e^{\lambda x}, y_2 = xe^{\lambda x}, \ldots, y_k = x^{k-1}e^{\lambda x} \\
&3)\text{如果}\lambda = \alpha \pm i\beta\text{是单复特征根,则方程基础解系中对应的}2\text{个解} \\
&y_1 = e^{\alpha x}\cos\beta x, y_2 = e^{\alpha x}\sin\beta x \\
&4)\text{如果}\lambda = \alpha \pm i\beta\text{是}k\text{重复特征根,则方程基础解系中对应的}2k\text{个解} \\
&y_1 = e^{\alpha x}\cos\beta x, y_2 = e^{\alpha x}\sin\beta x \\
&\quad xe^{\alpha x}\cos\beta x, \qquad xe^{\alpha x}\sin\beta x \\
&\quad \ldots \qquad\qquad \ldots \\
&\quad x^{k-1}e^{\alpha x}\cos\beta x, \quad x^{k-1}e^{\alpha x}\sin\beta x
\end{aligned}
$$

常系数非齐次线性微分方程

$$
\begin{aligned}
&y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x) \\
&1)\text{当}f(x) = P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \\
&\text{设}y^* = x^kQ_n(x), k\text{是}0\text{为方程特征根的重数} \\
&2)\text{当}f(x) = P_n(x)e^{\alpha x}, \text{设}y^* = x^kQ_n(x)e^{\alpha x} \\
&k\text{是}\alpha\text{为方程特征根重数} \\
&***3)\text{一般形式}: f(x) = e^{\alpha x}[P_n(x)\cos\beta x + Q_m(x)\sin\beta x] \\
&\text{设}y^* = x^ke^{\alpha x}[P_t(x)\cos\beta x + Q_t(x)\sin\beta x] \\
&k\text{为}\alpha \pm i\beta\text{是方程特征根的重数}, t = \max\{m, n\}
\end{aligned}
\tag{58}
$$

欧拉方程

$$
\begin{aligned}
&x^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \ldots + a_1xy' + a_0y = f(x) \\
&\text{令}x = e^t, \frac{dt}{dx} = \frac{1}{x}, D = \frac{d}{dt} \\
&\Rightarrow \quad x^ny^{(n)} = D(D-1)\ldots(D-n+1)y \\
&\text{带入原方程使其转换为常系数非齐次线性微分方程}
\end{aligned}
\tag{59}
$$

> 举例:求方程$x^2y'' + 4xy' + 2y = \frac{1}{x}$的通解
>
> 令$x = e^t$,原方程化为$[D(D-1) + 4D + 2]y = e^{-t} \Rightarrow [D^2 + 3D + 2]y = e^{-t}$
>
> 特征方程$\lambda^2 + 3\lambda + 2 = 0$, 特征根$\lambda_1 = -1, \lambda_2 = -2$
>
> 设特解$y^* = ate^{-t}, (y^*)' = ae^{-t} - ate^{-t}, (y^*)'' = -2ae^{-t} + ate^{-t}$
>
> 带入方程解出$a = 1$,则特解$y^* = te^{-t}$
>
> 通解$y = C_1e^{-t} + C_2e^{-2t} + te^{-t}$

第七章:向量代数与空间解析几何

混合积性质

$$
\begin{cases}
(\vec{a}, \vec{b}, \vec{c}) = (\vec{b}, \vec{c}, \vec{a}) = (\vec{c}, \vec{a}, \vec{b}) \\
(\vec{a}, \vec{b}, \vec{c}) = -(\vec{b}, \vec{a}, \vec{c})
\end{cases}
\tag{60}
$$

定义:平面

平面的一般方程

$$
\begin{aligned}
&Ax + By + Cz + D = 0 \\
&\text{法向量}\vec{n} = \{A, B, C\}
\end{aligned}
\tag{61}
$$

平面的点法式方程

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\text{法向量}\vec{n} = \{A, B, C\} \tag{62}$$

平面的截距式方程

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{63}$$

平面的参数方程

$$\begin{cases} x = X_1 t_1 + X_2 t_2 + x_0 \\ y = Y_1 t_1 + Y_2 t_2 + y_0 \\ z = Z_1 t_1 + Z_2 t_2 + z_0 \end{cases} \tag{64}$$

平面的向量方程

$$\vec{OP} - \vec{OM} = t_1 U_1 + t_2 U_2, \quad P\text{是平面}\Pi\text{上任意一点} \tag{65}$$

定义:直线

直线的一般方程

$$\begin{gathered} \begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases} \\ \text{法向量分别为}\vec{n_1} = \{A_1, B_1, C_1\},\ \vec{n_2} = \{A_2, B_2, C_2\} \\ \vec{n_1}\text{和}\vec{n_2}\text{不平行} \\ \text{方向向量} \quad \vec{S} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} \end{gathered} \tag{66}$$

直线的参数方程

$$\begin{gathered} \begin{cases} x = x_0 + ka \\ y = y_0 + kb \\ z = z_0 + kc \end{cases} \\ \text{方向向量} \quad \vec{S} = \{a, b, c\} \end{gathered} \tag{67}$$

直线的标准方程

$$\begin{gathered} \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \\ \text{方向向量} \quad \vec{S} = \{a, b, c\} \end{gathered} \tag{68}$$

求平面方程

$$\begin{gathered} \text{已知平面上一点}M_0(x_0, y_0, z_0)\text{和法向量}\vec{n} = \{A, B, C\} \\ \text{点法式} \quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \end{gathered} \tag{69}$$

$$\begin{gathered} \text{已知平面上一点}M(x_0, y_0, z_0)\text{, 以及与平面平行的两个不共线的向量} \\ \vec{U}_1 = \{X_1, Y_1, Z_1\}, \quad \vec{U}_2 = \{X_2, Y_2, Z_2\} \\ \text{则平面方程为} \quad \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0 \end{gathered} \tag{70}$$

求直线方程

$$
\begin{aligned}
&\text{求两个平面方程, 联立即为直线方程}\\
&\text{形如}\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
&\text{已知直线上一点和直线的方向向量}S=\{l,m,n\},\text{用参数式或标准式方程即可}\\
&\text{形如}\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}\\
&\text{或者}\begin{cases}x=x_0+tl\\y=y_0+tm\\z=z_0+tn\end{cases}
\end{aligned}
\tag{72}
$$

距离类

点到直线距离

$$
\begin{aligned}
&\text{点}P_0(x_0,y_0,z_0),\ \text{直线}L:\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\\
&\text{则}L\text{上定点}P_1(x_1,y_1,z_1),L\text{有方向向量}\vec{S}=\{l,m,n\}\\
&d=|P_0\vec{P}_1|\sin<P_0\vec{P}_1,S>=\frac{|P_0\vec{P}_1\times S|}{|S|}
\end{aligned}
\tag{73}
$$

点到平面距离

$$
\begin{aligned}
&\text{平面方程}:Ax+By+Cz+D=0,\ \text{点}P(x_1,y_1,z_1)\text{为平面外一点}\\
&d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}
\end{aligned}
\tag{74}
$$

夹角类

两平面的夹角

$$
\begin{aligned}
&\text{即为两个平面法向量}\vec{n_1}\text{和}\vec{n_2}\text{的夹角}\theta\quad(0\le\theta\le 90^\circ)\\
&\cos\theta=\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}
\end{aligned}
\tag{75}
$$

两直线的夹角

$$
\begin{aligned}
&\text{即为两个方向向量}\vec{S_1}\text{和}\vec{S_2}\text{的夹角}\theta\quad(0\le\theta\le 90^\circ)\\
&\cos\theta=\frac{|\vec{S_1}\cdot\vec{S_2}|}{|\vec{S_1}||\vec{S_2}|}
\end{aligned}
\tag{76}
$$

直线与平面的夹角

$$
\begin{gathered}
\text{即为}[\frac{\pi}{2}-\text{直线方向向量}\vec{S}\text{与平面法向量}\vec{n}\text{的夹角}] \\
\sin\theta=\frac{|\vec{S}\cdot\vec{n}|}{|\vec{S}||\vec{n}|}
\end{gathered}
\tag{77}
$$

旋转曲面与柱面

旋转曲面方程

> 🌈
>
> 求$\begin{cases} f(x,y)=0 \\ z=0 \end{cases}$绕$x$轴旋转所得的旋转曲面方程

绕哪个轴,哪个轴不动

$y=\pm\sqrt{y^2+z^2}$

得到旋转曲面方程$F(x,\pm\sqrt{y^2+z^2})=0$

> 🌈
>
> 参数形式:求$\begin{cases} x=f(t) \\ y=g(t) \\ z=h(t) \end{cases}$绕$z$轴旋转所得的旋转曲面方程

$$
\begin{cases}
x=\sqrt{f^2(t)+g^2(t)}\cos\theta \\
y=\sqrt{f^2(t)+g^2(t)}\sin\theta \\
z=h(t)
\end{cases}
\tag{78}
$$

投影问题TODO

常用技巧

$$
\begin{gathered}
\vec{\alpha},\vec{\beta}\text{是平面}\Pi\text{上两个不平行的向量,则该平面上的任一向量可用} \\
\vec{s}=x\vec{\alpha}+y\vec{\beta} \qquad \text{表示}
\end{gathered}
\tag{79}
$$

克莱姆法则

对三元一次方程组

$$
\begin{gathered}
\begin{cases}
a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1 \\
a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2 \\
a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3
\end{cases} \\
\text{如记}\alpha_1=\{a_{11},a_{21},a_{31}\},\alpha_2=\{a_{12},a_{22},a_{32}\},\alpha_3=\{a_{13},a_{23},a_{33}\},\beta=\{b_1,b_2,b_3\} \\
\text{则方程组可改写为}x_1\alpha_1+x_2\alpha_2+x_3\alpha_3=\beta \\
\text{因为}\alpha_2\times\alpha_3\text{与}\alpha_2,\alpha_3\text{都垂直,用}\alpha_2\times\alpha_3\text{对上式两端做点积,有} \\
x_1(\alpha_1,\alpha_2,\alpha_3)=(\beta,\alpha_2,\alpha_3) \\
\text{当}(\alpha_1,\alpha_2,\alpha_3)\neq 0\text{时},x_1=\frac{(\beta,\alpha_2,\alpha_3)}{(\alpha_1,\alpha_2,\alpha_3)}=\frac{D_{x_1}}{D} \\
\text{类似地},x_2=\frac{(\alpha_1,\beta,\alpha_3)}{(\alpha_1,\alpha_2,\alpha_3)}=\frac{D_{x_2}}{D},x_3=\frac{(\alpha_1,\alpha_2,\beta)}{(\alpha_1,\alpha_2,\alpha_3)}=\frac{D_{x_3}}{D}
\end{gathered}
\tag{80}
$$

> 🌈
>
> 若系数行列式$D \neq 0$,则方程组有唯一解,若还为齐次线性方程组,则方程组只有零解