1FA018: Exercise set 1

Leandro Morita

November 2025

The code for the solutions made on this document are also placed on a GitHub repository [1]

Question 1: Maximum likelihood

a) Derive the Maximum-Likelihood (ML) estimator for the lifetime τ of the ^{289}Fl isotope, calculate it and give the asymmetric uncertainties Δt+ and Δt-.

Given the exponential probability density function:

$$f(t_i, \tau) = \frac{1}{\tau} \cdot e^{\frac{-t_i}{\tau}} \tag{1}$$

where t_i is the i-th sample and τ is the lifetime parameter.

Since maximizing the logarithm of the function is maximizing the function itself, the logarithm of the likelihood function will be

$$\ln L(\tau) = \sum_{i=1}^{n} (\ln \frac{1}{\tau} - \frac{t_i}{\tau})$$

since in the logarithm space it can be represented as a summation. The likelihood function is no longer a function of the samples t_i, these are treated as constants, therefore

$$\ln L(\tau) = n \cdot \ln \frac{1}{\tau} - \frac{1}{\tau} \sum_{i=1}^{n} t_i \tag{2}$$

[1] https://github.com/Lemorita95/1FA018

```python
def likelihood(tau, samples):
    '''
        explicit form for the likelihood function of tau (parameter)
        input: single tau value and samples
        return logL(tau)
    '''
    tau = float(tau)
    if tau <= 0:
        return -np.inf

    n = len(samples)

    logL = n * np.log(1/tau) - (1/tau) * samples.sum()
    L = np.exp(logL)

    return logL
```

Figure 1: Likelihood function in python

Equation 2 is the objective function and its implementation in python is shown in Figure 1. Taking its first derivative and second derivative

$$\frac{\partial \ln L(\tau)}{\partial \tau} = \frac{-n}{\tau} + \frac{1}{\tau^2} \cdot \sum_{i=1}^{n} t_i \tag{3}$$

$$\frac{\partial^2 \ln L(\tau)}{\partial \tau^2} = \frac{n}{\tau^2} - \frac{2}{\tau^3} \cdot \sum_{i=1}^{n} t_i \tag{4}$$

Setting Equation 3 to zero and solving for τ yields

$$\hat{\tau} = \frac{1}{n} \cdot \sum_{i=1}^{n} t_i \tag{5}$$

Equation 5 is our Maximum likelihood estimator.
For the exercise set samples,

$$\hat{\tau} = 2.77 \tag{6}$$

and then evaluating $\hat{\tau}$ at Equation 4, equals -9.83, which tells us this is indeed a local maxima.

To find the 68.3% confidence interval, Equation 2 is evaluated at $\hat{\tau}$

$$lnL_{max} = \ln L(\hat{\tau})$$

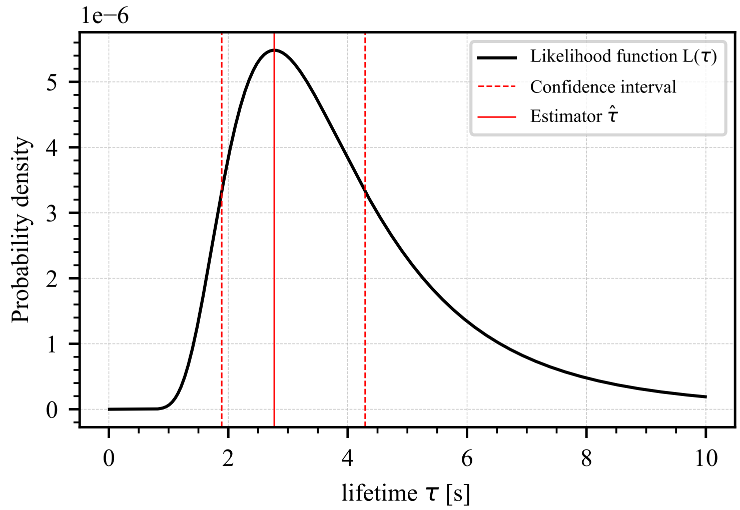


Figure 2: Likelihood function, estimator and confidence interval.

By definition the confidence interval boundaries are values of τ that satisfies

$$\ln L(\tau) = lnL_{max} - 0.5 \tag{7}$$

This is a root finding problem and for this function, there will be two real roots (a,b), Figure 2 show these roots as dashed lines. The resulting asymmetric uncertainties are then

$$\Delta t+ = b - \hat{\tau} = 1.52$$

$$\Delta t- = \hat{\tau} - a = 0.88$$

b) Check the consistency of the ML estimator by estimating the lifetime for a sample of 10, 100, 1000 and 10000 events. Please also calculate the uncertainties for N $\geq$ 100.

Generate N {10, 100, 1000, 10000} events from Equation 1 for $\tau = \tau_{true}$, evaluate consistency by check that $\hat{\tau} \to \tau_{true}$. In the context of this question $\hat{\tau}$ is the estimator obtained by the N events and τ_{true} is the value from Equation 6. This is shown in python at Figure 3

For each N:

1. Sample N samples from the exponential distribution with parameter $\tau = \tau_{true}$

```python
samples = np.random.exponential(tau_true, n)

# compute the estimator
tau_hat = ML_estimator(samples)

# find the boundaries as a root solving problem
lower, upper = find_boundaries(likelihood, samples, tau_hat)
```

Figure 3: Code in python for question 1.b

```python
def ML_estimator(samples):
    '''
        function to compute estimator for an exponential PDF
        the maximum likelihood when solving d/dx(likelihood) = 0 -> mean
    '''
    N = len(samples) # define number of samples
    tau_hat = 1/N * samples.sum()

    return tau_hat
```

Figure 4: The maximum likelihood estimator in python

2. Compute the Maximum likelihood estimator $\hat{\tau}$ from the samples using Equation 5. Figure 4

3. Compute the uncertainties using Equation 7. Figure 5

The results are shown in Figure 6 and it is observed that $\hat{\tau} \to \tau_{true}$ as N increases.

c) Calculate the bias of the Maximum-Likelihood estimator for a sample size of 50 events.

Produce 100 experiments, each containing a sample size of $N = 50$ and different seed. The approach is similar to the previous question 1.b) but a estimator $\hat{\tau}_i$ is computed for each experiment.

$$\bar{\hat{\tau}} = (\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_{100})$$

Take the expectation value of the estimator array $E[\bar{\hat{\tau}}]$. The bias is calculated as $E[\bar{\hat{\tau}}] - \tau_{true}$. For this number of experiments and sample size

$$bias = -0.0395$$

As shown in Figure 7 and the implementation in python in Figure 8

```python
def find_boundaries(f, samples, tau_hat, grid_factor=0.1, grid_points=10000, max_expand=10):
    '''
        find +delta_tau and -delta_tau for a 68.3% confidence interval
        it uses brentq root finding at lower boundary and grid search for upper boundary
        using only one of a kind result in errors at one side searching
    '''
    samples = np.asarray(samples)
    logL_max = f(tau_hat, samples)

    # find logF(x) value for confidence interval
    target = logL_max - 0.5

    # solve lower boundary as root finding problem
    tau_min = 1e-12
    # Define function for root: logL(tau) - target = 0
    func = lambda tau: f(tau, samples) - target
    ci_lower = brentq(func, tau_min, tau_hat)

    # solve upper boundary as grid search problem
    for _ in range(max_expand):
        tau_max = tau_hat * (1 + grid_factor)
        tau_grid = np.linspace(tau_hat, tau_max, grid_points)
        logL_grid = np.array([f(t, samples) for t in tau_grid])
        if logL_grid.min() <= target:
            ci_upper = np.interp(target, logL_grid[::-1], tau_grid[::-1])
            break
        grid_factor *= 2
    else:
        ci_upper = np.nan  # fallback


    return ci_lower, ci_upper
```

Figure 5: Code used for searching the boundaries. Brentq rootfinding algorithm on the left because of the steeper hill.

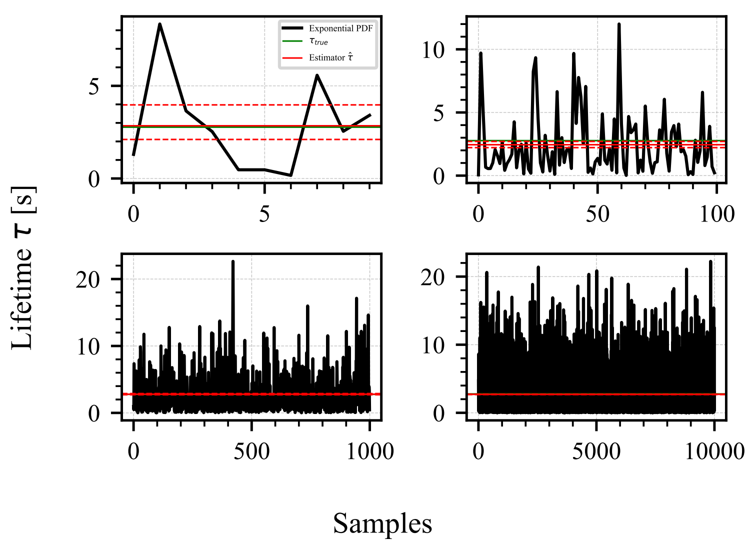


Figure 6: Consistency of maximum likelihood estimator.

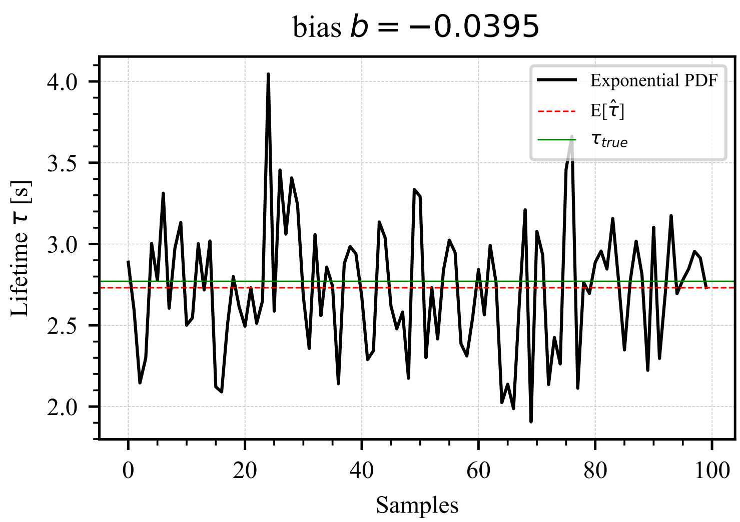


Figure 7: Bias of maximum likelihood estimator.

```python
estimator_array = []
for seed in range(0,100):
    np.random.seed(seed) # set individual seed for each experiment
    experiment = np.random.exponential(tau_true, 50)
    tau_hat = ML_estimator(experiment)
    estimator_array.append(tau_hat)
```

Figure 8: Python code for question 1.c

Question 2: Combining uncertainties

To solve this question, the following assumptions were made:

- Unreported systematic uncertainties included in the statistical
- Statistical and systematic uncertainties are summed in quadrature
- Experiments are independent
- Parent function is normal Gaussian
- Likelihood function is normal Gaussian

a) Combine all available measurements to calculate an "all-inclusive" world average.

Consider

$$y(x) = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i} \tag{8}$$

where

$$w_i = \frac{1}{\sigma_i^2}$$

In which $y(x)$ is the function we want to propagate the uncertainty, i.e. the all-incluse world average. N is the number of samples ($N = 16$) and x_i is the mass squared average value of each experiment.

The Jacobian

$$\mathbf{J} = \begin{vmatrix} \frac{\partial y(x)}{\partial x_1} & \frac{\partial y(x)}{\partial x_2} & \cdots & \frac{\partial y(x)}{\partial x_{16}} \end{vmatrix} \tag{9}$$

where

$$\frac{\partial y(x)}{\partial x_k} = \frac{\partial\left(\frac{w_1x_1+w_2x_2+\cdots+w_Nx_N}{w_1+w_2+\cdots+w_N}\right)}{\partial x_k} = \frac{w_k}{\sum_{i=1}^{N} w_i} \tag{10}$$

Since the experiments are independent

$$\mathbf{V}(\mathbf{x}) = \begin{bmatrix} \sigma x_1^2 & 0 & \cdots & 0 \\ 0 & \sigma x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma x_N^2 \end{bmatrix} \tag{11}$$

Then

$$V[y(x)] = J \cdot V(x) \cdot J^T \tag{12}$$

$V[y(x)]$ is a scalar

$$\sigma_y^2 = V[y(x)]$$
$$\sigma_y = \sqrt{V[y(x)]}$$

this results in an all-incluse world average of

$$m_v^2 = -0.02655 \pm 0.2057[eV^2]$$

b) What upper mass limits of this quantity do you obtain for a confidence level of 90% and 95%, respectively when using the classical frequentist method, i.e. by integrating the P.D.F.

Define the normal gaussian probability density function f as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{13}$$

Since the sampling distribution approaches the parent distribution for a large number of observations.

Define the probability $P(x \leq b) = \beta$ of a PDF $f(x)$ assume values less than or equal to b:

$$P(x \leq b) = \int_{-\infty}^{b} f(x)dx = \beta \tag{14}$$

Where the confidence level is defined as $1 - \beta$.

To find the b value that satisfies $1 - \beta = 0.90$ and $1 - \beta = 0.95$, Equation 14 is used to plot the CDF to find b, as shown in Figure 9 (b) and than the value for b, i.e. the upper mass limit:

$$b_{90\%} = 0.70\,\mathrm{e}V^2c^{-4}$$
$$b_{95\%} = 0.82\,\mathrm{e}V^2c^{-4}$$

c) What upper mass limits of this quantity do you obtain for a confidence level of 90% and 95%, respectively when using the Bayesian method.

Define the probability $P(x|\mu, \sigma)$ of observing a mass value of x given the known parameter μ and σ as a normal probability density function, similar to Equation 13. Truncate the function for non-physical mass values

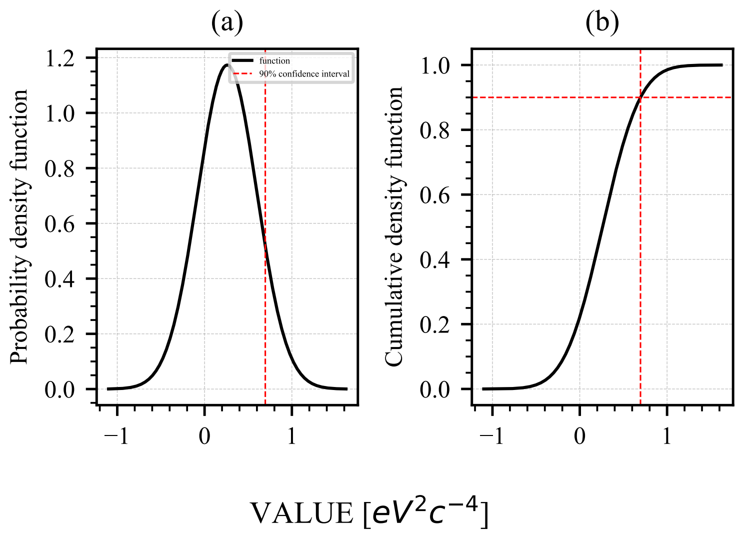


Figure 9: Frequentist approach for finding the 90% confidence interval.

$$P'(x|\mu,\sigma) = \begin{cases} 0, & \text{if } x < 0. \\ P(x|\mu,\sigma), & \text{otherwise.} \end{cases} \tag{15}$$

normalize $P'(x|\mu,\sigma)$ so its integral equals to 1, Figure 10 (a), then take the CDF to find the upper mass limit b.

$$Z = \int_0^\infty P'(x|\mu,\sigma)dx$$

$$G(x) = \frac{1}{Z}\int_{-\infty}^{b} P'(x|\mu,\sigma)dx \tag{16}$$

To find the b value that satisfies $1-\beta = 0.90$ and $1-\beta = 0.95$, Equation 16 is used to plot the CDF to find b, as shown in Figure 10 (b) and than the value for b, i.e. the upper mass limit:

$$b_{90\%} = 0.74\,\mathrm{e}V^2c^{-4}$$
$$b_{95\%} = 0.86\,\mathrm{e}V^2c^{-4}$$

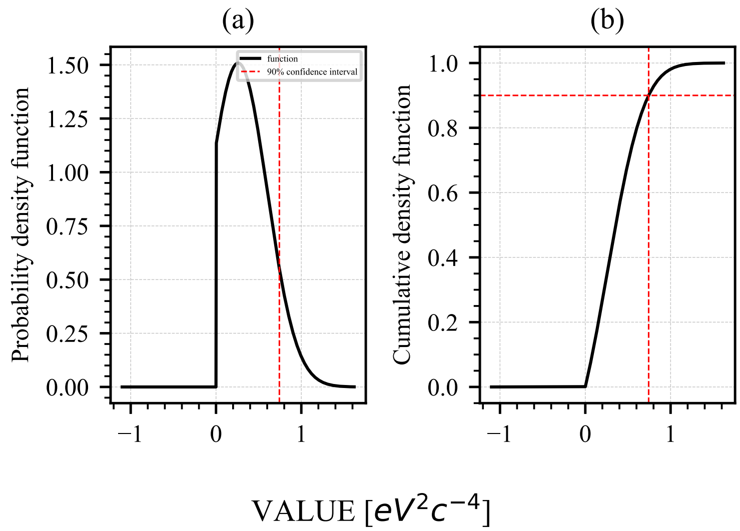


Figure 10: Bayesian approach for finding the 90% confidence interval.

Question 3: Error propagation

The answer of this question is structured as follow:

- Derive the covariance matrix for the cartesian coordinates
- Compute the covariance and means for cartesian coordinates
- Derive the variance matrix for spherical coordinates
- Compute the covariance and means for spherical coordinates
- The 2 previous steps simultaneously answer questions 3.a) and 3.b) from exercise set instructions
- Explain the meaning of the off-diagonal elements, the answer to 3.c) from exercise set instructions
- The hand written derivation are also presented in the appendix

To begin solving this problem, the approach is similar to the one in question 2.a) using the propagation Equation 12.

The assumptions made for this question are:

- Measurements x, y, z are independent i.e. V(x,y,z) is diagonal.

Consider the following function for a vector calculation between two points.

$$y(x, y, z) = V^0 = p2(x, y, z) - p1(x, y, z) \tag{17}$$

$$V^0 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Where *p2* is the decay point and *p1* is the production point.

Taking the partial derivative of V^0 with respect to the cartesian coordinates and for each point.

$$\frac{\partial V^0(x, y, z)}{\partial p1(x, y, z)} = -1$$

$$\frac{\partial V^0(x, y, z)}{\partial p2(x, y, z)} = 1$$

The previous are valid for coordinates on the same axis. The consequence of independence of measurements is that the derivative of coordinates on different axis are zero.

$$\frac{\partial V_x^0}{\partial y_1} = \frac{\partial V_x^0}{\partial z_1} = \frac{\partial V_x^0}{\partial y_2} = \cdots = 0$$

The Jacobian

$$\mathbf{J} = \begin{vmatrix} \frac{\partial V_x^0}{\partial x_1} & \frac{\partial V_x^0}{\partial y_1} & \frac{\partial V_x^0}{\partial z_1} & \frac{\partial V_x^0}{\partial x_2} & \frac{\partial V_x^0}{\partial y_2} & \frac{\partial V_x^0}{\partial z_2} \\ \frac{\partial V_y^0}{\partial x_1} & \frac{\partial V_y^0}{\partial y_1} & \frac{\partial V_y^0}{\partial z_1} & \frac{\partial V_y^0}{\partial x_2} & \frac{\partial V_y^0}{\partial y_2} & \frac{\partial V_y^0}{\partial z_2} \\ \frac{\partial V_z^0}{\partial x_1} & \frac{\partial V_z^0}{\partial y_1} & \frac{\partial V_z^0}{\partial z_1} & \frac{\partial V_z^0}{\partial x_2} & \frac{\partial V_z^0}{\partial y_2} & \frac{\partial V_z^0}{\partial z_2} \end{vmatrix} \tag{18}$$

is then

$$\mathbf{J} = \begin{vmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{vmatrix} \tag{19}$$

The covariance matrix

$$\mathbf{V}[\mathbf{p1}, \mathbf{p2}] = \begin{vmatrix} \sigma_{x_1}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{y_1}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{z_1}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{x_2}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{y_2}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{z_2}^2 \end{vmatrix} \tag{20}$$

Using Equation 12, the covariance matrix of the resulting vector

$$\mathbf{V^0}[\mathbf{cartesian}] = \begin{vmatrix} 0.0421 & 0 & 0 \\ 0 & 0.0545 & 0 \\ 0 & 0 & 0.1352 \end{vmatrix} \tag{21}$$

And the average direction vector

$$\mu_{\mathbf{V^0}}[\mathbf{cartesian}] = \begin{vmatrix} 15.60 \\ -3.50 \\ 0.20 \end{vmatrix} \tag{22}$$

Next, we want to make the transformation from cartesian to spherical $V^0(x, y, z) \to V^0(r, \theta, \phi)$.

Let the transformation functions be

$$r^2 = x^2 + y^2 + z^2$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

since $x > 0$

$$\phi = \tan^{-1} \frac{\rho}{z}$$

It is noted that the nomenclature definition for this solution is switched in relation to the conventional spherical coordinate system.

Since the spherical transformation, the previous functions are correlated, the Jacobian will have off-diagonal terms.

$$\mathbf{J} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \tag{23}$$

The Jacobian terms are

$$\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2 + y^2 + z^2}}{\partial x} = \frac{x}{r}$$

Similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{\rho^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{\rho^2}$$

$$\frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial \phi}{\partial x} = \frac{xz}{r^2 \cdot \rho}$$

$$\frac{\partial \phi}{\partial y} = \frac{yz}{r^2 \cdot \rho}$$

$$\frac{\partial \phi}{\partial z} = \frac{-\rho}{r^2}$$

Applying Equation 12 with $V(x) = V^0[cartesian]$ from Equation 21 and the previous Jacobian, Equation 23, Equation 25 is then the covariance matrix of vector V^0 in the spherical coordinates.

$$\mathbf{V^0[polar]} = \begin{vmatrix} 0.04271 & -0.0001657 & -7.236 \cdot 10^{-5} \\ -0.0001657 & 0.0002109 & -1.296 \cdot 10^{-7} \\ -7.236 \cdot 10^{-5} & -1.296 \cdot 10^{-7} & 0.0005288 \end{vmatrix} \tag{24}$$

$$\mathbf{V^0}[\mathbf{polar}] = \begin{vmatrix} 0.04271 & -0.0001657 & -7.236 \cdot 10^{-5} \\ -0.0001657 & 0.0002109 & -1.296 \cdot 10^{-7} \\ -7.236 \cdot 10^{-5} & -1.296 \cdot 10^{-7} & 0.0005288 \end{vmatrix} \tag{25}$$

The analytical expression of the covariance matrix is presented at Figure 11.

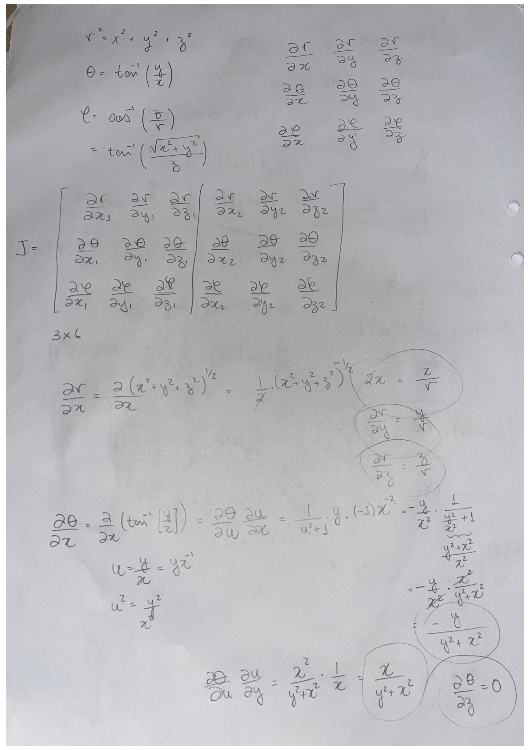
Finally, using the transformation function on the values from Equation 22, the resulting vector in the spherical coordinates (r, θ, ϕ)

$$\mu_{\mathbf{V^0}}[\mathbf{polar}] = \begin{vmatrix} 15.99 \text{ mm} \\ -12.65^\circ \\ 89.28^\circ \end{vmatrix} \tag{26}$$

The off-diagonal terms of the covariance matrix represents how the uncertainty of the random variables are related to each other, when equals to zero, the variables are independent, while a value $\neq 0$ means that the uncertainties of the i-th and j-th variable (for $i \neq j$) are related, this is proportional to the correlation length ρ since

$$cov(i, j) = \rho \cdot \sigma_i \cdot \sigma_j$$

Appendix



$$r^2 = x^2 + y^2 + z^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\varphi = \cos^{-1}\left(\frac{z}{r}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\begin{matrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{matrix}$$

$$J = \left[\begin{array}{ccc|ccc} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial y_1} & \frac{\partial r}{\partial z_1} & \frac{\partial r}{\partial x_2} & \frac{\partial r}{\partial y_2} & \frac{\partial r}{\partial z_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial y_1} & \frac{\partial \theta}{\partial z_1} & \frac{\partial \theta}{\partial x_2} & \frac{\partial \theta}{\partial y_2} & \frac{\partial \theta}{\partial z_2} \\ \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial y_1} & \frac{\partial \varphi}{\partial z_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial \varphi}{\partial y_2} & \frac{\partial \varphi}{\partial z_2} \end{array}\right]$$

3×6

$$\frac{\partial r}{\partial x} = \frac{\partial (x^2+y^2+z^2)^{1/2}}{\partial x} = \frac{1}{\cancel{2}} \cdot (x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}x = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left[\frac{y}{x}\right]\right) = \frac{\partial \theta}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{u^2+1} \cdot y \cdot (-1)x^{-2} = -\frac{y}{x^2} \cdot \frac{1}{\underbrace{\frac{y^2}{x^2}+1}_{\frac{y^2+x^2}{x^2}}}$$

$$u = \frac{y}{x} = yx^{-1}$$

$$u^2 = \frac{y^2}{x^2}$$

$$= -\frac{y}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{y^2+x^2}$$

$$= \frac{-y}{y^2+x^2}$$

$$\frac{\partial \theta}{\partial u}\frac{\partial u}{\partial y} = \frac{x^2}{y^2+x^2} \cdot \frac{1}{x} = \frac{x}{y^2+x^2}$$

$$\frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right)$$

$$\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{1}{u^2+1}\cdot\frac{1}{z}\cdot\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x$$

$$u = \frac{\sqrt{x^2+y^2}}{z}$$

$$= \frac{z^2}{r^2}\cdot\frac{1}{\cancel{z}}\cdot\frac{x}{\sqrt{(x^2+y^2)}} = \frac{xz}{r^2}\cdot\frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{1}{u^2+1} = \frac{1}{\frac{x^2+y^2}{z^2}+1}$$

$$= \frac{1}{\frac{x^2+y^2+z^2}{z^2}}$$

$$= \frac{z^2}{r^2}$$

$$\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{z^{\cancel{2}}}{r^2}\cdot\frac{1}{\cancel{z}}\cdot\frac{1}{2}\cdot\frac{2y}{\sqrt{x^2+y^2}}$$

$$= \frac{zy}{r^2}\cdot\frac{1}{\sqrt{x^2+y^2}}$$

$$z^{-1} = -z^{-2}$$

$$\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial z} = \frac{\cancel{z}^{\cancel{2}}}{r^2}\cdot\sqrt{x^2+y^2}\,(-1)\cdot\cancel{z}^{\cancel{-2}}$$

$$= -\frac{\sqrt{x^2+y^2}}{r^2}$$

(A) $\frac{\partial r}{\partial x} = \frac{x}{r}$ (D) $\frac{\partial \theta}{\partial x} = \frac{-y}{y^2+x^2}$ (G) $\frac{\partial \varphi}{\partial x} = \frac{xz}{r^2\sqrt{x^2+y^2}}$

(B) $\frac{\partial r}{\partial y} = \frac{y}{r}$ (E) $\frac{\partial \theta}{\partial y} = \frac{x}{y^2+x^2}$ (H) $\frac{\partial \varphi}{\partial y} = \frac{yz}{r^2\sqrt{x^2+y^2}}$

(C) $\frac{\partial r}{\partial z} = \frac{z}{r}$ (F) $\frac{\partial \theta}{\partial z} = 0$ (i) $\frac{\partial \varphi}{\partial z} = \frac{-\sqrt{x^2+y^2}}{r^2}$

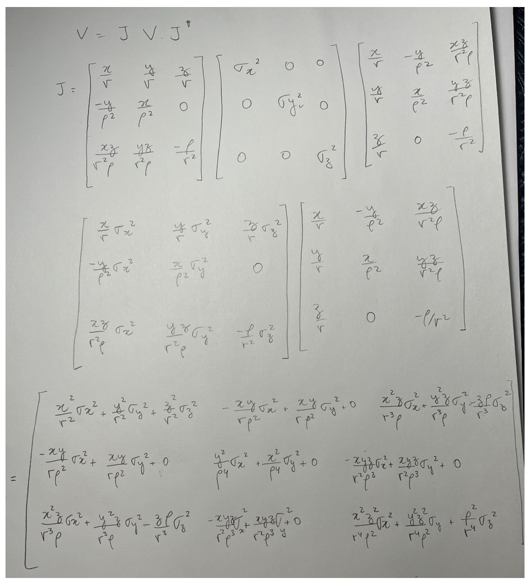


Figure 11: Analytical covariance matrix