Esercizi sulle matrici:

- Prodotto $A \cdot B = C$ $\qquad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix}$$

$A \in \mathbb{R}^{3\times 4} \quad B \in \mathbb{R}^{4\times 3}$

$C = A \cdot B \in \mathbb{R}^{3\times 3}$

$D = B \cdot A \in \mathbb{R}^{4\times 4}$

$$C = \begin{pmatrix} 4 & 2 & 5 \\ 0 & 0 & 1 \\ 5 & 3 & 8 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 2 & 2 & 3 & 4 \\ -3 & -2 & -2 & -3 \\ 5 & 4 & 8 & 11 \end{pmatrix}$$

- Proprietà commutativa in generale non valida:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 3 & 0 \\ 7 & 0 \end{pmatrix} \qquad B \cdot A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

- Matrici inverse:

$$A = \begin{pmatrix} -4 & 2 \\ -1 & 1 \end{pmatrix}$$

(1)

$$\det A = -4 + 2 = -2 \quad |A|$$

$$B = A^{-1} \quad : \quad b_{ij} = (-1)^{i+j} \frac{|A_{ji}|}{|A|}$$

$$b_{11} = (-1)^2 \cdot \frac{|A_{11}|}{|A|} = \frac{1}{-2} = -\frac{1}{2}$$

$$b_{12} = (-1)^3 \frac{|A_{21}|}{|A|} = \frac{-2}{-2} = 1$$

$$b_{21} = (-1)^3 \frac{|A_{12}|}{|A|} = - \frac{-1}{-2} = -\frac{1}{2}$$

$$b_{22} = (-1)^4 \frac{|A_{22}|}{|A|} = \frac{-4}{-2} = +2$$

$$B = \begin{pmatrix} -1/2 & 1 \\ -\frac{1}{2} & 2 \end{pmatrix} = A^{-1}$$

$$A \cdot A^{-1} = \begin{pmatrix} -4 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(2)

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \quad L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

③ $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ $\quad |A| = 1+12-2-3=8$

$A^{-1} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{8} & \frac{5}{8} \\ +\frac{3}{4} & -\frac{1}{8} & -3/8 \\ -\frac{1}{4} & 3/8 & \frac{1}{8} \end{pmatrix}$

- Matrice diag. dominante: $\quad |a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad \forall i$

$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 4 & 1 \\ 2 & 3 & 7 \end{pmatrix}$

Trovare α: $\begin{pmatrix} \alpha & 2 & -1 \\ 1 & 3 & 1 \\ 1 & -2 & 5 \end{pmatrix}$

$|\alpha| > 3$

$\cancel{-3<\alpha<3}$ $\quad \alpha > 3$

$\alpha < -3$

- Definita positiva:

$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$A_1 = 3$

$A_2 = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}$

$|A_2| = 5$

$|A_3| = 7$

- Vettori : $x = \begin{vmatrix} 1 \\ 3 \\ 0 \end{vmatrix}$ $y = \begin{vmatrix} 2 \\ 1 \\ 2 \end{vmatrix}$

$$x^T y = \begin{vmatrix} 1 & 3 & 0 \end{vmatrix} \begin{vmatrix} 2 \\ 1 \\ 2 \end{vmatrix} = 5$$

$$x y^T = \begin{vmatrix} 1 \\ 3 \\ 0 \end{vmatrix} \begin{vmatrix} 2 & 1 & 2 \end{vmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 6 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\|x\|_1 = 4 = \sum |x_i|$$
$$\|x\|_2 = \sqrt{\sum x_i^2} = \sqrt{10}$$
$$\|x\|_\infty = 3 = \max_i |x_i|$$

$$A = \begin{array}{cc} \begin{pmatrix} 1 & 3 & 2 \\ -1 & 5 & -6 \\ -2 & 0 & 1 \end{pmatrix} & \begin{array}{c} \Sigma_j \\ 6 \\ 12 \\ 3 \end{array} \\ \begin{array}{cccc} \Sigma_i & 4 & 8 & 9 \end{array} & \end{array}$$

$$\|A\|_1 = \max_j \Sigma_i |a_{ij}| = 9$$
$$\|A\|_\infty = \max_i \Sigma_j |a_{ij}| = 12$$

Non equivalenza delle norme in $C[a,b]$

Sia $f(x)=0$, $w(x)=1$ $\forall x \in [a,b]$, $a=0$, $b=3$

$\forall K>0$ sia $f_K(x)$:

$$f_K(x) = \begin{cases} K(Kx^2-1) & \frac{1}{K^2} \le x \le \frac{2}{K^2} \\ -K(K^2x-3) & \frac{2}{K^2} \le x \le \frac{3}{K^2} \\ 0 & \text{per altri } x \end{cases}$$

$\Rightarrow \|f-f_K\|_1 = \frac{1}{K}$

$\|f-f_K\|_2 = \frac{\sqrt{2}}{\sqrt{3}}$

$\|f-f_K\|_\infty = K$

$\Rightarrow \lim_{K\to\infty} \|f-f_K\|_1 = 0$

$\lim_K \|f-f_K\|_2 = \frac{\sqrt{2}}{\sqrt{3}}$

$\lim_K \|f-f_K\|_\infty = \infty$

Per cui $\{f_K(x)\}_{K=1}^{\infty} \to f(x)$ solo nella $\|\cdot\|_1$