Metodi iterativi per sistemi lineari

Eseguire il primo passo dei metodi di Jacobi e Gauss-Seidel per il sistema: $\quad Ax=b \quad$ con:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 12 \\ 0 \end{pmatrix} \qquad \underline{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

la cui sol. è: $\underline{x} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ e mostrare la convergenza/divergenza.

$$\text{J.} \quad \begin{cases} x_1^{(1)} = \dfrac{4+1-1}{2} = 2 \\ x_2^{(1)} = \dfrac{12-2-2}{2} = 4 \\ x_3^{(1)} = \dfrac{0+1+1}{2} = 1 \end{cases}$$

$$\text{G.S.} \quad \begin{cases} x_1^{(1)} = 2 \\ x_2^{(1)} = \dfrac{12-4-2}{2} = 3 \\ x_3^{(1)} = \dfrac{2+3}{2} = 5/2 \end{cases}$$

Troviamo le matrici M_J, M_{GS}.

$$D = \begin{pmatrix} 2 & & 0 \\ & 2 & \\ 0 & & 2 \end{pmatrix} \; ; \quad D^{-1} = \begin{pmatrix} 1/2 & & 0 \\ & 1/2 & \\ 0 & & 1/2 \end{pmatrix}$$

$$E+F = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_J = D^{-1}(E+F) = \begin{pmatrix} 0 & 1/2 & -1/2 \\ -1 & 0 & -1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$\det(M_J - \lambda I) = \det\begin{pmatrix} -\lambda & 1/2 & -1/2 \\ -1 & -\lambda & -1 \\ 1/2 & 1/2 & -\lambda \end{pmatrix} = -\lambda^3 + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{1}{4}\lambda - \frac{1}{2}\lambda - \frac{1}{2}\lambda =$$

$$= -\lambda^3 - \frac{5}{4}\lambda$$

$$\lambda\left(\lambda^2 + \frac{5}{4}\right) = 0 \rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = i\frac{\sqrt{5}}{2} \\ \lambda_3 = -i\frac{\sqrt{5}}{2} \end{cases} \qquad \rho(M_J) > 1$$

Pertanto il metodo di Jacobi diverge.

$$D-E = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{pmatrix} \quad ; (D-E)^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_{GS} = (D-E)^{-1}F = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{pmatrix}$$

$$\det(M_{GS} - \lambda I) = \det\begin{pmatrix} -\lambda & 1/2 & -1/2 \\ 0 & -1/2-\lambda & -1/2 \\ 0 & 0 & -1/2-\lambda \end{pmatrix} = -\lambda\left(-\frac{1}{2}-\lambda\right)^2$$

$$-\lambda\left(-\frac{1}{2}-\lambda\right)^2 = 0$$

$\lambda_1 = 0$, $\lambda_{2,3} = -\frac{1}{2}$ $\qquad \rho(M_{GS}) < 1 \Rightarrow$ Gauss-Seidel converge.