

IE533_Homework2

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NAME: Duo ZHOU

NETID: duozhou2

1 Question 1

Show that a directed graph G is acyclic if and only if we can renumber its nodes so that its node-node adjacency matrix is a lower triangular matrix.

1.1 Solution

Proof: A directed graph G is acyclic if and only if there are no directed cycles in the graph.

First, let's prove that if G is acyclic, we can renumber its nodes so that its node-node adjacency matrix is a lower triangular matrix.

An acyclic directed graph can be topologically sorted, which means that the nodes can be renumbered such that for every directed edge (u, v) , node u appears before node v in the ordering.

\therefore If there is an edge (u, v) and u is assigned a lower number than v , then the entry in the adjacency matrix for the edge (u, v) will be in a lower row and column than the entry for the edge (v, u) (if it exists).

\therefore If we use this topological ordering as the new numbering of the nodes, the node-node adjacency matrix will be a lower triangular matrix.

If the adjacency matrix of a graph is lower triangular, \rightarrow there is no edge from a node with a higher number to a node with a lower number, \rightarrow there are no directed cycles in the graph, and therefore the graph is acyclic.

\therefore A directed graph G is acyclic if and only if we can renumber its nodes so that its node-node adjacency matrix is a lower triangular matrix.

2 Question 2

Let M denote the node-node adjacency matrix of a network G . Show that G is strongly connected if and only if the matrix $M + M^2 + M^3 + \dots + M^n$ has no zero entry, for some integer n .

2.1 Solution

Proof:

Lemma 1: A graph G is strongly connected if there exists a path from every vertex to every other vertex.

Lemma 2: The adjacency matrix M of a graph G represents the edges between the vertices, where the entry $M_{i,j}$ is 1 if there is an edge from vertex i to vertex j , and 0 otherwise.

\therefore The matrix M^k for $k \geq 1$ represents the paths of length k between the vertices. \therefore If there exists an entry $M_{i,j}^k = 1$, it means that there is a path of length k from vertex i to vertex j .

The sum of matrices $M + M^2 + M^3 + \dots + M^n$ represents all possible paths of length 1, 2, 3, \dots , n between vertices. If all entries of the sum are non-zero, it means that there exists a path between every vertex and every other vertex, and therefore the graph is strongly connected.

Conversely, if the graph is strongly connected, there exists a path between every vertex and every other vertex. Hence, for any n , there will always exist a path of length n between every vertex and every other vertex, which means that all entries of the sum $M + M^2 + M^3 + \dots + M^n$ will be non-zero.

\therefore A graph G is strongly connected if and only if the matrix $M + M^2 + M^3 + \dots + M^n$ has no zero entry for some integer n .

3 Question 3

Consider an undirected network with an origin, a destination, and 100 additional nodes, with each pair of nodes connected by an arc. Show that the number of different paths from origin to destination is given by

$$\binom{100}{0} 100! + \binom{100}{1} 99! + \binom{100}{2} 98! + \dots + \binom{100}{98} 2! + \binom{100}{99} 1! + \binom{100}{100} 0!$$

3.1 Solution

Proof:

$$\therefore \binom{n}{i} = \binom{n}{n-i}, i < n$$

\therefore Reformulating as:

$$\binom{100}{100} 100! + \binom{100}{99} 99! + \binom{100}{98} 98! + \dots + \binom{100}{2} 2! + \binom{100}{1} 1! + \binom{100}{0} 0!$$

The last term, $\binom{100}{0} 0!$, represents the number of different paths with no intermediate nodes.

The second last term, $\binom{100}{1} 1!$, represents the number of different paths with one intermediate node.

And so on, until the first term, $\binom{100}{100} 100!$, which represents the number of different paths with 100 intermediate nodes.

Each binomial coefficient, $\binom{100}{k}$, gives the number of ways to choose k nodes out of 100, and the factorial, $k!$, gives the number of different orderings of these k nodes.

\therefore The expression represents the total number of different paths from the origin to the destination, taking into account all possible intermediate nodes and all possible orderings of these nodes.

4 Question 4

4.1 Show that every tree forms an undirected bipartite graph.

4.1.1 Solution

Proof:

Lemma1: A tree is a connected graph with no cycles.

Lemma2: An undirected bipartite graph is a graph whose vertices can be partitioned into two sets such that no two vertices in the same set are adjacent.

Given a tree, we can form an undirected bipartite graph by partitioning its vertices into two sets A and B . Start with an arbitrary vertex and assign it to set A . Then, for each vertex in set A , assign its neighbors to set B , and for each vertex in set B , assign its neighbors to set A .

Since a tree has no cycles, it follows that each vertex has at most two neighbors, and therefore, the partitioning of vertices into sets A and B ensures that no two vertices in the same set are adjacent.

\therefore A tree forms an undirected bipartite graph.

4.2 Does every undirected bipartite graph form a tree, prove or give a counter example?

4.2.1 Solution

No. For example, consider the bipartite graph with two vertices in set A and two vertices in set B , and two edges connecting the vertices in different sets. This graph is not a tree as it contains a cycle.

Therefore, it can be concluded that not every undirected bipartite graph forms a tree.

5 Programming Question:

For this part of the homework, you will implement some of the algorithms learnt in class. You may use Python, C++, or Java to execute this (Prefer C++ as it will be helpful with CUDA later). Custom packages cannot be used to replace the algorithms tested. The dataset for this can be downloaded from [here](#). This source contains an undirected graph with 112 nodes and 425 edges.

6 Solution

```
[1]: import time
import numpy as np
import pandas as pd
```

```
[2]: class VertexMatrix(object):
    """
    Vertex Class
    """
    def __init__(self, data):
```

```
self.data = data
self.info = None
```

```
[3]: class Graph(object):

    def __init__(self, kind):
        # kind: Undigraph, Digraph, Undinetwork, Dinetwork,
        self.kind = kind
        # vertex list: 1d
        self.vertices = []
        # Edge list: 2d
        self.arcs = []
        # number of vertices
        self.vexnum = 0
        # number of edges
        self.arcnum = 0

    def CreateGraph(self, vertex_list, edge_list, weights):
        self.vexnum = len(vertex_list)
        self.arcnum = len(edge_list)
        for vertex in vertex_list:
            vertex = VertexMatrix(vertex)
            self.vertices.append(vertex)
            # create adjacency matrix depends on weighted or not
            if weights:
                self.arcs.append([float('inf')] * self.vexnum)
            else:
                self.arcs.append([float(0)] * self.vexnum)

        for edge in edge_list:
            ivertex = self.LocateVertex(edge[0])
            jvertex = self.LocateVertex(edge[1])
            if weights:
                weight = edge[2]
            else:
                weight = 1
            self.InsertArc(ivertex, jvertex, weight)

        return self.InsertArc(ivertex, jvertex, weight)

    def LocateVertex(self, vertex):
        index = 0
        while index < self.vexnum:
            if self.vertices[index].data == vertex:
                return index
            else:
```

```

        index += 1

def InsertArc(self, ivertex, jvertex, weight):
    if self.kind == 'Undinetwork':
        self.arcs[ivertex][jvertex] = weight
        self.arcs[jvertex][ivertex] = weight
        return self.arcs

    if self.kind == 'Dinetwork':
        self.arcs[ivertex][jvertex] = weight
        #self.arcs[jvertex][ivertex] = weight
        return self.arcs

def ConnectTest(self, matrix):
    n_entrys = 0
    id_nodes = []

    for i in range(len(matrix)):
        n_entrys += sum(matrix[i])
        id_nodes.append(sum(matrix[i]))
    if self.kind == 'Undinetwork':
        n_entrys /= 2
    if n_entrys == self.arcnum:
        print(f'#Entries = {n_entrys} = #Edge = {self.arcnum}. This graph is_
↪connected. The median degree of the nodes is {np.median(id_nodes)}')
    else:
        print(f'This graph is NOT connected ')

def TriangleDetect(self, matrix):
    return np.linalg.matrix_power(matrix,3).trace()/6

def AddEdges(self):
    edges = []
    i = 0
    while i < self.vexnum:
        j = 0
        while j < self.vexnum:
            if self.arcs[i][j] != float('inf'):
                edges.append([self.vertices[i].data, self.vertices[j].data,
↪self.arcs[i][j]])
            j += 1
        i += 1
    return sorted(edges, key=lambda item: item[2])

def Kruskal(self):
    edges = self.AddEdges()
    flags = []

```

```

for index in range(self.vexnum):
    flags.append(index)
index = 0
while index < len(edges):
    ivertex = self.LocateVertex(edges[index][0])
    jvertex = self.LocateVertex(edges[index][1])
    if flags[ivertex] != flags[jvertex]:
        iflag = flags[ivertex]
        jflag = flags[jvertex]
        limit = 0
        while limit < self.vexnum:
            if flags[limit] == jflag:
                flags[limit] = iflag
            limit += 1
        index += 1
    else:
        edges.pop(index)
return edges

def GetMin(self, closedge):
    index = 0
    vertex = 0
    minweight = float('inf')
    while index < self.vexnum:
        if closedge[index][1] != 0 and closedge[index][1] < minweight:
            minweight = closedge[index][1]
            vertex = index
        index += 1
    return vertex

def Prim(self, start_vertex):
    k = self.LocateVertex(start_vertex)
    closedge = []
    arc = []
    for index in range(self.vexnum):
        closedge.append([k, self.arcs[k][index]])
    closedge[k][1] = 0
    index = 1
    while index < self.vexnum:
        minedge = self.GetMin(closedge)
        arc.append([self.vertices[closedge[minedge][0]].data, self.
→vertices[minedge].data, closedge[minedge][1]])
        closedge[minedge][1] = 0
        i = 0
        while i < self.vexnum:
            if self.arcs[minedge][i] < closedge[i][1]:
                closedge[i] = [minedge, self.arcs[minedge][i]]

```

```

        i += 1
        index += 1
    return arc

def Output(self, algorithm = 'Prim', firstedge = 1):
    print('=====')
    print('{0} MST: '.format(algorithm))
    weight = 0

    start = time.time()
    if algorithm == 'Prim':
        mst = self.Prim(firstedge)
    elif algorithm == 'Kruskal':
        mst = self.Kruskal()
    else:
        print('Wrong Algorithm')
    stop = time.time()

    for edge in mst:
        weight += edge[2]
        print('{0}-->{1}: {2}'.format(edge[0], edge[1], edge[2]))
    print('Sum of Weight:', weight)
    #print('Time Consuming of {0} = {1}'.format(algorithm, stop - start))
    print('=====')
    return stop - start

```

6.1 a. Store the graph in adjacency matrix form.

```

[4]: df = pd.read_table('/Users/claus/Downloads/adjnoun_adjacency/out.
    ↪adjnoun_adjacency_adjacency', sep=" ", names = ["node", "edge", "NA", "NA1"],
    ↪index_col=None)
df = df.drop(index=[0,1], columns=["NA", "NA1"])
df["w"] = df[["node", "edge"]].apply(lambda x: abs(int(x["node"]) -
    ↪int(x["edge"])), axis=1)

vertex_list = [i for i in range(1, 113)]
edge_list = df.to_numpy().astype(int)

```

```

[5]: graph1 = Graph(kind='Undinetwork')
G_unweighterd = graph1.CreateGraph(vertex_list, edge_list, False)
print(G_unweighterd)

```

```

[[0.0, 1, 1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1, 0.0, 0.0, 0.0, 0.0, 1, 0.0, 0.0, 0.0,
1, 1, 1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1, 0.0, 0.0, 0.0, 0.0, 0.0,
0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 1, 0.0, 0.0, 0.0, 1, 1, 0.0, 0.0, 0.0, 0.0,
1, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0,
0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0,

```

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

6.2 b. Is this graph connected? What is the median degree of the nodes?

```
[6]: graph1.ConnectTest(G_unweighterd)
```

```
#Entries = 425.0 = #Edge = 425. This graph is connected. The median degree of
the nodes is 6.0
```

6.3 c. Using the adjacency matrix, write a program to compute the number of triangles.

Note: A triangle is a connected subgraph of size three.

```
[7]: print('Number of Triangles: ', graph1.TriangleDetect(G_unweighterd))
```

Number of Triangles: 284.0

To set up instance of minimum spanning tree, add edge costs using the following procedure. For an edge between nodes with indices i and j , let $|i - j|$ be its edge cost. For example, the cost associated with edge $(31, 17)$ is 14. Using this modified graph:

```
[8]: graph2 = Graph(kind='Undinetwork')
      print(graph2.CreateGraph(vertex_list, edge_list, True))
```

```
[[inf, 1, 2, inf, inf, inf, inf, inf, 8, inf, inf, inf, inf, 13, inf, inf, inf,
17, 18, 19, inf, inf, inf, inf, inf, inf, inf, inf, 28, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, 41, inf, inf, inf, 45, 46, inf, inf, inf,
inf, 51, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, 91, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, 102, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf], [1, inf,
1, 2, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
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inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf], [2, 1, inf, 1,
inf, inf, inf, inf, 6, inf, inf, 9, inf, 11, inf, inf, inf, 15, inf, inf, inf,
19, 20, 21, 22, 23, 24, 25, inf, inf, inf, 29, inf, inf, 32, inf, 34, inf, inf,
inf, inf, 39, 40, inf, inf, 43, inf, inf, inf, inf, 48, 49, inf, inf, 52, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, 64, inf, inf, inf, inf, inf,
inf, 71, inf, inf, inf, inf, inf, inf, inf, inf, inf, 81, inf, inf, inf, inf,
inf, inf, inf, inf, inf, 91, inf, inf, inf, 95, 96, inf, inf, inf, 100, inf,
102, 103, inf, inf, inf, inf, inf, 109], [inf, 2, 1, inf, inf, inf, inf, inf, 5,
inf, 7, inf, inf, inf, inf, inf, inf, inf, 15, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
38, inf, 40, inf, inf, inf, inf, inf, inf, inf, 48, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf,
inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, inf, 82, inf, inf, inf,
```


[illegible]

[illegible]

[illegible]

[illegible]

6.4 d. Implement the Kruskal's algorithm to find the minimum spanning tree.

```
[9]: T1 = graph2.Output('Kruskal')
```

```
=====
Kruskal MST:
1-->2: 1
2-->3: 1
3-->4: 1
5-->6: 1
7-->8: 1
9-->10: 1
11-->12: 1
```


14-->15: 1
15-->16: 1
20-->21: 1
29-->30: 1
32-->33: 1
38-->39: 1
39-->40: 1
42-->43: 1
44-->45: 1
45-->46: 1
48-->49: 1
51-->52: 1
54-->55: 1
64-->65: 1
76-->77: 1
79-->80: 1
81-->82: 1
104-->105: 1
105-->106: 1
5-->7: 2
11-->13: 2
15-->17: 2
16-->18: 2
20-->22: 2
29-->31: 2
39-->41: 2
42-->44: 2
48-->50: 2
49-->51: 2
51-->53: 2
64-->66: 2
69-->71: 2
72-->74: 2
76-->78: 2
80-->82: 2
19-->22: 3
20-->23: 3
29-->32: 3
39-->42: 3
51-->54: 3
74-->77: 3
90-->93: 3
93-->96: 3
106-->109: 3
9-->13: 4
15-->19: 4
20-->24: 4
53-->57: 4

77-->81: 4
80-->84: 4
84-->88: 4
93-->97: 4
4-->9: 5
13-->18: 5
20-->25: 5
26-->31: 5
29-->34: 5
44-->49: 5
69-->74: 5
20-->26: 6
29-->35: 6
52-->58: 6
60-->66: 6
67-->73: 6
20-->27: 7
29-->36: 7
37-->44: 7
8-->16: 8
20-->28: 8
52-->60: 8
73-->81: 8
63-->72: 9
68-->77: 9
32-->42: 10
90-->100: 10
97-->107: 10
101-->111: 10
60-->71: 11
61-->72: 11
82-->95: 13
56-->71: 15
69-->85: 16
77-->93: 16
82-->99: 17
89-->107: 18
85-->105: 20
26-->47: 21
38-->59: 21
80-->110: 30
80-->111: 31
71-->103: 32
61-->102: 41
62-->105: 43
30-->75: 45
24-->70: 46
43-->91: 48

```

32-->87: 55
18-->83: 65
18-->86: 68
35-->108: 73
18-->94: 76
1-->92: 91
3-->98: 95
3-->112: 109
Sum of Weight: 1420
=====

```

6.5 e. Implement the Prim's algorithm to find the minimum spanning tree.

```
[10]: T2 = graph2.Output('Prim', 1) # edge start from index 1
```

```

=====
Prim MST:
1-->2: 1
2-->3: 1
3-->4: 1
4-->9: 5
9-->10: 1
9-->13: 4
13-->11: 2
11-->12: 1
13-->18: 5
18-->16: 2
16-->15: 1
15-->14: 1
15-->17: 2
15-->19: 4
19-->22: 3
22-->20: 2
20-->21: 1
20-->23: 3
20-->24: 4
20-->25: 5
20-->26: 6
26-->31: 5
31-->29: 2
29-->30: 1
29-->32: 3
32-->33: 1
29-->34: 5
29-->35: 6
20-->27: 7
29-->36: 7
16-->8: 8

```

8-->7: 1
7-->5: 2
5-->6: 1
20-->28: 8
32-->42: 10
42-->43: 1
42-->44: 2
44-->45: 1
45-->46: 1
42-->39: 3
39-->38: 1
39-->40: 1
39-->41: 2
44-->49: 5
49-->48: 1
48-->50: 2
49-->51: 2
51-->52: 1
51-->53: 2
51-->54: 3
54-->55: 1
53-->57: 4
52-->58: 6
44-->37: 7
52-->60: 8
60-->66: 6
66-->64: 2
64-->65: 1
60-->71: 11
71-->69: 2
69-->74: 5
74-->72: 2
74-->77: 3
77-->76: 1
76-->78: 2
77-->81: 4
81-->82: 1
82-->80: 2
80-->79: 1
80-->84: 4
84-->88: 4
81-->73: 8
73-->67: 6
72-->63: 9
77-->68: 9
72-->61: 11
82-->95: 13
71-->56: 15

```

69-->85: 16
77-->93: 16
93-->90: 3
93-->96: 3
93-->97: 4
90-->100: 10
97-->107: 10
82-->99: 17
107-->89: 18
85-->105: 20
105-->104: 1
105-->106: 1
106-->109: 3
26-->47: 21
38-->59: 21
80-->110: 30
80-->111: 31
111-->101: 10
71-->103: 32
61-->102: 41
105-->62: 43
30-->75: 45
24-->70: 46
43-->91: 48
32-->87: 55
18-->83: 65
18-->86: 68
35-->108: 73
18-->94: 76
1-->92: 91
3-->98: 95
3-->112: 109
Sum of Weight: 1420
=====

```

6.6 f. Report the performances for each algorithm (running time, optimal cost, and the optimal solution). Why is there a difference in the run time?

```

[11]: print('Optimal cost and the optimal Solution could be found above.\nTime_
      ↳Consuming of Krustal is {0},\nTime Consuming of Prim is {1},\nDifference of_
      ↳them is {2}'.format(T1,T2,T1-T2))

```

```

Optimal cost and the optimal Solution could be found above.
Time Consuming of Krustal is 0.007610797882080078,
Time Consuming of Prim is 0.005839824676513672,
Difference of them is 0.0017709732055664062

```

The prim algorithm complexity is related to the number of vertices, while the kruskal algorithm

complexity is related to the number of edges, and the number of edges describes the sparsity. That's why Prim algorithm is more suitable for dense graphs, Kruskal algorithm is more suitable for sparse graphs. In this graph there is not much difference between the two algorithms due to the number of both vertices and edges are small.

6.7 g. Formulate the minimum spanning tree problem as a polynomial sized mixed integer linear program (MILP).

6.7.1 Solution

Let $G = (V, E)$, w_e be the weight of edge $(i, j) \in E$, $E(S) \subseteq E$ is a subset of edges with both ends in subset $S \subseteq V$.

Introduce the variable $x_{ij} \in \{0, 1\}$, which indicate:

$$x_{ij} = \begin{cases} 1, & \text{If edge}(i, j) \text{ is in tree,} \\ 0, & \text{Otherwise} \end{cases}$$

$$\begin{aligned} \min : & \sum_{(i,j) \in E} w_{ij} x_{ij}, \\ \text{s.t.} & \sum_{(i,j) \in E(S)} x_{ij} \leq |S| - 1, \forall S \subseteq V, S \neq \emptyset, S \neq V \\ & \sum_{(i,j) \in E} x_{ij} = n - 1 \\ & x_{ij} \in \{0, 1\}, (i, j) \in E \end{aligned}$$

This MILP formulation ensures that the resulting tree T has $|V| - 1$ edges, and the sum of the weights of the edges in T is minimized.

6.8 e. For the given graph instance, solve the MILP formulation using a commercial solver of your choosing, and report the results (running time, optimal cost, and the optimal solution). How does this compare with part f?

```
[12]: import networkx as nx

G_e = nx.Graph()
for edge in edge_list:
    G_e.add_edge(edge[0], edge[1], weight = edge[2])

start = time.time()
T = nx.minimum_spanning_tree(G_e)
end = time.time()
```

```
[13]: sorted(T.edges(data=True))
```

```
[13]: [(1, 2, {'weight': 1}),
      (1, 92, {'weight': 91}),
      (2, 3, {'weight': 1}),
      (3, 4, {'weight': 1}),
      (3, 98, {'weight': 95}),
      (3, 112, {'weight': 109}),
      (5, 6, {'weight': 1}),
      (5, 7, {'weight': 2}),
      (7, 8, {'weight': 1}),
      (8, 16, {'weight': 8}),
      (9, 4, {'weight': 5}),
      (9, 10, {'weight': 1}),
      (9, 13, {'weight': 4}),
      (11, 13, {'weight': 2}),
      (12, 11, {'weight': 1}),
      (14, 15, {'weight': 1}),
      (15, 17, {'weight': 2}),
      (16, 15, {'weight': 1}),
      (18, 13, {'weight': 5}),
      (18, 16, {'weight': 2}),
      (18, 22, {'weight': 4}),
      (18, 83, {'weight': 65}),
      (18, 86, {'weight': 68}),
      (18, 94, {'weight': 76}),
      (19, 22, {'weight': 3}),
      (20, 21, {'weight': 1}),
      (20, 22, {'weight': 2}),
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      (20, 27, {'weight': 7}),
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      (29, 32, {'weight': 3}),
      (29, 34, {'weight': 5}),
      (29, 35, {'weight': 6}),
      (29, 36, {'weight': 7}),
      (30, 75, {'weight': 45}),
      (32, 33, {'weight': 1}),
      (32, 87, {'weight': 55}),
      (35, 108, {'weight': 73}),
      (37, 44, {'weight': 7}),
      (38, 39, {'weight': 1}),
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(40, 39, {'weight': 1}),
(42, 32, {'weight': 10}),
(42, 39, {'weight': 3}),
(42, 43, {'weight': 1}),
(42, 44, {'weight': 2}),
(43, 91, {'weight': 48}),
(44, 45, {'weight': 1}),
(44, 49, {'weight': 5}),
(46, 45, {'weight': 1}),
(47, 26, {'weight': 21}),
(49, 48, {'weight': 1}),
(51, 49, {'weight': 2}),
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(52, 50, {'weight': 2}),
(52, 51, {'weight': 1}),
(52, 55, {'weight': 3}),
(52, 58, {'weight': 6}),
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(60, 66, {'weight': 6}),
(61, 72, {'weight': 11}),
(61, 102, {'weight': 41}),
(63, 72, {'weight': 9}),
(64, 65, {'weight': 1}),
(64, 66, {'weight': 2}),
(67, 73, {'weight': 6}),
(69, 85, {'weight': 16}),
(71, 60, {'weight': 11}),
(71, 69, {'weight': 2}),
(73, 81, {'weight': 8}),
(74, 69, {'weight': 5}),
(74, 72, {'weight': 2}),
(74, 77, {'weight': 3}),
(76, 77, {'weight': 1}),
(76, 78, {'weight': 2}),
(77, 68, {'weight': 9}),
(77, 81, {'weight': 4}),
(77, 93, {'weight': 16}),
(80, 79, {'weight': 1}),
(80, 82, {'weight': 2}),
(80, 110, {'weight': 30}),
(80, 111, {'weight': 31}),
(81, 82, {'weight': 1}),


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(84, 80, {'weight': 4}),
(84, 88, {'weight': 4}),
(90, 93, {'weight': 3}),
(90, 100, {'weight': 10}),
(93, 96, {'weight': 3}),
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(97, 93, {'weight': 4}),
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(101, 111, {'weight': 10}),
(103, 71, {'weight': 32}),
(105, 62, {'weight': 43}),
(105, 85, {'weight': 20}),
(105, 104, {'weight': 1}),
(105, 106, {'weight': 1}),
(106, 109, {'weight': 3}),
(107, 89, {'weight': 18}),
(107, 97, {'weight': 10})]
```

```
[14]: all_w = 0
      for weight in T.edges(data=True):
          all_w += list(weight[2].values())[0]
      print(all_w)
```

1420

```
[15]: print('Optimal cost and the optimal Solution could be found above.\nTime_
      ↳Consuming of Krustal is {0},\nTime Consuming of Prim is {1},\nDifference of_
      ↳them is {2}\nTime Consuming of commercial solver is {3}'.format(T1,T2,T1-T2,_
      ↳end-start))
```

Optimal cost and the optimal Solution could be found above.
Time Consuming of Krustal is 0.007610797882080078,
Time Consuming of Prim is 0.005839824676513672,
Difference of them is 0.0017709732055664062
Time Consuming of commercial solver is 0.001027822494506836

Compared with part f, they shared the same optimal cost, and the optimal solution, but the efficiency of commercial solver is much more higher.