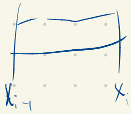
$$S_i''(x) = M_{i-1}\frac{x-x_i}{x_{i-1}-x_i} + M_i\frac{x-x_{i-1}}{x_i-x_{i-1}} = \frac{1}{h_i}\left(-M_{i-1}(x-x_i) + M_i(x-x_{i-1})\right)$$



$f''(x_{i-1}) = M_{i-1}$ $\quad f''(x_i) = M_i$

$S_i''(x_i) = M_i$

$S_i''(x_{i-1}) = M_{i-1}$

For natural spline. $M_0 = M_N = 0$

$$S_i'(x) = \frac{1}{h_i}\left(-M_{i-1}\frac{(x-x_i)^2}{2} - b_i + M_i\frac{(x-x_{i-1})^2}{2} + C_i\right)$$

$$S_i(x) = \frac{1}{h_i}\left(-M_{i-1}\frac{(x-x_i)^3}{6} - b_i(x-x_i) + M_i\frac{(x-x_{i-1})^3}{6} + C_i(x-x_{i-1})\right)$$

left-condition

$$S_i(x_{i-1}) = y_{i-1} = \frac{1}{h_i}\left(M_{i-1}\frac{h_i^3}{6} + b_i h_i\right) \quad \text{or} \quad y_{i-1} - M_{i-1}\frac{h_i^2}{6} = b_i$$

Right cond. $S_i(x_i) = y_i = S_i(x_i) = \frac{1}{h_i}\left(M_i\frac{h_i^3}{6} + C_i h_i\right) = \quad y_i - M_i\frac{h_i^2}{6} = C_i$

Plug this into $S_i'(x)$

$$S_i'(x) = \frac{1}{h_i}\left(M_i\frac{(x-x_{i-1})^2}{2} + y_i - M_i\frac{h_i^2}{6} - M_{i-1}\frac{(x-x_i)^2}{2} - y_{i-1} + M_{i-1}\frac{h^2}{6}\right) = \frac{1}{h_i}\left(-M_{i-1}\frac{(x-x_i)^2}{2} + M_i\frac{(x-x_{i-1})^2}{2} + (y_i - y_{i-1}) - (M_i - M_{i-1})h^2/6\right)$$

$$S_i'(x_i) = S_{i+1}'(x_i) \Rightarrow \frac{1}{\cancel{h_i}}\left(-b_i + \frac{M_i h_i^2}{2} + C_i\right) = \frac{1}{\cancel{h_{i+1}}}\left(-M_i\frac{h_{i+1}^2}{2} - b_{i+1} + C_{i+1}\right)$$

$$\frac{y_i - y_{i-1}}{h_i} + M_i\frac{h_i}{2} - (M_i - M_{i-1})\frac{h}{6} = -M_i\frac{h_{i+1}}{2} + \frac{(y_{i+1} - y_i)}{h_{i+1}} - (M_{i+1} - M_i)\frac{h}{6}$$

$y_{i+1} - 2y_i + y_{i-1}$

$$M_{i-1}\frac{h}{6} + M_i\underbrace{\left(\frac{h}{2} - \frac{h}{6} + \frac{h}{2} - \frac{h}{6}\right)}_{\frac{2}{3}h} + M_{i+1}\left(\frac{h}{6}\right) = \frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h_i}$$

divide by $2h$

$i = 1, \dots, n-1$

$$M_{i-1}\frac{1}{12} + M_i\frac{1}{3} + M_{i+1}\frac{1}{12} = \frac{y_{i+1} - 2y_i + y_{i-1}}{2h^2}$$

$i = 1 \quad \cancel{M_0}^{\,0}\frac{1}{2} + M_1\frac{1}{3} + M_2\frac{1}{12} = \frac{y_2 - 2y_1 + y_0}{2h^2}$ $\qquad i = 2 \quad M_1\frac{1}{12} + M_2\frac{1}{3} + M_3\frac{1}{12} = \frac{y_3 - 2y_2 + y_1}{2h^2}$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{12} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{12} & \frac{1}{3} & \frac{1}{12} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & & \frac{1}{12} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{N-1} \end{bmatrix} = \begin{bmatrix} \frac{y_2 - 2y_1 + y_0}{2h^2} \\ \frac{y_3 - 2y_2 + y_1}{2h^2} \\ \vdots \\ \frac{y_N - 2y_{N-1} + y_{N-2}}{2h^2} \end{bmatrix}$$