Homework 2, Numerical Analysis, Due to 15th Sept., Lena Kellner

Notation:
One = 1
Seven = 7
Using comma instead of point: $\frac{1}{2}=0{,}5$

① Show that ...

a) ... $(1+x)^n = 1+nx+o(x)$ as $x \to 0$

Soln.: Def. of "little o": Let $f(x)$ & $g(x)$ be defined in some neighbourhood of $x=0$.

We say $f(x) = o(g(x))$ as $x \to 0$

if $\lim\limits_{x\to 0} \left| \frac{f(x)}{g(x)} \right| = 0$.

Let's use Taylor series expansion: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$

We've got $f(x) = (1+x)^n$, $a=0$:

$$f(x) = (1+x)^n \quad \rightsquigarrow f(0) = 1$$

$$f'(x) = n(1+x)^{n-1} \quad \rightsquigarrow f'(0) = n$$

$$f''(x) = n\cdot(n-1)\cdot(1+x)^{n-2} \quad \rightsquigarrow f''(0) = n\cdot(n-1)$$

$$f'''(x) = n\cdot(n-1)\cdot(n-2)\cdot(1+x)^{n-3} \quad \rightsquigarrow f'''(0) = n\cdot(n-1)\cdot(n-2)$$

$$\vdots$$

$$\Rightarrow f(x) = 1+nx+\underbrace{\frac{n\cdot(n-1)}{2}x^2 + \frac{n\cdot(n-1)\cdot(n-2)}{6}x^3 + \ldots}_{=:\varphi(x)}$$

Now prove: $\varphi(x) = o(x)$ as $x \to 0$

Soln.:

$$\lim_{x\to 0}\left| \frac{\frac{n\cdot(n-1)}{2}\cdot x^2 + \frac{n\cdot(n-1)(n-2)}{6}x^3 + \ldots}{x} \right|$$

$$= \lim_{x\to 0}\left| \frac{n(n-1)x^2}{2x} + \frac{n(n-1)(n-2)x^3}{6x} + \ldots \right|$$

Exponent of x in numerator > Exponent of x in denominator

$$= 0$$

$\square$

$$\Rightarrow f(x) = 1+nx+\varphi(x)$$

$$= 1+nx+o(x) \quad \text{as } x \to 0$$

$\square$

b) ... $x \sin\sqrt{x} = O(x^{\frac{3}{2}})$ as $x \to 0$

Soln.: Def. big O:

We say $f(x) = O(g(x))$ as $x \to 0$ if $\exists$ a positive constant M s.t. $\left|\frac{f(x)}{g(x)}\right| \leq M \quad \forall x$ in a neighbourhood of 0.

Can also be shown by $\lim\limits_{x\to 0} \left|\frac{f(x)}{g(x)}\right| = k$ with k positive non-zero constant.

So, define $f(x) = x \sin\sqrt{x}$, $g(x) = x^{\frac{3}{2}}$.

Then $\lim\limits_{x\to 0} \left|\frac{x \sin\sqrt{x}}{x^{\frac{3}{2}}}\right| = \lim\limits_{x\to 0} \left|x^{-\frac{3}{2}} \cdot x \cdot \sin\sqrt{x}\right| = \lim\limits_{x\to 0} \left|x^{-\frac{1}{2}} \cdot \sin\sqrt{x}\right| = \lim\limits_{x\to 0} \left|\frac{\sin\sqrt{x}}{\sqrt{x}}\right|$

$\overset{\text{l'Hopital}}{=} \lim\limits_{x\to 0} \left|\frac{(\cos\sqrt{x}) \cdot x^{-\frac{1}{2}}}{\frac{1}{2} \cdot x^{-\frac{1}{2}}}\right| = \lim\limits_{x\to 0} \left|\frac{\cos\sqrt{x}}{\frac{1}{2}}\right| = \underline{\underline{2}} =: k$ non-zero, positive constant

$\Rightarrow f(x) = O(g(x)) \rightsquigarrow x \sin\sqrt{x} = O(x^{\frac{3}{2}})$ as $x \to 0$.

$\square$

c) ... $e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t \to \infty$

Soln.: Define $f(t) := e^{-t}$, $g(t) := \frac{1}{t^2}$

Then $\lim\limits_{t\to\infty} \left|\frac{f(t)}{g(t)}\right| = \lim\limits_{t\to\infty} \left|\frac{e^{-t}}{\frac{1}{t^2}}\right| = \lim\limits_{t\to\infty} \left|\frac{t^2}{e^t}\right| \overset{\text{l'H.}}{=} \lim\limits_{t\to\infty} \left|\frac{2t}{e^t}\right| \overset{\text{l'H.}}{=} \lim\limits_{t\to\infty} \left|\frac{2}{e^t}\right| = 0$

So, $f(t) = o(g(t)) \rightsquigarrow e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t \to \infty$.

$\square$

d) ... $\int_0^{\varepsilon} e^{-x^2}\,dx = O(\varepsilon)$ as $\varepsilon \to 0$.

Soln.: Define $f(\varepsilon) := \int_0^{\varepsilon} e^{-x^2}\,dx$, $g(x) := \varepsilon$.

Then $\lim\limits_{\varepsilon\to 0} \left|\frac{\int_0^{\varepsilon} e^{-x^2}\,dx}{\varepsilon}\right| \overset{\text{l'Hopital}}{=}$ (Fundamental Theorem of Calculus) $\lim\limits_{\varepsilon\to 0} \left|\frac{e^{-\varepsilon^2}}{1}\right| = \lim\limits_{\varepsilon\to 0} \left|\frac{1}{e^{\varepsilon^2}}\right| = 1 =: k$ non-zero, positive constant

$\Rightarrow f(\varepsilon) = O(g(x)) \rightsquigarrow \int_0^{\varepsilon} e^{-x^2}\,dx = O(\varepsilon)$ as $\varepsilon \to 0$.

$\square$

② Given: $A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

The exact solution of $Ax = b$ is $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

The inverse of A: $A^{-1} = \begin{pmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{pmatrix}$

There is a perturbation in b of $\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}$

a) Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx

Sol.: The solution of the exact problem is $x = A^{-1}b = \begin{pmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-10^{10}+10^{10} \\ 1+10^{10}+(-10^{10}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

—— " —— perturbed problem is $\Delta x = A^{-1}\left(b + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right) = \begin{pmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{pmatrix}\begin{pmatrix} 1+\Delta b_1 \\ 1+\Delta b_2 \end{pmatrix}$

$$= \begin{pmatrix} 1+\Delta b_1 \cancel{-10^{10}} - 10^{10}\Delta b_1 + \cancel{10^{10}} + 10^{10}\cdot\Delta b_2 \\ 1+\Delta b_1 + \cancel{10^{10}} + 10^{10}\Delta b_1 - \cancel{10^{10}} - 10^{10}\Delta b_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+(1-10^{10})\Delta b_1 + 10^{10}\Delta b_2 \\ 1+(1+10^{10})\Delta b_1 - 10^{10}\Delta b_2 \end{pmatrix}$$

Change in the solution $\delta x := \Delta x - x = \begin{pmatrix} 1+(1-10^{10})\Delta b_1 + 10^{10}\Delta b_2 \\ 1+(1+10^{10})\Delta b_1 - 10^{10}\Delta b_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \underline{\underline{\begin{pmatrix} (1-10^{10})\Delta b_1 + 10^{10}\Delta b_2 \\ (1+10^{10})\Delta b_1 - 10^{10}\Delta b_2 \end{pmatrix}}} = A^{-1}\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}$$

b) What is the condition # of A?

Sol.: The numerator of the relative condition # is $\dfrac{\left\| A^{-1}\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|}{\| A^{-1} b\|}$ (relative error)

The nominator of —— " —— is $\dfrac{\left\| b + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|}{\|b\|}$

$$\Rightarrow \kappa = \frac{\dfrac{\left\|A^{-1}\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|}{\|x\|}}{\dfrac{\left\|\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|}{\|b\|}} = \frac{\left\|A^{-1}\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|}{\|x\|} \cdot \frac{\|b\|}{\left\|\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|} \underset{\text{Cauchy-Schwarz}}{\leq} \frac{\|A^{-1}\| \cdot \left\|\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|}{\|x\|} \cdot \frac{\|b\|}{\left\|\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}\right\|}$$

$$= \frac{\|A^{-1}\| \cdot \|Ax\|}{\|x\|} \leq \|A^{-1}\| \cdot \|A\| =: \kappa(A)$$

Lecture?

$$\Rightarrow \kappa(A) = \left\| \begin{pmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{pmatrix} \right\|_2 \cdot \left\| \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{pmatrix} \right\|_2$$

$$= \sigma_{1_{A^{-1}}} \cdot \sigma_{1_A}$$

Python-code in Github?

$$= 2\cdot 10^{10} \cdot 1$$

$$= \underline{\underline{2\cdot 10^{10}}}$$

□

c) Let $\Delta b_1, \Delta b_2$ be of magnitude 10^{-5}.

i) What is the relative error of the solution?

$$\frac{|x-\Delta x|}{|x|} \overset{a)}{=} \frac{A^{-1}\begin{pmatrix}\Delta b_1\\ \Delta b_2\end{pmatrix}}{A^{-1}b} = \frac{\begin{pmatrix}\Delta b_1\\ \Delta b_2\end{pmatrix}}{b}$$

ii) What is the relationship between relative error, condition #, perturbation?

The condition # is independent of the perturbation since this is truncated when deriving the condition #.

When the perturbations are exactly the same, then the behavior should be the same. However, getting perturbation which are the same seems improbable.

□

③ Let $f(x) = e^x - 1$.

a) i) What is the relative condition # $K(f(x))$?

Since f is differentiable we can use the formula of the lecture of 8^{th} September:

$$K(x) = \frac{|f'(c)|\cdot|x|}{|f(x)|}.$$

$f'(x) = e^x$, so $K(x) = \dfrac{e^c \cdot x}{e^x - 1}$ for some $c \in (x, x+\Delta x)$ □

ii) Are there any values of x for which this is ill-conditioned?

As $K(x)$ approaches ∞ for x-values near 0, $K(x)$ is ill-conditioned there □

b) Consider computing f(x) via

1: y = math.e^x

2: return y-1.

Is this algorithm stable?

Soln.: Yes, it is stable. e^x is producing accurate results even if there are small variations in the input value x. The subtraction of 1 also is insensitive for small variations. So, also the algorithm of both steps is well-defined and stable.

□

c) Let x have the value $9.99999999500000 \cdot 10^{-10}$. Then $f(x) = 10^{-9}$ up to 16 decimal places.

How many correct digits does the algorithm give?

It's one correct digit since 1 just has one digit.

No, I would not have expected that.

④ See in Github.