Useful things from past math courses

Intermediate value theorem
Consider an interval $I = [a, b]$ for real numbers and continuous function $f : I \to \mathbb{R}$. Then if c is a number between $f(a)$ and $f(b)$ then there exist a value $w \in (a, b)$ such that $f(w) = c$.

Mean value theorem
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and differentiable on (a, b) where $a < b$. Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, there is a point c in the interval (a, b) where the tangent line at c has the same slope as the secant line through the points $(a, f(a))$ and $(b, f(b))$.

Taylor theorem
Let $f(x)$ be infinitely differentiable in an interval around the value $x = a$. Then

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Taylor polynomial approximation Let $T_N(x)$ denote the Taylor polynomial truncated at the N^{th} term; i.e.

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then there are two forms for the remainder

- *infinite sum*

$$R_N(x) = \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- *Concise formula thanks to MVT* There exists a c in the interval containing x and a such that

$$R_N(x) = \frac{f'^{N+1}(c)}{(N+1)!}(x-a)^{N+1}$$

The second expression tends to be the most useful. This is the one that yeilds the bound on the error

$$|R_N(x)| \leq \max_{\eta} |f^{N+1}(\eta)| \frac{1}{(N+1)!} |x-a|^{N+1}$$

where η is in the interval containing x and a.

Property of norms in vector spaces
Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ (n can equal 1) and $\alpha, \beta \in \mathbb{R}$, then

$$\|\alpha\mathbf{x} + \beta\mathbf{y}\| \leq |\alpha|\|\mathbf{x}\| + |\beta|\|\mathbf{y}\|.$$