Prelab 6

- Taylor-expansion of f(x) about point $x_0 = a$:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots$$

evaluated at $x = x_0 + h$:

$$f(x_0+h) = f(x_0) + f'(x_0)(x_0+h-x_0) + \frac{f''(x_0)}{2}(x_0+h-x_0)^2 + \frac{f'''(x_0)}{3!}(x_0+h-x_0)^3 + \dots$$

$$= f(x_0) + h \cdot f'(x_0) + \frac{h^2}{2} \cdot f''(x_0) + \frac{h^3}{3!} \cdot f'''(x_0) + \dots$$

- Taylor expansion of f(x) about point $x_0 = a$, evaluated at $x = x_0 - h$:

$$f(x_0-h) = f(x_0) + f'(x_0)(x_0-h-x_0) + \frac{f''(x_0)}{2}(x_0-h-x_0)^2 + \frac{f'''(x_0)}{3!}(x_0-h-x_0)^3 + \dots$$

$$= f(x_0) - h \cdot f(x_0) + \frac{h^2}{2} \cdot f''(x_0) - \frac{h^3}{3!} \cdot f'''(x_0) + \dots$$

- Put both expansions into formula for forward derivative

$$f'(x) = \frac{f(x+h) - f(x)}{h} = \frac{h \cdot f'(x) + \frac{h^2}{2} \cdot f''(x) + \frac{h^3}{3!} f'''(x) + \dots}{h} = f'(x) + \frac{h}{2} \cdot f''(x) + \frac{h^2}{6} \cdot f'''(x) + \dots$$

$\Rightarrow$ error term $O(h)$ converges linearly

- Put both expansions into formula for center derivative

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(x_0) + h \cdot f'(x_0) + \frac{h^2}{2} \cdot f''(x_0) + \frac{h^3}{3!} \cdot f'''(x_0) + \dots - \left(f(x_0) - h \cdot f(x_0) + \frac{h^2}{2} \cdot f''(x_0) - \frac{h^3}{3!} \cdot f'''(x_0) + \dots\right)}{2h}$$

$$= \frac{2h \cdot f'(x_0) + \frac{h^3}{3} \cdot f'''(x_0) + \dots}{2h}$$

$$= f'(x_0) + \frac{h^2}{3} \cdot f'''(x_0) + \dots$$

$\Rightarrow$ error term $O(h^2)$ converges quadratically