Homework 10, Numerical Analysis, Lena Kellner, due 10th Nov

(2) There are 3 variables $\Rightarrow$ 3 degree of freedom

$\Rightarrow$ • Set $f(x) = 1$: $\int_0^1 1\,dx = \frac{1}{2} f(x_0) + c_1 \cdot f(x_1)$

$$1 = \frac{1}{2} \cdot 1 + c_1 \cdot 1$$

$$1 = \frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

• Set $f(x) = x$: $\int_0^1 x\,dx = \frac{1}{2} x_0 + \frac{1}{2} x_1$

$$\frac{1}{2} = \frac{1}{2}(x_0 + x_1)$$

$$1 = x_0 + x_1$$

$$x_0 = 1 - x_1$$

• Set $f(x) = x^2$: $\int_0^1 x^2\,dx = \frac{1}{2} x_0^2 + \frac{1}{2} x_1^2$

$$\frac{1}{3} = \frac{1}{2}(x_0^2 + x_1^2)$$

$$\frac{2}{3} = x_0^2 + x_1^2$$

$$\frac{2}{3} = (1 - x_1)^2 + x_1^2$$

$$\frac{2}{3} = 1 - 2x_1 + x_1^2 + x_1^2$$

$$0 = 2x_1^2 - 2x_1 + \frac{1}{3}$$

case 1: $x_1 \approx 0{,}78868 \Rightarrow x_0 = 1 - x_1 = 0{,}21132$

case 2: $x_1 \approx 0{,}21132 \Rightarrow x_0 = 1 - x_1 = 0{,}78868$

$\rightarrow$ Choose case 1 without loss of generality.

$\Rightarrow$ The three variables are : $c_1 = \frac{1}{2}$

$$x_0 = 0{,}21132$$

$$x_1 = 0{,}78868$$

(1)

a) Both num. & denom. of Padé approx. are cubic : $P(x) = \dfrac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3}$

$\Rightarrow$ Need $\overbrace{6^{th} \text{ Taylor approximation at } x=0}^{\text{Maclaurin polynomial}}$ of $f(x) = \sin x$:

$$f(x) = \sin x \xrightarrow{x:=0} 0$$

$$f'(x) = \cos x \qquad 1$$

$$f^{(2)}(x) = -\sin x \qquad 0$$

$$f^{(3)}(x) = -\cos x \qquad -1$$

$$f^{(4)}(x) = \sin x \qquad 0$$

$$f^{(5)}(x) = \cos x \qquad 1$$

$$f^{(6)}(x) = -\sin x \qquad 0$$

$$T_6(f(0)) = \sum_{i=0}^{6} \frac{f^{(i)}(0)}{i!}(x - x_0)^i$$

$$= 0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$\Rightarrow$ Set both approximations equal :

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = (1 + b_1 x + b_2 x^2 + b_3 x^3)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5\right)$$

$$= 0 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + b_1 x^2 - b_1 \cdot \frac{1}{3!}x^4 + b_1 \cdot \frac{1}{5!}x^6$$

$$+ b_2 x^3 - b_2 \cdot \frac{1}{3!} \cdot x^5 + b_2 \cdot \frac{1}{5!} \cdot x^7 + b_3 x^4 - b_3 \cdot \frac{1}{3!}x^6 + b_3 \cdot \frac{1}{5!} \cdot x^8$$

$$= 0 + x + b_1 x^2 + \left(b_2 - \frac{1}{3!}\right)x^3 + \left(b_3 - \frac{1}{3!}b_1\right)x^4 + \left(\frac{1}{5!} - \frac{1}{3!}b_2\right)x^5$$

$$+ \left(\frac{1}{5!}b_1 - \frac{1}{3!}b_3\right)x^6 + \frac{1}{5!}b_2 \cdot x^7 + \frac{1}{5!}b_3 x^8$$

$\Rightarrow$ $a_0 = 0$

$a_1 = 1$

$a_2 = b_1 \overset{(*)}{=} 0$

$a_3 = -\frac{1}{3!} + b_2 \overset{(**)}{=} -\frac{1}{3!} + 0 = -\frac{1}{3!}$

$$0 = \left(b_3 - \frac{1}{3!} b_1\right) \quad \longrightarrow \quad b_3 = \frac{1}{3!} \cdot b_1$$

$$0 = \frac{1}{5!} - \frac{1}{3!} b_2 \quad \longrightarrow \quad b_2 = \frac{3!}{5!} = \frac{1}{20}$$

$$0 = \frac{1}{5!} b_1 - \frac{1}{3!} b_3 \quad \longrightarrow \quad b_1 = \frac{5!}{3!} b_3 \quad \longrightarrow \quad b_1 = 0 \quad (*)$$

$$0 = \frac{1}{5!} b_2 \quad \longrightarrow \quad b_2 \overset{(**)}{=} 0$$

$$0 = \frac{1}{5!} b_3 \quad \longrightarrow \quad b_3 = 0$$

$$\Rightarrow P(x) = \frac{0 + 1x + 0x^2 - \frac{1}{3!} x^3}{1 + 0x + 0x^2 + 0x^3} = \frac{x - \frac{1}{3!} x^3}{1}$$

b) num. is quadratic, denom. is forth degree: $P(x) = \dfrac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}$

$\Rightarrow$ Same Taylorapprox as in a): $T_6(f(0)) = x - \frac{1}{6} x^3 + \frac{1}{120} x^5$

$\Rightarrow$ equal:

$$\frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} = x - \frac{1}{6} x^3 + \frac{1}{120} x^5$$

$$\begin{aligned}
a_0 + a_1 x + a_2 x^2 &= \left(1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4\right)\left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5\right) \\
&= 0 + x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + b_1 x^2 - b_1 \cdot \frac{1}{3!} x^4 + b_1 \cdot \frac{1}{5!} x^6 \\
&\quad + b_2 x^3 - b_2 \cdot \frac{1}{3!} x^5 + b_2 \cdot \frac{1}{5!} \cdot x^7 + b_3 x^4 - b_3 \cdot \frac{1}{3!} x^6 + b_3 \cdot \frac{1}{5!} \cdot x^8 \\
&\quad + b_4 x^5 - \frac{1}{3!} b_4 x^7 + \frac{1}{5!} b_4 x^9 \\
&= 0 + x + b_1 x^2 + \left(b_2 - \frac{1}{3!}\right) x^3 + \left(b_3 - \frac{1}{3!} b_1\right) x^4 + \left(\frac{1}{5!} - \frac{1}{3!} b_2 + b_4\right) x^5 \\
&\quad + \left(\frac{1}{5!} b_1 - \frac{1}{3!} b_3\right) x^6 + \left(\frac{1}{5!} b_2 - \frac{1}{3!} b_4\right) \cdot x^7 + \frac{1}{5!} b_3 \cdot x^8 + \frac{1}{5!} b_4 \cdot x^9
\end{aligned}$$

$$\Rightarrow a_0 = 0$$

$$a_1 = 1$$

$$a_2 = b_1 \quad \longrightarrow \quad a_2 = 0$$

$$0 = b_3 - \frac{1}{3!} b_1 \quad \xrightarrow{(*)} \quad b_1 = 0$$

$$0 = \frac{1}{5!} - \frac{1}{3!} b_2 + b_4$$

$$0 = \frac{1}{5!} b_1 - \frac{1}{3!} b_3$$

$$0 = \frac{1}{5!} b_2 - \frac{1}{3!} b_4 \quad \xrightarrow{(**)} \quad b_2 = 0$$

$$0 = \frac{1}{5!} b_3 \quad \longrightarrow \quad b_3 \overset{(*)}{=} 0$$

$$0 = \frac{1}{5!} b_4 \quad \longrightarrow \quad b_4 \overset{(**)}{=} 0$$

$$\Rightarrow P(x) = \frac{0 + 1x + 0x^2}{1 + 0x + 0x^2 + 0x^3 + 0x^4} = \frac{x}{1}$$

c) num. fourth deg & denom quadr : $P(x) = \dfrac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2}$

$\Rightarrow$ Same Taylorapprox : $T_6(f(0)) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

Set equal: $\dfrac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = \left(1 + b_1 x + b_2 x^2\right)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)$$
$$= 0 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + b_1 x^2 - b_1 \cdot \frac{1}{3!}x^4 + b_1 \frac{1}{5!}x^6$$
$$+ b_2 x^3 - b_2 \frac{1}{3!} \cdot x^5 + b_2 \cdot \frac{1}{5!} \cdot x^7$$
$$= 0 + x + b_1 x^2 + \left(b_2 - \frac{1}{3!}\right)x^3 - \frac{1}{3!} b_1 x^4 + \left(\frac{1}{5!} - b_2 \frac{1}{3!}\right)x^5 + \frac{1}{5!} b_1 \cdot x^6 + \frac{1}{5!} b_2 x^7$$

$\rightarrow a_0 = 0$

$a_1 = 1$

$a_2 = b_1 \longrightarrow a_2 \overset{(*)}{=} 0$

$a_3 = b_2 - \frac{1}{3!} \xrightarrow{(**)} a_3 = -\frac{1}{3!}$

$a_4 = -\frac{1}{3!} b_1 \xrightarrow{(*)} a_4 = 0$

$0 = \frac{1}{5!} - \frac{1}{3!} b_2$

$0 = \frac{1}{5!} b_1 \longrightarrow b_1 \overset{(*)}{=} 0$

$0 = \frac{1}{5!} b_2 \longrightarrow b_2 \overset{(**)}{=} 0$

$\Rightarrow P(x) = \dfrac{0 + 1x + 0x^2 - \frac{1}{3!}x^3 + 0x^4}{1 + 0x + 0x^2} = \underline{\underline{\dfrac{x - \frac{1}{3!}x^3}{1}}}$

Accuracy : Most accurate approximation in ~[0, 4.5] is Maclaurin

In [4.5 , 5] Padé of b) is best approximation

$\rightarrow$ Plot see below!

```python
import matplotlib.pyplot as plt
import numpy as np
import math
from numpy.linalg import inv
from numpy.linalg import norm

def driver():
    x_values = np.linspace(0, 5, 100)  # Define a range of x val

    f = lambda x: math.sin(x)
    T = lambda x: x - (1/6)*x**3 + (1/120)*x**5
    Pa = lambda x: x - (1/6)*x**3
    Pb =  lambda x: x
    Pc = lambda x: x - (1/6)*x**3

    errT = [abs(f(x) - T(x)) for x in x_values]  # Calculate err
    errPaPc = [abs(f(x) - Pa(x)) for x in x_values]
    errPb = [abs(f(x) - Pb(x)) for x in x_values]

    plt.figure()
    plt.plot(x_values, errT)  # Plot x_values on x-axis and errT
    plt.plot(x_values, errPaPc)
    plt.plot(x_values, errPb)
    plt.legend(['error Maclaurin', 'error a) c)', 'error b)'])
    plt.show()

driver()
```