HOMEWORK NUMERICAL ANALYSIS

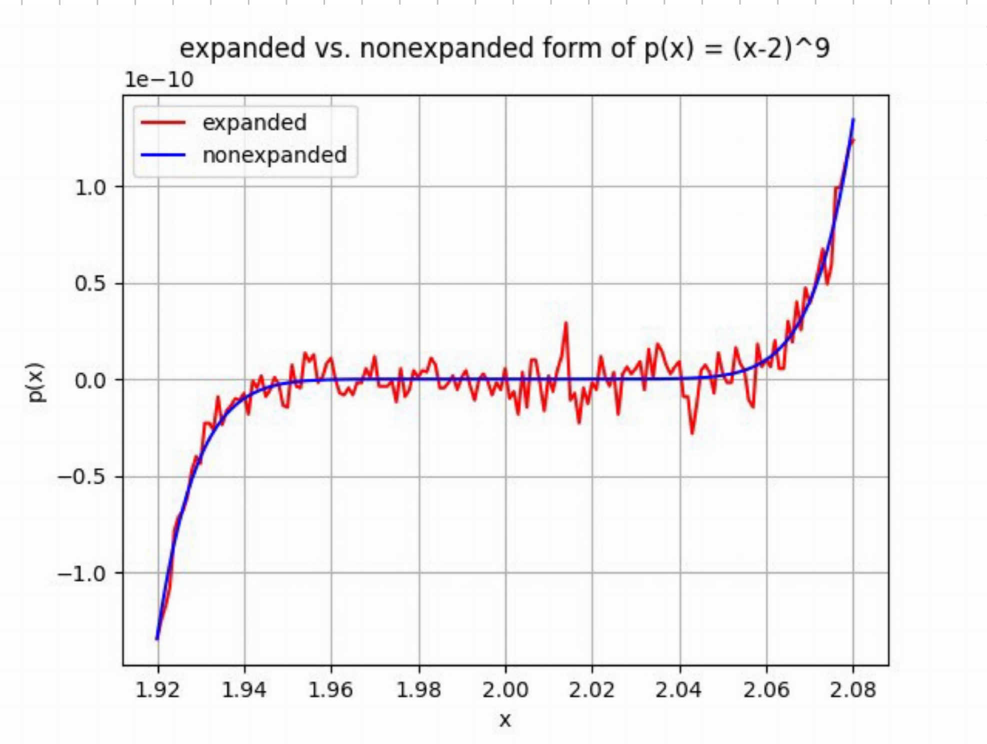
HW 1 - due to 8. Sept. 2023

Notation:
One = 1
Seven = 7
Using comma instead of point: $\frac{1}{2} = 0{,}5$

a) Given: $p(x) = (x-2)^9$

i) Exercise: plot p(x) by evaluating via its coefficients

ii) Exercise: plot p(x) by evaluating via $(x-2)^9$



iii) What's the difference between i) and ii)?

It should plot the same function, but it does not.

At i) the expanded form of the term is used and at ii) the nonexp. form is used. The second plot is smooth whereas the plot of the expanded form is not.

Reason for this discrepancy is cancellation. The smooth plot is correct

② How would you preform the calculations to avoid cancellation?

i) Given: $\sqrt{x+1} - 1$ for $x \approx 0$

Steps: $\sqrt{x+1} - 1 \quad \Big| \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$\iff \frac{\sqrt{x+1}^2 - 1^2}{\sqrt{x+1} + 1}$

$\iff \frac{x+1-1}{\sqrt{x+1}+1}$

$\iff \frac{x}{\sqrt{x+1}+1}$

Explaination: Since $\sqrt{} \geq 0$, adding 1 in the denominator should not cause a loss of digits. However, the original term seems riskful because $\sqrt{1} = 1$.

ii) Given: $\sin(x) - \sin(y)$ for $x \approx y$ $\qquad \sin^2 x + \cos^2 x = 1$

Steps: $\sin(x) - \sin(y) \quad \Big| \cdot \frac{\sin(x)+\sin(y)}{\sin(x)+\sin(y)}$

$\iff \frac{\sin^2(x) - \sin^2(y)}{\sin(x) + \sin(y)}$

$\iff \frac{\sin(x-y)\cdot\sin(x+y)}{\sin(x) + \sin(y)}$

Explaination: Since $\sin(x+y) > 0$ is multiplied with $\sin(x-y)$, there should not be a significant loss of digits despite $\sin(x-y) \approx 0$

iii) Given: $\frac{1-\cos(x)}{\sin(x)}$ for $x \approx 0$

Steps: $\frac{1-\cos(x)}{\sin(x)} \quad \Big| \cdot \frac{1+\cos(x)}{1+\cos(x)}$

$\iff \frac{1-\cos^2(x)}{\sin(x)\cdot(1+\cos(x))}$

$\iff \frac{\sin(x)}{1+\cos(x)}$

Explaination: Since $\cos(x) \approx 1$ for $x \approx 0$, $1+\cos(x)$ in the denumerator will not cause a loss of digits. However, $1-\cos(x)$ in the numerator could have done.

③ Given: $f(x) = (1+x+x^3)\cdot\cos(x)$, $x_0 = 0$

Statement: The 2nd-degree Taylor polynomial $P_2(x)$ for $f(x)$ about $x_0 = 0$

is $P_2(x) = 1 + x - \frac{1}{2}x^2$

Proof: Formula of Taylor Polynomial: $P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

$$\rightarrow P_2(x) = f(0) + \frac{f'(0)}{1}(x-0) + \frac{f''(0)}{2}(x-0)^2 \overset{(*)}{=} 1 + \frac{1}{1}x + \frac{-1}{2}\cdot x^2 = \underline{\underline{1 + x - \frac{1}{2}x^2}}$$

$$(*)\left\{\begin{aligned}
f'(x) &= (1+3x^2)\cos(x) + (1+x+x^3)(-\sin x)\\
f''(x) &= 6x\cdot\cos(x) + (1+3x^2)(-\sin x) + (1+3x^2)(-\sin x) + (1+x+x^3)(-\cos x)\\
&= 6x\cdot\cos(x) - (1+x+x^3)(\cos x) + 2\cdot(1+3x^2)(-\sin x)\\
&= (-1+5x-x^3)\cdot\cos(x) - (2+6x^2)\cdot\sin x\\
f(0) &= 1\cdot\cos(0) = 1\\
f'(0) &= 1\cdot\cos(0) + 1\cdot(-\sin 0) = 1 - 0 = 1\\
f''(0) &= -1\cdot\cos(0) - 2\cdot\sin(0) = -1 - 0 = -1
\end{aligned}\right.$$

(a) Statement: $P_2(0{,}5) = 1{,}375$

Upper bound for the error $|f(0{,}5) - P_2(0{,}5)|$: $0{,}0625$

Comparison to actual error: $0.0510 \leq 0{,}0625$

Proof: • $P_2(0{,}5) = 1 + 0{,}5 - \frac{1}{2}\cdot(0{,}5)^2 = 1{,}5 - \frac{0{,}25}{2} = \underline{\underline{1{,}375}}$

• Error formula for Taylor polynomials: $|R_{n,a}(x)| \leq \frac{f^{(n+1)}(z)}{(n+1)!}\cdot(x-a)^{n+1}$ for all $z \in [a,x]$.

$$\Rightarrow |R_{2,0}(0{,}5)| \leq \frac{f^3(z)}{3!}(0{,}5)^3 \quad \forall z \in [0,\, 0{,}5]$$

We've got: $n = 2$

$a := x_0 = 0$

$x = 0{,}5 \quad \Rightarrow |z| \leq 0{,}5$

$$f^3(x) = (5-3x^2)\cos(x) - (-1+5x-x^3)\sin(x) - \left(12x\cdot\sin x + (2+6x^2)\cos x\right)$$

$$= (5-3x^2)\cos(x) + (1-5x+x^3)\sin(x) - 12x\cdot\sin x + (-2-6x^2)\cos x$$

$$= \underbrace{(3-9x^2)}_{\substack{\text{max if}\\ z=0}}\underbrace{\cos(x)}_{\substack{\text{max if}\\ z=0}} + \underbrace{(1-17x+x^3)}_{\substack{\text{has to be } z<0{,}1\\ \text{to be a positive}\\ \text{term}}}\sin(x)$$

$\rightarrow$ consider $z = 0$

$$\Rightarrow |R_{2,0}(0{,}5)| \le \frac{(3-9z^2)\cos(z) + (1-17z+z^3)\sin(z)}{3!} \cdot (0{,}5)^3$$

$$= \left((3-9z^2)\cos(z) + (1-17z+z^3)\sin(z)\right) \cdot \frac{1}{48}$$

$$\overset{z=0}{\le} (3\cdot\cos(0) + 1\cdot\sin(0)) \cdot \frac{1}{48}$$

$$= 3 \cdot \frac{1}{48}$$

$$= \frac{1}{16} = 0{,}0625$$

- The actual error $E = |f(0{,}5) - P_2(0{,}5)|$

$$= |(1+0{,}5+0{,}5^3)\cdot\cos(0{,}5) - 1+0{,}5 - \frac{0{,}5^2}{2}|$$

$$= |1{,}625\cdot\cos(0{,}5) - 1{,}375|$$

$$\approx |1{,}4260 - 1{,}375|$$

$$= 0{,}0510$$

$\Rightarrow 0{,}0510 \le 0{,}0625$

b) Exercise: Find an error bound for $|f(x) - P_2(x)|$.

Steps: Following through a): $|R_{2,0}(x)| \le \frac{f^3(z)}{3!}(x)^3 \quad \forall z \in [0,x]$

$$\Rightarrow |R_{2,0}(x)| \le \frac{(3-9z^2)\cos(z) + (1-17z+z^3)\sin(z)}{6} \cdot x^3$$

c) Given: $\int_0^1 f(x)\,dx \approx \int_0^1 P_2(x)\,dx$

Statement: $\int_0^1 f(x)\,dx \approx \frac{4}{3}$

Main theorem of differential and integral calculus

Proof: $\int_0^1 P_2(x)\,dx = \int_0^1 1+x-\frac{1}{2}x^2\,dx = \left[x+\frac{1}{2}x^2-\frac{1}{6}x^3\right]_0^1$

$$= 1+\frac{1}{2}\cdot 1^2 - \frac{1}{6}\cdot 1^3 - \left(0+\frac{1}{2}\cdot 0^2 - \frac{1}{6}\cdot 0^3\right) = \frac{8}{6} = \underline{\underline{\frac{4}{3}}}$$

d) Estimate the error in the integral

$$\rightsquigarrow \int_0^1 f(x)\,dx = \int_0^1 \underbrace{(1+x+x^3)}_{v}\underbrace{\cos(x)}_{u'}\,dx$$

partial integration

$$\overset{!}{=} [\underset{v}{(1+x+x^3)}\cdot\underset{u}{\sin(x)}]_0^1 - \int_0^1 \underset{\substack{v' \\ v_2}}{(1+3x^2)}\cdot\underset{\substack{u \\ u_2'}}{\sin(x)}\,dx$$

partial integration

$$= 3\cdot\sin(1) - 1\cdot\sin(0) - \left(\left[\underset{v_2}{(1+3x^2)}\cdot\underset{u_2}{(-\cos(x))}\right]_0^1 - \int_0^1 \underset{v_2'}{6x}\cdot\underset{u_2}{(-\cos(x))}\,dx\right)$$

$$= 3\sin(1) - (-4\cos(1) + 1\cdot\cos(0)) + 6\cdot\int_0^1 \underset{v_3}{x}\cdot\underset{u_3'}{\cos x}\,dx$$

partial integration

$$= 3\cdot\sin(1) + 4\cdot\cos(1) + 1 + 6\cdot[\underset{v_3}{x}\cdot\underset{u_3}{\sin x}]_0^1 - \int_0^1 \underset{v_3'}{1}\cdot\underset{u_3}{\sin x}\,dx$$

$$= 3\sin(1) + 4\cdot\cos(1) + 1 + 6\sin(1) - 0 - [-\cos x]_0^1$$

$$= 9\sin(1) + 4\cos(1) + 1 + \cos(1) + \cos(0)$$

$$= 9\sin(1) + 5\cos(1) + 2$$

$$\approx 12{,}274$$

$$\text{Error}: \left|\int_0^1 f(x) - \int_0^1 P_2(x)\right| = |12{,}274 - 1{,}333| = \underline{\underline{10{,}941}}$$

seems too big. I've probably made a mistake by calculating $\int_0 f(x)\,dx$

4 a) Given: $x^2 - 56x + 1$

Statement: By calculating the square root with 3 correct decimals

Proof: $r_{1,2} = \frac{+56 \pm \sqrt{(-56)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{56 \pm \sqrt{56^2 - 4}}{2} = \frac{56 \pm \sqrt{3132}}{2} \approx \frac{56 \pm 55{,}964}{2}$

$\Rightarrow r_1 = \frac{111{,}964}{2} = 55{,}982$

$r_2 = \frac{0{,}036}{2} = \frac{0{,}018}{2} = \frac{18}{200}$