Numerical Analysis, HW3, Due to 22. Sept., Lena Kellner

① Consider the equation $2x-1 = \sin x$.

(a) Find a closed intervall $[a,b]$ on which the equation has a root r

Use the Intermediate Value Theorem to prove that r exists

Soln.: Denote $f(x) = \sin x - 2x + 1$.

Then $f(x)$ is continuous because it's a sum of $\sin x$, $-2x$, 1 which are continuous itself.

An intervall where $f(x)$ has a root is $[0, \frac{\pi}{2}]$.

That is can be shown with the IVT:

- $f(x)$ is continuous
- $f(0) = \sin 0 - 2 \cdot 0 + 1 = 0 - 0 + 1 = 1 > 0$
- $f(\frac{\pi}{2}) = \sin \frac{\pi}{2} - 2 \cdot \frac{\pi}{2} + 1 = 1 - \pi + 1 = 2 - \pi < 0$

$\overset{IVT}{\Rightarrow} \exists$ a point $r \in [0, \frac{\pi}{2}]$ s.t. $f(r) = 0$

$\square$

(b) Prove that r is the only root of the equation on all of $\mathbb{R}$

Soln.: $f'(x) = \underbrace{\cos x}_{\leq |1|} - 2 < 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is strictly monotonically decreasing on all of $\mathbb{R}$

$\Rightarrow$ $f(x)$ can cross the x-axis just once

$\Rightarrow$ We know that $f(x)$ crosses the x-axis at r, so this is the only root

(c) Use the bisection code to get a solution with an accuracy of 8 digits

Soln.: The approximate root is 0,8878622154822129

The error message reads: 0

f(astar) = $-5,354352961006725e^{-9}$

# of iterations: 27

2 a)

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PS C:\Users\kelln> & C:/Users/kelln/AppData/Local/Programs/Python/Python311/python.exe "c:/Users/kelln/OneDrive/Dokumente/GitHub/APPM 4600 Numerical Analysis Fall23/Homework/HW3/Bisection HW3.2a.py"
the approximate root is 5.000073242187501
the error message reads: 0
f(astar) = 6.065292655789404e-38
PS C:\Users\kelln>
```

b)

```
PS C:\Users\kelln> & C:/Users/kelln/AppData/Local/Programs/Python/Python311/python.exe "c:/Users/kelln/OneDrive/Dokumente/GitHub/APPM 4600 Numerical Analysis Fall23/Homework/HW3/Bisection HW3.2b.py"
the approximate root is 5.12875
the error message reads: 0
f(astar) = 0.0
PS C:\Users\kelln>
```

c) The solution in a) is correct and the solution in b) is not. That is because the expanded form of the term causes many more rounding errors. So, the machine cannot find the correct root in b).

③ (a) Find upper bound on the # of iterations in the bisection needed to approximate the solution of $x^3 + x - 4 = 0$ in $[1,4]$ with accuracy of 10^{-3}.

Soln.: Theorem 2.1:

Suppose $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. Then the Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n} \quad \text{when } n \geq 1.$$

Since $f(x) := x^3 + x - 4$ consists of continuous terms, it is continuous itself. Also, $f(1) = 1^3 + 1 - 4 = -2$ and $f(4) = 4^3 + 1 - 4 = 64 + 1 - 4 = 67$ which results in $f(1) \cdot f(4) < 0$.

So, we can apply the definition: $|p_n - p| \leq \frac{4-1}{2^n}$

When we use the wanted accuracy of 10^{-3}, we get $10^{-3} \leq \frac{3}{2^n}$.

$$\iff 2^n \leq 3 \cdot 10^3$$

$$\iff n \log(2) \leq \log(3) + \log(10^3)$$

$$n \leq \frac{\log(3) + 3}{\log(2)} \approx 11{,}55074$$

b) Find an approximation of the root with accuracy 10^{-3}.

How does the # of iterations compare with the upper bound in a)?

Soln.: Approx. root: 1,37866210

f(astar) = -0,0009021

iterations: 11

$\Rightarrow$ The # of iterations fits the upper bound which I've found.

So I assume it's correct.

(4) Do the following iterations converge to the indicated fixed point x_*?

If they do: Which order of convergence?

Which rate of convergence (if linear)?

(a) Given: $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$

$x_* = 2$

Soln.: Denote $f(x) = -16 + 6x + \frac{12}{x}$

Then $f'(x) = 6 - \frac{12}{x^2}$

$f'(2) = 3 \quad > 1$

$\Rightarrow$ The iteration does not converge

(b) Given: $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $x_* = 3^{\frac{1}{3}}$

Soln.: Denote $f(x) = \frac{2}{3}x + \frac{1}{x^2}$

Then $f'(x) = \frac{2}{3} - \frac{2}{x^3}$

$f'(3^{\frac{1}{3}}) = \frac{2}{3} - \frac{2}{(3^{\frac{1}{3}})^3} = 0$

$\Rightarrow$ The convergence is not linear

So, $f''(x) = \frac{6}{x^4}$

$f''(3^{\frac{1}{3}}) = \frac{6}{3^{\frac{4}{3}}} \neq 0$

$\Rightarrow$ The convergence is quadratic (order $\alpha = 2$)

(c) Given: $x_{n+1} = \frac{12}{1+x_n}$, $x_* = 3$

Soln.: Denote $f(x) = \frac{12}{1+x}$

Then $f'(x) = \frac{-12}{(1+x)^2}$

$f'(3) = \frac{-12}{(1+3)^2} = -\frac{12}{16} = -\frac{3}{4} \neq 0$

$\Rightarrow$ The convergence is linear (order $\alpha = 1$)

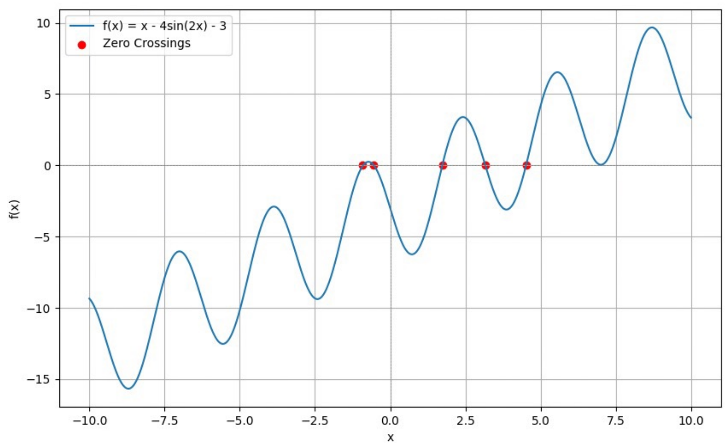
$\Rightarrow$ The convergence rate $\lambda = |f'(x_*)| = |-\frac{3}{4}| = \frac{3}{4}$

5. All roots of $x - 4\sin(2x) - 3 = 0$ are to be determined with ≥ 10 accurate digits.

Tried Bisection method (uploaded at Github) but only found:

- 1,732069502140
- 4,517789514 1678

(a)




```python
import numpy as np
import matplotlib.pyplot as plt

# Define the function
def f(x):
    return x - 4*np.sin(2*x) - 3

# Generate x values
x_values = np.linspace(-10, 10, 500)

# Calculate corresponding y values
y_values = f(x_values)

# Find zero crossings (where f(x) = 0)
zero_crossings = x_values[np.where(np.diff(np.sign(y_values)))[0]]

# Plot the function
plt.figure(figsize=(10, 6))
plt.plot(x_values, y_values, label='f(x) = x - 4sin(2x) - 3')
plt.axhline(0, color='gray', linewidth=0.5, linestyle='--')
plt.axvline(0, color='gray', linewidth=0.5, linestyle='--')
plt.scatter(zero_crossings, [0]*len(zero_crossings), color='red', label='Zero Crossings')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
plt.grid(True)
plt.show()

# Print the number of zero crossings
print(f'There are {len(zero_crossings)} zero crossings.')
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
PS C:\Users\kelln> & C:/Users/kelln/AppData/Local/Programs/Python/Python311/python.exe "c:/Users/kelln/OneDrive/Dokumente/GitHub/APPM 4600 Numerical Analysis Fall23/Homework/HW3/HW3.5a.py"
There are 5 zero crossings.
PS C:\Users\kelln>
```