# Lab 2 Solutions

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## 1 Bounds

Is there a single comparison-based sorting algorithm that will sort:

a) every sequence of 10 elements in at most 21 comparisons? No, because the decision tree would have $10! = 3,628,800$ leaves and height 21, but a binary tree with height 21 has at most $2^{21} = 2,097,152$ leaves.

b) every sequence of $n$ elements in at most $\frac{n}{2}log_2(n)$ comparisons? No. For example, for $n = 100$, we have $log_2(n!) > \frac{n}{2}log_2(n)$, so no decision tree could exist for an algorithm that did this.

c) Yes, e.g. bubble-sort, insertion sort, selection sort...

(Note: $10! = 3,628,800$ and $2^{21} = 2,097,152$.)

d) What is the minimum number of comparisons needed to sort a single list of $n$ elements? Suppose the input is already sorted. Then we still need $n - 1$ comparisons just to verify that it is sorted.

e) What is the minimum number of comparisons necessary to merge two sorted lists of 10 integers each? Best-case input and best-case comparisons: 1. Worst-case input and best-case comparisons: $2 * 10 - 1 = 19$.

f) Given sorted lists $A = (1, 3, 5, 7, 9)$ and $B = (6, 8)$, what is the minimum number of comparisons needed to merge the lists? 4.

g) Give a sequence of comparisons meeting the bound in $f$). 5:6,6:7,7:8,8:9.

## 2 Posets

a) Given a poset $\{a, b, c, d, e\}$ with $a \leq b, c \leq d, d \leq e$, how many total orders are possible? $\binom{2+5}{2}$

b) Given a poset $\{a, b, c, d\}$, $a \leq b, a \leq c, a \leq d$, how many total orders are possible? $3! = 6$.

c) Given a poset $\{a, b, c, d, e\}$, $a \leq b, b \leq d, a \leq c, c \leq d$, how many total orders are possible? $2*5 = 10$. First there are 2 ways to order everything except $e$. After that is done, there are 5 places to insert $e$.

## 3 Red-black Trees

a) Draw a red-black tree with black-height 3, a leaf at distance 3 from the root, and a leaf at distance 6 from the root. Multiple answers are possible.

b) Is there a red-black tree with a leaf at distance 3 from the root and a leaf at distance 7 from the root? No, because $7 > 2 * 3$. If a leaf has distance 3 from the root, the tree has black-height at most 3. If the black-height of the tree is at most 3, then the max. possible length of a path from the root to any leaf is at most $2 * 3 = 6$ (a path of alternating red and black nodes).