Generating Functions

In how many ways can 20 identical ballons be distributed to 4 children so that each child gets at least 3 ballons but no one gets more than 7.

$3 \leqslant ? \leqslant 7$ Bounds

$(x^3 + x^4 + x^5 + x^6 + x^7)^4 \rightarrow$ kids

We need to look for the coefficient x^{20}, as the number of ballons

Move x^3 outside the parentheses

$(x^3)^4(1+x+x^2+x^3+x^4)^4 = x^{12}(1+x+x^2+x^3+x^4)^4$

$\dfrac{1-x^{n+1}}{1-x}$

$1+x+x^2+\dots+x^n$

Using $(1-x^{n+1})/1-x$ formula

$x^{12}\left(\dfrac{1-x^5}{1-x}\right)^4$ Move the nominator outside

$\dfrac{1-x^{n+1}}{1-x}$ it's the same

$1+x+x^2+\dots+x^n$

$x^{12}(1-x^5)^4\left(\dfrac{1}{1-x}\right)^4$ use the Generating function to expand $(1-x^5)^4$

$(1+x^r)^n$

$\sum_{k=0}^{n}\binom{n}{k}x^{rk}$

$x^{12}\left(\binom{4}{0}x^0 - \binom{4}{1}x^{5\cdot1} + \binom{4}{2}x^{5\cdot2} - \binom{4}{3}x^{5\cdot3} - \binom{4}{4}x^{5\cdot4}\right) =$

$x^{12}(1-4x^5+12x^{10}-4x^{15}+x^{20}) =$

$x^{12}-4x^{17}+12x^{22}-4x^{27}+x^{40}$

Remember we're looking for coefficien x^{20} and we still have to solve half equation. this are not important $> x^{20}$

use Generating functions to expand $\left(\dfrac{1}{1-x}\right)^4$

$\dfrac{1}{(1-x)^4} = \binom{4+0-1}{0}x^0 + \binom{4}{1}x^1 + \binom{5}{2}x^2 + \binom{6}{3}x^3 + \dots + \binom{11}{8}x^8$

$\dfrac{1}{(1-x)^n}$

$\sum \binom{n+k-1}{k}x^k$

Possible combinations of all the solution that sum x^{20}:

$x^{12} + \binom{11}{8}x^8 \rightarrow \binom{11}{8}x^{20}$ $\quad -4x^{17}\binom{6}{3}x^3 \rightarrow -4\binom{6}{3}x^{20}$

Solution: $\binom{11}{8} - 4\binom{6}{3}$

Generating Functions

You have to buy 25 bagels. they are three types: plain, tomato and blueberry. the restrisctions are: At least five of each sort but not more than 11 of any sort.

$$5 \le x \le 11$$

$$Gf(x) = (x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11})^3$$

$$x^{15}(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

$$= x^{15} \left(\frac{1-x^7}{1-x}\right)^3 \rightarrow \text{extract } (1-x^7)^3$$

$$= x^{15} (1-x^7)^3 \left(\frac{1}{1-x}\right)^3 \rightarrow \text{expand } (1-x^7)^3 \text{ using Gf}$$

$$= x^{15}\left(\binom{3}{0}(-x^7)^0 + \binom{3}{1}(-x^7)^1 + \binom{3}{2}(-x^7)^2 + \binom{3}{3}(-x^7)^3 \right.$$

$$= x^{15}(1 - 3x^7 + 3x^{14} - x^{21}) \left(\frac{1}{1-x}\right)^3 \rightarrow \text{expand } \left(\frac{1}{1-x}\right)^3$$

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n$$

$$=$$

$$(1+x^r)^n$$

$$\sum_{k=0}^{n} \binom{n}{k} x^{rk}$$

$$=$$

$$\frac{1}{(1-x)^n}$$

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

$$\left(\frac{1}{1-x}\right)^3 = \binom{2}{0}x^0 + \binom{3}{1}x^1 + \dots + \binom{5}{3}x^3 + \dots + \binom{12}{10}x^{10} + \dots$$

find solutions of those exponents that sum 25.

$$\left.\begin{aligned} x^{15}(-3x^7)\binom{5}{3}x^3 &= -3\binom{5}{3} \\ x^{15} \cdot 1 \cdot \binom{12}{10} x^{10} &= \binom{12}{10} \end{aligned}\right\} \binom{12}{10} - 3\binom{5}{3}$$

More Generating Functions

Possible solutions of $x_1 + x_2 + x_3 = 14$ Using GF and I-E. $2 < x_1 < 6$, $6 < x_2 < 10$, $0 < x_3 < 5$

Generating Function

$$GF(x) = \overbrace{(x^3 + x^4 + x^5)}^{x_1}\overbrace{(x^7 + x^8 + x^9)}^{x_2}\overbrace{(x + x^2 + x^3 + x^4)}^{x_3}$$

$$= x^3(1+x+x^2)\,x^7(1+x+x^2)\,x(1+x+x^2+x^3)$$

$= x^{11}(1+x+x^2)^2(1+x+x^2+x^3) \rightarrow$ fold with table page 526

$$= x^{11}\left(\frac{1-x^{2+1}}{1-x}\right)^2\left(\frac{1-x^4}{1-x}\right) = x^{11}\frac{(1-x^3)^2(1-x^4)}{(1-x)^2(1-x)} = x^{11}\frac{(1-x^3)^2(1-x^4)}{(1-x)^3}$$

$= x^{11}(1-x^3)^2(1-x^4)\dfrac{1}{(1-x)^3} \rightarrow$ Expand with the table 526

$$\underbrace{(1-x^3)^2 = \sum_{k=0}^{2}\binom{2}{0}x^{3\cdot 0} + \binom{2}{1}(-x^3)^1 + \binom{2}{2}(-x^3)^2 = 1-2x^3+x^6}$$

$$x^{11}(1-x^4) = (x^{11}-x^{15})(1-2x^3+x^6) = x^{11} - 2x^{14} + x^{17} - x^{15} + 2x^{18} - x^{21}$$

We want coefficients from exponentials that sum 14. We still have the last part to solve but from this part we take x^n $n \le 14$, for instance x^{11} and $-2x^{14}$. We need exp. ^{3} from the last part. Solve with 'the' table

$$\frac{1}{(1-x)^3} = \sum_{\substack{k=0\\ n=3}}^{\infty}\binom{n+k-1}{k}(+1)^k x^k \xrightarrow{x^3} \binom{3+3-1}{3}(+1)^3x^3 = \binom{5}{3}x^3$$

$$\binom{5}{3} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot3\cdot2\cdot1} = \frac{20}{2} = 10$$

$10x^3\,x^{11}$ = coeff. 10

$-2x^{14} = -2$

Total $= 10 - 2 = 8$

Inclusion - Exclusion

$x_a + x_b + x_c = 25$ $\qquad$ $5 \leq x \leq 11$

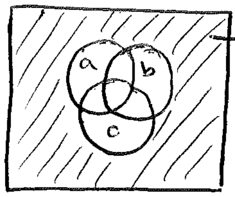
$x_a = 5 + y_a$ $\qquad$ At least 5 of each are sure. the combinations

$x_b = 5 + y_b$ $\qquad$ are reduced to: $y_a + y_b + y_c = 25 - 15$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 10$

$x_c = 5 + y_c$ $\qquad$ $0 \leq y \leq 11 - 5 = 6$

the solution is about $y_a + y_b + y_c = 10$ for $y \leq 6$
but it is easier to look for the
combinations of $y \leq 7$ for $y_a, y_b, y_c \geq 7$ and then
calculating the $\bar{y}_a \bar{y}_b \bar{y}_c$



$\to U = \forall$ combinations

$a, b, c =$ combination
for $y_{a,b,c} \geq 7$

$\to \overline{y_a y_b y_c}$

$$N(\bar{a}\bar{b}\bar{c}) = U - (N(a) + N(b) + N(c)) + N(ab) + N(ac) + N(bc) - N(abc)$$

$N(a)$ $\qquad$ $x'_a + x_b + x_c = 10$ $\qquad$ $x'_a \geq 7$

$x_b + x_c = 10 - 7 = 3 \longrightarrow \binom{3+3-1}{3} = \binom{5}{3}$ $\quad$ number of x's

$N(a) = N(b) = N(c) \to 3\binom{5}{3}$

$N(ab)$ $\qquad$ $x'_a + x'_b + x_c = 10$ $\qquad$ $x'_a \geq 7$ $\quad$ $x'_b \geq 7$

$x_c = 10 - 7 - 7 = -4$ $\to$ it is not possible because we only have 10 muffins

$N(ab) = N(bc) = N(ac)$

$$N(\bar{a}\bar{b}\bar{c}) = \binom{12}{10} - 3\binom{5}{3}$$

More Inclusion - Exclusion

$x_1 + x_2 + x_3 = 17 \qquad x_i \leq 7 \qquad 1 \leq i \leq 3$

$C_i = x_i \geq 8 \qquad \overline{C_i} = x_i < 8 == x_i \leq 7$

$$N(\overline{C_1}, \overline{C_2}, \overline{C_3}) = N - (N(C_1) + N(C_2) + N(C_3)) + N(C_1C_2) + N(C_1C_3) + N(C_2C_3) - N(C_1C_2C_3)$$

Total without restrictions $\binom{17+3-1}{17} = \binom{19}{17}$

$N(C_1) \qquad x_1 + x_2 + x_3 = 17 \qquad x_1 \geq 8$

$x_1' + x_2 + x_3 = 17 - 8 \qquad x' \geq 0$

$= 9 \longrightarrow \binom{9+3-1}{9} = \binom{11}{9}$ $\quad x_1\ x_2\ x_3$

$N(C_1) = N(C_2) = N(C_3) \rightarrow \binom{3}{1}\binom{11}{9} = 3\binom{11}{9}$

$N(C_1C_2) \qquad x_1' + x_2' + x_3 = 17 - 8 - 8 = 1 \qquad x_1 \geq 8 \quad x_2 \geq 8$

$\binom{3}{1}$

$N(C_1C_2) = N(C_2C_3) = N(C_1C_3) \rightarrow 3\binom{3}{1}$

$N(C_1C_2C_3)$ it is not possible because it is bigger than 17

$$N(\overline{C_1}\overline{C_2}\overline{C_3}) = \binom{19}{17} - 3\binom{11}{9} + 3\binom{3}{1}$$

Relations: Equivalent

A relation on a set is called an equivalence relation if its:

- Reflexive : xRx for $\forall x$
- Symmetric:
- transitive: $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

Let R be an equivalence relation on a set. the set of all elements that are related to an element a of A is called the **equivalence class** of a, $\boxed{[a]_R}$.
When only one relation is under consideration $[a]$.

Example

R: same number of 1's sRt $s,t \in \Sigma$ $\Sigma = \{0,1\}$

$A = A^4 = \{0000, 0001, 0010, 0100, 1000, \ldots\}$

All equivalence classes

$[0] = \{0000\}$

$[1] = \{0001, 0010, 0100, 1000\}$

$[2] = \{0001, 0010, 0011, 0101, 0110, 1010, 1100\}$

$[3] = \{0111, 1110, 1011, 1101\}$

$[4] = \{1111\}$

Relations Examples

Prove this poset $\in \mathbb{Z}$ $\quad R: xRy \leftrightarrow x|y \quad x,y\ \mathbb{Z}^+$

$2/2, 3/3 \quad aRa$ Reflexive ✓

$aRb, bRa \rightarrow a=b$

$b = k \cdot a \qquad b = k \cdot a = k \cdot l \cdot b$

$a = l \cdot b \qquad k \cdot l = b/b = 1 \rightarrow k=1, l=1$

$a=b$ antisymmetric

$aRb \qquad bRc$

$c = k \cdot b = k \cdot l \cdot a \rightarrow c = j \cdot a \rightarrow aRc$ Transitive ✓

Relations Examples

$\Sigma = \{0,1\}$. $A = \{A^4\}$ sRt s,t $\in A$ R: Same number of 1's

n = number of 1's $A\{0000, 0001, 0010, 1100, 0100, \dots\}$

$[0] = \{0000\}$
$[1] = \{0001, 0010, 0100, 1000\}$

$[2] = \{0011, 0101, 1001, 1010, 1100\}$

$[3] = \{0111, 1110, 1101, 1011\}$

$[4] = \{1111\}$

$\Sigma = \forall \mathbb{Z}$ xRy $x, y \in \mathbb{Z}$ $R: x \equiv y \pmod{13}$

$[4]$ $4/13 = 0 + 4$ $17 - 4 = 13k \rightarrow k = 13/13$ or $17/13 = 1 + 4$

$17 \equiv 30 \rightarrow 13k = 30 - 17 \rightarrow 30/13 = 2 + 4$

$[4] = \{\dots, -22, -9, 4, 17, 30, 43, \dots\}$

Show the relation R: xRy $x, y \in \mathbb{Z}$ iff $x \equiv y \pmod{7}$

$x \underset{7}{\equiv} x$ $14 \underset{7}{\equiv} 14$ Reflexive ✓

$a \underset{7}{\equiv} b \leftrightarrow b \underset{7}{\equiv} a$ $a \underset{7}{\equiv} b \rightarrow b - a = k \cdot 7 \rightarrow b \underset{7}{\equiv} a$ Symmetric ✓

$a - b = k \cdot 7$, $b - c = l \cdot 7 \rightarrow a - c = (k + l) \cdot 7$ transitive ✓

$a \equiv b \leftrightarrow a \equiv c$

POSET

A relation R on a set S is called a partial ordering if it is:

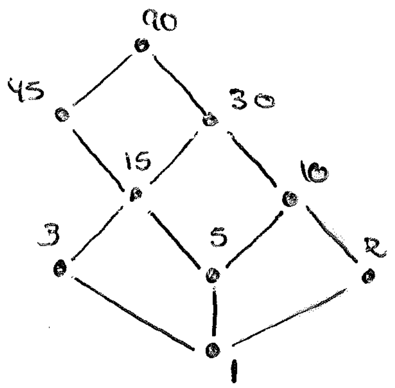
- Reflexive: xRx for $\forall x$
- Antisymmetric: $\forall a \forall b \quad (a,b) \in R \wedge (b,a) \in R \rightarrow a = b$
- transitive: $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

A set together with a partial ordering R is called a poset, an is denoted by (S,R)

Hasse diagrams of a poset don't show loops, because they are innherent to a poset (reflexive), do not show transitive edges because a poset is transitive.
Finalli arrange each edge so that its initial vertex is below its terminal vertex. Remove arrows, because all edges point "forward" to the terminal edges.

Lattice

A partially ordered set in which every pair of elements has both a least upper bound (join) and a greatest lower bound (meet) is called a lattice.



the set is lattice because, for example $\forall (a,b)$ have a join and a meet

join $\{15, 10\} = 30$
meet $\{15, 10\} = 5$

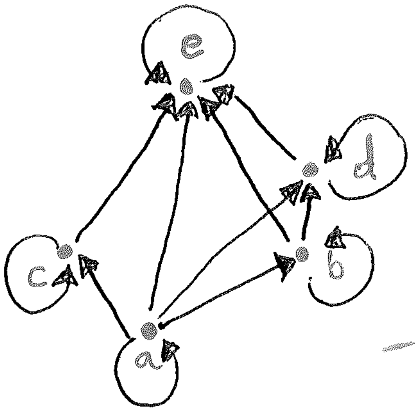
Making a Hasse Diagram

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflexive

	a	b	c	d	e
a	x	x	x	x	x
b		x		x	x
c			x		x
d				x	x
e					x

edges between points/verteces



a edges = (a,a), (a,b), (a,c), (a,d)
(a,e)

b edges = (b,b), (b,d), (b,e)

c edges = (c,c), (c,e)

d edges = (d,d), (d,e)

e edges = (e,e)

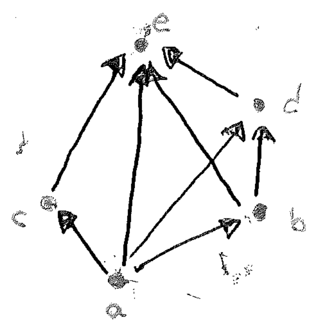
A Hasse Diagram doesn't show loops.

e_a = (a,b) (a,c) (a,d) (a,e)

e_b = (b,d) (b,e)

e_c = (c,e)

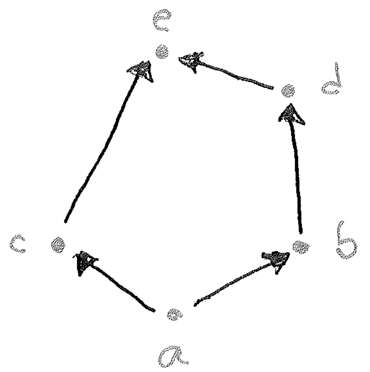
e_d = (d,e)

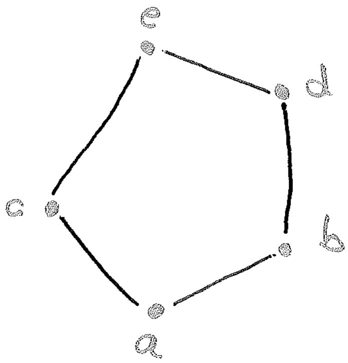


Hasse Diagrams doesn't show transitivity.

$$\underbrace{(a,b)\ (b,d)}_{=(a,d)}\ (d,e) = (a,e)$$

$$(a,c)\ (c,e) = (a,e)$$

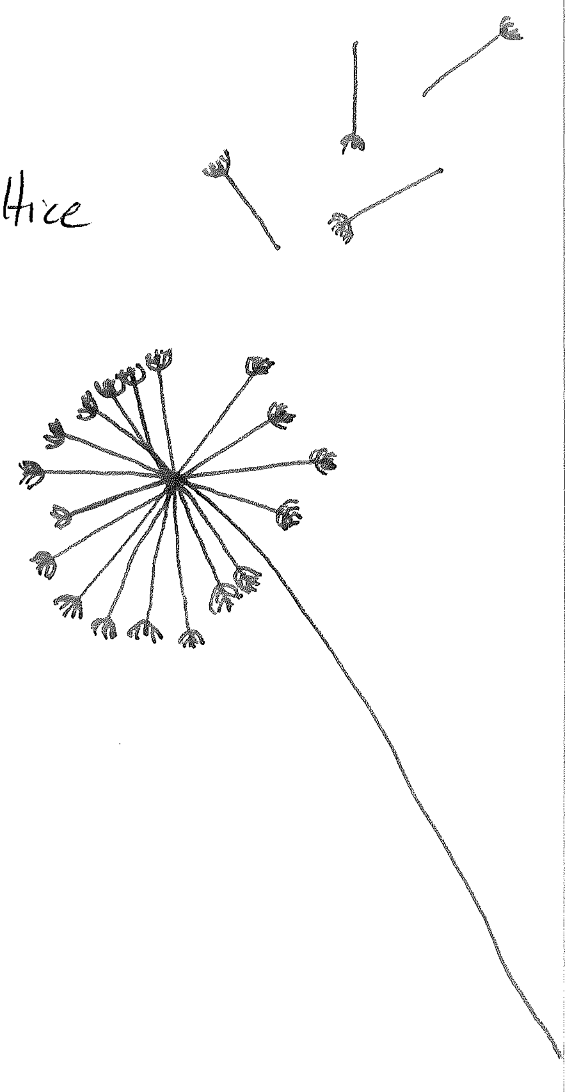




Hasse Diagrams doesn't need arrows because it points allway forward

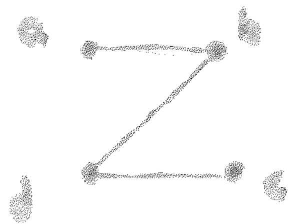
Lattice

$$\left.\begin{array}{l} (c,d) \text{ join } a \text{ , meet } e \\ (c,b) \text{ join } a \text{ , meet } e \\ (a,d) \text{ join } a \text{ , meet } e \end{array}\right\} \text{Lattice}$$



Graph theory

A graph G consist of two types of elements, namely vertices and edges. Every edge has two endpoints in the set of vertices and is said to connect or join the two endpoints. An edge can thus be defined as a set of two vertices (or an ordered pair, in the case of directed graphs) the two endpoints of an edge are also said to be adjacent to each other.
Alternative models of graphs exist. For example, a graph may be thought of as square (0,1) - matrix



$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 1 \\ d & 0 & 1 & 1 & 0 \end{array}$$

A vertex is simply drawn as a node or a dot. the vertex set of G is usually denoted by V(G), or V when there is no danger of confusion. the order of a graph is the number of vertices, $|V(G)|$.

An edge (a set of two elements) is drawn as a line connecting two vertices, called endpoints or endvertices. An edge with endvertices x and y is denoted by xy (without any symbol in between). the edge set of G is usually denoted by E(G) or E when there is no danger of confusion.

Graph Theory

The Handshaking Theorem

$2m = \sum_{v \in V} deg(v)$

How many edges there are in a graph.

An undirected graph has an even number of vertices of odd degree.

All the edges are conted twice ⟶ $1+1$: that's why $m = \frac{2}{2}$

$\frac{1+1}{2} = 1$ edge

this applies even if multiple edges and loops are presented.

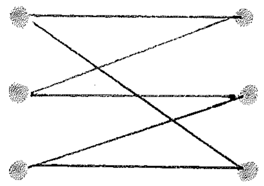
Ex:

$2m = 5$ vertices $deg(4)$ + 10 vertices $deg(1) = (5 \times 4) + (10 \times 1) = 30$

$m = 30/2 = 15$ edges.

Can be used to know the vertices of regions given some degree

V_1 V_2



$K_{3,3}$

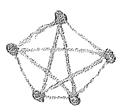
Bipartite Graphs

A simple graph is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 and every edge conects between this two subset.

A simple graph is bipartite iff it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Special Graphs

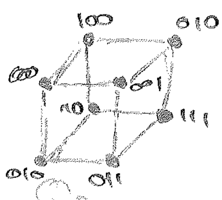
COMPLETE K_n | CYCLE C_n | WHEELS W_n | Cube Q_n



K_5

C_5

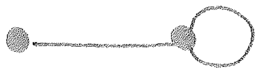
W_6



Q_3

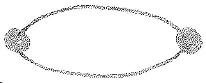
Graph Theory

A loop is an edge whose endpoints are the same vertex.

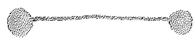


loop

An edge is multiple if there is another edge with the same endvertices; otherwise is simple. A graph is simple graph if it has no multiple edges or loops, a multigraph if it has multiple edges but no loops.

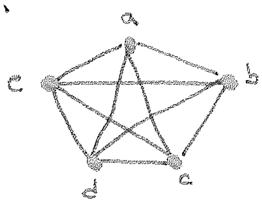


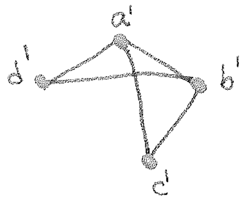
Multiple

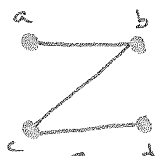


Simple

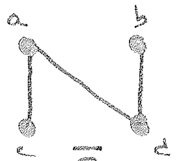
A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$. A subgraph of G is a proper subgraph of G if $H \neq G$







G

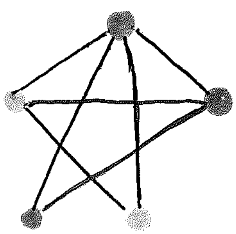


$\overline{G}$

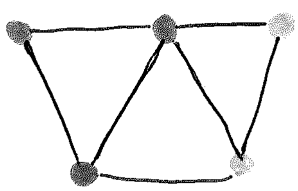
$E(G) = ab, bc, cd$ $\quad E(\overline{G}) = ac, ad, bd$

The complementary graph $\overline{G}$ is a graph with the same vertex set as G but with an edge set such as xy is an edge in $\overline{G}$ iff xy is not an edge in G

Two graphs G and H are isomorphic if there exists a one-to-one and onto function (surjective function, $f(x) = y \ \forall y$) f from V_1 to V_2 with the property that two vertices are adjacent in G iff their corresponding vertices are adjacent in H. Two simple graphs that are not isomorphic are called non isomorphic



A



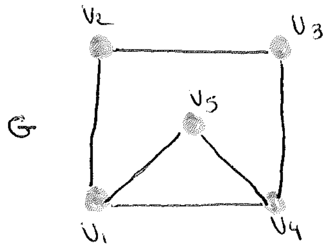
B

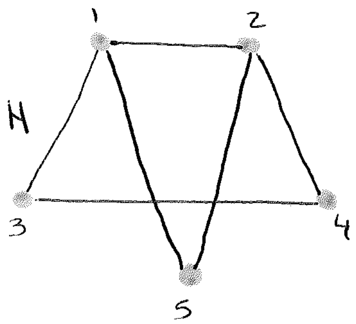
$A \sim B$

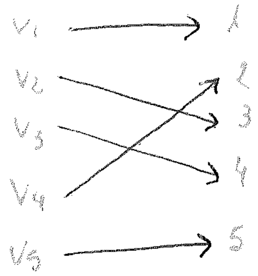
Isomorphic graphs

Example Isomorphism

Isomorphism needs
- Bijection







Bijection from G to H

$$\theta : \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$$

$v_1 v_2$ are adjacent = 13 are adjacent in H

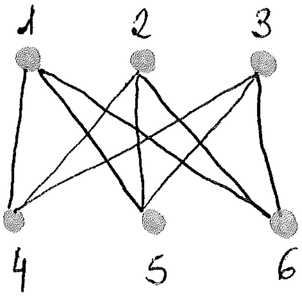
And so all the edges

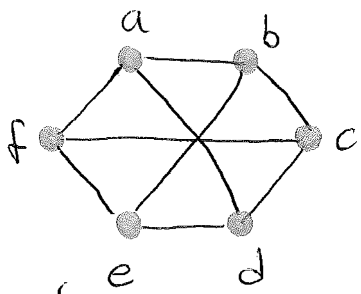
- onto. Every vertex in H is covered by one vertex in G
- one to one. No two vertices from G go to the same vertex in H

for each V(G) there is a differen V(H)

Isomorphic or not?

- [ ] Same # vertices?
- [ ] Same # edges?
- [ ] Structural similarities or differences?





1	2	3	4	5	6
b	d	f	c	e	a

$[H] \neq [G]$

6 vertices
9 edges
3 grade
bipartite
} G

H {
6 vertices
9 edges
3 grade
bipartite

$$G = \begin{bmatrix} 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \end{bmatrix} \qquad H = \begin{bmatrix} 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \\ 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \\ 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \end{bmatrix}$$

One-to-one ¬ onto

$1 \longrightarrow A$
$2 \longrightarrow B$
$3 \longrightarrow D$
C

INJECTIVE

Onto ¬ one-to-one

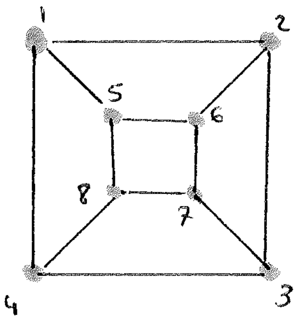
$1 \longrightarrow A$
$2 \longrightarrow B$
$3 \longrightarrow C$
$4 \longrightarrow C$

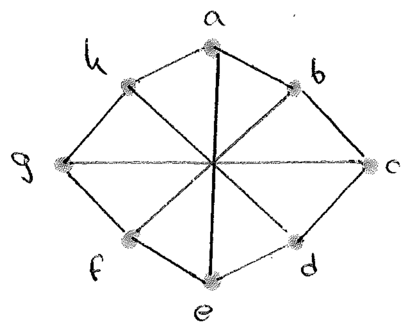
SURJECTIVE

Bijection

$1 \longrightarrow A$
$2 \longrightarrow B$
$3 \longrightarrow C$

EXAMPLE ISOMORPHISM





— G —

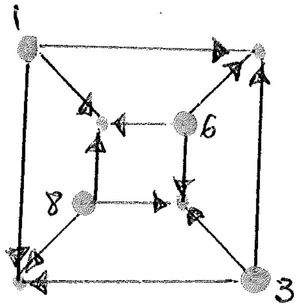
- 8 vertices
- 12 edges
- deg (3)

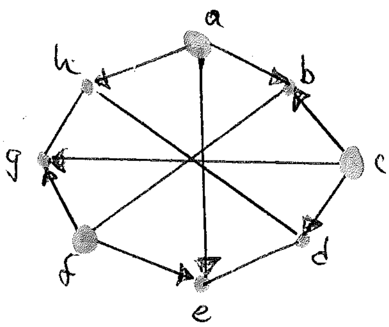
— H —

- 8 vertices
- 12 edges
- deg (3)

Structure

If we pick a vertex from G (G1) and a non adjacent vertex (G3) we can search for more non adjacent vertices, G6 and G8 are also non adjacent between each other. the |set of non adjacent| = 4

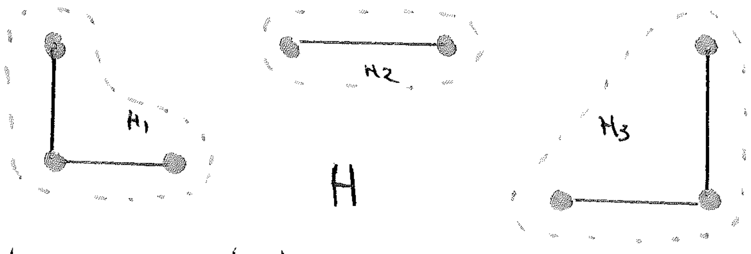




If we try to do this↑ with H
we see differences. We pick Ha, adjacents are eliminated (Gh, Gb, Ge)
Next non adjacent is Hc, its adjacents are Gd and Gg
the next non adjacent to Ga ∧ Gc is Gf, its adjacents are Gg and Ge
the set of non adjacent vertices is {Ga, Gc, Gf} = 3
there is a structural difference between G and H, so they are non isomphic.

Connectedness

An undirected graph is called connected if there is a path (a sequence of edges) between every pair of distinct vertices of the graph



H - No directed graph, no connected

H_1, H_2, H_3 - Connected components

The existence of a simple circuit can be used to show that two graphs are not isomorphic (Pag. 3)

A path is a circuit if begins and ends in the same vertex

Euler

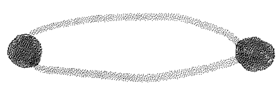
An Euler circuit in a graph G is a simple circuit containing every edge of G.

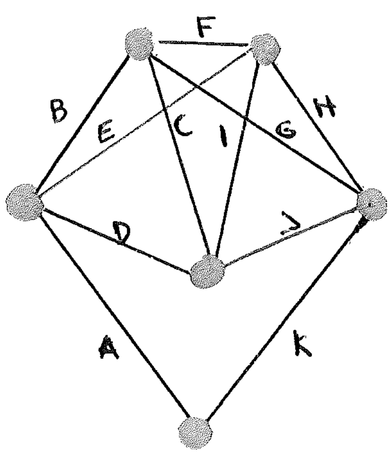
An Euler path is a simple path containing every edge of G.

A connected multigraph (multiple / parallel edges) with at least two vertices iff each of its vertices has even degree.

$Q_n \rightarrow$ Euler

$n \geq 2 \leftrightarrow deg(v) 2K$





Every vertex has even degree. Following the edges in alphabetical order gives an Eulerian circuit.

More about Euler in planar graphs

HAMILTON

A simple path or cycle is called Hamiltonian if it uses all vertices exactly one

Dirac's Theorem → Ore's Theorem

Dirac's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Vertices $n \geq 3$

$\deg(v) \geq n/2$

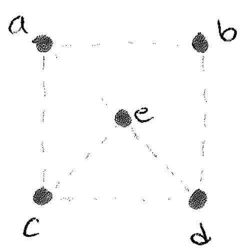
Ore's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G is Hamiltonian

Ore's gives sufficient conditions to be Hamiltonian, some graphs may still be Hamiltonian

vertices $n \geq 3$

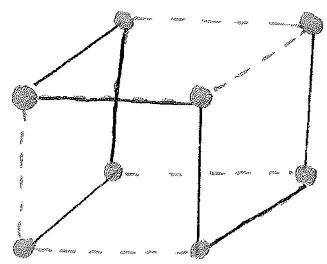
$\deg(u) + \deg(v) \geq n$



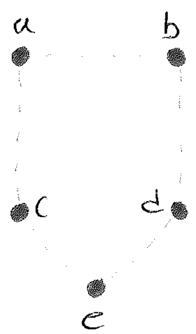
$$\left.\begin{aligned} ad &= 6 \\ be &= 5 \\ bc &= 5 \\ da &= 6 \end{aligned}\right\} \geq 5$$

vertices 5

Circuit = A B D E C A



Hamilton circuit for Q_3

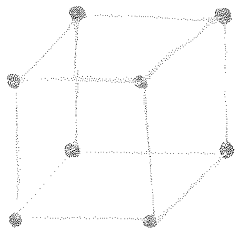


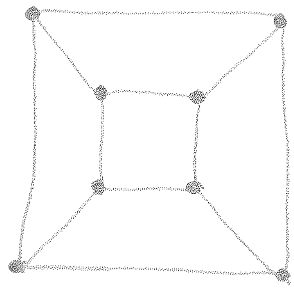
$$\left.\begin{aligned} ad &= 4 \\ ae &= 4 \\ be &= 4 \\ bc &= 4 \\ dc &= 4 \\ da &= 4 \end{aligned}\right\} < 5$$

But! it is Hamiltonian

PLANAR GRAPHS

A graph is planar if can be drawn in the plane without any edges crossing.





Platonic Solids are the 3D shape where each face is the same regular polygon

EULER'S FORMULA

A planar representation of a graph splits the plane into regions, including an unbounded region.

G is a simple connected graph with e edges and v vertices, let r the number of regions.

$$\boxed{r = e - v + 2}$$

For example:

X = 20 vertices deg(3)

Handshaking theorem

edges $= \frac{3 \cdot 20}{2} = 30$ edges

regions $= 30 - 20 + 2 = 12$ regions

this formula stablishes that if $v \geq 3$, $e \leq 3v - 6$

Or in other words, if G is planar, there exists a vertex of degree not exceeding 5.

Ex:

$K_5 \to 5v\ 10e$

$e \leq 3 \cdot v - 6 = 9$

$\Big\}$ K_5 is not Planar

Ex:

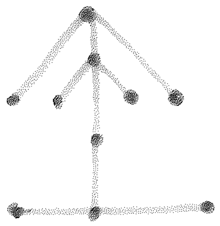
$K_{3,3} \to 6v, 9v$

$e \leq 3 \cdot 6 - 6 = 12$

BUT! it is NOT planar

TREES

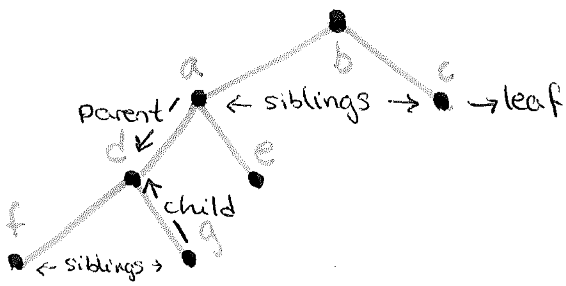
A tree is a connected undirected graph with no simple circuits and there is a unique simple path between any two of its vertices



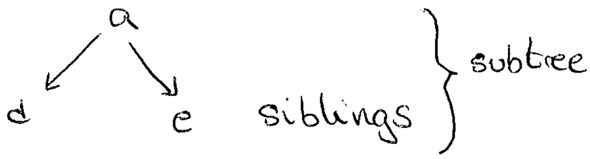
a tree with n vertices has $n-1$ edges

$h \rightarrow$ the height of a tree is the length of the longest path from root to any vertex

A ~~rooted~~ tree has one vertex that has been designated as the root and every edge is directed away from the root



a $\longleftrightarrow$ d
parent child



Ancestors (f) = d, a, b

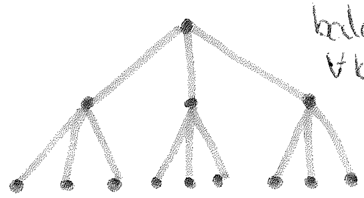
Descendants(a) = d, e, f, g

Leaf ≡ no children

v with childs ≡ internal vertices
ex: a, d

A rooted tree is called an m-ary tree if every internal vertex has no more than m children.

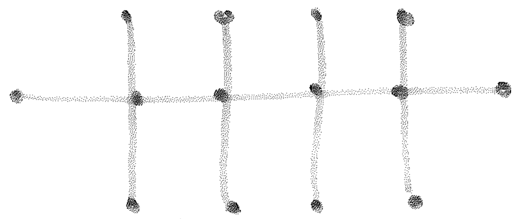
A tree is called a full m-ary tree if every internal vertex has exactly m children.



full 3-ary

balanced iff $\forall$ leaves are $h \vee h-1$

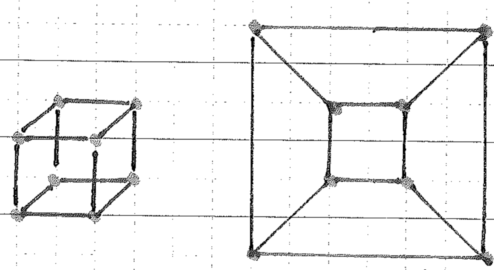
An ordered rooted tree is a tree where the children of each internal vertex are ordered, from left to right.



an m-ary tree with i internal vertices contains $n = mi + 1$ vertices

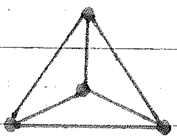
Platonic Solids

A platonic solid is a 3D shape were each side has same shape, size, vertex degree and region degree. P.S. can be shown as planar graphs

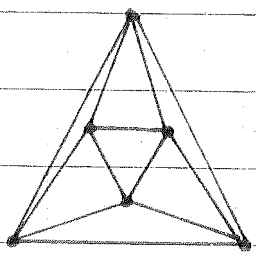


Cube (planar)

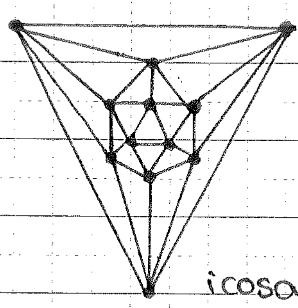
Polyhedron	Vertices	Edges	Regions	deg(v)	deg(r)
tetrahedron	4	6	4	3	3
cube	8	12	6	3	4
octahedron	6	12	8	4	3
dodecahedron	20	30	12	3	5
icosahedron	12	30	20	5	3



Tetrahedron



Octahedron



icosahedron

OBS! Hamiltonian

Why only five?

All other combinatorial information such as V, E, R can be determined from p and q. Since any edges join two vertices and has two adjacent faces we must have:

R(regions) = F (faces)

$$pF = 2E = qV$$

Platonic Solids P. 2

why only 5 P.S?

$$pF = 2E = qv$$

Ex: Dodecahedron

$$5 \cdot 12 = 30 \cdot 2 = 30 \cdot 2$$

Or in other words (and not wikipedia)

$$\deg(r) \cdot r = 2 \cdot e = \deg(v) \cdot v$$

$$r = \frac{2e}{\deg(r)} \qquad v = \frac{2e}{\deg(v)}$$

If we use the Euler's formula: $V - E + F = 2$

$$\frac{2e}{\deg(r)} + \frac{2e}{\deg(v)} - e = 2$$

$$\frac{1}{\deg(r)} + \frac{1}{\deg(v)} - \frac{1}{2} = \frac{1}{e}$$

e cannot be less than zero, so $\frac{1}{e} > 0$

$$\frac{1}{\deg(r)} + \frac{1}{\deg(v)} > \frac{1}{2}$$

Only the 5 Platonics work!

deg(r)	deg(v)	$1/d(r) + 1/d(v)$	
3	3	0,666 ...	tetahedron
3	4	0,583..	cube
4	3	0,583...	Octahedron
4	4	0,5	×
5	3	0,533...	Icosahedron
3	5	0,533...	dodecahedron
5	4	0,45	×
5	5	0,4	×