Language and Logic: Assigment #3

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Problem 1

Give a deductive proof for the following:

1.1

If $\forall x \quad (P(x) \to (Q(x) \land S(x)))$ and $\forall x \quad (P(x) \land R(x))$, then $\forall x \quad (R(x) \land S(x))$.

In this exercise of propositional logic, we have a group of premises and we have to prove the conclusion. In order to do that, I will use a dummy element from x. The idea is that if the natural logic can show that the premises are true for an element from xm then the conclusion has to be also true for all the elements. First we eliminate $\forall x \quad$ and focus con the dummy element x_0. After using the following logic rules, then the conclusion can be general $\forall x \quad$.

Proof.

1.		$\forall x \quad (P(x) \to (Q(x) \land S(x)))$	Premise
2.		$\forall x \quad (P(x) \land R(x))$	Premise
3.	x_0	$P(x_0) \to (Q(x_0) \land S(x_0))$	$\forall x \quad _e 1$
4.		$P(x_0) \land R(x_0)$	$\forall x \quad _e 2$
5.		$P(x_0)$	$\land_e 4$
6.		$Q(x_0) \land S(x_0)$	$\to_e$ 3,5
7.		$S(x_0)$	$\land_{e1} 6$
8.		$R(x_0)$	$\land_{e2}$ 4
9.		$S(x_0) \land R(x_0)$	$\forall x \quad _i$ 3-9
10.		$\forall x \quad (R(x) \land S(x))$	Conclusion

□

1.2

If $\forall x \qquad (P(x) \lor Q(x))$ and $\forall x \qquad ((\neg P(x) \land Q(x)) \to R(x))$, then $\forall x \qquad (\neg R(x) \to P(x))$.

This exercise is similar to the first part. The quantifier is taken by using a dummy variable, for instance x_0, and the conclusion. after being proved to work with a general element, adds a quantifier for all cases of x. The difference is that the rules applied are more complex than those in the first exercise. In this exercise, and in subsequent ones, we are using a derived argument called *Disjunctive Syllogism.* In order to give a better explanation for this, I prove it below.

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Disjunctive Syllogism proof

Proof.

1.	$a \vee b$	Premise
2.	$\neg b$	Premise
3.	$\neg a$	Assumption
4.	a	Assumption
5.	$a \wedge \neg a$	$\wedge_i$3,4,
6.	$\perp$	Contradiction
7.	b	Assumption
8.	$b \wedge \neg b$	$\wedge_i$ 2,6
9.	$\perp$	Contradiction
10.	$\perp$	$\vee e_1$ 1,4-6,7-9
11.	a	$\perp_e$ 3-10

□

After showing the truth of the Disjunctive Syllogism, the proof of the exercise 1.2 is below.

Proof.

1.		$\forall(P(x) \vee Q(x))$	Premise
2.		$\forall x \quad ((\neg P(x) \wedge Q(x)) \to R(x))$	Premise
3.	x_0	$P(x_0) \vee Q(x_0)$	$\forall x \quad _e 1$
4.		$(\neg P(x_0) \wedge Q(x_0) \to R(x_0)$	$\forall x \quad _e 2$
5.		$\neg R(x_0)$	Assumption
6.		$\neg(\neg P(x_0) \wedge Q(x_0))$	MT 4,5
7.		$P(x_0) \vee Q(x_0)$	$\neg_e 6$
8.		$\neg Q(x_0)$	Assumption
9.		$P(x_0) \vee Q(x_0)$	Copy, 7
10.		$P(x_0)$	Disjunctive Syllogism, 5-9
11.		$P(x_0)$	$\vee_e, 6-8$
12.		$R(x_0) \to P(x_0)$	$\to_i$5-11
13.		$\forall x \quad R(x) \to P(x)$	$\forall x \quad _i$ 3-12

□

1.3

$(p \vee q) \wedge (\neg p \vee r) \to (p \vee r)$

The tactic to solve this exercise was to construct an implication from an assumption. The first thing to do is find the implication symbol and assume the first part. During this exercise I used another rule called Constructive Dilemma that I will prove below. This rule uses the proof for *or elimination*, for instance, proving that the two propositions that form the or end up in with the same solution, to prove that an specific *or formula* is true and that it has an implication.

Proof.

1.	$(p \lor q) \land (\neg p \lor r)$	Assumption
2.	$p \lor q$	$\land_{e1}$ 1
3.	$\neg p \lor r$	$\land_{e2}$ 1
4.	p	Assumption
5.	r	Disjunctive Syllogism 3,4
6.	$q \lor r$	$\lor_i 5$
7.	q	Assumption
8.	$q \lor r$	$\lor_i$ 7
9.	$q \lor r$	$\lor_e$ 2,4-6,7-8
10.	$(p \lor q) \land (\neg p \lor r) \to (q \lor r)$	$\to_i$ 1,9

□

1.4

$[(p \to q) \land (q \to r)] \to [p \to r]$

The solution by deduction of this exercise is showed in the proof below, but it is important to mention that this formula is a tautology and for that, it is always true as it can be seen in the table 1 .

p	q	r	$(p \to q)$	$(q \to r)$	$[(p \to q) \land (q \to r)]$	$[p \to r]$	$[(p \to q) \land (q \to r)] \to [p \to r]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Table 1: Truth table of possible combinations

After showing that this is a tautology, now I will show by using deduction, that this still holds.

Proof.

1.	$[(p \to q) \wedge (q \to r)] \to [p \to r]$	Premise
2.	$(p \to q) \wedge (q \to r)$	Assumption
3.	$p \to q$	$\wedge_{e1}$ 2
4.	$q \to r$	$\wedge_{e2}$ 2
5.	p	Assumption
6.	q	$\to_e$ 2
7.	r	$\to_e$ 3
8.	$p \to r$	$\to_i$ 5,7
9.	$[(p \to q) \wedge (q \to r)] \to [p \to r]$	$\to_i$ 2,8

□

1.5

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge q$	$p \vee (\neg p \wedge q)$	$\neg(p \vee (\neg p \wedge q))$	$\neg p \wedge \neg q$	$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$
T	T	F	F	F	T	F	F	T
T	F	F	T	F	T	F	F	T
F	T	T	F	T	T	F	F	T
F	F	T	T	F	F	T	T	T

Table 2: Truth table of possible combinations

As can be seen in table 2 this is yet another tautology where all the possibilities seen in the truth table end up in a true value. In order to prove this I decided to prove implications in both directions, this is, from teh first part of the coimplication I shall deduce the second part, and from the second part, the first part. The process is shown below.

Proof.

1.	$\neg(p \vee (\neg p \wedge q))$	Assumption
2.	$\neg p \wedge \neg(\neg p \wedge q)$	DM 1
3.	$\neg(\neg p \wedge q)$	$\wedge_{e2}$ 2
4.	$\neg\neg p \vee \neg q$	DM 3
5.	$\neg p$	$\wedge_{e2}$ 2
6.	$\neg q$	Disjunctive Syllogism 4-5
7.	$\neg p \wedge \neg q$	$\wedge_i$ 5,6
8.	$\neg(p \vee (\neg p \wedge q)) \rightarrow (\neg p \wedge \neg q)$	$\rightarrow_i$ 1,7
9.	$\neg p \wedge \neg q$	Assumption
10.	$p \vee (\neg p \wedge q)$	Assumption
11.	p	Assumption
12.	$\neg p$	$\wedge_{e1}$ 9
13.	$\perp$	Contradiction
14.	$\neg p \wedge q$	Assumption
15.	q	$\wedge_{e2}$ 14
16.	$\neg q$	$\wedge_{e2}$ 9
17.	$\perp$	Contradiction
18.	$\perp$	$\vee_e$ 10,11-13,14-17
19.	$\neg(p \vee (\neg p \wedge q))$	$\perp_e$ 10-18
20.	$(\neg p \wedge \neg q) \rightarrow \neg(p \vee (\neg p \wedge q))$	$\rightarrow_i$ 9-19
21.	$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$	$\equiv_i$ 8,20

$\square$

Problem 2

Prove the following using induction:

$$p_1 \wedge p_2 \wedge ... \wedge p_n \to q \neg p_1 \vee \neg p_2 \vee ... \vee \neg p_n \vee q$$

Induction allows us to prove equations, and show that every natural number satisfies a certain property. In this case, I made a proof for a base case, for instance p_1, seen below.

Proof.

1.	p_1	Assumption
2.	q	Assumption
3.	$\neg p_1 \vee q$	$\vee_i$ 3
4.	$q \to (\neg p_1 \vee q)$	$\to_i$ 2-3
5.	$\neg p_1 \vee q$	Assumption
6.	q	Disjunction Syllogism 5,1
7.	$(\neg p_1 \vee q) \to q$	$\to_i$ 5-6
8.	$q \equiv (\neg p_1 \vee q$	$\equiv_i$ 4,7
9.	$p_1 \to (q \equiv (\neg p_1 \vee q)$	Proof for p_1

□

Once proved that the base case holds. The next step is to prove that if n is a natural number then we can show that n+1 also has that property. The next proof shows that from p_k we can find an implication to p_{k+1}. Because this two conditions holds, I can say that this formula has been proved true by induction.

Problem 2 continued on next page. . .

Proof.

1.	$p_k \to (q \equiv (\neg p_k \vee q))$	Assumption
2.	p_{k+1}	Assumption
3.	q	Assumption
4.	$\neg p_{k+1} \vee q$	$\vee_i$ 3
5.	$q \to (\neg p_{k+1} \vee q)$	$\to_i$ 3-4
6.	$\neg P_{k+1} \vee q$	Assumption
7.	q	Disjunction Syllogism 6,2
8.	$(\neg p_{k+1} \vee q) \to q$	$\to_i$ 6-7
9.	$q \equiv (\neg P_{k+1} \vee q)$	$\equiv_i$ 2,9
10.	$p_{k+1} \to (q \equiv (\neg P_{k+1} \vee q))$	$\to_i$ 1,9
11.	$(p_k \to (q \equiv (\neg p_k \vee q))) \to (p_{k+1} \to (q \equiv (\neg P_{k+1} \vee q)$	$\to_1$ 1,10

$\square$

Problem 3

In order to complete this exercise I used the next premises based on the text.

- A: The component A is in the system
- B: The component B is in the system
- C: The component C is in the system

The propositions would be:

- The component A is not compatible with the component B

$$\neg(A \wedge B)$$

- Component B can be used only if component C will be included

$$B \rightarrow C$$

- Component C *cannot* be added unless component A is present

$$C \rightarrow A$$

A good way to see the possible combinations that resolve in a truth value for this exercise is by using a truth table. After filling the table 3 we reach that the three possible combinations of the three elements in order to make the system work are:

- $A \wedge \neg B \wedge \neg C$
- $A \wedge \neg B \wedge C$
- $\neg A \wedge \neg B \wedge \neg C$

A	B	C	$\neg(A \wedge B)$	$B \rightarrow C$	$C \rightarrow A$	Combinations
T	T	T	F	T	T	F
T	T	F	F	F	T	F
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	T	T	T

Table 3: Truth table of possible combinations A-B-C

Problem 4

In order to solve this exercise I created a model based in the predicates and domain of the graph presented. Because the graph is undirected, I will assume that is asymmetric, for instance, the connection of two nodes (a, b) is the same as the connection of (b, a).
Model $\mathcal{M}(\mathcal{F}, \mathcal{P})$ where:
$\mathcal{F} = \{1, 2, 4, 5, 6\}$
$\mathcal{P} = N(n), E(x, y), P(x, y)$
Predicates
$N(n)$ - Node host for $n \in \mathcal{F}$.
$E(x, y)$ - Connection between $N(x)$ and $N(y)$ for $xandy \in \mathcal{F}$ has a value of true.
$P(x, y)$ - Set of connections between $N(x)$ and $N(y)$ for x and y$\in \mathcal{F}$.
The truth value of the predicate$E(x, y)$is showed in the table 4.

	1	2	4	5	6
1	T	T	T	F	F
2	T	T	T	F	F
4	T	T	T	T	F
5	F	F	T	T	T
6	F	F	F	T	T

Table 4: Truth table of E(x,y) for the network

The predicate$P(x, y)$ states that there is a path between $H(x)$ and $H(y)$ when the connection between $H(x)$ and $H(y)$ exists or there is a path such as $H(x)$ and a node $H(z)$ with true value and a connection between$H(z)$ and $H(y)$ is also true. For instance:

$$P(x, y) \rightarrow E(x, y) \vee (\exists z \quad P(x, z) \wedge E(z, y))$$

Path from node 1 to 6
For a specific case of H(1) and H(6) we can apply this predicate as P(q,6). It is important to remember that in a model, $\varphi_1 \vee \varphi_2$ holds if the model holds φ_1 or φ_2.

$$P(1, 6) \rightarrow E(1, 6) \vee (\exists y \quad P(1, y) \wedge E(y, 6))$$

This formula holds because there is at least a node y that holds true, such as the next calculus.

$$P(1, 6) \rightarrow E(1, 6) \vee (P(1, 5) \wedge E(5, 6) \tag{1}$$
$$P(1, 5) \rightarrow E(1, 5) \vee (P(1, 4) \wedge E(4, 5) \tag{2}$$
$$P(1, 4) \rightarrow E(1, 4) \tag{3}$$

The whole formula holds true because all the steps done are also true. So the solution of the set of connections is the next one:

$$P(1, 6) = \{(1, 4), (4, 5), (5, 6)\}$$

Well-connected network
In the case of a well-connected network, there should be a path between each two nodes of the network. Our model will need a new interpretation of a new predicate:

$$WC(G) \rightarrow \forall x \forall y \in N \quad P(x, y)$$

Problem 4 continued on next page. . .

We can apply this predicate to a network X with hosts a,b and c and two links between its hosts and prove that it is well-connected.

$$X =< \{a, b, c\}, \{(a, b), (b, c)\} > \quad (4)$$
$$WC(X) = \forall x \forall y P(x, y) \quad (5)$$
$$x = a \quad (6)$$
$$P(a, b) \rightarrow E(a, b) \quad (7)$$
$$P(a, c) \rightarrow (a, c) \vee (P(a, b) \wedge E(b, c)) \quad (8)$$
$$x = b \quad (9)$$
$$P(b, a) \rightarrow E(b, a) \quad (10)$$
$$P(b, c) \rightarrow E(b, c) \quad (11)$$
$$x = c \quad (12)$$
$$P(c, a) \rightarrow E(c, a) \vee (P(c, b) \wedge E(b, a)) \quad (13)$$

In all cases, in all nodes, all their connections have a direct connection or a path that connects the two nodes. After this steps, the conclusion is that the network is well-connected.