2016 - 1

Linnaeus University
Department of Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA462, 7,5 hp
Thursday 26th of May 2016, Time 08.00-13.00.

To obtain maximal points a complete solution, presented in such a way that calculations and reasoning are easy to follow, is demanded. If nothing else is given you can assume that all graphs are undirected, loopfree and not multigraphs.
Aid: Sheet with formulas. No calculators.

1. Five short questions on graphs and trees:(One point on each.)
   a) What is the chromatic number for $K_{4,3}$?
   b) Draw a full, rooted, binary tree with 8 leaves.
   c) Find a Hamilton cycle in K_5.
   d) For which n have Q_n an Euler circuit?
   e) Draw the non-isomorphic simple graphs with 4 vertices and 2 edges. (5p)

2. How many elements are in the union of four sets if the sets have 45, 60, 70 and 100 elements, respectiveley, each pair of sets share 25 elements, each three of the sets share 10 elements and no element is in all four sets? (4p)

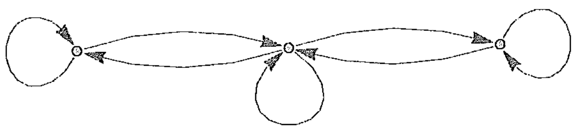
3. Draw the Hasse diagram for divisibility on the set $\{1, 2, 3, 5, 10, 15, 30, 45, 90\}$. Is it a lattice? (3p)

4. a) Determine whether the relation represented by this zero-one matrix is an equivalence relation.

$$\mathrm{M}_R = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(2p)

   b) Determine whether the relation represented by this directed graph is an equivalence relation. (2p)



5. Find the generating function, $G(x)$ for the following problem: In how many ways can 20 identical ballons be distributed to 4 children so that each child gets at least 3 ballons but no one gets more than 7 ballons. Express $G(x)$ on closed form. Explain how to proceed to solve the problem. (4p)

6. Investigate simple connected graphs, G, with 15 vertices such that five of them have degree 4 and the remaining 10 vertices have degree 1. Use the following questions:

   How many edges?
   Is G planar? If so, how many regions?
   Are there non-isomorphic graphs of this type?

   Finally you make drawings of the possible graphs. (5p)

7. The *odd graph* O_k, k is an integer ≥ 2, is defined in the following way:
   The vertices represent the subsets with $k-1$ elements that can be obtained from a set with $2k-1$ elements. Two vertices are joined with an edge if and only if the corresponding subsets are disjoint.
   a) Draw O_2 and O_3 (2p)
   b) Show that O_k is k-regular for all $k \geq 2$, that is, show that all vertices in O_k has degree k. (3p)

Good Luck!

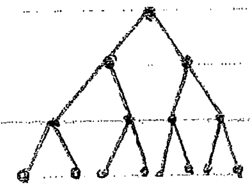
2016-11

1MA462, Discrete Math

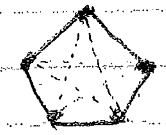
26/5 2016

1 a) $\chi(K_{4,3}) = 2$ since it is bipartite

b)



c) C_5 for example



d) $\deg(v) = n$ for all vertices in Q_n so n must be even.

e)

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2. $45 + 60 + 70 + 100$

$- \binom{4}{2} \cdot 25$

$+ \binom{4}{3} \cdot 10$

$- \binom{4}{4} \cdot 0$

165

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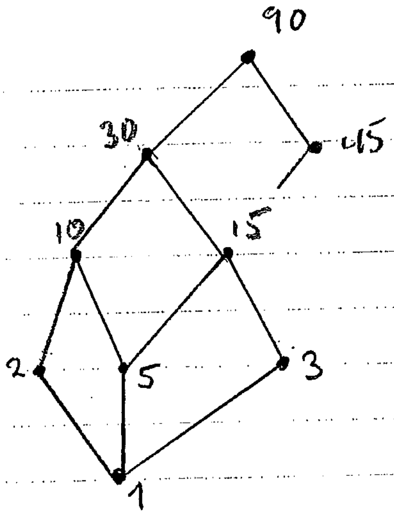
3)



Yes it is a lattice.
For example:

$lub\{10,15\} = 30$
$glb\{10,15\} = 5$

4a) Yes. 2 equivalence classes
$\{1,2,5\}$ and $\{3,4\}$

b) No, it is not transitive.

5) $G(x) = \left(x^3 + x^4 + x^5 + x^6 + x^7\right)^4$

$= x^{12} \cdot \dfrac{(1-x^5)^4}{(1-x)^4}$. Look for coefficient in front of x^{20}.

Nominator: $x^{12} - 4x^{17} + \ldots$

Denominator: $\dfrac{1}{(1-x)^4} = 1 + 4x + \binom{5}{2}x^2 + \binom{6}{3}x^3 + \ldots + \binom{11}{8}x^8 + \ldots$

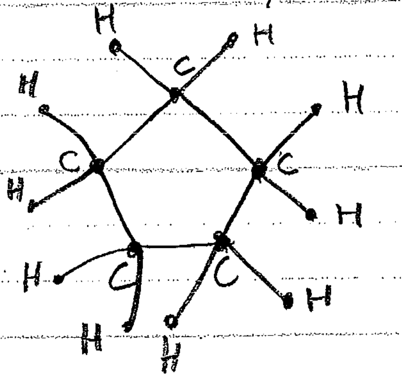
The coefficient is $\binom{11}{8} - 4\binom{6}{3}$.

$C - \deg(v) = 4, \quad H - \deg(v) = 1$

6) $2e = 5 \cdot 4 + 10 \cdot 1 \Leftrightarrow e = 15 = v$

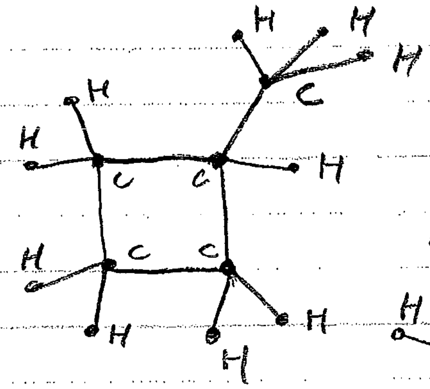
If planar: $r = 15 - 15 + 2$

Note $e = v$ so they are not trees. There are cycles. They must contain the C-vertices. Cycle length can be 3, 4 or 5

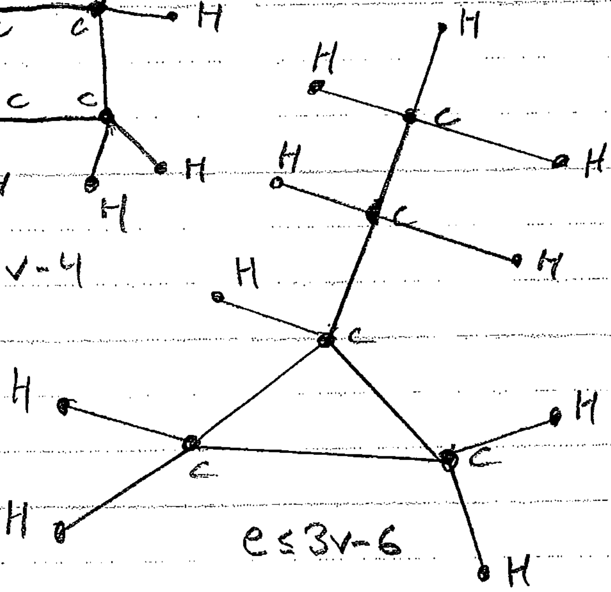


C_5H_{10} - cyclopentane

$e \leq \frac{5}{3}(v-2)$



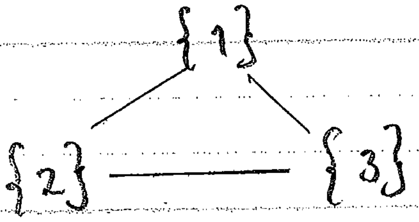
$e \leq 2v - 4$



$e \leq 3v - 6$

Kuratowski:
All graphs of this type must be planar!

7) a) $k = 2$. 1 element from $\{1, 2, 3\}$

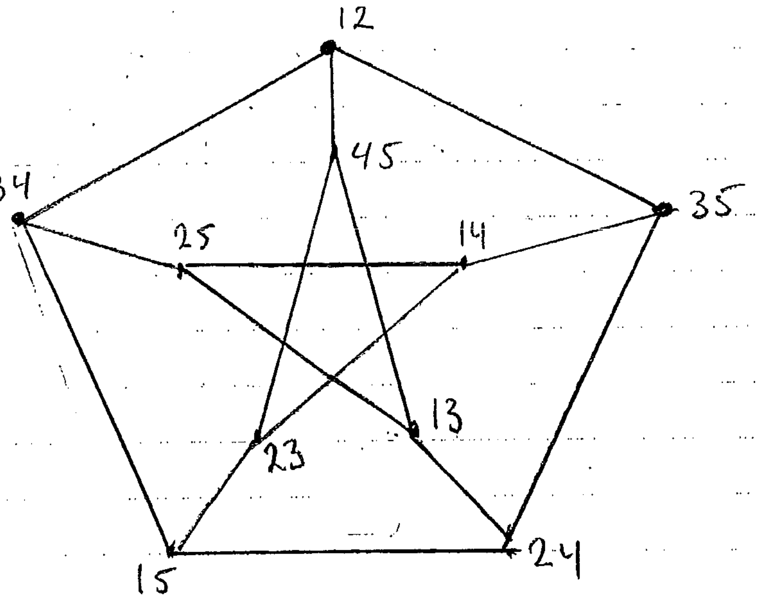


$O_2 = C_3$

O_3 : from $\{1,2,3,4,5\}$ pick 2

$\binom{5}{2} = 10$ vertices , $2e = 10 \cdot 3$

$e = 15$



O_3

b) In O_3 $5-2=3$ remaining elements when you have selected a vertex.

$\binom{3}{2} = 3$ vertices to connect to.

In O_k $(2k-1)-(k-1) = k$ remaining when you have selected a vertex that is a subset with $k-1$ elements.

$\binom{k}{k-1} = k$ different vertices to connect to.

4

2016-2

Linnaeus University
Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA462, 7,5 hp
Thursday 9th of June 2016, Time 08.00-13.00.

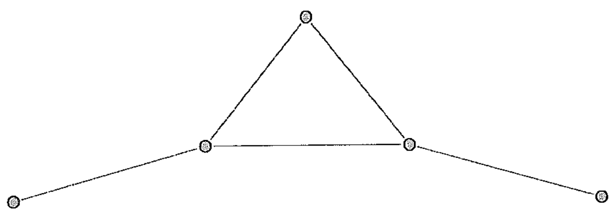
Note: To obtain maximal points a complete solution, presented in such a way that calculations and reasoning are easy to follow, is demanded. If nothing else is given you can assume that all graphs are undirected, connected, loopfree and not multigraphs.

Aid: Sheet with formulas and concepts.

1. Five short questions on counting techniques (one point on each).
   a) What is the sum of the n:th row in Pascal's triangle?
   b) How many six letter strings can be formed by 2 U, 2 R and 2 D?
   c) How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 13 \quad ?$$

   d) Give on closed form the generating function for an infinite string of ones, that is 1, 1, 1, 1, ...
   e) In how may ways can you make a proper coloring of the graph below if you have 5 colors available? (5p)



2. You have to buy 25 bagels. They are of three types: plain, tomato and blueberry. The restrictions are: At least five of each sort but not more than 11 of any sort.
   In how many ways can you do this? Solve the problem with inclusion and exclusion or by using generating function. (5p)

3. a) Show that the divides relation is a partial order on the positive integers. The divides relation, $\mathcal{R}$, is defined by: $x\mathcal{R}y$ if and only if $x|y$. Here x and y are positive integers. (3p)
   b) Define the relation $\mathcal{R}$ on the the integers $\mathbb{Z}$ by $x\mathcal{R}y$, for $x, y \in \mathbb{Z}$, if and only if $x \equiv y \pmod 7$. Show that this is an equivalence relation on $\mathbb{Z}$. (3p)

4. Draw all non-isomorphic trees with five vertices. Note trees are simple connected graphs. (4p)

5. Let G be a simple graph with $|V| = n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all nonadjacent $x, y \in V$, then G contains a Hamilton cycle. This is a sufficient condition for the existence of a Hamilton cycle proved by Ore 1960.
   a) Draw a graph for which the condition holds and find the Hamilton cycle. Show also that it is not a necessary condition, that is, find a graph with a Hamilton cycle but for which

the condition is not fulfilled. (3p)

b) A group of 12 people meet for dinner at a big circular table. In this group everyone knows at least 6 others. Can they be seated around the table in such a way that each person knows the person to the left and to the right? (2p)

6. a) Consider connected simple graphs, G, with 11 vertices. Prove that either G or its complement $\overline{G}$ must be nonplanar. (3p)

b) This theorem does not hold for eight vertices. Find a counterexample to part a above, that is, find a planar G for which also the complement is planar. (2p)

Good luck!

1MA462 9/6 2016

1. a)

1	Σ
1 1	2
1 2 1	4
1 3 3 1	8

Sum is $2^n =$

$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \dots \binom{n}{n}$

b) $\dfrac{6!}{2!\,2!\,2!} = \binom{6}{2} \cdot \binom{4}{2} \cdot \binom{2}{2}$

c) $\binom{15}{13} = \binom{15}{2} = 105$. 13 sticks
2 +

d) $1 + x + x^2 + x^3 + \dots = \dfrac{1}{1-x}$ $|x|<1$

e) $5 \cdot 4 \cdot 4 \cdot 3 \cdot 4 = 5 \cdot 4^3 \cdot 3$

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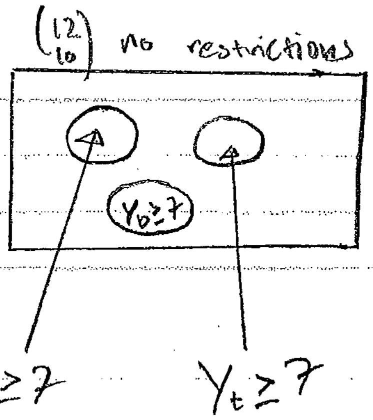
2. $X_p + X_t + X_b = 25$

$5 \le X_t \le 11$, $5 \le X_p \le 11$, $5 \le X_b \le 11$

With $X_t = 5 + Y_t$ etc. we get

$Y_p + Y_t + Y_b = 10$, $0 \le Y_p \le 6$
$0 \le Y_t \le 6$
$0 \le Y_b \le 6$

Answer: $\binom{12}{10} - 3\binom{5}{3}$



With GF:

$$G(x) = \left(x^5 + x^6 + \dots \; x^{11}\right)^3 =$$

$$x^{15} \cdot \left(1 + x + x^2 + \dots x^6\right)^3 =$$

$$x^{15} \frac{\left(1 - x^7\right)^3}{(1-x)^3} = \left(x^{15} - 3x^{22} + \dots\dots\right)$$

$$\cdot \left(1 + 3x + \binom{4}{2}x^2 + \binom{5}{3}x^3 + \dots + \binom{12}{10}x^{10} + \dots\right)$$

Two ways to get x^{25}: $x^{15} \cdot x^{10}$ and $x^{22} \cdot x^3$

Answer: $\binom{12}{10} \cdot 1 + \binom{5}{3} \cdot (-3)$.

	a	b
3		
Reflexive?	$x \mid x$ since $x = 1 \cdot x$ so xRx YES!	$x \equiv x$ since $(x - x) = 0 \cdot 7$ so xRx YES!

2

a	b
Anti-symmetric? $x \mid y \Leftrightarrow y = kx$ $y \mid x \Leftrightarrow x = ly$ $y = kx = kly$ so $k \cdot l = 1$ $k, l \in \mathbb{Z}^+$ so $k = l = 1$ and $y = x$ YES!	Symmetric? $xRy \Leftrightarrow$ $x \equiv y \Leftrightarrow (x-y) = k \cdot 7$ but then $(y-x) = (-k) \cdot 7$ so $y \equiv x \Leftrightarrow yRx$ YES!
Transitive? $x \mid y$ and $y \mid z$ means $y = kx$ and $z = l \cdot y$ k, l are positive integers. $z = ly = lk \cdot x$ so xRz YES!	xRy and yRz: $x - y = k \cdot 7$ and $y - z = l \cdot 7$. Add the 2 equations! $x - y + y - z = (k+l) \cdot 7$ $x - z = (k+l) \cdot 7$ so xRz. YES!

4. $v = e + 1$ for trees.

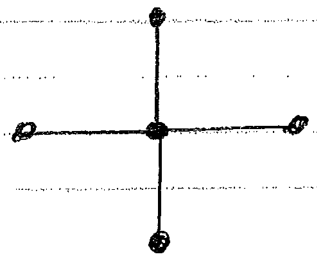
$v = 5$ and $e = 4$.

$$2e = \sum_{v \in V} \deg(v) =$$

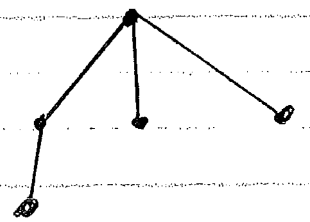
$$8 = \deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4) + \deg(v_5)$$

Max degree is 4

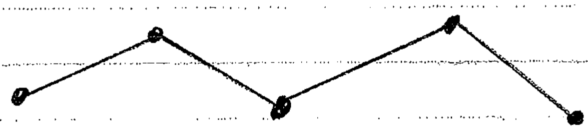
$8 = 1+1+1+1+4$



$8 = 1+1+1+2+3$



$8 = 1+1+2+2+2$



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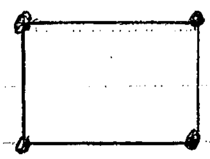
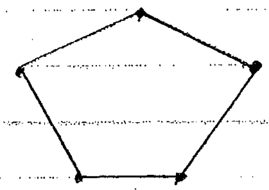
5 a)

C_4



$2+2 \geq 4$
fulfilled.

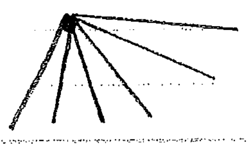
C_5

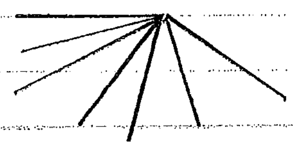


$2+2 < 5$ not fulfilled.

But both graphs have Hamilton cycles.

b)





$$6+6 \geq 12$$

So there is a Hamilton cycle according to the condition.

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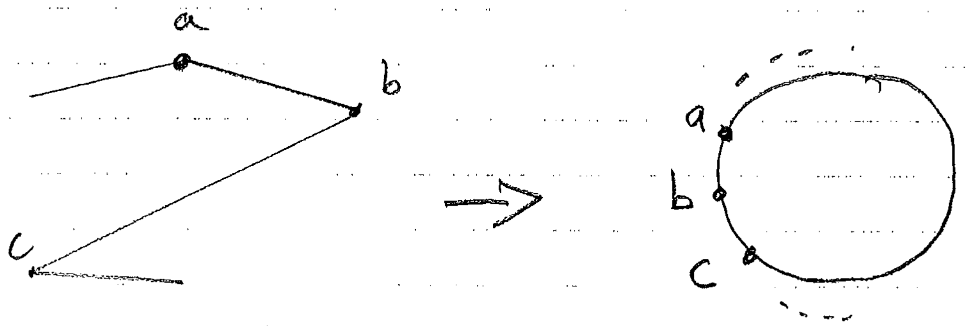
Use it for seating the people



part of HC

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6. a) For planar graphs

$$e \leq 3v - 6$$

is a necessary condition.

K_{11} has $\binom{11}{2} = \frac{11 \cdot 10}{2} = 55$ edges

To be planar with $v = 11$ there can not be more than

$$3 \cdot 11 - 6 = 27 \text{ edges}$$

$$55 = 27 + 28 = 26 + 29 =$$
$$28 + 27 = 29 + 26 = \ldots$$

If G has 27 edges $\bar{G}$ has 28 and vice versa. $\Rightarrow$ One of them must be non-planar.

5

6b) K_8 has $\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$ edges

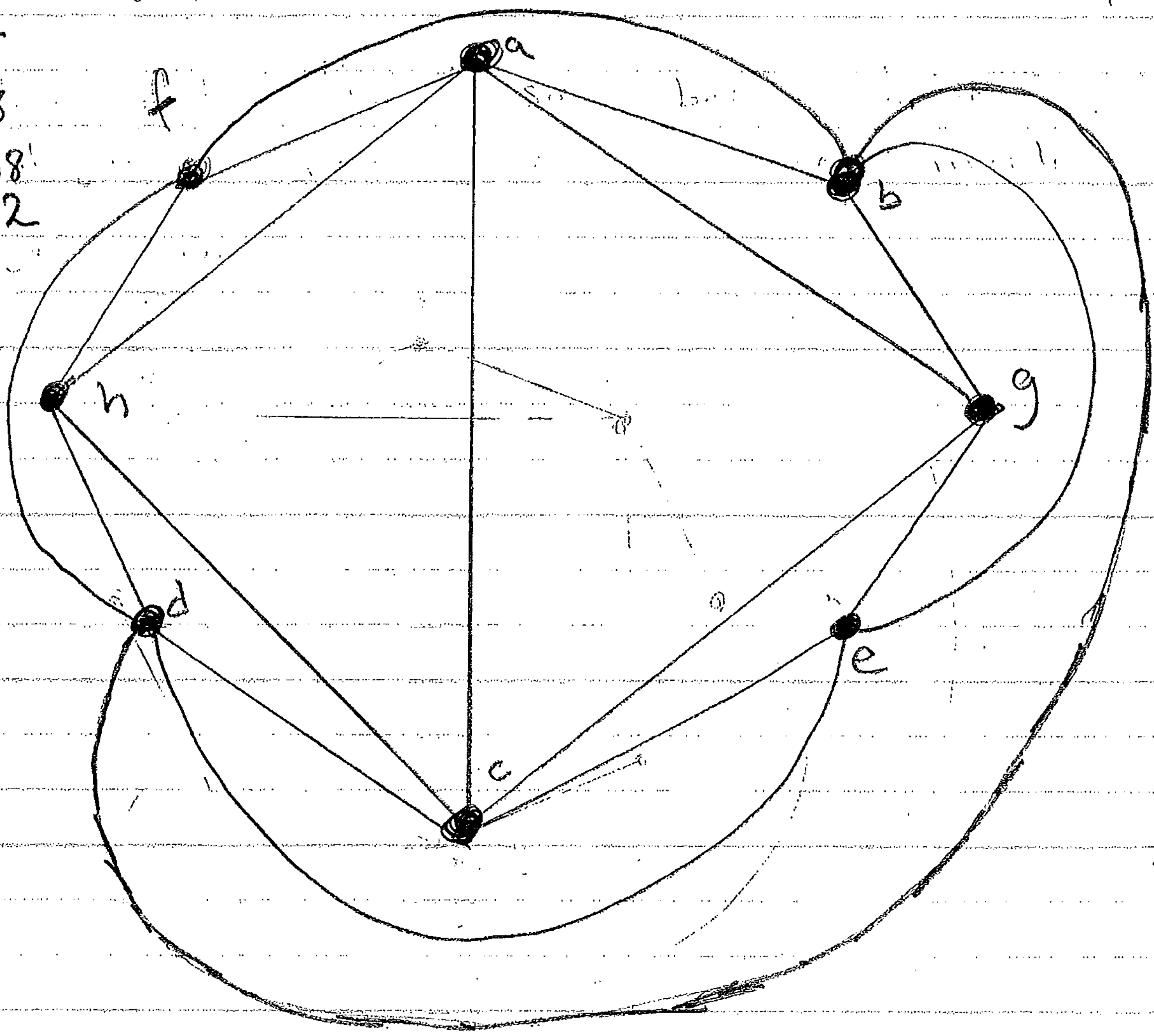
$3 \cdot 8 - 6 = 18$ is the maximal number of edges, Then the number of regions is 12

G

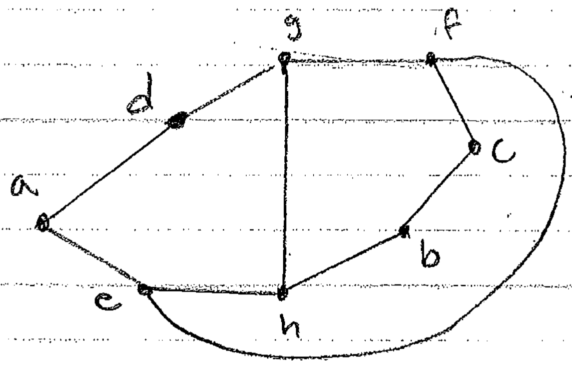
$v = 8$

$e = 18$

$r = 12$



$\overline{G}$



$v = 8$

$e = 10$

$r = 4$

2015 - 1

Linnaeus University
Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA162, 4 hp
Thursday, 28th of May 2015, 08.00-13.00.

1. a) Draw the Hasse diagram for the poset represented by following zero-one matrix. Label the elements in the set 1, 2, 3, 4 and 5 as in the matrix.

$$\mathrm{M}_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(2p)

b) Is this poset a lattice? (2p)

2. This question concerns complete bipartite graphs which are denoted $K_{m,n}$ where n and m are positive integers.
a) How many edges and vertices are there in $K_{m,n}$? (2p)
b) What is the chromatic number for $K_{m,n}$? (1p)
c) For which m and n is $K_{m,n}$ a tree? (2p)
d) For which m and n has $K_{m,n}$ a Hamilton cycle? (2p)
e) Show that $K_{2,3}$ is isomorphic to the graph G on next page. (2p)

3. Define the equivalence relation $\mathcal{R}$ on the the integers $\mathbb{Z}$ by $x\mathcal{R}y$, for $x, y \in \mathbb{Z}$, if and only if $x \equiv y \pmod{13}$. Give the elements in the equivalence class [4]. (3p)

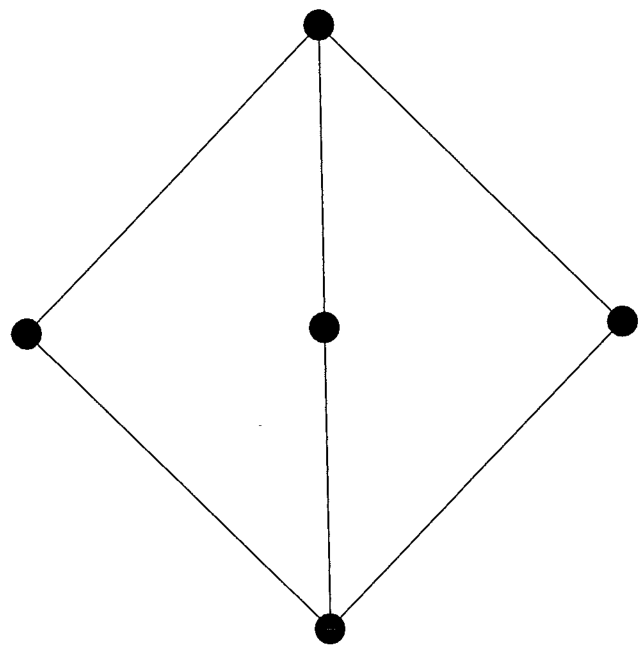
4. In how many ways can the vertices of C_3 (the triangle graph) be given a proper coloring if 5 colors are available?
a) Solve the problem directly using the product rule. (1p)
b) Solve it using the inclusion-exclusion principle. From all possible colorings (no restrictions) you exclude the forbidden ones etc. (4p)

5. Give the generating function, $G(x)$, for the following problem:

You have to distribute 20 cakes to 5 children in such a way that each child gets at least 2 cakes but no one gets more than 7 cakes

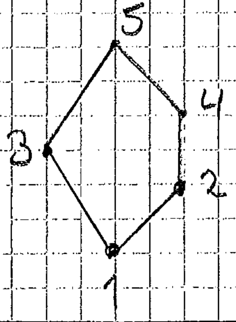
The generating function must be on closed form. Explain then in words how $G(x)$ can be used to solve the problem. (4p)

6. The cube, Q_3, is a *regular* planar graph. The degree of all regions, m, is 4 and the degree of all vertices, n, is 3. Find other combinations of m and n for regular planar graphs and draw them. Hint: $2 > 0$. (5p)



Answers/Solutions IMA162 28/5 2015

1. a)



b) Yes. lub and glb always exist.

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2. a) $v = m+n$, $e = m \cdot n$

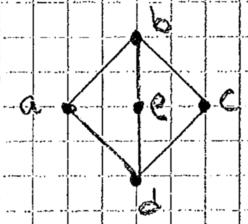
b) $\chi(K_{m,n}) = 2$

c) $n=1$ or $m=1$ or both.

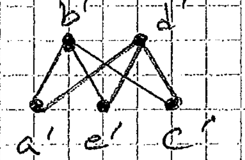
$K_{1,3}$

d) $m=n$ and $m>1$.

e)



$a \to a'$
$b \to b'$
$c \to c'$
$d \to d'$
$e \to e'$



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3. $[4] = \{\ldots, -22, -9, 4, 17, 30, 43, \ldots\}$

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4. a) $5 \cdot 4 \cdot 3 = 60$

b) $5^3 - 3 \cdot 5^2 + 3 \cdot 5^1 - 1 \cdot 5 = 60$

5)

$$G(x) = \left(x^2+x^3+x^4+x^5+x^6+x^7\right)^5$$

$$= x^{10}\left(1+x+x^2+x^3+x^4+x^5\right)^5 =$$

$$= x^{10} \cdot \left(\frac{1-x^6}{1-x}\right)^5 = x^{10} \cdot \frac{(1-x^6)^5}{(1-x)^5}$$

We solve the problem by finding the coefficient in front of x^{20}.

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6)

$$2 = r - e + v \quad \text{(Euler's formula)}$$

$$2e = \sum_i \deg(R_i) = mr \Leftrightarrow r = \frac{2e}{m}$$

$$2e = \sum_j \deg(V_j) = nv \Leftrightarrow v = \frac{2e}{n}$$

so $\quad 2 > 0$

$$r - e + v > 0$$

$$\frac{2e}{m} - e + \frac{2e}{n} > 0$$

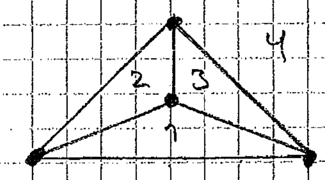
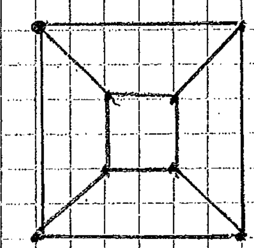
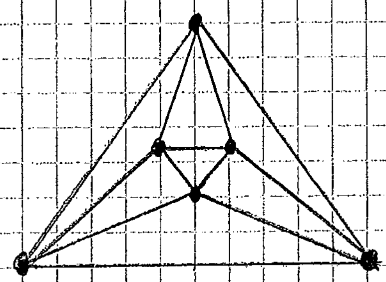
$e > 0$ so $\quad \left(\frac{2}{m} - 1 + \frac{2}{n}\right) > 0$

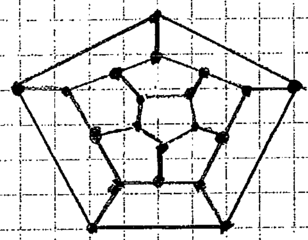
or $\quad \dfrac{4-(m-2)(n-2)}{mn} > 0$

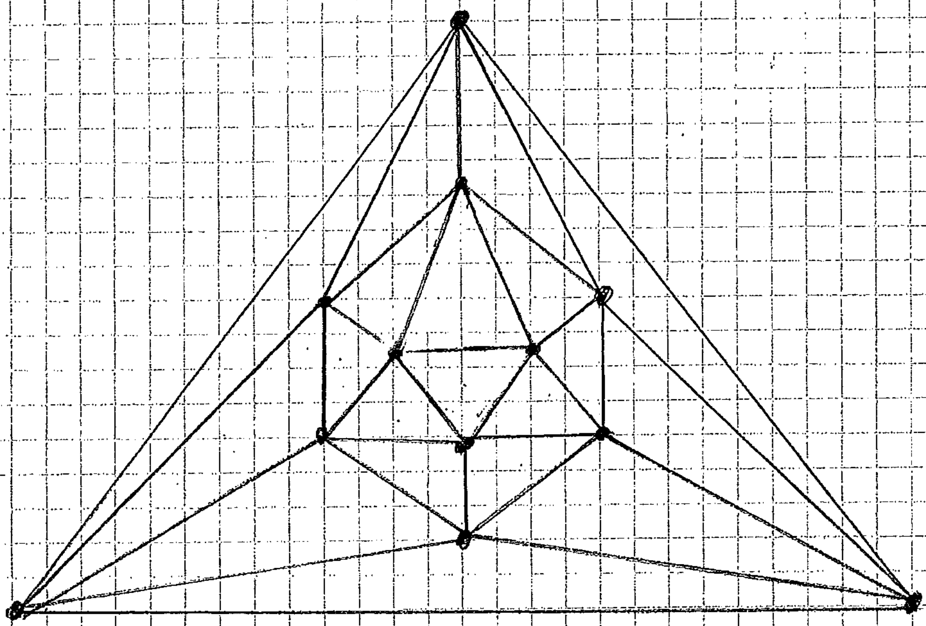
So for these regular planar graphs

$$\boxed{(m-2)(n-2) < 4}$$

$m \geq 3$
$n = 2 \to C_n$
so below
$n \geq 3$

m	n	G
3	3	$e=6$, $r=4$, $v=4$ 
4	3	$e=12$, $v=8$, $r=6$ 
3	4	$e=12$ $r=8$ $v=6$ 

m	n	G
5	3	$e = 30$, $r = 12$ $V = 20$ 
3	5	$e = 30$, $V = 12$ $r = 20$

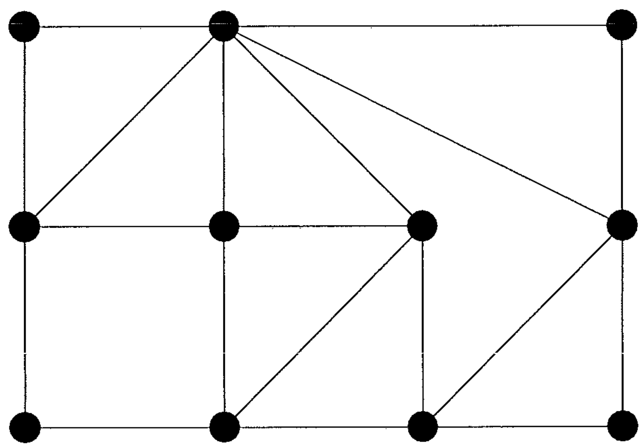


2015-2

Linnaeus University
Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA162, 7,5 hp
Wednesday, 10th of June 2015, 08.00-13.00.

1. Is there an Euler circuit in the graph below? If so, try to find it.



Find also a Hamilton cycle. (4p)

2. How many positive integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 14$$

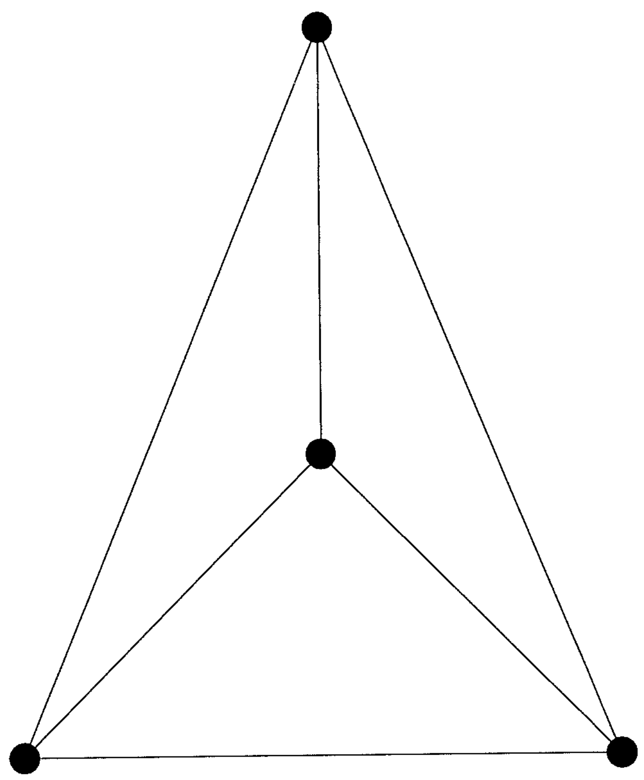
with $2 < x_1 < 6$, $6 < x_2 < 10$ and $0 < x_3 < 5$? To get full point you have to solve the problem with both inclusion-exclusion method and with help of generating function. (6p)

3. Define the relation $\mathcal{R}$ on the the integers $\mathbb{Z}$ by $x\mathcal{R}y$, for $x, y \in \mathbb{Z}$, if and only if $x \equiv y \pmod 3$. Show that this is an equivalence relation on $\mathbb{Z}$ and give the equivalence classes. (5p)

4. Let G be a planar simple graph with eight regions where all vertices have degree 4. Determine the number of vertices and edges. Draw such a graph. (4p)

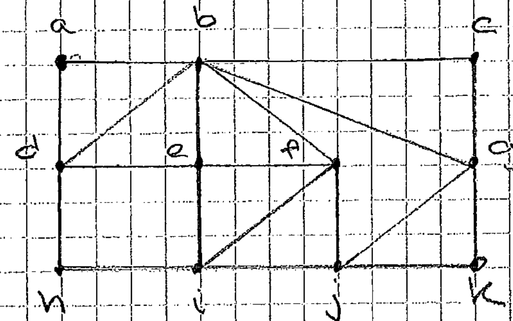
5. An **edge coloring** of a graph is an assignment of colors to the edges so that edges incident with a common vertex are assigned different colors. The **edge chromatic nummber** of a graph is the smallest number of colors that can be used in an edge coloring of the graph.
a) Find the edge chromatic number of W_3. See graph on next page. (2p)
b) Show that if G is a graph with n vertices, then no more than $n/2$ edges can be colored the same in an edge coloring of G. (4p)

6. In a tree with 17 vertices five of the vertices have degree 4 and the remaining 12 vertices have degree 1. Find and draw two non-isomorphic trees of this type. (5p)



IMA162 10/6 2015

1.



All degrees are even so there is an EC.

EC: $a \to b \to c \to g \to k \to j \to g \to b \to f \to j \to i \to f \to e \to i \to h \to d \to e \to b \to d \to a$.

HC: $a \to b \to c \to g \to k \to j \to f \to e \to i \to h \to d \to a$.

---

2. $3+7+4 = 3+8+3 = \dots = 5+8+1 = 14$.

8 possibilities

$X_1 = 3 + Y_1 \qquad 0 \leq Y_1 \leq 2$

$X_2 = 7 + Y_2 \qquad 0 \leq Y_2 \leq 2$

$X_3 = 1 + Y_3 \qquad 0 \leq Y_3 \leq 3$

$$(3+y_1)+(7+y_2)+(1+y_3)=14$$

$$y_1+y_2+y_3=3$$

No restrictions give $\binom{5}{3}=10$ solutions.

Exclude $y_1 \geq 3$ and $y_2 \geq 3$. There are two solutions of this type. $y_3 \geq 4 \rightarrow$ No solutions. In total

$$10-1-1=8 \text{ solutions.}$$

With generating function.

$$G(x)=(x^3+x^4+x^5)\cdot(x^7+x^8+x^9)\cdot(x+x^2+x^3+x^4)=$$

$$\frac{x^{11}\cdot(1-x^3)^2\cdot(1-x^4)}{(1-x)^3}$$

$$=\frac{x^{11}-2x^{14}-x^{15}+\cdots}{(1-x)^3}$$

Tabel gives $\frac{1}{(1-x)^3}=1+3x+6x^2+10x^3+\ldots$

Coefficient in front of x^{14}: $10-2=8$.

3. Reflexive ?

$x - x = 0 \cdot 3 = 0$ Yes. xRx

Symmetric ?

If $x - y = k \cdot 3$ then $y - x = (-k) \cdot 3$.

Yes. $xRy \Rightarrow yRx$

Transitive ?

If $x - y = k \cdot 3$ and $z - y = l \cdot 3$

then $(x-y) - (z-y) = k \cdot 3 - l \cdot 3$

$$\Leftrightarrow$$

$$x - z = (k - l) \cdot 3$$

Yes. xRy and yRz

$\Rightarrow xRz$

$[0] = \{ \ldots -6, -3, 0, 3, 6, \ldots \}$

$[1] = \{ \ldots -5, -2, 1, 4, 7, \ldots \}$

$[2] = \{ \ldots -4, -1, 2, 5, 8, \ldots \}$

---

4. $r = 8 = e - v + 2 \Leftrightarrow e = 6 + v$

$$2e = \sum_{v \in V} \deg(v) = 4 \cdot v$$

so also $e = 2v$.

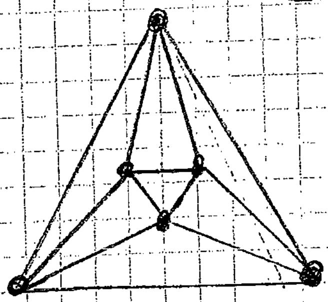
This gives $e = 2v = 6 + v \Leftrightarrow v = 6$

3

$r = 8$, $v = 6$ and $e = 12$

If all regions have the same degree it must be 3.

$$8 \cdot x = 2 \cdot 12$$
$$x = 3.$$



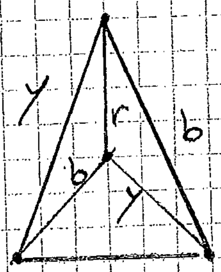
This is the only possibility.

octahedron

---

5. a)

r = red, b = blue
y = yellow



3 colors needed.

b) We know that

$$2e = \sum_{v \in V} \deg(v)$$

If we consider the subgraph with, say, blue edges then

$$2e_B = \sum_{v_B \in V_B} \deg(v_B)$$

All degrees are here 0 or 1 since it is an edge coloring.

4

$n = 2k$: $$2e_B \le (1+1+1+\dots+1) = 2k$$
$$e_B \le k$$

$n = 2k+1$: $$2e_B \le (1+1+\dots+1+0) = 2k$$
$$e_B \le k.$$

Note!

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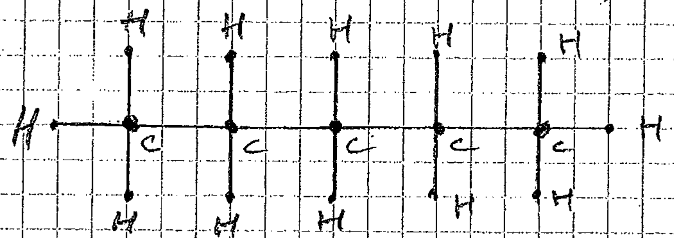
6. $$2e = 5\cdot 4 + 12\cdot 1 = 32$$
$e = 16 = v-1$ as it should be in a tree.

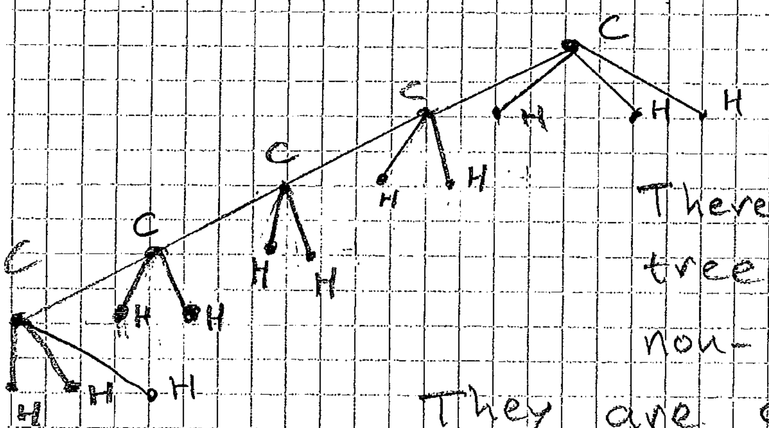
G_1



Pentane

C for degree 4 vertices and H for degree 1 vertices. As for the molecules

G_1 is isomorphic to



(root)

level 1

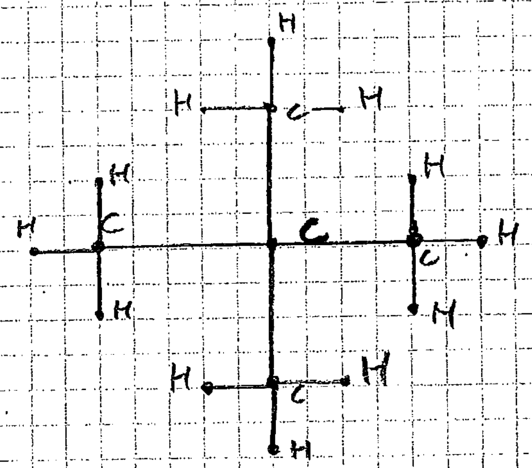
There are 2 other trees of this type non-isomorphic to G_1.

They are called <u>isomers</u> in chemistry.

You get them by putting 3 or 4 C on level 1.

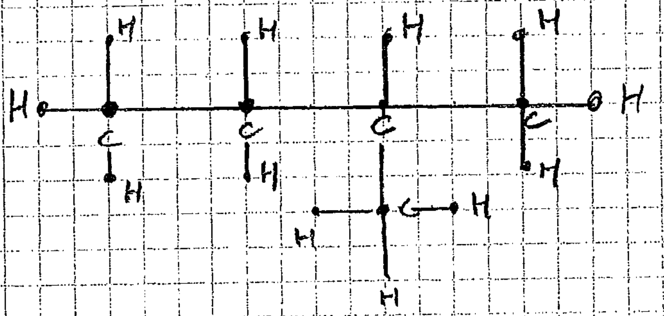
neopentane

4 C on level 1:



isopentane

3C on level 1

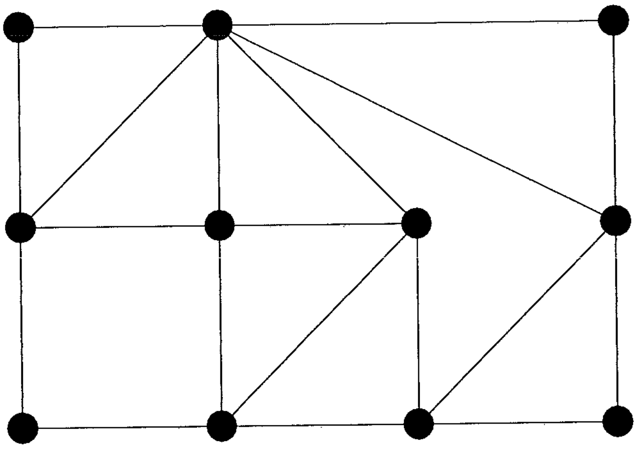


2015 - 3

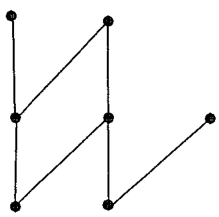
Linnaeus University
School of Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA162, 7,5 hp
Saturday 29th of August 2015, Time 09.00-14.00.

1. Determine the chromatic number, $\chi(G)$, for the following graph G. Give a proper coloring of G using $\chi(G)$ colors. (4p)



2. a) A Hasse diagram is shown below. Unfortunately the labels at the vertices are missing. Give suggestions of what they can be. The relation $\mathcal{R}$ is the inclusion relation $\subseteq$ on the power set, $P(S)$, of the set $S = \{1, 2, 3, 4\}$. That is, $A\mathcal{R}B$ if and only if $A \subseteq B$, where $A \in P(S)$ and $B \in P(S)$. (3p)



b) Define the relation $\mathcal{R}$ on the set A of all bit strings of length four such that $s\mathcal{R}t$, for $s, t \in A$, if and only if s and t contain the same number of 1s. Thus $A = \{0000, 0001, 0010, \ldots\ldots, 0111, 1111\}$. What are the equivalence classes? Give the elements in them. (3p)

3. You have to buy 25 bagels. They are of four types: plain, onion, tomato and blueberry. The restrictions are: At least three of each sort. Not more than six with onions.
In how many ways can you do this? Solve the problem with inclusion and exclusion and then by using generating function. (6p)

4. Some different questions on graphs.
   a) How many edges does a simple graph have if the degrees of the vertices are 5,2,2,2,2,1? Draw such a graph. (3p)
   b) For which values of n do K_n have an Euler circuit? (2p)
   c) Draw the complement graph of $K_{2,3}$. For the definition of complement graph see sheet with formulas and theory. (2p)
   d) How many edges does a tree with 150 vertices have? (1p)

5. Suppose that a connected planar simple graph with e edges and v vertices contains no simple cycles of length 4 or less. Show that

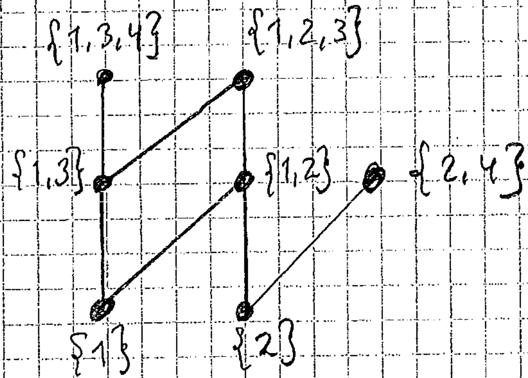
$$e \leq \frac{5v - 10}{3}$$

when $v \geq 5$. Draw such a graph with $v = 7$ and maximal number of edges. (6p)

1MA162 29/8 2015

1. There are triangles as subgraphs so at least 3 colors needed. And it is possible to do it with 3 colors!

2. a) For example



b) $[0000] = \{0000\}$

$[0001] = \{1000, 0100, 0010, 0001\}$

$[0011] = \{1100, 1010, 1001, 0110, 0101, 0011\}$

$[0111] = \{1110, 1101, 1011, 0111\}$

$[1111] = \{1111\}$

3) p - plain
o - onion
b - blueberry

$$X_p + X_o + X_t + X_b = 25$$

$$X_p \geq 3, \; 3 \leq X_o \leq 6,$$

$$X_t \geq 3, \; X_b \geq 3$$

Put $X_p = 3 + Y_p$ etc.

$$Y_p + Y_o + Y_t + Y_b = 13$$

$y_p \geq 0$, $0 \leq y_o \leq 3$, $y_t \geq 0$, $y_b \geq 0$

No upper restriction $\binom{16}{3}$

With $y_o \geq 4$ we have $\binom{12}{3}$ possibilities.

So there are $\binom{16}{3} - \binom{12}{3}$ integer solutions.

with GF:

$$G(x) = (x^3 + x^4 + \dots)^3 (x^3 + x^4 + x^5 + x^6)$$

$$= x^{12} \cdot \frac{1}{(1-x)^3} \cdot \frac{1-x^4}{1-x} =$$

$\frac{x^{12} - x^{16}}{(1-x)^4}$. What is the coefficient in front of x^{25}?

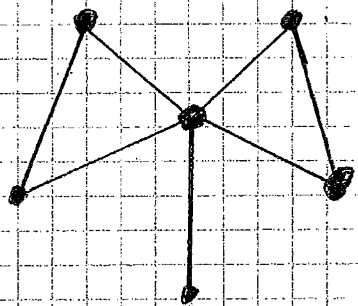
$$\frac{1}{(1-x)^4} = 1 + \binom{4}{1} x + \binom{5}{2} x^2 + \dots + \binom{12}{9} x^9 +$$

$$\dots + \binom{16}{13} x^{13} + \dots$$

So we get $\binom{16}{13} - \binom{12}{9}$ for the coefficient. Note $\binom{16}{13} = \binom{16}{3}$, $\binom{12}{9} = \binom{12}{3}$

4a) $2e = 5 + 4 \cdot 2 + 1 = 14$

$e = 7$

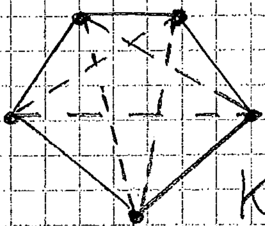


b) $\deg(v) = n-1$ in K_n

$n-1$ must be even so

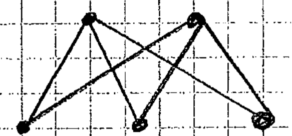
n must be odd

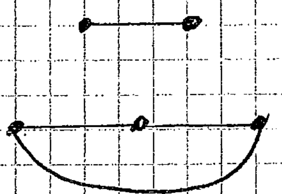
K_3



K_5

c) $G = K_{2,3}$ $\qquad \overline{G}$





d) $V = e+1$ for trees so $e = 149$

3

5)

$$r = e - v + 2$$

$$2e = \sum_{R_i} \deg(R_i) \geq 5r$$

since there are no cycles of length 4 or less.

$$r \leq \frac{2e}{5} \quad \text{but} \quad r = e - v + 2$$

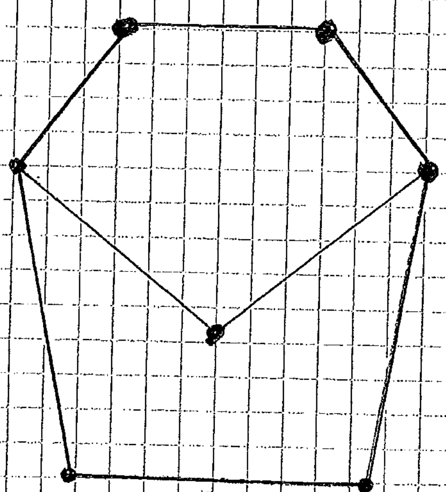
so

$$e - v + 2 \leq \frac{2e}{5}$$

$$\frac{3e}{5} \leq v - 2$$

$$e \leq \frac{5v - 10}{3}$$

If $v = 7$ $e \leq \frac{25}{3}$ for these graphs. $e = 8$ is the maximum number of edges



$v = 7$
$e = 8$
Shortest cycle is of length 5

4

2014 - 1

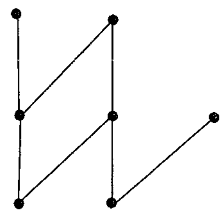
Linnaeus University
Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA162, 7,5 hp
Tuesday 27th of May 2014, Time 08.00-13.00.

Note: To obtain maximal points a complete solution, presented in such a way that calculations and reasoning are easy to follow, is demanded. If nothing else is given you can assume that all graphs are undirected, loopfree and not multigraphs.

Aid: Sheet with formulas and concepts.

1. a) A Hasse diagram is shown below. Unfortunately the labels at the vertices are missing. Give suggestions of what they can be. The relation is the divides relation, that is $x\mathcal{R}y$ if and only if $x|y$, and the set is some subset of the positive integers. (2p)



   b) Let A be a non empty set and fix the set B, where $B \subseteq A$. Define the relation $\mathcal{R}$ on the power set $\mathcal{P}(A)$ by $X\mathcal{R}Y$, for $X, Y \subseteq A$, if and only if $B \cap X = B \cap Y$. This is an equivalence relation on $\mathcal{P}(A)$. If $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$ find the equivalence class $[X]$ if $X = \{1,3,5\}$. The power set $\mathcal{P}(A)$ is the set of all subsets of A. (3p)

2. 25 *identical* cakes are distributed to 10 children. In how many ways can this be done if....
   a) there are no other restrictions? (1p)
   b) each child must get at least one cake but not more than four cakes? Use inclusion-exclusion method. (3p)
   c) Solve problem b above using generating function. (3p)

3. How many regions have a planar graph with six vertices which is 3-regular, i.e. all vertices have degree 3? Draw such a graph. (3p)

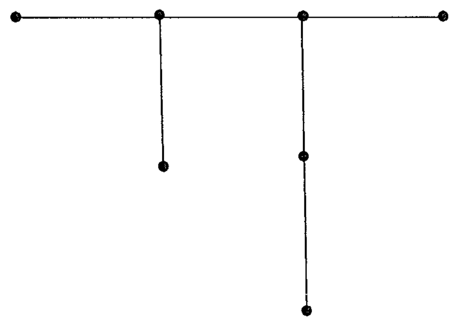
4. Are the two trees below isomorphic to each other? Motivate! (3p)

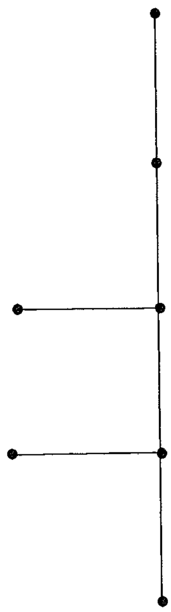
5. a) How many edge disjoint Hamilton cycles do you find in K_5? Draw them. (2p)
   b) Show that a necessary condition for K_n, $n \geq 3$, to be decomposable into edge disjoint Hamilton cycles is $n = 2k + 1$ (k is an integer). That is, n must be odd. (4p)

6. Prove by mathematical induction the following formula for the *chromatic polynomial* for a cycle, C_n, with n vertices.

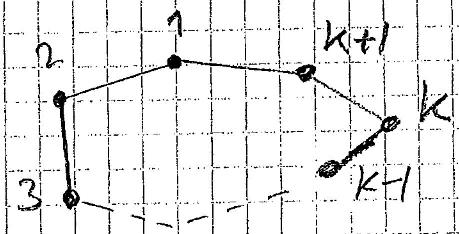
$$p(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1), \quad n \geq 3.$$

C_3 is a triangel, C_4 is a square and so on. The chromatic polynomial $p(G, \lambda)$ tells us in how many ways we can do a (proper) coloring of graph G with λ colors.
Hint: Start with the triangle. Continue with the square. To obtain $p(C_4, \lambda)$ split the problem into two parts. First when opposite vertices have the same color and, secondly, when opposite vertices have different colors. Compare with formula. Go on with $p(C_5, \lambda)$. If you order the vertices 1,2,3,4,5 in clockwise sense split the problem into: 1) vertex 1 and 4 have the same color. 2) vertex 1 and 4 have different colors. How can the results for triangle and square be used in the pentagon? (6p)





Lycka till! Good Luck! Bonne Chance! Buena Suerte! Succes Gewenst!



C_{k+1}

$$P(C_{k+1}) = (\lambda-1)\,P(C_{k-1}) + (\lambda-2)\,P(C_k)$$

For corner k+1

Join corners 1 and k since they have same colors.

For corner k+1

Since 1 and k have different colors

$$\overset{\text{using induction assumption}}{=} (\lambda-1)\left((\lambda-1)^{k-1} + (-1)^{k-1}(\lambda-1)\right) + (\lambda-2)\left((\lambda-1)^{k} + (-1)^{k}(\lambda-1)\right) =$$

Using induction assumption

Note $(-1)^{k-1} = (-1)^{k+1}$

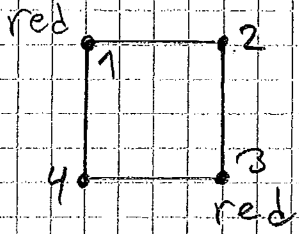
$(-1)^{k} = -(-1)^{k+1}$

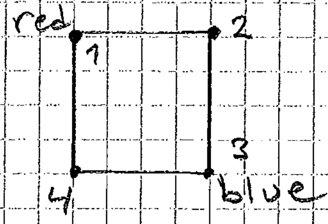
$$= (\lambda-1)^{k} + (\lambda-2)(\lambda-1)^{k} + (-1)^{k+1}\left((\lambda-1)^{2} - (\lambda-2)(\lambda-1)\right)$$

$$= (\lambda-1)^{k}\cdot(1+\lambda-2) + (-1)^{k+1}\cdot(\lambda-1)\,(\lambda-1-(\lambda-2)) = (\lambda-1)^{k+1} + (-1)^{k+1}\cdot(\lambda-1)$$

$$= P(C_{k+1}, \lambda).$$ And we are done!

Two cases for square:





$\lambda \cdot (\lambda-1)^2$

possibilities for corner 1 and 3.

possibilities for corner 2 and 4

$$\underbrace{\lambda(\lambda-1)}_{\text{corner 1 and 3}} \underbrace{(\lambda-2)^2}_{\text{corner 2 and 4.}}$$

Totally $\lambda \cdot (\lambda-1)(\lambda-2)^2 + \lambda(\lambda-1)^2$

$$= \lambda \cdot (\lambda-1)\left((\lambda-2)^2 + (\lambda-1)\right) =$$

$$\lambda \cdot (\lambda-1)(\lambda^2 - 3\lambda - 3)$$

The formula tells us

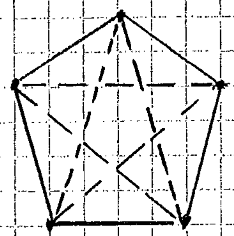
$$P(C_4, \lambda) = (\lambda-1)^4 + (\lambda-1) =$$

$$(\lambda-1)\left((\lambda-1)^3 + 1\right) = (\lambda-1)(\lambda^3 - 3\lambda^2 + 3\lambda)$$

$$= \lambda \cdot (\lambda-1)(\lambda^2 - 3\lambda + 3). \quad \text{ok!}$$

Assume the formula is valid up to C_k. Consider now C_{k+1}

5 a) Two. Pentagon + Pentagram.



b) X is the number of HC:s.

Then $X \cdot n = e = \frac{n(n-1)}{2}$

Since each HC has n edges

since we consider K_n

$$X \cdot n = \frac{n(n-1)}{2}$$

$$X = \frac{n-1}{2}, \quad X \text{ is an integer.}$$

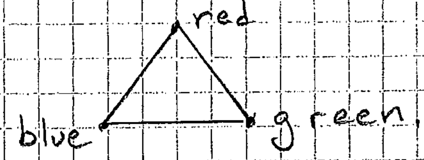
$$n = 2X + 1 \quad \blacksquare$$

---

6. $P(C_3, \lambda) = (\lambda-1)^3 - (\lambda-1) =$

$(\lambda-1)\left((\lambda-1)^2-1\right) = (\lambda-1)(\lambda^2-2\lambda) =$

$\lambda \cdot (\lambda-1) \cdot (\lambda-2)$. New color for each new vertex.



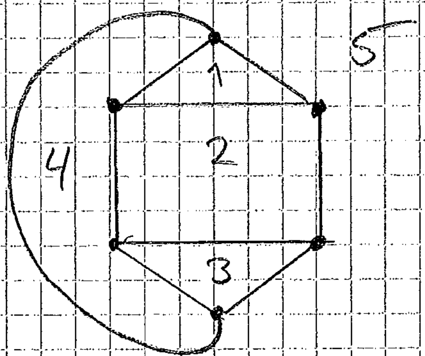
Let's do the square also

$$G(x) = x^{10} \cdot \left(1 - \binom{10}{1}x^4 + \binom{10}{2}x^8 - \binom{10}{3}x^{12} + \cdots\right)$$
$$\cdot \left(1 + \binom{10}{1}x + \binom{11}{2}x^2 + \binom{12}{3}x^3 + \cdots\right).$$

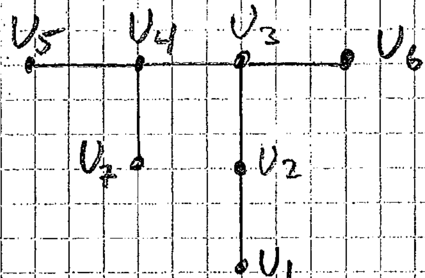
What is the coefficient in front of x^{25}? Check $x^{10} \cdot x^{15}$, $x^{14} \cdot x^{11}$, $x^{18} \cdot x^7$ and $x^{22} \cdot x^3$. It gives

$$\binom{24}{15} - \binom{20}{11} \cdot \binom{10}{1} + \binom{16}{7} \cdot \binom{10}{2} - \binom{12}{3}\binom{10}{3}.$$

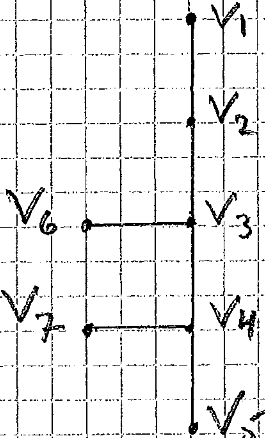
3. $2e = \sum_{v \in V} \deg(v) = 3v = 18$
so $e = 9$.
$r = e - v + 2 = 9 - 6 + 2 = 5$



4. Yes!



$U_1 \to V_1$
$U_2 \to V_2$
$U_3 \to V_3$
$U_4 \to V_4$
$U_5 \to V_5$
$U_6 \to V_6$
$U_7 \to V_7$



$\overline{3|}$

b) $1 \le x_i \le 4 \qquad i = 1, 2, \ldots, 10$

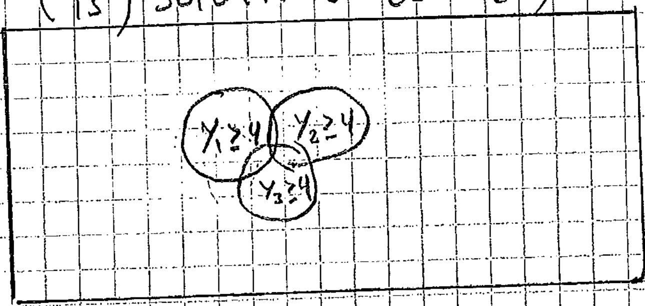
Put $x_i = 1 + y_i \qquad i = 1, 2, \ldots, 10$

Then $0 \le y_i \le 3$.

$(1+y_1) + (1+y_2) + \cdots + (1+y_{10}) = 25$

$$y_1 + y_2 + y_3 + \cdots + y_{10} = 15 \qquad (*)$$

$\binom{24}{15}$ solutions to $(*)$



I just show $y_1 \ge 4$, $y_2 \ge 4$, $y_3 \ge 4$.

Using inclusion-exclusion we get the number of solutions

$$\binom{24}{15} - \binom{10}{1}\binom{20}{11} + \binom{10}{2}\binom{16}{7} - \binom{10}{3}\binom{12}{3}.$$

$15-4=11 \qquad 15-8=7 \qquad 15-12=3$

left to distribute

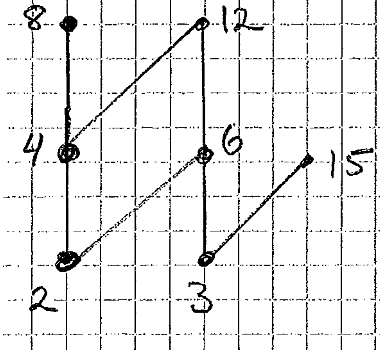
c) $G(x) = \left(x + x^2 + x^3 + x^4\right)^{10} =$

$$x^{10}\frac{(1-x^4)^{10}}{(1-x)^{10}}$$

Discrete Mathematics

27/5 2014

1. a) For example



b) $X = \{1, 3, 5\}$

$[X] = \{ \{1,3\}, \{1,3,4\}, \{1,3,5\}, \{1,3,4,5\} \}$

The power set $P(A)$ has $2^5 = 32$ elements. Each equivalence class has 4 elements. To X we can add no element, 4, 5 and 4 and 5.

There are 8 equivalence classes. $[\emptyset]$, $[\{1\}]$, $[\{2\}]$, $[\{3\}]$, $[\{1,2\}]$, $[\{1,3\}]$, $[\{2,3\}]$, $[\{1,2,3\}]$.

$8 \cdot 4 = 32$.

---

2. $$X_1 + X_2 + X_3 + \ldots + X_9 + X_{10} = 25$$

a) $\binom{34}{25}$. 25 sticks and 9 plus signs.

2014-2

Linnaeus University
Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA162, 7,5 hp
Monday 9th of June 2014, Time 08.00-13.00.

Note: To obtain maximal points a complete solution, presented in such a way that calculations and reasoning are easy to follow, is demanded. If nothing else is given you can assume that all graphs are undirected, loopfree and not multigraphs.

Aid: Sheet with formulas and concepts.

1. You have to buy 20 bagels. They are of four types: plain, onion, tomato and blueberry. The restrictions are: At least two of each sort. Not more than five with onions.
   In how many ways can you do this? Solve the problem with inclusion and exclusion or by using generating function. (6p)

2. a) Show that the divides relation is a partial order on the positive integers. The divides relation, $\mathcal{R}$, is defined by: $x\mathcal{R}y$ if and only if $x|y$. Here x and y are positive integers. (3p)
   b) Define the relation $\mathcal{R}$ on the the integers $\mathbb{Z}$ by $x\mathcal{R}y$, for $x, y \in \mathbb{Z}$, if and only if $x \equiv y \pmod 7$. Show that this is an equivalence relation on $\mathbb{Z}$. (3p)

3. Let G be a planar graph with eight regions which is 3-regular, i.e. all vertices have degree 3. Determine the number of vertices and edges. Draw two such graphs that are nonisomorphic. (6p)

4. What is $\chi(T)$ and $\chi(C_n)$? That is, what is the chromatic number for a tree T and for cycles of length n? (3p)

5. The *girth* of a graph G is the length, i.e. the number of edges, of the *shortest* cycle it is possible to find in G and it is denoted $g(G)$. For graphs with n vertices and $n+1$ edges, $n \geq 4$, it is possible to show that

$$g(G) \leq \left\lfloor \frac{2(n+1)}{3} \right\rfloor$$

   Draw a graph with *maximal* girth for $n = 5$. The floor function , $\lfloor x \rfloor$, gives the largest integer smaller or equal to x. (3p)

6. The *odd graph* O_k, k is an integer ≥ 2, is defined in the following way:
   The vertices represent the subsets with $k-1$ elements that can be obtained from a set with $2k-1$ elements. Two vertices are joined with an edge if and only if the corresponding subsets are disjoint.
   a) Draw O_2 and O_3 (2p)
   b) Show that O_k is k-regular for all $k \geq 2$, that is, show that all vertices in O_k has degree k. (4p)

Lycka till! *Good Luck*! *Bonne Chance*! *Buena Suerte*! *Succes Gewenst*!

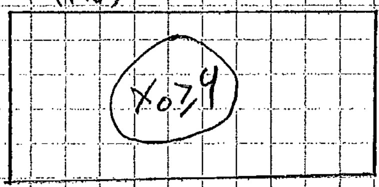
Discrete Math 9/6 2014

1) $$X_p + X_o + X_t + X_b = 12$$

$$0 \leq X_p,\ 0 \leq X_o \leq 3,\ 0 \leq X_t,\ 0 \leq X_b$$

Note: I removed directly the $2 \cdot 4 = 8$ bagels I have to take.

$\binom{15}{12}$ solutions



$\binom{15}{12} - \binom{11}{8}$ possibilities with not more than five onion bagels.

---

or with generating function

$$G(x) = \left(x^2 + x^3 + x^4 + \cdots\right)^3 \left(x^2 + x^3 + x^4 + x^5\right)$$

$$= x^8 \cdot \left(\frac{1}{1-x}\right)^3 \frac{1-x^4}{(1-x)}$$

$$= \frac{x^8 - x^{12}}{(1-x)^4}$$

We seek the coefficient in front of x^{20}.

It is $1 \cdot \binom{15}{12} - \binom{11}{8}$

2 a) aRa since $a|a$. Reflexive ✓

aRb and bRa means $b = ka$ and $a = l \cdot b$, k and l are integers, so
$b = ka = k \cdot l \cdot b$,
$k \cdot l = 1 \Rightarrow k = l = 1$ Anti-symmetric ✓

aRb and bRc
$c = k \cdot b = k \cdot l a \Rightarrow$
aRc. Transitive ✓

b) $a \equiv a \pmod 7$ Reflexive ✓

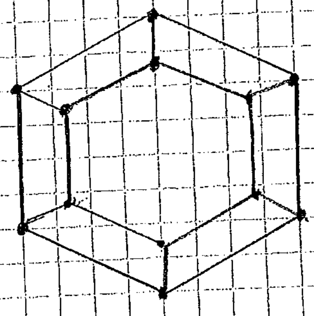
$a \equiv b \Rightarrow b - a = k \cdot 7$.
$a - b = (-k) \cdot 7$ so
$b \equiv a \pmod 7$ Symmetric ✓

$a - b = k \cdot 7$, $b - c = l \cdot 7$
$a - c = (k+l) \cdot 7$ Transitive ✓
$a \equiv c \mod 7$

---

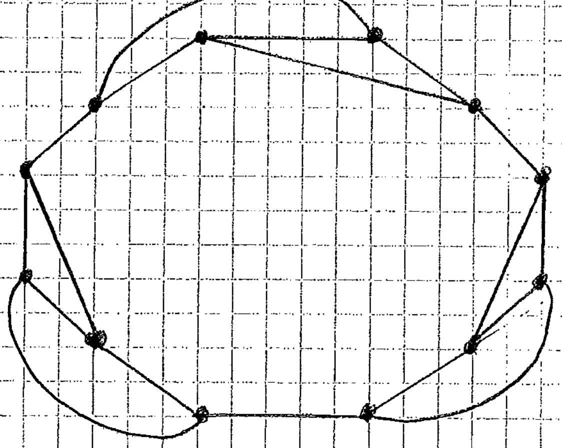
3) $r = 8$
$2e = 3v$
$r = e - v + 2$
$8 = \frac{3v}{2} - v + 2$
$v = 12$
$e = 18$

G_1



2

G_2



G_1 has no 3-cycles, G_2 has six of them.

---

4) $\chi(T) = 2$, Trees are bipartite

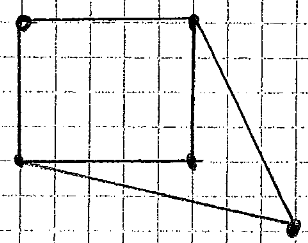
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

---

5) $n = 5$, $g(G) \leq \left\lfloor \frac{12}{3} \right\rfloor = 4$

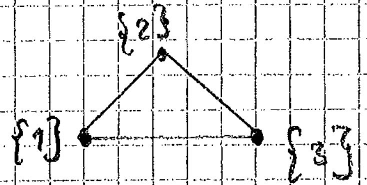
$v = 5$

$e = 6$. The shortest cycle must be of length 4 if the girth is maximal.



6) a) $k = 2$, $\{1, 2, 3\}$

O_2

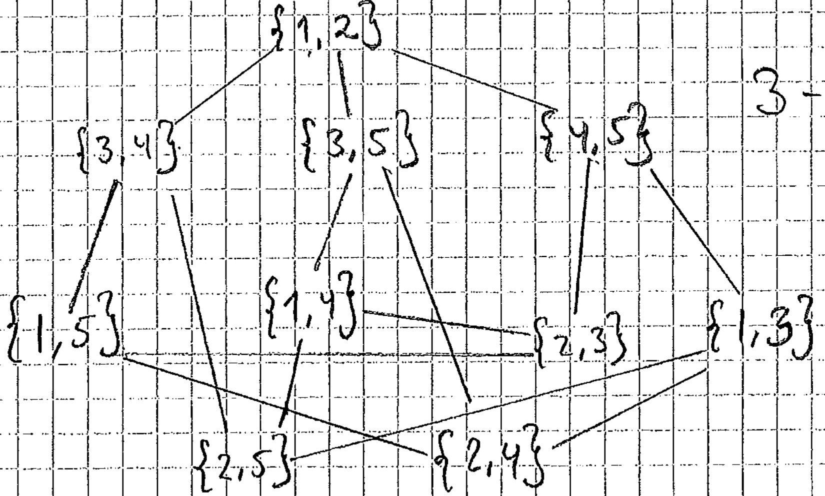


2-regular

$k = 3$, $\{1, 2, 3, 4, 5\}$

$\binom{5}{2} = 10$ vertices

O_3



3-regular

b) If you pick $k-1$ elements from $2k-1$ elements you are left with

$$(2k-1) - (k-1) = k \text{ elements}$$

How many subsets with $k-1$ elements can be formed from these k elements? Answer: $\binom{k}{k-1} = k$

So O_k is k-regular.

One edge to each of the k subsets (vertices).

4/7

2014 - 3

Linnaeus University
Mathematics
Hans Frisk

Exam in Discrete Mathematics, 1MA162, 7,5 hp
Saturday, 30th of August 2014, 09.00-14.00.

Note: To obtain maximal points a complete solution, presented in such a way that calculations and reasoning are easy to follow, is demanded. If nothing else is given you can assume that all graphs are connected, undirected, loopfree and not multigraphs.

Aid: Sheet with formulas and concepts.

1. In how many ways can you distribute 24 cakes to 6 children in such a way that each child gets at least 2 cakes but no one gets more than 6 cakes?
   a) Solve the problem with inclusion-exclusion. (3p)
   b) Solve the problem with generating function. (3p)

2. a) Determine whether the relation, R, represented by the following zero-one matrix is a *total order*. A total order is a partial order for which for every pair of elements i and j either iRj or jRi. In a zero-one matrix the matrix element $M_{ij} = 1$ if iRj, $M_{ij} = 0$ otherwise.

$$\mathrm{M}_R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3p)

   b) A set A consists of 21 elements, $|A|$=21. An equivalence relation R on A partitions A into three disjoint equivalence classes A_1, A_2 and A_3, where $|A_1|$=$|A_2|$=$|A_3|$. Determine the number of elements, $|R|$, of the relation. (3p)

3. Consider the two upper graphs on next page. Are they isomorphic to each other? Calculate also the *diameter* for each of them. This concept is defined in the following way:
   Let $G = G(V, E)$ be a connected graph. The distance between two vertices v and w in G is the the number of edges for the shortest path from v to w. We denote it $\text{dist}(v, w)$. The diameter for G, $d(G)$, is then defined in the following way

$$d(G) = \max\{\text{dist}(v, w)|v, w \in V\}.$$

(6p)

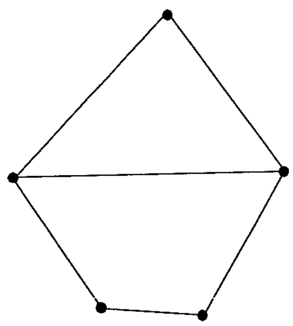
4. Let G be a planar graph with seven regions which is 3-regular, i.e. all vertices have degree 3. Determine the number of vertices and edges. Draw such a graph. (6p)

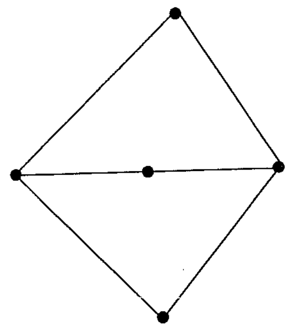
5. Consider the cube Q_3 on next page (lower graph). Is there an Euler circuit? Find a Hamilton cycle. A sufficient condition for the existence of a Hamilton cycle is given by Ore's theorem:

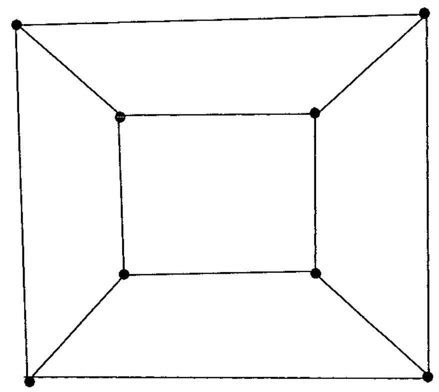
   Let G be a simple graph with $n \geq 3$ vertices. If for every pair of nonadjacent vertices u and v of G, $\deg(u) + \deg(v) \geq n$, then G has a Hamilton cycle.

   Gives Ore's theorem a necessary condition for the existence of a Hamilton cycle? Finally, try to find a cubic graph, that is all vertices have degree 3, without a Hamilton cycle. (6p)

6. a) Prove that every tree is a bipartite graph. (3p)
   b) Prove that every tree with at least two vertices contains at least two vertices of degree 1. (3p)







Discrete Math 30/8 2014.

1a) $X_i = 2 + Y_i \ , \quad i = 1, 2 \dots 6$

Then $Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 = 12$

$$Y_i \leq 4.$$

No restrictions: $\binom{17}{12}$

$Y_1 \geq 5$: $\binom{12}{7}$ same for $i = 2, \dots 6$

$Y_1 \geq 5$ and $Y_2 \geq 5$: $\binom{7}{2}$. same for all other pairs.

Put all together we get

$$\binom{17}{12} - \binom{6}{1} \cdot \binom{12}{7} + \binom{6}{2}\binom{7}{2}.$$

b) $$G(x) = \left(x^2 + x^3 + x^4 + x^5 + x^6\right)^6$$

$$= x^{12} \frac{(1-x^5)^6}{(1-x)^6} = x^{12} - 6x^{17} + 15x^{22} \dots$$

$$\cdot \left(1 + 6x + \binom{7}{2}x^2 + \dots + \binom{12}{7}x^7 + \dots \binom{17}{12}x^{12} + \dots\right)$$

1

Using formula for geometric sum and expansion of $\frac{1}{(1-x)^6}$.

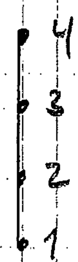
Coefficient in front of x^{24} becomes

$$\binom{17}{12} - 6 \cdot \binom{12}{7} + 15 \cdot \binom{7}{2}.$$

(Turns out to be 1751 possibilities.)

2a) yes it is a total order.

For them the Hasse diagram is just a straight line



b)

$$M_R = \begin{array}{c} A_3 \\ A_2 \\ A_1 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \end{array}$$

21 x 21 matrix consists of
6 blocks with zero-matrices and
3 blocks with 1-matrices.

In total the matrix contains
$3 \cdot 7^2 = 147$ ones.

$|R| = 147.$

---

3) No they are not isomorphic. There is a 3-cycle in left graph but not in right one.

$d(G) = 2$ for both graphs.

---

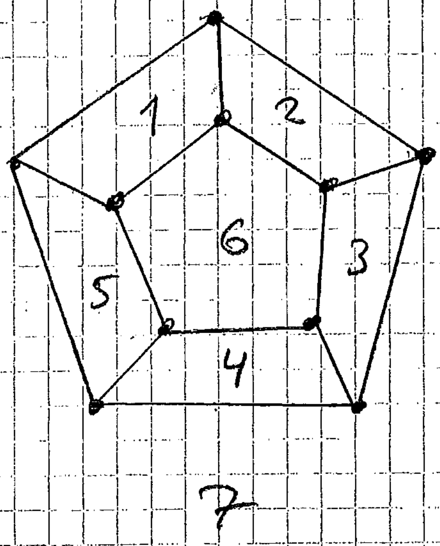
4)
$$V - e + r = 2$$
$$r = 7$$
$$2e = \sum_{r \in V} \deg(v) = 3V$$
$e = \frac{3V}{2}$. Put this into Euler formula

$$V - \frac{3V}{2} + 7 = 2$$
$$V = 10$$

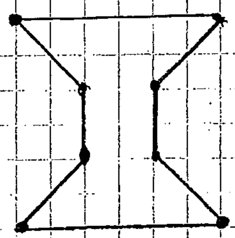
so $e = 15$

3

4 cont)



5) No EC since all vertices have odd degree.

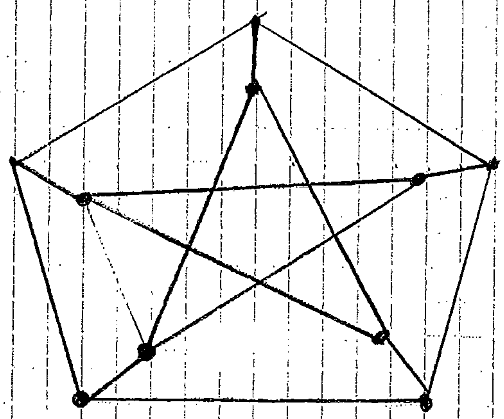


Is a HC.

No Ore's theorem is not necessary

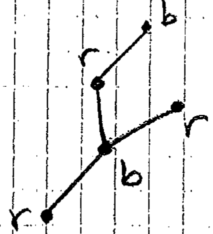
$3 + 3 < 8$ but it was anyhow easy to find a HC. $2e = 3v$ so V must be even. For $v = 6$ and 8 we can find HC. Let's try $v = 10$

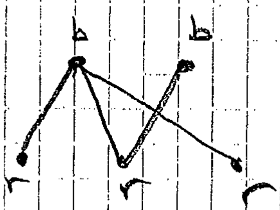
Then $e=15$. We can change the inner pentagon on page 4 to a pentagram



This is Petersens graph and it has not a HC.

6) a) Trees have no cycles. For every new vertex drawn we can use a used color when doing a vertex coloring. That means $\chi(T) = 2 \implies$ it is bipartite.





6b) $V = e + 1$ for trees. (1)

$$2e = \sum_{v \in V} \deg(v) \quad \text{always.}$$

i) $\deg(v) \geq 2, \ \forall v \in V$

Then $2e \geq 2v$

$\iff$

$v \leq e$. Contradiction with (1)

ii) $\deg(v) \geq 2$ except for one degree 1 vertex. Then

$$2e \geq (v-1)2 + 1 = 2v - 1$$

$$v \leq e + \tfrac{1}{2}$$

also contradiction with (1).

iii) $2e = 2(v-2) + 2 \iff e = v-1$

Two degree one vertices and the rest degree 2.

$$2e = 2(v-4) + 3 + 3 = 2v - 2$$

3 degre 1 $\to$ 1 degree 3 vertex

etc.

6