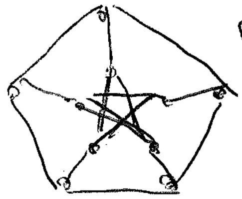
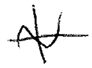
Discrete Math

counting
relations
graph theory →



pentagram inside pentagon

↓
not graphs
like



group project: 1.) mathematical & Latex
2.) Four programs
lecture 4/4
3.) Project problems
Presentations 17/5

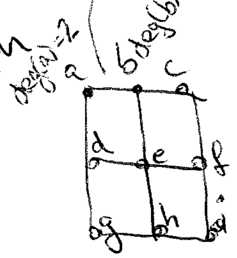
1. Recurrence
2. Relations

Game of Life

3. Cellular Automata
4.

→ Network

graph



vertex
$V = \{a, b, c, \dots i\}$

find all the pairs
$E = \{(a,b), (a,d) \dots (h,i)\}$

vertices with degree ~~2~~ 4 → which are attached to it

2: a, c, g, i $\quad \deg(a) = 2$
3: b, d, f, h
4: e

Handshake theorem: $2e = \sum_{v \in V} \deg(v) = 4 \cdot 2 + 4 \cdot 3 + 4 = 24$

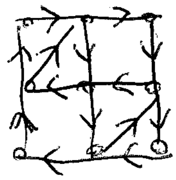
$e = \#$ edges $\qquad e = 12$
we have 12 edges

vertices with large degrees are nowadays called

hub

The postman wants to go over an edge exactly once!

Euler circuit



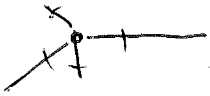
sufficient and necessary conditions

Theorem: The graph has an Euler circuit

proof by induction ⇐ sufficient condition ⇒ ⇐

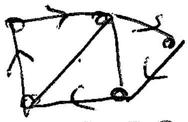
all degrees of the vertices are even numbers

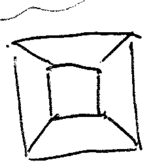
Proof:



→ we come in and we go out

Travelling salesman wants to visit each town exactly once



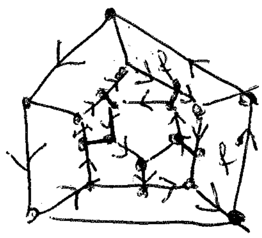


$f = 6$

$e = 12$

$V = 8$

$v + f - e = 2$



$V = 20$ → vertices → •

$e = 30$ → edges

$f = 12$ → faces

$$\boxed{e - v + f = 2}$$

He wants to make a

Hamilton cycle

Sufficient condition for a Hamilton cycle

$\deg(u) + \deg(v) \geq V$ where u and v are two non-adjacent vertices

Necessary condition

→ 6 pentagons inside

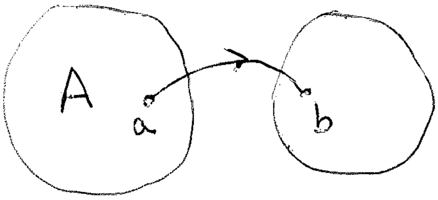
6 pentagons outside

Discrete Math

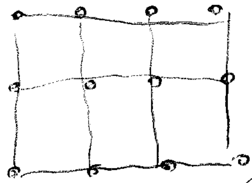
28.3.2017

1

Bijection – $\begin{cases} \text{one-to-one} \\ \text{on to} \end{cases}$



Graph



$v = 12$

vertices in G

Edges in HC

Ex.) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ → is this function on to?

$f(m,n) = m^2 - n^2 = (m-n)(m+n)$

Can you find pairs (m,n) such that $f(m,n) = 0$ | $f(2,2) = 0$

$f(m,n) = 1$ | $f(1,0) = 1$

$f(m,n) = 2$

$2 \cdot 1 = (m+n)(m-n)$ $\quad m+n = 2$, $m-n = 1$ $\quad 2m = 3$ $\quad \cancel{m = \tfrac{3}{2}}$

If there is a HC in a graph with v vertices then the HC has 66... edges

$v = 4$

Answer: we can't ~~get~~ 2 with such a function!

splitting up into primes: $60 = 2^2 \cdot 15 = 2^2 \cdot 3 \cdot 5$

Arithmetic sum: $S_n = 1+2+3+4+\dots+n = \boxed{\dfrac{n \cdot (n+1)}{2}}$ formula

$S_1 = 1$

$S_2 = 1+2 = 3$

$S_3 = 1+2+3 = 6$

Proof 1: $S_n = 1+2+\dots+n$

$S_n = n + (n-1) + \dots + 1$

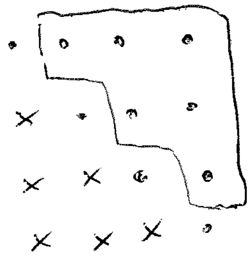
Add them $2S_n = (n+1) + (n+1) + \dots (n+1)$

$= n \cdot (n+1)$

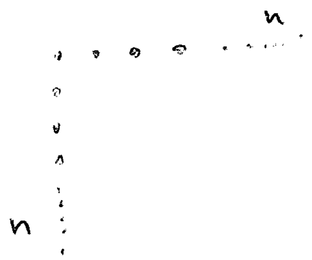
$$\boxed{S_n = \frac{n \cdot (n+1)}{2}}$$

Proof 2:

$n = 4$



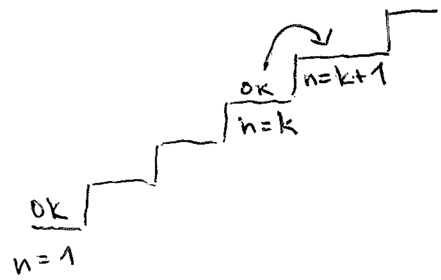
$$S_4 = \underbrace{\frac{(4^2-4)}{2}}_{\text{\# points above diagonal}} + \underbrace{4}_{\text{on the diagonal}} = \frac{4^2+4}{2} = \frac{4(4+1)}{2}$$



$$S_n = \frac{(n^2-n)}{2} + n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

Proof 3: by Induction

> well-ordering principle
> Every subset of the positive integers has a smallest element



$$S_n = \frac{n(n+1)}{2} = 1+2+\dots+n$$

$$S_1 = 1 = \frac{1\cdot(1+1)}{2} = 1 \quad \text{OK}$$

$$S_2 = 1+2 = 3 = \frac{2\cdot(2+1)}{2} = 3 \quad \text{OK}$$

Assume $S_k = \dfrac{k(k+1)}{2}$ Induction Assumption

$$S_{k+1} = \underbrace{1+2+3+\dots+k}_{\frac{k(k+1)}{2}} + (k+1) \stackrel{!}{=} \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\left[\frac{k}{2}+1\right]$$

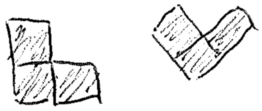
$$= \frac{(k+1)(k+2)}{2} \quad \text{Yes!}$$

$$= \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2} \leftarrow \text{dream result}$$

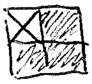
28. 3. 2017

3

Ex.)

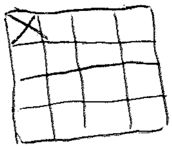


exclude →



2×2

exclude →



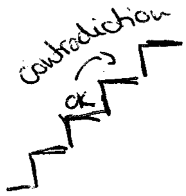
4×4

8×8

16×16

⋮

Prove by Induction, that it is always possible to tile $2^m \times 2^m$ boards with L-Shaped figures.



Big-oh notation

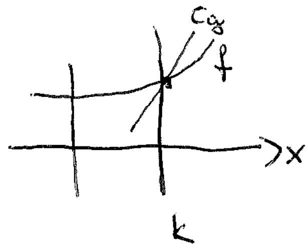
$$\frac{n^2}{2} < S_n = \frac{n^2+n}{2} < n^2$$

So S_n is $O(n^2)$

other function

f is $O(g)$ if $f \leq C g$ if $x > k$

2 constants



$$f(n) = \frac{n(n+1)}{2}$$

$g(n) = n^2$

$C = 1$

$k = 1$

For large n:

$n! > 2^n > n^2 > n \log n > \log n > 1$

Bit

A bit operation takes 10^{-9} s. 2^{100} bit operations takes ≈ 4000 years

Multiplication

$(1101)_2 = 13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$

$(1010)_2 = 10$

$13 \cdot 10 = 130$

T(n) = the time for multiplication of two n digit numbers.

T(n) is O(?)

n^2

```
    1101
    1010
   -----
    0000
   1101
  0000
 1101
--------
10000010
```

$10000010 = 2^7 + 2^1 = 128 + 2 = 130$

T(n) is $O(n^2)$

$1101 = 2^2 \cdot \underbrace{11}_{A_1} + \underbrace{01}_{A_0}$ $\qquad A_1 = (11)_2 = 2^1 + 2^0 = (3)_{10}$ $\qquad$ 28.3.2017

4

$T(2n) = 3T(n) + C_1 \cdot n$

$\vdots$

$T(n) = O(n^{\log_2 3 = 1.6})$

$1010 = 2^2 \cdot 10 + 10$

$$(2^2 A_1 + A_0) \cdot (2^2 B_1 + B_0) = (2^4 A_1 B_1 + 2^2 A_1 B_1) + 2^2 ((A_1 - A_0)(B_0 - B_1)) + (2^2 + 1) A_0 B_0$$

Fermat's little theorem

$a^p \equiv a \pmod{p}$ if p is a prime
$p \nmid a$

$35 \equiv 25 \pmod{5}$

since $\frac{35-25}{5} = 2$

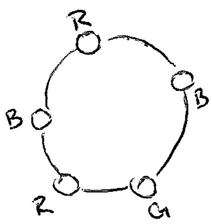
Is $36 \equiv 20 \pmod{17}$?

No!

$a = 3$
$p = 5$

$3^5 \equiv 3 \mod 5$

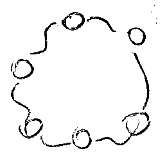
$\frac{3^5 - 3}{5} = \frac{243 - 3}{5} = 48$



5 bricks $\quad$ red blue green

3 colors $\quad$ R, B, G

Total number of colorings $= 3^5$



does it work if it is not with a prime number?

$6 = 3 \cdot 2$

think of it!

Exclude the 3 uni-colored necklaces

Left is $3^5 - 3$ necklaces

There are only $\frac{3^5 - 3}{5}$ unique necklaces $\rightarrow$ because rotated would'nt be unique

$\downarrow$

48

30.3.2017 ①

$$\boxed{S} \xrightarrow{901} \boxed{R} \quad ABC$$

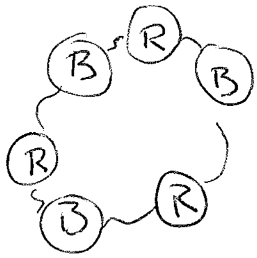
$ABC \to 123$

$a^p \equiv a \pmod p$

$$\frac{3^5 - 3}{5} = 48$$

$$\frac{5^3 - 5}{3} = 40$$

$$\frac{2^6 - 2}{6} = \frac{62}{6}$$

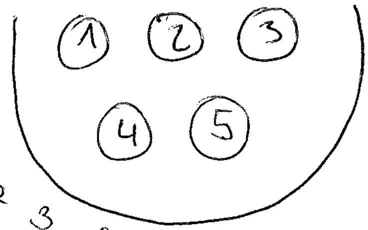


In how many ways ....?

Combinatorics

* Product rule
* Sum rule

Balls in a bowl



Take 3 of them

with respect to order?

Allow for repetitions? → putting the ball back in

Allow for repetitions?	Yes	No
Yes	5 51 5^3 using the product rule	New! $\binom{7}{3}$
No	542, 245 one outcome, another outcome possibilities $5 \cdot 4 \cdot 3 = \frac{5!}{2!} = P(5,3)$	$\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = \frac{5!}{2!\,3!} = \binom{5}{3}$ $= C(5,3) =$ 10 [redacted] $= \binom{5}{2}$

Here we are only interested of how many of each sort

two fives & one three

553 ~~355~~

So our problem is really about the number of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3$$

$x_i \in \mathbb{N}$

$0 + 0 + 1 + 0 + 2 = 3$

$x_1 = \#$ of 1s. (Number of Balls with ① on it)

$\vdots$

$x_5 = \#$ of 5s.

Back to the stone age! ~~||||~~ ||||||

$$\underbrace{+ + | + + ||}_{7 \text{ positions}}$$

$1 + 1 + 0 + 1 + 0$

$\Updownarrow$

$| + | + + | + \Rightarrow 7$ pos

$0 + 0 + 3 + 0 + 0 \Leftrightarrow + + ||| + + \Rightarrow 7$p

30.3.2017 (2)

In how many ways can we put 3 sticks on 7 positions?

_ _ _ _ _ _ _

$$\frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = \binom{7}{3}$$

---

13 balls, 6 repetitions
we don't care about order.
In how many ways? (Sticks, ...)

$$\binom{12+6}{6} = \binom{18}{6} = \binom{18}{12}$$

Ex: 6 people shake hands with each other

\# How many handshakes?

$n = 6$
$r = 2$

$\binom{6}{2} = 15$

6 people

1 2 3
4 5 6

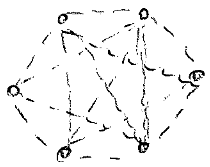
we take out 2

combinatorics → no repetition
no respect to order

$$\frac{6!}{2!\,4!} = \frac{6 \cdot 5}{2} = 15$$

$= 5+4+3+2+1$

Ex: K_6



coloring of the edges with red or blue.
How many colorings?

$n = 2$
$r = 15$

(R B)

2^{15}

Sum rule

How many bit strings of length 8 have at least 3 zeros?

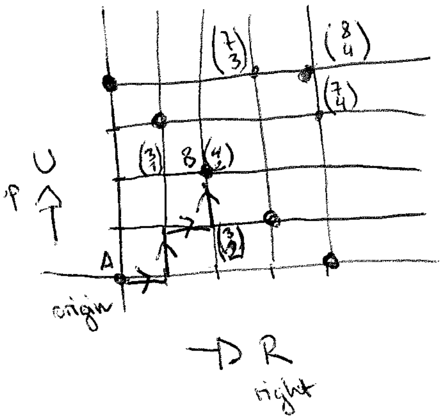
00111110
01111111

"universe" in total we have 2^8 bit strings

0 zero	At least 3 zeroes
1 zero	
2 zeros	

$$N(\overset{\text{three zeroes}}{\text{at least } 3\ 0s}) = 2^8 - \underbrace{N(\overset{\text{zero zeroes}}{0\ 0s})}_{1} - \underbrace{N(1 \text{ zero})}_{8 = \binom{8}{1}} - \underbrace{N(2 \text{ zeroes})}_{\binom{8}{2} = \frac{8 \cdot 7}{2} = 28}$$

$$= 256 - 1 - 8 - 28 = 256 - 37 = 219$$



ways to go from

A to B $= \binom{4}{2} = 6$

RURU (right, up, right, up) $\frac{4 \cdot 3}{2} = 6$

_ _ _ _

_ _ _ _

_ _ _ _

come from low come from left

$\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$ $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$\binom{8}{4} = \binom{7}{3} + \binom{7}{4}$

4 ways

$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 = 16$

Box-principle

P

$10 \cdot 10 \cdot 10 = 10^3$

KAF 077

← Parking lot

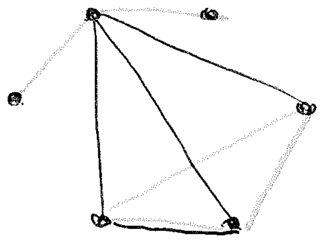
1001 swedish crowns

At least two cars have the same number combination!

Understand this!

Ramsey theory

$5 = 4+1 = 3+2 = 2+3$



Edge coloring of K_6 with 2 colors, green and black

A green or a black triangle is unavoidable

in a group of 6 people you can find 3 who are complete strangers ← $R(3,3) = 6$

$R(4,4) = 18$

$43 \leq R(5,5) \leq 49$ → impossible

$102 \leq R(6,6) \leq 165$ → to compute

Recurrence Relations

RR + CA

I + moves!

Geometric Series

$$S = 1 + k + k^2 + k^3 = \frac{k^4 - 1}{k - 1}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3)$$

$\vdots$

(i) Linear RR

external influence

memory	NO homogenous	YES non-homogenous
1 first-order	$a_0 = 100 IV$ $a_{n+1} = 1.03 a_n$ $a_n = 100 \cdot 1.03^n$ $n = 0, 1, 2, 3, \dots$	$a_{n+1} = 1.03 a_n + 100$
2	$a_n = a_{n-1} + 6a_{n-2}$ $a_0 = 2$ $a_1 = 5$ $a_2 = 17$	$a_n = a_{n-1} + a_{n-2} + 2^n$

How to solve ~~it~~ the RR?

1.) R Solve
2.) Find a pattern. Prove your formula by induction
3.) Matrices
4.) Two solutions is also a solution! Add them and fix the constants by initial values.

Ex.) $a_{n+1} = 2a_n + 2^n$

$a_0 = 1$

Use Method (2) → find pattern

Ex.) $a_{n+1} = 3a_n + 2^n$

$a_0 = 1$

$a_1 = 3 \cdot 1 + 2^0 = 4$

$a_2 = 3 \cdot 4 + 2^1 = 14$

$a_2 = 3(3 \cdot 1 + 2^0) + 2^1$

$= 3^2 + 3 \cdot 2^0 + 2^1 = 14$

$a_3 = 3(3^2 + 3 \cdot 2^0 + 2^1) + 2^2$

$= 3^3 + 3^2 \cdot 2^0 + 3^1 \cdot 2^1 + 2^2 \Rightarrow$

$3^3 + \frac{3^3 - 2^3}{3 - 2} = 2 \cdot 3^3 - 2^3 =$

$54 - 8 = \underline{\underline{46}}$

Guess:

$a_n = 2 \cdot 3^n - 2^n$!!

4 i) Fix the particular solution

$a_n = c \cdot 2^n$

plug it in

$c \cdot 2^{n+1} = 3c \cdot 2^n + 2^n$

$(2c - 3c)2^n = 2^n$

$c = -1$ $\qquad a_n = -2^n$

ii) $a_{n+1} = 3a_n$ $\qquad a_n = c_1 3^n$

iii) put them together

$a_n = c_1 3^n - 2^n$

Determine C_1 by IV:

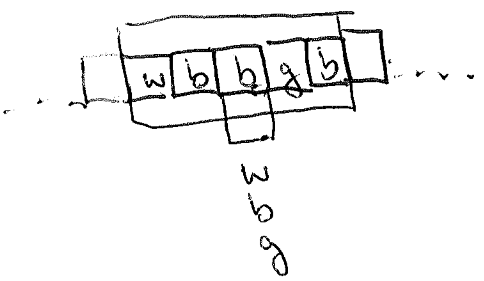
$a_0 = 1 = c_1 3^0 - 2^0$

$\Leftrightarrow$

$1 = c_1 - 1$

$c_1 = 2$

The Feigenbaum Constant - YouTube Numberphile



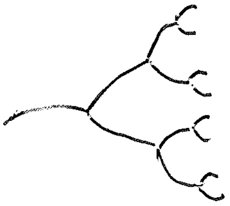
$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$

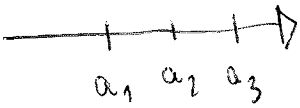
3^{243} -> so many rules

Game of life!
YouTube Numberphile

6.4.2017

(1)

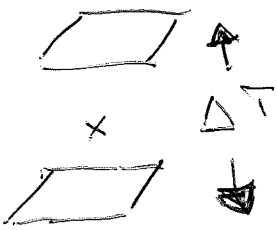


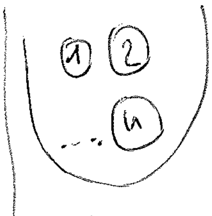


$$\frac{a_3 - a_2}{a_2 - a_1} \approx 4.66\ldots$$

for all humps!

Turbulence





pick r

care about order?

Put them back	Y	N
Y	n^r	$\binom{n+r-1}{r}$
N	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$

Sum Rule

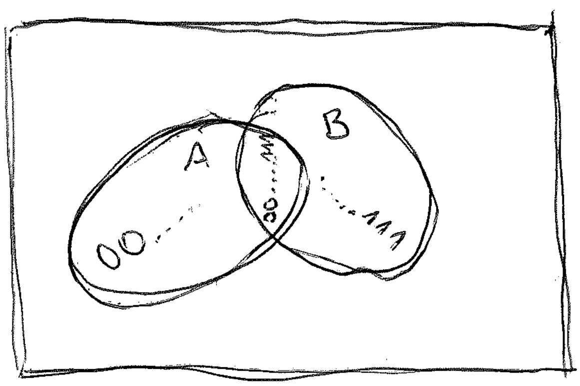
Product Rule

Today Inclusion - Exclusion + vertex colorings of graphs

\# bit strings of length 10 that start with 00......

or end with ......111 = ?

All bit strings of length $10 = 2^{10}$



$|A \cup B| =$

$|A| + |B| - |A \cap B|$

$= 2^8 + 2^7 - 2^5 =$

$2^5(8+4-1)$

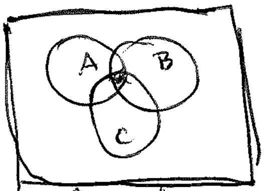
$= 11 \cdot 32$

$= 352$

6.4.2017

2

3-sets



$$|A\cup B\cup C| = |A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

consider an element in $A\cap B\cap C$:

$1+1+1-1-1-1$

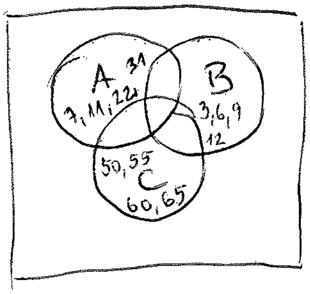
8.5.19 Pick 4 numbers from one to 100

A - all 4 ~~numbers~~ numbers odd

B - all 4 multiplies of 3

C - -||- 5

$|A\cup B\cup C| = ?$



Beware! Maybe wrong figure!!!

$15, 30, 45, 60, 75, 90$

only 3

wrong figure because we can't pick 4

odd number = 50

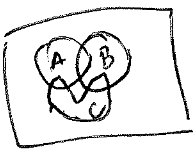
multiples of 3 = 33

multiples of 5 = 20

$$|A\cup B\cup C| = \overbrace{\frac{50\cdot49\cdot48\cdot47}{1\cdot2\cdot3\cdot4}}^{\binom{50}{4}} + \binom{33}{4} + \binom{20}{4} - \binom{17}{4} - \binom{10}{4}$$

$$-\binom{6}{4} + 0$$

correct figure



onto functions

$\{a,b,c,d,e\} \to \{1,2,3\}$

put them into containers

functions $= 3^5$

But here all boxes are not empty!

First without Inclusion - Exclusion

sum: $3+1+1$ | abe | c | d | How many of this $\binom{5}{3}\cdot3\cdot2$

$2+2+1$ | b | dc | ae | How many of this $+\dfrac{\binom{5}{2}\cdot\binom{3}{2}\cdot3!}{2} = 60 + \dfrac{10\cdot3\cdot6}{2}$

$= 60+90 = 150$

with I-E

$3^5 = 243$

here are the onto functions

1 - the set where container 1 is empty

2 - container 2 empty

3 - -||- 3 empty

→ containers continued

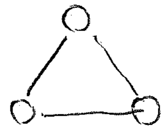
6.4.2017

3

$$3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 243 - 96 + 3$$
$$= 246 - 96 = 150$$

We want to keep things apart

⇒ vertex-coloring of graphs



with 2 colors we can do 2^3 colorings of the Δ

3 colors 3^3

x colors x^3

Proper colorings (PC): Vertices with an edge in common must have different colors

The smallest number of colors needed for a PC is called the chromatic number χ

#PC of Δ using 2 colors = 0

3 colors = $3 \cdot 2 \cdot 1 = 6$

x $= x \cdot (x-1)(x-2) = P_\Delta(x)$

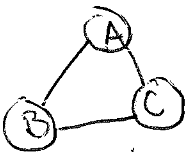
chromatic polynomial

$P(5) = 5 \cdot 4 \cdot 3 = 60$

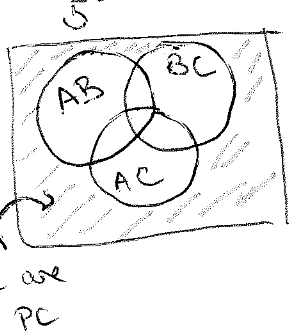
ways of coloring

by I-E

Δ and 5 colors



5^3



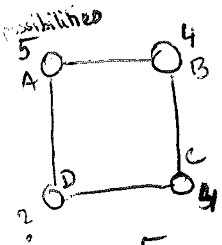
here are the PC

AB - vertex A and B have the same color

intersections all unicolored Δ

$$5^3 - 3 \cdot 5^2 + 3 \cdot 5^1 - 5 = 60$$
$$= 125 - 75 + 15 - 5 = 60$$

so in set AB vertex A and B have same color



→ because we can reuse

$\#$PC with same color on A and C = $\overset{A}{5}\cdot\overset{B}{4}\cdot\overset{D}{4} = 80$

$\#$PC with different colors on A and C $= 5\cdot 4\cdot 3^2 = 180$

$$\frac{180}{260}$$

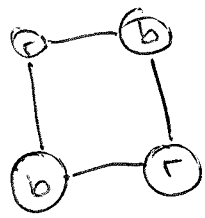
$x = 5$

colors

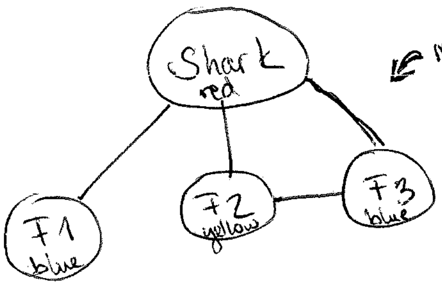
How many PCs?

$P_{\square}(5) = 260$

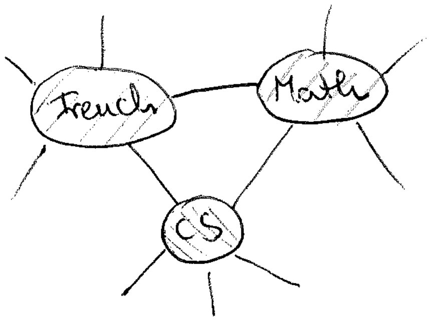
$\chi_{\square} = 2$ ← minimum number needed to make a proper coloring



Do it with I-E!



← must be apart! not together in aquarium → $\chi_G = 3$



χ_G gives the minimum number of exam days

Generating Functions (GF) an alternative to I-E.

11.4.2017 | 1

$\boxed{x\text{ - very small}}$

$S_3 = 1 + x + x^2 + x^3$ ← polynomial with coefficient 1

$x \cdot S_3 = x + x^2 + x^3 + x^4$

we don't care about x, only about the number in front of it

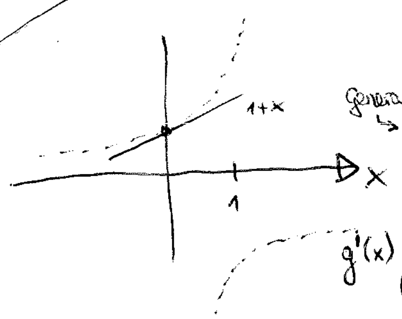
$S_3 - x \cdot S_3 = 1 + x - x + x^2 - x^2 + x^3 - x^3 - x^4 = 1 - x^4$

$S_3 = 1 - x$

$S_3 = \dfrac{1 - x^4}{1 - x}$ ← $x \neq 1$

$S_n = \dfrac{1 - x^{n+1}}{1 - x}$ if $n \to \infty$ (goes to infinity) $S_\infty = \dfrac{1}{1-x}$

$= 1 + x + x^2 + x^3 + \dots$ "polynomial of infinite degree"



Generating function

$g(x) = 1 + x + x^2 + \dots = \dfrac{1}{1-x}$

$g'(x)\ \dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

$g''(x) = \dfrac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \dots$

$\dfrac{1}{(1-x)^3} = 1 + 3x + 6x^2 + \dots$

Ex: $x = \frac{1}{2}$

$\dfrac{1}{1 - \frac{1}{2}} = 2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{10} + \dots$

GF	Number sequence
$\dfrac{1-x^4}{1-x}$	$1,1,1,1,0,0,\dots$
$1/1-x$	$1,1,1,1,1,\dots$
$1/(1-x)^2$	$1,2,3,4,5,\dots$
$1/(1-x)^3$	$1,3,6,\dots,\binom{3+n-1}{n},\dots$

Question 1: Generating function for $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}, \dots 0 \dots 0 \dots$

Question 2: Number sequence for $\dfrac{1}{1-2x}$ is

Question 3: Generating function for $1,1,0,1,1,1,1 \dots$ is

1: $\binom{3}{0} + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3 = (1+x)^3$

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$$ Binomial theorem

2: $1, 2, 4, 8, 16, 32, \dots$ → is the answer! think about it

3: $\dfrac{1}{1-x} - x^2 = \dfrac{1 - x^2 + x^3}{1-x}$

reminder: power laws

$x^5 \cdot x^6 \cdot x^3 = x^{5+6+3} = x^{14}$

Ex: $x^4 \cdot x^6 \cdot x^7 = x^{4+6+7} = x^{17}$

Ex: $(*) \begin{cases} e_1+e_2+e_3=17 \\ e_i \geq 0 \quad i=1,2,3 \end{cases}$

In how many ways can we distribute 17 cakes to 3 kids.

This is the number of integer solutions to $(*)$ $\to 17+2=19$

Ans. $\binom{19}{17} = \binom{19}{2} = \frac{19 \cdot 18}{2} = 171$

$\frac{19!}{2!\,19-2!}$ cancel out the common factorial factors

The GF for this problem is:

$$G(x) = \underbrace{(1+x+x^2+x^4\ldots)}_{\text{child 1}} \underbrace{(1+x+x^2+x^6\ldots)}_{\text{child 2}} \underbrace{(1+x+x^2+x^7\ldots)}_{\text{child 3}}$$

What is the coefficient in front of x^{17}?

$$G(x) = \frac{1}{(1-x)^3} = 1+3x+6x^2+\ldots\ldots\binom{19}{17}x^{17}$$

$\binom{3+n-1}{n}$

$n=17$

$\binom{19}{17}$

Ex.) $e_1+e_2+e_3=17$

$2 \leq e_1 \leq 5$

$3 \leq e_2 \leq 6$ sum of maximum values: 18

$4 \leq e_3 \leq 7$

Q: In how many way can we distribute these 17 cakes with the restrictions above?

Ans.: $\underline{3}$

$$GF: G(x) = \underbrace{\overbrace{(x^2+x^3+x^4+x^5)}^{x^2(1+x+x^2+x^3)}}_{\text{child 1}} \underbrace{\overbrace{(x^3+x^4+x^5+x^6)}^{x^3(1+x+x^2+x^3)}}_{\text{child 2}} \underbrace{(x^4+x^5+x^6+x^7)}_{\text{child 3}} =$$

what is the coefficient in front of x^{17}?

$$= x^2 \cdot x^3 \cdot x^4 \cdot (1+x+x^2+x^3)^3 =$$

$$= x^9 \frac{(1-x^4)^3}{(1-x)^3} = x^9(1-3x^4+3x^8-x^{12}) \cdot$$

$$\left(1+3x+6x^2+\ldots\binom{n+2}{n}x^n+\ldots\right)$$

How can we get x^{17}?

i) $x^9 \cdot 1 \cdot \binom{10}{8}x^8$

ii) $x^9 \cdot (-3x^4) \cdot \binom{6}{4}x^4$

iii) $3 \cdot x^9 \cdot x^8$

In total: $\boxed{\binom{10}{8} - 3\binom{6}{4} + 3} = 45-45+3 = 3$

11.4.2017 3

Previous example

with I-E instead

$e_1 + e_2 + e_3 = 17$

$2 \le e_1 \le 5$

$3 \le e_2 \le 6$

$4 \le e_3 \le 7$

$e_1 = 2 + y_1$

$e_2 = 3 + y_2$

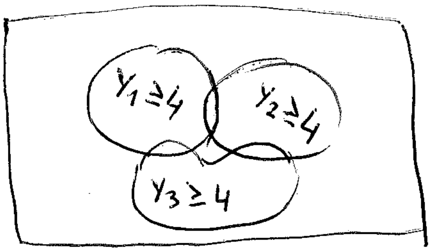
$e_3 = 4 + y_3$

$2 + y_1 + 3 + y_2 + 4 + y_3 = 17$

$0 \le y_i \le 3$

$$\boxed{\begin{array}{c} y_1 + y_2 + y_3 = 8 \\ 0 \le y_i \le 3 \end{array}}$$

All solutions without upper restrictions $= \binom{10}{2} = \binom{10}{8}$



I) $y_1 \ge 4$

$y_1 = 4 + z_1$

$4 + z_1 + y_2 + y_3 = 8$

$z_1 + y_2 + y_3 = 4$

$z_1 \ge 0$

$y_2 \ge 0$ Ans.: $\binom{6}{2} = \binom{6}{4}$

$y_3 \ge 0$

E: Throw 5 dice

⚀ ..... ⚅

$s = x_1 + x_2 + \dots x_5$

$5 \le s \le 30$

in how many ways can we get 17?

in Function + I-E !

Ex.) → not important! will not come to exam!!

$a_{n+1} = 2a_n + 2^n$

$a_0 = 1$

$$\boxed{a_n = (n+2) \cdot 2^{n-1}}$$

$a_0 = 1, \ a_1 = 3, \ a_2 = 8 \dots$

$$\sum_{n=0}^{\infty} a_{n+1} \cdot x^{n+1} = \sum_{n=0}^{\infty} 2a_n \cdot x^{n+1} + 2^n \cdot x^{n+1}$$

$$(G(x) - 1) = 2xG(x) + \frac{x}{1-2x}$$

$$G(x) \cdot (1-2x) = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x} \qquad G(x) = \frac{1-x}{(1-2x)^2}$$

18.4.2017

1

\# ways to get sum 17 throwing 5 dice =

$$\binom{16}{12} - 5\binom{10}{6} + \binom{5}{2}\cdot 1$$

Block chains "A new kind of trust"

10 likes ≥ colleague

70 likes ≥ friend

300 likes ≥ partner!

Cambridge Analytica

Relations (This week)

1.) Reflexive

2.) Symmetric

3.) Anti-symmetric

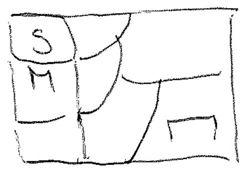
4.) Transitive

Posets: 1,3,4 $\geq$ $\supseteq$

Hasse diagram, lattice

Equivalence relation 1,2,4

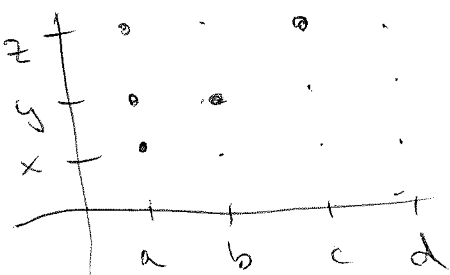
"vote on the same party"



A relation from A to B is a subset of $A\times B$

$A = \{a,b,c,d\}$ $\quad |A\times B| = |A|\cdot|B| = 4\cdot 3 = 12$

$B = \{x,y,z\}$



$R = \{(a,x),(a,y),(a,z),$
$(b,y),(c,z)\}$

On $A\times B$ we have 2^{12} relations

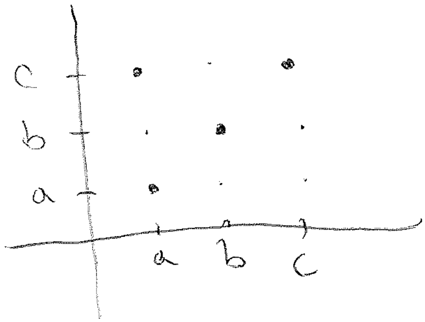
$\hookrightarrow$ 4096

A relation on A is a subset of $A \times A$ 2

1.) A relation is reflexive if $(a,a) \in R$ for every $a \in A$.

$A = \{a, b, c\}$

$R = \{(a,a), (b,b), (c,c), (a,c)\}$



2.) A relation R is symmetric if $(a,b) \in R \to (b,a) \in R$ for all $a, b \in A$

3.) Anti-symetric relation: $aRb \wedge bRa \to a=b$ (or $(a,b) \in R$)

4.) A relation is ~~transitive~~ if whenever $(a,b) \in R \overset{\text{and}}{\wedge} (b,c) \in R \to (a,c) \in R$

Ex 1.) A - webpages www....

$(a,b) \in R$ Relations	Reflexive?	Symmetric?	Anti-symm?	Transitive
$\Leftrightarrow$ Everyone who has visited webpage a has also visited webpage b	Y	N	N	Y
There are no common links on a and b	N	Y	N	N
$x \cdot y \geq 1$	No $0 \cdot 0 < 1$	Y	N	Y
$x \equiv y \pmod 7$				

none of them $= 0$

They have to have the same sign

$x \cdot y \geq 1$ Is then $x \cdot z \geq 1$?

$y \cdot z \geq 1$

$(x \cdot y)(y \cdot z) \geq 1$

$x z y^2 \geq 1$

≥ 1

$x \cdot z \geq 1$

A relation on $\to \mathbb{Z}$ $(a,b) \in R \Leftrightarrow$

$\frac{17-10}{7} = 1$

10 R 17   17 R 51

18.4.2017

3

$x \equiv y \pmod 7$

1.) Reflexive $x - x = 0 \cdot 7$ Yes!

2.) Symmetric $x - y = k \cdot 7 \quad k \in \mathbb{Z}$

$y - x = (-k) \cdot 7$ Yes

$y - x = l\,7$

$\frac{24-17}{7} = 1 \qquad 24 - 17 = 1 \cdot 7$

$17 - 24 = (-1) \cdot 7$

4.) $\overset{x}{77} - \overset{y}{49} = \overset{k}{4} \cdot 7 \quad (1)$

$\overset{y}{49} - \overset{z}{35} = \overset{l}{2} \cdot 7 \quad (2)$

$(1) + (2) \quad 77 - 35 = (4+2) \cdot 7 = 6 \cdot 7$

$x - z = (k + l) \cdot 7 \quad xRz$

Grand-parents,

Train trips with one stop

From R we can form the composite relation.

$R \cdot R = R^2$

composite Relation

$(a,b) \in R^2$ if there is a c such that $(a,c) \in R$ and $(c,b) \in R$

Ex.) R - parent relation

H parent to E, E is parent to S

HRE

ERS

$H\cancel{R}S$

This means HR^2S

grandparent

Theorem: R is transitive $\Leftrightarrow R^n \subseteq R$

$n = 1, 2, 3, 4, 5 \ldots$

Ex.) VR^2S

grand-grand-parent

$R^3 = R^2 \cdot R$

$\vdots$

Representing relations by

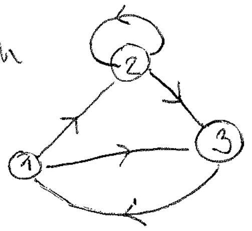
1.) 0-1 matrices

2.) Directed graphs (Digraphs)

Ex.)
$A = \{1,2,3\}$

$R = \{(1,2), (1,3), (2,2), (2,3), (3,1)\}$

Digraph



0-1 matrix
$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Is R reflexive? No — you can see it in diagonals

Is R symmetric? No

Is R anti-symm.? No

Is R transitiv? No

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Ex.) $A = \{1,2,3,8,12,21,24\}$

$aRb \overset{\text{if}}{\Leftrightarrow} a \mid b$ $\qquad$ $a \mid b$ means $b = k \cdot a$ for some integer k.

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 8 & 12 & 21 & 24 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 8 \\ 12 \\ 21 \\ 24 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

symmetric: yes $\quad$ 1 1 1 1 $\to$ diagonal

anti-sym: $b = k \cdot a$ I

$a = l \cdot b = k \cdot l \cdot a$

$k \cdot l = 1$

$k = l = 1$

20.4.2017

(1)

Next week: graph theory until 11/5

16/5, 18/5 old exams repetition

17/5 presentation Deadline 12/5

Ex.) $A = \{x, y, z\}$

A relation on a set A is a subset of $A \times A$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad yRz$$

the diagonal

on A we have $2^9 = 512$ relations

- How many of them are <u>reflexive</u>? $\rightarrow 2^6 = 64$

diagonal would have 1's everywhere on the diagonal

$$M_{R^2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- How many of them are <u>symmetric</u>? If $aRb \rightarrow bRa$ then $\rightarrow 2^6 = 64$

$R^2 = R \cdot R$

- How many are anti-symmetric $\rightarrow$ $2^3 \cdot 3^3 = 216$

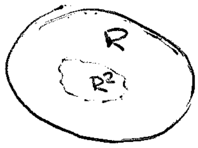
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\leftarrow$ anti-symm.
Think of divides relation $a|b$ and $b|a \Rightarrow a = b$

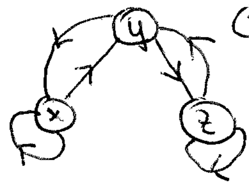
- How many are reflexive and anti-symm.? $\rightarrow 3^3$

Too difficult: How many transitive relations on A

Theorem: The relation R on A is transitive $\Leftrightarrow R^n \subseteq R$
$n = 1, 2, 3, 4, \dots$



<u>Digraph:</u>



$\Leftarrow$ sufficient

$\Rightarrow$ necessary
Do it with induction

it is not possible to go from x to z without stopping in y

$R^1 = R \subseteq R$ ok

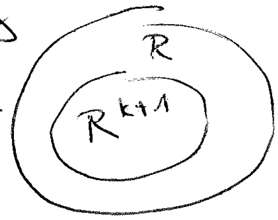
Induction Assumption (IA):
Assume $R^k \subseteq R$
Take an element (a, c) in R^{k+1}. Is it also an element in R. $(a, c) \in R$?

$R^{k+1} = R^k \cdot R$
There is, by definition, an element b such that aR^kb and bRc
But $R^k \subseteq R$ due to IA
so <u>aRb</u>

continued $\Rightarrow$

→ we are allowed to use the transitivity

$aRb \wedge bRc \Rightarrow (a,c) \in R$ OK



Equivalence relations are

1, reflexive
2, symmetric
3, transitive

Ex: Vote on some party

S	MP	PD
M	L	
V	C	FI

Ex.) $A = \mathbb{Z}$

$xRy \Leftrightarrow x \equiv y \mod 4$

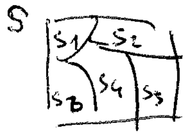
$5R9$ since $\frac{9-5}{4} = 1$

$13 \not R 18$ $\quad \frac{18-13}{4} = \frac{5}{4}$

-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
⋮	⋮	⋮	⋮
16			

Partition of a set

A partition : 5 subsets



every element in S is in a subset S_i

$S_i \cap S_j = \emptyset \quad i \neq j$

Equivalence classes: is the same set, are related

$[0] = \{ \ldots -8, -4, 0, 4, 8, \ldots \} = [20] = [100]$

$[1] = \{ \ldots -3, 1, 5 \ldots \} = [17]$

$[2] = \{ \ldots -2, 2, 6 \ldots \} = [38]$

$[3] = \{ \ldots -1, 3, 7, \ldots \} = [51]$

all elements related to three

Equivalence rel. ⇒ Partition of the set

⇐ from Partition we get E. relation

20.4.2017

③

Theorem 3 equivalent statements
about an equiv. rel R on A

i) aRb
ii) $[a]=[b]$
iii) $[a] \cap [b] \neq \emptyset$

i ⇒ ii

$\boxed{\begin{array}{l}\text{if we can prove } [a] \subseteq [b] \\ \text{AND} \qquad\qquad [b] \subseteq [a] \\ \qquad \Rightarrow [a]=[b]\end{array}}$

i) implies ii) ⇒ iii) ⇒ i)

Take an element c in $[a]$.
cRa and also aRc → here we use symmetry
Remember aRb
Don't forget transitivity

$cRa \wedge aRb \Rightarrow cRb$

That is $c \in [b]$

$[a] \subseteq [b]$

Ex.) $A=\{x,y,z\}$

How many equiv. rel. on A?
Remember partitions!
How many partitions do we have on A?
How many
Symmetric and reflexive 0-1 matrices

$\begin{bmatrix} 1 & \cdot & \triangle \\ \cdot & 1 & \square \\ \triangle & \square & 1 \end{bmatrix}$ They are 8.

1.) $\{x,y,z\}$
2.) $\{x\},\{y\},\{z\}$
3.) $\{x,y\},\{z\}$
4.) $\{y,z\},\{x\}$
5.) $\{x,z\}, \{y\}$

Partial orders (posets)

1.) reflexive

2.) anti-symmetric

3.) Transitivity

Examples $\leq$

$\subseteq$

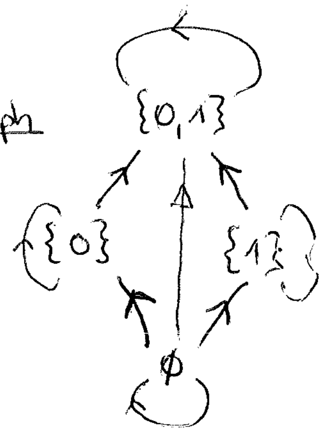
Ex.)

$\subseteq$ on the subsets of $\{0,1\} = A$

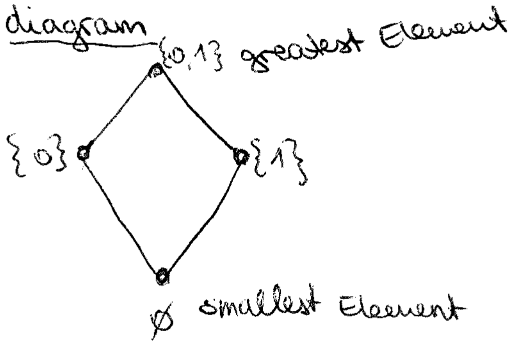
$P(A) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

empty set

Digraph



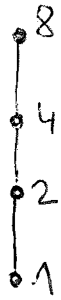
Hasse-diagram



Ex.) Draw the Hasse-diagram for the divides relation on $A = \{1,2,4,8\}$

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hasse-diagram



→ total Order

Digraph

25.4.2017

(1)

A relation on a set A is a subset of $A \times A$

Partial order: $a \leq b$ like $\subseteq, \leq, \mid$	reflexive anti-symmetric transitive
Equivalence relations $a \sim b$ gives a partion of the set 	reflexive symmetric transitive

E .) Divides relation on the positive integers

Is it a partial order?

$a R b \Leftrightarrow a \mid b$

i) reflexive?

$a \mid a$ since $a = 1 \cdot a$ Yes!

ii) anti-symm?

$6 \mid 30 \qquad 30 \nmid 6$

$a \mid b \qquad b = k \cdot a$

$b \mid a \qquad a = l \cdot b$

$a = l \cdot k a$

$l \cdot k = 1$

$l = 1$

$k = 1$

$k, l \in \mathbb{Z}^+$ ← positive Integers

iii) Transitive?

$\overset{x}{6} \mid \overset{y}{30}$ since $\overset{y = k \cdot x}{30 = 5 \cdot 6}$ $\quad k \in \mathbb{Z}^+$, $l \in \mathbb{Z}^+$

$\overset{y}{30} \mid \overset{z}{90}$ since $\overset{z = l \cdot y}{90 = 3 \cdot 30} = 3 \cdot 5 \cdot 6 = 15 \cdot 6$

$\Rightarrow 6 \mid 90$

$z = l \cdot k \cdot x$

$z = (l \cdot k) x$

$\left.\begin{matrix} x \mid y \\ y \mid z \end{matrix}\right. \Rightarrow x \mid z$

Total order on $A = \{x, y, z\}$

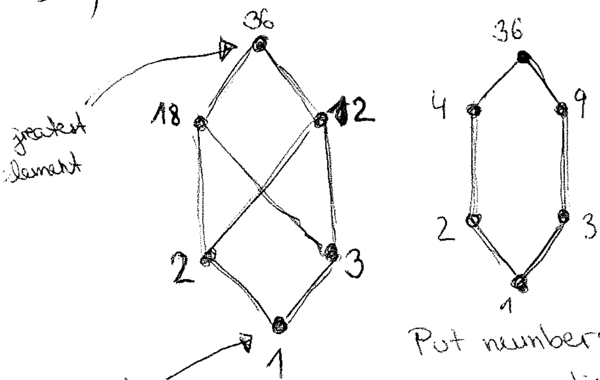
Q: How many total orders on $\{x, y, z\}$?

Ans: $3! = 6$

$xRy \qquad x \leq y$

$yRz \qquad y \leq z$

Ex.) Divides relation on a subset of the positive integer



least element

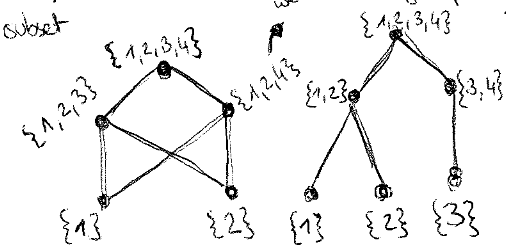
because: 12 is not related to 18!

Put numbers on the vertices

Ex.) $\subseteq$ relation on the subset of the powerset of some set $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \ldots \{1,2,3\}\}$

subset

we have to go up to 4 elements! Impossible with 3



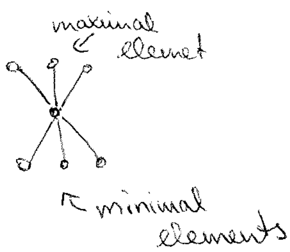
Put subsets at the vertices

→ both not LATTICE

minimal elements nothing is related to them

$\emptyset R \{2\}$

we can not use the empty set in this Hasse-Diagrams because of the Relation



25.4.2017 3

LCM - least common multiple $\rightarrow$ lub$\{a,b\}$ least upper bound for a and b

GCD - greatest common divisor $\rightarrow$ glb$\{a,b\}$ greatest lower bound for a and b

LCM (12, 21) ·

$12 = 2^2 \cdot 3^1 \cdot \cancel{7}^0$

$21 = \cancel{2}^0 \cdot 3^1 \cdot 7^1$

$LCM(12,21) = 2^2 \cdot 3^1 \cdot 7^1 = 84$

maximum of each power

GCD (30, 12) = 6

$30 = 2 \cdot 12 + 6$

$12 = 2 \cdot 6$

$30 = 2 \cdot 3 \cdot 5$

$12 = 2^2 \cdot 3 \cdot 5^0$

$GCD(30,12) = 2^1 \cdot 3^1 \cdot 5^0 = 6$

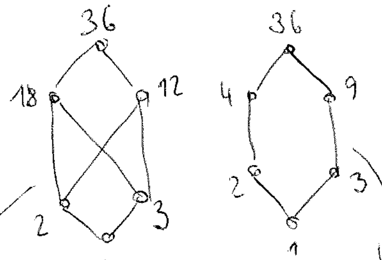
we take minimum ~~maximum~~ of each power

<u>LATTICE</u>: lub$\{a,b\}$ and glb$\{a,b\}$ exist for all pair of elements in the <u>poset</u>.

gitter

glb - greatest lower bound
lub - least upper bound

Ex.) Divides relation on $\mathbb{Z}^+$ is a lattice.



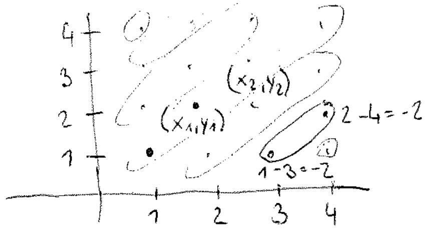
glb$\{2,3\} = 1$ ← least
lub$\{2,3\} = 36$ ← greatest

LATTICE!

upper boundaries
ub$\{2,3\} = 12, 18, 36$
No one is the least
<u>not</u> a lattice

Ex: $S = \{1,2,3,4\}$

A relation on $S \times S$ that is a subset of $(S\times S)\times(S\times S)$



$(x_1,y_1) R (x_2,y_2) \Leftrightarrow y_1 - x_1 = y_2 - x_2$

Show that this is an equivalence relation!
Give the equivalence classes

i) Reflexive? Yes

$y_1 - x_1 = y_1 - x_1$

ii) symmetric? assume $(x_1,y_1) R (x_2,y_2) \Rightarrow y_1 - x_1 = y_2 - x_2$
is then
$(x_2,y_2) R (x_1,y_1)$? $y_2 - x_2 = y_1 - x_1$
yes!

iii) $(x_1\, y_1) R (x_2, y_2)$
$(x_2\, y_2) R (x_3, y_3)$ $\Rightarrow (x_1,y_1) R (x_3, y_3)$?

Yes!

Ex.)

$S = \{ (a,b) \mid a,b \in \mathbb{Z} \wedge b \neq 0 \}$

such that — and

$(a,b) \, R \, (c,d) \Leftrightarrow a/b = c/d$ → ex: $(1,3) R (2,6)$ since $1/3 = 2/6$

Check that it is an equivalence relation on S. (answer is on Hans page)

$[(2,3)] = \{ (4,6), (6,9), (8,12), \dots \}$

$\frac{2}{3} \neq \frac{3}{4}$

$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15}$

think of Rational number as equivalence class

27.4.2017

1

Graphs

Almost everything but Not

10.6 shortest paths ...

10.3 matrices

Just know the concept connectivity

19 partial orders on

$A = \{x, y, z\}$

(Binary relations - wikipedia)

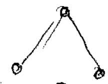
$\underline{6} = 3!$ $3 \cdot 2 \cdot 1$

$x \le y$

$y \le x$

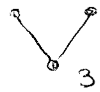
$\Rightarrow x = y$

$\underline{1}$

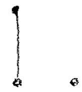


$\underline{3}$

How many of this structure?

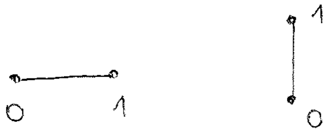


$\underline{3}$



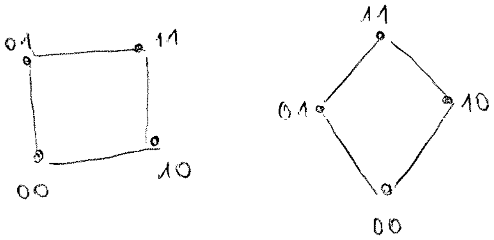
$\underline{6} = 3 \cdot 2$

Q1:



(Hasse-diagram wikipedia)

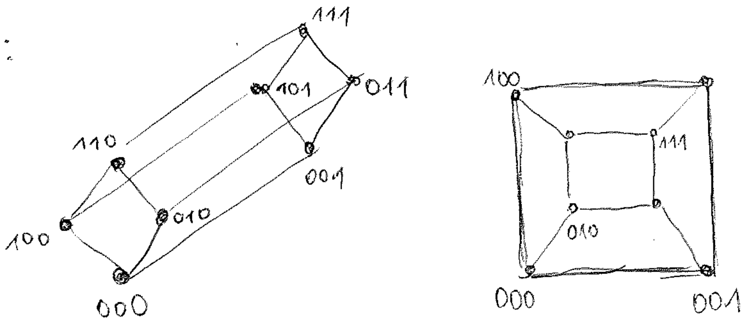
Q2:



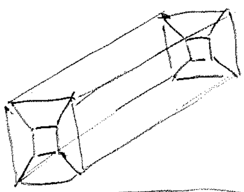
$xy \, R \, zw$
$\Leftrightarrow$
$x \le z$ and
$y \le w$

Cubes

Q3:



Hypercube Q4:



Graph vertex Edge

Def: $G = (V, E)$

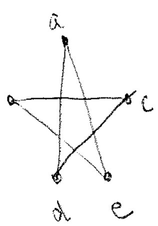
$E \subseteq V \times V$

$v = 5$

$|V| = 5$

$e = 5$

$|E| = 5$



$\deg(a) = 2$

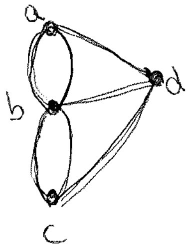
$2e = 5 \cdot 2$

$e = 5$

Handshake-theorem:

$$\boxed{2e = \sum_{v \in V} \deg(v)}$$

Multigraphs   more than one edge between 2 vertices

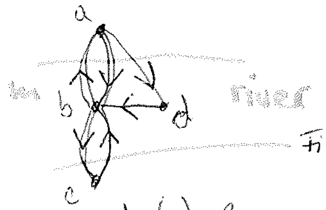


$deg(a) = 3$
$deg(b) = 5$
$deg(c) = 3$
$deg(d) = 3$

$2e = 14$
$e = 7$

The people in Königsberg wanted to find an Euler circuit → closed (go over all edges once)

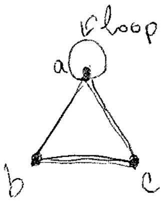
path not repeating edges



Find an euler path

$deg(a) = 3$
$deg(b) = 5$
$deg(c) = 2$
$deg(d) = 2$

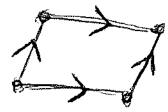
$a \to d \to b \to c \to b \to a \to b$



$deg(a) = 4$
$deg(b) = 2$
$deg(c) = 2$

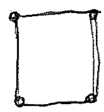
$2e = 8$
$e = 4$

Directed graphs

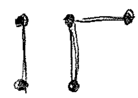


www (1999)
$|V| = 3 \cdot 10^9$
70% was connected?

A graph is connected if there is a path between all pairs of vertices



connected

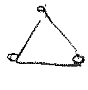


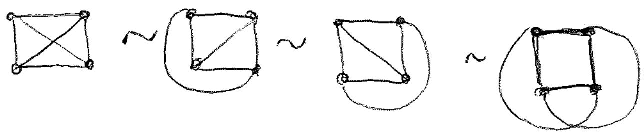
not connected

we consider mainly simple graphs (not multigraphs) without loops and undirected.

complete graphs

K_2

K_3 

K_4 

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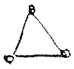
3

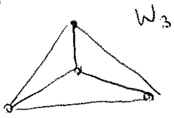
C - cycles no vertices repeated

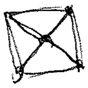
C_3 

C_4 

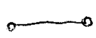
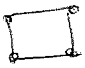
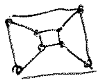
C_5 

wheels:


W_3


W_4

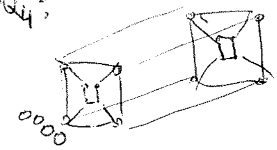
Hypercubes

	e
Q_1 	1
Q_2 	4
Q_3 	12

can we see a pattern? How many edges in Hypercube

How many edges in Q_n?

Q_4:



$12+12+8=32$

$|V| = 2^4 = 16$

$\deg(v) = 4$

Handshake theorem

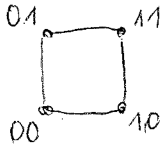
$2e = 16 \cdot 4$

$e = \frac{16 \cdot 4}{2} = 32$

Ex.) # edges in Q_n

$$2e = \sum_{v \in V} \deg(v)$$

Q_2



bit strings

$|V| = 2^2 = 4$

$\deg(v) = 2 \quad \forall v \in V$

$2e = 4 \cdot 2$

$e = 4$

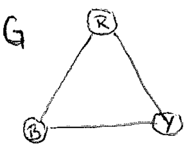
Q_n $\underbrace{0000000\ldots0}_{n}$

$|V| = 2^n$

$\deg(v) = n$

$$\boxed{\begin{aligned} 2e &= 2^n \cdot n \\ e &= 2^{n-1} \cdot n \end{aligned}} = \#\text{ edges in } Q_n$$

Proper coloring of a graph G

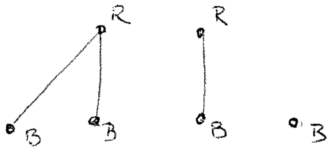
G 

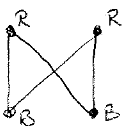
chi

$\chi(G) = 3$

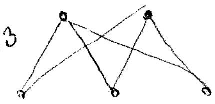
the chromatic number

Bipartite graphs



$K_{2,2}$ 

Theorem: A graph G is bipartite $\iff$ $\chi(G) = 2$

$K_{2,3}$ 

$2e = 3+3+2+2+2$

$e =$

edges in $K_{m,n}$

$m = 1,2,3,\ldots$

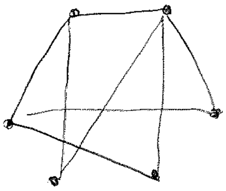
$n = 1,2,3,\ldots$

1 2 3 ... m

1 2 3 ... n

$2e = m \cdot n + m \cdot n \rightarrow \boxed{e = m \cdot n}$

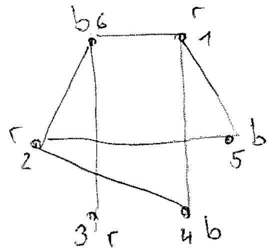
Ex.) G



Is G bipartite?

Answer: No!

would be if:



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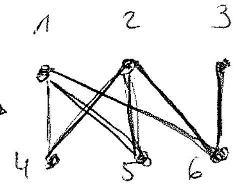
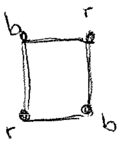


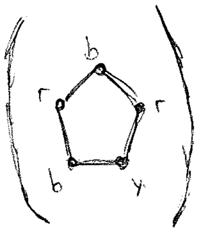
bipartite: because we can do the proper coloring with 2 colors

Ex.) for which n are C_n bipartite



C_4 ?

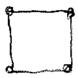
Answer: even n



Ex.) Draw $\overline{Q_2}$ complementary graph to Q_2

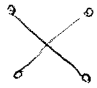
The graph has the same number of vertices as Q_2

Two vertices are adjacent in $\overline{Q_2}$ $\Leftrightarrow$ They are not adjacent in Q_2

Q_2:  $\overline{Q_2}$: ?

Answer:

Q_2:  $\overline{Q_2}$: ?



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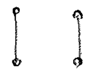
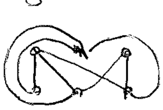


Ex.) Try draw $K_{3,3}$ without crossings



Answer:
-> impossible here

2.5.2017

1

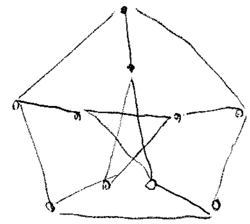
* path (walk) vandring

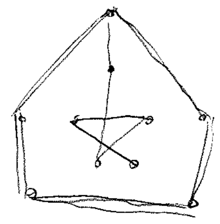
* simple path - (trail)
no repetitions of edges (väg)

* circuit (closed walk)
path that starts and ends (krets)
at the same point

* cycle - simple closed path
without repetitions
of vertices

G

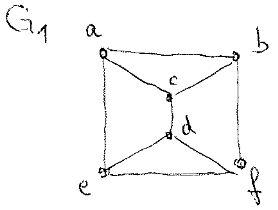




a subgraph
of G

Isomorphism
(same structure)

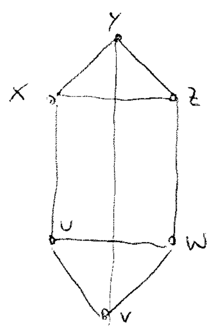
G_1



$v_1 = 6$

$e_1 = 9$

G_2



$v_2 = 6$

$e_2 = 9$

$f(a) = x$
$f(b) = z$
$f(c) = y$
$f(d) = v$
$f(e) = u$
$f(f) = w$

G_1 G_2

Def: 2 graphs are isomorphic if there is a bijection* f

$f: V_1 \rightarrow V_2$ such that

* onto and one-to-one

$\triangle$

$\{a,b\}$ an edge in G_1 $\Leftrightarrow$ $\{f(a), f(b)\}$ is an edge in G_2

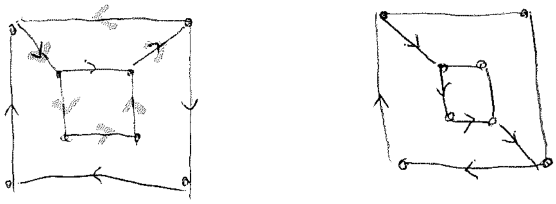
Q: Draw all non-isomorphic graphs with 3 vertices.

opposite would be isomorphic

1) 

$e=0$ $e=1$ $e=2$ $e=3$

Preperation:
G_1



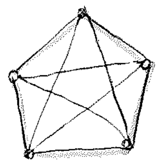
must be invariants

$v=8$	$v=8$
$e=10$	$e=10$
# vertices with degree $2 = 4$ in □	$=4$ in □
# cycles of length 6 $= 2$	$=4$

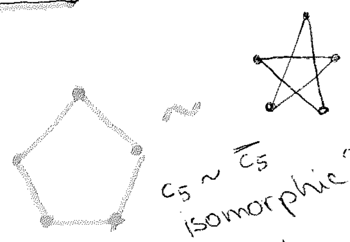
this two graphs are not the same, they differ in cycles.
Steps: check for vertices
check for cycles → we found that they differ here → we don't have to continue

(isomorphic: same number of vertices + same # edges)

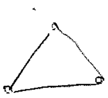
Q: C_5

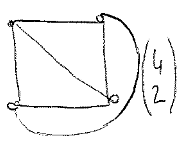


Is it isomorphic to its complement $\overline{C_5}$

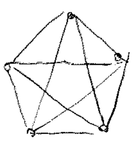


$C_5 \sim \overline{C_5}$ isomorphic? Yes!

K_3  $\binom{3}{2} = 3$ edges ~~[illegible]~~ K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges

K_4  $\binom{4}{2} = 6$

If a graph G is isomorphic to its complement ~~[illegible]~~ we must have
$n \equiv 0, 1 \mod 4$

K_5  $\binom{5}{2} = 10$

To be isomorphic G and $\overline{G}$ must have

$$\underbrace{\frac{n \cdot (n-1)}{2 \cdot 2}}$$ edges each

$$\frac{n(n-1)}{4}$$

(For example $n=5$: $\frac{5 \cdot 4}{4} = 5$)

Handshake theorem: $2e = \sum_{v \in V} \deg(v)$

$n=5$ $2e = 5 \cdot 4$
$e = \frac{5 \cdot 4}{...}$

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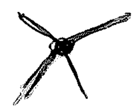
(3)

Euler circuit: go over all edges once (allowed to repeat vertices)
(EC)

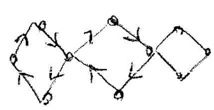
EC exists $\Leftrightarrow$ $deg(v_i)$ is even for all vertices $v_i \in V$

$\Rightarrow$  $2+2+\ldots$ will always be an even number

$\Leftarrow$ Proof by induction. Over the number of edges

Assume it is checked up to $e=k$

Consider a graph with $e=k+1$

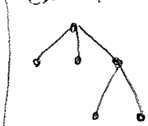


Since all degrees are even there must be cycles in the Graph! This needs an explanation

A tree is a graph without cycles.

$\boxed{V = e+1}$ for trees.

Example: tree



$V=6$

$e=5$

$6=5+1$

For trees

$\boxed{V = e+1}$

On the other hand
The Handshake-theorem tells us

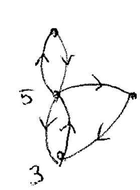
$$2e = \sum_{v_i \in V} deg(v_i) \geq 2 \cdot V$$

so $V \leq e$

so they are not trees!

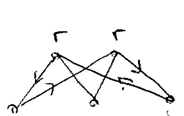
$\hookrightarrow$ There are cycles in the graph!

Euler path exists $\overset{\text{if and only if}}{\Leftrightarrow}$ Exactly two vertices are of odd degree



Hamilton cycle/path

Q: For which m,n has $K_{m,n}$ a Hamilton cycle

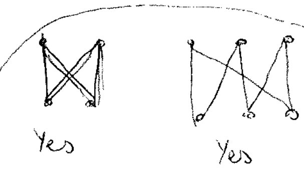


$\chi(K_{2,3}) = 2$

$\rightarrow$ no HC here!

Answer: $m=n\geq 2$

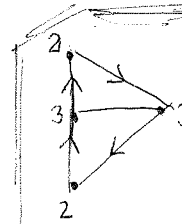
≥ 2 because in we are not allowed to repeat edges



See pattern so Answer

④

Q: Construct a graph with 4 vertices without a HC where $\deg(x) + \deg(y) \geq 3$ for all non-adjacent vertices

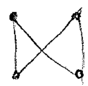


with a HC ?

Ore's theorem: If $\deg(x) + \deg(y) \geq V$ then there is a HC

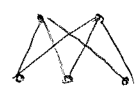
*Q: Show that $e \leq V^2/4$ for bipartite graphs

Ex: $K_{2,2}$



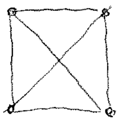
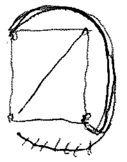
$e = 4 \leq \frac{4^2}{4}$ OK

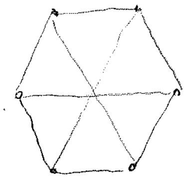
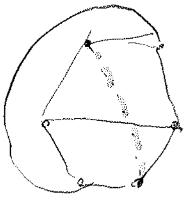
$K_{2,3}$

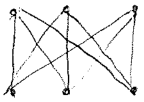


$e = 6 \leq 5^2/4 = 6\frac{1}{4}$

Planar graphs

Is  planar? Yes! isomorphic ~ 

Is  planar? ≁  No! non-planar

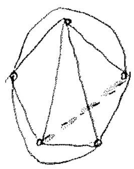
 $\sim K_{3,3}$

For a given V we can not have too many edges if G should be planar.

Kuratowski's theorem:

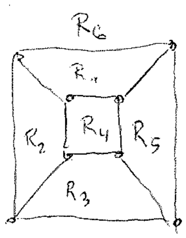
In all non-planar graphs $K_{3,3}$ or K_5 are "hidden"

K_5



Eulers formula for planar graphs

Q_3



Regions

$r = 6$

$e = 12$

$v = 8$

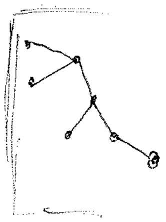
$12 - 8 + 2 = 6$

$$\boxed{r = e - v + 2}$$ ⇒ always for planar Graphs

Proof: Induction proof over the number of regions.

Ex: $r = 1$

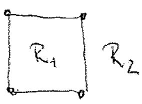
$1 = e - (e+1) + 2$



Tree! $\boxed{v = e + 1}$

Ex: $r = 2$ v e ?

$v = e$



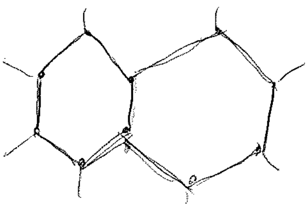
$2 = e - e + 2$

OK!

Assume the formula holds for K regions (IA)

Consider a graph with $r = k+1$ regions

Fall 1989



$r \to r-1$

$e \to e-1$

Using IA:

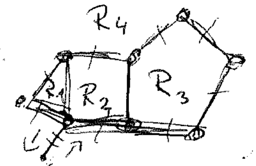
$(r-1) = (e-1) - v + 2$

$$\boxed{r = e - v + 2}$$

Handshake - formula: $2e = \sum_{v_i \in V} \deg(v_i)$

For planar graphs: $2e = \sum_{i=1,\dots,r} \deg(R_i)$

Ex:



$\deg(R_1) = 3$ ← # of edges

$\deg(R_2) = 4$

$\deg(R_3) = 5$

$\deg(R_4) = 10$

$10 + 5 + 4 + 3 = 22$

$2e = 22$

$e = 11$

Ex.) $v = \underline{\underline{6}}$

$\deg(v) = 3$

for all vertices.

It is planar. How many regions? Draw it.

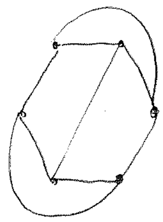
1.) Handshake theorem

2.) E. formula $r = e - v + 2$

$2e = 6 \cdot 3$

$e = 9$

$r = 9 - 6 + 2 = \underline{\underline{5}}$



$r = \underline{\underline{5}}$

$v = \underline{\underline{6}}$

$e = \underline{\underline{9}}$

instead $\deg(v) = 4$

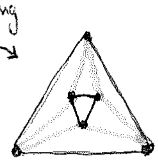
$2e = 6 \cdot 4$

$e = 12$

$r = 12 - 6 + 2$

$= 8$

2D drawing of the octahedron →



$v = \underline{6}$

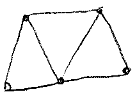
$e = \underline{12}$

(cube )

Not too many edges...

$2e = \sum \deg(R_i) \geq 3 + 3 + 3 + \dots = 3 \cdot r$

$r = e - v + 2$



$\deg(R_i) \geq 3$

$2e \geq 3r$

$r \leq \frac{2e}{3}$

$e - v + 2 \leq \frac{2e}{3}$

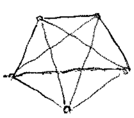
$\frac{e}{3} \leq v - 2$

$e \leq 3v - 6$

Ex: K_5 $v = 5$

$e = 10 = \binom{5}{2}$

$= 4 + 3 + 2 + 1$



Non-planar! Is $10 \leq 3 \cdot 5 - 6 = 9$?

No!

Simple graphs

4.5.2017

3

$K_{3,3}$ $v = 6$

$e = 3 \cdot 3 = 9$

Is $9 \leq 3 \cdot 6 - 6 = 12$?

Yes !? It is not a sufficient condition

↳ bipatite graph

Conclusion: $e \leq 3v - 6$ is necessary but not sufficient condition for planarity.

In bipatite graphs the minimum cycle length is 4 ~

↳ $2e \geq 4 + 4 + 4 + \ldots + 4 = 4 \cdot r$

$e - v + 2 = r \leq \frac{e}{2}$

$\frac{e}{2} \leq v - 2$

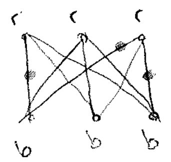
$e \leq 2v - 4$

$K_{3,3}$ $e = 9$, $v = 6$

Is $9 \leq 2 \cdot 6 - 4 = 8$?

No! $K_{3,3}$ is non-planar

K_5 or $K_{3,3}$ is hiding in every non planar graph



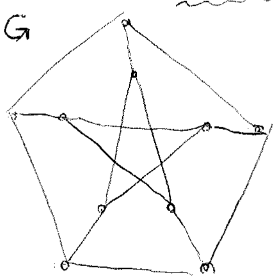
→ new vertices are all deg 2

→ is homeomorphic to $K_{3,3}$

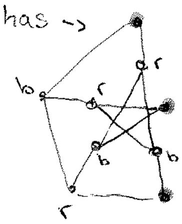
Theorem by Kuratowski

Find subgraph with structure of $K_{3,3}$ or K_5

G



has →



→ as a subgraph

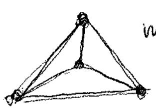
Check that this subgraph is homeomorphic to $K_{3,3}$

5 platonic solids

(degree of every Region)

$\deg(R_i) = m$ $i = 1, 2, \ldots r$

$\deg(v_i) = n$ $i = 1, \ldots |V|$



$m = 3$
$n = 3$

$r = e - v + 2$

$2e = \sum \deg(R_i) = r \cdot m$ (1)

$2e = \sum \deg(v_i) = v \cdot n$ (2)

$2 = \boxed{r - e + v > 0}$ (3)

(1) and (2) into (3)

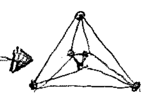
$\frac{2e}{m} - e + \frac{2e}{n} > 0$

$e \left[\frac{2n - mn + 2m}{mn} \right] > 0$

$2n - mn + 2m > 0$

m	n	name
3	3	tetrahedron (4 faces)
3	4	octahedron
4	3	cube
3	5	
5	3	

symmetry (



Ex: $m = 3$
$n = 4$

$r = e - v + 2$

$\frac{2e}{3} = e - \frac{2e}{4} + 2$

$\frac{2e}{3} + \frac{e}{2} - e = 2$

$\frac{4e + 3e - 6e}{6} = 2$

$e = 12$
$r = 8$
$v = 6$

9.5.2017 1

Vertex colorings of graphs

$\chi(\triangle) = 3$

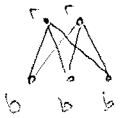
r, b, y

$\chi(K_n) = n$

K_4 

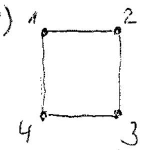
Proper colorings (P.C.)
two vertices with edge in common must have different colors. The <u>minimum</u> number of colors needed for a p.c. of G is called the chromatic number $\chi(G)$.

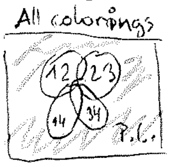
The G bipartite $\Leftrightarrow \chi(G) = 2$



$K_{2,3}$ (complete graph)

Ex.) How many proper colorings of C_4 with 3 colors?

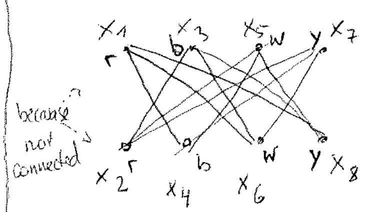




$\rightarrow$ try with I.E.

$3 \cdot 2 \cdot 2 + 3 \cdot 2 \cdot 1^2 = 12 + 6 = 18$

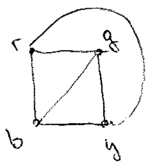
Greedy algorithm

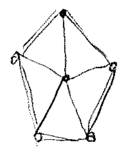


r
b
w
y

4 colors needed

Q: Find two different planar graphs with chromatic number $= 4$





Theorem: $\chi \leq 6$

Q: Can all vertices in a <u>planar</u> graph have degree 6 or more?

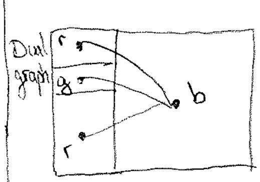
$$2e = \sum_{v_i \in V} \deg(v_i) \geq 6 + 6 + 6 + \ldots = 6v$$

$$\boxed{e \geq 3v}$$

$\rightarrow$ Answer: No!

There ~~are~~ vertices with smaller degree, like 5

4-color theorem: 1976 with computer



Note: The dual graph is planar.

χ(The dual graph) $= 4$ at most. (No 5^{th} needed)

red	blue
blue	red

one point is not enough!

But from last week: $2e = \sum \deg(R_i) \geq 3 + 3 + 3 + \ldots$

$= 3r$

so $r \leq \frac{2e}{3}$

Euler's formula: $r = e - v + 2 \leq \frac{2e}{3}$

$\frac{e}{3} \leq v - 2$

$$\boxed{e \leq 3v - 6}$$

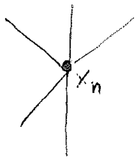
$|V| = n$

G x_n

$G - \{x_n\}^{*}$ x_{n-1}

$G - \{x_n, x_{n-1}\}$ x_{n-2}

$\vdots$



(G without x_n and

* The subgraph of all edges attached to it)

In this way we get a list of the vertices

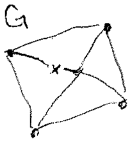
$\overset{r}{x_1}, \overset{b}{x_2}, \overset{y}{x_3}, \overset{g}{x_4}, \overset{w}{x_5}, \overset{v}{x_6}, \overset{v}{x_7}, \dots\dots \; x_n$

x_7 is connected with at most 5 of the vertices $x_1, x_2, x_3, x_4, x_5, x_6$

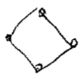
Let's do the greedy algorithm

I can reuse a color for vertex x_7

G

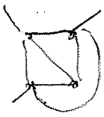


$G - \{x\}$



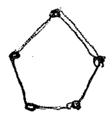
Q: Show that a graph G has at least $\binom{\chi(G)}{2}$ edges.

$\chi(G)$	$e \geq$
2	$\binom{2}{2} = 1$
3	$\binom{3}{2} = 3$
4	$\binom{4}{2} = 6$
5	$\binom{5}{2} = 10$
6	$\binom{6}{2} = 15$



K_n has $\binom{n}{2}$ edges

$\chi(K_n) = n$



$\chi(G) = 3$

edges $\hookrightarrow 5 > \binom{3}{2}$

Today trees

11.5.2017

1

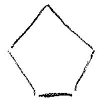
16/5 Tuesday: K-building Problems 1-3

18/5 Thursday: The rest

23/5 Last minute questions

$\chi = 3$

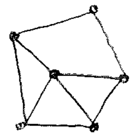
$e=3$



$e=5$

$\chi = 4$

$e=6$



$e=10$

Extremal problems

Given a v

$e \leq \lfloor \frac{v^2}{4} \rfloor$ if we want to avoid

and the graph is $K_{\lfloor \frac{v}{2} \rfloor, \lceil \frac{v}{2} \rceil}$

$v = 7$ (floor function)

$\lfloor \frac{7^2}{4} \rfloor = \lfloor 12\frac{1}{4} \rfloor = 12$

$\lfloor \frac{v}{2} \rfloor = 3$

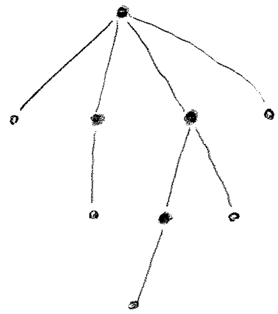
ceiling function $\lceil \frac{v}{2} \rceil = 4$

$e = 12$

Trees - no cycles

$v = e+1$

root



l • leaves

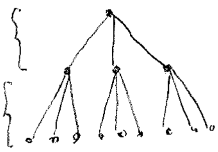
i • internal vertices

$v = l + i$

$l=5$
$i=4$
$v=9$
$e=8$

root = internal

A full 3-ary



$$\overset{\text{root}}{1} + \overset{\text{children}}{3} + \overset{\text{grandchildren}}{3^2}$$

$$1 + 3 \cdot \underbrace{(1+3)}_{i} = 1 + 3i$$

$v = l + i = 1 + 3i$

$\Leftrightarrow$

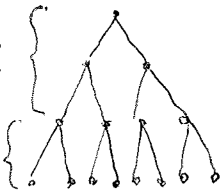
only for full $\boxed{\frac{l-1}{3-1} = i}$

General formula for m-ary

$$\boxed{\frac{l-1}{m-1} = i}$$

exp: $m=2$ $\quad i = l-1$

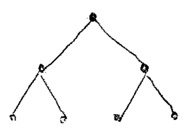
full binary tree



$l = 8$

$i = 7$

Tournaments

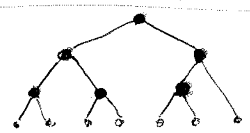


1 2 3 64

How many games will be played during the whole tournament?

$$\boxed{\underset{\#\text{games}}{i} = \overset{\text{leaves}}{l} - 1}$$

games $= 64 - 1 = 63$



1 2 3 4 5 6 7

← not a full tree

How many games will be played?

games $= 7 - 1 = 6$

Everyone is a loser exactly once except the winner!!!

There is a bijection between the set of losers and the set of games.

Q: How many edges in a full binary tree with 1023 internal vertices?

$l = i+1 = 1024$ $\quad v = 2047$

$e = 2046$

Q: C_4H_{10} is it a tree?

Q: Which complete bipartite graphs are trees?

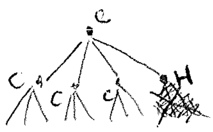
$v = 14$

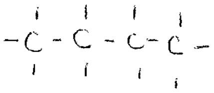
$2e = 26$

$e = 13$

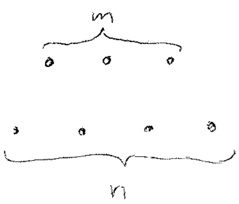
$v = e+1$

Yes!





$K_{m,n}$



$\boxed{v = e+1}$

$v = m+n$

$e = m \cdot n$

$m+n = m \cdot n + 1$

$m = 1$

and/or $n = 1$

Ex.) 25 bagels

x_3, x_4, x_1, x_2

$3 \le x_i \quad i = 2,3,4,\dots$

$3 \le x_i = 6$

GF: $G(x) = (x^3+x^4+x^5+\dots)^3 \cdot (x^3+x^4+x^5+x^6)$

$= \dots\dots\dots + 2x^{25} + x^9 \cdot \dfrac{1}{(1-x)^3} \cdot \dfrac{x^3 \cdot (1-x^4)}{1-x}$

$= \dfrac{x^{12} - x^{16}}{(1-x)^4}$

$G(x) = (x^{12} - x^{16})(1 + 4x + ax^9 + 6x^{13}) = \dots + 2x^{25}$

Ans. $b - a =$

$y_1, y_2, y_3, y_4 = 13$

In total (restrictions) $\binom{16}{13}$

$b = \binom{16}{13}$

$a = \binom{12}{9}$

$\binom{12}{9}$ $\quad y_4 \ge 4$

$y_1 \ge 0$	$z_4 + 4 = y_4$
$y_2 \ge 0$	$y_1 + y_2 + y_3 + \cancel{y_4} + z_4 = 13$
$y_3 \ge 0$	
$z_4 \ge 0$	$y_1 + y_2 + y_3 + z_4 = 9$

Inclusion-Exclusion

$x_1 = 3 + y_1$

$x_2 = 3 + y_2$

$x_3 = 3 + y_3$

$x_4 = 3 + y_4$

$y_i \ge 0 \quad 0 \le y_4 \le 3$

Chemistry ②

isomers - non isomorphic graphs

Ex: C_3H_8

$deg(C) = 4$

$deg(H) = 1$

$v = 3 + 8 = 11$

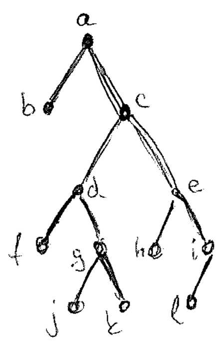
Is $v = e+1$?

if so $e = 10$

let's check with handshake - theorem

$2e = \sum_{v \in V} deg(v) = 3 \cdot 4 + 8 \cdot 1 = 20$

$e = 10$



center = the vertices with the smallest eccentricity of a vertex is the length of the longest path that can be found starting at the vertex.

vertex	ecc.
a	4
b	5
c	3
d	4
e	4
f	5
g	5
h	5
i	5
j	6
k	6
l	6

center (the smallest number)

Q: with two centers? think of it

16.5.2017

1

Today
Combinatorics
&
Relations

Thursday: Graphs & Trees

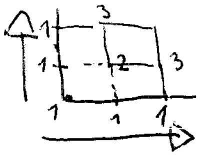
3 exams: 150829
160526
160609

Combinatorics

* 4 cases
* sum rule
* multiplication rule

1.) 9/6 2016

a) sum of the n:th row in Pascal's Δ



Σ	
1	1
2	1 1
4	1 2 1
8	1 3 3 1
$2^4 = 16$	1 4 6 4 1

Ans $= 2^n$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^k \cdot y^{4-k}$$

$$(1+1)^4 = \sum_{k=0}^{4} \binom{4}{k} = 2^4$$

b) How many six-letter words can be formed with 2U, 2R and 2D

$\boxed{R, G, C, A}$

D U R D U R

1.) $\dfrac{6!}{2!2!2!}$

2.) First decide the positions for the U's

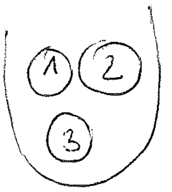
D U R U D R

$\binom{6}{2}$ way to do this $\cdot \binom{4}{2} \cdot \binom{2}{2} = \dfrac{6!}{2!\cancel{4!}} \cdot \dfrac{\cancel{4!}}{2!2!} \cdot \dfrac{2!}{2!0!}$

$= \dfrac{6!}{2!2!2!}$

$0! = 1$

c.)



Pick them 13 times
with repetition
Don't care about
order

$x_1 + x_2 + x_3 = 13$ How many integer solutions if $\underline{\underline{x_i \geq 0}}$?

$\downarrow$

non-negative

$$\binom{15}{13} = \binom{15}{2} =$$

$6+7+0 \leftrightarrow \underbrace{|||||| + ||||||| +}_{15 \text{ positions}}$ $\to$ 1 + ...

$\to$ 13 sticks

$\to$ 2 +

not exam:

$x_1 + x_2 + x_3 = 13$ # integer solutions?

$x_i \geq -4$

$i = 1,2,3$

$x_i = -4 + y_i \quad y_i \geq 0$

$y_1 + y_2 + y_3 = 25 \quad y_i \geq 0$

Ans $\binom{27}{25}$

d.) $1,1,1,1,1,1,\ldots$ (infinitiv string of ones)

$G(x) = 1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 + 1\cdot x^4 + \ldots$

$= \frac{1}{1-x} \qquad |x|<1$

$x \cdot s = (1 + x + x^2 + x^3)\cdot x$

$s = 1 + \ldots x^3$

$s_3 - x \cdot s_3 = 1 - x^4$

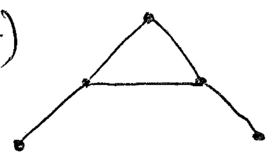
$\frac{s_3(1-x)}{1-x} = \frac{1-x^4}{(1-x)}$

$s_3 = \frac{1-x^4}{1-x}$

$s_n = \frac{1-x^{n+1}}{1-x}$

$s_\infty = \frac{1}{1-x} \qquad n \to \infty$

e.)



- 5 colors available

\# ways to do a proper coloring with 5 colors

1.) Go from left to right $\to 5\cdot4\cdot4\cdot3\cdot4 = 5\cdot4^3\cdot3$

2.) Start with triangle

o o o o o

We can make ___ colorings of the Δ

$$\binom{5}{3}\cdot 3! = \frac{5!}{\cancel{3!}\,2!}\cdot\cancel{3!}$$

$= 5\cdot4\cdot3$

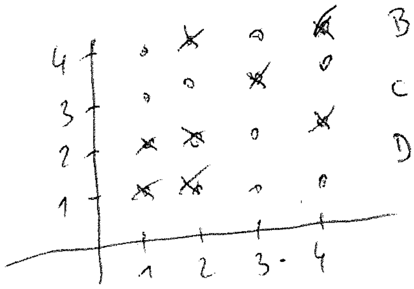
Totally $5\cdot4\cdot3\cdot4\cdot4 = 5\cdot4^3\cdot3$

16.5.2017 (3)

Relations

$A = \{1,2,3,4\}$

A relation on A is a subset of $A \times A$



A. reflexive

B. symmetric

C. anti-symmetric

D. transitive

A, B, D : Equivalence relations $[x] = \{x, \dots\}$

Partition

A, C, D : Partial order

Hasse diagram

Lattice

Ex) GCD(a,b) | glb(a,b)

LCM(a,b) | lub(a,b)

least common multiple

---

(3) 26/5 - 16

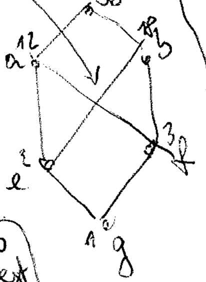
$\{1,2,3,5,10,15,30,45,90\}$

aRb

$\Leftrightarrow$

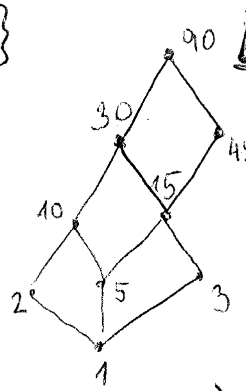
if and only if

$a \mid b$



lb(a,b) = e, f, g

No one is the greatest



$\boxed{\text{Lattice}}$

lower bounds

lb(10,15) = 1, 5

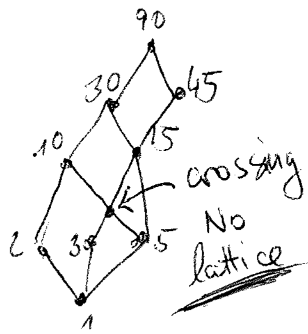
greatest

glb(10,15) = 5

1 divides 2, 5, 3

2, 5 | 10

⋮



No lattice

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\
M_R = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

3R4

equivalence Relation

$[1] = \{1, 2, 5\}$

$[4] = \{3, 4\}$

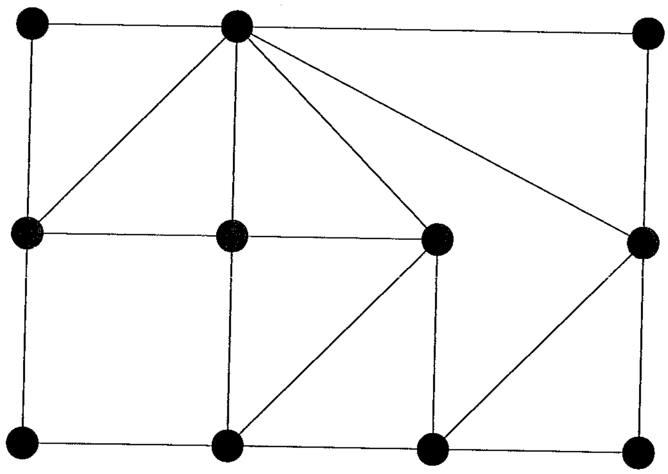
$$
\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)

Ruth

Linnaeus University
School of Mathematics
Hans Frisk

Exam Questions on Graph Theory, 1MA462, 7,5 hp

1. Determine the chromatic number, $\chi(G)$, for the following graph G. Give a proper coloring of G using $\chi(G)$ colors. (4p)



2. Some different questions on graphs.
   a) How many edges does a simple graph have if the degrees of the vertices are 5,2,2,2,2,1? Draw such a graph. (3p)
   b) For which values of n do K_n have an Euler circuit? (2p)
   c) Draw the complement graph of $K_{2,3}$. For the definition of complement graph see sheet with formulas and theory. (2p)
   d) How many edges does a tree with 150 vertices have? (1p)

3. Suppose that a connected planar simple graph with e edges and v vertices contains no simple cycles of length 4 or less. Show that

$$e \leq \frac{5v-10}{3}$$

when $v \geq 5$. Draw such a graph with $v = 7$ and maximal number of edges. (6p)

4. The *odd graph* O_k, k is an integer ≥ 2, is defined in the following way:
   The vertices represent the subsets with $k-1$ elements that can be obtained from a set with $2k-1$ elements. Two vertices are joined with an edge if and only if the corresponding subsets are disjoint.
   a) Draw O_2 and O_3 (3p)
   b) Show that O_k is k-regular for all $k \geq 2$, that is, show that all vertices in O_k has degree k. (3p)

5. a) Draw a planar graph with six vertices which is 3-regular (i.e. all vertices have degree 3). (2p)
   b) Let G be a graph with 11 vertices and let $\bar{G}$ be its complement. Show that not both G and $\bar{G}$ can be planar. (4p)

6. The *girth* of a graph G is the length, i.e. the number of edges, of the shortest cycle it is possible to find in G and it is denoted $g(G)$. For graphs with n vertices and $n+1$ edges, $n \geq 4$, it is possible to show that

$$g(G) \leq \left\lfloor \frac{2(n+1)}{3} \right\rfloor$$

a) Draw a graph with maximal girth for $n = 7$. (1p)
b) To show the inequality above start by first showing that a graph with n vertices and $n+1$ edges always has at least two cycles. Consider then separateley the two cases where the two cycles are edge disjoint and when they have edges in common. What is the maximal girth if the cycles are edge disjoint? Note that when the two cycles have edges in common there must be three different paths between two vertices in the graph. (5p)

7. Suppose that a connected planar simple graph with e edges and v vertices contains no simple cycles of length 4 or less. Show that

$$e \leq \frac{5v - 10}{3} \qquad (1)$$

8. Draw all non-isomorphic trees with five vertices. Note trees are simple connected graphs. (4p)

9. Let G be a simple graph with $|V| = n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all nonadjacent $x, y \in V$, then G contains a Hamilton cycle. This is a sufficient condition for the existence of a Hamilton cycle proved by Ore 1960.
a) Draw a graph for which the condition holds and find the Hamilton cycle. Show also that it is not a necessary condition, that is, find a graph with a Hamilton cycle but for which the condition is not fulfilled. (3p)
b) A group of 12 people meet for dinner at a big circular table. In this group everyone knows at least 6 others. Can they be seated around the table in such a way that each person knows the person to the left and to the right? (2p)

10. a) Consider connected simple graphs, G, with 11 vertices. Prove that either G or its complement $\overline{G}$ must be nonplanar. (3p)
b) This theorem does not hold for eight vertices. Find a counterexample to part a above, that is, find a planar G for which also the complement is planar. (2p)

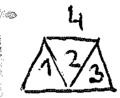
18.05.2017

1

Graphs

* Handshake theorem
* Euler circuit & Hamilton Cycles
* isomorphisms
* Trees, $\boxed{v = e+1}$, no cycles, $v = l + i$
* Planar graphs

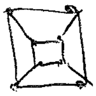
$$\boxed{2e = \sum_i \deg(R_i)}$$



Euler's formula → $\boxed{r = e - v + 2}$

$r = 1$ for trees

$r = 6$ for Q_3



3 = 7

5b = 10a

4b *

5b *

6b ***

2a.) 5, 2, 2, 2, 2, 1

$v = 6$

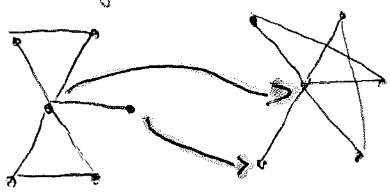
$$2e = \sum_{v_i \in V} \deg(v_i) = 14$$

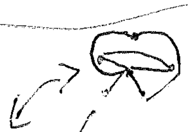
$e = 7$

planar G
↳ no crossings

or

–//–





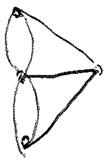
(Regions is something different)

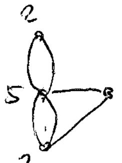
both isomorphic → same structure to each other

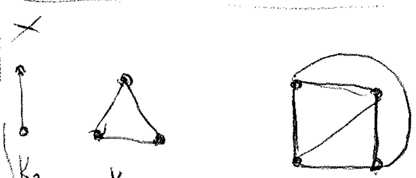
$v_1 \xrightarrow{f} v_2$

vertice mapped to v_2

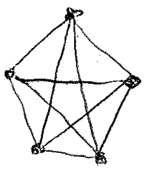
b.) For which n have K_n an Euler circuit?
go over all edges
(↳ didn't want to repeat edges)





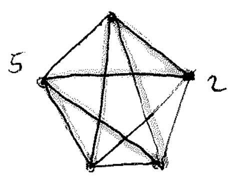


In K_n all vertices have degree $n-1$
(if it's 4 it's 3
if it's 3 it's 2)
so $n-1$ must be even
n must be odd, $n \geq 3$



K_5

c.)



-> handshake swedes with chinese

-> black is the handshake between swedes themselves and chines among them

$K_{2,3}$ is a subgraph of K_5

$\overline{K_{2,3}}$

3.) Necess. cond. for planar graphs

$e \leq 3v-6$

$e \leq 2v-4$

$e \leq \frac{5v-10}{3}$

Shortest cycle in the graph

3 -> we do this one!

4

5 -> Pentagon

$r = e - v + 2$

$2e = \sum \deg(R_i) \geq 3+3+3+\ldots+3 = 3 \cdot r$ <- Regions

or: $\boxed{2e \geq 3r \\ \frac{2e}{3} \geq r}$

$e - v + 2 = r \leq \frac{2e}{3}$

$\frac{e}{3} \leq v - 2$

$e \leq 3v-6$

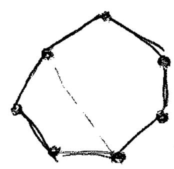
Do 4 or 5 on your own! Same way!

$v = 7$

$e \leq \frac{7 \cdot 5 - 10}{3} = 8\frac{1}{3}$

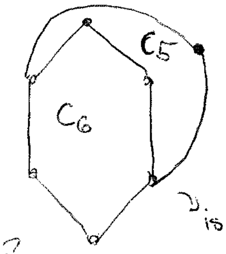
edges = 8
vertices = 7
avoid squares or triangles

No C_3
No C_4



-> something wrong here

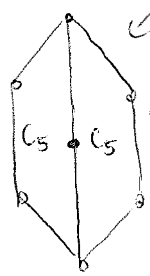
$v = 7$
$e = 8$



-> is it now possible to avoid C_3 & C_4

<- or

both! <- work.



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$2e \geq 4+4+4+\dots+4 = 4r$

$2e \geq 4r$

$e \geq 2r$

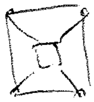
$e - v + 2 = r \leq \frac{e}{2}$

$e - v + 2 \leq \frac{e}{2}$

$\frac{e}{2} \leq v - 2$

$\boxed{e \leq 2v - 4}$

Q_3



or degrees are 4

$v = 8$

$e = 12$

---

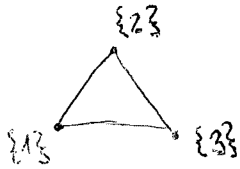
4) odd graph

$O_k \quad k = 2, 3, \dots$

$k = 2$ subsets with $2-1$ elements that can be obtained from a set with $2 \cdot 2 - 1 = 3$ elements

$\{1, 2, 3\}$

So O_2 has ...3... vertices.

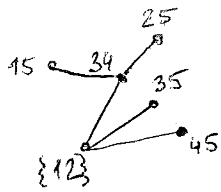


O_2 isomorphic to K_3 or C_3

$k = 3$ subsets with $3 - 1 = 2$ elements

...... from a set with $2 \cdot 3 - 1 = 5$ elements

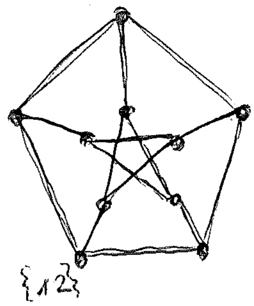
So O_3 has $\binom{5}{2} = 10$ vertices.



so O_3 is 3-regular that is degree $= 3$ for all vertices

O_3 has ...15... edges $\to \frac{3 \cdot 10}{2} \to$ handshake theorem

O_3

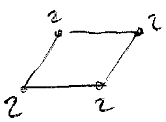


9. $|V| = n \geq 3$

If there are many edges we should find a HC

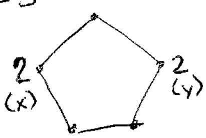
$$\boxed{\deg(x) + \deg(y) \geq n}$$

$n = 4$



we find it directly. Every vertex has a degree 2

$n = 5$ C_5



non-adjacent

$2+2<5$ → condition not fulfilled but cycle is there!

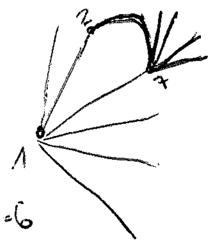
A graph with 66 vertices has a Hamilton cycle

This cycle has how many

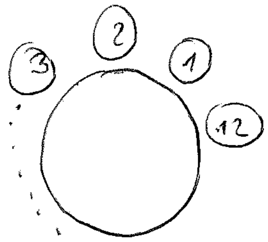
...66... edges.

---

b.) 12 people



$\deg(v) = 6$



$6+6 \geq 12$ Yes! There is a HC.

Put the HC around the table

6.) girth = umfang
length = # edges

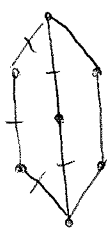
$v = n$
$e = n+1$

There are cycles in the Graph!

$$g(G) \leq \left[\frac{2(n+1)}{3}\right] \quad n \geq 4$$

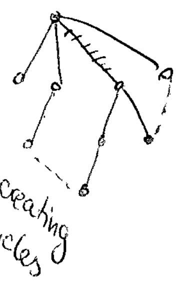
$n = 7$ $\quad g \leq \left[\frac{16}{3}\right] = \underline{\underline{5}}$

$n=7$
$e=8$



girth = shortest cycle $= \underline{\underline{5}}$

$v = n = 9$
$e = 8$



creating cycles

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29/8 - 2 Bagels I-E

$x_p + x_t + x_b = 25$

$11 \geq x_p \geq 5$

$11 \geq x_t \geq 5$

$11 \geq x_b \geq 5$

$x_p = 5 + y_p$

$x_t = 5 + y_t$

$x_b = 5 + y_b$

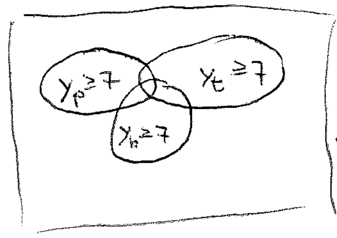
$y_p + y_t + y_b = 10$

$0 \leq y_p \leq 6$

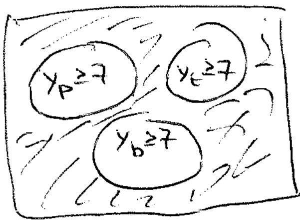
$0 \leq y_t \leq 6$

$0 \leq y_b \leq 6$

All poss. no upper restriction $= \binom{12}{10}$
$= 66$



wrong figure



← correct!

← we are interested what is outside

$y_p \geq 7$:

$y_p = 7 + z_p$

$7 + z_p + y_t + y_b = 10$

$z_p + y_t + y_b = 3$

$\binom{5}{2} = \binom{5}{3}$

Answer: $\binom{12}{10} - 3 \cdot \binom{5}{3}$

$66 - 3 \cdot 10 = 36$

with GF (generating function)

$G(x) = (\underline{x^5} + x^6 + \dots x^{11}) \cdot (\underline{x^5} + \dots x^{10} + x^{11}) \cdot (\underline{x^5} + \dots x^{10} + x^{11})$

p t b

$= x^{15} + \dots + x^{25} + \qquad + x^{33}$

↑ what is in front of x^{25}

$G(x) = x^5 \cdot x^5 \cdot x^5 (1 + x^2 + \dots x^6)^3$

$= x^{15} \cdot \left(\frac{1 - x^7}{1 - x}\right)^3 = x^{15} \cdot (1 - 3 \cdot x^7 + 3 \cdot x^{14} - x^{21}) \cdot \frac{1}{(1-x)^3}$ → table formulas on the exam

$= (x^{15} - 3x^{22} + 3x^{29} + \dots)$ ← not interesting only until 25 $\cdot \left(1 + 3x + \binom{4}{2}x^2 + \binom{5}{3}x^3 + \dots + \binom{12}{10}x^{10}\right)$

Ans $\binom{12}{10} - 3\binom{5}{3}$