Design and Analysis of Algorithms

LECTURE 5

Divide and Conquer

- Merge & Quick Sort
- Binary search
- Powering a number
- Matrix Multiplication(Strassen's Algorithm)

Decrease and Conquer

- Insertion Sort
- DFS, BFS
- Topological Sort

The divide-and-conquer design paradigm

1. ***Divide*** the problem (instance) into subproblems.
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.

Merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

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$$T(n) = 2\,T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

work dividing and combining

Master theorem (reprise)

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$, constant $\varepsilon > 0$

$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, constant $k \geq 0$

$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$, constant $\varepsilon > 0$, and regularity condition

$\Rightarrow T(n) = \Theta(f(n))$.

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Merge sort: $a = 2,\ b = 2 \ \Rightarrow\ n^{\log_b a} = n^{\log_2 2} = n$

$\Rightarrow$ CASE 2 $(k = 0)$ $\Rightarrow$ $T(n) = \Theta(n \lg n)$.

Quick sort

1. ***Divide:*** Select a pivot and Divide the array into sub-arrays.
2. ***Conquer:*** Recursively sort the subarrays.
3. ***Combine:*** Linear-time combine.

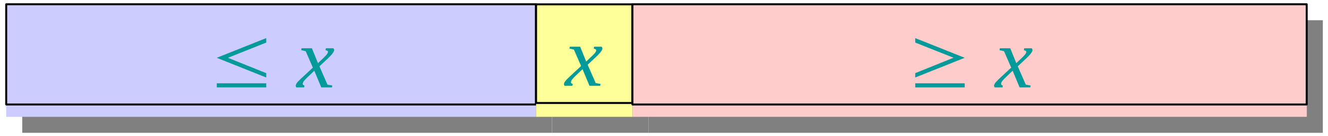
Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts “in place” (like insertion sort, but not like merge sort).
- Very practical (with tuning).

Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*

Partitioning subroutine

PARTITION(A, p, q) ⊳ $A[p \,..\, q]$

$x \leftarrow A[p]$ ⊳ pivot = $A[p]$

$i \leftarrow p$

for $j \leftarrow p + 1$ **to** q

do if $A[j] \leq x$

then $i \leftarrow i + 1$

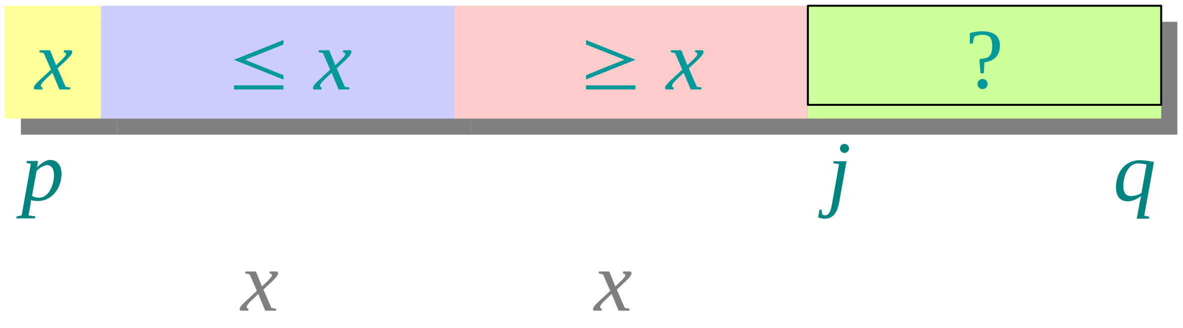
exchange $A[i] \leftrightarrow$

$A[j]$ exchange $A[p] \leftrightarrow A[i]$

return i

Running time $= O(n)$ for n elements.

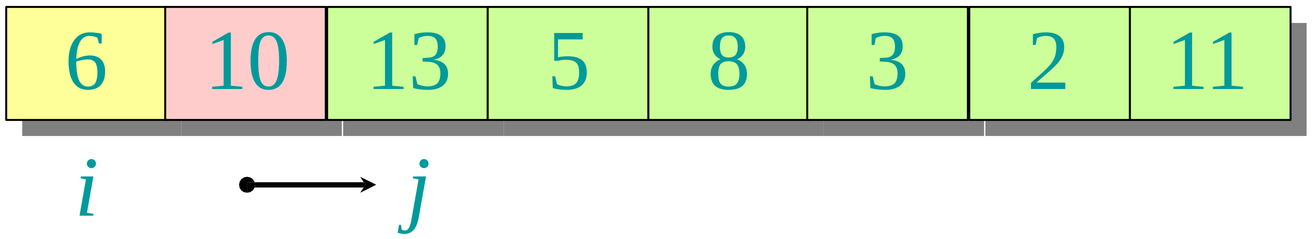
Invariant:



Example of partitioning

6	10	13	5	8	3	2	11
i	j						

Example of partitioning



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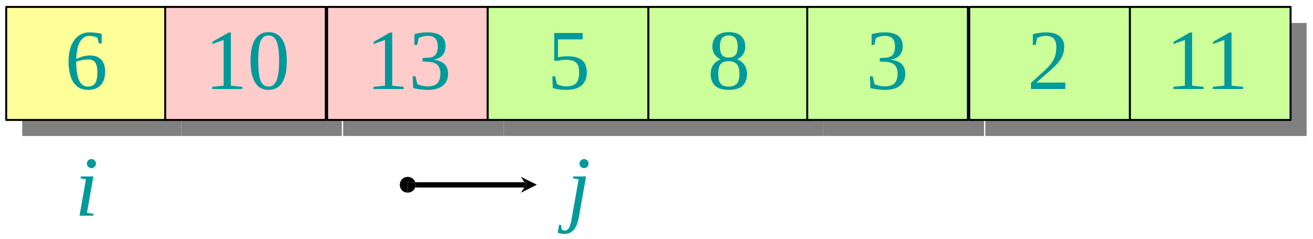

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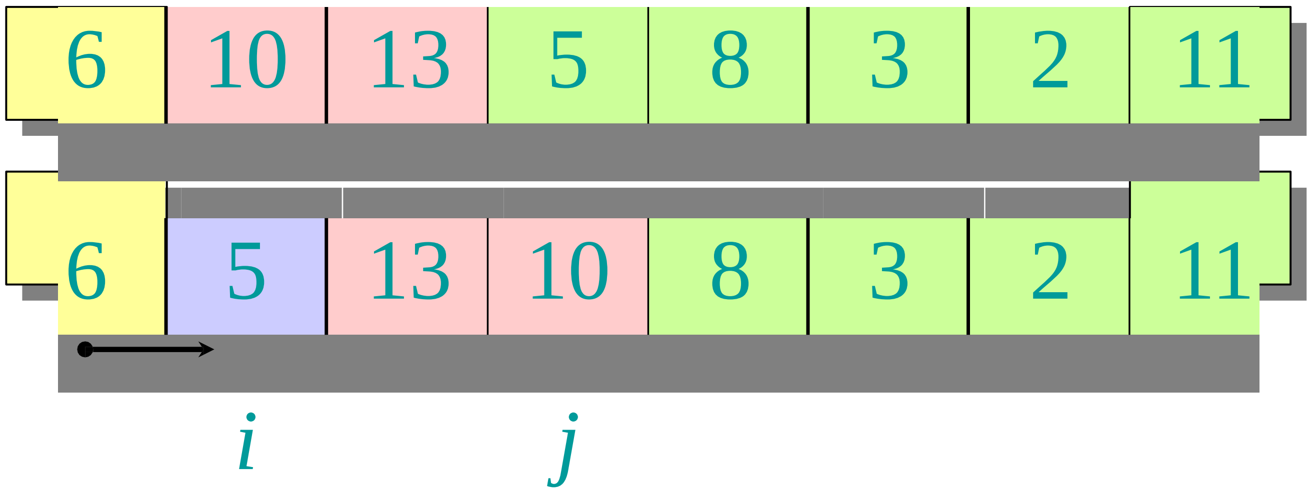

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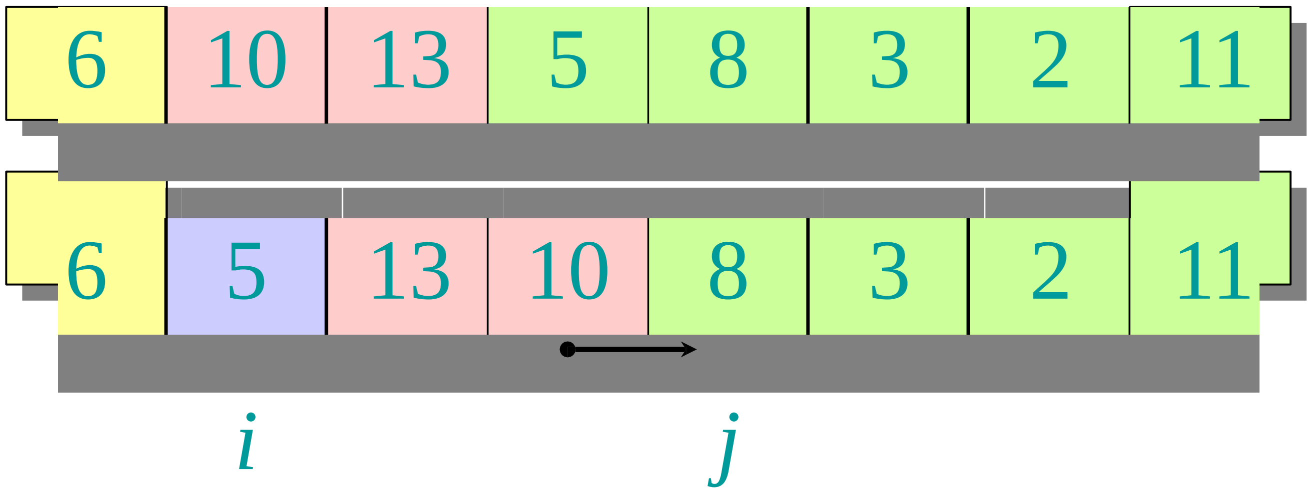

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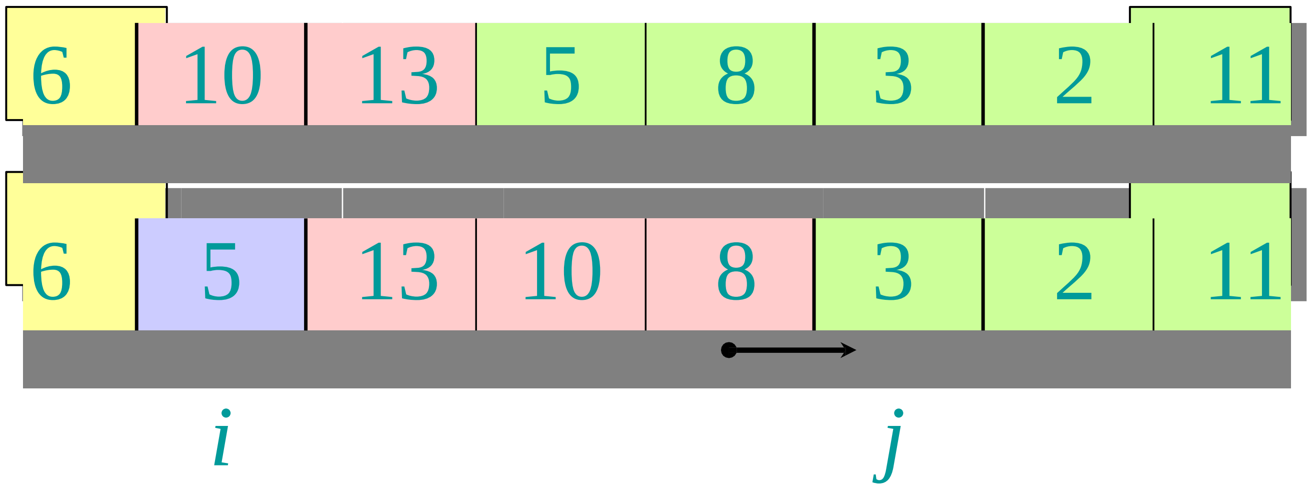

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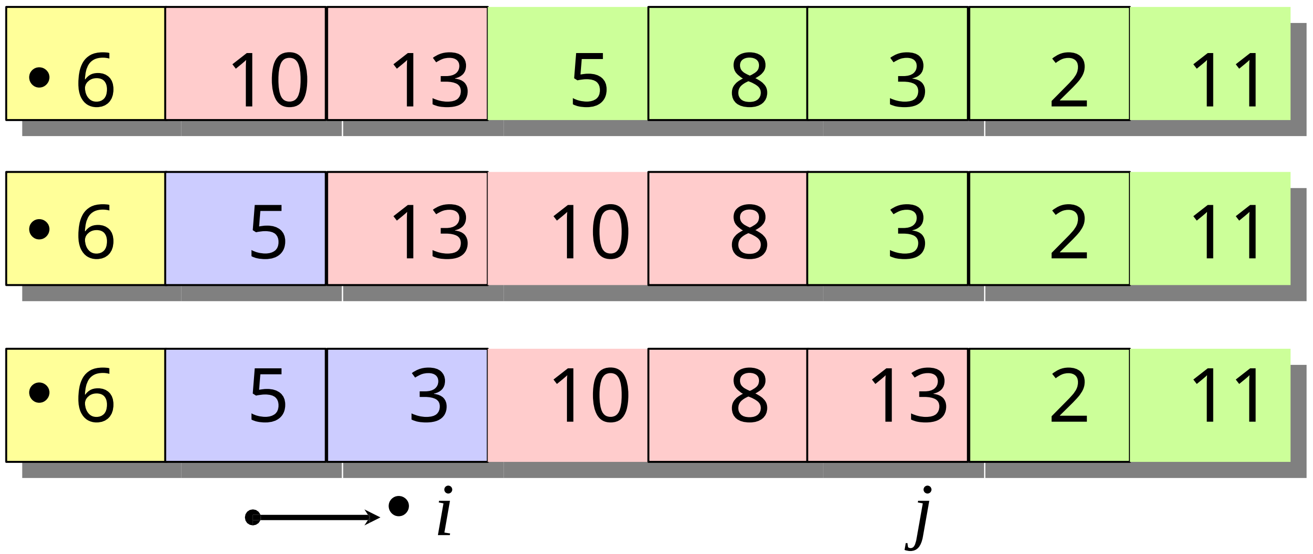

Example of partitioning

• 6	10	13	5	8	3	2	11
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- i

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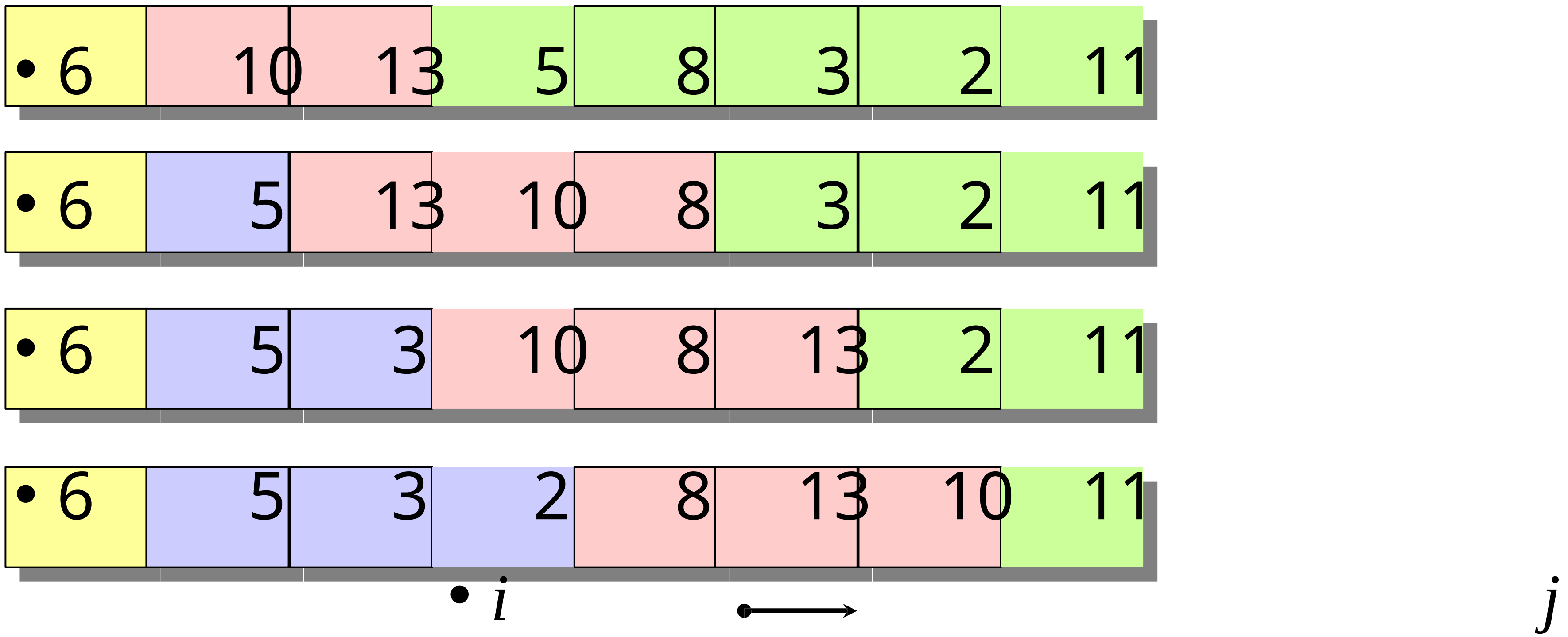

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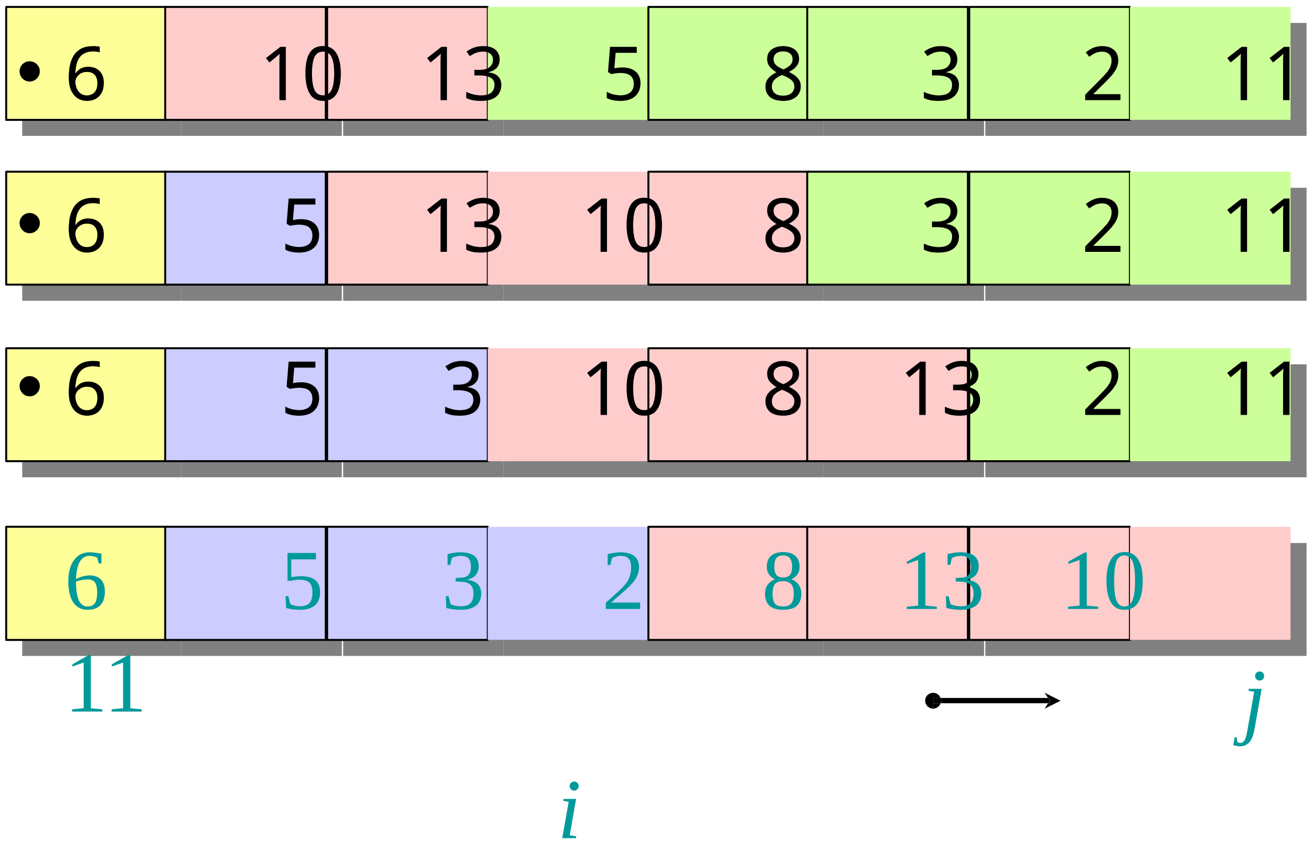

Example of partitioning

- 6 10 13 5 8 3 2 11
- 6 5 13 10 8 3 2 11
- 6 5 3 10 8 13 2 11
- 6 5 3 2 8 13 10 11
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Pseudocode for quicksort

QUICKSORT(A, p, r)
  if $p < r$
    then $q \leftarrow$ PARTITION(A, p, r)
      QUICKSORT$(A, p, q-1)$
      QUICKSORT$(A, q+1, r)$

Initial call: QUICKSORT$(A, 1, n)$

Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \quad \textit{\textbf{(arithmetic series)}}
\end{aligned}
$$

Binary search

- Find an element in a sorted array:

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Recurrence for binary search

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 # subproblems

subproblem size

work dividing and combining

Recurrence for binary search

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 # subproblems

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$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow$ CASE 2 ($k = 0$)

$\Rightarrow \quad T(n) = \Theta(\lg n)$.

Powering a number

Problem: Compute a^n, where $n \in N$.

Naive algorithm: $\Theta(n)$.

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Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

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$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\lg n) .$$

Matrix multiplication

Input: $A = [a_{ij}], B = [b_{ij}].$

Output: $C = [c_{ij}] = A \cdot B.$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \; i, j = 1, 2, \ldots, n.$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Standard algorithm

for $i \leftarrow 1$ **to** n
do for $j \leftarrow 1$ **to** n
do $c_{ij} \leftarrow 0$
for $k \leftarrow 1$ **to** n
do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

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for $i \leftarrow 1$ **to** n

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Running time $= \Theta(n^3)$

Divide-and-conquer algorithm

IDEA:

$n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$C \quad = \quad \cdot$

$A \quad B$

$r = ae + bg$

$s = af +$

$bh \quad t = ce$

$+ dg \quad u =$

$cf + dh$

8 mults of $(n/2) \times (n/2)$ submatrices 4 adds of $(n/2) \times (n/2)$ submatrices

Divide-and-conquer algorithm

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$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dh \\ u &= cf + dg \end{aligned}\right\}$$

recursive

8 mults of $(n/2) \times (n/2)$ submatrices

4 adds of $(n/2) \times (n/2)$ submatrices

Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices

submatrix size

work adding submatrices

Analysis of D&C algorithm

$T(n) = 8\,T(n/2) + \Theta(n^2)$

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$n^{\log_b a} = n^{\log_2 8} = n^3 \quad \Rightarrow \quad$ CASE 1 $\quad \Rightarrow \quad T(n) = \Theta(n^3).$

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$n^{\log_b a} = n^{\log_2 8} = n^3 \quad \Rightarrow \quad$ CASE 1 $\quad \Rightarrow \quad T(n) = \Theta(n^3)$.

No better than the ordinary algorithm.

Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

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$$P_1 = a \cdot (f - h)$$
$$P_2 = (a + b) \cdot h$$
$$P_3 = (c + d) \cdot e$$
$$P_4 = d \cdot (g - e)$$
$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$

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$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e +$$

$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$

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$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!

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$$r = P_5 + P_4 - P_2 + P_6$$
$$= (a + d)(e + h) + d(g - e) - (a + b)h + (b - d)(g + h)$$
$$= ae + ah + de + dh + dg - de - ah - bh$$

Strassen's algorithm

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using $+$ and $–$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $–$ on $(n/2)\times(n/2)$ submatrices.

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The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.

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Best to date (of theoretical interest only): $\Theta(n^{2.376\ldots})$.

Dicrease and Conquer

- Based on exploting the relationship between a solution to a given instance of a problem and a solution to a smaller instance of the same problem.
- After establishing the relationship it can exploited using
  - Top Down and Bottom Up approaches

  - Decrease:- reduce problem instance to smaller instance of the same problem and extend solution.
  - Conquer:- the problem by solving smaller instance of the problem.
  - Extend solution of smaller instance to obtain solution to original problem .

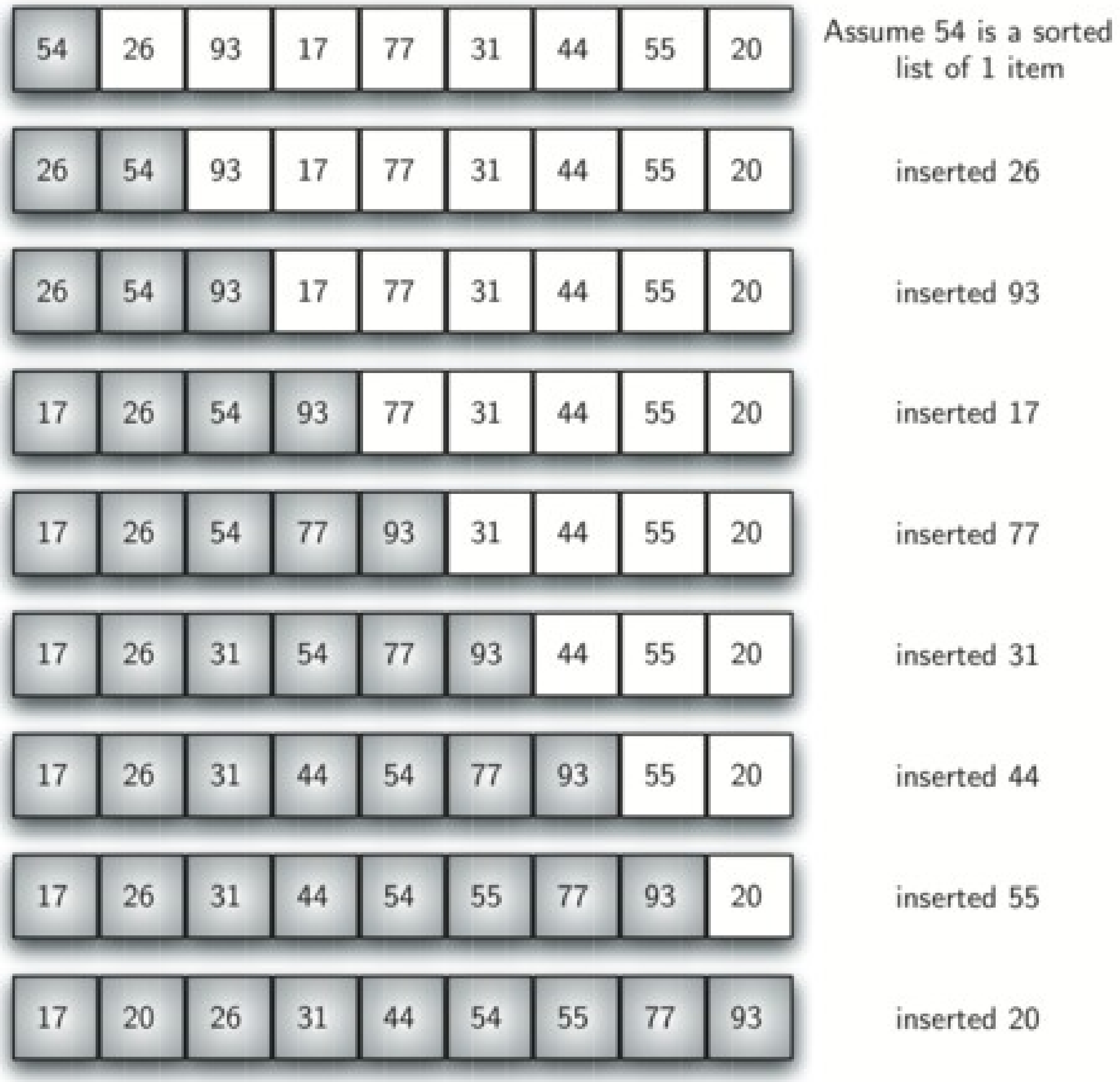
Variations of Decrease and Conquer

- Decrease by Constant
- reduced by the same constant on each iteration of the algorithm. Typically, this constant is equal to one.

- Decrease by Constant factor
- reducing a problem instance by the same constant factor on each iteration of the algorithm. In most applications, this constant factor is equal to two.

- Variable Size Decrease
- the size-reduction pattern varies from one iteration of an algorithm to another.

Insertion Sort



Algorithm of Insertion Sort

```
//Insertion sort
   int arr[]={89,2,67,37,72,17,4};
   int n=sizeof(arr)/sizeof(arr[0]);
   for(int i=1;i<n;i++)
   {
      for(int j=i;j>0;j--)
      {
         if(arr[j]<arr[j-1])
         {
            int temp =arr[j];
            arr[j]=arr[j-1];
            arr[j-1]=temp;
         }
      }
   }
```

Class work

Find the time complexity analysis of Insertion sort?

Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method.
- The divide-and-conquer strategy often leads to efficient algorithms.

Reading Assignment

- Graph search algorithms: DFS, BFS
- Topological sorting