$$Y^{comm} = \left[y_1^{comm}, \ldots, y_A^{comm} \right] \quad K \times A$$

$$Y^{micro} = \left[y_1^{micro}, \ldots, y_M^{micro} \right]_{N_0 \times M}$$

$$vec(Y_{A \times B}) = (\underset{\sim}{y_1}^T, \underset{\sim}{y_2}^T, \ldots, y_B^T)^T_{(AB \times 1)}$$

Similarly,

$$Z = \left[z_1, \ldots, z_R \right]_{N_1 \times R}$$

comm matrix? aggd $\underset{\text{comm}}{\leftarrow}$ x

$$\text{for } a = 1, \ldots, A \quad \overset{\text{comm}\atop\text{matrix?}}{y_a} \sim MVN\left(\underset{\sim}{\alpha} 1 + \underset{K \times P}{X_0} \underset{\sim}{\theta}, (\underset{\sim}{w_a^{comm}})^{-1} \underset{K \times 1}{I}\right)$$

$$\text{for } m = 1, \ldots, M \quad y_m^{micro} \overset{ind}{\sim} MVN(\underset{\sim}{\alpha} 1 + X_0 \underset{\sim}{\theta}, (w_m^{micro})^{-1} I)$$

$$Z_r \overset{ind}{\sim} MVN\left(\alpha \underset{\sim}{1} + X_1 \underset{\sim}{\theta}_1, (\tilde{W}_r^{rank})^{-1}\right)$$

$$\underbrace{\begin{pmatrix} vec(Y^{comm}) \\ vec(Y^{micro}) \\ vec(Z) \end{pmatrix}}_{(AK+MN_0+RN_1)\times 1} \sim MVN\left(\alpha \underset{\sim}{1} + X\underset{\sim}{\theta} + \Sigma\right)$$

call this vector $\underset{\sim}{u}$, for calculations

$$X = \begin{bmatrix} \underset{\sim}{1}_{A\times 1} \otimes X_0^{comm} \\ \underset{\sim}{1}_{M\times 1} \otimes X_0 \\ \underset{\sim}{1}_{R\times 1} \otimes X_1 \end{bmatrix}$$

$$\Sigma^{-1} = diag\left((w^{comm} \otimes 1)^T, (w^{micro} \otimes \underset{\sim}{1}\right)$$

$$\Sigma = \mathrm{diag}\left((w^{comm} \otimes \underset{\sim}{1}), (w^{micro} \times \underset{\sim}{1}), (w^{rank} \otimes \underset{\sim}{1}) \right)$$

Posterior for $\alpha \mid \underset{\sim}{\theta}, \Sigma, Y, Z$

α is a scalar. $\alpha \sim N(0, \sigma_\alpha^2)$

$$f(\alpha \mid \cdot) \propto \exp\left\{ -\frac{1}{2} \left((\underset{\sim}{u} - X\underset{\sim}{\theta} - \alpha \underset{\sim}{1})^T \Sigma^{-1} (\underset{\sim}{u} - X\underset{\sim}{\theta} - \alpha \underset{\sim}{1}) \right) \right\} \times \exp\left(-\frac{1}{2\sigma_\alpha^2} \alpha^2 \right)$$

$$\propto \exp\left\{ -\frac{1}{2} \left[-2 \underset{1\times T}{(\underset{\sim}{u} - X\underset{\sim}{\theta})^T} \underset{T\times T}{\Sigma^{-1}} \underset{T\times 1}{\underset{\sim}{1}} \alpha + \alpha \underset{1\times T}{\underset{\sim}{1}^T} \underset{T\times T}{\Sigma^{-1}} \underset{T\times 1}{\underset{\sim}{1}} \alpha + \frac{1}{\sigma_\alpha^2} \alpha^2 \right] \right\}$$

$$\propto \exp\left\{ -\frac{1}{2} \left[-2 \underbrace{(\underset{\sim}{u} - X\underset{\sim}{\theta})^T \mathrm{diag}(\Sigma^{-1})}_{A} \alpha + \underbrace{\left((\mathrm{diag}(\Sigma^{-1})^T \underset{\sim}{1}) + \frac{1}{\sigma_\alpha^2} \right)}_{B} \alpha^2 \right] \right\}$$

$$\rightarrow \alpha \mid \cdot \sim N(\overline{\mu}_\alpha, \overline{V}_\alpha)$$

$$\overline{V}_\alpha^{-1} = B = K \sum_{a=1}^{A} w_a^{comm} + N_0 \sum_{m=1}^{M} w_m^{micro} + N_1 \sum_{r=1}^{R} w_r^{rank} + \sigma_\alpha^{2^{-1}}$$

$\overline{\mu} =$

$$\bar{\mu}_\alpha = A B =$$

Posterior for $\underset{\sim}{\theta} \mid \alpha, \Sigma, Y, Z$

$\underset{\sim}{\theta}$ is a $P \times 1$ vector of coefficients

$$\underset{\sim}{\theta} \sim MVN(\underset{\sim}{0}, \sigma_\theta^2 I)$$

$$f(\underset{\sim}{\theta} \mid \cdot) \propto \exp\left\{-\frac{1}{2}(\underset{\sim}{u} - \alpha \underset{\sim}{1} - X\underset{\sim}{\theta})^T \Sigma^{-1} (\underset{\sim}{u} - \alpha \underset{\sim}{1} - X\underset{\sim}{\theta})\right\}$$

$$\times \exp\left\{-\frac{1}{2\sigma_\theta^2} \underset{\sim}{\theta}^T \underset{\sim}{\theta}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[-2(\underset{\sim}{u} - \alpha \underset{\sim}{1})^T \Sigma^{-1} X \underset{\sim}{\theta} + \underset{\sim}{\theta}^T X^T \Sigma^{-1} X \underset{\sim}{\theta} + \frac{1}{\sigma_\theta^2} \underset{\sim}{\theta}^T \underset{\sim}{\theta}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[-2\overbrace{(\underset{\sim}{u} - \alpha \underset{\sim}{1})^T \Sigma^{-1} X}^{A} \underset{\sim}{\theta} + \underset{\sim}{\theta}^T \underbrace{\left(X^T \Sigma^{-1} X + \frac{1}{\sigma_\theta^2} I\right)}_{B} \underset{\sim}{\theta}\right]\right\}$$

$$\underset{\sim}{\theta} \mid \cdot \sim MVN(\bar{\mu}_\theta, \bar{V}_\theta)$$

$$\bar{V}_\theta^{-1} = B$$

$$A B^{-1}$$

$$\overline{u}_\theta = AB^{-1}$$