Single-Output Prediction Model

Bayesian additive regression tree (BART)

- Univariate, nonparametric prediction tool
- Sum of T 'weak learner' trees
- Dynamically learns important predictors via a sparsity inducing prior

Mathematically, a BART model looks like:

$$Y_i \sim N(\mu_i, \sigma^2)$$

$$\mu_i = \sum_{t=1}^{T} g(x_i; \tau_t, M_t)$$

- $g()$ tree function which inputs x and outputs prediction
- τ_t: the t^{th} tree structure in the forest
- M_t: node parameters for t^{th} tree

Multi-Output Prediction Model

Shared Forest

- An extension of BART for multivariate responses
- Correlation between responses is utilized via shared tree structures
- Assumption: variables that predict Y_1 are the same variables that predict Y_2 (though the nature of the relationship may be different!)
- Information sharing $\Rightarrow$ better predictions for Y_1 *and* Y_2
- Can model binary and/or continuous outcomes

Multi-Output Prediction Model

1 binary, 1 continuous

The shared forest package accomodates heterogeneous outcomes.

ex) Y = hours worked, δ = gender

$$Y_i \sim N(\mu_i, \sigma_i^2)$$
$$\delta_i \sim Bernoulli(\pi_i)$$

- $(\mu_i \quad \pi_i)$ modeled jointly using a shared forest model

$$\begin{pmatrix} \mu_i \\ \Phi^{-1}(\pi_i) \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{T} g(x_i; \tau_t, M_t^{\mu}) \\ \sum_{t=1}^{T} g(x_i; \tau_t, M_t^{\theta}) \end{pmatrix}$$

where Φ^{-1} is the probit link function.

Multi-Output Prediction Model

1 binary, 1 continuous

- "Information sharing": likelihood for τ_t involves two dimensional response: $\{Y_i, \delta_i\}_{i=1}^N$
- Algorithm's decisions about tree structure (τ_t) considers what is beneficial to both Y and δ
- Predictions of δ are improved by utilizing information provided by Y, vice versa

In our application, we may have two binary responses

ex) δ_1 = gender, δ_2 = employed

$$\delta_{1i} \sim Bernoulli(\pi_{1i}) \tag{1}$$
$$\delta_{2i} \sim Bernoulli(\pi_{2i}) \tag{2}$$

As before, the tree structures will be built using information shared between δ_1 and δ_2, while the location parameters are estimated separately.

$$\begin{pmatrix} \Phi^{-1}(\pi_{1i}) \\ \Phi^{-1}(\pi_{2i}) \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^{T} g(x_i; \tau_t, M_t^1) \\ \sum_{t=1}^{T} g(x_i; \tau_t, M_t^2) \end{pmatrix}$$

We run 100 simulations. In each:

- $n_{train} = 500$, $n_{test} = 500$
- Predictors: $x_1, \ldots, x_{150} \sim Unif(0, 1)$
- True underlying means based on a modification of the "Friedman function"

$$y_i = 10\sin(\pi x_{1i} x_{2i}) + 20(x_{3i} - 0.5)^2 + 10x_{4i} + 5x_{5i} + \epsilon_i^Y$$

$$\delta_i = \begin{cases} 1 & \text{if } 5\sin(\pi x_{1i} x_{2i}) + 25(x_{3i} - 0.5)^2 + 5x_{4i} + 10x_{5i} + \epsilon_i^\delta > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $(\epsilon_i^Y, \epsilon_i^\delta) \overset{iid}{\sim} N(0, 1)$

Simulation Study

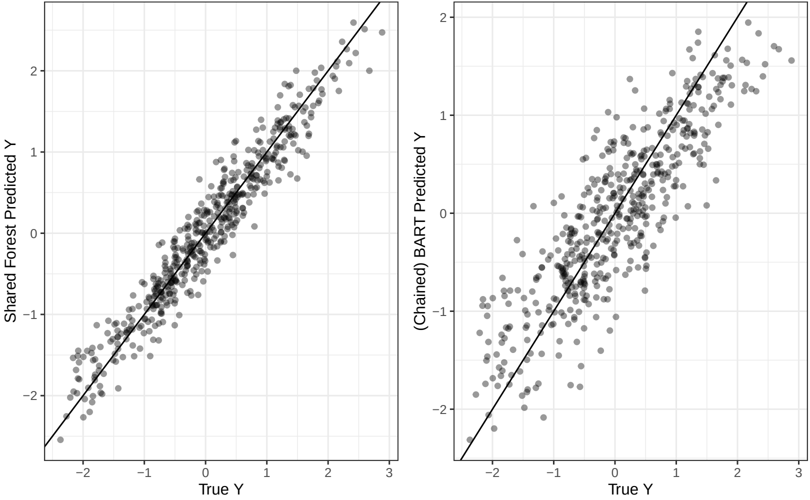
1 binary, 1 continuous

Two quantities may be of interest for prediction

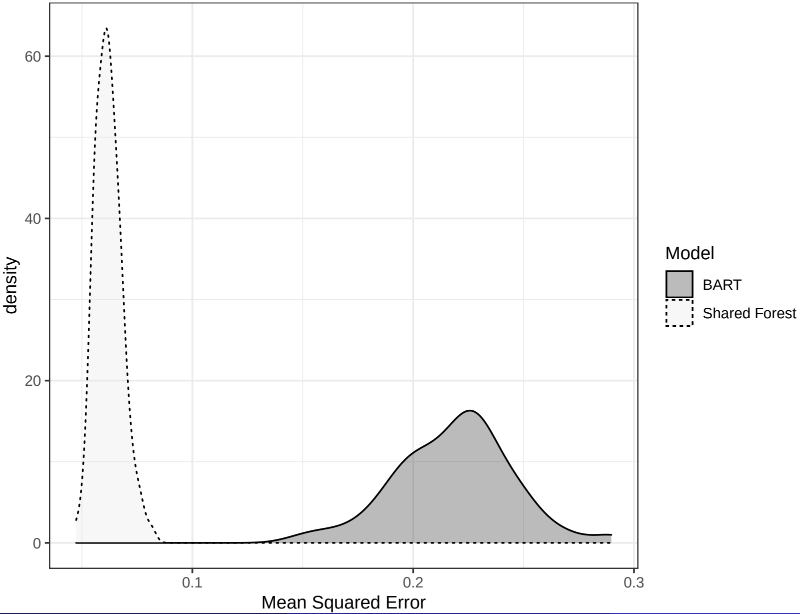
1. $E(y_i^* \mid x_i)$,$E(\delta_i^* \mid x_i)$, where * indicates those observations come from a test / hold-out sample
   - ex) Predict hours worked for individual, conditional on cellular traits x_i
   - ex) Predict gender of individual, conditional on cellular traits x_i
2. $E(y^* \mid \delta = 1) = \frac{1}{\sum I(\delta_i=1)} \sum_{i:\delta_i=1} E(y_i^* \mid x_i)$
   - ex) Predict mean hours worked for females (avg. across traits observed for females, i.e., $\delta = 1$)
   - ex) Predict mean hours worked for males (avg. across traits observed for males, i.e., $\delta = 0$)

Predicting Individual Y^*

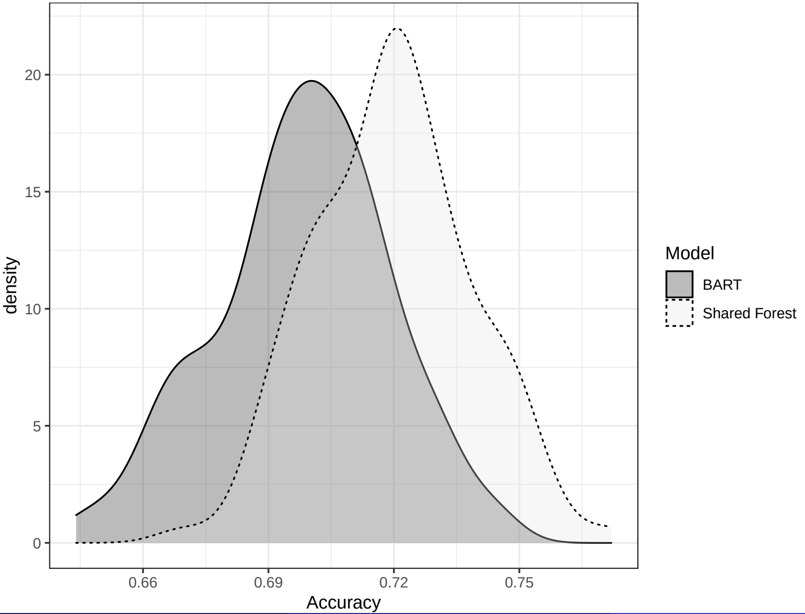
One typical simulation



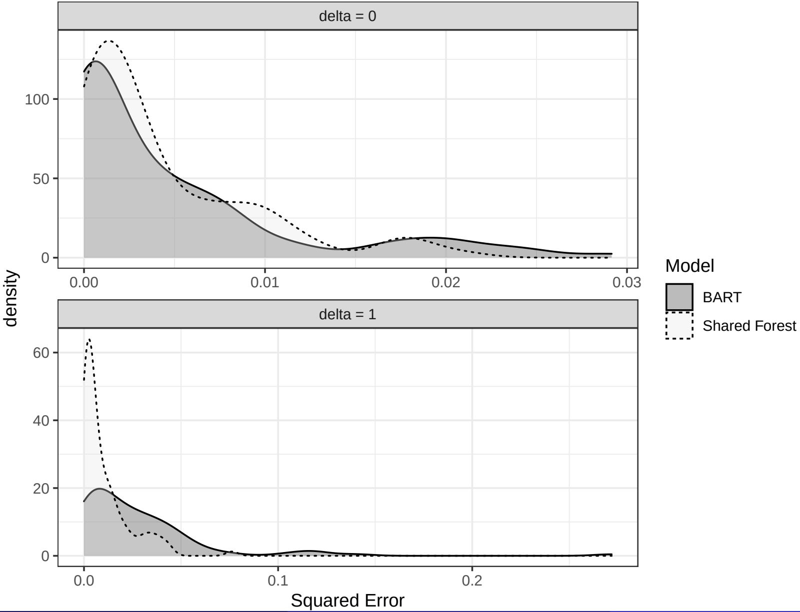
Predicting Individual Y^*



Predicting Individual $\delta*$



Predicting $E(Y^* \mid \delta^*)$



Predicting $E(Y^* \mid \delta^*)$

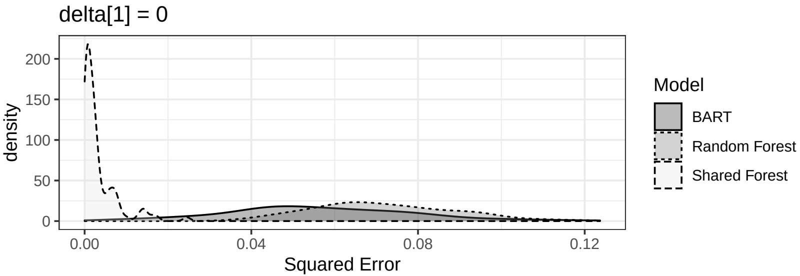
δ	Model	MSE
0	BART	0.004875
0	Shared Forest	0.004330
1	BART	0.028464
1	Shared Forest	0.010399

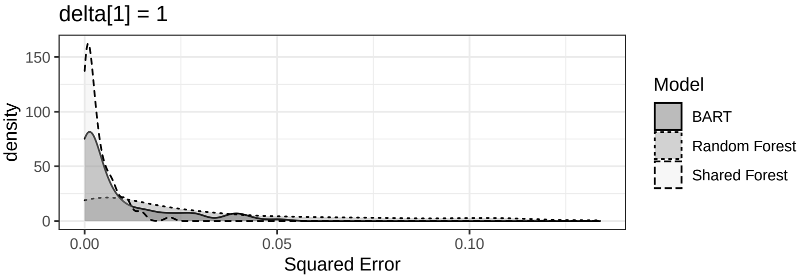
Table: Mean of the squared errors (across simulations) comparing the true $E(Y^* \mid \delta^*)$ to the estimated $\hat{E}(Y^* \mid \hat{\delta})$.

2 binary responses

- In this context, we are interested in predicting $P(\delta_2^* = 1 \mid \delta_1^*)$.
  - ex) Probability a randomly selected female is employed. (δ_1 = gender, δ_2 = employment)
- We measure model performance by looking at the squared errors comparing the true population (conditional) means to the estimated.
- The simulation study is identical, but with two binary responses. Importantly, these responses again have differing mean functions.

Predicting $E(\delta_2^* \mid \delta_1^*)$





Predicting $E(\delta_2^* \mid \delta_1^*)$

Model	MSE	δ_1
BART	0.056580	0
Random Forest	0.071952	0
Shared Forest	0.003252	0
BART	0.008327	1
Random Forest	0.027744	1
Shared Forest	0.003514	1