

Kalman Filter

Assume that $\alpha_t | y_{1:t-1} \sim N(\hat{\alpha}_{t|t-1}, P_{t|t-1})$

Let $v_t = y_t - \hat{y}_{t|t-1}$

Then $\begin{pmatrix} \alpha_t \\ v_t \end{pmatrix} | y_{1:t-1} \sim N \left(\begin{pmatrix} E[\alpha_t | y_{1:t-1}] \\ E[v_t | y_{1:t-1}] \end{pmatrix}, \begin{pmatrix} \text{Cov}(\alpha_t | y_{1:t-1}) & \text{Cov}(\alpha_t, v_t | y_{1:t-1}) \\ \text{Cov}(\alpha_t, v_t | y_{1:t-1})^T & \text{Cov}(v_t | y_{1:t-1}) \end{pmatrix} \right)$

with:

$E[\alpha_t | y_{1:t-1}] = \hat{\alpha}_{t|t-1} \leftarrow \text{From assumption}$

$E[v_t | y_{1:t-1}] = E[y_t - \hat{y}_{t|t-1} | y_{1:t-1}] = E[Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1} | y_{1:t-1}]$
 $= Z E[\alpha_t | y_{1:t-1}] - Z\hat{\alpha}_{t|t-1} = 0.$

$\text{Cov}(\alpha_t | y_{1:t-1}) = E[(\alpha_t - \hat{\alpha}_{t|t-1})(\alpha_t - \hat{\alpha}_{t|t-1})^T | y_{1:t-1}] = P_{t|t-1} \leftarrow \text{From assumption}$

$\text{Cov}(v_t | y_{1:t-1}) = E[v_t v_t^T | y_{1:t-1}] = E[(Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1})(Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1})^T | y_{1:t-1}]$
 $= Z \cdot E[(\alpha_t - \hat{\alpha}_{t|t-1})(\alpha_t - \hat{\alpha}_{t|t-1})^T | y_{1:t-1}] Z^T = Z P_{t|t-1} Z^T$

$\text{Cov}(\alpha_t, v_t | y_{1:t-1}) = E[(\alpha_t - \hat{\alpha}_{t|t-1}) v_t^T | y_{1:t-1}] = E[(\alpha_t - \hat{\alpha}_{t|t-1})(Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1})^T | y_{1:t-1}]$
 $= E[\alpha_t (\alpha_t - \hat{\alpha}_{t|t-1})^T | y_{1:t-1}] Z^T = P_{t|t-1} Z^T$

This gives us

$\begin{pmatrix} \alpha_t \\ v_t \end{pmatrix} | y_{1:t-1} \sim N \left(\begin{pmatrix} \hat{\alpha}_{t|t-1} \\ 0 \end{pmatrix}, \begin{pmatrix} P_{t|t-1} & P_{t|t-1} Z^T \\ Z P_{t|t-1} & \underbrace{Z P_{t|t-1} Z^T}_{= F_{t|t-1}} \end{pmatrix} \right)$

Using the conditional lemma we get that

$\alpha_t | v_t, y_{1:t-1} \sim N(\hat{\alpha}_{t|t-1} + P_{t|t-1} Z^T (F_{t|t-1})^{-1} v_t, P_{t|t-1} - P_{t|t-1} Z^T F_{t|t-1}^{-1} Z P_{t|t-1})$

where $\alpha_t | v_t, y_{1:t-1} = \alpha_t | y_{1:t}$ is The filter distribution.