Exercise session 1

1a) Derive $\hat{y}_{t+2|t}$ for $AR(p)$

$$\hat{y}_{t+2|t} \triangleq \mathbb{E}[y_{t+2} \mid y_{1:t}]$$

zero mean, ind. of $y_{1:t}$

$$= \mathbb{E}[a_1 y_{t+1} + a_2 \underbrace{y_t} + \dots + a_p \underbrace{y_{t+2-p}} + \varepsilon_{t+2} \mid y_{1:t}]$$

known variables

$$= a_1 \underbrace{\mathbb{E}[y_{t+1} \mid y_{1:t}]}_{\triangleq\, \hat{y}_{t+1|t}} + a_2 y_t + \dots + a_p y_{t+2-p}$$

$$= a_1 \hat{y}_{t+1|t} + a_2 y_t + \dots + a_p y_{t+2-p}$$

b) $\hat{y}_{t+k|t} \triangleq \mathbb{E}[y_{t+k} \mid y_{1:t}]$

$$= \sum_{j=1}^{p} a_j \, \mathbb{E}[y_{t+k-j} \mid y_{1:t}]$$

where

$$\mathbb{E}[y_{t+k-j} \mid y_{1:t}] = \begin{cases} \hat{y}_{t+k-j|t} & \text{if } k > j \\ y_{t+k-j} & \text{if } k \leq j \end{cases}$$

if we define $\hat{y}_{s|t} = y_s$ for $s \leq t$ then

$$\hat{y}_{t+k|t} = \sum_{j=1}^{p} a_j \hat{y}_{t+k-j|t}$$

c) From 1b) we have that the prediction of $\hat{y}_{t+k|t}$ depends on the p previous predictions.

Let $\Upsilon_{t+1} = (y_t, \dots, y_{t+1-p})^T$

For $k = 1, \dots, n$

$$\hat{y}_{t+k|t} = \theta^T \Upsilon_{t+k}$$

$$\Upsilon_{t+k-1} = \begin{pmatrix} \hat{y}_{t+k|t} \\ [\Upsilon_{t+k}]_{1:p-1} \end{pmatrix}$$

Here $\theta = (a_1, \dots, a_p)^T$

2a) $y_t = a y_{t-1} + \varepsilon_t$

$\mu_t = \mathbb{E}[y_t] = \mathbb{E}[a y_{t-1} + \varepsilon_t] = a \cdot \mu_{t-1}$

Since $\mu_1 = \mathbb{E}[y_1] = c$ we get that

$\mu_t = a^{t-1} \cdot c$

b) $V[y_1] = 0$ since y_1 is deterministic

$V[y_t] = V[a y_{t-1} + \varepsilon_t] = a^2 V[y_{t-1}] + \sigma_\varepsilon^2$

iid.

$= a^2 \left(a^2 V[y_{t-2}] + \sigma_\varepsilon^2 \right) + \sigma_\varepsilon^2 = \dots = \sigma_\varepsilon^2 \sum_{j=1}^{t-1} a^{2(j-1)}$

c) No, if $a \neq 1$ then the mean is changing.

If $a = 1$ then the variance is not a constant

d) Mean: if $|a| < 1$ then $\mu_t \to 0$

if $|a| > 1$ then $|\mu_t| \to \infty$

Variance:

When $t \to \infty$ we get a geometric series

$$\sigma_\xi^2 \sum_{j=1}^{\infty} a^{2(j-1)} = \frac{\sigma_\xi^2}{1-a^2} \quad \text{if } |a| < 1$$

if $|a| \geq 1$ then $V[y_t] \to \infty$

3 a) $y_t = y_{t-1} + \varepsilon_t$

$= y_{t-2} + \varepsilon_{t-1} + \varepsilon_t$

$= \dots\dots = y_1 + \varepsilon_2 + \dots + \varepsilon_t$

$= y_1 + \sum_{j=2}^{t} \varepsilon_j$ ← all ε_i are mutually independant

b) $\mathbb{E}[y_t] = \mathbb{E}[y_1 + \sum_{j=2}^{t} \varepsilon_j] = \underbrace{\mathbb{E}[y_1]}_{=c} + \sum_{j=2}^{t} \underbrace{\mathbb{E}[\varepsilon_j]}_{=0}$

$= c$

$V[y_t] = V[y_1 + \sum_{j=2}^{t} \varepsilon_j] = \underbrace{V[y_1]}_{=0} + \sum_{j=2}^{t} \underbrace{V[\varepsilon_j]}_{=\sigma_\varepsilon^2}$

$= (t-1)\sigma_\varepsilon^2$