# Time Series and Sequence Learning

## Stationarity, Empirical ACF

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## Stationary time series

**Def.** A stochastic process $\{y_t\}_{t \geq 1}$ is said to be strictly stationary if, for all $t_1, \ldots, t_n$ and all $h \geq 0$

$$p(y_{t_1}, \ldots, y_{t_n}) = p(y_{t_1+h}, \ldots, y_{t_n+h}).$$

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**Def.** A stochastic process $\{y_t\}_{t \geq 1}$ is said to be (weakly) stationary if, for all $t$,

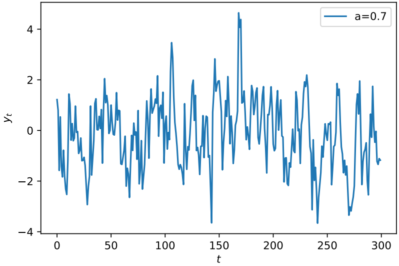
1. $\mathrm{Var}(y_t) < \infty$,
2. $\mu(t) = \text{const.}$,
3. The autocovariance function depends only on the time lag,

$$\gamma(t, t+h) =: \gamma(h) \qquad \text{for all } h.$$

# First-order AR

**ex)** A first-order AR model $y_t = ay_{t-1} + \varepsilon_t$ is **(weakly) stationary** iff

1. $|a| < 1$ $\leftarrow$ ***key requirement!***
2. $\mu(1) = 0$ and $\gamma(1,1) = \frac{\sigma_\varepsilon^2}{1-a^2}$



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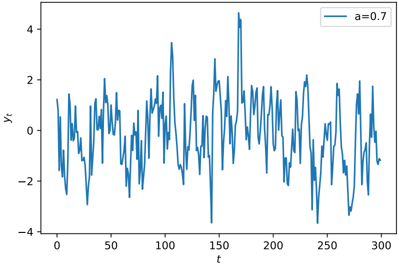


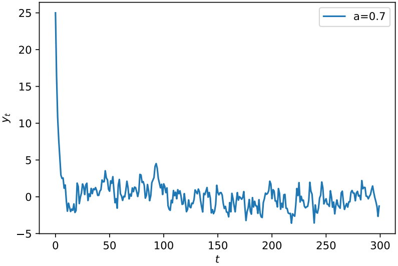


**N.B.** If the second requirement is not fulfilled, the process will still **converge to stationarity** for large $t$.

# First-order AR

**ex)** For a first-order AR model $y_t = ay_{t-1} + \varepsilon_t$ with

$$y_1 \sim \mathcal{N}\left(0, \frac{\sigma_\varepsilon^2}{1-a^2}\right), \qquad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_\varepsilon^2\right),$$

all marginal distributions are Gaussian.

If $|a| < 1$, then the process is strictly stationary.

If $|a| \geq 1$, then the variance of the process grows without bound at a rate which is

- Linear if $|a| = 1$,
- Exponential if $|a| > 1$.

Such a process is said to be **unstable!**

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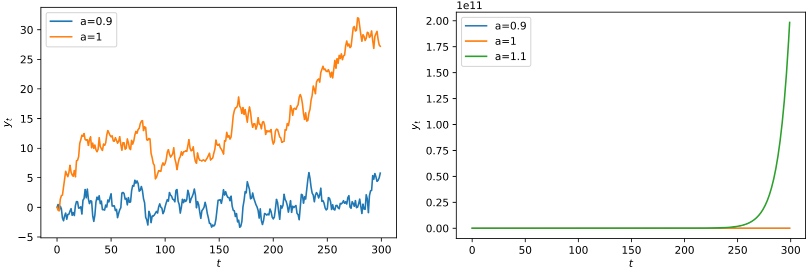

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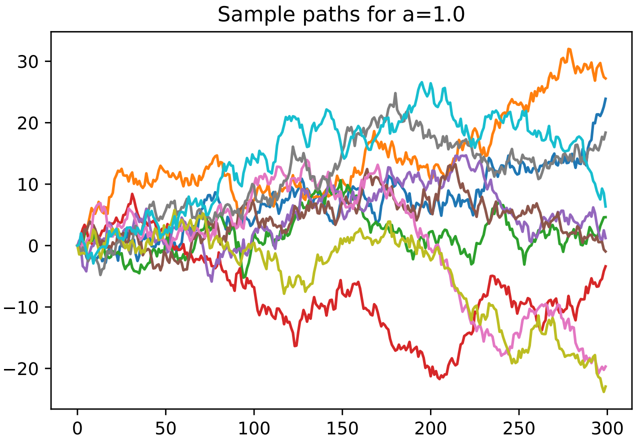

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# Autocorrelation function of AR(1)

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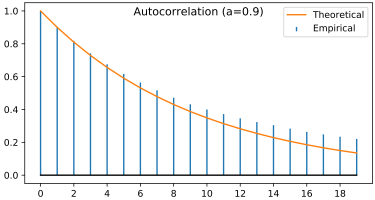

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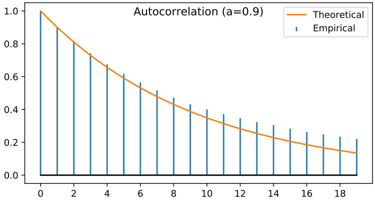


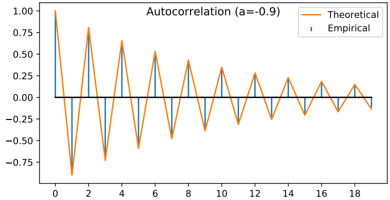

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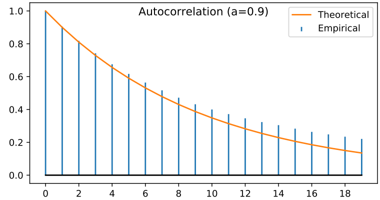


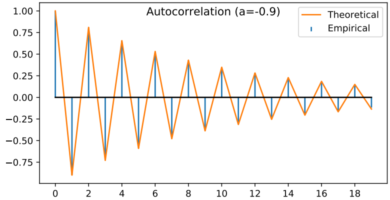


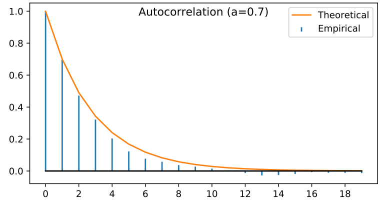

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