Cooperative games and Shapley value

In this report, we look at cooperative games and how to solve them using the core and the Shapley value.

Besides introducing terms, theorems and definitions, we try to decode everything in an intuitive way by giving explanation and examples.

1. Cooperative games and core

In cooperative games, players are allowed to communication, form coalitions and make binding agreements. We focus on what groups of players can achieve rather than individuals.

1. 1 Cooperative games with transferable utility

Cooperative games

For some games, outcomes are specified in terms of a utility that can be divided among a coalition. These are games with transferable utility. The following assumptions are made in a coalitional game with transferable utility:

1. the payoffs may be freely redistributed among its members
2. a universal currency is used for exchange in the system
3. each coalition can be assigned a single value as its payoff

The coalitional form and characteristic function

The coalition form of an cooperaative game is given by the pair (N, v), where

1. $N = \{1, 2, \ldots, n\}$, a set of players
2. v is the characteristic function that associate each coalition with a payoff.

   It is a real-valued function defined on the set 2^N (all subsets of N) and satisfying:

   1. $v(\emptyset) = 0$, meaning that the empty set has payoff 0
   2. if $S \bigcap T = \emptyset$, then $v(S) + v(T) \leq v(S + T)$, meaning that the larger the coalition, the higher the payoff. A game satifies this constraint is superadditive.

Example:

A group of customers must be connected to a power plant provided by a central facility. A customer can either connect to the facility directly, or connect to some other connected customer.

In this coalitional game (N, v), N is the set of customers, and $v(S) =$(the cost of connecting all customers directly to the facility)$-$(the cost of the minimum spanning tree that connects all customers in S and the facility)

1.2 Payoff vector and the core

For a coalition game, we are interested in which coalition to form, and how that coalition divides its payoff among its players. According to superadditivity, this coalition should be N. We then need to figure out how the payoff $v(N)$ should be split.

Payoff vector

Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ denote the payoff division, where player i 's payoff is x_i. Then $\vec{x}$ has two properties:

1. $\sum_{i=1}^{n} x_i = v(N)$: in a superadditive game, coalition N has the highest overall payoff
2. $x_i \geq v(\{i\})$ for all i: a player will only receive payoff that is at least the same as the payoff he could obtain by himself.

Suppose we have a coalition S with payoff $v(S)$. For a $\vec{x}$, let the total return be $x(S) = \sum_{i \in S} x_i$.

A payoff vector $\vec{x}$ is unstable through a coliation S if $v(S) > \sum_{i \in S} x_i$. The payoff vector is only stable when $x(S) \geq v(S)$; otherwise there must exist $y(S) = v(S)$ where the members could receive more than in $\vec{x}$.

Core

The **core** of a coalitional game is the set C of all stable payoff vectors:

$$C = \{\vec{x} = (x_1, \ldots, x_n) : \sum_{i \in N} x_i = v(N) \text{ and } x(S) \geq v(S) \text{ forall } S \in N\}$$

In words, the core is the set of feasible payoff divisions for the grand coalition that no coalition can upset.

Example:

Consider the following three-person game. Three players can together obtain \$1 to share, any two can obtain α , and a singer player gets nothing.

$$v(N) = 1,\ v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = \alpha,\ v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

The core of the game is
$$\{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_1 + x_2 \geq \alpha, x_1 + x_3 \geq \alpha, x_2 + x_3 \geq \alpha, x_1 > 0, x_2 > 0, x_3 > 0\}$$

If $\alpha > \frac{2}{3}$, the core is empty. If $\alpha = \frac{2}{3}$, the core is $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$. Otherwise the core consists of infinite points.

2. The Shapley Value and its computation

The core of a game only presents a set of payoff distribution. It doesn't distinguish one point of the set as preferable to another, i.e., it doesn't show which the best distribution is. Besides, the core can sometimes be empty or very large.

The Shapley value solves the following problem: how the payoff should be split among players. It's a function from the game space to outcomes. It considers the contribution of each player in the coalition, and a game always has a solution.

2.1 The Shapley axioms

Value function

A value function $\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ is a function that assigns each (N, v) an n-tuple of real numbers, where $\phi_i(v)$ is the payoff of player i in the game.

The Shapley Axioms for $\phi(v)$:

1. Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$. The payoff of each player add up to $v(N)$.
2. Symmetry: if $i, j \notin S$ and $v(S \bigcup i) = v(S \bigcup j)$, then $\phi_i(v) = \phi_j(v)$. If two players contribute the same to each coalition, they should get the same payoff
3. Dummy Axiom: if $v(S \bigcup i) = v(S)$, then $\phi_i(v) = 0$. If a player contributions nothing to any coalition, he gets nothing.
4. Additivity: for payoff function u and v, $\phi(u + v) = \phi(u) + \phi(v)$. The solution to the sum of two games must be equal to the sum of payoff of each game.

2.2 The Shapley value

Theorem

Given a coalitional game (N, v), there is a unique payoff division that satisfies the Shapley axioms. It is what we call the Shapley value, a function that assigns to each player i the payoff

$\phi_i(v) = \sum_{S \subset N} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} [v(S) - v(S - \{i\})]$

Suppose that all players are randomly ordered in a line, all orders being equally likely. This formula assigns to each player his average marginal contribution to the grand coalition.

Interpretation

Imagine that the coalition is assembled by starting with $\emptyset$ and adding one random player at a time.

1. The average of player i's marginal contribution to S = (i's marginal contribution to coalition S) $\times$ (probability that when i is added the coalition becomes S)
    1. When player i is added and he forms S, his marginal contribution is $v(S) - v(S - \{i\})$
    2. Before i is added, there are $|S| - 1$ players and $(|S| - 1)!$ ways for them to be added. After i is added, there are $|N| - |S|$ players left and therefore $(|N| - |S|)!$ ways to add them.

       The total number of sequence to add all players is $|N|!$

       Hence, the probability S if formed the when i joins is $\frac{(|S|-1)!(|N|-|S|)!}{|N|!}$
2. In the end, consider all possible coalitions S that is formed and sum over them.

Proof

The proof consists of two parts.

First, we show that the formula satisfies all axioms.

1. As long as we fix an ordering, the sum of the marginal contributions of all players must be $v(N)$, so the sum of the average marginal contributions over all sequences must also be $v(N)$.
2. If two players contribute the same to each S, then their marginal contribution to each S are also the same, and thus their payoff are the same.
3. If a player contribution nothing to any coalition, then his marginal contribution is always 0, so he gets nothing.
4. The average marginal contribution of a sum of two games is always the same as the sum of marginal contributions in each game.

Then, we show that there exists exactly one mapping ϕ for v that satisfies the axioms. We prove this by prooving that the results from the formula is a basis of $v(N)$ through mapping ϕ.

1. Prove that ϕ is a linear transformation.

   From the addivity axiom, $\phi(u + v) = \phi(u) + \phi(v)$. Next, $\phi(pv) = \phi(v + \ldots + v) = p\phi(v)$. Therefore, the mapping ϕ is a linear transformation.
2. Prove that vectors in the formula are linearly independent.

   Let $L(N)$ denote the set all of non-empty subsets of N. $|L(N)| = 2^n - 1$ and hence $\mathbb{R}^{|L(N)|}$ is a $(2^n - 1)$-dimensional vector space.

   Let $S \subseteq N$ be any non-empty coalition. Define a coalitional game (N, w_S):

   $$w_S(T) = \begin{cases} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise} \end{cases}$$

   This implies that a coalition T has payoff 1 in the characteristic function if it contains all players in S and has payoff 0 otherwise.

   From axiom 2, if both i, j are in S, then $\phi_i(w_S) = \phi)j(w_S)$. From axiom 1, $\sum_{i \in N} \phi_i(w_S) = w_S(N) = 1$, so $\forall i \in S, \phi_s(w_S) = \frac{1}{|S|}$. From axiom 3, if $i \notin S$, $\phi_i(w_S) = 0$. Therefore,

$$\phi_i(w_S) = \begin{cases} \frac{1}{|S|}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

For each coalition S, there is a game (N, w_S) and therefore we have $2^n - 1$ games. Each game is represented by a characteristic function which is a vector of $2^n - 1$ elements. Thus we have $2^n - 1$ vectors, each of size $2^n - 1$. We now can show that these are linearly independent in the space $\mathbb{R}^{2^n-1}$.

To show linear independence, we have to prove that $\sum_{S \in L(N)} \lambda_S w_S = \vec{0} \implies \forall S \in L(N), \lambda_S = 0$. We prove this by contradiction.

Suppose the above implication is not true. Then $\sum_{S \in L(N)} \lambda_S w_S = \vec{0} \implies \forall T \subseteq N, \lambda_S w_S(T) = 0$.

Let $T \subseteq N$ be any coalition of minimal size for which $\lambda_T \neq 0$. Since S is either a subset of T or not,

$\sum_{S \subseteq T} \lambda_S w_S(T) + \sum_{\neg S \subseteq T} \lambda_S w_S(T) = 0 \implies \forall T \subseteq N, \sum_{S \in L(N)} \lambda_S w_S(T) = 0$

Since $\forall S \subseteq T, w_S(T) = 1$ and $w_S(T) = 0$ otherwise, $\sum_{S \subseteq T} \lambda_S w_S(T) = \sum_{S \subseteq T} \lambda_S = 0$.

According to our assumption, because T is a minimal size subset for which $\lambda_T \neq 0$, we have $\forall S \subset T, \lambda_S = 0$.

However, $\sum_{S \subseteq T} \lambda_S = \lambda_T$, which means $\lambda_T = 0$. This is a contradiction.

3. Hence, set $\{w_S : S \in L(N)\}$ is a basis of $\mathbb{R}^{2^n-1}$, and the mapping ϕ is unique.

3. A comparison between core and Shapley value

Core

- Based on coalitional threats - each coalition must get at least what it can get alone
- Under what payment divisions would the agents want to form the grand coalition

Shapley value

- Based on marginal contribution: what each player contributes to each possible coalition
- A fair way of dividing the grand coalition's payment among its members

Example: Compare core and Shapley value

Suppose there is a security council of 3 members: 1 permanent member with a veto and 2 temporary members. To pass a resolution, the permanent member must agree and a majority is needed.

Now the council tries to pass a $1 million bill. We represent this as an cooperative game. Let the permanent member be player 1, and temporary members are player 2 and 3.

$v(S) = 1$ if $1 \in S$ and $|S| \geq 2$; $v(S) = 0$ otherwise.

- Core:
  $\{(x_1, x_2, x_3) : x_i \geq 0, x_1 + x2 \geq 1, x_1 + x_3 \geq 1, x_1 + x_2 + x_3 = 1\} \implies x_1 = 1, x_2 = x_3 = 0$
- Shapley value:

  Player 1:
  $(v(\{1,2,3\}) - v(\{2,3\})) \cdot \frac{4}{6} + (v(\{1,2\}) - v(\{2\})) \cdot \frac{1}{6} + (v(\{1,3\}) - v(\{3\})) \cdot \frac{1}{6} = \frac{2}{3}$

  Player 2:
  $(v(\{1,2,3\}) - v(\{1,3\})) \cdot \frac{4}{6} + (v(\{1,2\}) - v(\{1\})) \cdot \frac{1}{6} + (v(\{2,3\}) - v(\{3\})) \cdot \frac{1}{6} = \frac{1}{6}$

  Player 3:
  $(v(\{1,2,3\}) - v(\{1,2\})) \cdot \frac{4}{6} + (v(\{1,3\}) - v(\{1\})) \cdot \frac{1}{6} + (v(\{2,3\}) - v(\{2\})) \cdot \frac{1}{6} = \frac{1}{6}$
  $\implies x_1 = \frac{2}{3}, x_2 = x_3 = \frac{1}{6}$