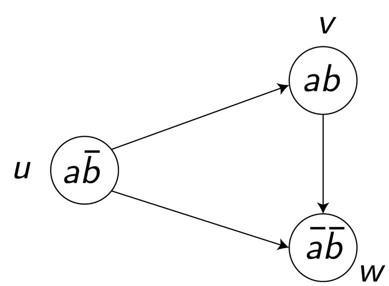
Lab session 9: An overview of other logics

Pierre-Alexandre Murena

June 14, 2024

1 Kripke models

We consider the following Kripke model:



where the possible worlds are denoted $W = \{u, v, w\}$ and the accessibility relation R is denoted by the arrows.

Question 1. Which of the following statements are true with this model $\mathcal{I}$?

1. $\mathcal{I}, u \models \Diamond(a \wedge \Diamond\neg a)$
2. $\mathcal{I}, v \models \Box\neg a$
3. $\mathcal{I}, w \models \Diamond a \vee b$
4. $\mathcal{I}, w \models \Box a \vee b$
5. $\mathcal{I}, u \models \Box((a \leftrightarrow b) \wedge \Diamond\neg a)$

2 Temporal logic

In this exercise, the goal is to build a temporal logic where the modal operators read:

- $\Box\phi$: Always ϕ
- $\Diamond\phi$: Sometimes ϕ.

Question 2. For each of the following statement, propose two sentences in the language expressing the statement, one containing only $\Box$ as a modal operator and the other one containing only $\Diamond$ as a modal operator: (S-1) ϕ is always false; and (S-2) ψ is always true, but $\phi \rightarrow \psi$ is true only from time to time.

In order to define the semantic of this logic, we need to choose a Kripke model. We consider worlds in the space $W = \mathbb{Z}$. This can be interpreted as having the world w corresponding to the date. We define the Kripke relational frame by defining the accessibility relations, i.e. choosing which words are accessible from which world. Here, we consider that, from w, one can access only to worlds with a larger value.

Question 3. Express this as a mathematical condition on vRw.

In order to build the Kripke model from the chosen Kripke relational frame, one needs to attribute an interpretation of each propositional variable in each world. Given the set of propositional variables $\mathcal{V}$, we denote by π_w the function $\mathcal{V} \rightarrow \{True, False\}$ which applies in world w. For instance π_{42} is the interpretation of variables at time 42.

We denote by $\mathcal{I}$ the tuple $\mathcal{I} = \langle W, R, \pi \rangle$.

Question 4. What would be an interpretation (in English) of $\mathcal{I}, w \models \phi$? (see slide 21 of the lecture)

Unlike for propositional logic, it is not possible to say that a sentence is true *in general*. The truth of a sentence now depends on the structure of the chosen Kripke relational frame. Given a Kripke relational frame $\mathcal{F}$ (i.e. the world space and the relation between worlds), we say that $\mathcal{F} \models \phi$ if ϕ is true for all interpretations $\mathcal{I}$ based on $\mathcal{F}$.

Question 5. If $\mathcal{V} = \{a, b\}$, do we have $\mathcal{F} \models a$? $\mathcal{F} \models \Diamond a$? $\mathcal{F} \models a \vee \neg a$?

We will now show that for any ϕ, we do not have $\mathcal{F} \models \Box\phi \rightarrow \phi$. To do so, we first notice that $\Box\phi \rightarrow \phi$ is logically equivalent to $\phi \vee \neg\Box\phi$. Now, we will construct an interpretation based on $\mathcal{F}$ that falsifies $\Box\phi \rightarrow \phi$.

Let us fix a world w.

Question 6. Consider the case $\mathcal{V} = \{a\}$. Fix a world w. Can you define a π such that $\mathcal{I}, w \models \Box a$ and $\mathcal{I}, w \not\models a$? Conclude.