Proof Sketch: Irrationality of e

1 The sequence $a_n = 1/n!$

Lemma 1 (Key bound on the factorial function)**.** *For all $n > 0$ and $k \in \mathbb{N}$,*

$$(n+k)! \geq 2^k \cdot n!.$$

Proof. The proof proceeds by induction on k. For the base case $k = 0$, we have $(n+0)! = 2^0 \cdot n!$. For the inductive step, assume $(n+k)! \geq 2^k \cdot n!$. Then

$$\begin{aligned}
(n+k+1)! &= (n+k+1) \cdot (n+k)! \\
&\geq (n+k+1) \cdot 2^k \cdot n! \\
&\geq 2 \cdot 2^k \cdot n! = 2^{k+1} \cdot n!,
\end{aligned}$$

where the last inequality uses that $n+k+1 \geq 2$ when $n > 0$. □

Remark: the condition that $n > 0$ cannot be dropped.

Definition 1. *Set $a_n = 1/n!$. This defines a function $a \colon \mathbb{N} \to \mathbb{R}, n \mapsto a_n$.*

Lemma 2 (Key bound on a_n)**.** *For all $n \geq 1$ and $k \in \mathbb{N}$, we have*

$$a_{n+k} \leq \left(\frac{1}{2}\right)^k \cdot a_n.$$

Proof. Follows from Lemma 1, using the fact that both sides in the inequality in Lemma 1 are positive. □

2 The partial sums $s_n = a_0 + \cdots + a_{n-1}$

Define the partial sums

$$s_n = \sum_{i=0}^{n-1} a_i = \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{(n-1)!} \in \mathbb{R}$$

for $n \in \mathbb{N}$. Note that there are n terms in the sum. We follow the usual convention that $s_0 = 0$ (empty sum).

The key inequality for the partial sums is:

Definition 2. *Consider the partial sum of the geometric series*

$$g_k = 1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{k-1},$$

where again $g_0 = 0$ (empty sum).

Lemma 3. *$g_k < 2$.*

Proof. Indeed, we have $g_k = 2 - (1/2)^{k-1}$. □

Lemma 4. *For all m and all $n \geq 1$ and $k \in \mathbb{N}$, we have*

$$s_{n+k} \leq s_n + a_n \cdot g_k.$$

Proof. The proof proceeds by induction on k. For $k = 0$, we have $s_{n+0} = s_n + a_n \cdot g_0$.

For the inductive step, assume $s_{n+k} \leq s_n + a_n \cdot g_k$. Then

$$\begin{aligned}
s_{n+k+1} &= s_{n+k} + a_{n+k} \\
&\leq s_n + a_n \cdot g_k + a_{n+k} \\
&\leq s_n + a_n \cdot g_k + \left(\frac{1}{2}\right)^k \cdot a_n \\
&= s_n + a_n \cdot \left(g_k + \left(\frac{1}{2}\right)^k\right) \\
&= s_n + a_n \cdot g_{k+1},
\end{aligned}$$

where we used the bound $a_{n+k} \leq (1/2)^k \cdot a_n$ from Lemma 2. □

Lemma 5. *For all $n \geq 1$ and $k \in \mathbb{N}$,*

$$s_{n+k} < s_n + 2 \cdot a_n.$$

Proof. Follows from Lemma 4 and Lemma 3. □

It will be convenient to establish the following variant:

Lemma 6. *For all $n \geq 1$ and all $m \geq 0$ we have*

$$s_m < s_n + 2 \cdot a_n$$

Proof. Follows by case distinction: if $m > n$ then this is Lemma 5, and if $m \leq n$ then this follows from $s_m \leq s_n$ and $a_n > 0$. □

3 e as limit of s_n

The number e is defined as the limit of the sequence (s_n):

$$e = \lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Since (s_n) is a strictly increasing sequence (as $a_n > 0$ for all n) and bounded above (by Lemma 6), the limit exists.

Lemma 7 (Characterisation of e)**.** *The number e satisfies:*

1. *For all n, we have $s_n < e$;*
2. *If $c \in \mathbb{R}$ and $s_n \leq c$ for all n, then $e \leq c$.*

Moreover, e is the only *real number that satisfies these two properties.* □

Combining the second point with Lemma 6 we obtain:

Theorem 1 (Key bound on e)**.** *For all $n > 0$ we have $e \leq s_n + 2 \cdot a_n$*

4 Proof that $n! \cdot s_{n+1}$ is an integer, but $n! \cdot e$ cannot be

Define the tail of the series:

$$t_n = e - s_n = \sum_{i=n}^{\infty} \frac{1}{i!}.$$

Note that $t_n > 0$ for all n (since $s_n < e$), and from the bound above, we have $t_n \leq 2 \cdot a_n$ for $n \geq 1$.

The key ingredients are:

Lemma 8 (Integrality of $n! \cdot s_{n+1}$)**.** *For all $n \in \mathbb{N}$, the number $n! \cdot s_{n+1}$ is an integer.*

Proof sketch. We have

$$n! \cdot s_{n+1} = n! \cdot \sum_{i=0}^{n} \frac{1}{i!} = \sum_{i=0}^{n} \frac{n!}{i!}.$$

For $i \leq n$, we have $n!/i! = n \cdot (n-1) \cdots (i+1) \in \mathbb{N}$, so the sum is an integer. □

Lemma 9 (Bound on $n! \cdot t_{n+1}$)**.** *For all $n \geq 2$,*

$$0 < n! \cdot t_{n+1} < 1.$$

Proof sketch. The positivity follows from $t_{n+1} > 0$. For the upper bound, we use:

$$\begin{aligned} n! \cdot t_{n+1} &\leq n! \cdot 2 \cdot a_{n+1} \\ &= n! \cdot 2 \cdot \frac{1}{(n+1)!} \\ &= \frac{2}{n+1} \\ &\leq \frac{2}{3} < 1, \end{aligned}$$

where the last inequality uses $n \geq 2$. □

Theorem 2 (Non-integrality of $n! \cdot e$)**.** *For all $n \geq 2$ and any integer N, we have $n! \cdot e \neq N$.*

Proof sketch. Suppose for contradiction that $n! \cdot e = N$ for some integer N. Since $n! \cdot s_{n+1}$ is an integer (say M), we have

$$n! \cdot t_{n+1} = n! \cdot (e - s_{n+1}) = N - M \in \mathbb{Z}.$$

But $0 < n! \cdot t_{n+1} < 1$, and there are no integers strictly between 0 and 1, which is a contradiction. □

5 Proof that e is not rational

Theorem 3 (Irrationality of e)**.** *The number e is irrational.*

Proof sketch. Suppose for contradiction that $e = p/q$ for some natural numbers p, q with $q > 0$.

Then for $n = q + 1 \geq 2$, we have:

$$(q+1)! \cdot e = (q+1)! \cdot \frac{p}{q} = (q+1) \cdot q! \cdot \frac{p}{q} = (q+1) \cdot (q-1)! \cdot p \in \mathbb{N}.$$

But by the previous theorem, $(q+1)! \cdot e$ cannot be an integer, which is a contradiction. □