Proof Sketch: Irrationality of e

1 Bound on the factorial function

Theorem 1 (`fac_bound`). *For all $n > 0$ and $k \in \mathbb{N}$,*

$$(n+k)! \geq 2^k \cdot n!.$$

Proof. Induction on k. For $k = 0$, clear. Assume $(n+k)! \geq 2^k \cdot n!$. Then

$$\begin{aligned}(n+k+1)! &= (n+k+1) \cdot (n+k)! \\ &\geq (n+k+1) \cdot 2^k \cdot n! \\ &\geq 2 \cdot 2^k \cdot n! = 2^{k+1} \cdot n!,\end{aligned}$$

where the last inequality uses that $n + k + 1 \geq 2$ when $n > 0$. □

2 The sequence $a_n = 1/n!$

Definition 1. *Set $a_n = 1/n!$. This defines a function $a\colon \mathbb{N} \to \mathbb{R}, n \mapsto a_n$.*

Theorem 2 (`a_bound`). *For all $n \geq 1$ and $k \in \mathbb{N}$, we have*

$$a_{n+k} \leq \left(\frac{1}{2}\right)^k \cdot a_n.$$

Proof. Follows from Theorem 1, using the fact that both sides in the inequality in Theorem 1 are positive. □

3 Bounding the geometric series g_n

Define

$$g_n = \sum_{i=0}^{n-1} (\frac{1}{2})^i = 1 + \cdots + (\frac{1}{2})^{n-1}$$

Note that there are n terms in the sum. By convention, $g_0 = 0$ (empty sum).

Lemma 1 (`g_formula`). $g_n = 2 - 2 \cdot (1/2)^n$. □

Corollary 1 (`g_lt_2`). $g_n < 2$. □

4 The partial sums s_n

Define the partial sums

$$s_n = \sum_{i=0}^{n-1} a_i = \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{(n-1)!} \in \mathbb{R}$$

Note that there are n terms in the sum.

Theorem 3 (`s_geometric_bound`)**.** *For all m and all $n \geq 1$ and $k \in \mathbb{N}$, we have*

$$s_{n+k} \leq s_n + a_n \cdot g_k.$$

Proof. The proof proceeds by induction on k. For $k = 0$, this is clear. Assume $s_{n+k} \leq s_n + a_n \cdot g_k$. Then

$$\begin{aligned} s_{n+k+1} &= s_{n+k} + a_{n+k} \\ &\leq s_n + a_n \cdot g_k + a_{n+k} \\ &\leq s_n + a_n \cdot g_k + \left(\frac{1}{2}\right)^k \cdot a_n \\ &= s_n + a_n \cdot \left(g_k + \left(\frac{1}{2}\right)^k\right) \\ &= s_n + a_n \cdot g_{k+1}, \end{aligned}$$

where we used the bound $a_{n+k} \leq (1/2)^k \cdot a_n$ from Theorem 2. □

It will be convenient to establish the following variant:

Theorem 4 (`s_key_bound`)**.** *For all $n \geq 1$ and all $m \geq 0$ we have*

$$s_m < s_n + 2 \cdot a_n$$

Proof. Follows by case distinction: if $m \geq n$ then $m = n + k$ for some k and this follows from Theorem 3 and Corollary 1. □

5 Integrality and rationality

A real number x is called *integral* if x lies in $\mathbb{Z}$ and *rational* if it lies in $\mathbb{Q}$.

Theorem 5 (`isInt_fac_mul_s`)**.** *If $m \geq n - 1$, then $m! \cdot s_n$ is integral*

Proof. Follows from the fact that $m! \cdot a_k$ is integral for $k \leq m$. □

Theorem 6 (`rationality_criterion`)**.** *If x is rational, then there exists an $n \geq 2$ such that $n! \cdot x$ is integral.*

Proof. For n sufficiently large, the denominator of x will divide $n!$. □

6 The number $e = \lim_{n\to\infty} s_n$

The number e is defined as the limit of the sequence (s_n):

$$e = \lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Since (s_n) is an increasing sequence and bounded above (by Lemma 4), the limit exists.

Lemma 2 (`s_lt_e`, `e_le_of_s_le`)**.** *The number e satisfies:*

1. *For all n, we have $s_n < e$;*
2. *If $c \in \mathbb{R}$ and $s_n \leq c$ for all n, then $e \leq c$.*

□

Note that e is the *only* real number satisfying this, so you can use this lemma as a *definition* of e. Combining the second point with Theorem 4 we obtain:

Theorem 7 (`key_bound_e`)**.** *For all $n > 0$ we have $e \leq s_n + 2 \cdot a_n$*

7 The tail t_n of the series

Define

$$t_n = e - s_n = \sum_{i \geq n} a_i = \frac{1}{n!} + \frac{1}{(n+1)!} + \cdots$$

Lemma 3 (`t_le_twice_a`)**.** *For $n \geq 1$ we have $t_n \leq 2 \cdot a_n$.*

Proof. Follows from Theorem . □

Theorem 8 (`fac_mul_t_succ_pos`, `fac_mul_t_succ_lt_one`)**.** *For $n \geq 2$, we have*

$$0 < (n!) \cdot t_{n+1} < 1$$

Proof. For the upper bound, we use:

$$\begin{aligned} n! \cdot t_{n+1} &\leq n! \cdot 2 \cdot a_{n+1} \\ &= n! \cdot 2 \cdot \frac{1}{(n+1)!} \\ &= \frac{2}{n+1} \leq \frac{2}{3} < 1, \end{aligned}$$

where we use $n \geq 2$. □

Corollary 2. *If $n \geq 2$, then $n! \cdot t_{n+1}$ is not integral.*

Proof. Direct consequence of Theorem 8: there is no integer strictly between 0 and 1. □

8 Irrationality of e

Theorem 9 (Irrationality of e)**.** *The number e is irrational.*

Proof sketch. Suppose for contradiction that e is rational. By theorem there is an $n \geq 2$ such that $n! \cdot e$ is integral. By Theorem 5 also $n! \cdot s_{n+1}$ is integral. But then we have that also

$$n!t_{n+1} = n! \cdot e - n! \cdot s_{n+1}$$

is integral, which contradicts Corollary 2. $\square$