QUBITS AND QUANTUM STATES

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Exercise 2.1

Given the state:

$$|\psi\rangle = \frac{1-i}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

The measurement probabilities are:

$$P_0 = \left|\frac{1-i}{\sqrt{3}}\right|^2 = \frac{|1-i|^2}{3} = \frac{2}{3}, \qquad P_1 = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}.$$

Checking if probabilities add up to one:

$$P_0 + P_1 = \frac{2}{3} + \frac{1}{3} = 1$$

Exercise 2.2

Let

$$|a\rangle = \begin{pmatrix} -4i \\ 2 \end{pmatrix} \qquad |b\rangle = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}.$$

(A) Find $|a+b\rangle$.

$$|a+b\rangle = |a\rangle + |b\rangle = \begin{pmatrix} -4i \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \begin{pmatrix} 1-4i \\ 1+i \end{pmatrix}$$

(B) Calculate $3|a\rangle - 2|b\rangle$.

$$3|a\rangle - 2|b\rangle = \begin{pmatrix} -12i \\ 6 \end{pmatrix} + \begin{pmatrix} -2 \\ 2-2i \end{pmatrix} = \begin{pmatrix} -2+8i \\ 8-2i \end{pmatrix}$$

(C) Normalize $|a\rangle, |b\rangle$.

$$\langle a|a\rangle = |a\rangle^\dagger * |a\rangle = \begin{pmatrix} 4i & 2 \end{pmatrix} * \begin{pmatrix} -4i \\ 2 \end{pmatrix} = 16 + 4 = 20$$

$$||a|| = \sqrt{\langle a|a\rangle} = \sqrt{20}$$

$$|\tilde{a}\rangle = \frac{|a\rangle}{||a||} = \frac{\begin{pmatrix} -4i \\ 2 \end{pmatrix}}{\sqrt{20}} = \frac{1}{\sqrt{20}} \begin{pmatrix} -4i \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{2i}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\langle b|b\rangle = |b\rangle^\dagger * |b\rangle = \begin{pmatrix} 1 & -1-i \end{pmatrix} * \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = 1 + 1 + 1 + i - i = 3$$

$$||b|| = \sqrt{\langle b|b\rangle} = \sqrt{3}$$

$$|\tilde{b}\rangle = \frac{|b\rangle}{||b||} = \frac{\begin{pmatrix} 1 \\ -1+i \end{pmatrix}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1+i}{\sqrt{3}} \end{pmatrix}$$

Exercise 2.3

Another basis for $\mathbb{C}^2$ Given:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|+\rangle + |-\rangle = \frac{2|0\rangle}{\sqrt{2}} = \sqrt{2}|0\rangle$$

so

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|+\rangle - |-\rangle = \frac{2|1\rangle}{\sqrt{2}} = \sqrt{2}|1\rangle$$

so

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

These are inverted relations.

Exercise 2.4

a quantum system is in the state

$$|\psi\rangle = \frac{3i|0\rangle + 4|1\rangle}{5}$$

(A) Is the state normalized

To show that it is normalized we must show: $\langle\psi|\psi\rangle = 1$

$$|\psi\rangle^{\dagger} = -\frac{3i}{5}\langle 0| + \frac{4}{5}\langle 1|$$

$$\langle\psi|\psi\rangle = (\frac{-3i}{5}\langle 0| + \frac{4}{5}\langle 1|) * (\frac{3i}{5}|0\rangle + \frac{4}{5}|1\rangle) =$$

$$= \frac{9}{25}\langle 0|0\rangle + \frac{-12}{25}\langle 0|1\rangle + \frac{12i}{25}\langle 1|0\rangle + \frac{16}{25}\langle 1|1\rangle$$

Because of the orthonormality relations the only terms left are: $\langle 0|0\rangle = \langle 1|1\rangle = 1$ so

$$\langle\psi|\psi\rangle = \frac{9}{25}\langle 0|0\rangle + \frac{16}{25}\langle 1|1\rangle = \frac{9}{25} + \frac{16}{25} = 1$$

therfore state is normalized

(B) Express the state in the $|+\rangle, |-\rangle basis$

$$|\psi\rangle = \frac{1}{5}(3i|0\rangle + 4|1\rangle) \Rightarrow |\psi\rangle = \frac{1}{5}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{5}[\begin{pmatrix} 3i \\ 4 \end{pmatrix}] = \frac{1}{5}[\begin{pmatrix} 3i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}] = \frac{1}{5}[\frac{1}{2}\begin{pmatrix} 3i+3i \\ 3i-3i \end{pmatrix} + [\frac{1}{2}\begin{pmatrix} 4+4 \\ 4-4 \end{pmatrix}] =$$

$$= \frac{1}{5}[\frac{1}{2}\begin{pmatrix} 3i \\ 3i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3i \\ -3i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 4 \\ -4 \end{pmatrix}] =$$

$$= \frac{1}{5}[\frac{4+3i}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{-4+3i}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}] =$$

$$\frac{1}{5}[\frac{4+3i}{\sqrt{2}}(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}) + \frac{-4+3i}{\sqrt{2}}(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}) =]$$

$$= \frac{4+3i}{5\sqrt{2}}|+\rangle + \frac{-4+3i}{5\sqrt{2}}|-\rangle$$

Exercise 2.5

Gram-Shmidt on $\mathbb{R}^4$ Given:

$$|u_1\rangle = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \qquad |u_2\rangle = \begin{pmatrix}1\\2\\4\\5\end{pmatrix}, \qquad |u_3\rangle = \begin{pmatrix}1\\-3\\-4\\-2\end{pmatrix},$$

We use tilde to denote normalized vectors

$$|v_1\rangle = |u_1\rangle$$

Norm:

$$\langle u_1|u_1\rangle = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix} * \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 4$$

Normalize:

$$|\tilde{v_1}\rangle = \frac{|v_1\rangle}{\sqrt{\langle u_1|u_1\rangle}} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Second vector:

$$|v_2\rangle = \begin{pmatrix}1\\2\\4\\5\end{pmatrix} - \frac{12}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}$$

Norm:

$$\langle V_2|V_2\rangle = \begin{pmatrix}-2 & -1 & 1 & 2\end{pmatrix} * \begin{pmatrix}-2\\-1\\1\\2\end{pmatrix} = 4+1+1+4 = 10$$

Normalize:

$$|\tilde{v_2}\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix} = \begin{pmatrix}\frac{-2}{\sqrt{10}}\\\frac{-1}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\\\frac{2}{\sqrt{10}}\end{pmatrix}$$

Third vector:

$$|v_3\rangle = \begin{pmatrix}1\\-3\\-4\\-2\end{pmatrix} - \frac{-8}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{-7}{10}\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix} = \begin{pmatrix}\frac{16}{10}\\\frac{-17}{10}\\\frac{-13}{10}\\\frac{14}{10}\end{pmatrix}$$

Norm:

$$\langle V_3|V_3\rangle = \begin{pmatrix}\frac{16}{10} & \frac{-17}{10} & \frac{-13}{10} & \frac{14}{10}\end{pmatrix} * \begin{pmatrix}\frac{16}{10}\\\frac{-17}{10}\\\frac{-13}{10}\\\frac{14}{10}\end{pmatrix} = \frac{910}{100}$$

Normalize:

$$|\tilde{v_3}\rangle = \frac{1}{\sqrt{\frac{910}{100}}}\begin{pmatrix}\frac{16}{10}\\\frac{-17}{10}\\\frac{-13}{10}\\\frac{14}{10}\end{pmatrix} = \frac{10}{\sqrt{910}}\begin{pmatrix}\frac{16}{10}\\\frac{-17}{10}\\\frac{-13}{10}\\\frac{14}{10}\end{pmatrix} = \frac{1}{\sqrt{910}}\begin{pmatrix}16\\-17\\-13\\14\end{pmatrix}$$

The orthonormal basis for the span is:

$$e_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \qquad e_2 = \frac{1}{\sqrt{10}}\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix} \qquad e_3 = \frac{1}{\sqrt{910}}\begin{pmatrix}16\\-17\\-13\\14\end{pmatrix}$$

Exercise 2.6

Photon horizontal and vertical polarization states are written as $|h\rangle$ and $|V\rangle$ respectively. Suppose:

$$|\psi_1\rangle = \frac{1}{2}|h\rangle + \frac{\sqrt{3}}{2}|v\rangle$$

$$|\psi_2\rangle = \frac{1}{2}|h\rangle - \frac{\sqrt{3}}{2}|v\rangle$$

$$|\psi_3\rangle = |h\rangle$$

$$\langle\psi_1|\psi_2\rangle = (\frac{1}{2}\langle h| + \frac{\sqrt{3}}{2}\langle v|)(\frac{1}{2}|h\rangle - \frac{\sqrt{3}}{2}|v\rangle) = (\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2 = -\frac{1}{2}$$

$$|\langle\psi_1|\psi_2\rangle|^2 = (-\frac{1}{2})^2 = \frac{1}{4}$$

$$\langle\psi_1|\psi_3\rangle = (\frac{1}{2}\langle h| + \frac{\sqrt{3}}{2}\langle v|)|h\rangle = \frac{1}{2}$$

$$|\langle\psi_1|\psi_3\rangle|^2 = (\frac{1}{2})^2 = \frac{1}{4}$$

$$\langle\psi_3|\psi_1\rangle = \langle h|(\frac{1}{2}|h\rangle + \frac{\sqrt{3}}{2}|v\rangle) = \frac{1}{2}$$

$$|\langle\psi_3|\psi_2\rangle|^2 = (\frac{1}{2})^2 = \frac{1}{4}$$