CS215: Discrete Math (H)
2021 Fall Semester Written Assignment # 4
Due: Dec. 8th, 2021, please submit at the beginning of class

Q.1 Prove by induction that, for any integer $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\left(1-\frac{1}{4^2}\right)\cdot\dots\cdot\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Q.2 Use induction to prove that 4 divides $2n^2+6n$ whenever n is a positive integer.

Q.3 Let $x \in \mathbb{R}$ and $x \neq 1$. Using mathematical induction, prove that for all integers $n \geq 0$,

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}.$$

Q.4 Prove that if $A_1, A_2, \dots, A_n$ and B are sets, then

$$\begin{aligned} & (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \\ = \; & (A_1 \cap A_2 \cap \cdots \cap A_n) - B. \end{aligned}$$

Q.5 Prove that if $h > -1$, then $1+nh \leq (1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

Q.6 Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

(a) Show statements $P(18)$, $P(19)$, $P(20)$ and $P(21)$ are true, completing the basis step of the proof.

(b) What is the inductive hypothesis of the proof?

(c) What do you need to prove in the inductive step?

(d) Complete the inductive step for $k \geq 21$.

(e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Q.7 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Q.8 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

(a) Find $f(16)$

(b) Find a big-O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Q.9 The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in *closed form* (in terms of b if necessary) and should *not* use summation.

(b) Find a solution for $T(n)$. Your solution should be in *closed form* (in terms of a and c if necessary) and should *not* use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Q.10 Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Q.11 How many bit strings of length 19 contain at least 9 1's and at least 9 0's? You may leave your answer as an equation.

Q.12 Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding n that are relatively prime to n, i.e., the Euler function $\phi(n)$.

Q.13 Alice is going to choose a selection of 12 chocolates. There are 25 different brands of them and she can have as many as she wants of each brand (but can only choose 12 pieces). How many ways can she make this selection?

Q.14 16 points are chosen inside a 5×3 rectangle. Prove that two of these points lie within $\sqrt{2}$ of each other.

Q.15 Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Q.16 Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Q.17

Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.

Q.18 Solve the recurrence relation

$$a_n = 4a_{n-2},$$

with initial conditions $a_0 = 3$, $a_1 = 2$.

Q.19

(a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

(b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

Q.20 A computer system considers a string of decimal digits $(0, 1, \ldots, 9)$ to be a **valid** code word if and only if it contains an **odd number of zero digits**. For example, 12030 and 11111 are **not** valid, but 29046 is. Let $V(n)$ denote the number of valid n-digit code words. Find a recurrence relation for $V(n)$ with initial cases, and give a closed-form solution to this recurrence relation. Please explain how you find the recurrence relation. (Hint: notice that the number of non-valid code words is equal to $10^n - V(n)$.)

Q.21 Use generating functions to prove Pascal's identity: $C(n, r) = C(n - 1, r) + C(n - 1, r - 1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1 + x)^n = (1 + x)^{n-1} + x(1 + x)^{n-1}$.]