CS201H: Discrete Math for Computer Science
2021 Fall Semester   Written Assignment # 2
Due: Oct. 27th, 2021, please submit at the beginning of class

Q.1 Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A \cap B \neq \emptyset) \to ((A - B) \subset A)$

(b) $(A \subseteq B) \to (|A \cup B| \geq 2|A|)$

(c) $\overline{(A - B)} \cap (B - A) = B$

Q.2 The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. Give an example of two uncountable sets A and B such that the intersection $A \oplus B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.3 Give an example of two uncountable sets A and B such that the difference $A - B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.4 Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Q.5 The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

(a) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Q.6 For each set A, the *identity function* $1_A : A \to A$ is defined by $1_A(x) = x$ for all x in A. Let $f : A \to B$ and $g : B \to A$ be the functions such that $g \circ f = 1_A$. Show that f is one-to-one and g is onto.

Q.7 Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the *composition* function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must f be onto? Explain your answer.

(e) If $f \circ g$ is onto, must g be onto? Explain your answer.

Q.8 Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Q.9 Derive the formula for $\sum_{k=1}^{n} k^3$.

Q.10 Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when m is a positive integer.

Q.11 Show that a subset of a countable set is also countable.

Q.12 Show that if A, B, C and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

Q.13 Show that if A and B are sets with the same cardinality, then $|A| \leq |B|$ and $|B| \leq |A|$.

Q.14 Show that if A, B and C are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

Q.15 Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m + n - 2)(m + n - 1)/2 + m$ is one-to-one and onto.

Q.16 By the Schröder-Bernstein theorem, prove that $(0, 1)$ and $[0, 1]$ have the same cardinality.

Q.17 Suppose that $f(x), g(x)$ and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.

Q.18 If $f_1(x)$ and $f_1(x)$ are functions from the set of positive integers to the set of positive real numbers and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, is $(f_1 - f_2)(x)$ also $\Theta(g(x))$? Either prove that it is or give a counter example.

Q.19 Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.

Q.20 Prove that $n \log n = \Theta(\log n!)$ for all positive integers n.

Q.21

(a) Show that this algorithm determines the number of 1 bits in the bit string S:

Algorithm 1 bit count (S: bit string)

```
count := 0
while S ≠ 0 do
    count := count + 1
    S := S ∧ (S − 1)
end while
return count {count is the number of 1's in S}
```

Here $S - 1$ is the bit string obtained by changing the rightmost 1 bit of S to a 0 and all the 0 bits to the right of this to 1's. [Recall that $S \wedge (S - 1)$ is the bitwise AND of S and $S - 1$.]

(b) How many bitwise AND operations are needed to find the number of 1 bits in a string S using the algorithm in part a)?

Q.22

(1) Show that $(\sqrt{2})^{\log n} = O(\sqrt{n})$, where the base of the logarithm is 2.

(2) Arrange the functions

$$n^n,\ (\log n)^2,\ n^{1.0001},\ (1.0001)^n,\ 2^{\sqrt{\log_2 n}},\ n(\log n)^{1001}$$

in a list such that each function is big-O of the next function.

Q.23 Give an example of two increasing functions $f(n)$ and $g(n)$ from the set of positive integers to the set of positive integers such that neither $f(n)$ is $O(g(n))$ nor $g(n)$ is $O(f(n))$.