南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iterating recurrences
- Three different behaviors

Divide and conquer algorithms

- We just analyzed recurrences of the form

$$T(n) = \begin{cases} b & \text{if } n = 0 \\ r \cdot T(n-1) + a & \text{if } n > 0 \end{cases}$$

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Divide and conquer algorithms

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Divide and conquer algorithms

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These corresponded to the analysis of recursive algorithms in which a problem of size n is solved by recursively solving a problem of size $n-1$.

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

We will now look at recurrences of the form

$$T(n) = \begin{cases} \text{something given} & \text{if } n \leq n_0 \\ r \cdot T(n/m) + a & \text{if } n > n_0 \end{cases}$$

Binary Search

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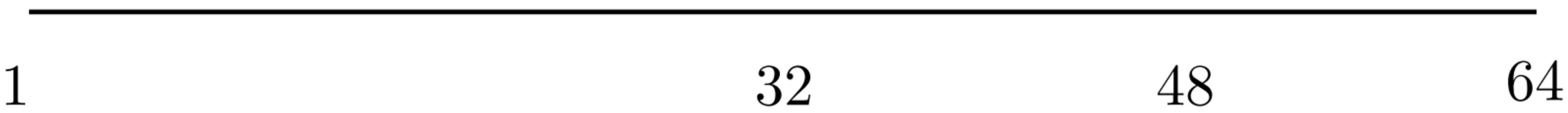
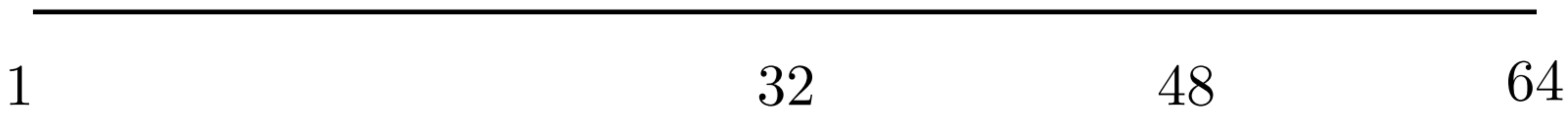
  Our strategy will be to always ask greater than questions, at each step halving our search range, until the range only contains one number, when we ask a final equal to question.

Binary Search Example

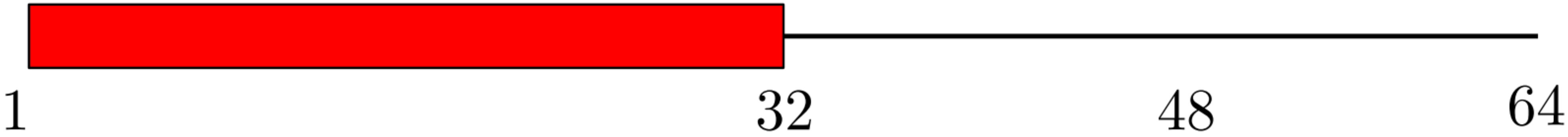


Binary Search Example



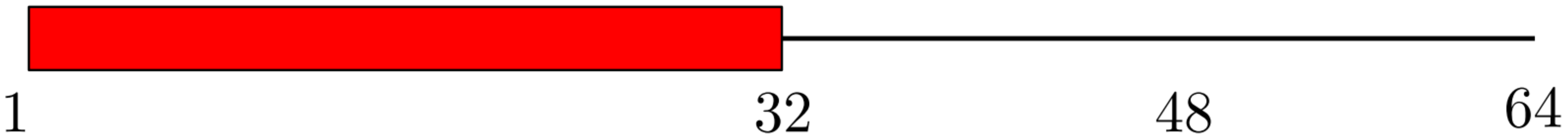
Is $x > 32$?

Binary Search Example



Is $x > 32$?   Answer: Yes

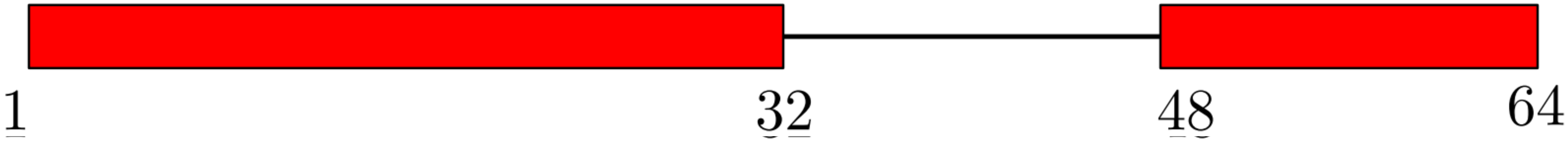
Binary Search Example



Is $x > 32?$ Answer: Yes

Is $x > 48?$

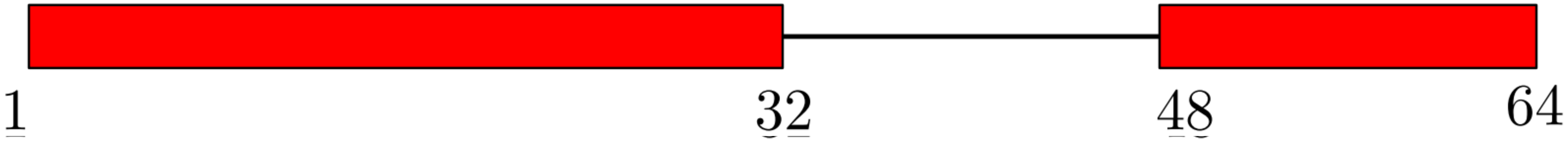
Binary Search Example



Is $x > 32?$ Answer: Yes

Is $x > 48?$ Answer: No

Binary Search Example

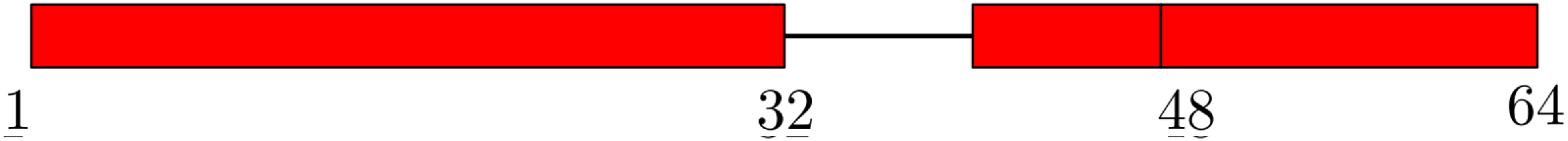


Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Is $x > 40$?

Binary Search Example

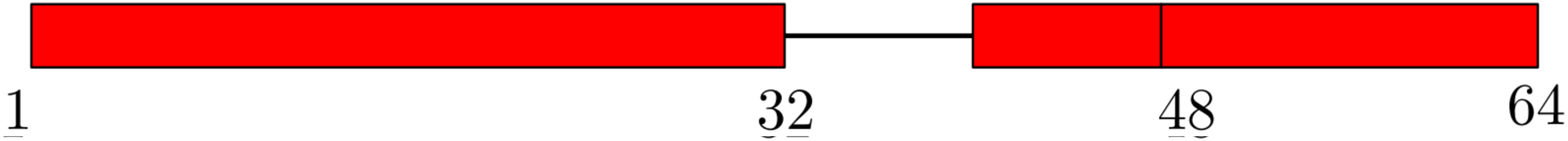


Is $x > 32?$ Answer: Yes

Is $x > 48?$ Answer: No

Is $x > 40?$ Answer: No

Binary Search Example



Is $x > 32?$ Answer: Yes

Is $x > 48?$ Answer: No

Is $x > 40?$ Answer: No

Is $x > 36?$

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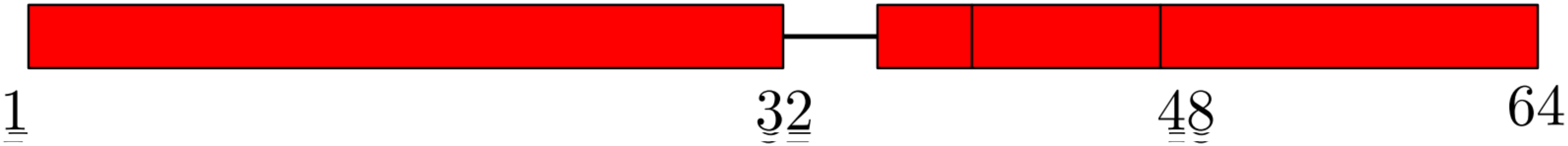

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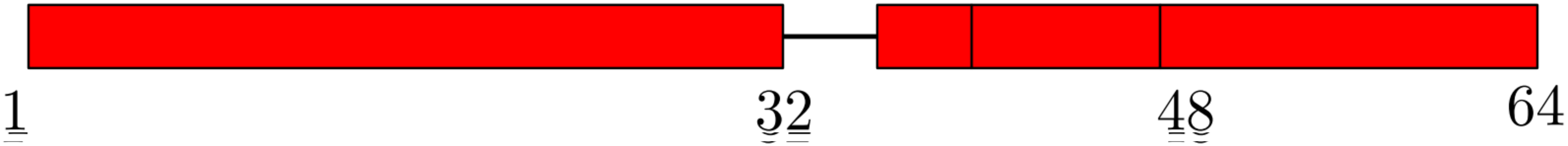

Is $x > 32$? Answer: Yes

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Is $x > 34$?

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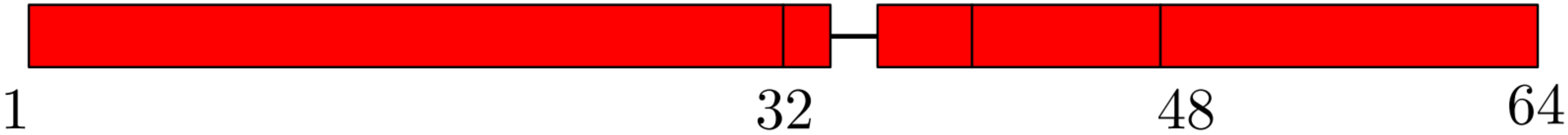

Is $x > 32?$ Answer: Yes

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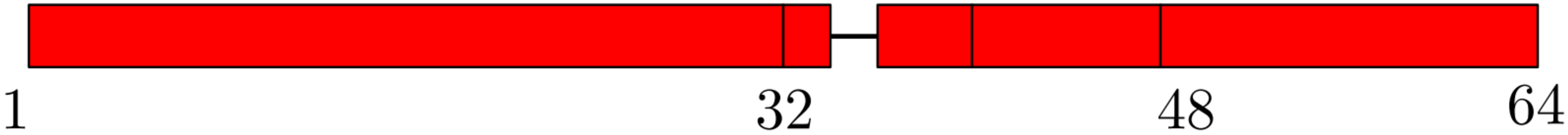

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Is $x > 35$?

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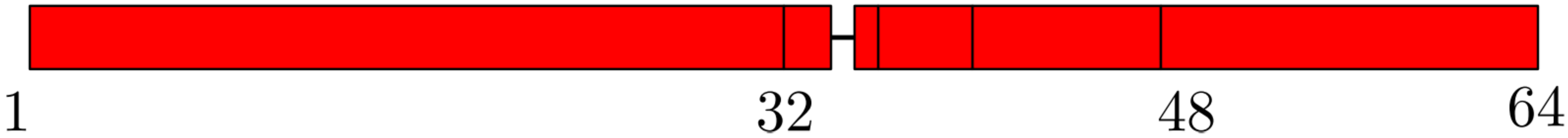

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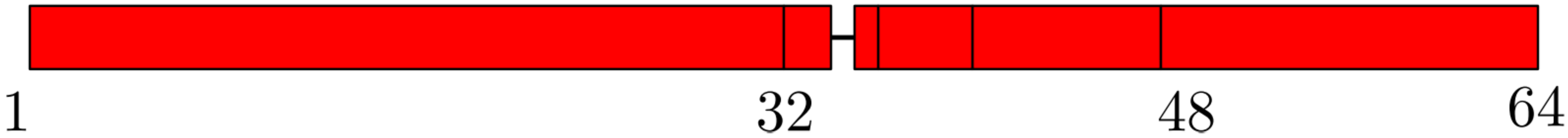
Is $x > 36$? Answer: No

Is $x > 34$? Answer: Yes

Is $x > 35$? Answer: No

Binary Search Example



Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

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Is $x > 36$? Answer: No

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Is $x > 35$? Answer: No

Is $x = 35$?

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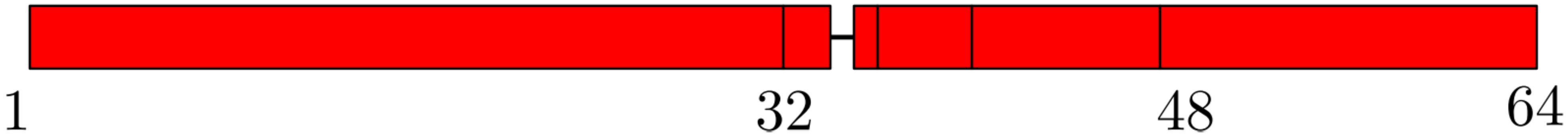

Is $x > 32$? Answer: Yes

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Is $x > 36$? Answer: No

Is $x > 34$? Answer: Yes

Is $x > 35$? Answer: No

Is $x = 35$? Answer: BINGO!

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  Note: When n is a power of 2, $T(n)$, the number of questions in a binary search on $[1, n]$, satisfies

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

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  This can also be proved inductively, similar to the tower of Hanoi recurrence.

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Base case (1 item): $T(1) = 1$ to ask: "Is the number k?"

Binary Search Example

- $$(*) \qquad T(n) = \begin{cases} C_1 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + C_2 & \text{if } n \geq 2 \end{cases}$$

  For simplicity, we will (usually) assume that n is a power of 2 (or sometimes 3 or 4) and also often that constants such as C_1, C_2 are 1. This will let us replace a recurrence such as (*) by one such as (**).

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  In practice, the solution of (*) will be very close to that of (**) (this can be proved mathematically). Hence, we can restrict attention to (**).

Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms

- Iterating recurrences

- Three different behaviors

Iterating Recurrences: Example 1

- $$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

Iterating Recurrences: Example 1

- $$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

  This corresponds to solving a problem of size n, by

  (i) solving 2 subproblems of size $n/2$ and
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  or using $T(1)$ work for "bottom" case of $n = 1$

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We now see how to solve (*) by algebraically iterating the recurrence.

Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

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  Assume that n is a power of 2

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Iterating Recurrences: Example 1

- Algebraically iterating the recurrence

  Assume that n is a power of 2

$$\begin{aligned} T(n) = 2T\left(\tfrac{n}{2}\right) + n \quad &= 2\left(2T\left(\tfrac{n}{4}\right) + \tfrac{n}{2}\right) + n \\ = 4T\left(\tfrac{n}{4}\right) + 2n \quad &= 4\left(2T\left(\tfrac{n}{8}\right) + \tfrac{n}{4}\right) + 2n \end{aligned}$$

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$$
\begin{aligned}
T(n) = 2T\left(\tfrac{n}{2}\right) + n \quad & = 2\left(2T\left(\tfrac{n}{4}\right) + \tfrac{n}{2}\right) + n \\
= 4T\left(\tfrac{n}{4}\right) + 2n \quad & = 4\left(2T\left(\tfrac{n}{8}\right) + \tfrac{n}{4}\right) + 2n \\
= 8T\left(\tfrac{n}{8}\right) + 3n \quad & \\
\vdots \qquad \vdots \quad & \\
= 2^i T\left(\tfrac{n}{2^i}\right) + in \quad &
\end{aligned}
$$

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$$= 4T\left(\frac{n}{4}\right) + 2n \qquad = 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

$$= 8T\left(\frac{n}{8}\right) + 3n$$

$$\vdots \qquad \vdots$$

$$= 2^i T\left(\frac{n}{2^i}\right) + in$$

$$\vdots \qquad \vdots$$

$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + (\log_2 n)n$$

End when $i = \log_2 n$

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  Assume that n is a power of 2

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$$= nT(1) + n\log_2 n$$

Iterating Recurrences: Example 1

- We just iterated the recurrence to derive that the solution to

$$(*) \qquad T(n) = \begin{cases} T(1) & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

is $nT(1) + n\log_2 n$.

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is $nT(1) + n\log_2 n$.

Note: Technically, we still need to use **induction** to prove that our solution is correct. Practically, we never explicitly perform this step, since it is obvious how the induction would work.

Iterating Recurrences: Example 2

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n \geq 2 \end{cases}$$

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Iterating Recurrences: Example 2

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$$T(n) = T\left(\frac{n}{2}\right) + 1 \qquad = \left(T\left(\frac{n}{2^2}\right) + 1\right) + 1$$

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$$= T\left(\frac{n}{2^2}\right) + 2$$

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&= T\left(\frac{n}{2^3}\right) + 3 \\
&\ \ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^i}\right) + i \\
&\ \ \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n & &= 1 + \log_2 n
\end{aligned}$$

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Iterating Recurrences: Example 3

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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&= T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \\
&\quad \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^i}\right) + \frac{n}{2^{i-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&\quad \vdots \qquad \vdots \\
&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n
\end{aligned}$$

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&= T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{2^{\log_2 n-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \\
&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n
\end{aligned}$$

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&= 1 + 2 + 2^2 + \cdots + \frac{n}{2^2} + \frac{n}{2} + n \qquad = \Theta(n)
\end{aligned}$$

Iterating Recurrences: Example 4

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 3T(n/3) + n & \text{if } n \geq 3 \end{cases}$$

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$$= 3^2 T\left(\frac{n}{3^2}\right) + 2n$$

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Iterating Recurrences: Example 5

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&\vdots \qquad \vdots \\
&= 4^{\log_2 n}T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{4^{\log_2 n-1}}{2^{\log_2 n-1}}n + \cdots + \frac{4}{2}n + n
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&\vdots \qquad \vdots \\
&= 4^{\log_2 n}T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{4^{\log_2 n-1}}{2^{\log_2 n-1}}n + \cdots + \frac{4}{2}n + n \\
&= 2n^2 - n
\end{aligned}$$

Growth Rates of Solutions to Recurrences

- Divide and conquer algorithms
- Iteration recurrences
- Three different behaviors

Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

Three Different Behaviors

- Compare the iteration for the recurrences

$$T(n) = 2T(n/2) + n$$

$$T(n) = T(n/2) + n$$

$$T(n) = 4T(n/2) + n$$

◇ all three recurrences iterate $\log_2 n$ times

◇ in each case, size of subproblem in next iteration is half the size in the preceding iteration level

Three Different Behaviors

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/2) + n,$$
where a is a positive integer and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

  1. If $a < 2$, then $T(n) = \Theta(n)$.
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Proof

We already proved Case 1 when $a = 1$ in Example 3.

(will not prove it for $1 < a < 2$)

We already proved Case 2 in Example 1.

We will now prove Case 3.

Iterating Recurrences

- $T(n) = aT(n/2) + n$, where $a > 2$. Assume that $n = 2^i$.

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$$T(n) = a^{\log_2 n} T(1) + n \textstyle\sum_{i=0}^{\log_2 n-1} \left(\tfrac{a}{2}\right)^i$$

Work at
"bottom"

Iterated
Work

Total work

- The total work is

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Since $a > 2$, the geometric series is Θ of the largest term.

$$n \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{2}\right)^i = n \frac{1-(a/2)^{\log_2 n}}{1-a/2} = n\Theta((a/2)^{\log_2 n - 1})$$

Total work

- n times the largest term in the geometric series is

$$n\left(\frac{a}{2}\right)^{\log_2 n-1} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{2^{\log_2 n}} = \frac{2}{a}\cdot\frac{n\cdot a^{\log_2 n}}{n} = \frac{2}{a}\cdot a^{\log_2 n}$$

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Notice that

$$a^{\log_2 n} = \left(2^{\log_2 a}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 a} = n^{\log_2 a}$$

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So the total work is

$$a^{\log_2 n}T(1) + n\sum_{i=0}^{\log_2 n-1}\left(\frac{a}{2}\right)^i$$

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$$\Theta\left(n^{\log_2 a}\right) \qquad \Theta\left(n^{\log_2 a}\right)$$

Example 5 Recap

- $$(*) \qquad T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

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$a = 4$, so the Theorem says that

$$T(n) = \Theta\left(n^{\log_2 a}\right) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

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$a = 4$, so the Theorem says that

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This matches with the exact answer of $2n^2 - n$.

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The Master Theorem

- **Theorem** Suppose that we have a recurrence of the form
$$T(n) = aT(n/b) + cn^d,$$
where a is a positive integer, $b \geq 1$, c, d are real numbers with c positive and d nonnegative, and $T(1)$ is nonnegative. Then we have the following big Θ bounds on the solution:

  1. If $a < b^d$, then $T(n) = \Theta(n^d)$.
  2. If $a = b^d$, then $T(n) = \Theta(n^d \log n)$.
  3. If $a > b^d$, then $T(n) = \Theta(n^{\log_b a})$

Counting

- Assume we have a set of objects with certain properties

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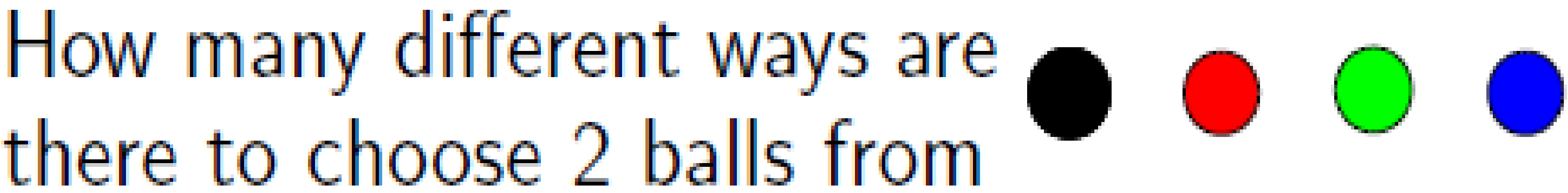

How many different ways are there to choose 2 balls from

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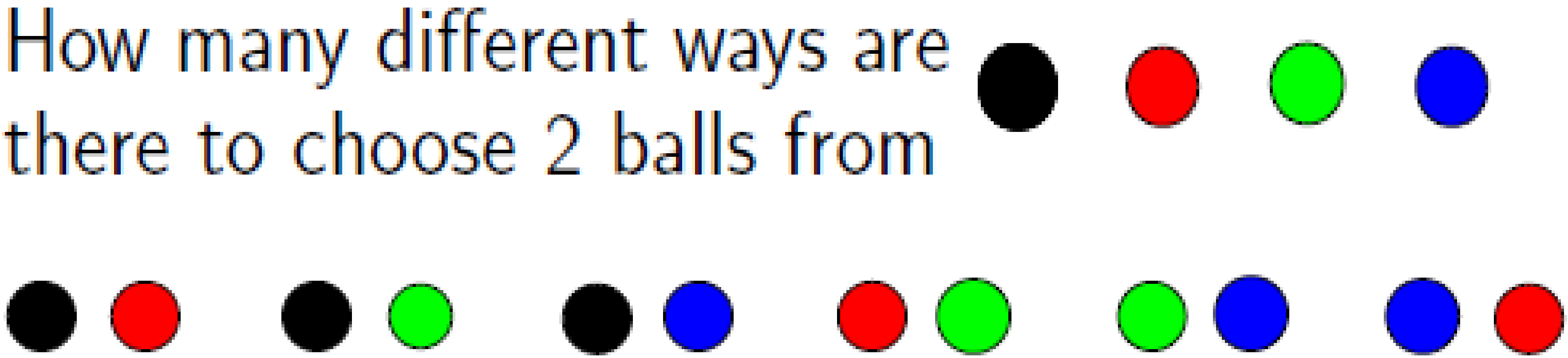

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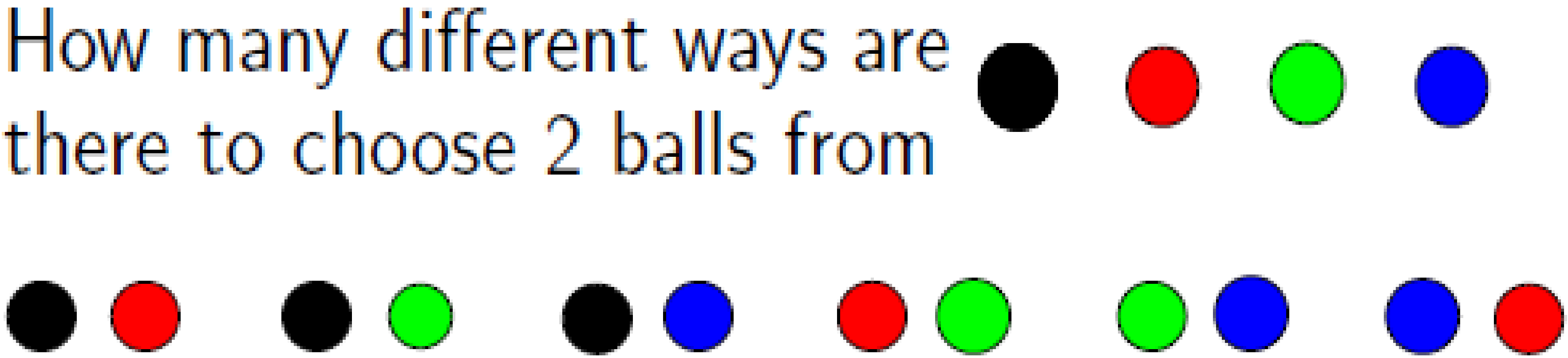

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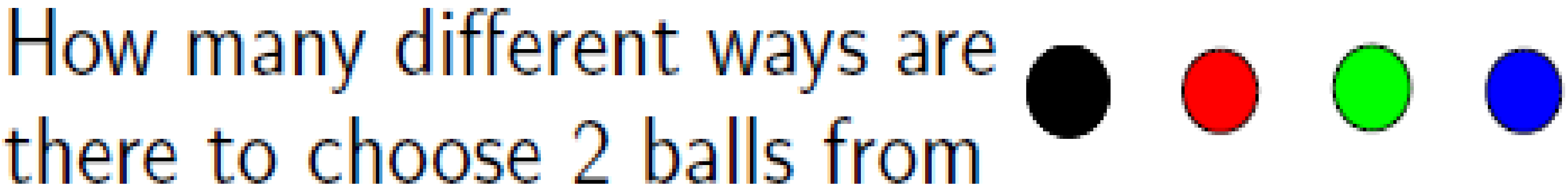
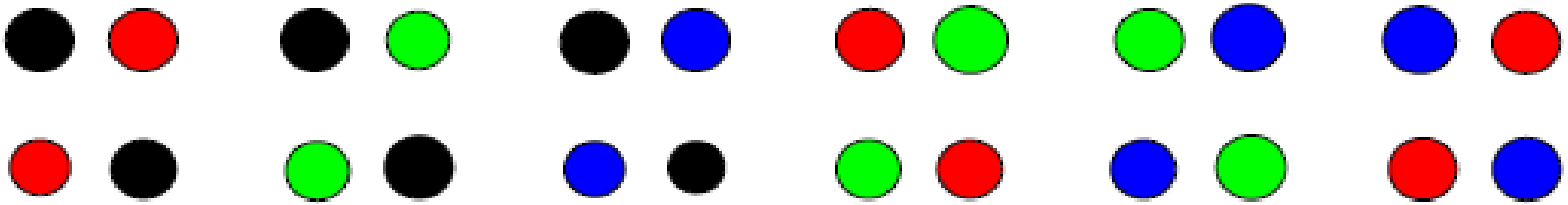


How many different ways are there to choose 2 balls from

What about when order counts?



Counting

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  – simplify the solution by decomposing the problem

Basic Counting Rules

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In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

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Example

In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

We may either list all or use the product rule.

$$26 \times 50 = 1300$$

The Product Rule

- **Product Rule**: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is

$$n = n_1 \cdot n_2 \cdot \dots \cdot n_k$$

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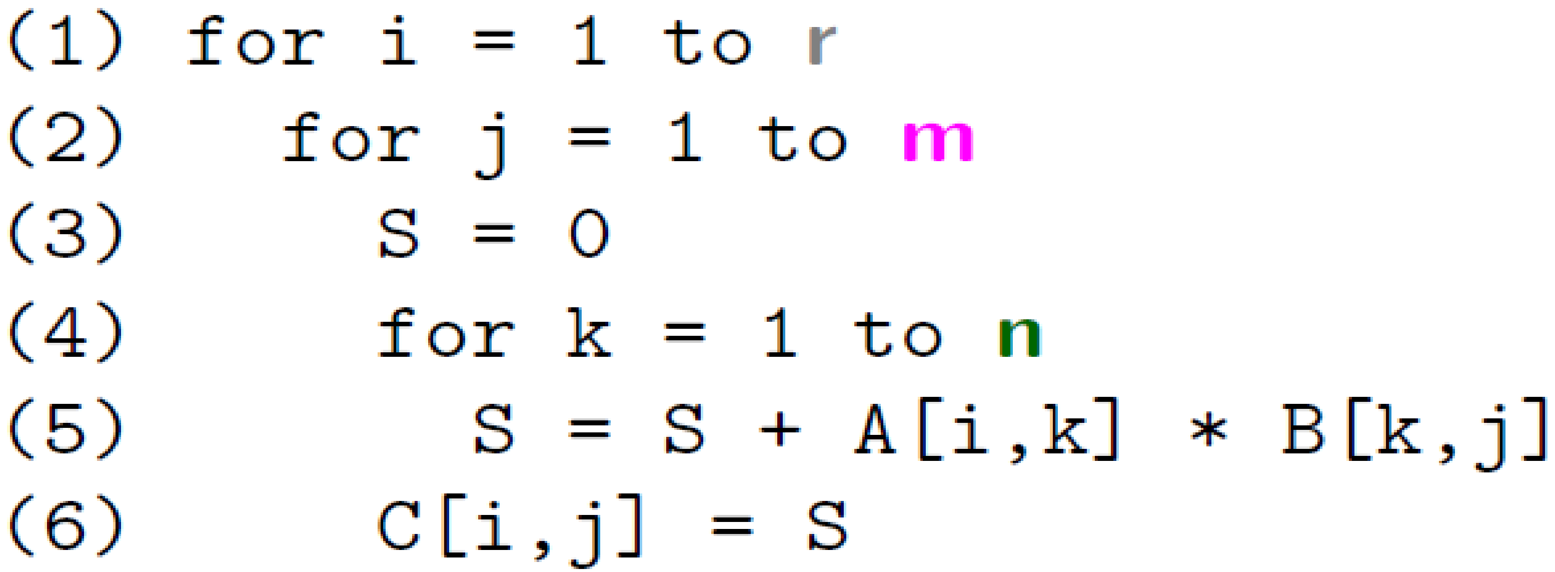
How many onto functions?

The Product Rule

- The following loop is a part of program computing the product of two matrices.



```
for i = 1 to r
    for j = 1 to m
        S = 0
        for k = 1 to n
            S = S + A[i,k] * B[k,j]
        C[i,j] = S
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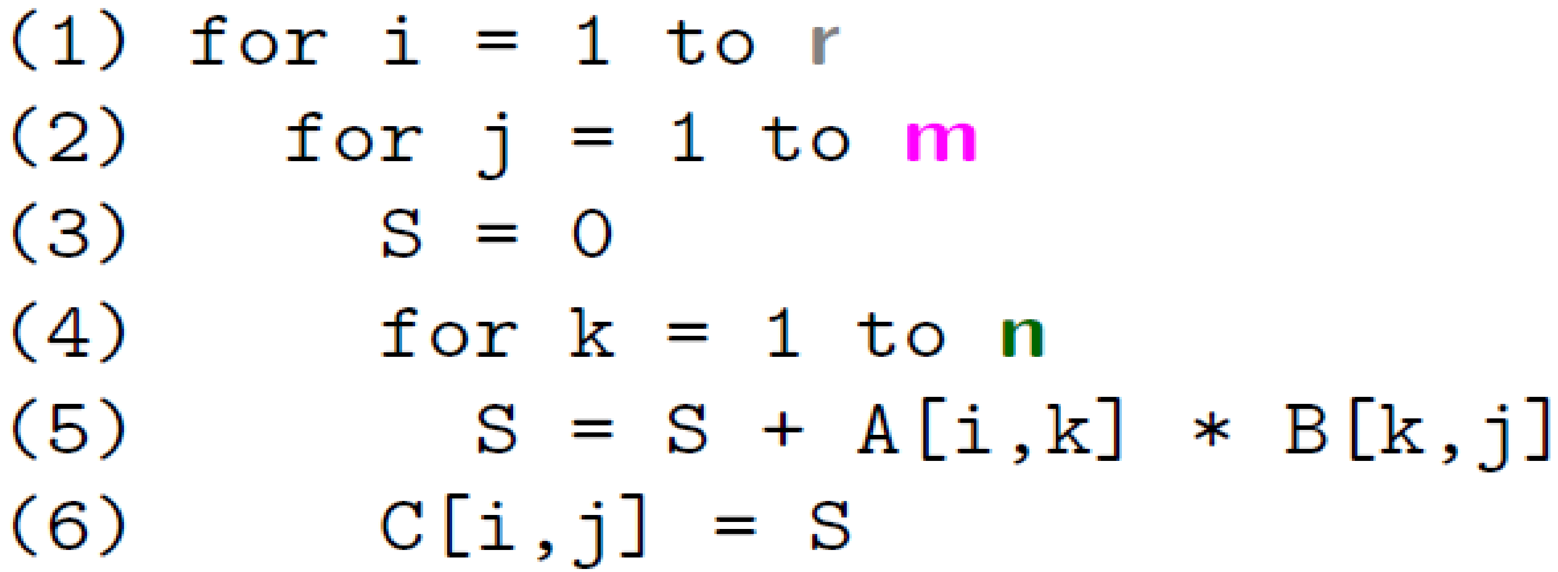

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How many multiplications (in terms of r, m, n) does this program carry out in total among all iterations of line 5?

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You need to travel from city A to B. You may either fly, take a train, or a bus. There are 12 different flights, 5 different trains and 10 buses. How many options do you have to get from A to B?

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We may use the sum rule.

$$12 + 5 + 10$$

The Sum Rule

- **Sum Rule**: If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is
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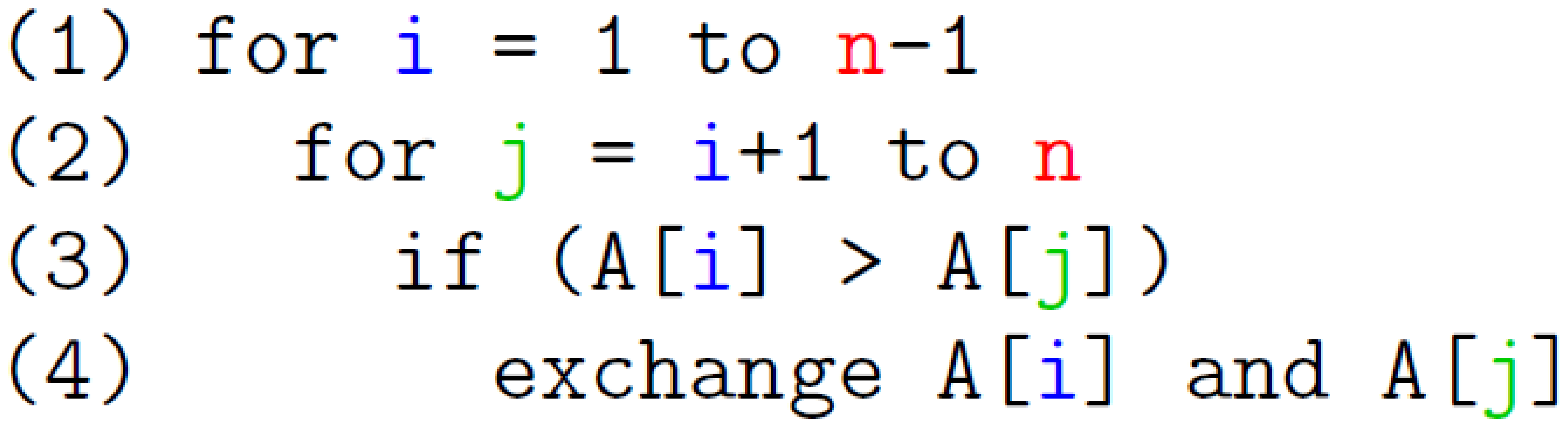

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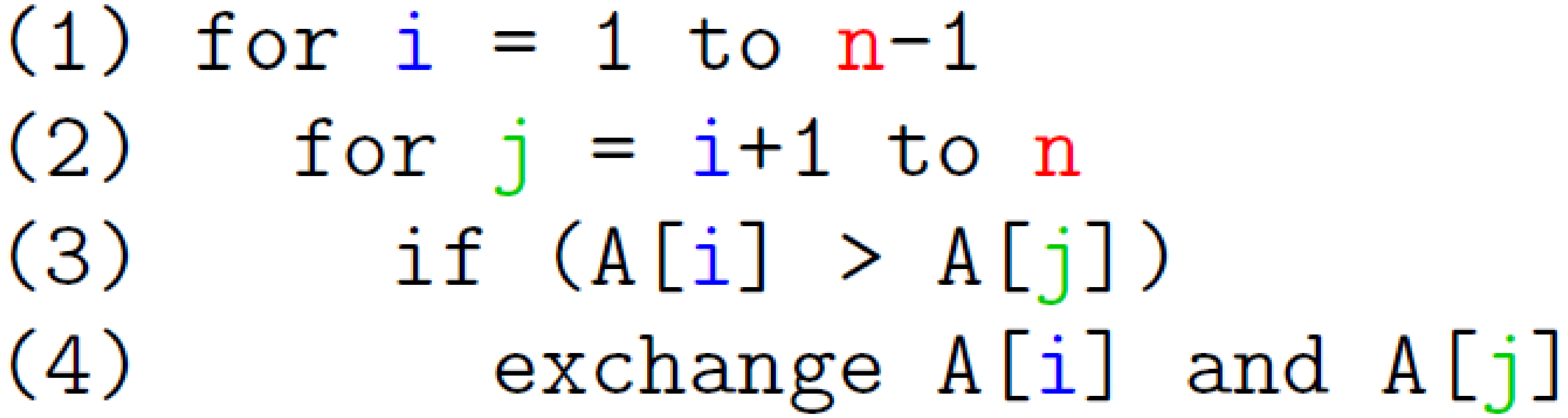

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How many comparisons (in terms of *n*) does this program carry out in total among all iterations of line 3?

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Each password is 6 to 8 characters long, where each character is an lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

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$$P = P_6 + P_7 + P_8$$

Tree Diagrams

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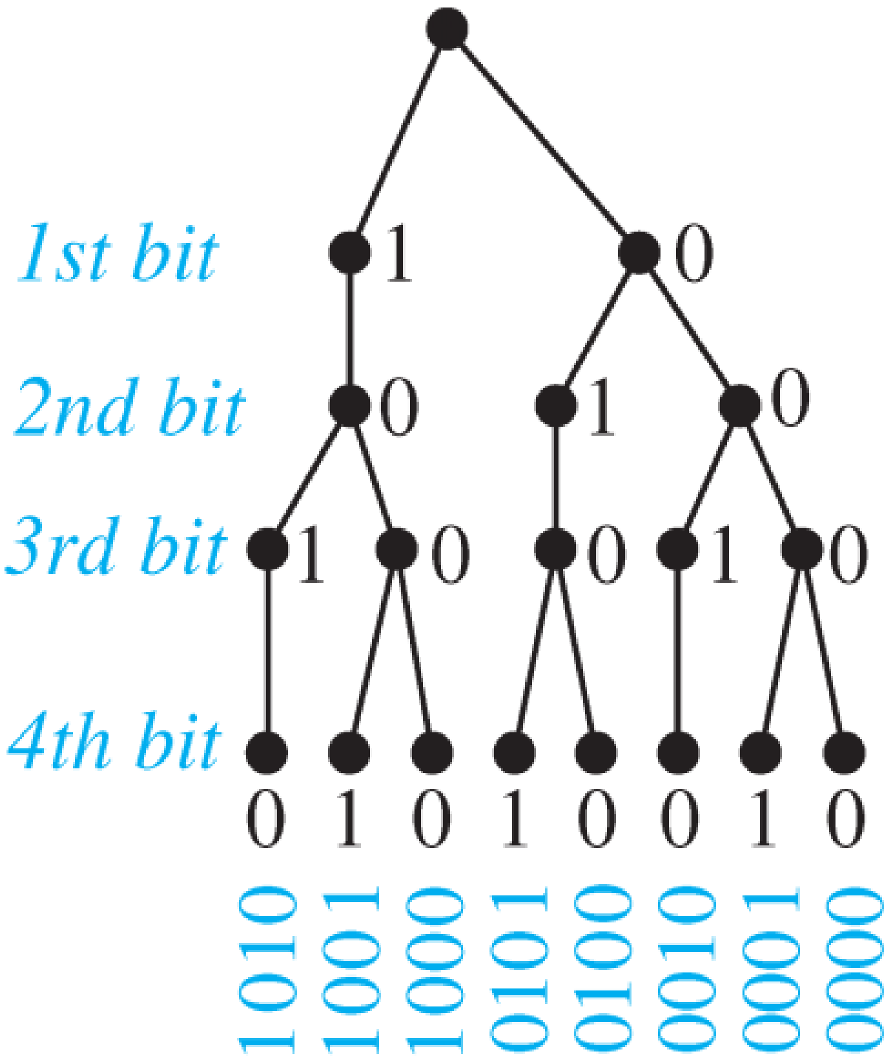

Tree Diagram

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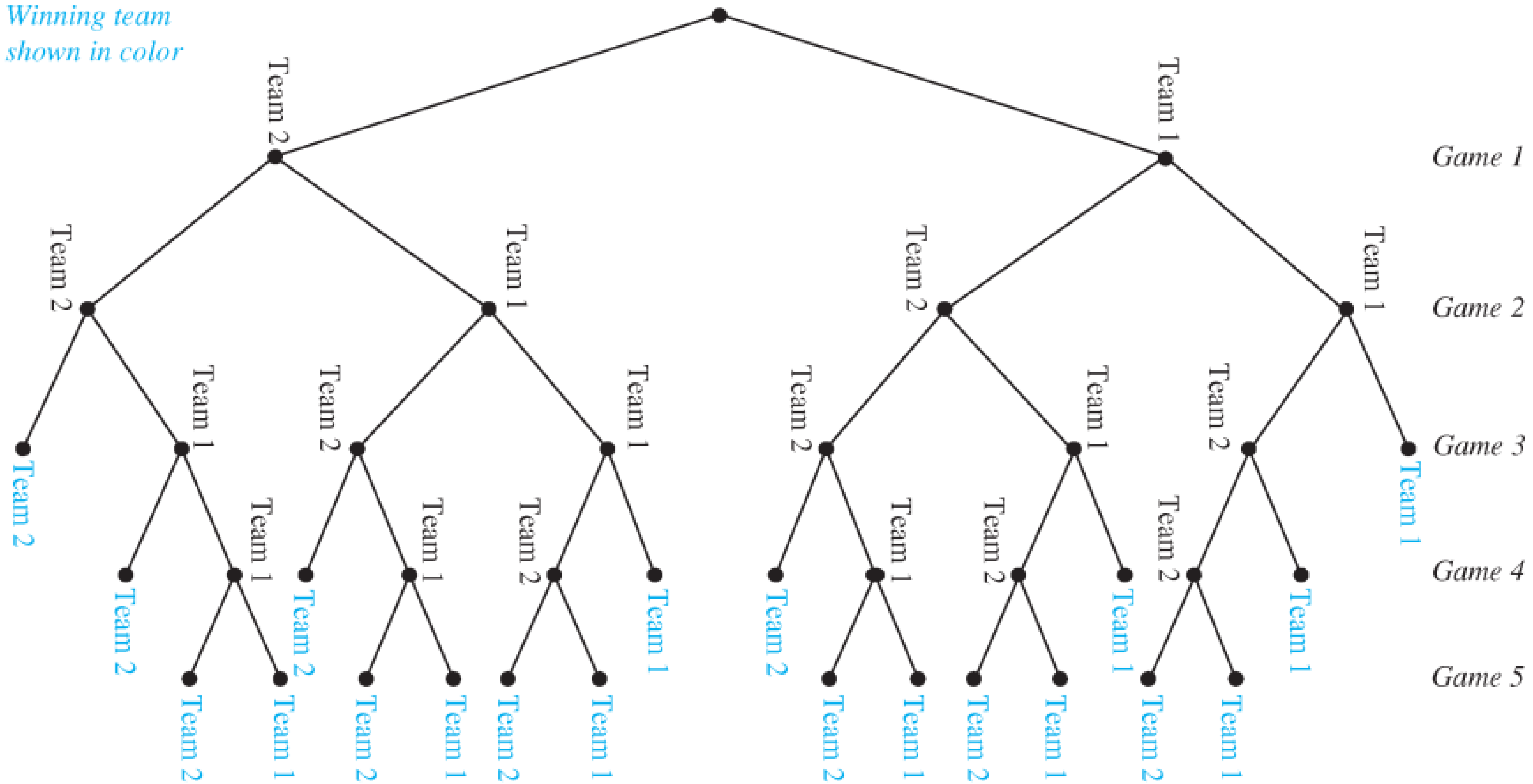

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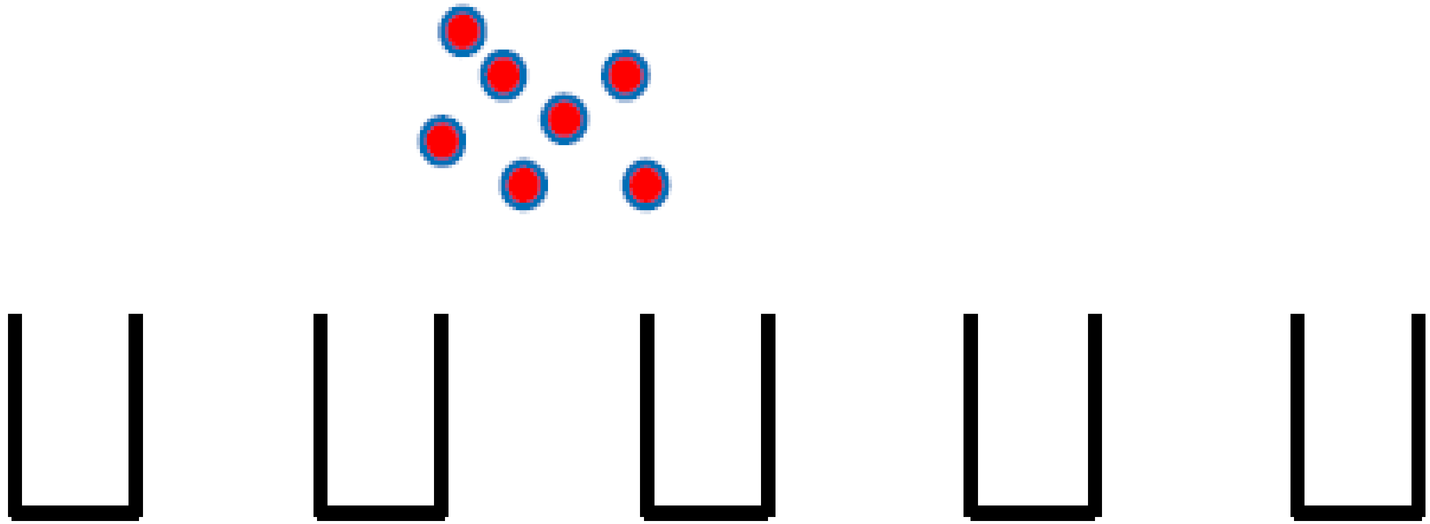

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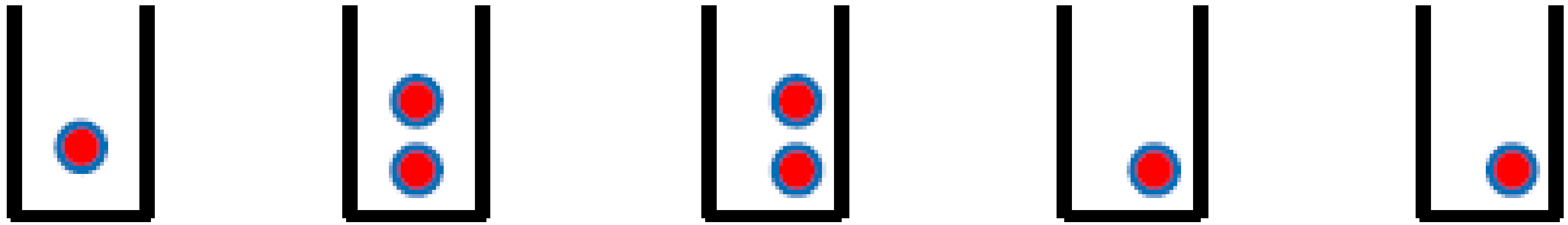

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  Example

  Assume that there are 367 students. Are there any two people who have the same birthday?

  There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?

Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

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Example

Assume there are 100 students. How many of them were born in the same month?

Bijections and Permutations

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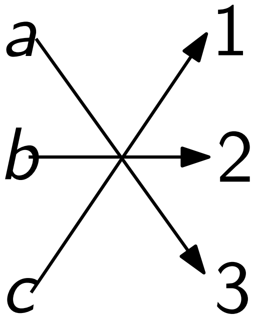

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  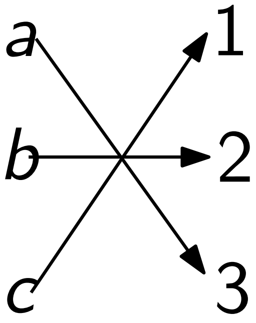

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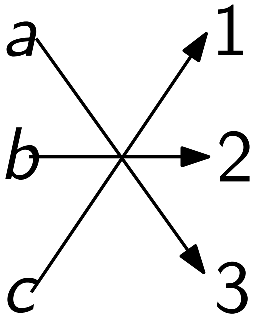

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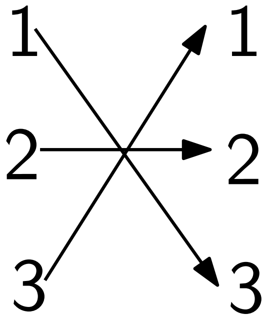

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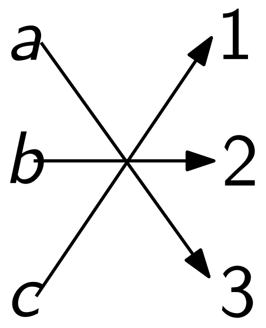

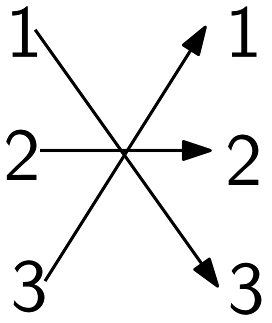

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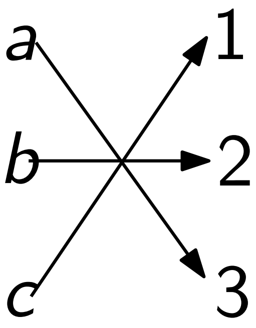
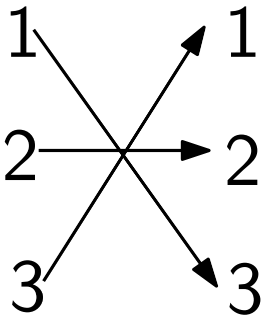
  Thus,
  the left and right sides must have the same size.





The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by n points in the plane.

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trianglecount = 0
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Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

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- 3 points form a triangle if and only if they are non collinear

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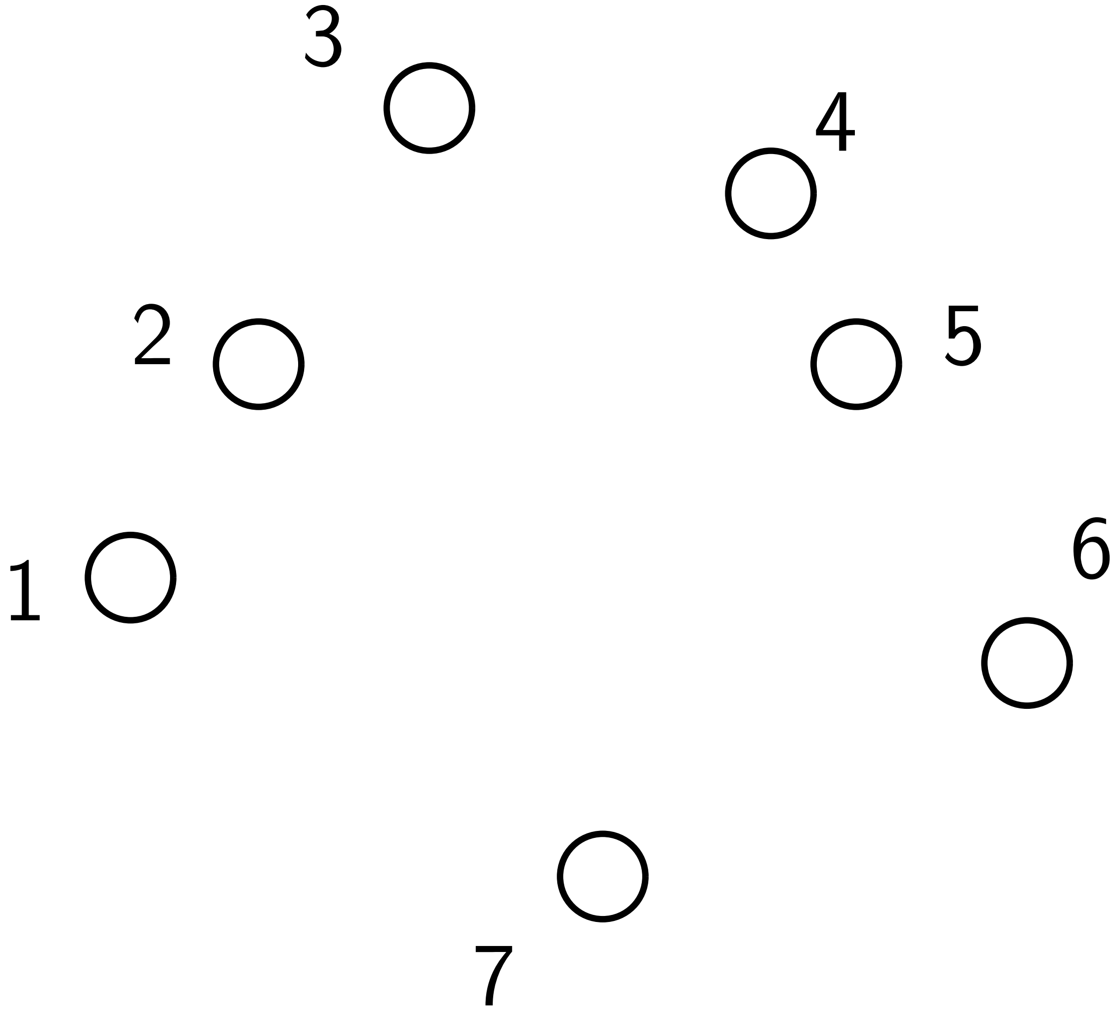

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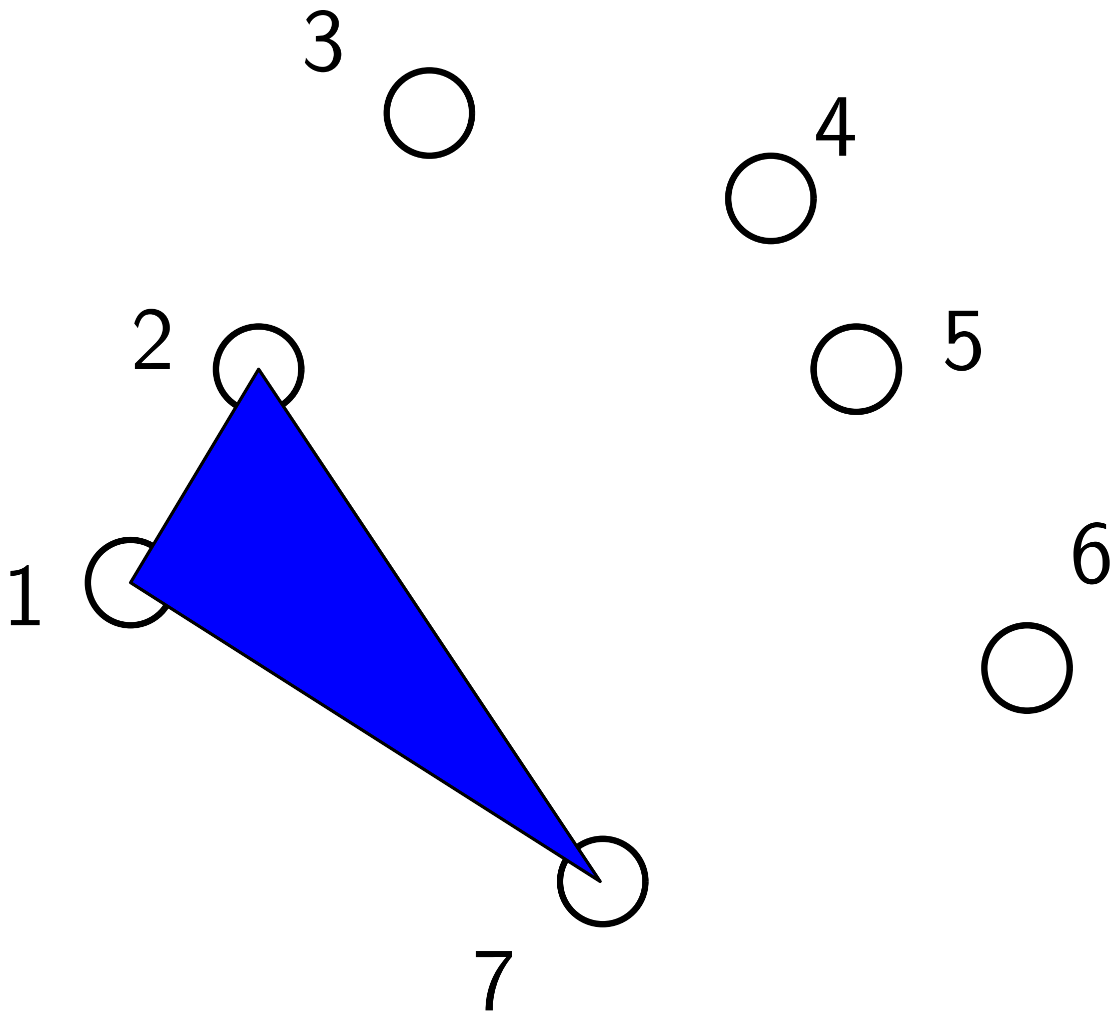

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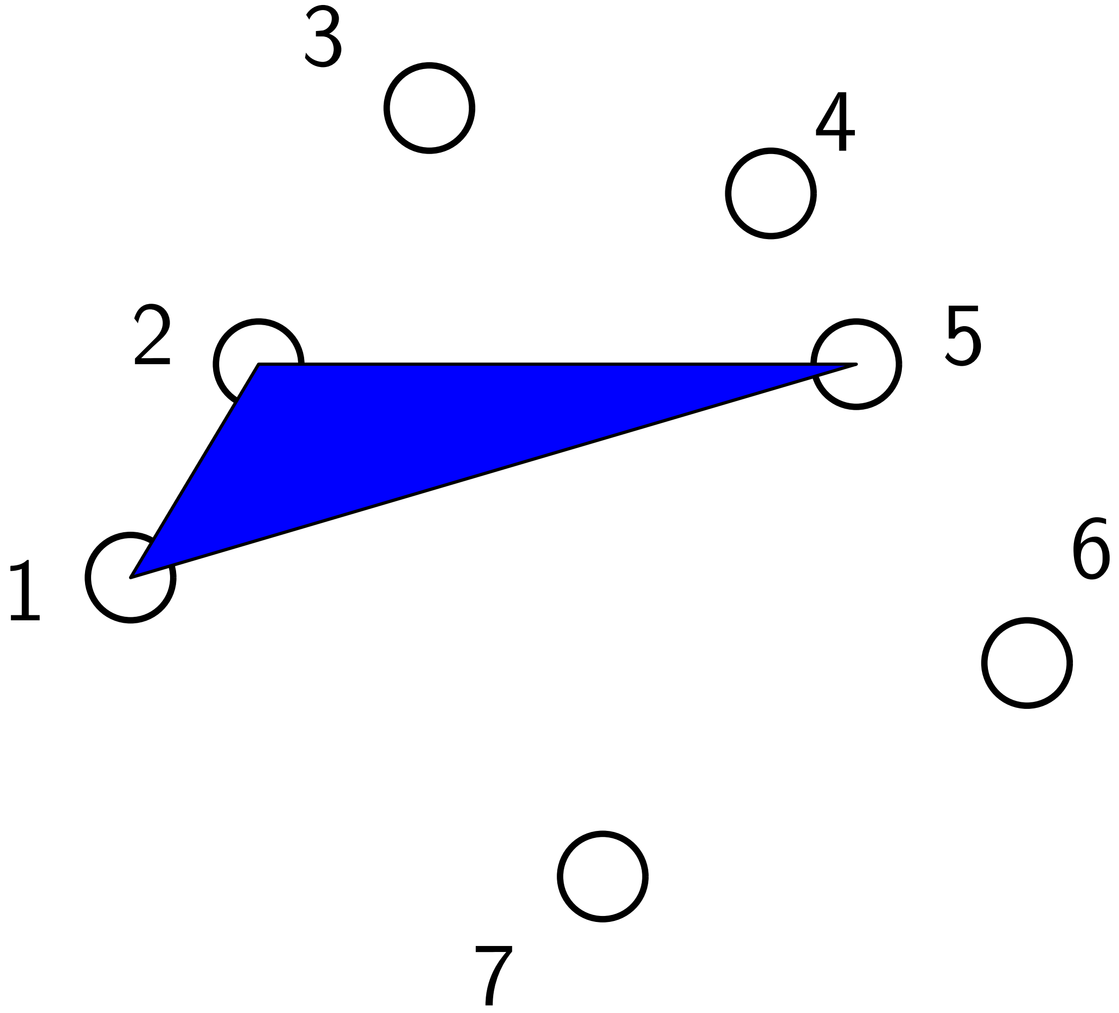

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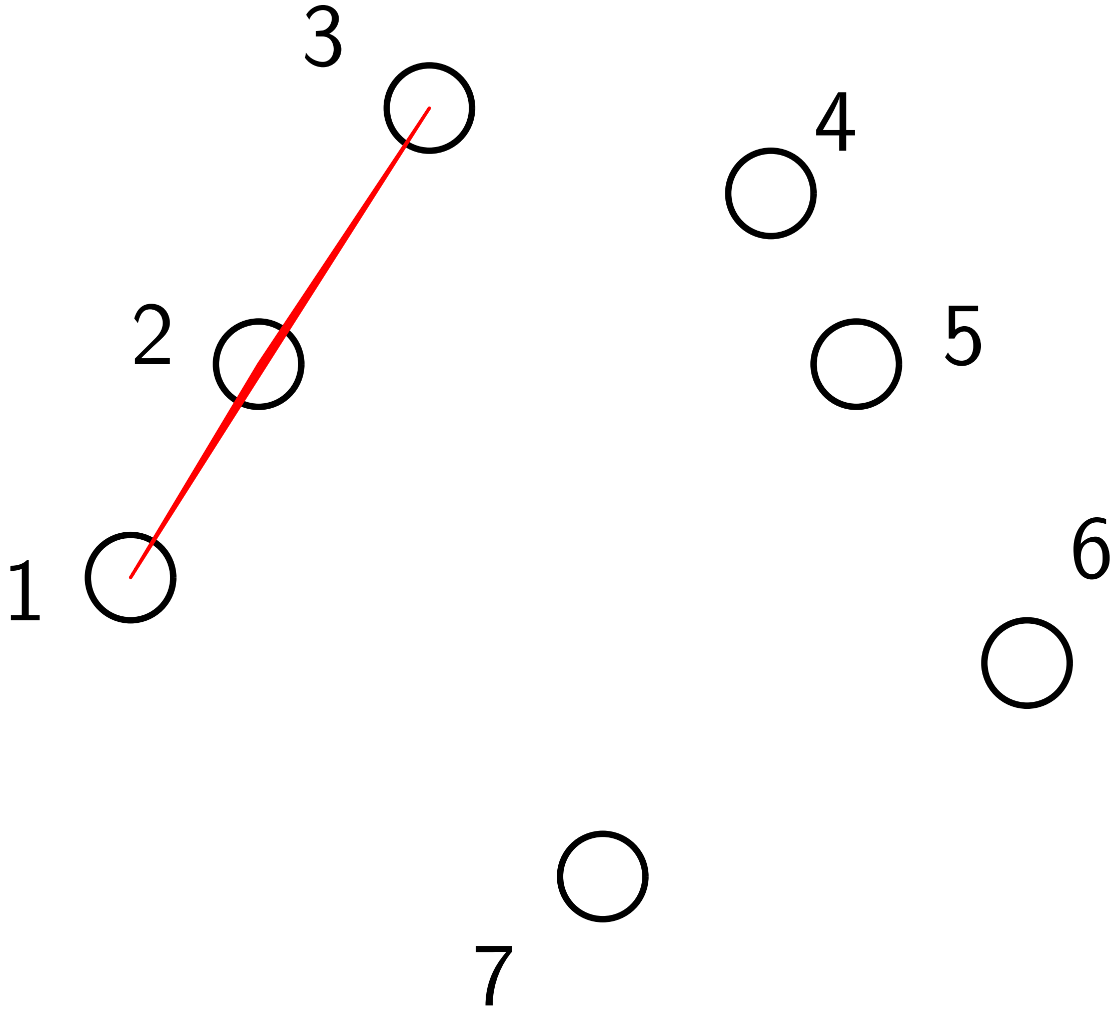

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1 – 2 – 3: no

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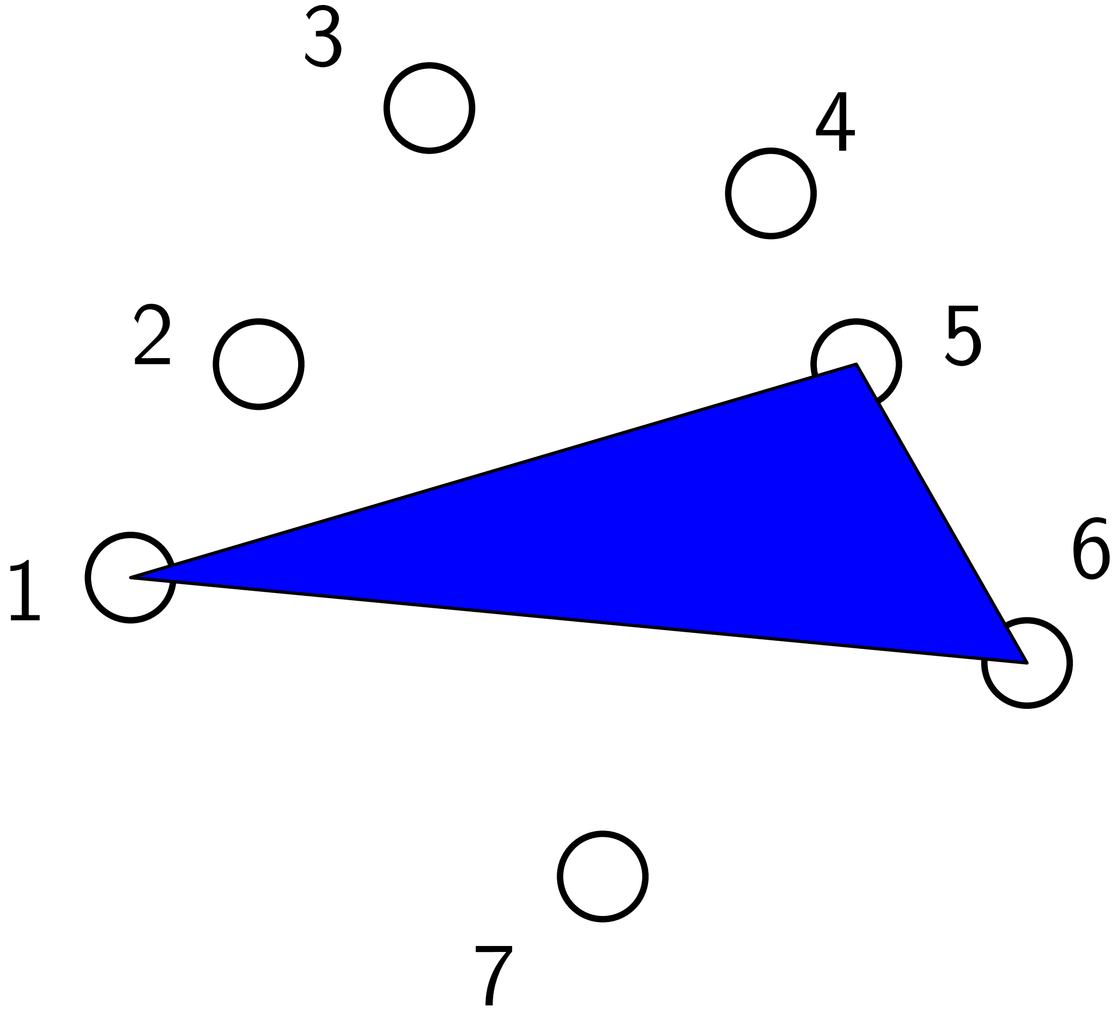

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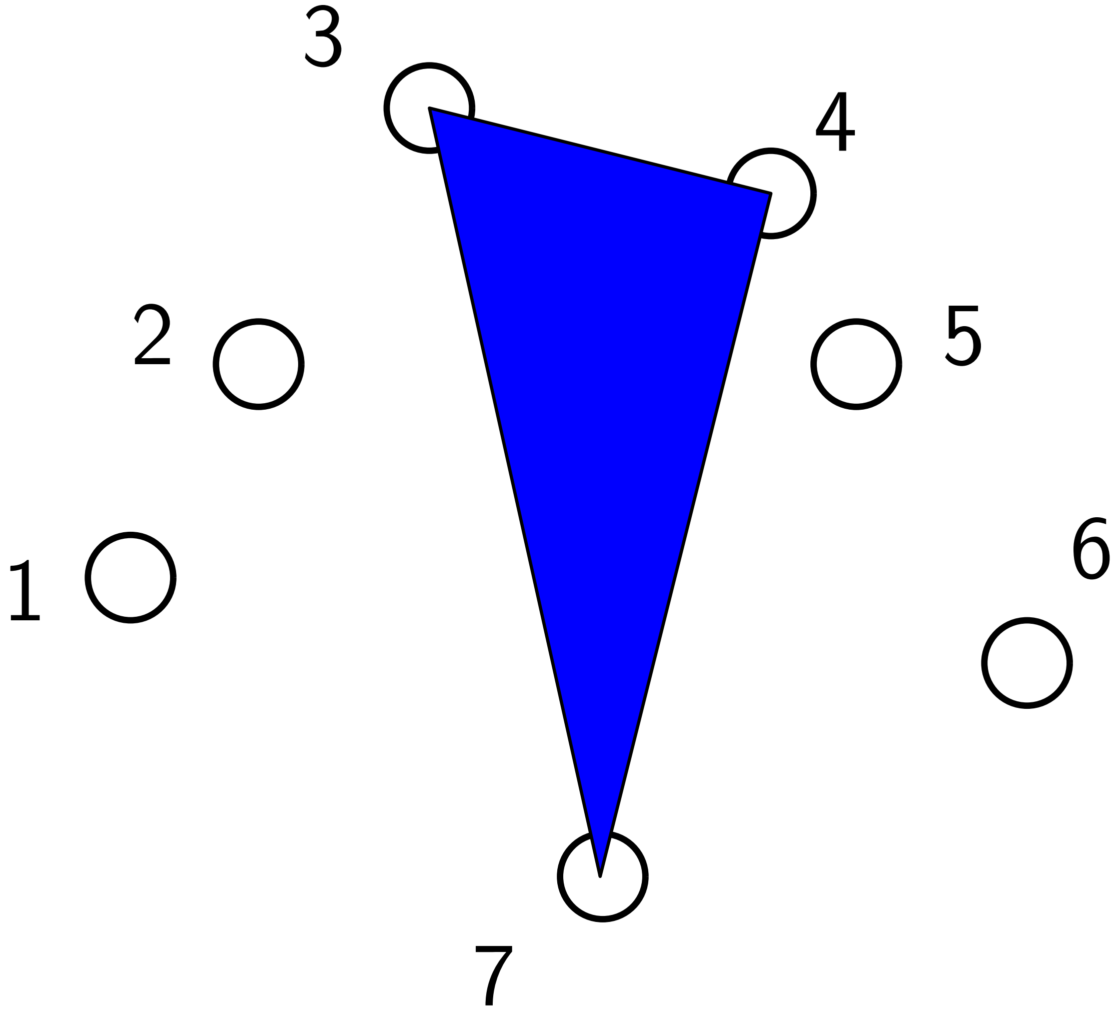

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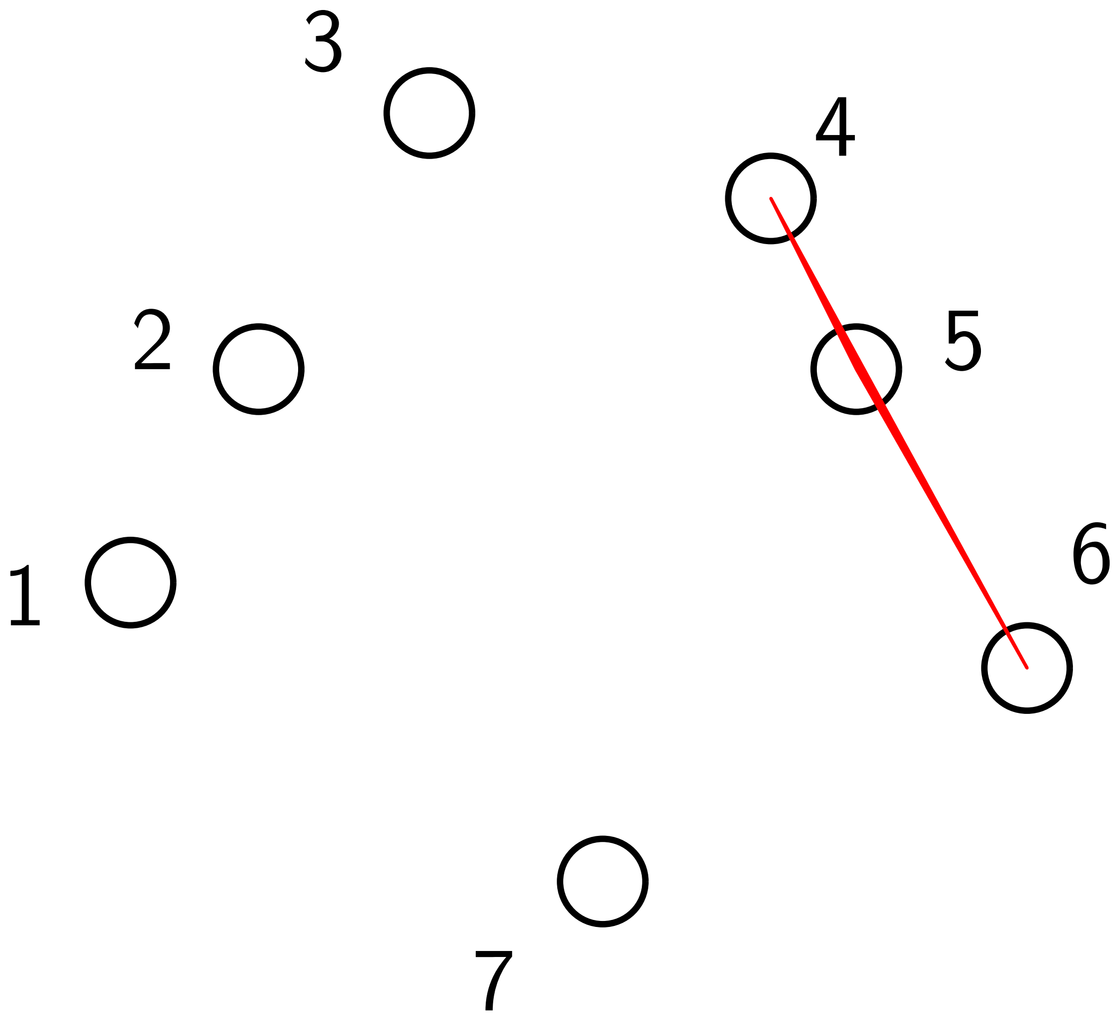

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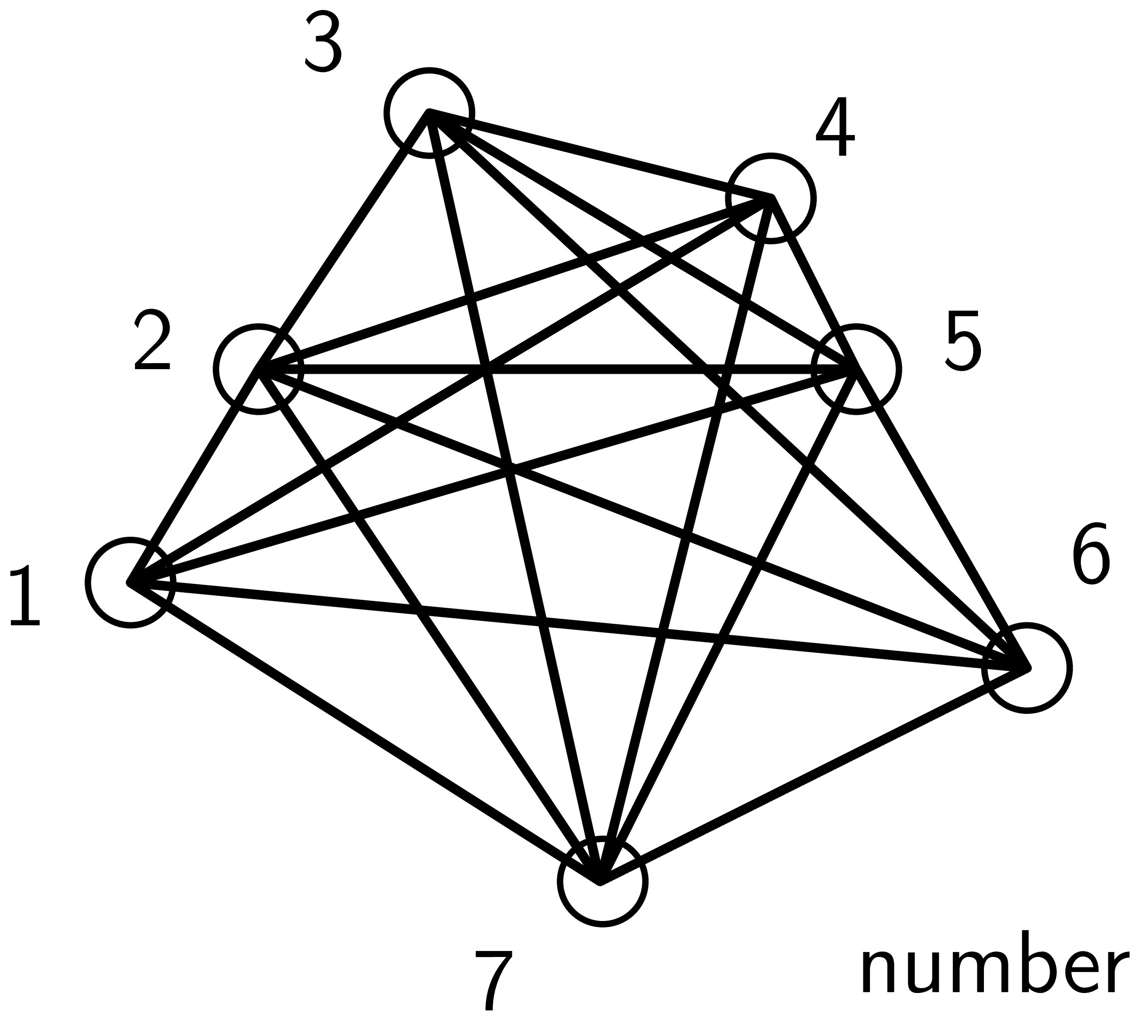

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1 – 5 – 6: yes

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number of triangles: 33

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For example, if $n = 4$, then triples (i, j, k) used by algorithm are (1,2,3), (1,2,4), (1,3,4), and (2,3,4).

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Claim: f is a **bijection** (why) so $|X| = |Y|$

f is a bijection because
f is one-to-one
if $(i, j, k) \neq (i', j', k') \Rightarrow f(\,(i, j, k)\,) \neq f(\,(i', j', k'))$
f is onto
if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$
where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

Counting Pairs

- The number of

  increasing pairs (i, j) with $1 \leq i < j \leq n$

  is the same as the number of

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  We actually already saw that $|X| = |Y| = \binom{n}{2}$

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  In practice, in real problems we often only *implicitly* use the bijection and don't *explicitly* describe it

  Currently, we started with the problem of counting the $\#$ of increasing triples and changed it to the problem of counting the $\#$ of 3-element sets from $\{1, 2, \ldots, n\}$

Inclusion-Exclusion Principle

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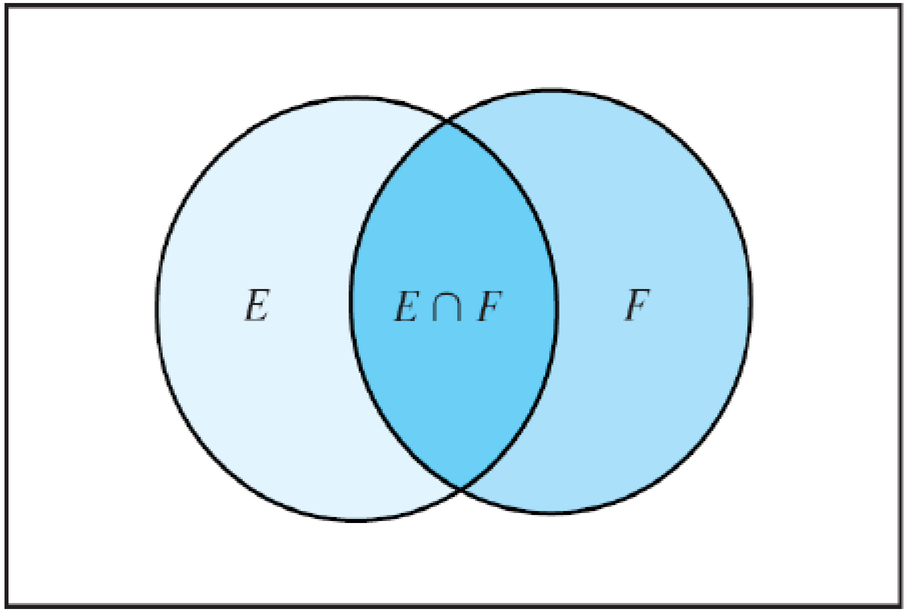
- Two sets

$$|E \cup F| = |E| + |F| - |E \cap F|$$



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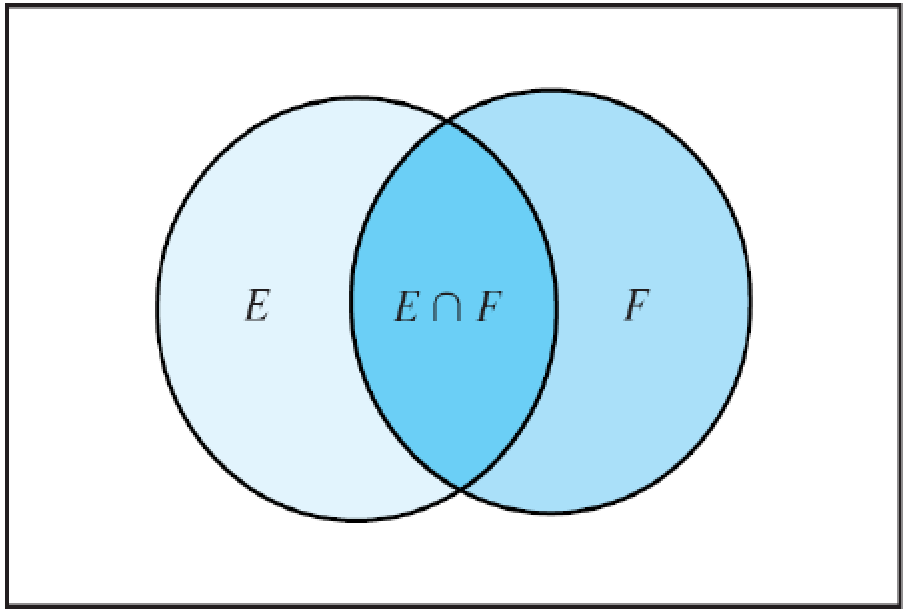
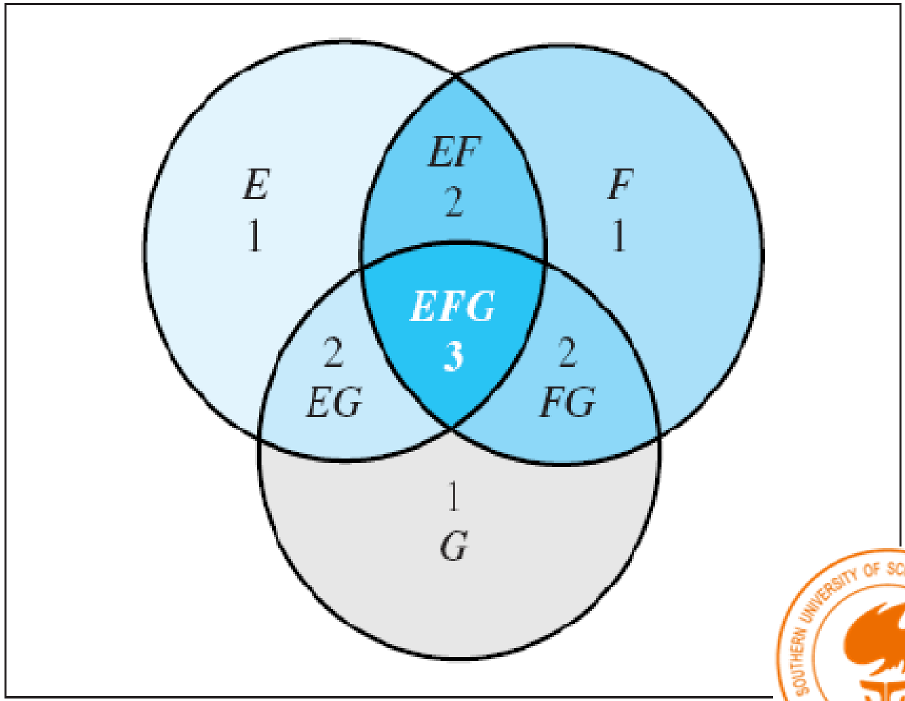


Three sets



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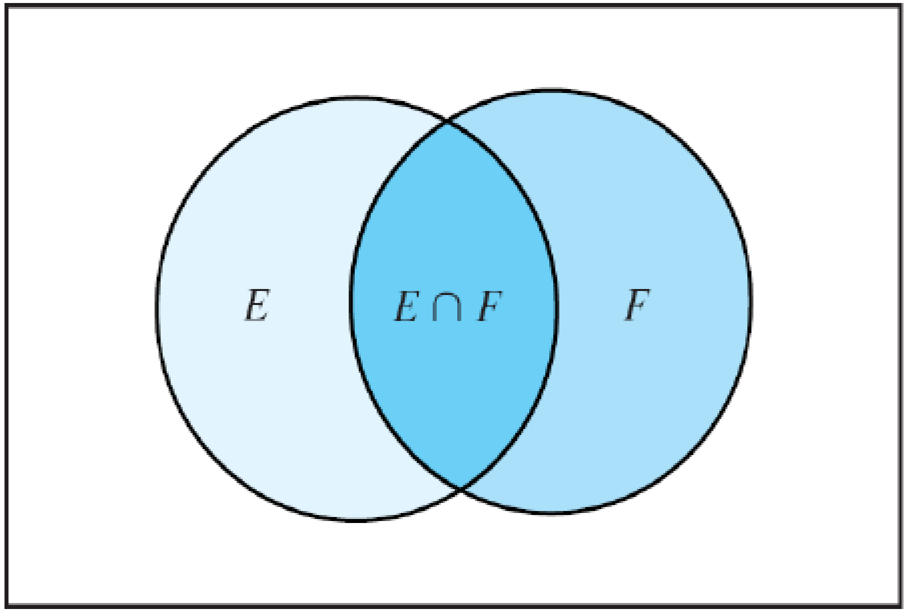
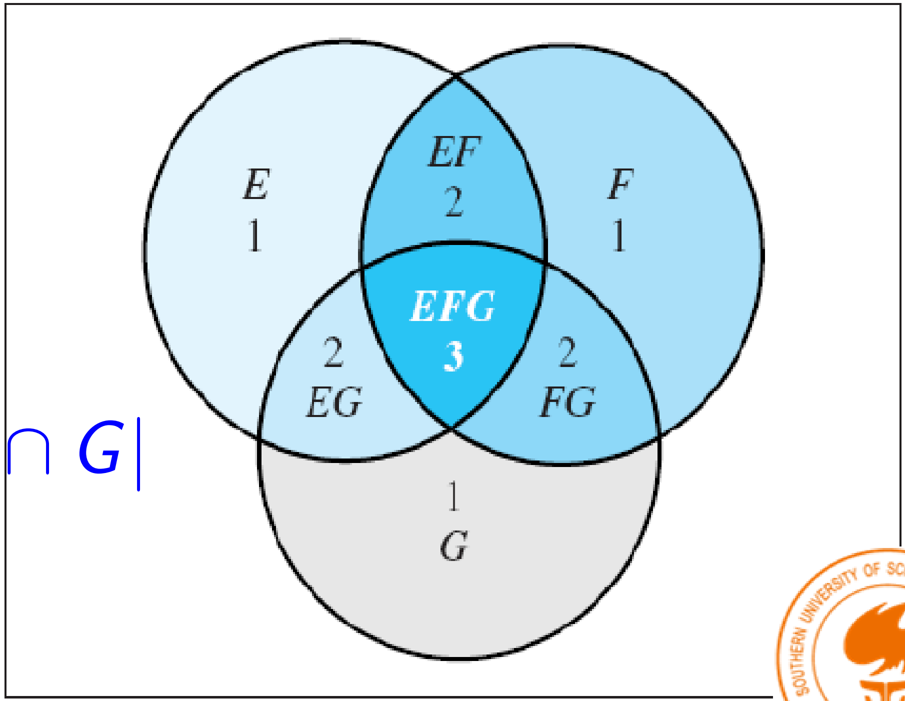


Three sets

$$\begin{aligned} &|E \cup F \cup G| \\ &= |E| + |F| + |G| \\ &\quad -|E \cap F| - |E \cap G| - |F \cap G| \\ &\quad +|E \cap F \cap G| \end{aligned}$$



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Inductive Hypothesis

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For the third term, by distributive law,

$$\left|(\cup_{i=1}^{n-1} E_i) \cap E_n\right| = |\cup_{i=1}^{n-1} (E_i \cap E_n)| = |\cup_{i=1}^{n-1} G_i|$$

where $G_i = E_i \cap E_n$.

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- So far

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Note that (why?)

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Some discussion:
first summation sums $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do not contain n
$|E_n|$ and second summation together sum $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do contain n

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Next Lecture

- recurrence ...