

CS215: Discrete Math

2021 Fall Semester Written Assignment # 6

Due: Dec. 31st, 2021, please submit at the beginning of class

Q.1 Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G .

Q.2 The *complementary graph* \overline{G} of a simple graph G has the same vertices as G . Two vertices are adjacent in \overline{G} if and only if they are not adjacent in G . If the degree sequence of the simple graph G is 4, 3, 3, 2, 2, what is the degree sequence of \overline{G} ?

Q.3 The complementary graph of a simple graph $G = (V, E)$ is the graph $(V, \{(x, y) : x, y \in V, x \neq y\} \setminus E)$. A graph is *self-complementary* if it is isomorphic to its complement.

- (a) Prove that no simple graph with two or three vertices is self-complementary, without enumerating all isomorphisms of such simple graphs.
- (b) Find examples of self-complementary simple graphs with 4 and 5 vertices.

Q.4 Let G be a *simple* graph with n vertices. Show that if the degree of any vertex of G is $\geq (n - 1)/2$, then G must be connected.

Q.5 Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \dots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

- (a) Draw G_5 .
- (b) Find the degree of each vertex in G_n .

Q.6 Let G be a simple graph with n vertices and k connected components.

- (a) What is the minimum possible number of edges of G ?
- (b) What is the maximum possible number of edges of G ?

Q.7 Suppose that G is a graph on a finite set of n vertices. Prove that if G is disconnected, then its complement is connected.

Q.8 In an n -player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner – there are no ties. The results of such a tournament can then be represented with a *tournament directed graph* where the vertices correspond to players and there is an edge $x \rightarrow y$ iff x beats y in their game.

- (a) Explain why a tournament directed graph cannot have cycles of length 1 or 2.
- (b) Is the “beats” relation for a tournament graph always/sometimes/never: antisymmetric? reflexive? irreflexive? transitive?
- (c) Show that a tournament graph represents a total ordering iff there are no cycles of length 3.

Q.9 Let G be a connected simple graph. Show that if an edge in a connected graph is not traversed by any simple cycle, then this edge is a *cut edge*.

Q.10 Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G . Let G be a bipartite k -regular graph (the degree of every vertex is k) for $k \geq 2$. Prove that G has no bridge.

Q.11 Let G be a connected graph, with the vertex set V . The *distance* between two vertices u and v , denoted by $\text{dist}(u, v)$, is defined as the *minimal* length of a path from u to v . Show that $\text{dist}(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

- (i) $\text{dist}(u, v) \geq 0$ and $\text{dist}(u, v) = 0$ if and only if $u = v$.
- (ii) $\text{dist}(u, v) = \text{dist}(v, u)$.
- (iii) $\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$.

Q.12 Given a connected graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices u, v in G is defined as the length of a shortest path between u and v . The *diameter* $\text{diam}(G)$ of G is defined as the greatest distance among all pairs of vertices in G . That is, $\max_{u, v \in V} d_G(u, v)$. The *eccentricity* $\text{ecc}(v)$ of

a vertex v of G is defined as $\max_{u \in V} d_G(u, v)$. Finally, the *radius* $\text{rad}(G)$ of G is defined as the minimal eccentricity of a vertex in G , namely $\min_{v \in V} \text{ecc}(v)$. Prove the following.

- (a) $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$.
- (b) For every positive integer n , there are connected graphs G_1 and G_2 with $\text{diam}(G_1) = \text{rad}(G_1) = n$ and $\text{diam}(G_2) = 2\text{rad}(G_2) = 2n$.

Q.13 Show that isomorphism of simple graphs is an equivalence relation.

Q.14 Suppose that G_1 and H_1 are isomorphic and that G_1 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.

Q.15 Given a graph G , its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding edges of G share a common endpoint. Prove that if G is regular (all vertices have the same degree) and connected, then $L(G)$ has an Euler circuit.

Q.16 Show that if G is simple graph with at least 11 vertices, then either G or its complement graph \bar{G} , the complement of G , is nonplanar.

Q.17 Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

Q.18 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2 . The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

- (1) K_6
- (2) $K_{4,5}$
- (3) Q_3
- (4) C_6

Q.19 Let G be a graph in which all vertices have degree at least d . Prove that G contains a path of length d .

Q.20 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly two* vertices of degree i for each $i = 1, 2, \dots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Q.21 Consider the two graphs G and H . Answer the following three questions, and explain your answers.

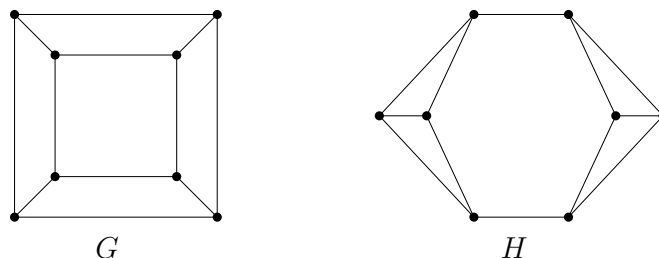


Figure 1: Q.21

- (1) Which of the two graphs is/are *bipartite*?
- (2) Are the two graphs *isomorphic* to each other?
- (3) Which of the two graphs has/have an *Euler circuit*?

Q.22 There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Q.23 Prove that $G = (V, E)$ is a tree if and only if $|V| = |E| + 1$ and G has no cycles.

Q.24 The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and

for $n = 3, 4, \dots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?

Q.25

What is the value of each of these postfix expressions?

(a) $5\ 2\ 1\ -\ -\ 3\ 1\ 4\ +\ +\ *$

(b) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

Q.26

Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

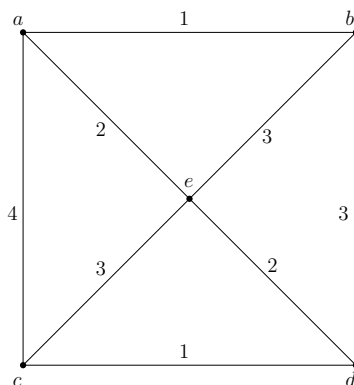


Figure 2: Q.26

Q.27

Use Kruskal's algorithm to find a minimum spanning tree for the weighted graph in Q.26.