南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Solving Linear Recurrence Relations of degree k

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  Theorem If this CE has k distinct roots r_i, then the solutions to the recurrence are of the form

  $$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

  for all $n \geq 0$, where the α_i's are constants.

The Case of Degenerate Roots

- **Theorem** If the CE $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0, then

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- **Theorem** [Theorem 4, p.519] Suppose that there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then
  $$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n,$$
  for all $n \geq 0$ and constants $\alpha_{i,j}$.

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  We get $\alpha_1 = 1$ and $\alpha_2 = -1$. Thus,

  $a_n = 2^n - n2^n$

Linear Nonhomogeneous Recurrence Relations

- **Definition** A *linear nonhomogeneous relation* with constant coefficients may contain some terms $F(n)$ that depend only on n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$$

The recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the *associated homogeneous recurrence relation*.

Linear Nonhomogeneous Recurrence Relations

- **Theorem** If $a_n = p(n)$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

  Then all its solutions are of the form

  $$a_n = p(n) + h(n),$$

  where $a_n = h(n)$ is any solution to the associated homogeneous recurrence relation

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  Let $p(n) = cn + d$, then
  $cn + d = 3(c(n-1) + d) + 2n$, which means
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  We get $c = -1$ and $d = -3/2$. Thus,
  $$p(n) = -n - 3/2$$

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Definition The *generating funciton* for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k$$

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  $1, 2, 3, 4, 5, \ldots$

Operations of Generating Functions

- **Theorem** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

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Useful Generating Functions

$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$

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$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

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$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

- **Problem 1** Find the number of solutions of

$$x_1 + x_2 + x_3 = 17,$$

where x_1, x_2, x_3 are nonnegative integers with $2 \leq x_1 \leq 5$, $3 \leq x_2 \leq 6$, $4 \leq x_3 \leq 7$.

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where x_1, x_2, x_3 are nonnegative integers with $2 \le x_1 \le 5$, $3 \le x_2 \le 6$, $4 \le x_3 \le 7$.

Using *generating functions*, the number is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$

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The coefficient of x^8 in the expansion

$$(x^2 + x^3 + x^4)^3$$

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$$C(n+r-1, r) = C(19, 17) = C(19, 2)$$

r-Combinations from a Set

- **Definition** An *r-combination* with repetition allowed, or a *multiset of size r*, chosen from a set of n elements, is an unordered selection of elements with repetition allowed.

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  Read more on pp. 537-548.

- **Problem 4** Use generating functions to find the number of k-combinations of a set with n elements, $C(n, k)$.

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Each of the n elements in the set contributes the term $(1+x)$ to the generating function $f(x) = \sum_{k=0}^{n} a^k x^k$.
Hence, $f(x) = (1+x)^n$.

Then by the binomial theorem, we have $a_k = \binom{n}{k}$.

Cartesian Product

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

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Cartesian product defines a set of all ordered arrangements of elements in the two sets.

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  Example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$

  ◇ Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B?

  ◇ Is $Q = \{(1, a), (2, b)\}$ a relation from A to B?

  ◇ Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A?

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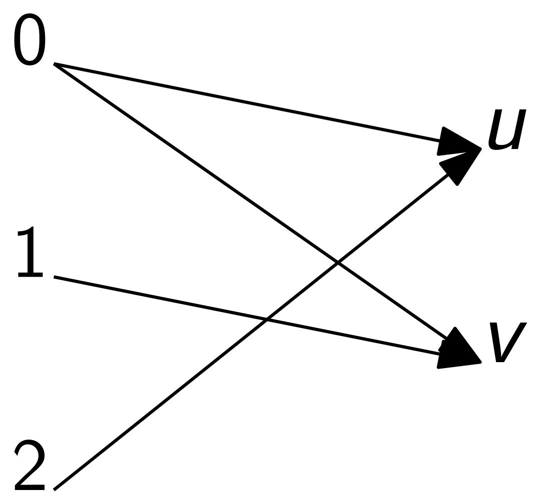

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Representing Binary Relations

- We can also represent a binary relation R by a table showing the ordered pairs of R.

Example: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, u), (2, v)\}$. ($R \subseteq A \times B$)

R	u	v
0	×	×
1	×	
2		×

Relations and Functions

- Relations represent one to many relationships between elements in A and B.

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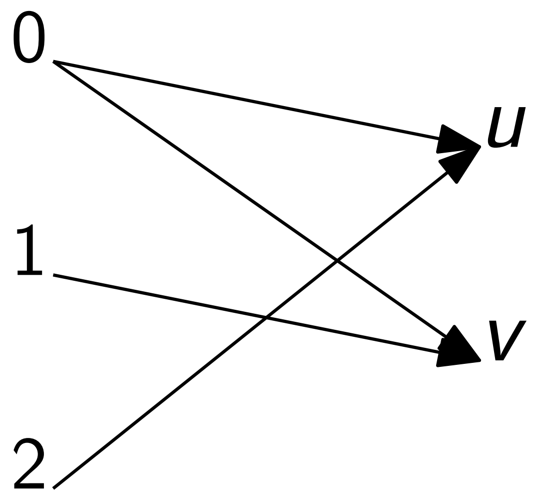

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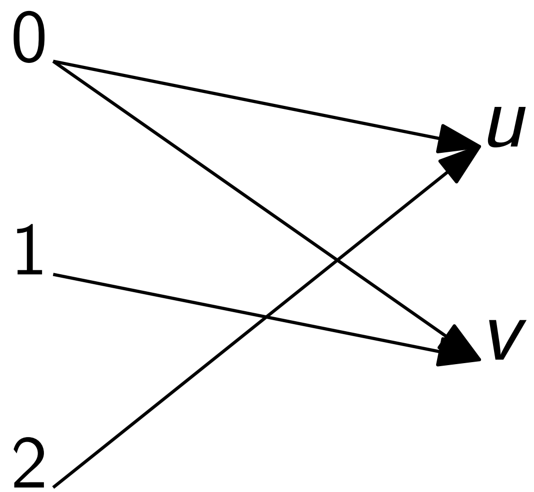

What is the difference between a relation and a function from A to B?

Relation on the Set

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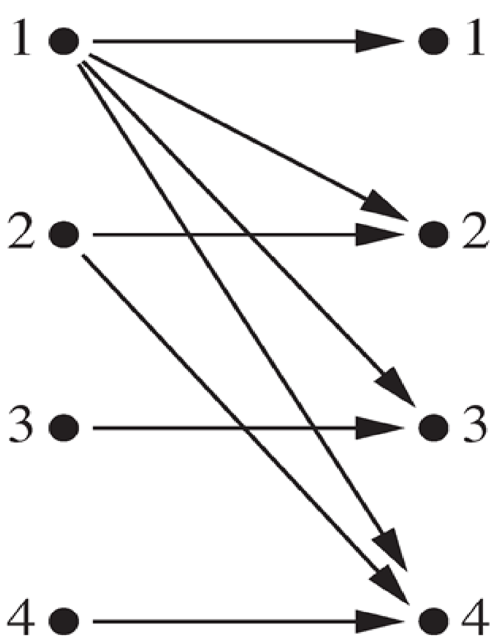

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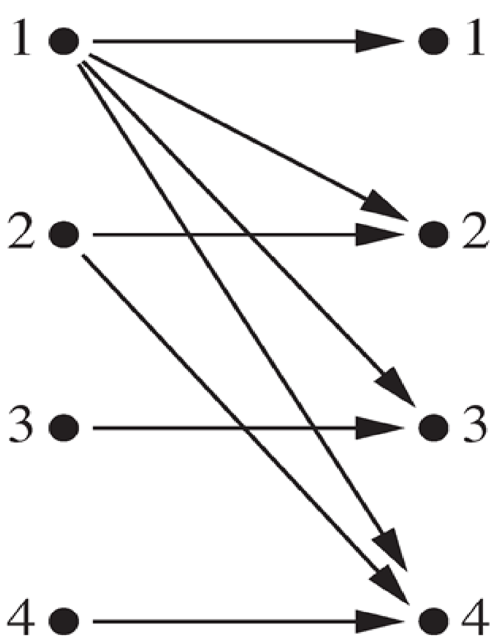

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Number of Binary Relations

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  The number of subsets of a set with k elements is 2^k

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

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Combining Relations

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

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  Set operations: union, intersection, difference, etc.

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What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

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  We may also combine relations by matrix operations.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

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$R^k = ?$ for $k > 3$

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Next Lecture

- relation II...