CS215: Discrete Math (H)
2021 Fall Semester Written Assignment # 5
Due: Dec. 17th, 2021, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b)|a$ and b have no letters in common$\}$

(2) $R_2 = \{(a, b)|a$ and b are not the same length$\}$

(3) $R_3 = \{(a, b)|a$ is longer than $b\}$

Solution:

(1) Irreflexive, symmetric

(2) Irreflexive, symmetric

(3) Irreflexive, antisymmetric, transitive

$\square$

Q.2 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Solution:

(a) $2^{n(n+1)/2}$

(b) $2^n 3^{n(n-1)/2}$

(c) $2^{n(n-1)}$

(d) $2^{n(n-1)/2}$

(e) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

(f) $3^{n(n-1)/2}$

(g) 2^n

$\square$

Q.3 Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?
Solution: R^2 might not be irreflexive. For example, $R = \{(1,2),(2,1)\}$.

$\square$

Q.4 Give an example of a relation R such that its transitive closure R^* satisfies $R^* = R \cup R^2 \cup R^3$, but $R^* \neq R \cup R^2$.
Solution: We fix the ground set $S = \{a, b, c, d\}$, and we consider the relation $R = \{(a,b),(b,c),(c,d)\}$. Then the transitive closure of R equals $R^* = \{(a,b),(b,c),(c,d),(a,c),(b,d),(a,d)\}$. On the other hand, $R^2 = \{(a,c),(b,d)\}$, and $R^3 = \{(a,d)\}$. Hence, R^3 is necessary to get R^*.

$\square$

Q.5 Suppose that R_1 and R_2 are both *reflexive* relations on a set A.

(1) Show that $R_1 \oplus R_2$ is *irreflexive.*

(2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.

(3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Solution:

(1) Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \oplus R_2$ for all $a \in A$. Thus, $R_1 \oplus R_2$ is irreflexive.

(2) Yes. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \cap R_2$

(3) Yes. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \cup R_2$

$\square$

Q.6 Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$.

(a) Show that R is an equivalence relation.

(b) What is the equivalence class of $(1,2)$ with respect to the equivalence relation R?

(c) Give an interpretation of the equivalence classes for the equivalence relation R.

Solution:

(a) For reflexivity, $((a,b),(a,b)) \in R$ because $a \cdot b = b \cdot a$. If $((a,b),(c,d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c,d),(a,b)) \in R$; this tells us that R is symmetric. Finally, if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a,b),(e,f)) \in R$; this tells us that R is transitive.

(b) The equivalence classes of $(1,2)$ is the set of all pairs (a,b) such that the fraction a/b equals $1/2$.

(c) The equivalence classes are the positive rational numbers.

$\square$

Q.7 Show that the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined on $(a,b)\mathbb{R}(c,d)$ if and only if $a+d=b+c$ is an *equivalence* relation.

Solution: $((a,b),(a,b)) \in R$ because $a+b=a+b$. Hence R is reflexive.

If $((a,b),(c,d)) \in R$ then $a+d=b+c$, so that $c+b=d+a$. It then follows that $((c,d),(a,b)) \in R$. Hence R is symmetric.

Suppose that $((a,b),(c,d))$ and $((c,d),(e,f))$ belong to R. Then $a+d=b+c$ and $c+f=d+e$. Adding these two equations and subtracting $c+d$ from both sides gives $a+f=b+e$. Hence $((a,b),(e,f))$ belongs to R. Hence, R is transitive.

$\square$

Q.8 How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?

Solution: 25. There are two possibilities to form exactly three different equivalence classes with 5 elements. One is 3, 1, 1 elements for each equivalence class, and the other is 2, 2, 1 elements for each equivalence class. By counting techniques, there are $\binom{5}{3} + \binom{5}{1} \cdot \binom{4}{2}/2 = 25$.

$\square$

Q.9 Show that $\{(x,y)|x-y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Solution: This relation is reflexive, since $x-x=0 \in \mathbb{Q}$. To see that it is symmetric, suppose that $x-y \in \mathbb{Q}$. Then $y-x=-(x-y)$ is again a rational number. For transitivity, if $x-y \in \mathbb{Q}$ and $y-z \in \mathbb{Q}$, then their sum, namely $x-z$, is also rational (the rational numbers are closed under addition). The equivalence class of 1 and of 1/2 are both just the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number, in other words, $\{\pi + r | r \in \mathbb{Q}\}$.

$\square$

Q.10 Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x,y) \in T$ if and only if $(x,y) \in R$ and

$(x, y) \in S$. Prove or disprove: If R and S are both *equivalence relations*, then T is also an equivalence relation.

Solution: We need to show that T is reflexive, symmetric, and transitive.

Reflexive: For any x, we have $(x, x) \in R$ and $(x, x) \in S$, then $(x, x) \in T$.

Symmetric: Suppose that $(x, y) \in T$. This means $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both symmetric, we have $(y, x) \in R$ and $(y, x) \in S$. Then $(y, x) \in T$.

Transitive: Suppose that $(x, y) \in T$ and $(y, z) \in T$. Then $(x, y) \in R$ and $(y, x) \in R$ imply that $(x, z) \in R$. Similarly, we have $(x, z) \in S$. This will imply that $(x, z) \in T$.

$\square$

Q.11 Let $\sim$ be a relation defined on $\mathbb{N}$ by the rule that $x \sim y$ if $x = 2^k y$ or $y = 2^k x$ for some $k \in \mathbb{N}$. Show that $\sim$ is an equivalence relation.

Solution: We first show the following lemma.

Lemma For any $x, y \in \mathbb{N}$, $x \sim y$ if and only if there exists some $k \in \mathbb{Z}$ such that $x = 2^k y$ in $\mathbb{Q}$.

Proof. Suppose that $x \sim y$. Then either $x = 2^k y$ for some $k \in \mathbb{N} \subseteq \mathbb{Z}$ and we are done, or $y = 2^{k'} x$ for some $k' \in \mathbb{N}$. In the latter case, solve for $x = 2^{-k'} y$ and let $k = -k'$. In the other direction, if $x = 2^k y$, and $k \geq 0$, then $x = 2^k y$ for some $k \in \mathbb{N}$, giving $x \sim y$. If instead $k < 0$, then $y = 2^{-k}$, again giving $x \sim y$.

To show $\sim$ is an equivalence relation, we show the following three properties.

Reflexive For any $x \in \mathbb{N}$, $x = 2^0 x$ so $x \sim x$.

Symmetric If $x \sim y$, then from **Lemma** there exists $k \in \mathbb{Z}$ such that $x = 2^k y$. But then $y = 2^{-k} x$, so applying the lemma again, gives $y \sim x$.

Transitive If $x \sim y \sim z$, then $x = 2^k y$ and $y = 2^\ell z$ for some $k, \ell \in \mathbb{Z}$ by **Lemma**. Solve to get $x = 2^{k+\ell} z$, which gives $x \sim z$.

$\square$

Q.12 Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

(a) Prove that $\preceq$ is a partial ordering.

(b) Prove or disprove: $\preceq$ is a total ordering.

Solution:

(a) **Reflexive** For all $x \in \mathbb{R}$, $f(x) \leq f(x)$, so $f \preceq f$.

Antisymmetric Let $f \preceq g$ and $g \preceq f$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq f(x)$ and thus $f(x) = g(x)$. Since this holds for all x, we have $f = g$.

Transitive Let $f \preceq g \preceq h$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq h(x)$, giving $f(x) \leq h(x)$. So, $f \preceq h$.

(b) It is not a total ordering. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\leq -1 = g(1)$ and $g(-1) = 1 \not\leq -1 = f(-1)$. So it is not the case that for all x, $f(x) \leq g(x)$, and it is not the case that for all x, $g(x) \leq f(x)$. That is, these two functions are incomparable.

$\square$

Q.13 Which of these are posets?

(a) $(\mathbf{R}, =)$

(b) $(\mathbf{R}, <)$

(c) $(\mathbf{R}, \leq)$

(d) $(\mathbf{R}, \neq)$

Solution:

(a) Yes. (It is the smallest partial order: reflexivity ensures that very partial order contains at least all pairs (a, b).)

(b) No. It is not reflexive.

(c) Yes.

(d) No. The relations is not reflexive, not antisymmetric, not transitive.

$\square$

Q.14 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Solution:

(a) No. $\propto$ is not symmetric. Let $f(n) = n$ and $g(n) = n^2$. Here $f = O(g)$ but $g \neq O(f)$.

(b) No. $\propto$ is not antisymmetric. Let $f(n) = n$ and $g(n) = 2n$. Then $f = O(g)$ and $g = O(f)$, but $f \neq g$.

(c) No. It is not partial ordering, then not a total ordering.

$\square$

Q.15 For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3 + 5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Solution:

(a) Yes, because the product of positive integers greater than or equal to 2 is less than their sum.

(b) No, because $33 \preceq 26$ and $26 \preceq 33$, but $26 \neq 33$.

(c) No, because $33 \preceq 35$ and $35 \preceq 13$, but we do not have $33 \preceq 13$.

$\square$

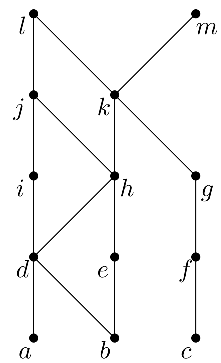

Figure 1: Q.16

Q.16 Answer these questions for the partial order represented by this Hasse diagram.

(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) Find all lower bounds of $\{f, g, h\}$.

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

(a) The maximal elements are the ones with no other elements above them, namely l and m.

(b) The minimal elements are the ones with no other elements below them, namely a, b and c.

(c) There is no greatest element, since neither l nor m is greater than the other.

(d) There is no least elements, since neither a nor b is less than the other.

(e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l and m are the elements fitting this description.

(f) Since k is less than both l and m, it is the least upper bound of a, b and c.

(g) No element is less than both f and h, so there are no lower bounds.

(h) Since there is no lower bound, there cannot be greatest lower bound.

□

Q.17 We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0,1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0,1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.

Solution:

(a) There are many choices here. One is to let $R = \{A_0, A_1, A_2, \ldots\}$ where $A_i = \{j \in \mathbb{N} | j < i\}$. Then R has no maximal element, because for any $A_i \in R$, we have $A_i \not\supseteq A_{i+1} \in R$.

(b) For this we will do the same thing as above in reverse. Let $S = \{B_0, B_1, B_2, \ldots\}$ where $B_i = \{j \in \mathbb{N} | j \geq i\}$. Then S has no minimal element, because for any $B_i \in S$, we have $B_i \not\subseteq B_{i+1}$.

(c) Here we can combine the previous two results. Let $T = \{C_{ij} | i \in \mathbb{N},\ j \in \mathbb{N}\}$ where each $x \in \mathbb{N}$ is in C_{ij} if and only if $x = 2k$ and $k < i$, or $x = 2k+1$ and $K \geq j$. Now T has no minimal or maximal elements, because for any $C_{ij} \in T$, $C_{i,j+1} \nsubseteq C_{ij} \nsubseteq C_{i+1,j}$.

$\square$