



CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Department of Computer Science and Engineering

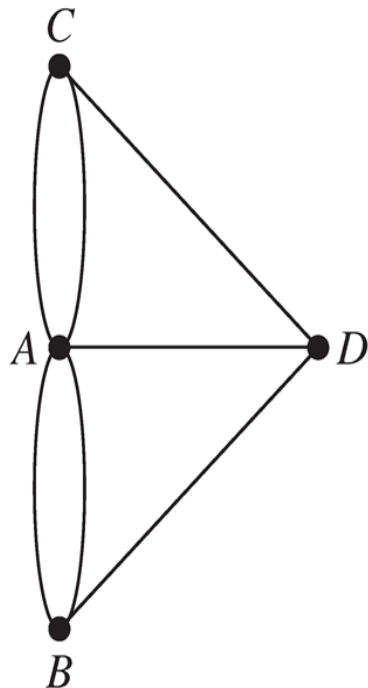
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Euler Circuits and Euler Paths

- **Theorem** A connected multigraph with at least two vertices has an *Euler circuit* if and only if **each of its vertices has even degree**.

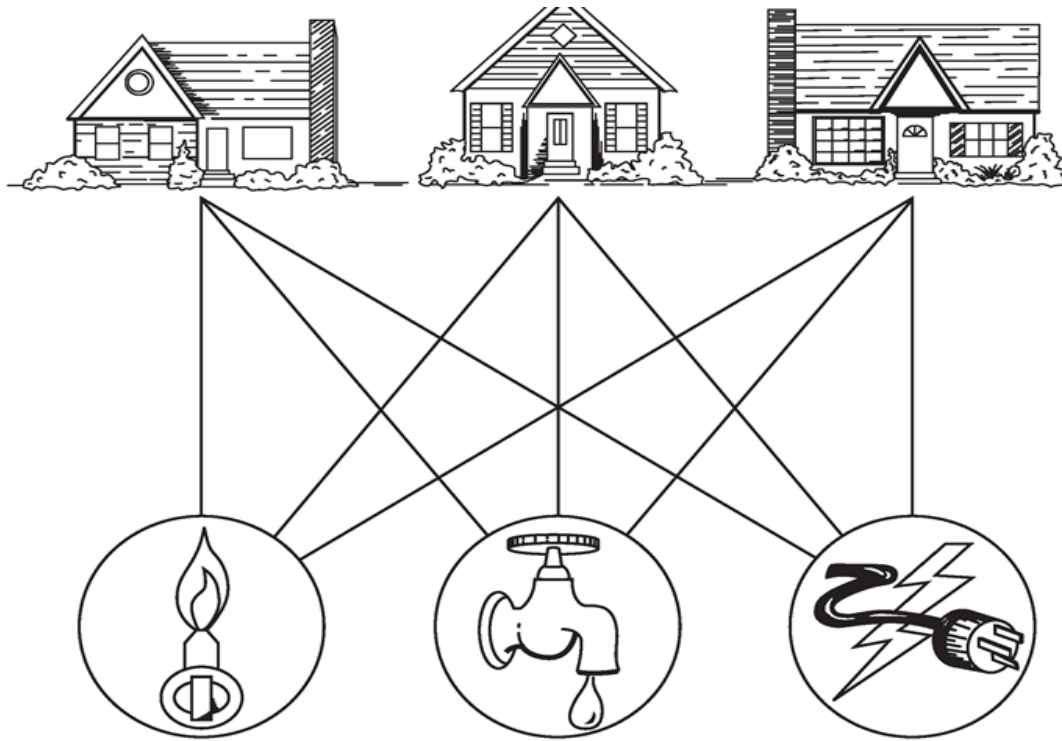
Theorem A connected multigraph has an *Euler path* but not an *Euler circuit* if and only if **it has exactly two vertices of odd degree**.



No Euler circuit

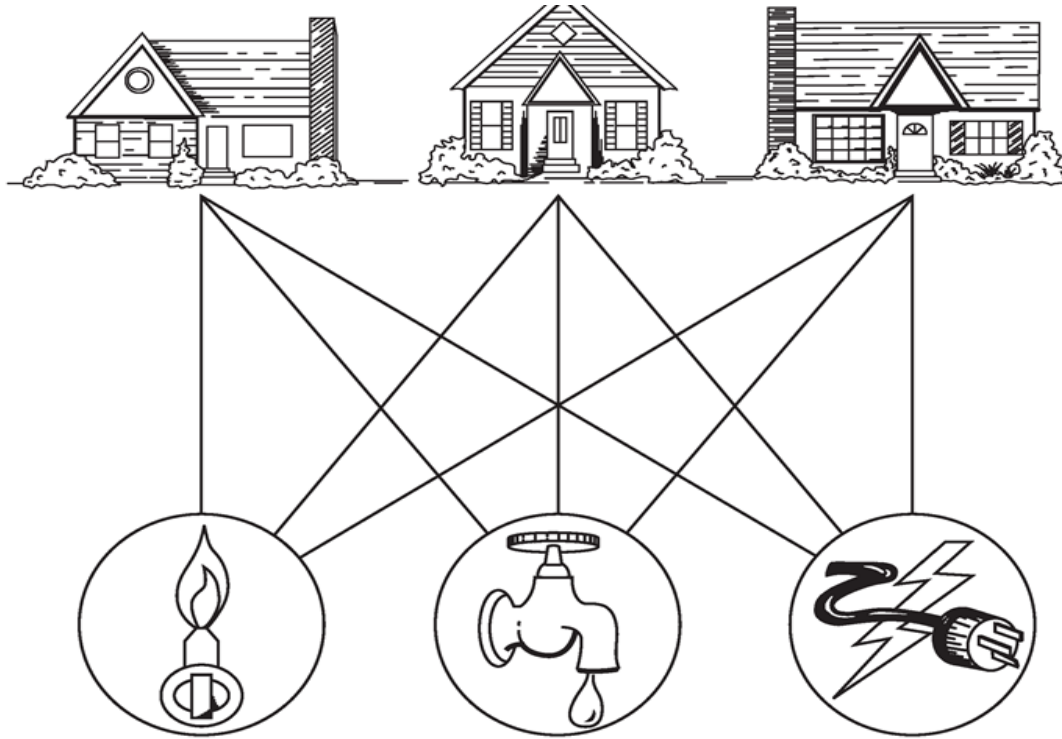
Planar Graphs

- Join three houses to each of three separate utilities.



Planar Graphs

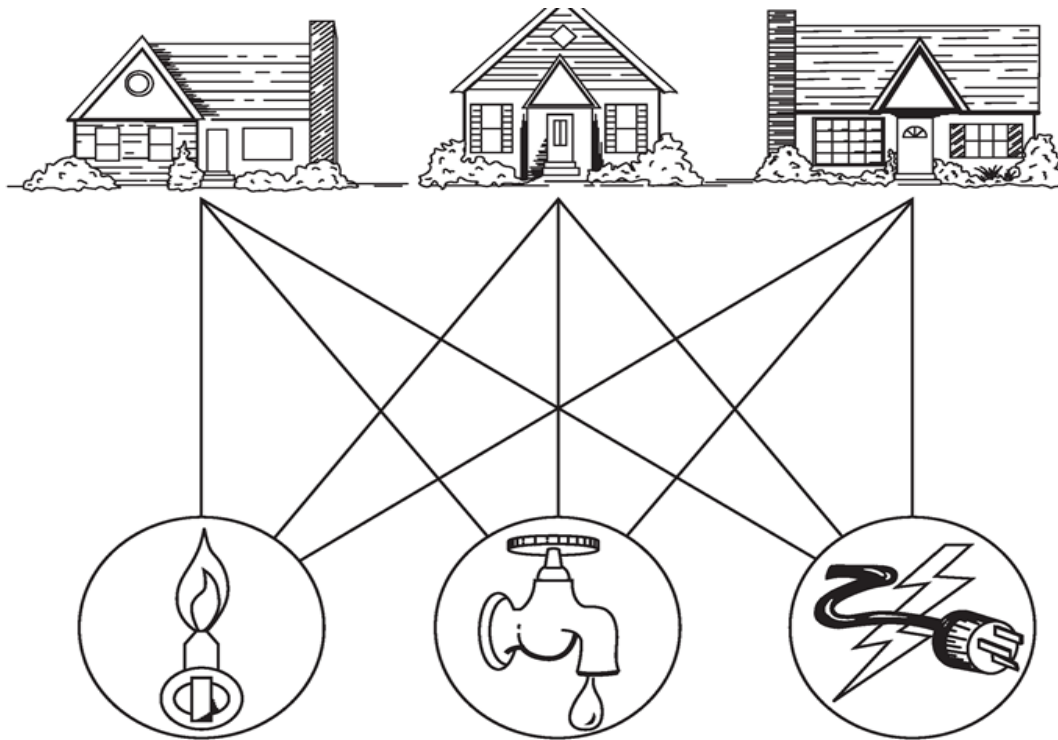
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Can this graph be drawn in the plane s.t. no two of its edges cross?

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Can this graph be drawn in the plane s.t. no two of its edges cross? $K_{3,3}$

Planar Graphs

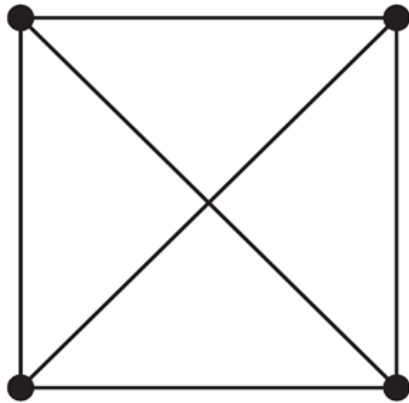
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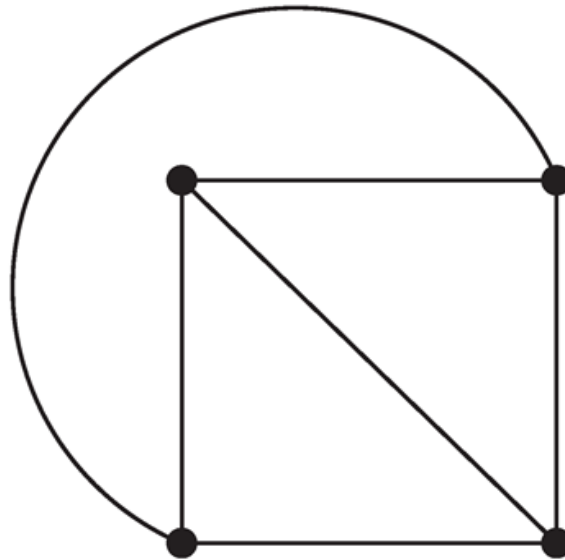
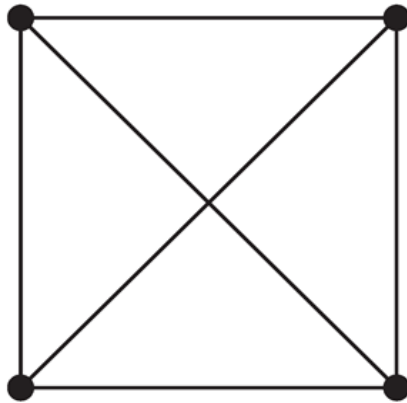
Example Is K_4 *planar*?



Planar Graphs

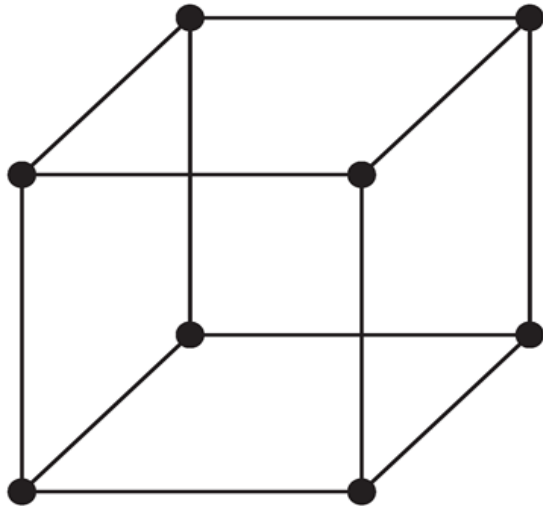
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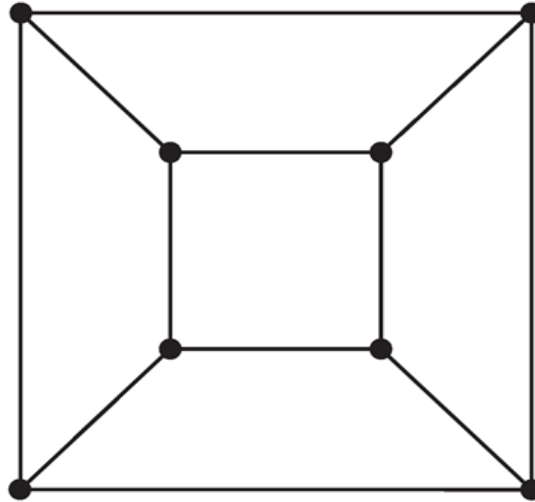
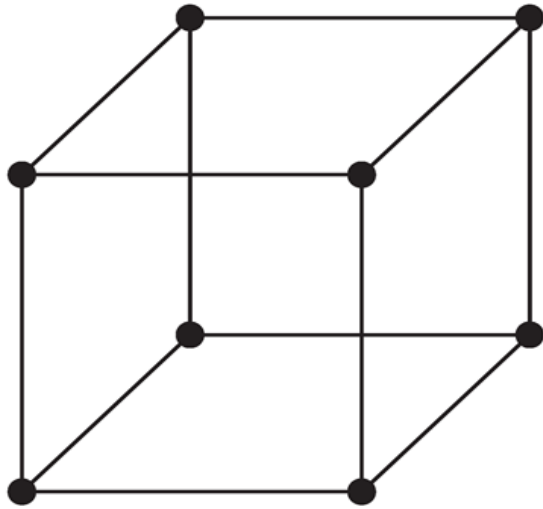
Planar Graphs

■ Example



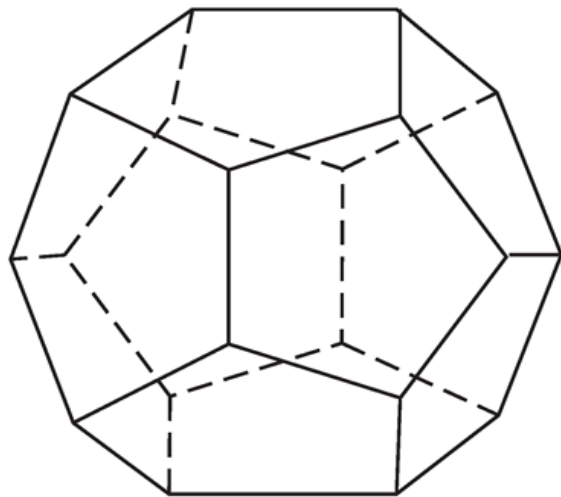
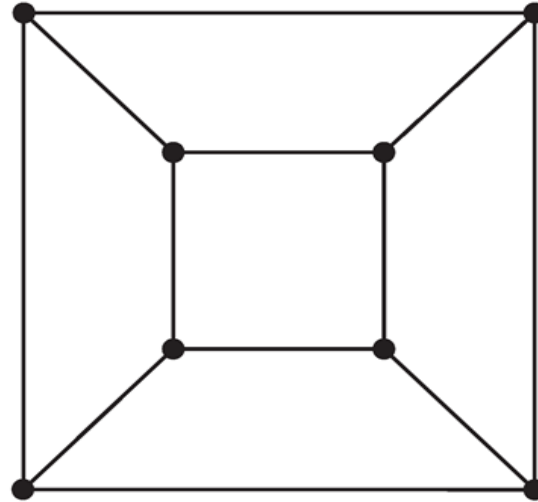
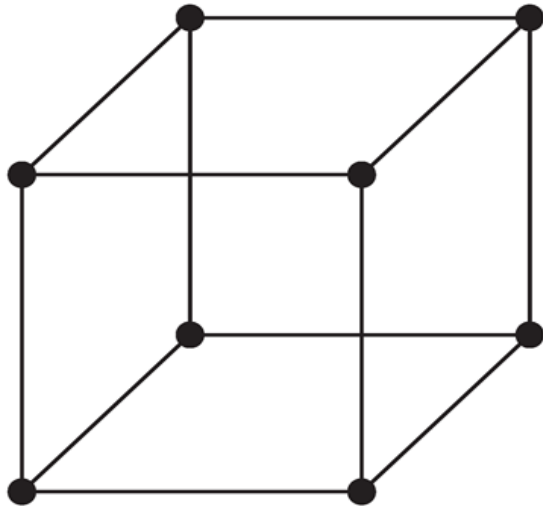
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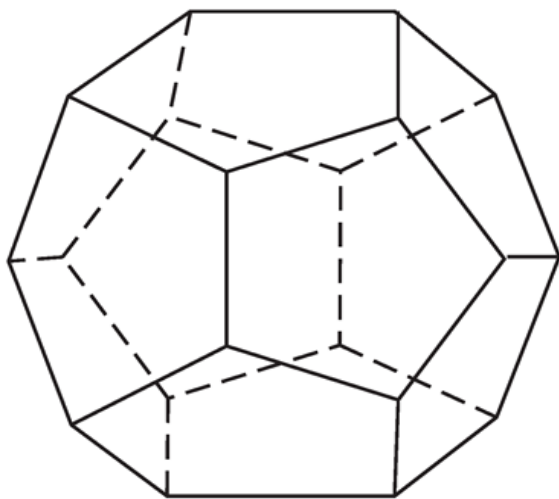
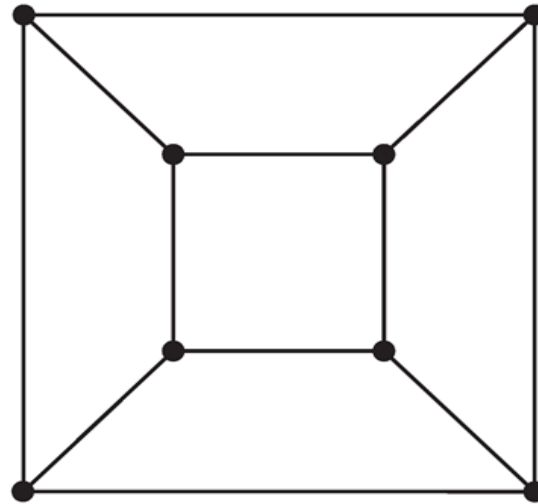
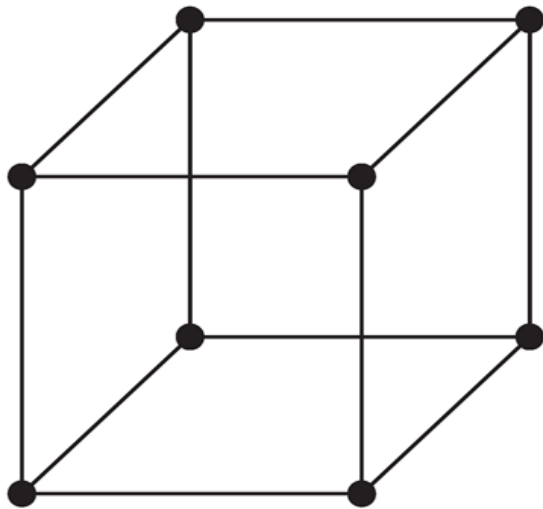
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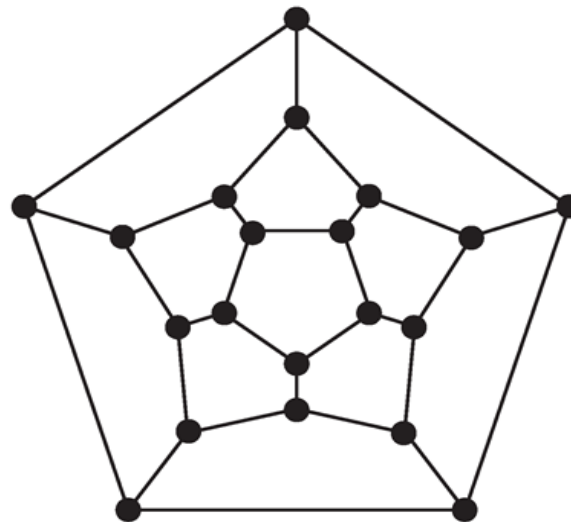
(a)

Planar Graphs

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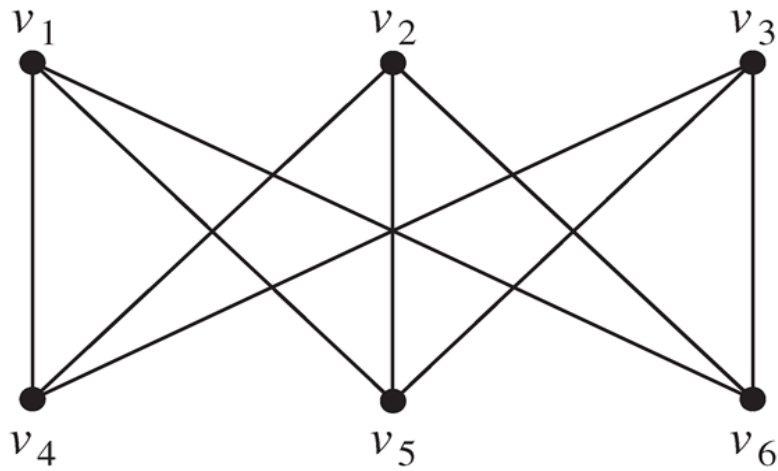
(a)



(b)

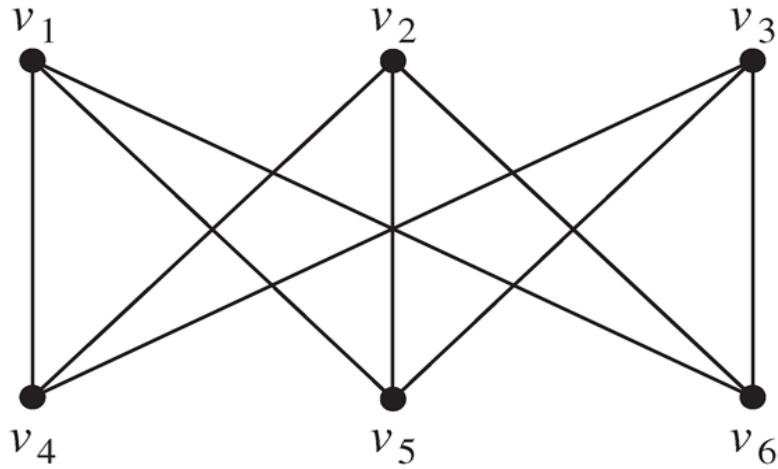
Planar Graphs

■ Example



Planar Graphs

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Applications

- ◇ IC design
- ◇ design of road networks



Euler's Formula

- **Theorem (Euler's Formula)** Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G . Then $r = e - v + 2$.

Proof (by induction)



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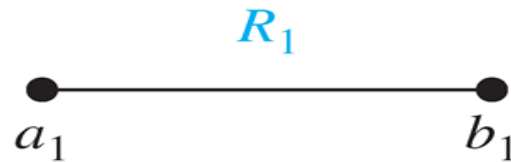


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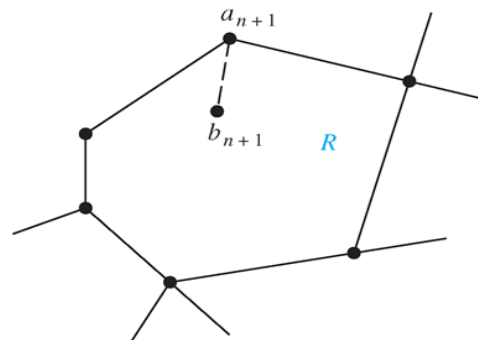
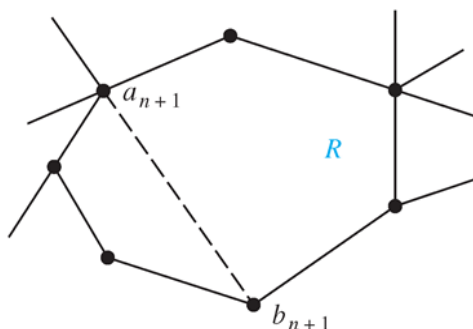
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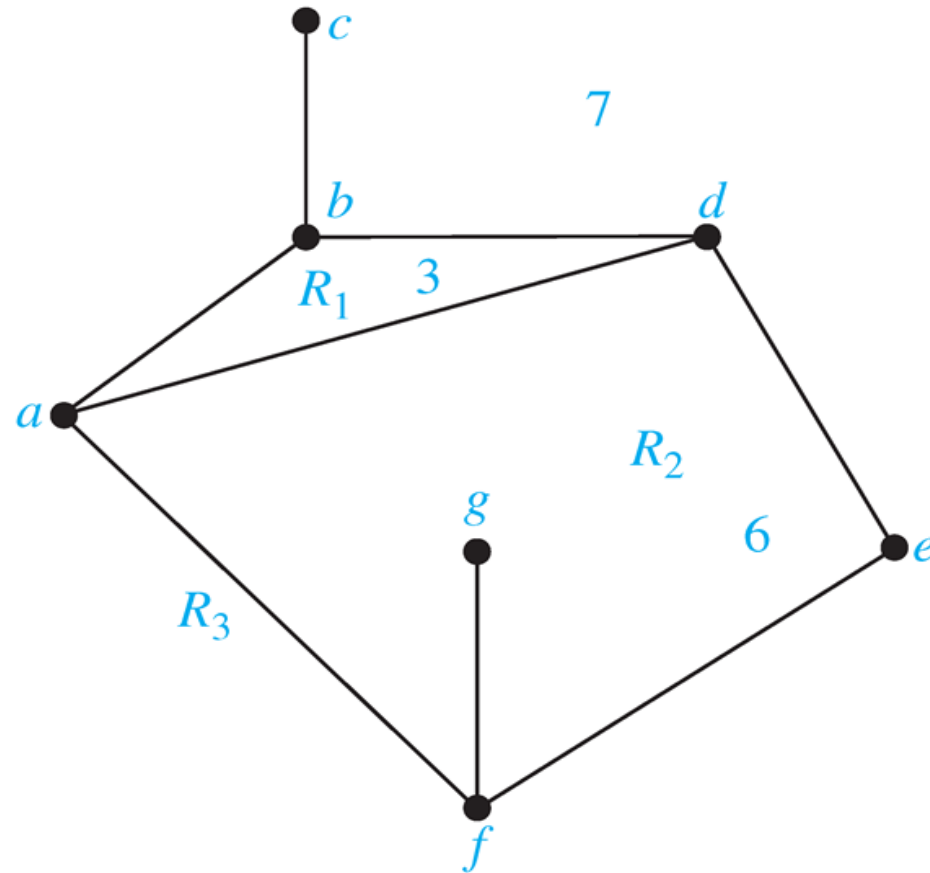
The Degree of Regions

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By Euler's formula, the proof is completed.



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Proof similar to that of Corollary 1.



Examples

- Show that K_5 is nonplanar.



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Using Corollary 1



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Using Corollary 3



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Using Corollary 1

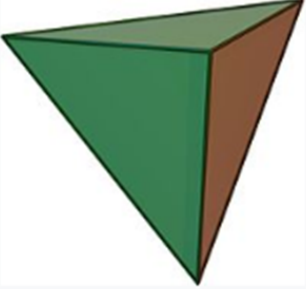
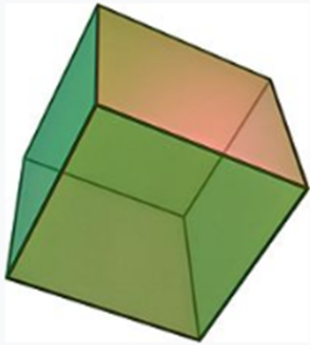
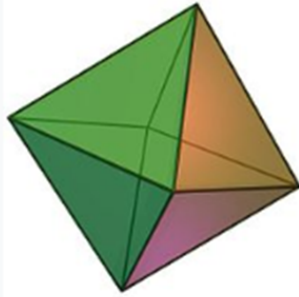
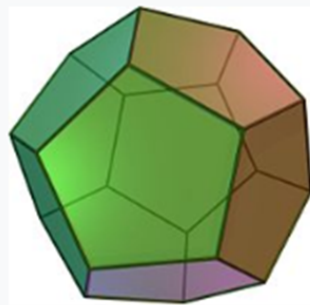
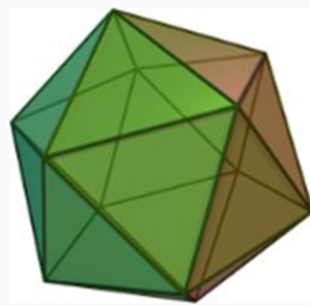
Show that $K_{3,3}$ is nonplanar.

Using Corollary 3

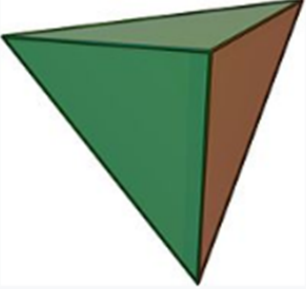
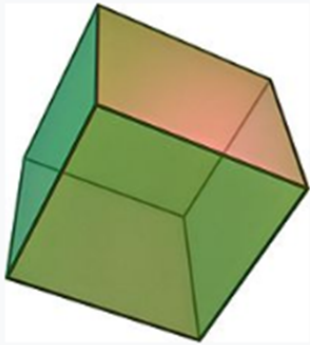
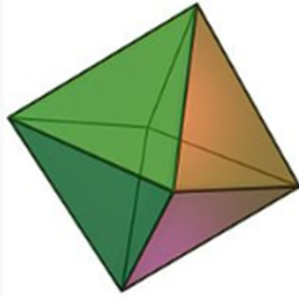
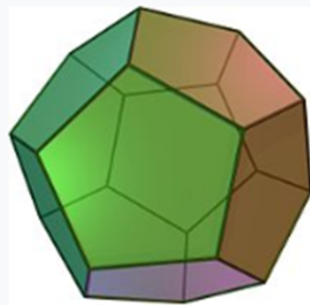
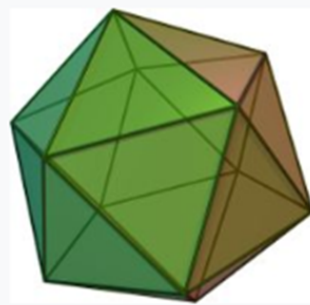
Corollary 2 is used in the proof of Five Color Theorem.

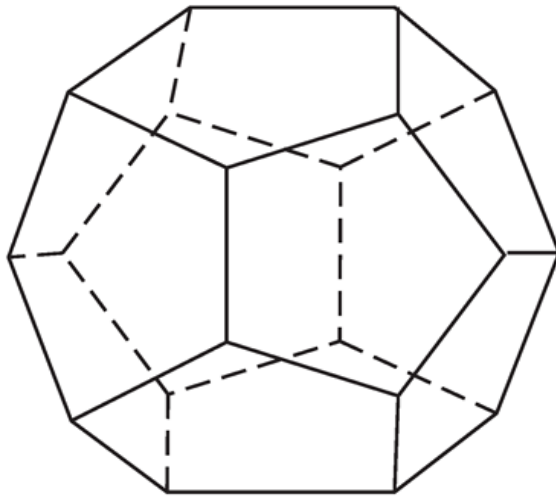


Only 5 Platonic Solids

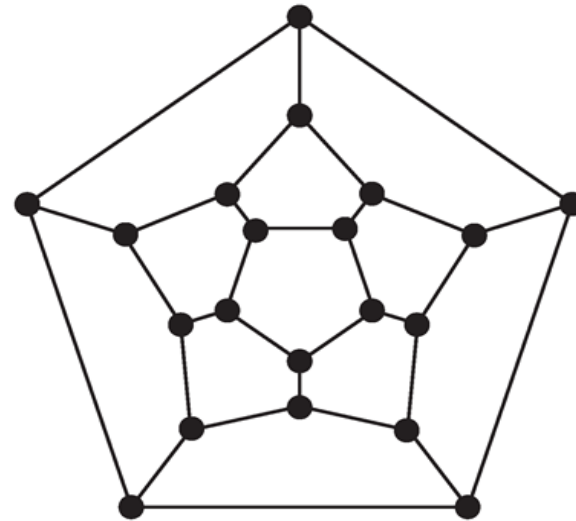
				
<p>Tetrahedron $\{3, 3\}$</p>	<p>Cube $\{4, 3\}$</p>	<p>Octahedron $\{3, 4\}$</p>	<p>Dodecahedron $\{5, 3\}$</p>	<p>Icosahedron $\{3, 5\}$</p>

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(a)



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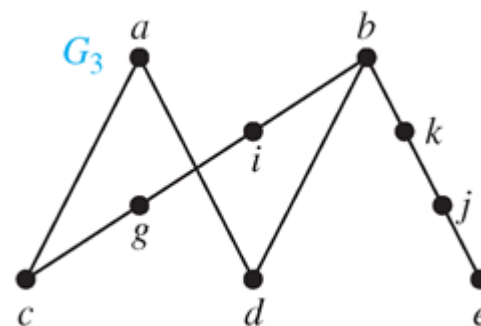
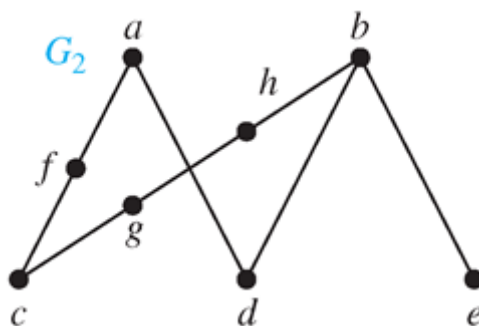
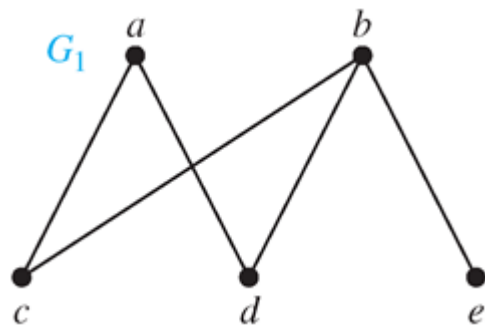
Kuratowski's Theorem

- **Definition** If a graph is planar, so will be **any graph** obtained by **removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$** . Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from **the same graph** by a sequence of elementary subdivisions.



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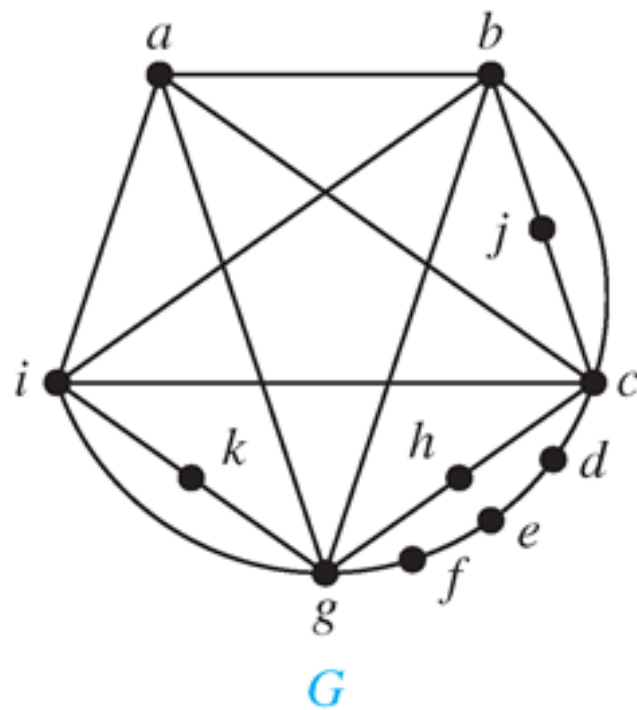
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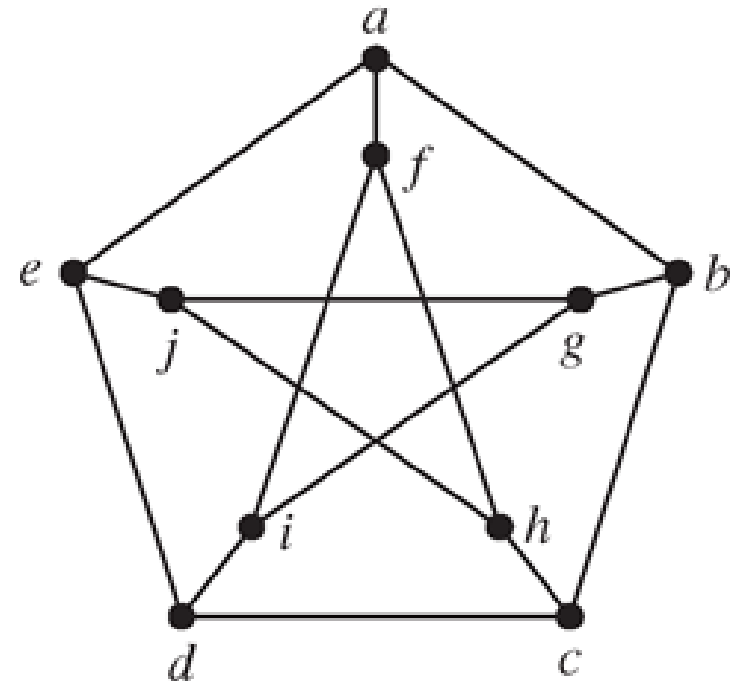
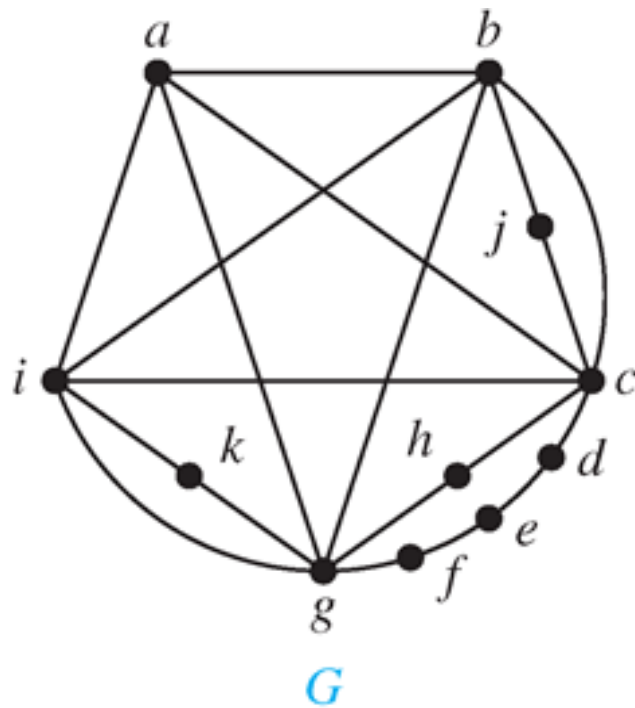
Theorem A graph is **nonplanar** if and only if it contains a subgraph **homomorphic to $K_{3,3}$ or K_5** .



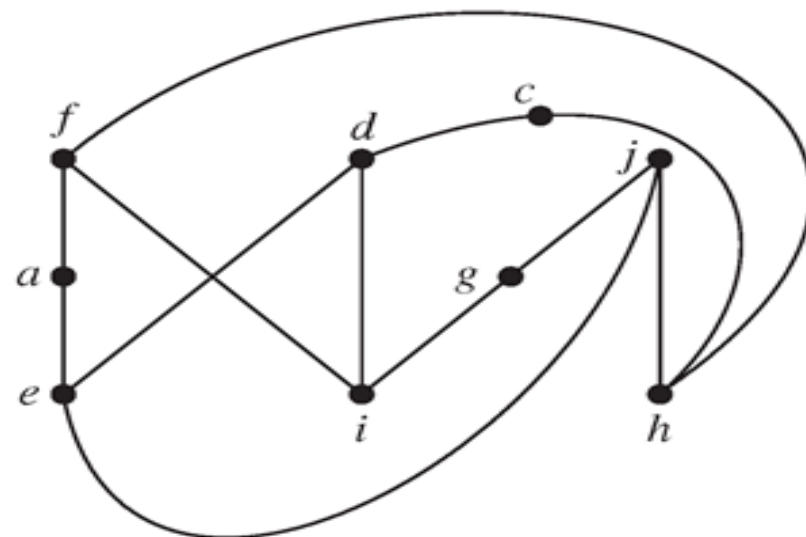
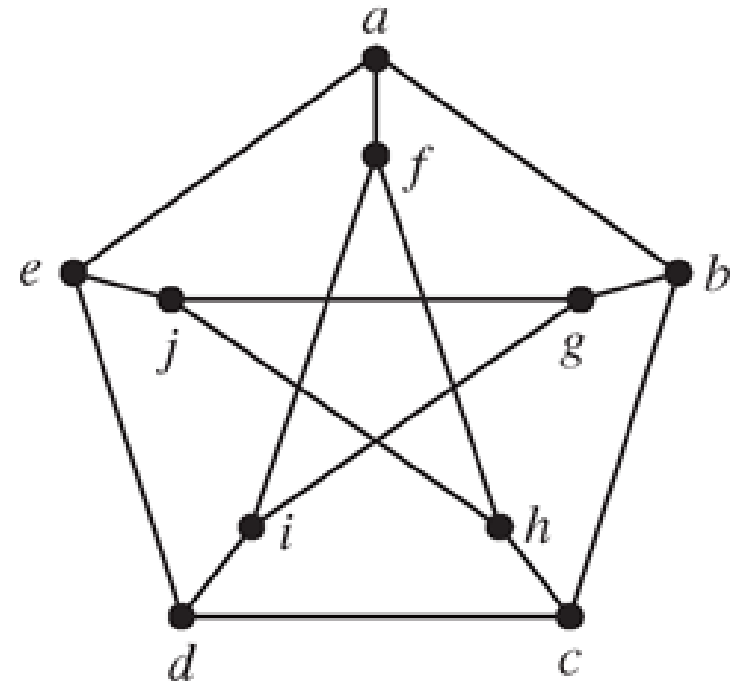
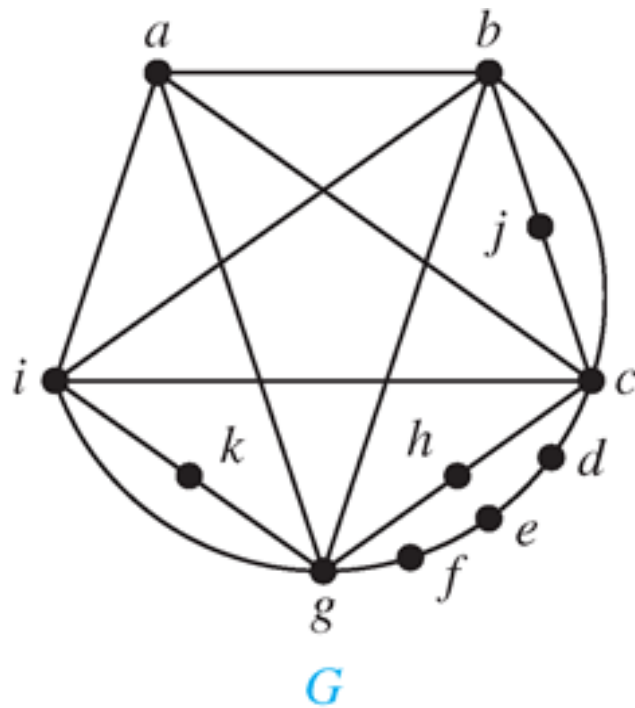
Examples



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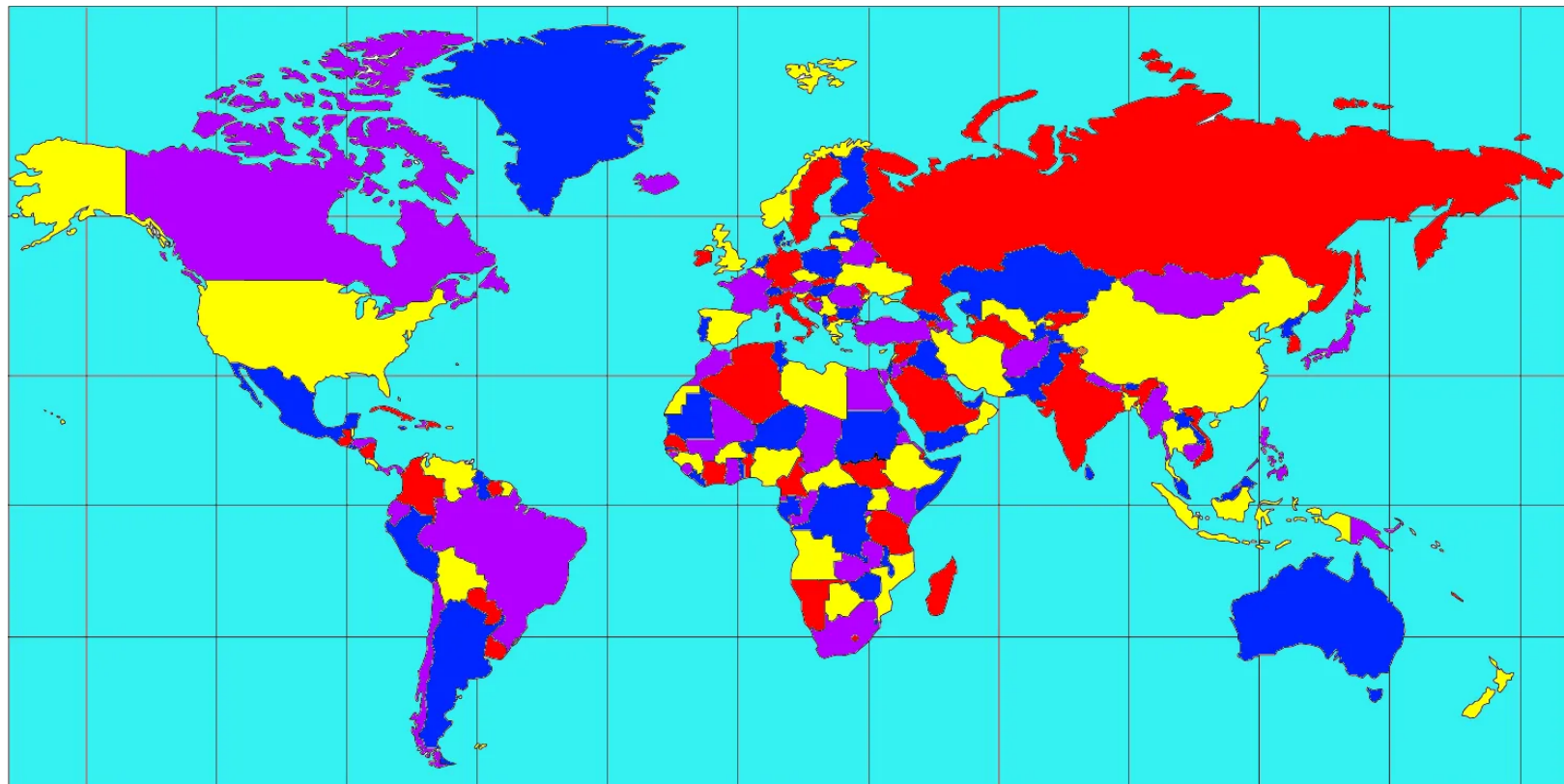


Examples



Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



■ Four-color theorem

- ◇ first proposed by Francis Guthrie in 1852
- ◇ his brother Frederick Guthrie told Augustus De Morgan
- ◇ De Morgan wrote to William Hamilton
- ◇ Alfred Kempe proved it **incorrectly** in 1879
- ◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*
- ◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)
- ◇ Kempe's incorrect proof serves as a basis



Graph Coloring

- A *coloring* of a simple graph is the *assignment* of a color to *each vertex* of the graph so that *no two adjacent vertices* are assigned the same color.



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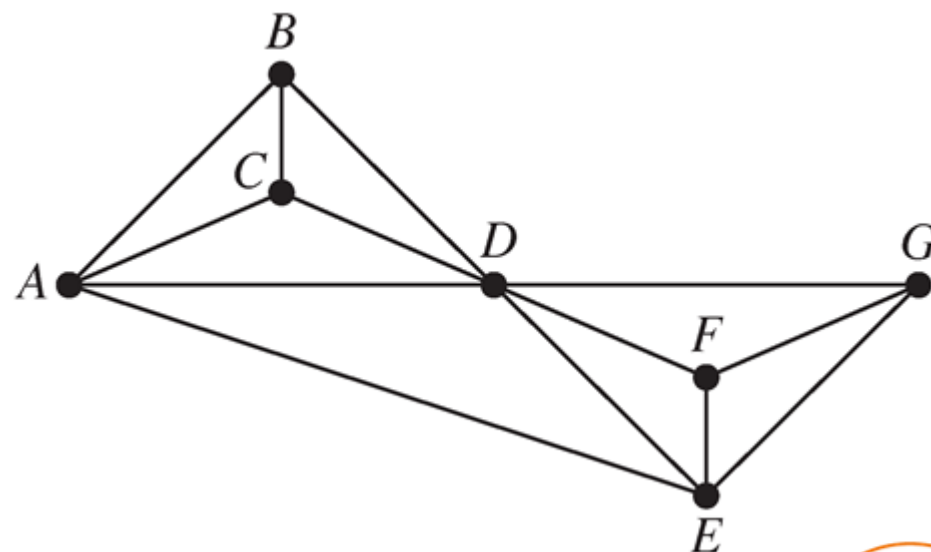
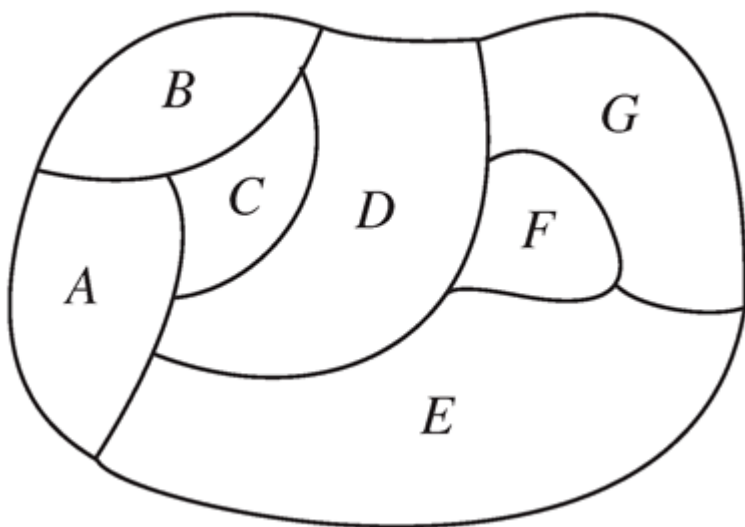
The *chromatic number* of a graph is the *least number* of colors needed for a coloring of this graph, denoted by $\chi(G)$.



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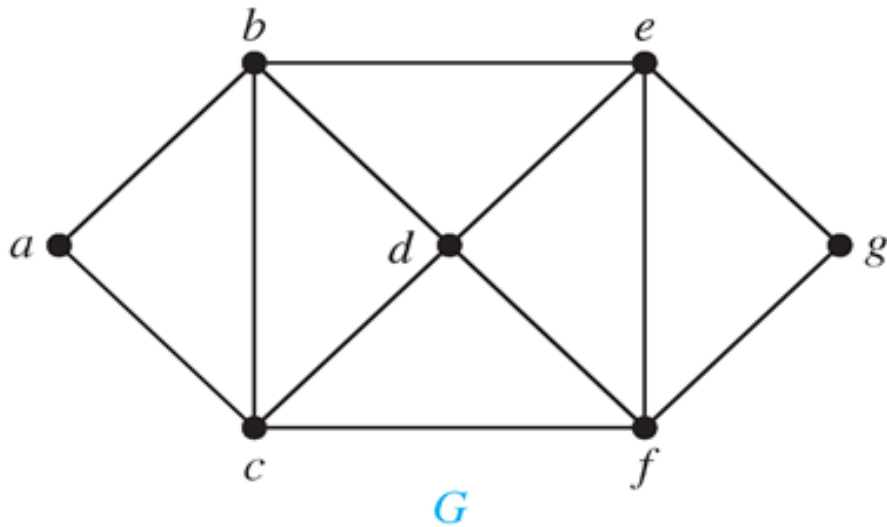
Graph Coloring

- **Theorem** (Four Color Theorem) The chromatic number of a planar graph is no greater than four.



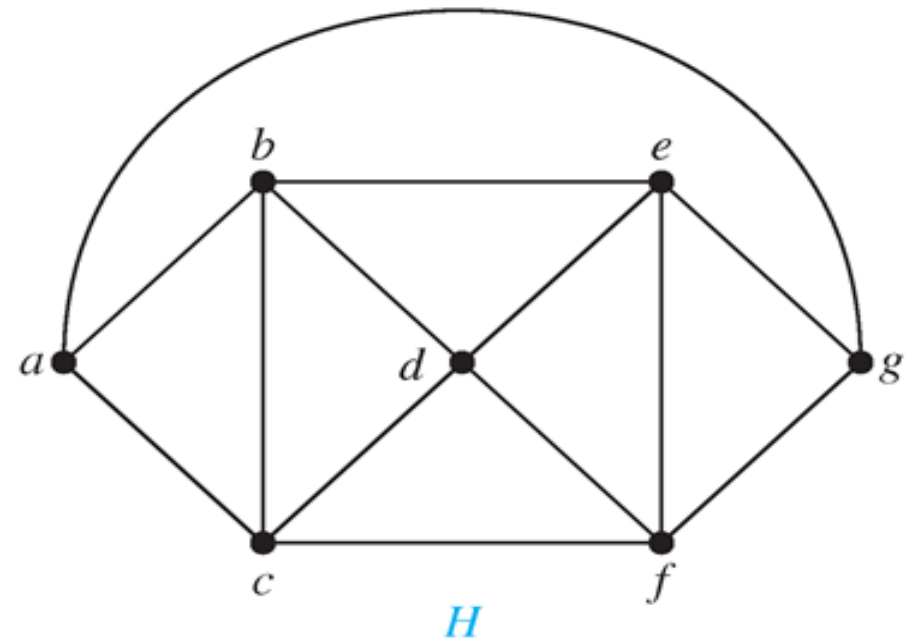
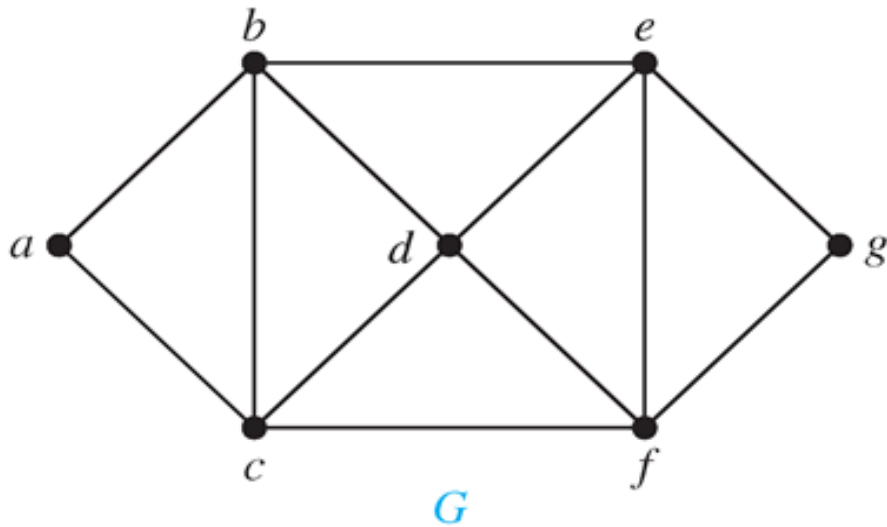
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Proof (by induction on the number of vertices)
w.l.o.g., assume that the graph is connected.



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Inductive step: Consider a planar graph with $k + 1$ vertices.



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w.l.o.g., assume that the graph is connected.

Basic step: For one single vertex, pick an arbitrary color.

Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 6-colored.

Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in. Since there are at most 5 colors adjacent, so we have at least one color left.



Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.



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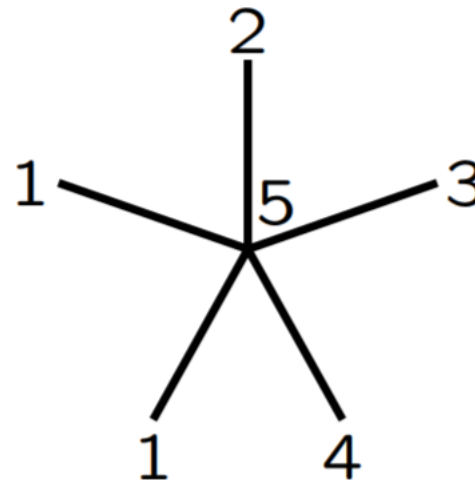
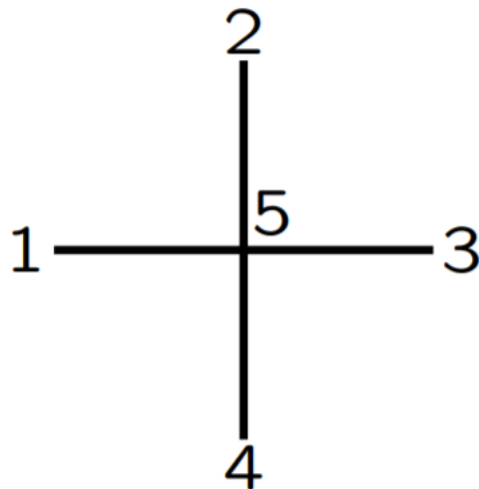


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If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we can pick an available color for it.

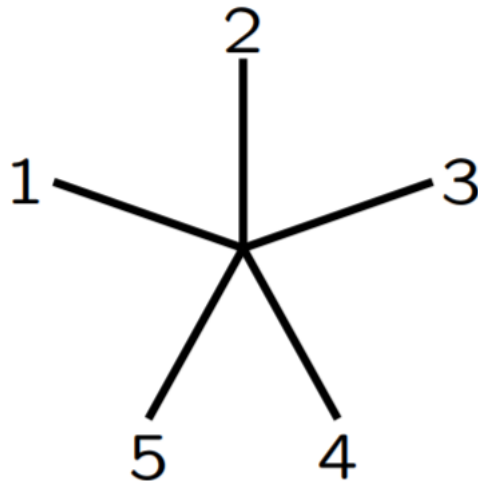


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If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the “special” vertex (degree 5) 1 to 5 (in order).



Graph Coloring

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Proof (by induction on the number of vertices)

We make a subgraph out of all the vertices colored 1 or 3. If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.

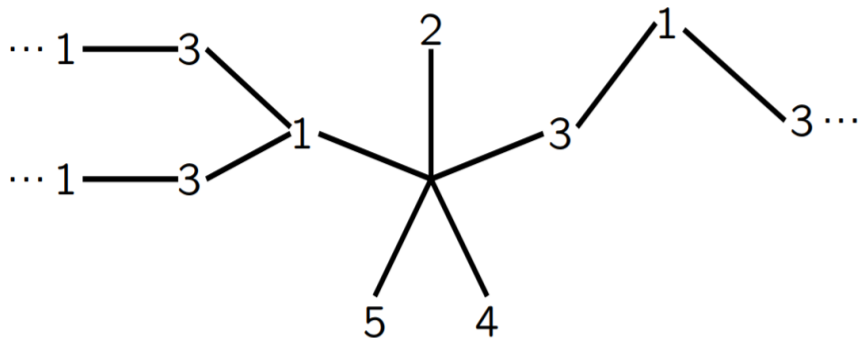


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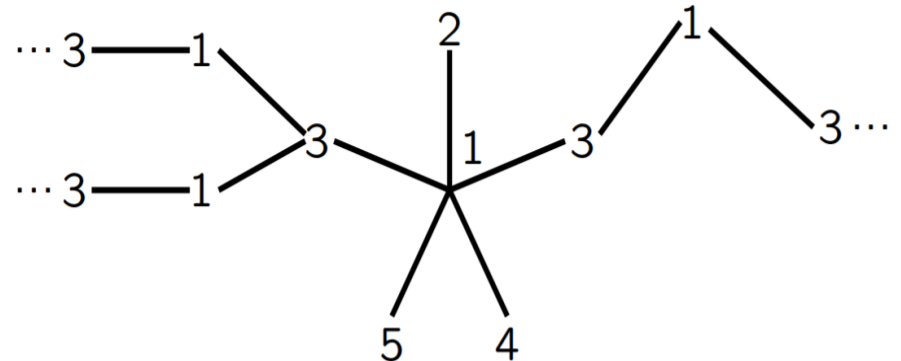
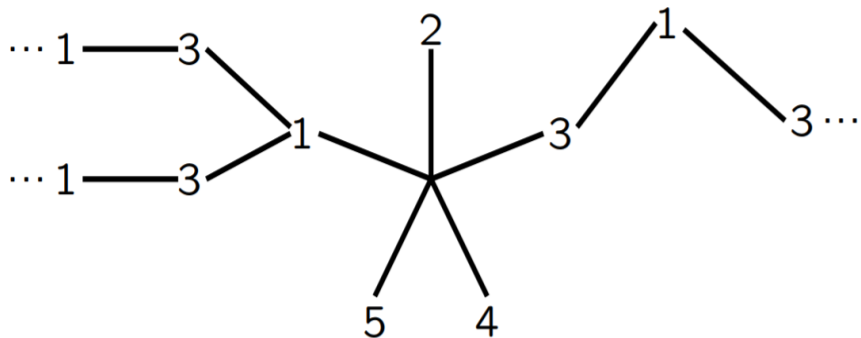


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On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the **the same** for the vertices colored 2 and 4. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3 (**Why?**)

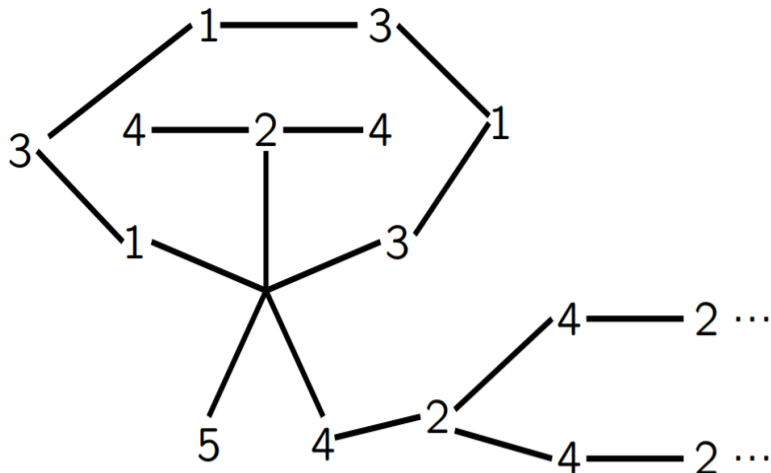


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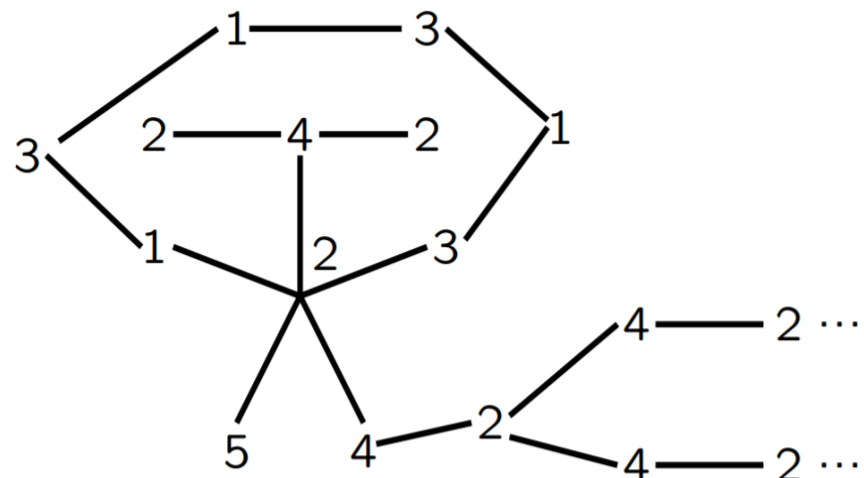
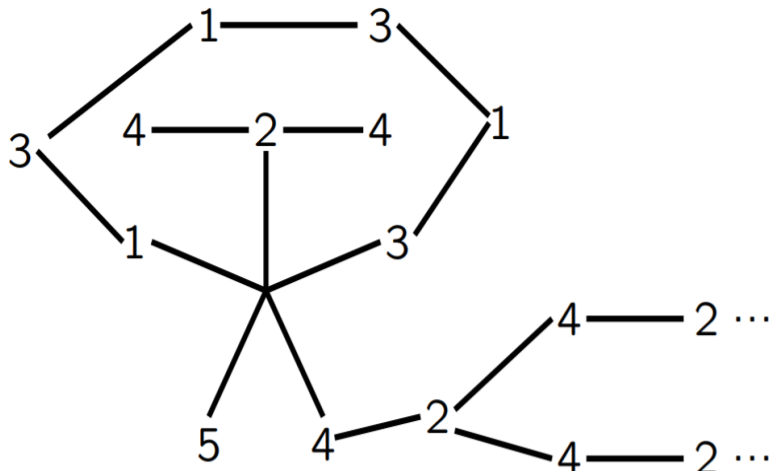


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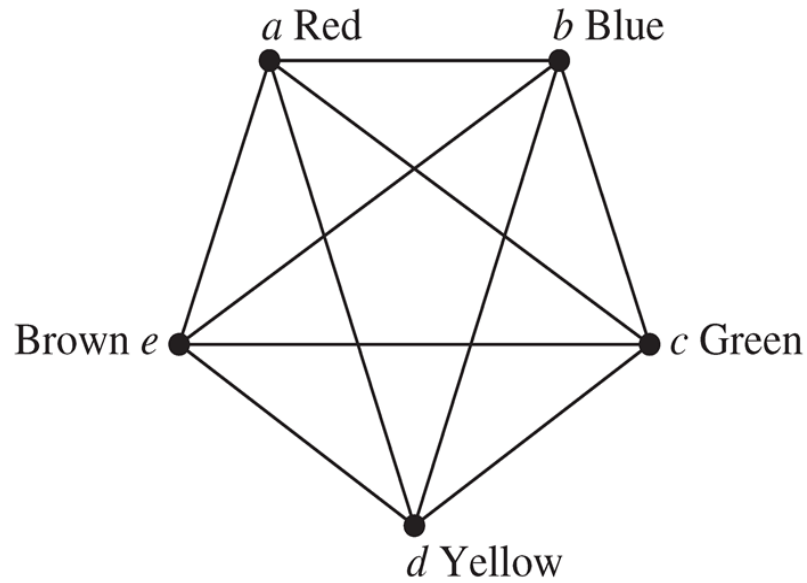
Examples

- What is the chromatic number of K_n , $K_{m,n}$, C_n ?



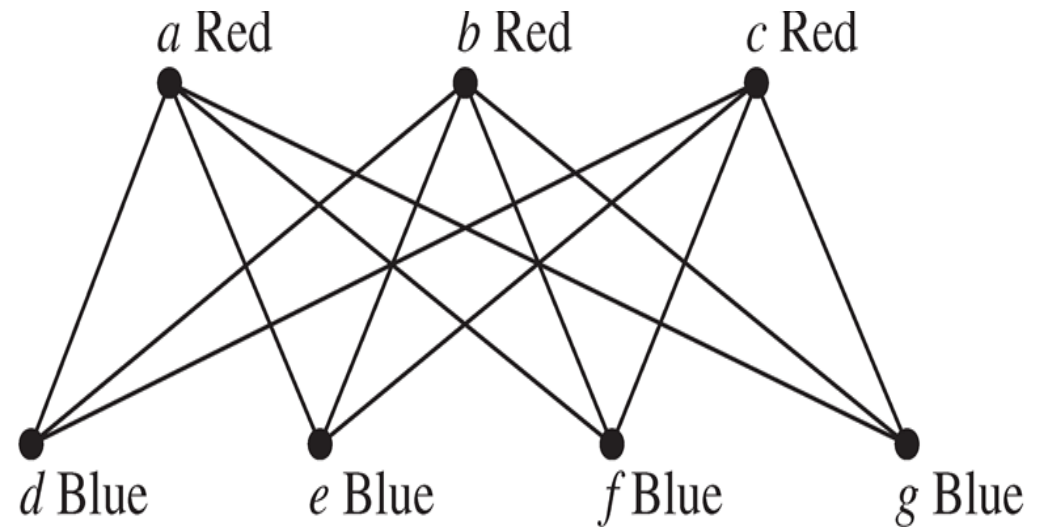
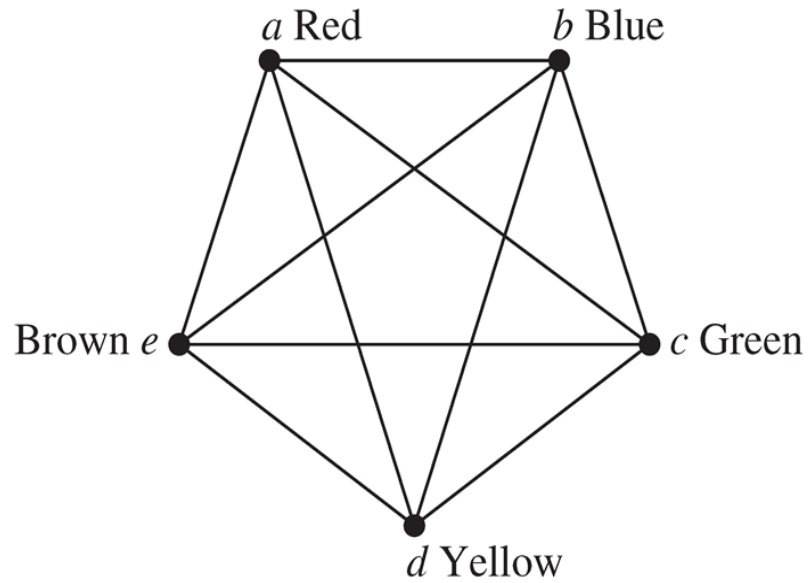
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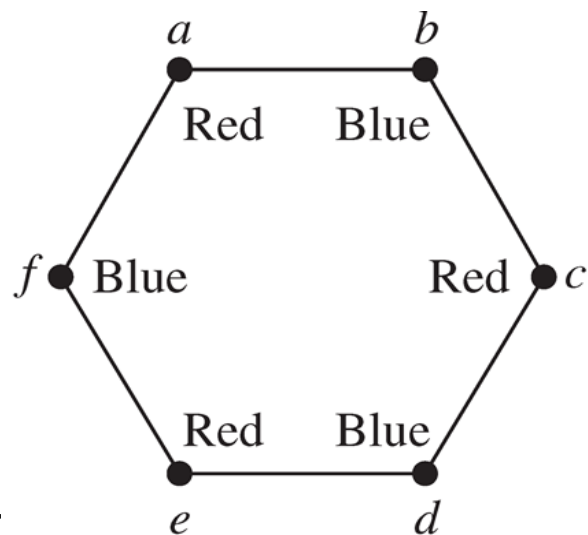
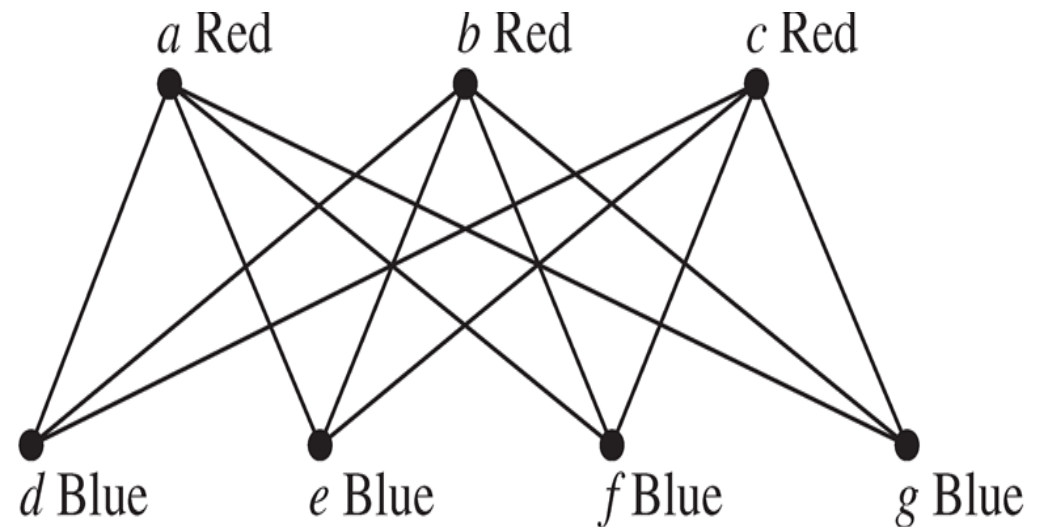
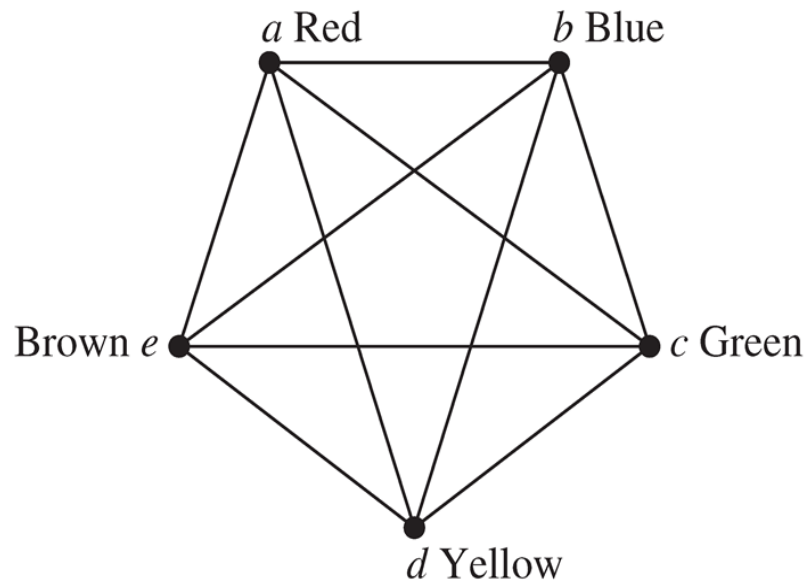
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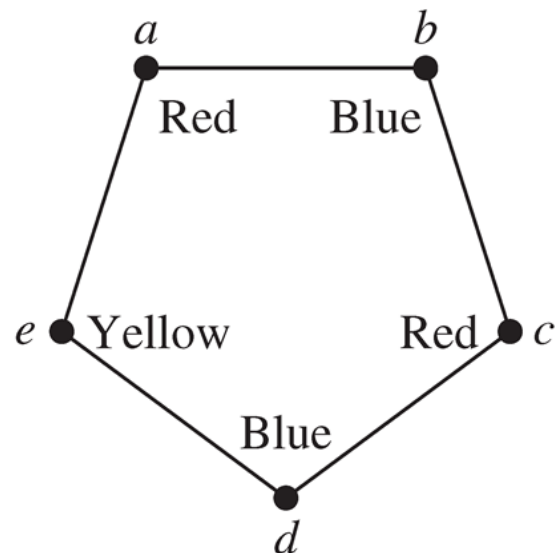
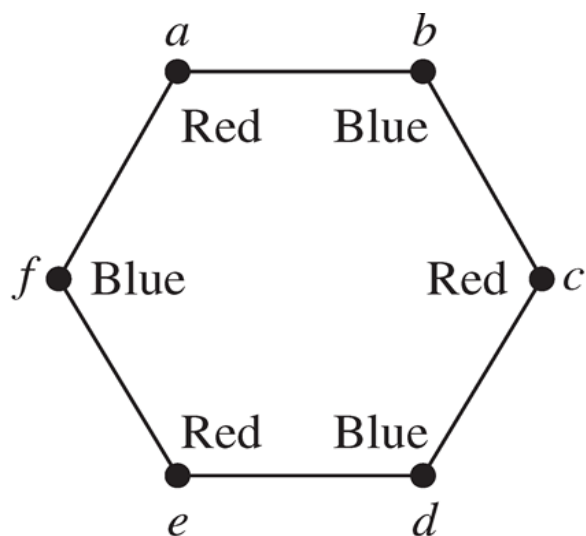
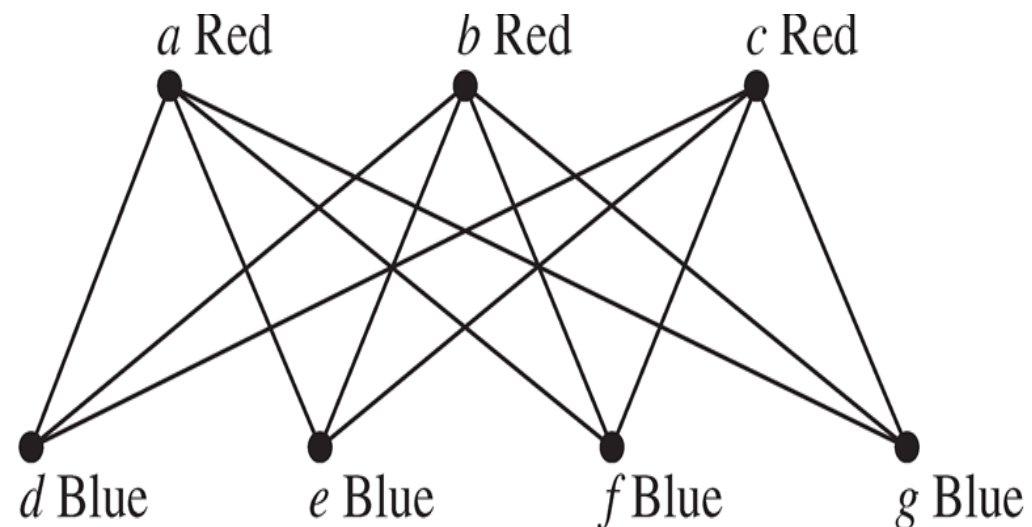
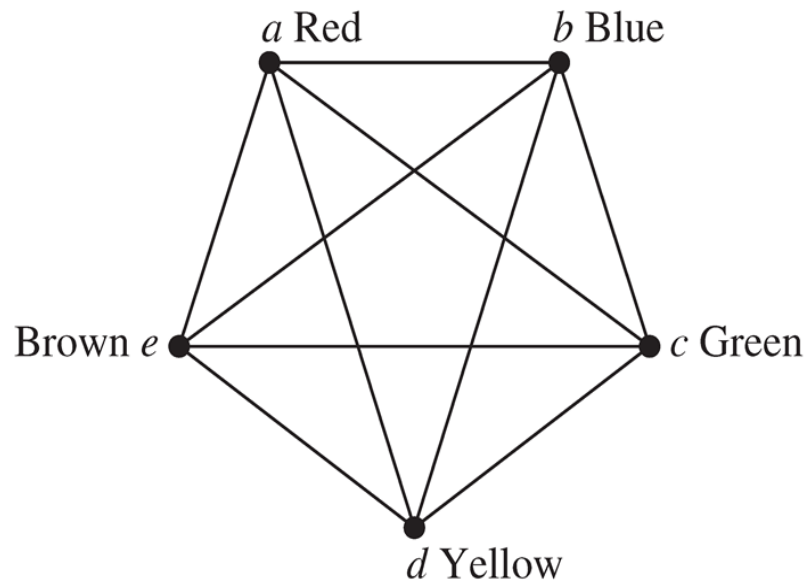
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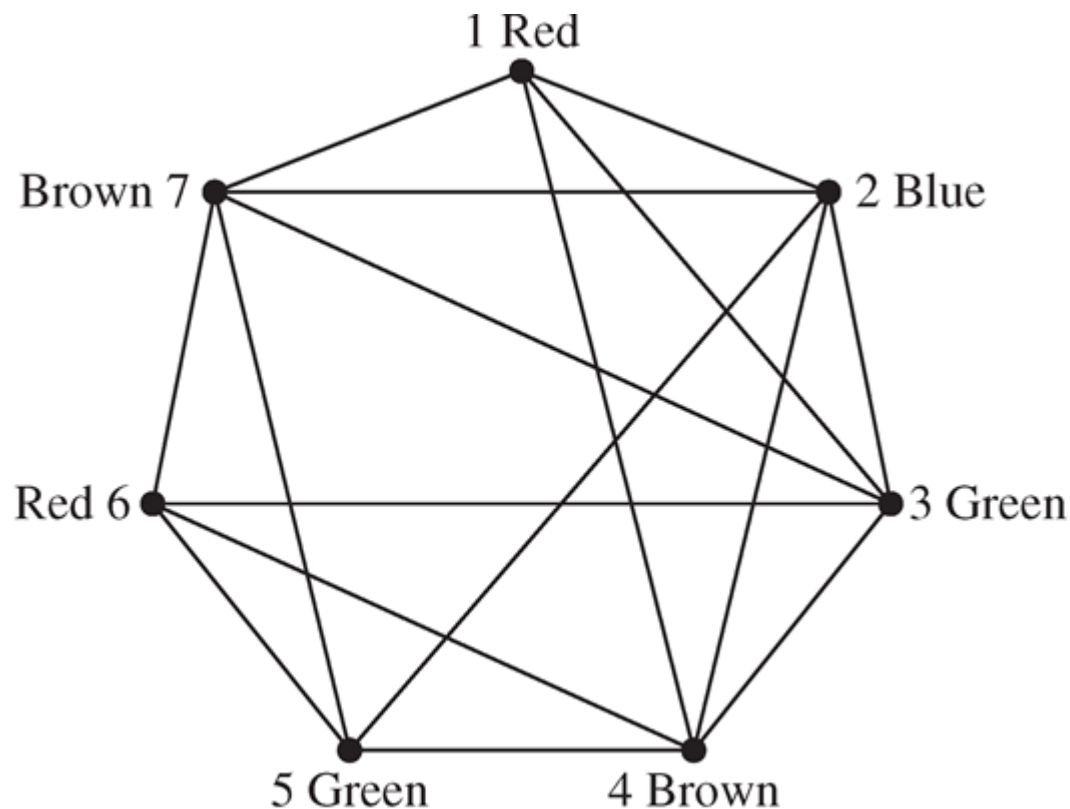
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Applications of Graph Coloring

■ Scheduling Final Exams

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period

I

Courses

1, 6

II

2

III

3, 5

IV

4, 7

Applications of Graph Coloring

■ Channel Assignments

Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?



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Graph Coloring \in NPC



Next Lecture

- tree ...

