CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Zero knowledge proofs

- S. Goldwasser, S. Micali, C. Rackoff, STOC'85

The Knowledge Complexity of Interactive Proof-Systems

(Extended Abstract)

Shafi Goldwasser, MIT
Silvio Micali, MIT
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Shafi, with Micali (and later Rackoff) [6], had been thinking for a while about expanding the traditional notion of "proof" to an interactive process in which a "prover" can convince a probabilistic "verifier" of the correctness of a mathematical proposition with overwhelming probability if and only if the proposition is correct. They called this interactive process an "interactive proof" (a name suggested by Mike Sipser). They wondered if one could prove some non-trivial statement (for example, membership of a string in a hard language) without giving away any knowledge whatsoever about why it was true. They defined that the verifier receives no knowledge from the prover if the verifier could simulate on his own the probability distribution that he obtains in interacting with the prover.The idea that "no knowledge" means simulatability was a very important contribution. They also gave the first example of these "zero knowledge interactive proofs" using quadratic residuosity. This paper won the first ACM SIGACT Gödel Prize. This zero-knowledge work led to a huge research program in the community that continues to this day, including results showing that (subject to an assumption such as the existence of one-way functions) a group of distrusting parties can compute a function of all their inputs without learning any knowledge about other people's inputs beyond that which follows from the value of the function.

`https://amturing.acm.org/award_winners/goldwasser_8627889.cfm`

Applications of ZKPs

- *Protocol design*. A *protocol* is an algorithm for interactive parties to achieve a certain goal. However, in crypto, we often want to design protocols that should achieve security even when one of the parties is "cheating". Alice can prove in *zero knowledge* that she followed the instructions.

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Proofs that Yield Nothing But their Validity and a Methodology of Cryptographic Protocol Design

(Extended Abstract)

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Applications of ZKPs

- *Identification scheme*. How should Alice prove to Bob that she is who she claimed to be? For example, how to design a control access system to the CSE dept.?

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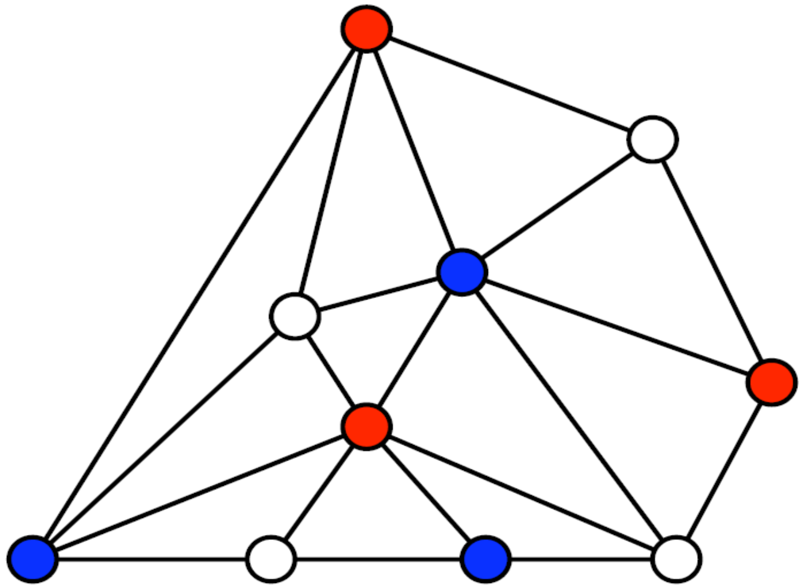
- *Identification scheme*. How should Alice prove to Bob that she is who she claimed to be? For example, how to design a control access system to the CSE dept.?
- A direct solution is to have a box on the door and give authorized people a secret PIN number. However, a drawback is that the box remains outside all the time and if someone could examine the box, they would perhaps be able to view its memory and extract the secrets keys of all people.
- **Ideas using ZKPs**:
  - Let the box contain an *instance* of a hard problem.
  - Give the authorized people the *solution* to the instance.
  - The authorized people will *prove* to the box that they know the solution in zero knowledge.

An example



- Alice knows how to 3-color a graph: no two adjacent vertices have the same color; this is an NPC problem.

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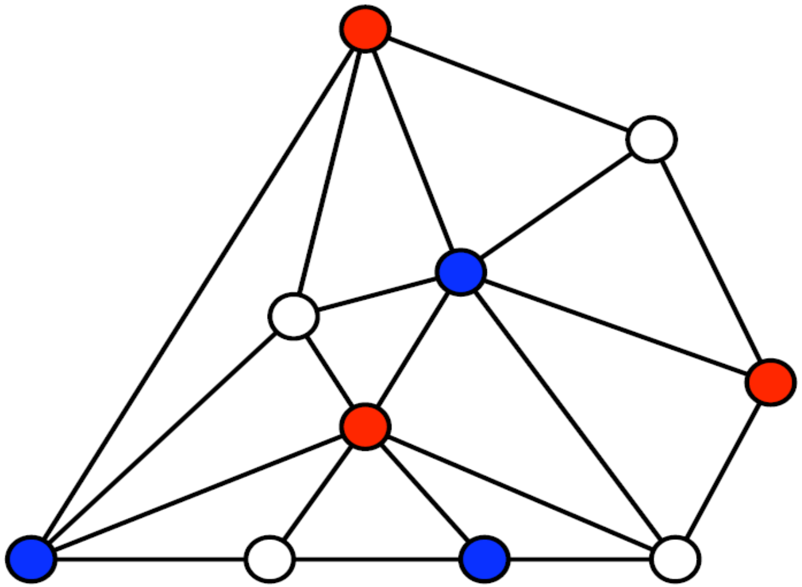

- Alice knows how to 3-color a graph: no two adjacent vertices have the same color; this is an NPC problem.
  - can impress your friends
  - useful for identification

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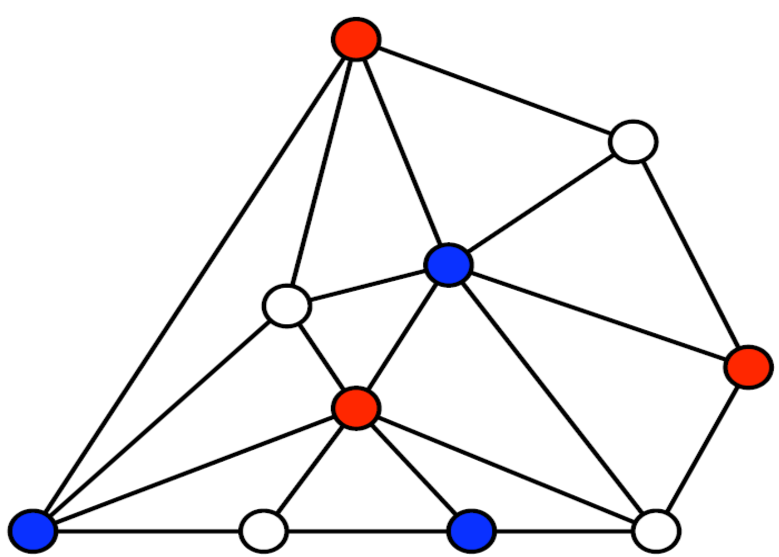
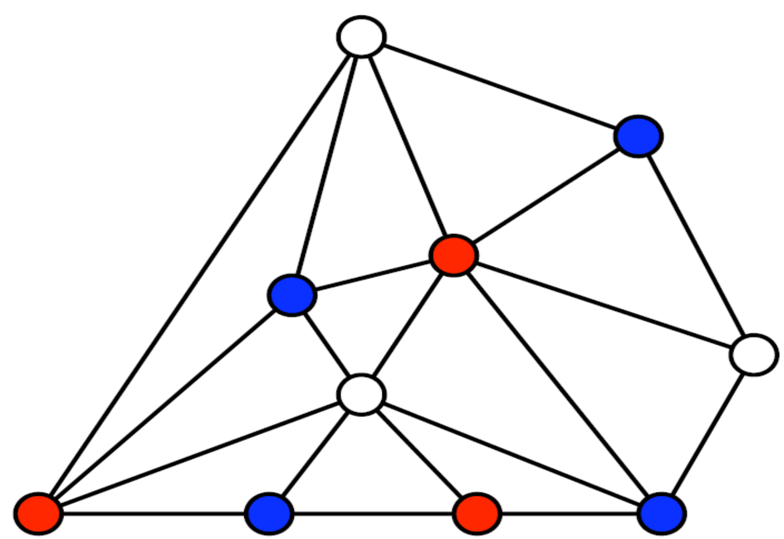
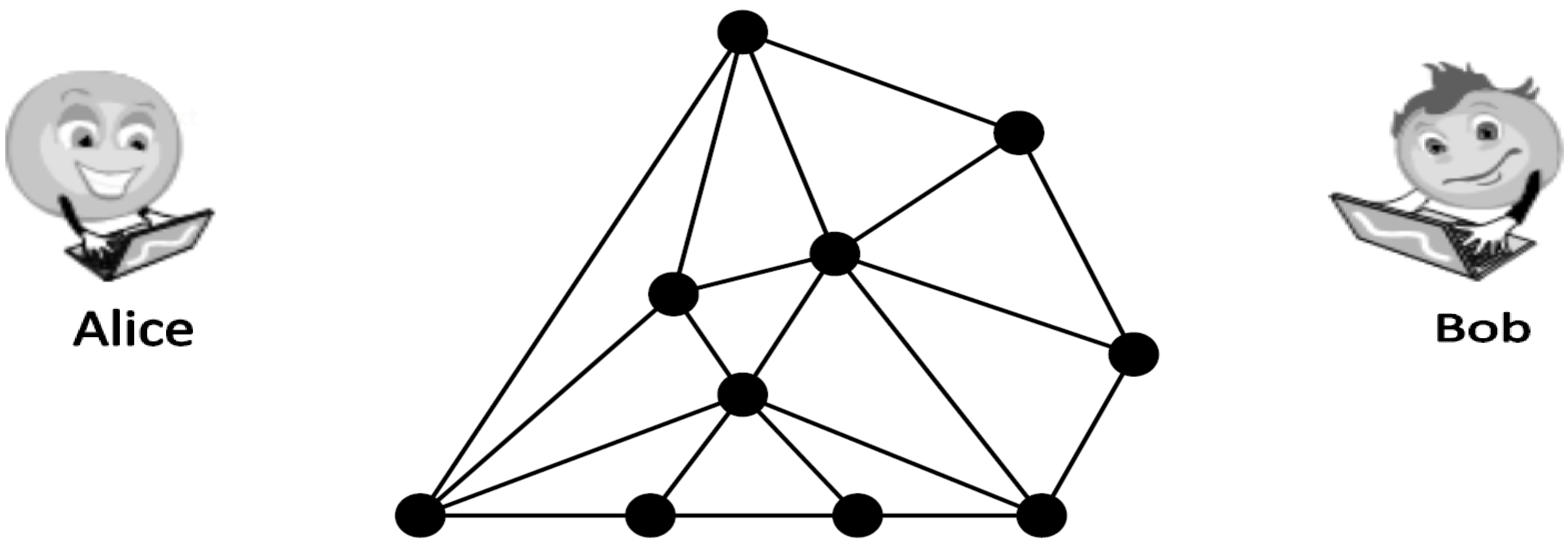
Alice may permute the vertex colors.





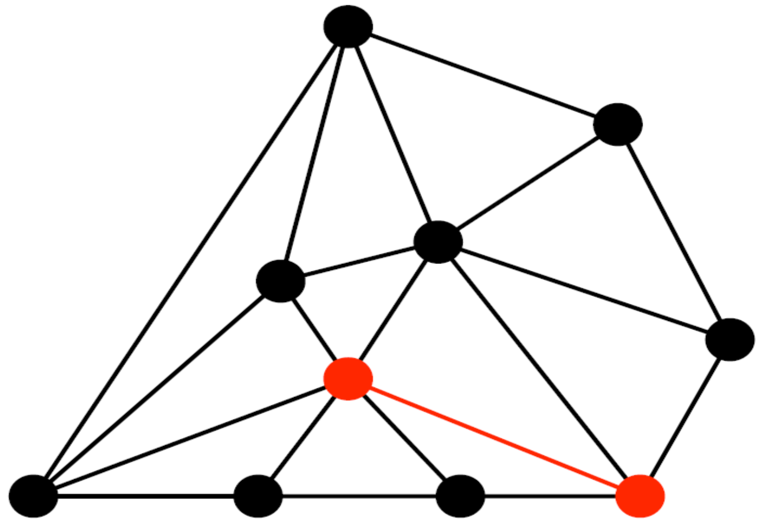
An example

- Alice then encrypts all vertex colors (one key per vertex), and sends the graph to Bob.



An example

- Bob picks an edge at random.



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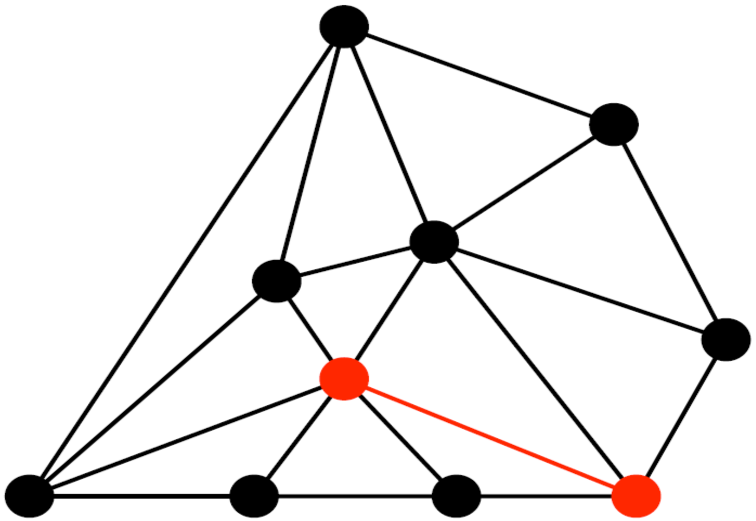
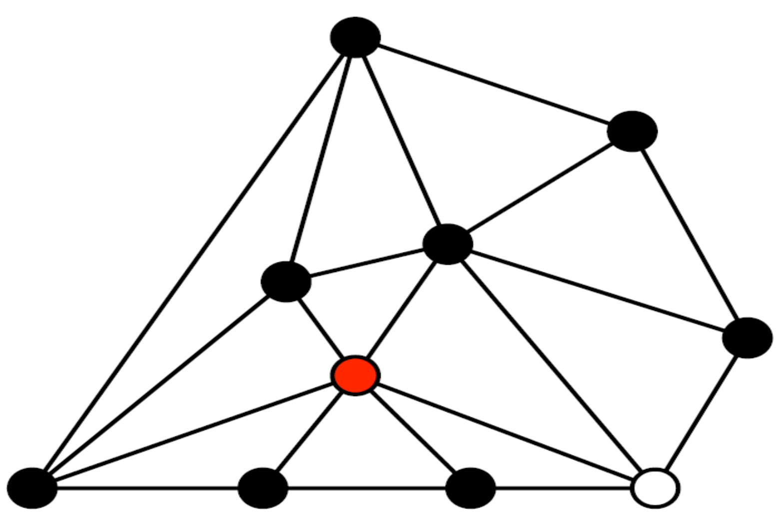


Alice reveals colors of those two keys.



An example

- Repeat as much as needed:
  - Alice permutes graph coloring
  - Alice enrypts all vertices with distinct keys
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After k repetitions, the probability she fools Bob is $(1 - \frac{1}{|E|})^k$.

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  Because Bob could have generated those keys and colors by himself, he learns nothing from the graph coloring.

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  (a) If the statement $P(b)$ is true

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  (a) – *Basic Step* *Inductive Hypothesis*

  (b) – *Inductive Step* *Inductive Conclusion*

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  - ◇ Iterating gives us a proof of $P(n)$ for all n

Strong Induction

- **Principle** (*Strong Principle of Mathematical Induction*)

  (a) If the statement $P(b)$ is true

  (b) for all $n > b$, the statement
  $P(b) \wedge P(b+1) \wedge \cdots \wedge P(n-1) \to P(n)$ is true.

  then $P(n)$ is true for all integers $n \geq b$.

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  ◇ Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime or a product of powers of primes.

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- In reality, they are equivalent to each other in that the weak form is a special case of the strong form, and the strong form can be derived from the weak form.

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  3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Recursion

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- A classical example of *recursion* is the **Towers of Hanoi** Problem.

Towers of Hanoi

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- 3 pegs; n disks of different sizes
- A *legal move* takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk
- **Problem**: Find a (efficient) way to move all of the disks from one peg to another

Towers of Hanoi

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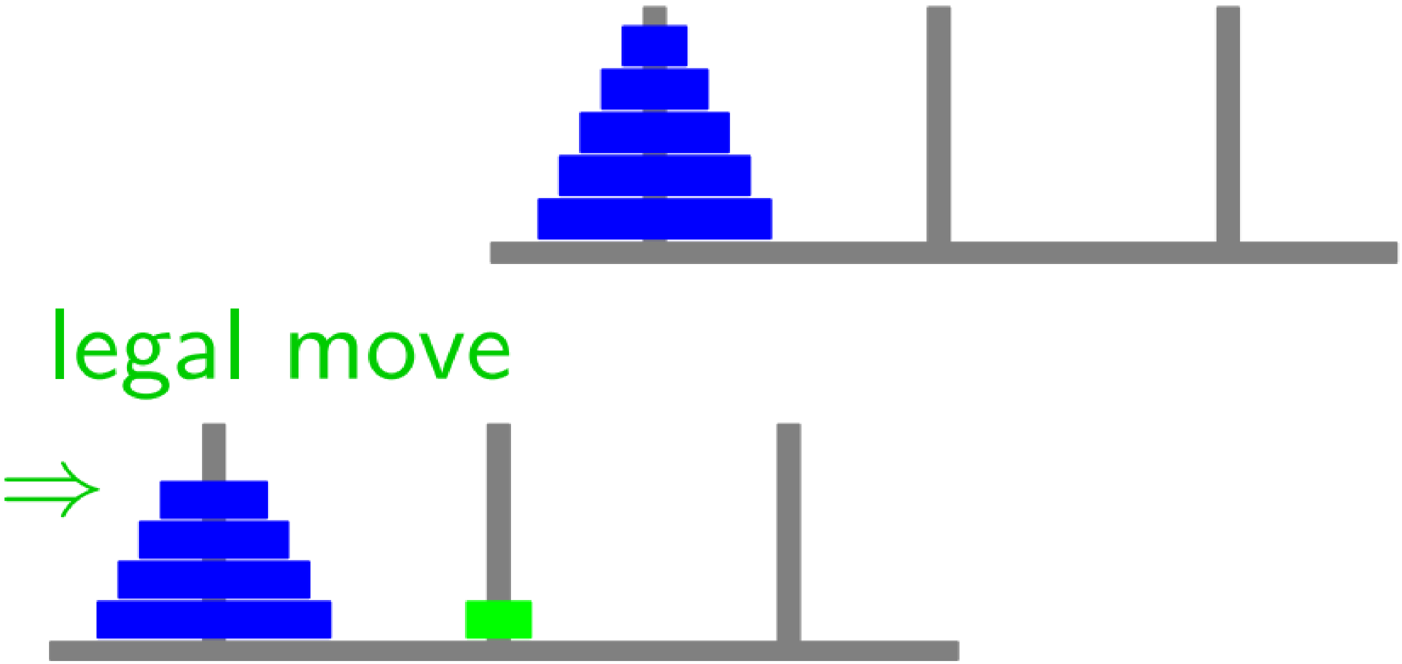
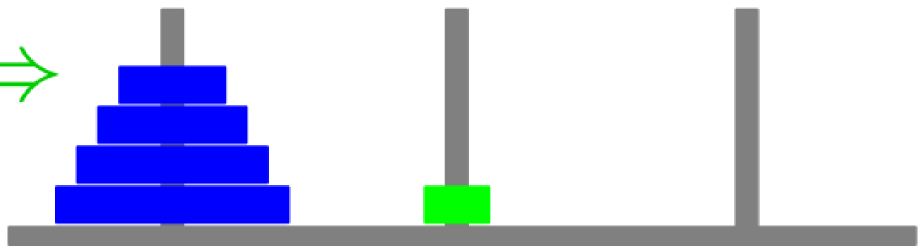


Towers of Hanoi

legal move

$\Rightarrow$



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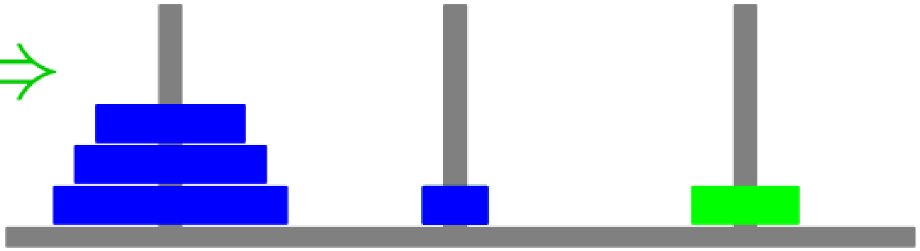

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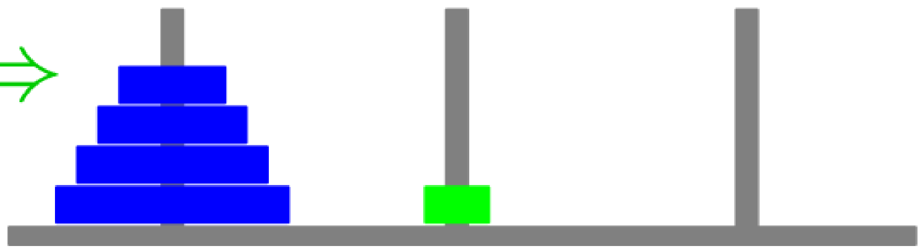

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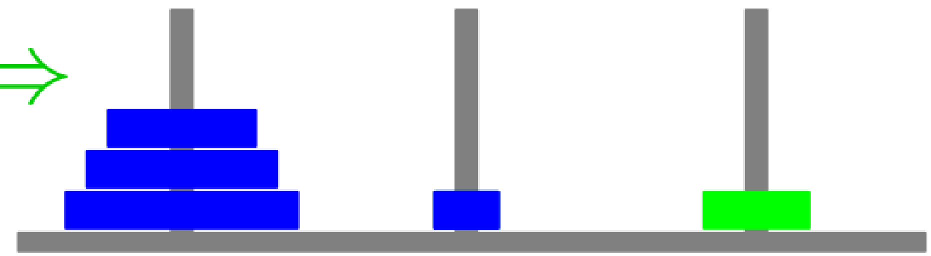

not legal

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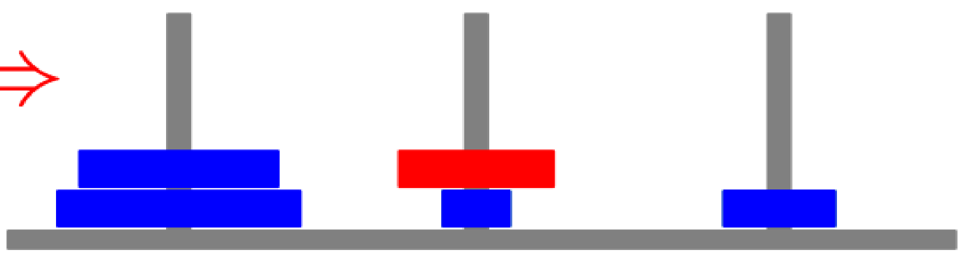

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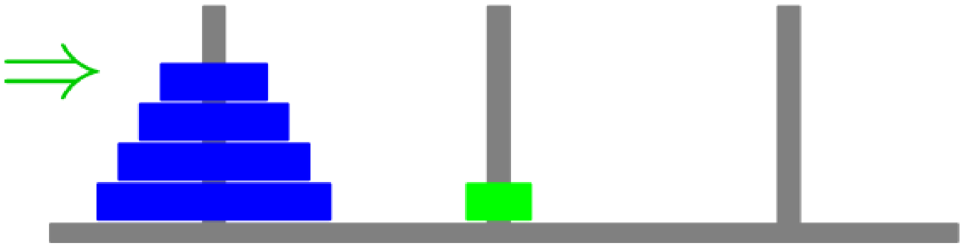

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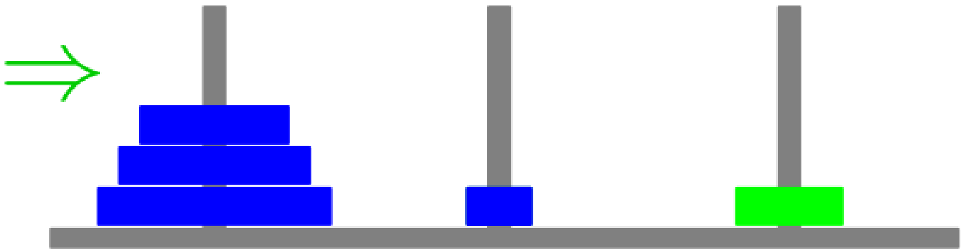

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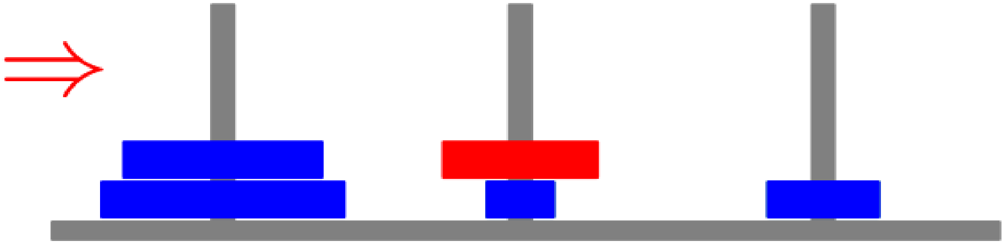

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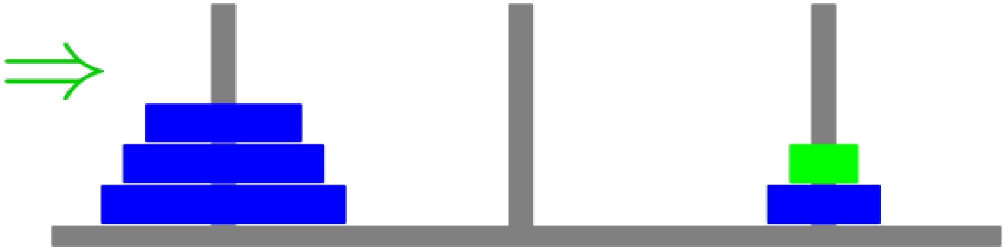

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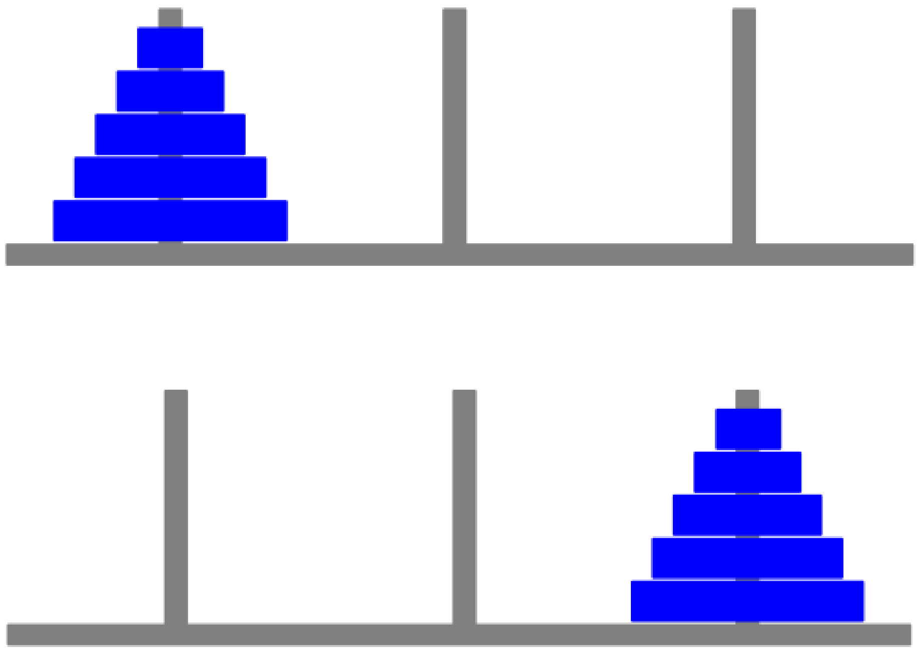

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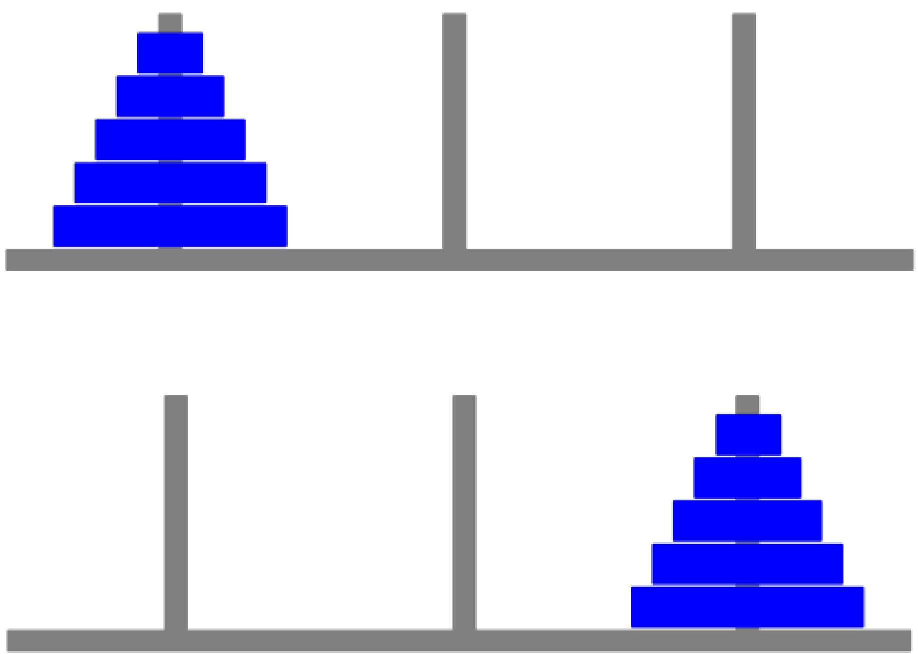

Given $i, j \in \{1,2,3\}$, let
$\overline{\{i,j\}} = \{1,2,3\} - \{i\} - \{j\}$,
i.e., $\overline{\{1,2\}} = \{3\}$, $\overline{\{1,3\}} = \{2\}$,
$\overline{\{2,3\}} = \{1\}$.

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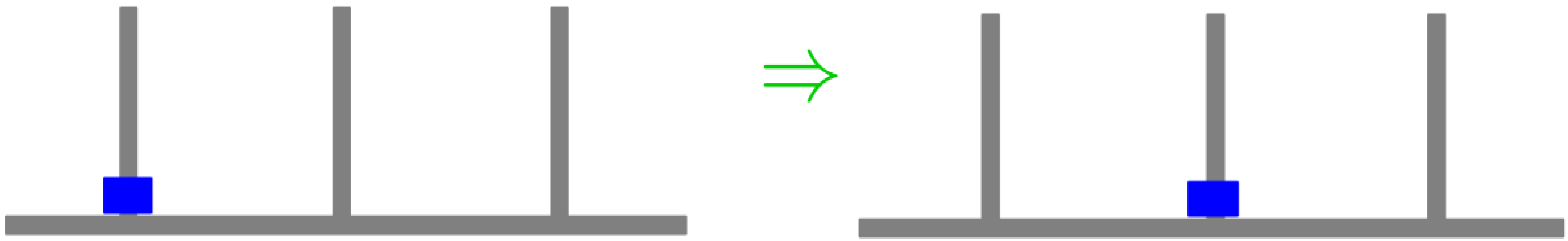

Towers of Hanoi

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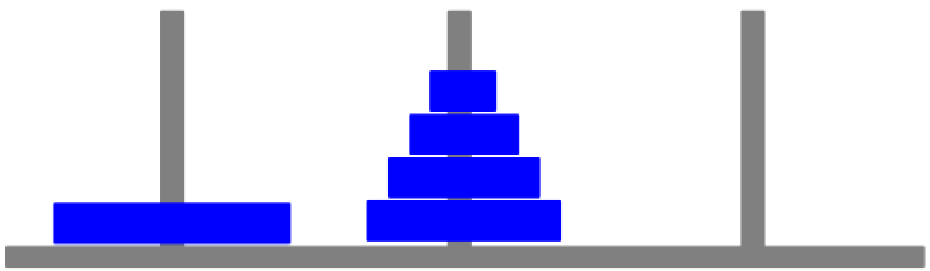

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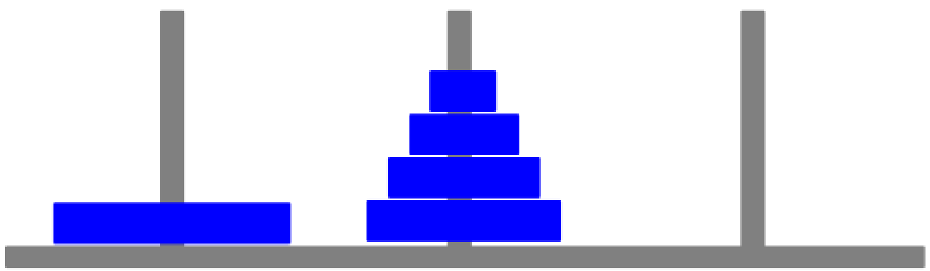
1) 

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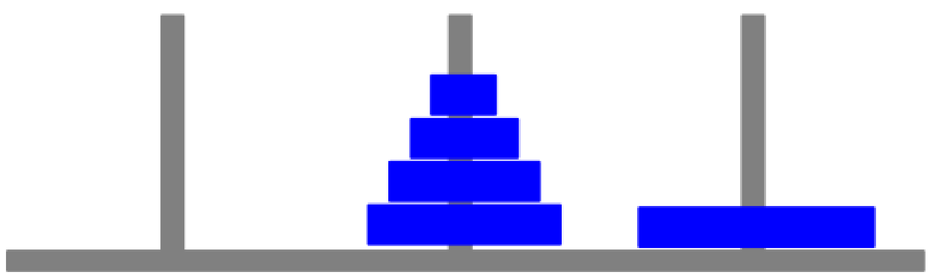
2) 

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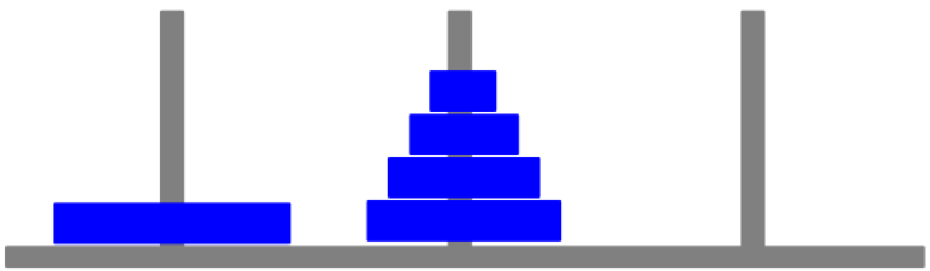
1) 

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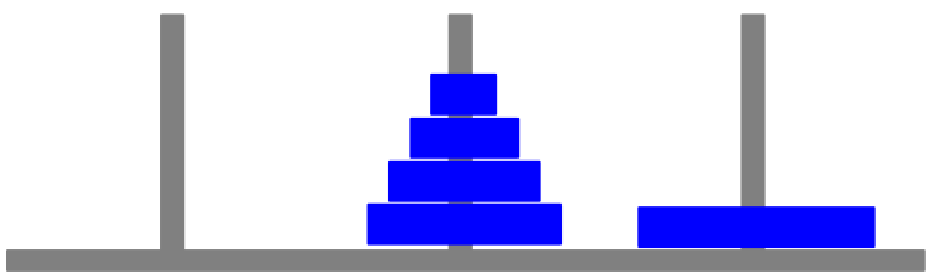
2) 

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3) 

move top $n-1$ disks from $\overline{\{i,j\}}$ to j

Towers of Hanoi

```java
public class Hanoi
{

    public void move(int n, char a, char b, char c)
    {
        if (n == 1)
            System.out.println("plate " + n + " from " + a + " to " + c);
        else
        {
            move(n-1,a,c,b);
            System.out.println("plate " + n + " from " + a + " to " + c);
            move(n-1,b,a,c);
        }

    }

```

Towers of Hanoi

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

ii) move largest disk from i to j

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- $p(1)$ is statement that algorithm works for $n = 1$ disks, which is obviously true

Towers of Hanoi

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- $p(n)$ is statement that algorithm is correct for n

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

ii) move largest disk from i to j

iii) move top $n-1$ disks from $\overline{\{i,j\}}$ to j

- $p(1)$ is statement that algorithm works for $n = 1$ disks, which is obviously true

- $p(n-1) \rightarrow p(n)$ is *recursion* statement that

  if our algorithm works for $n-1$ disks, then we can build a correct solution for n disks

Towers of Hanoi

- Running time

  $M(n)$ is number of disk moves needed for n disks

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  if $n > 1$, then $M(n) = 2M(n-1) + 1$

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  Later, we'll also see how to solve without guessing

Towers of Hanoi

- Formally, given

$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1) + 1 & \text{otherwise} \end{cases}$$

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Proof. (by induction)

The base case $n = 1$ is true, since $2^1 - 1 = 1$.

For the inductive step, assume that $M(n-1) = 2^{n-1} - 1$ for $n > 1$.

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Then $M(n) = 2M(n-1) + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$

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$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1)+1 & \text{otherwise} \end{cases}$$

- The second time was to derive the closed form solution $M(n) = 2^n - 1$ of the recurrence.

Recurrences

- A *recurrence equation* or *recurrence* for a function defined on the set of integers $\geq b$ is one that tells us how to compute the nth value from some or all the first $n-1$ values.

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Towers of Hanoi

Fibonacci Sequence

$$F(n) = \begin{cases} 1 & \text{if } n = \\ F(n-1)+F(n-2) & \text{other} \end{cases}$$

- **Example 2**: Let $S(n)$ be the number of subsets of a set of size n. What is the formula for $S(n)$?

  The empty set, of size $n = 0$ has only one subset (itself), so $S(0) = 1$.

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  We "guess" that $S(n) = 2^n$. But, in order to prove formula, we'll need to think recursively.

Recurrences

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  So, we get a subset of $\{1,2,3\}$ either by taking a subset of $\{1,2\}$ or by adjoining 3 to a subset of $\{1,2\}$.

  This suggests that the recurrence for the number of subsets of an n-element set $\{1,2,\ldots,n\}$ is

  $$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2S(n-1) & \text{if } n \geq 1 \end{cases}$$

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  Proof by induction is easy.

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  Can we generalize this to find a closed-form solution?

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Guess $T(n) = r^n T(0) + a \sum_{i=0}^{n-1} r^i$

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  $$\begin{aligned} &T(0) = b \\ &T(1) = rT(0) + a = rb + a \\ &T(2) = rT(1) + a = r(rb + a) + a = r^2b + ra + a \\ &T(3) = rT(2) + a = r^3b + r^2a + ra + a \end{aligned}$$

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  This would lead to the same guess

  $$T(n) = r^n b + a \sum_{i=0}^{n-1} r^i.$$

Formula of Recurrences

- **Theorem** If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1 - r^n}{1 - r}$$

for all nonnegative integers n.

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Proof by induction

The base case:

$$T(0) = r^0 b + a\frac{1-r^0}{1-r} = b.$$

So the formula is true when $n = 0$.

Now assume that $n > 0$ and

$$T(n-1) = r^{n-1} b + a\frac{1-r^{n-1}}{1-r}.$$

- **Proof by induction**

$$
\begin{aligned}
T(n) &= rT(n-1) + a \\
&= r\left(r^{n-1}b + a\frac{1-r^{n-1}}{1-r}\right) + a \\
&= r^n b + \frac{ar - ar^n}{1-r} + a \\
&= r^n b + \frac{ar - ar^n + a - ar}{1-r} \\
&= r^n b + a\frac{1-r^n}{1-r}.
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$T(n) = 3T(n-1) + 2$ with $T(0) = 5$

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Example:
$T(n) = 3T(n-1) + 2$ with $T(0) = 5$

Plugging $r = 3, a = 2, b = 5$ in the formula, gives

$$T(n) = 3^n \cdot 5 + 2\frac{1-3^n}{1-3} = 3^n \cdot 6 - 1$$

First-Order Linear Recurrences

- A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a *first-order linear recurrence*.

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  ◇ Linear because $T(n-1)$ only appears to the first power.

  Something like $T(n) = (T(n-1))^2 + 3$ would be a non-linear first-order recurrence relation.

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- $T(n) = f(n)T(n-1) + g(n)$

  When $f(n)$ is a constant, say r, the general solution is almost as easy as we derived before. Iterating the recurrence gives

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\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r(rT(n-2) + g(n-1)) + g(n) \\
&= r^2 T(n-2) + rg(n-1) + g(n) \\
&= r^3 T(n-3) + r^2 g(n-2) + rg(n-1) + g(n) \\
&\vdots \\
&= r^n T(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}
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- **Theorem** For any positive constants a and r, and any function g defined on nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i).$$

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\begin{aligned}
T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + (1 - \frac{1}{2^n}) \cdot 4^n \\
&= 7 \cdot 4^n - 2^n.
\end{aligned}
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Theorem. For any real number $x \neq 1$,

$$\sum_{i=1}^{n} ix^i = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}.$$

Examples

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T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\
&= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \\
&= 10 \cdot 3^n + 3^n \left(-\frac{3}{2}(n+1)3^{-(n+1)} - \frac{3}{4}3^{-(n+1)} + \frac{3}{4} \right. \\
&= \frac{43}{4}3^n - \frac{n+1}{2} - \frac{1}{4}.
\end{aligned}
$$

Next Lecture

- recurrence ...