南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Algorithm Design and Analysis (H)

CS216

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Divide and Conquer

4. Integer Multiplication

Integer Addition

- Addition. Given two n-bit integers a and b, compute $a + b$.
- Grade-school. $\Theta(n)$ bit operations.

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

- Remark. Grade-school addition algorithm is optimal.

Integer Multiplication

- Addition. Given two n-bit integers a and b, compute $a + b$.
- Grade-school. $\Theta(n)$ bit operations.

```
                1 1 0 1 0 1 0 1
              × 0 1 1 1 1 1 0 1
              -----------------
                1 1 0 1 0 1 0 1
              0 0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1 0
          1 1 0 1 0 1 0 1 0
        1 1 0 1 0 1 0 1 0
      1 1 0 1 0 1 0 1 0
    1 1 0 1 0 1 0 1 0
  0 0 0 0 0 0 0 0 0
-------------------------------
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1
```

- Q. Is grade-school multiplication algorithm optimal?

Divide-and-Conquer Multiplication: Warmup

To multiply two n-bit integers a and b:

- Multiply four $\frac{1}{2}n$-bit integers, recursively.
- Add and shift to obtain result.

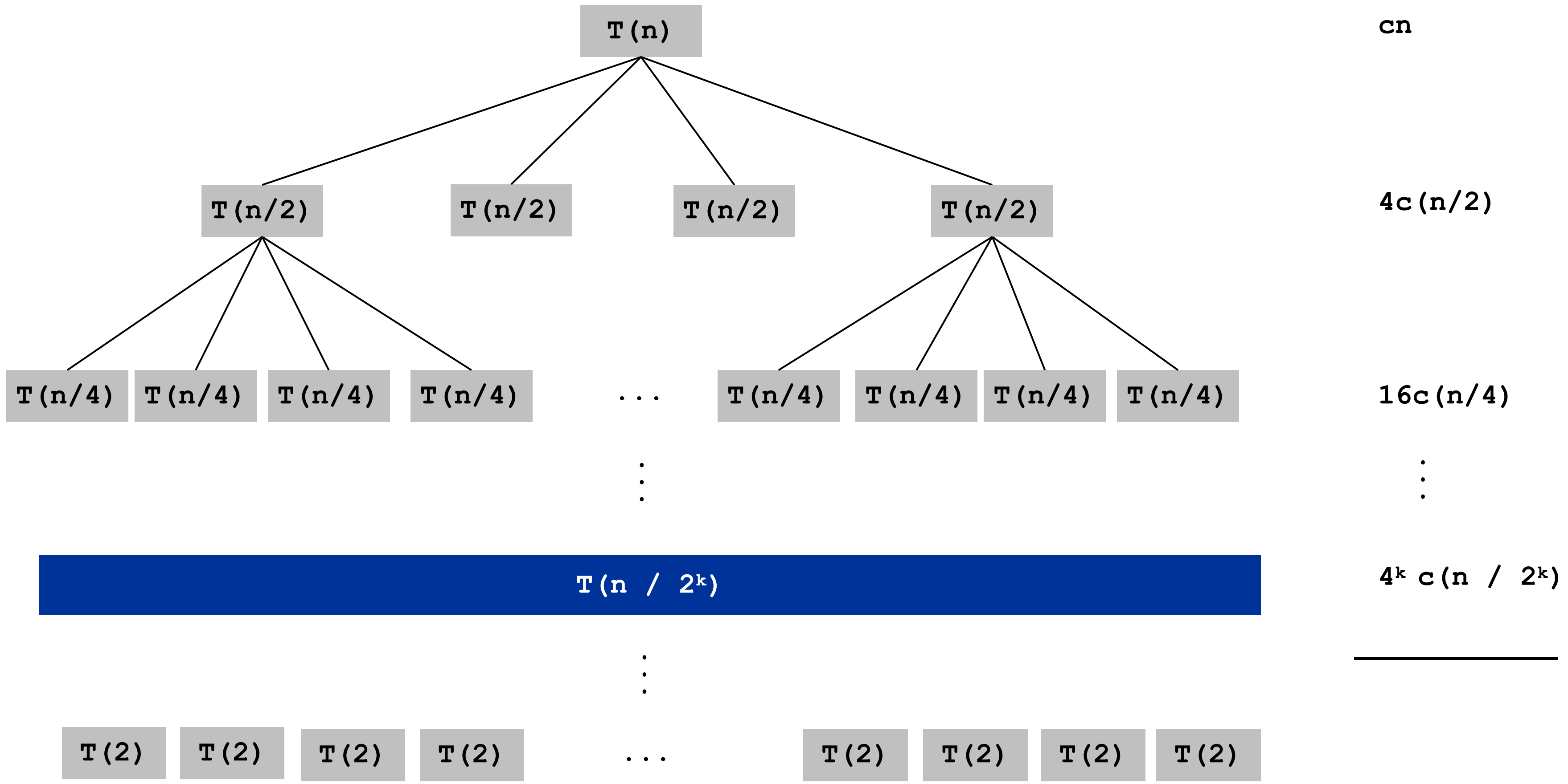
$$
\begin{aligned}
xy &= (x_1 \cdot 2^{n/2} + x_0)(y_1 \cdot 2^{n/2} + y_0) \\
&= x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0
\end{aligned}
$$

Ex. $\boldsymbol{a} = \underbrace{1000}_{a_1}\underbrace{1101}_{a_0}$ $\boldsymbol{b} = \underbrace{1110}_{b_1}\underbrace{0001}_{b_0}$

Recursion Tree

$$T(n) \leq 4T\left(\frac{n}{2}\right) + cn$$

$$T(n) \leq O(n^2)$$



Karatsuba Multiplication

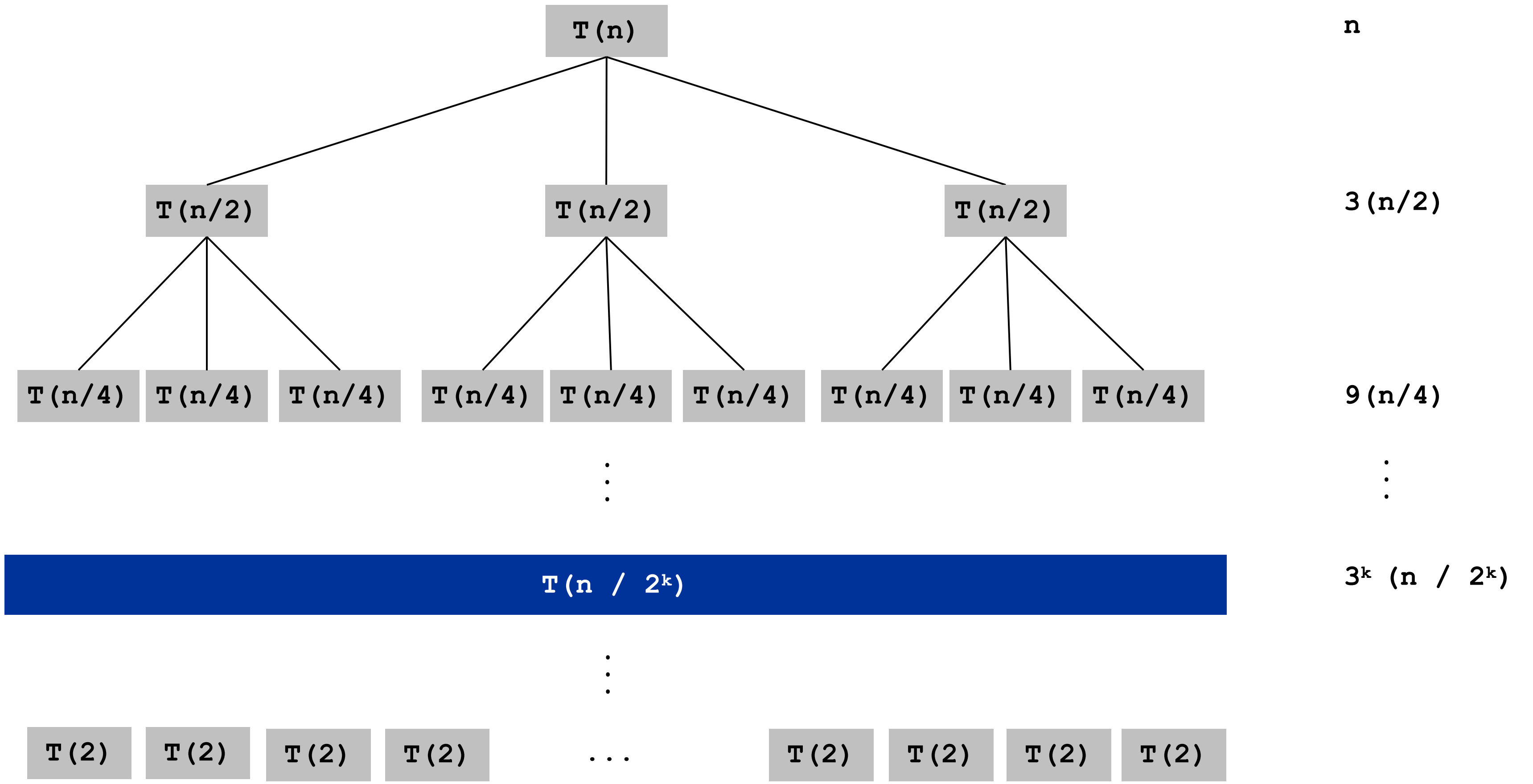
- To multiply two n-bit integers a and b:
  - Add two ½n bit integers.
  - Multiply three ½n-bit integers, recursively.
  - Add, subtract, and shift to obtain result.

```
Recursive-Multiply(x,y):
  Write x = x_1 · 2^{n/2} + x_0
        y = y_1 · 2^{n/2} + y_0
  Compute x_1 + x_0 and y_1 + y_0
  p = Recursive-Multiply(x_1 + x_0, y_1 + y_0)
  x_1y_1 = Recursive-Multiply(x_1, y_1)
  x_0y_0 = Recursive-Multiply(x_0, y_0)
  Return x_1y_1 · 2^n + (p − x_1y_1 − x_0y_0) · 2^{n/2} + x_0y_0
```

- Theorem. [Karatsuba-Ofman 1962] Can multiply two n-bit integers in $O(n^{1.585})$ bit operations.

Karatsuba: Recursion Tree

$$T(n) \leq 3T\left(\frac{n}{2}\right) + cn \qquad T(n) \leq O(n^{1.585})$$



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5. Convolution and FFT

Fast Fourier Transform: Applications

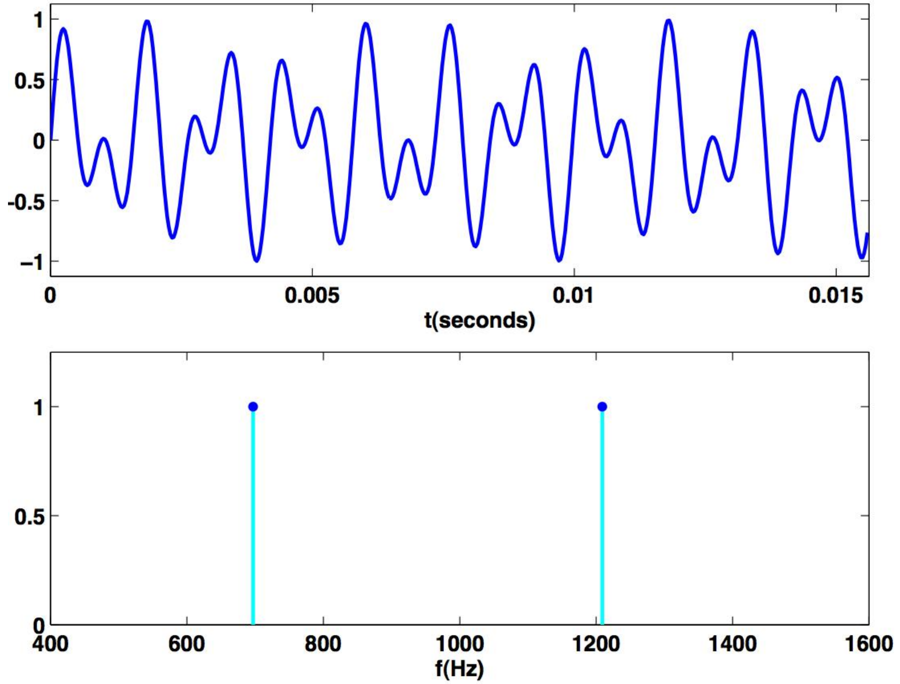
- Applications.
  - Optics, acoustics, quantum physics, telecommunications, control systems, signal processing, speech recognition, data compression, image processing.
  - DVD, JPEG, MP3, MRI, CAT scan.
  - Numerical solutions to Poisson's equation.

> The FFT is one of the truly great computational developments of this [20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. -Charles van Loan

Touch Tone

$$y_j = \sum_{k=0}^{n-1} a_k e^{-jk\frac{2\pi}{n}i}$$

- Button 1 signal.

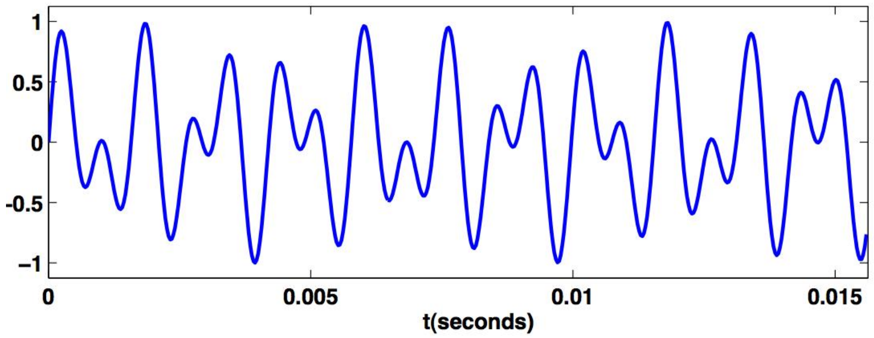


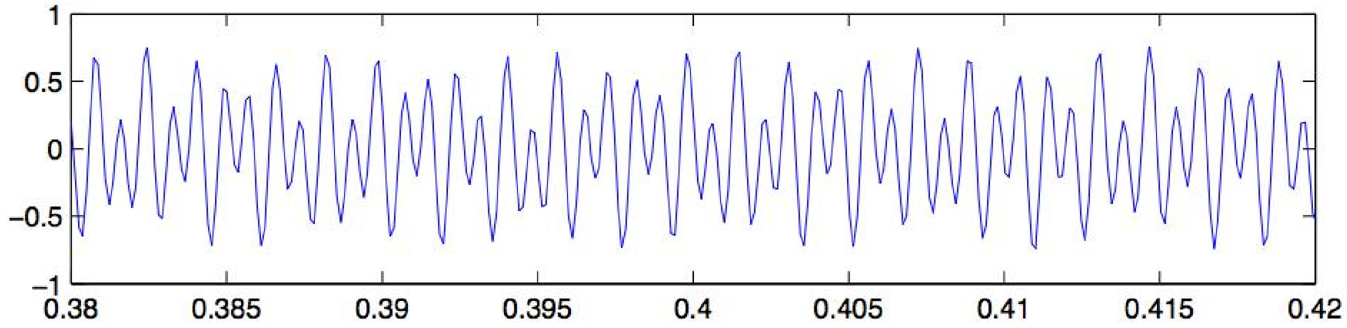
- Magnitude of Fourier transform of button 1 signal.

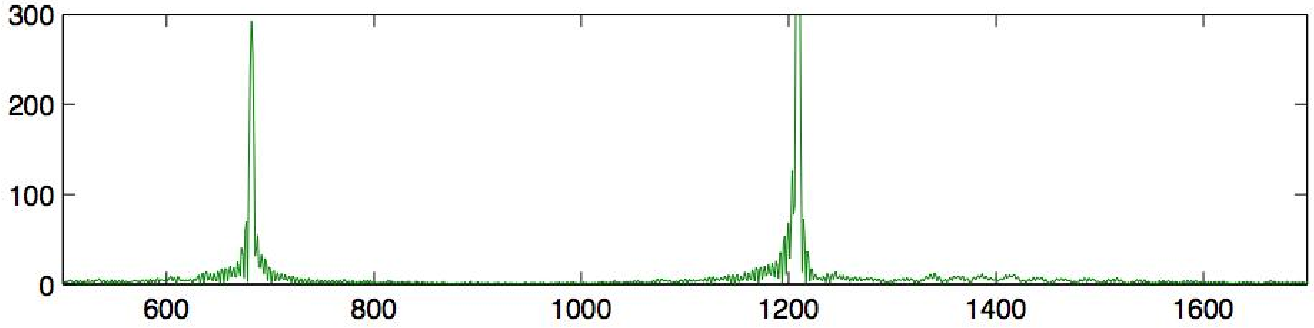
Reference: Cleve Moler, Numerical Computing with MATLAB

Touch Tone

- Button 1 signal. [recorded, 8192 samples per second]







- Magnitude of FFT.

Reference: Cleve Moler, Numerical Computing with MATLAB

Fast Fourier Transform: Brief History

- Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

- Runge-König (1924). Laid theoretical groundwork.

- Danielson-Lanczos (1942). Efficient algorithm.

- Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.

- Importance not fully realized until advent of digital computers.

Convolution

$(a_0, a_1, a_2, \ldots, a_{n-1})$

$(b_0, b_1, b_2, \ldots, b_{n-1})$

- What's convolution?
- $O(n^2)$?

Polynomials: Coefficient Representation

- Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

- Add: O(n) arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

- Evaluate: O(n) using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

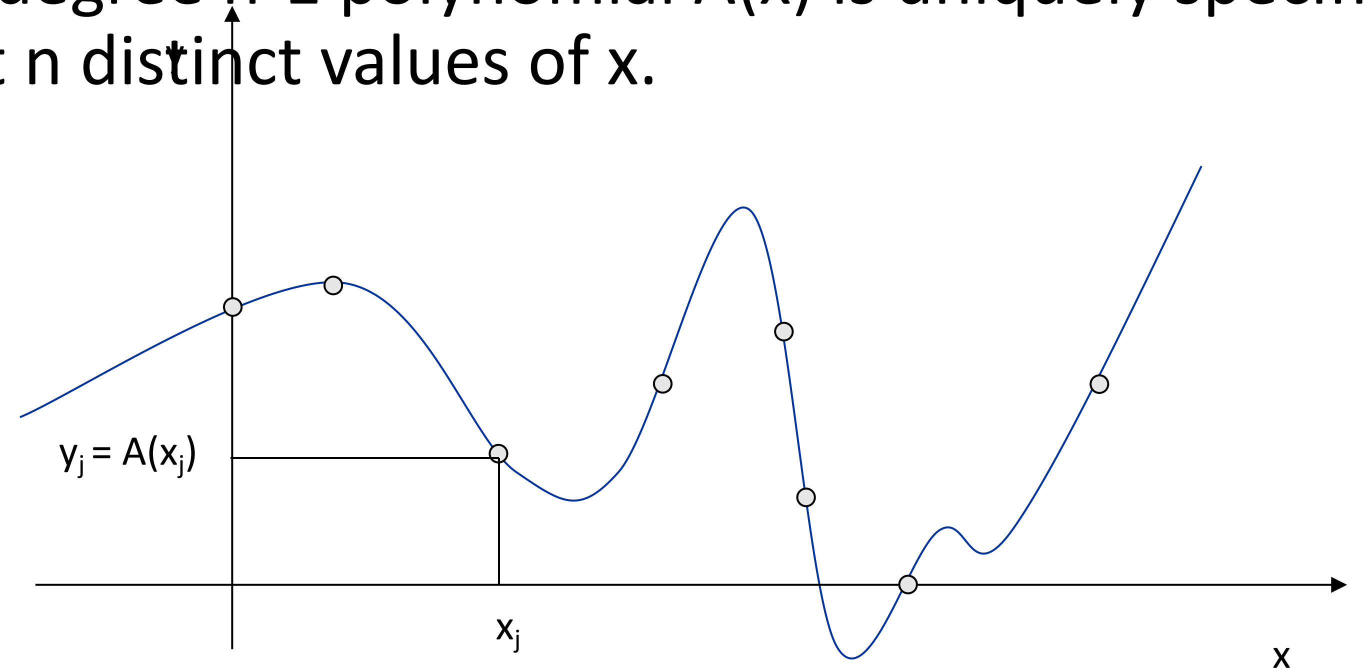
- Multiply (convolve): O(n^2) using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Polynomials: Point-Value Representation

- Fundamental theorem of algebra. [Gauss, PhD thesis] A degree n polynomial with complex coefficients has n complex roots.

- Corollary. A degree n-1 polynomial A(x) is uniquely specified by its evaluation at n distinct values of x.



Polynomials: Point-Value Representation

- Polynomial. [point-value representation]

$$A(x): \ (x_0, y_0), \ \ldots, (x_{n-1}, y_{n-1})$$

$$B(x): \ (x_0, \ z_0), \ \ldots, (x_{n-1}, z_{n-1})$$

- Add: O(n) arithmetic operations.

$$A(x) + B(x): \ \ (x_0, y_0 + z_0), \ldots, (x_{n-1}, y_{n-1} + z_{n-1})$$

- Multiply: O(n), but need 2n-1 points.

$$A(x) \times B(x): \ \ (x_0, y_0 \times z_0), \ldots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

- Evaluate: O(n^2) using Lagrange's formula.

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting Between Two Polynomial Representations

- Tradeoff. Fast evaluation or fast multiplication. We want both!

Representation	Multiply	Evaluate
Coefficient	$O(n^2)$	$O(n)$
Point-value	$O(n)$	$O(n^2)$

- Goal. Make all ops fast by efficiently converting between two representations.

$$a_0, a_1, \ldots, a_{n-1} \rightleftarrows (x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$

coefficient
representation

point-value
representation

Converting Between Two Polynomial Representations: Brute Force

- Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

$O(n^2)$ for matrix-vector multiply

$O(n^3)$ for Gaussian elimination

Vandermonde matrix is invertible iff x_i distinct

- Point-value to coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

Coefficient to Point-Value Representation: Intuition

- Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

- Divide. Break polynomial up into even and odd powers.
  - $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.
  - $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$.
  - $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$.
  - $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
  - $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

Can evaluate polynomial of degree $\leq n$ at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 1 point.

- Intuition. Choose two points to be ± 1.
  - $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
  - $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.

Coefficient to Point-Value Representation: Intuition

- Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

- Divide. Break polynomial up into even and odd powers.
  - $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
  - $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3.$
  - $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3.$
  - $A(x) = A_{even}(x^2) + x A_{odd}(x^2).$
  - $A(-x) = A_{even}(x^2) - x A_{odd}(x^2).$

Can evaluate polynomial of degree $\leq n$ at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 1 point.

- Intuition. Choose four points to be $\pm 1, \pm i$.
  - $A(1) = A_{even}(1) + 1\, A_{odd}(1).$
  - $A(-1) = A_{even}(1) - 1\, A_{odd}(1).$
  - $A(i) = A_{even}(-1) + i\, A_{odd}(-1).$
  - $A(-i) = A_{even}(-1) - i\, A_{odd}(-1).$

Discrete Fourier Transform

- Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

- Key idea: choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

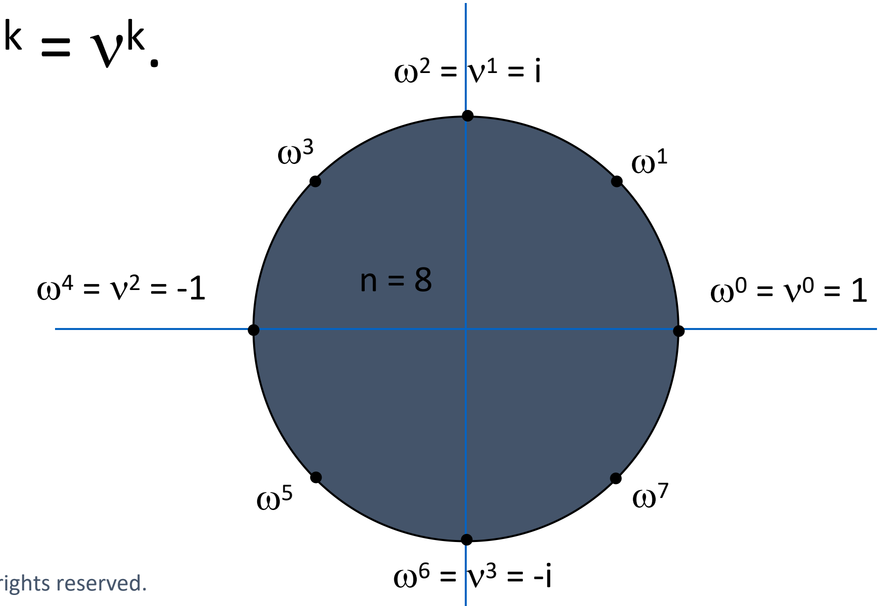
$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{\text{Discrete Fourier transform}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}}_{\text{Fourier matrix } F_n} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Roots of Unity

- Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

- Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.
- Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

- Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = e^{4\pi i / n}$.
- Fact. $\omega^2 = \nu$ and $(\omega^2)^k = \nu^k$.



Fast Fourier Transform

- Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + \ldots + a_{n-1} x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

- Divide. Break polynomial up into even and odd powers.
  - $A_{even}(x) = a_0 + a_2x + a_4x^2 + \ldots + a_{n/2-2}\, x^{(n-1)/2}$.
  - $A_{odd}(x) = a_1 + a_3x + a_5x^2 + \ldots + a_{n/2-1}\, x^{(n-1)/2}$.
  - $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

- Conquer. Evaluate degree $A_{even}(x)$ and $A_{odd}(x)$ at the ½n^{th} roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

$$\nu^k = (\omega^k)^2 = (\omega^{k+½n})^2$$

- Combine.
  - $A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$
  - $A(\omega^{k+n/2}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\omega^{k+½n} = -\omega^k$

FFT Algorithm

```
fft(n, a₀,a₁,…,aₙ₋₁) {
   if (n == 1) return a₀

   (e₀,e₁,…,e_{n/2-1}) ← FFT(n/2, a₀,a₂,a₄,…,aₙ₋₂)
   (d₀,d₁,…,d_{n/2-1}) ← FFT(n/2, a₁,a₃,a₅,…,aₙ₋₁)

   for k = 0 to n/2 - 1 {
      ωᵏ ← e^{2πik/n}
      y_k     ← e_k + ωᵏ d_k
      y_{k+n/2} ← e_k - ωᵏ d_k
   }

   return (y₀,y₁,…,yₙ₋₁)
}
```

FFT Summary

- Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the n^{th} roots of unity in O(n log n) steps.

assumes n is a power of 2

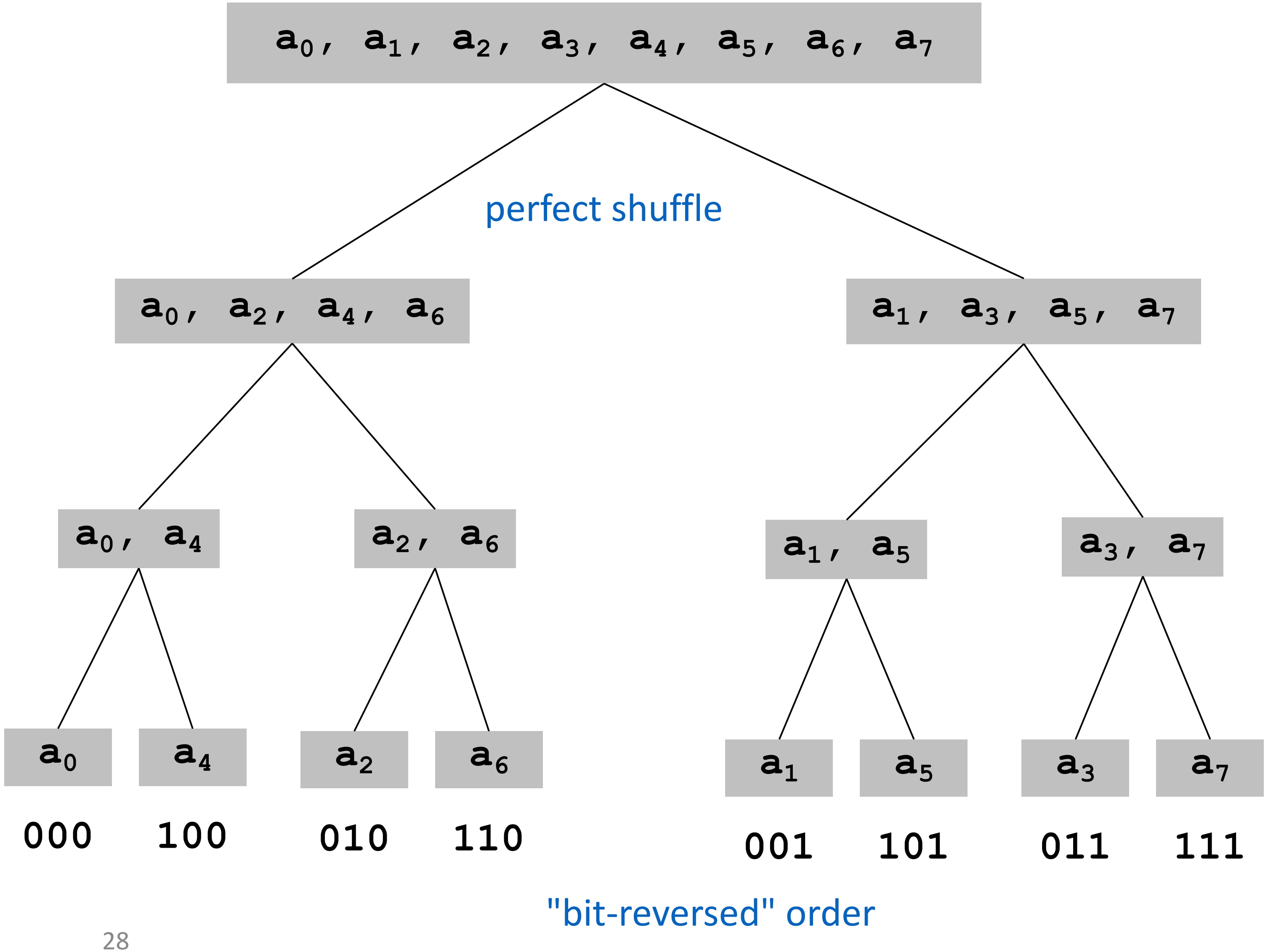
- Running time. $T(2n) = 2T(n) + O(n) \Rightarrow T(n) = O(n \log n)$.

$$a_0, a_1, \ldots, a_{n-1} \underset{}{\overset{O(n \log n)}{\rightleftarrows}} (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$$

coefficient representation

point-value representation

Recursion Tree



Point-Value to Coefficient Representation: Inverse DFT

- Goal. Given the values $y_0, \ldots, y_{n-1}$ of a degree n-1 polynomial at the n points $\omega^0, \omega^1, \ldots, \omega^{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Inverse DFT

Fourier matrix inverse $(F_n)^{-1}$

Inverse FFT

- Claim. Inverse of Fourier matrix is given by following formula.

$$G_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

- Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide by n).

Inverse FFT: Proof of Correctness

- Claim. F_n and G_n are inverses.
- Pf.

$$\left(F_n\, G_n\right)_{k k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{kj}\, \omega^{-jk'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')\,j} \underset{\uparrow}{=} \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

- Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

- Pf.
  - If k is a multiple of n then $\omega^k = 1 \Rightarrow$ sums to n.
  - Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{n-1})$.
  - if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \Rightarrow$ sums to 0.

■

Inverse FFT: Algorithm

```
ifft(n, a0,a1,…,an-1) {
   if (n == 1) return a0

   (e0,e1,…,en/2-1) ← FFT(n/2, a0,a2,a4,…,an-2)
   (d0,d1,…,dn/2-1) ← FFT(n/2, a1,a3,a5,…,an-1)

   for k = 0 to n/2 - 1 {
      ωk ← e-2πik/n
      yk     ← (ek + ωk dk) / n
      yk+n/2 ← (ek - ωk dk) / n
   }

   return (y0,y1,…,yn-1)
}
```

Inverse FFT Summary

- Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the n^{th} roots of unity in O(n log n) steps.

assumes n is a power of 2

$$a_0, a_1, \ldots, a_{n-1} \xrightarrow{O(n \log n)} (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$$

$$a_0, a_1, \ldots, a_{n-1} \xleftarrow[O(n \log n)]{} (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$$

coefficient representation

point-value representation

Polynomial Multiplication

- Theorem. Can multiply two degree n-1 polynomials in O(n log n) steps.

coefficient representation

$a_0, a_1, \ldots, a_{n-1}$
$b_0, b_1, \ldots, b_{n-1}$

FFT O(n log n)

$A(x_0), \ldots, A(x_{2n-1})$
$B(x_0), \ldots, B(x_{2n-1})$

point-value multiplication O(n)

$C(x_0), C(x_1), \ldots, C(x_{2n-1})$

inverse FFT O(n log n)

$c_0, c_1, \ldots, c_{2n-2}$

coefficient representation

Integer Multiplication

- Integer multiplication. Given two n bit integers $a = a_{n-1} \ldots a_1 a_0$ and $b = b_{n-1} \ldots b_1 b_0$, compute their product $c = a \times b$.

- Convolution algorithm.
  - Form two polynomials.
  - Note: a = A(2), b = B(2).
  - Compute C(x) = A(x) $\times$ B(x).
  - Evaluate C(2) = a $\times$ b.
  - Running time: O(n log n) complex arithmetic operations.

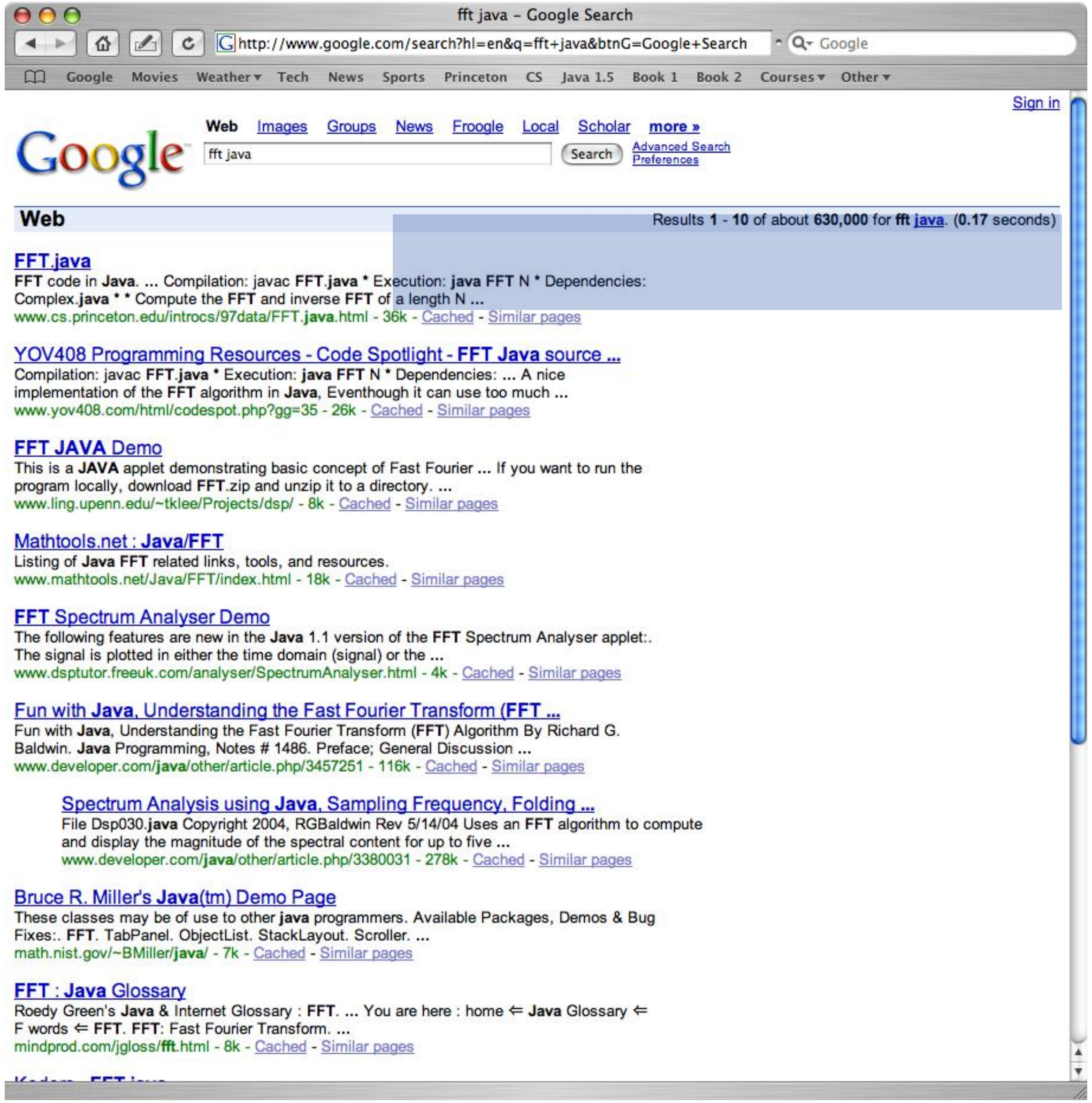
$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

- Theory. [Schönhage-Strassen 1971] O(n log n log log n) bit operations.
- Theory. [Fürer 2007] O(n log n $2^{O(\log^* n)}$) bit operations.

- Practice. [GNU Multiple Precision Arithmetic Library] GMP proclaims to be "the fastest bignum library on the planet." It uses brute force, Karatsuba, and FFT, depending on the size of n.

FFT in Practice ?



FFT in Practice

- Fastest Fourier transform in the West. [Frigo and Johnson]
  - Optimized C library.
  - Features: DFT, DCT, real, complex, any size, any dimension.
  - Won 1999 Wilkinson Prize for Numerical Software.
  - Portable, competitive with vendor-tuned code.

- Implementation details.
  - Instead of executing predetermined algorithm, it evaluates your hardware and uses a special-purpose compiler to generate an optimized algorithm catered to "shape" of the problem.
  - Core algorithm is nonrecursive version of Cooley-Tukey radix 2 FFT.
  - O(n log n), even for prime sizes.

Reference: http://www.fftw.org