CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

Dr. QI WANG

Department of Computer Science and Engineering
Office: Room413, CoE South Tower
Email: wangqi@sustech.edu.cn

$$a = (a_{n-1}a_{n-2}\ldots a_1 a_0)_2, \; b = (b_{n-1}b_{n-2}\ldots b_1 b_0)_2$$

$$
\begin{aligned}
ab &= a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1} 2^{n-1}) \\
&= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1} 2^{n-1})
\end{aligned}
$$

procedure *multiply*(*a, b*: positive integers)
{the binary expansions of a and b are $(a_{n-1}, a_{n-2}, \ldots, a_0)_2$ and $(b_{n-1}, b_{n-2}, \ldots, b_0)_2$, respectively}
for $j := 0$ to $n-1$
    if $b_j = 1$ **then** $c_j = a$ shifted j places
    else $c_j := 0$
{$c_0, c_1, \ldots, c_{n-1}$ are the partial products}
$p := 0$
for $j := 0$ to $n-1$
  $p := p + c_j$
return p {p is the value of ab}

$$a = (a_{n-1}a_{n-2}\ldots a_1 a_0)_2, \; b = (b_{n-1}b_{n-2}\ldots b_1 b_0)_2$$

$$\begin{aligned}
ab &= a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1}2^{n-1}) \\
&= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1}2^{n-1})
\end{aligned}$$

procedure $multiply(a, b$: positive integers)

{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}

for $j := 0$ to $n - 1$

    if $b_j = 1$ **then** $c_j = a$ shifted j places

    else $c_j := 0$

{$c_0, c_1, \ldots, c_{n-1}$ are the partial products}

$p := 0$

for $j := 0$ to $n - 1$

  $p := p + c_j$

return p {p is the value of ab}

$O(n^2)$ shifts and $O(n^2)$ bit additions

2 - 2

procedure *division algorithm* (a: integer, d: positive integer)

$q := 0$

$r := |a|$

while $r \geq d$

$\quad r := r - d$

$\quad q := q + 1$

if $a < 0$ and $r > 0$ **then**

$\quad r := d - r$

$\quad q := -(q+1)$

return (q, r) {$q = a$ **div** d is the quotient, $r = a$ **mod** d is the remainder }

procedure *division algorithm* (a: integer, *d*: positive integer)

$q := 0$

$r := |a|$

while $r \geq d$

$\quad r := r - d$

$\quad q := q + 1$

if $a < 0$ and $r > 0$ **then**

$\quad r := d - r$

$\quad q := -(q+1)$

return (q, r) {$q = a$ **div** d *is the quotient*, $r = a$ **mod** d *is the remainder* }

$O(q \log a)$ bit operations. But there exist more efficient algorithms with complextiy $O(n^2)$, where $n = \max(\log a, \log d)$

3 - 2

- **procedure** *division*2 $(a, d \in \mathbb{N}, \; d \geq 1)$
  if $a < d$
    return $(q, r) = (0, a)$
  $(q, r) = division2(\lfloor a/2 \rfloor, d)$
  $q = 2q, \; r = 2r$
  if a is odd
    $r = r + 1$
  if $r \geq d$
    $r = r - d$
    $q = q + 1$
  return (q, r)

- **procedure** *division*2 $(a, d \in \mathbb{N},\ d \geq 1)$
  if $a < d$
     return $(q, r) = (0, a)$
  $(q, r) = division2(\lfloor a/2 \rfloor, d)$
  $q = 2q,\ r = 2r$
  if a is odd
     $r = r + 1$
  if $r \geq d$
     $r = r - d$
     $q = q + 1$
  return (q, r)

  $O(\log q \log a)$ bit operations.

Algorithm: Binary Modular Exponentiation

$$b^n = b^{a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0} = b^{a_{k-1} \cdot 2^{k-1}} \cdots b^{a_1 \cdot 2} \cdot b^{a_0}$$

Successively finds $b \bmod m$, $b^2 \bmod m$, $b^4 \bmod m$, $\ldots$, $b^{2^{k-1}} \bmod m$, and multiplies together the terms b^{2^j} where $a_j = 1$.

procedure *modular exponentiation*(b: integer, $n = (a_{k\text{-}1}a_{k\text{-}2}...a_1a_0)_2$, m: positive integers)

$x := 1$

power := b **mod** m

for $i := 0$ to $k - 1$

    if $a_i = 1$ **then** $x := (x \cdot power)$ **mod** m

    power := (*power·power*) **mod** m

return x {x equals b^n **mod** m }

$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}} \cdots b^{a_1\cdot 2} \cdot b^{a_0}$$

Successively finds $b \bmod m$, $b^2 \bmod m$, $b^4 \bmod m$, $\ldots$, $b^{2^{k-1}} \bmod m$, and multiplies together the terms b^{2^j} where $a_j = 1$.

procedure *modular exponentiation*(b: integer, $n = (a_{k\text{-}1}a_{k\text{-}2}...a_1a_0)_2$, m: positive
   integers)
$x := 1$
power := b **mod** m
for $i := 0$ to $k-1$
    if $a_i = 1$ **then** $x := (x \cdot power)$ **mod** m
    power := $(power \cdot power)$ **mod** m
return x {x equals b^n **mod** m }

$O((\log m)^2 \log n)$ bit operations

5 - 2

- A positive integer p that is greater than 1 and is divisible only by 1 and by itself is called a *prime*.

Primes

- A positive integer p that is greater than 1 and is divisible only by 1 and by itself is called a *prime*.

- A positive integer p that is greater than 1 and is not a prime is called a *composite*.

- A positive integer p that is greater than 1 and is divisible only by 1 and by itself is called a *prime*.

- A positive integer p that is greater than 1 and is not a prime is called a *composite*.

- **Fundamental Theorem of Arithmetic** Every integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Primes and Composites

- How to determine whether a number is a prime or a composite?

- How to determine whether a number is a prime or a composite?

  Approach 1: test if each number $x < n$ divides n.

Primes and Composites

- How to determine whether a number is a prime or a composite?

  Approach 1: test if each number $x < n$ divides n.

  Approach 2: test if each prime number $x < n$ divides n.

Primes and Composites

- How to determine whether a number is a prime or a composite?

  Approach 1: test if each number $x < n$ divides n.

  Approach 2: test if each prime number $x < n$ divides n.

  Approach 3: test if each prime number $x \leq \sqrt{n}$ divides n.

- If n is composite, then n has a prime divisor less than or equal to $\sqrt{n}$.

- If n is composite, then n has a prime divisor less than or equal to $\sqrt{n}$.

Proof.

$\diamond$ if n is composite, then it has a positive integer factor a such that $1 < a < n$ by definition. This means that $n = ab$, where b is an integer greater than 1.

$\diamond$ assume that $a > \sqrt{n}$ and $b > \sqrt{n}$. Then $ab > n$, contradiction. So either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

$\diamond$ Thus, n has a divisor less than $\sqrt{n}$.

$\diamond$ By the Fundamental Theorem of Arithmetic, this divisor is either prime, or is a product of primes. In either case, n has a prime divisor less than $\sqrt{n}$.

- There are infinitely many primes.

Proof (by contradiction)

Greatest Common Divisor (GCD)

- Let a and b be integers, not both 0. The largest integer d such that $d \mid a$ and $d \mid b$ is called the *greatest common divisor* of a and b, denoted by $\gcd(a, b)$.

- Let a and b be integers, not both 0. The largest integer d such that $d|a$ and $d|b$ is called the *greatest common divisor* of a and b, denoted by $\gcd(a, b)$.

  The integers a and b are *relatively prime* if their greatest common divisor is 1.

- Let a and b be integers, not both 0. The largest integer d such that $d \mid a$ and $d \mid b$ is called the *greatest common divisor* of a and b, denoted by $\gcd(a, b)$.

  The integers a and b are *relatively prime* if their greatest common divisor is 1.

  A systematic way to find the gcd is **factorization**. Let $a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$. Then $\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_n^{\min(a_k, b_k)}$

Least Common Multiple (LCM)

- Let a and b be integers. The *least common multiple* of a and b is the smallest positive integer that is divisible by both a and b, denoted by $\text{lcm}(a, b)$.

- Let a and b be integers. The *least common multiple* of a and b is the smallest positive integer that is divisible by both a and b, denoted by $\text{lcm}(a, b)$.

  We can also use **factorization** to find the lcm. Let $a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$. Then $\text{lcm}(a, b) = p_1^{\max(a_1, b_1)} p_2^{\max(a_2, b_2)} \cdots p_n^{\max(a_k, b_k)}$

- Factorization can be **cumbersome** and **time consuming** since we need to find all factors of the two integers.

Euclidean Algorithm

- Factorization can be **cumbersome** and **time consuming** since we need to find all factors of the two integers.

- Luckily, we have an efficient algorithm, called **Euclidean algorithm**. This algorithm has been known since ancient times and named after the ancient Greek mathmaticain Euclid.

- For two integers 287 and 91, we want to find $\gcd(287, 91)$.

- For two integers 287 and 91, we want to find $\gcd(287, 91)$.

    Step 1: $287 = 91 \cdot 3 + 14$

- For two integers 287 and 91, we want to find $\gcd(287, 91)$.

  Step 1: $287 = 91 \cdot 3 + 14$

  Step 2: $91 = 14 \cdot 6 + 7$

- For two integers 287 and 91, we want to find $\gcd(287, 91)$.

  Step 1: $287 = 91 \cdot 3 + 14$

  Step 2: $91 = 14 \cdot 6 + 7$

  Step 3: $14 = 7 \cdot 2 + 0$

- For two integers 287 and 91, we want to find $\gcd(287, 91)$.

  Step 1: $287 = 91 \cdot 3 + 14$

  Step 2: $91 = 14 \cdot 6 + 7$

  Step 3: $14 = 7 \cdot 2 + 0$

  $\gcd(287, 91) = \gcd(91, 14) = \gcd(14, 7) = 7$

- **The Euclidean algorithm in pseudocode**

ALGORITHM 1 The Euclidean Algorithm.

procedure $gcd(a, b$: positive integers)
$x := a$
$y := b$
while $y \neq 0$
    $r := x \bmod y$
    $x := y$
    $y := r$
return $x\{\gcd(a, b) \text{ is } x\}$

Euclidean Algorithm

- **The Euclidean algorithm in pseudocode**

ALGORITHM 1 The Euclidean Algorithm.

procedure $gcd(a, b$: positive integers)
$x := a$
$y := b$
while $y \neq 0$
    $r := x \bmod y$
    $x := y$
    $y := r$
return $x\{gcd(a, b)$ is $x\}$

The number of divisions required to find $gcd(a, b)$ is $O(\log b)$, where $a \geq b$. (this will be proved later.)

- **Lemma** Let $a = bq + r$, where a, b, q and r are integers. Then $\gcd(a, b) = \gcd(b, r)$.

■ **Lemma** Let $a = bq + r$, where a, b, q and r are integers. Then $\gcd(a, b) = \gcd(b, r)$.

Proof.

⋄ suppose that $d|a$ and $d|b$. Then d also divides $a - bq = r$. Hence, any common divisor of a and b must also be any common divisor of b and r.

⋄ suppose that $d|b$ and $d|r$. Then d also divdes $bq + r = a$. Hence, any common divisor of b and r must also be a common divisor of a and b.

⋄ Therefore, $\gcd(a, b) = \gcd(b, r)$.

- Suppose that a and b are positive integers with $a \geq b$. Let $r_0 = a$ and $r_1 = b$.

- Suppose that a and b are positive integers with $a \geq b$. Let $r_0 = a$ and $r_1 = b$.

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 & 0 \leq r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 & 0 \leq r_3 < r_2, \\
& \quad \cdot & \\
& \quad \cdot & \\
& \quad \cdot & \\
r_{n-2} &= r_{n-1} q_{n-1} + r_n & 0 \leq r_n < r_{n-1}, \\
r_{n-1} &= r_n q_n \,. &
\end{aligned}
$$

- Suppose that a and b are positive integers with $a \geq b$. Let $r_0 = a$ and $r_1 = b$.

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 & 0 &\leq r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 & 0 &\leq r_3 < r_2, \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
r_{n-2} &= r_{n-1} q_{n-1} + r_n & 0 &\leq r_n < r_{n-1}, \\
r_{n-1} &= r_n q_n.
\end{aligned}
$$

$$
\gcd(a, b) = \gcd(r_0, r_1) = \cdots = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n
$$

- **Bezout's Theorem** If a and b are positive integers, then there exist integers s and t such that $\gcd(a, b) = sa + tb$. This is called *Bezout's identity*.

- **Bezout's Theorem** If a and b are positive integers, then there exist integers s and t such that $\gcd(a, b) = sa + tb$. This is called *Bezout's identity*.

  We may use *extended Euclidean algorithm* to find Bezout's identity.

- **Bezout's Theorem** If a and b are positive integers, then there exist integers s and t such that $\gcd(a, b) = sa + tb$. This is called *Bezout's identity*.

We may use *extended Euclidean algorithm* to find Bezout's identity.

Example: Express 1 as the linear combination of 503 and 286.

GCD as Linear Combinations

- **Bezout's Theorem** If a and b are positive integers, then there exist integers s and t such that $\gcd(a, b) = sa + tb$. This is called *Bezout's identity*.

We may use *extended Euclidean algorithm* to find Bezout's identity.

Example: Express 1 as the linear combination of 503 and 286.

$$503 = 1 \cdot 286 + 217$$
$$286 = 1 \cdot 217 + 69$$
$$217 = 3 \cdot 69 + 10$$
$$69 = 6 \cdot 10 + 9$$
$$10 = 1 \cdot 9 + 1$$

GCD as Linear Combinations

- **Bezout's Theorem** If a and b are positive integers, then there exist integers s and t such that $\gcd(a,b) = sa + tb$. This is called *Bezout's identity*.

We may use *extended Euclidean algorithm* to find Bezout's identity.

Example: Express 1 as the linear combination of 503 and 286.

$$503 = 1 \cdot 286 + 217$$
$$286 = 1 \cdot 217 + 69$$
$$217 = 3 \cdot 69 + 10$$
$$69 = 6 \cdot 10 + 9$$
$$10 = 1 \cdot 9 + 1$$

$$
\begin{aligned}
1 &= 10 - 1 \cdot 9 \\
&= 7 \cdot 10 - 1 \cdot 69 \\
&= 7 \cdot 217 - 22 \cdot 69 \\
&= 29 \cdot 217 - 22 \cdot 286 \\
&= 29 \cdot 503 - 51 \cdot 286
\end{aligned}
$$

Corollaries of Bezout's Theorem

- If a, b, c are positive integers such that $\gcd(a, b) = 1$ and $a | bc$, then $a | c$.

- If a, b, c are positive integers such that $\gcd(a, b) = 1$ and $a|bc$, then $a|c$.

  Proof. Since $\gcd(a, b) = 1$, by Bezout's Theorem there exist s and t such that $1 = sa + tb$. This yields $c = sac + tbc$. Since $a|bc$, we have $a|tbc$, and then $a|(sac + tbc) = c$.

- If a, b, c are positive integers such that $\gcd(a, b) = 1$ and $a|bc$, then $a|c$.

  Proof. Since $\gcd(a, b) = 1$, by Bezout's Theorem there exist s and t such that $1 = sa + tb$. This yields $c = sac + tbc$. Since $a|bc$, we have $a|tbc$, and then $a|(sac + tbc) = c$.

- If p is prime and $p|a_1 a_2 \cdots a_n$, then $p|a_i$ for some i.

Corollaries of Bezout's Theorem

- If a, b, c are positive integers such that $\gcd(a, b) = 1$ and $a|bc$, then $a|c$.

  Proof. Since $\gcd(a, b) = 1$, by Bezout's Theorem there exist s and t such that $1 = sa + tb$. This yields $c = sac + tbc$. Since $a|bc$, we have $a|tbc$, and then $a|(sac + tbc) = c$.

- If p is prime and $p|a_1 a_2 \cdots a_n$, then $p|a_i$ for some i.

  Proof. by induction. Will be given later.

- We prove that a prime factorization of a positive integer where the primes are in nondecreasing order is unique.

■ We prove that a prime factorization of a positive integer where the primes are in nondecreasing order is unique.

Proof. (by contradiction) Suppose that the positive integer n can be written as a product of primes in two distinct ways:

$n = p_1 p_2 \cdots p_s$ and $n = q_1 q_2 \cdots q_t$

Remove all common primes from the factorizations to get

$p_{i_1} p_{i_2} \cdots p_{i_u} = q_{j_1} q_{j_2} \cdots q_{j_v}$

It then follows that p_{i_1} divides q_{j_k} for some k, contradicting the assumption that p_{i_1} and q_{j_k} are distinct primes.
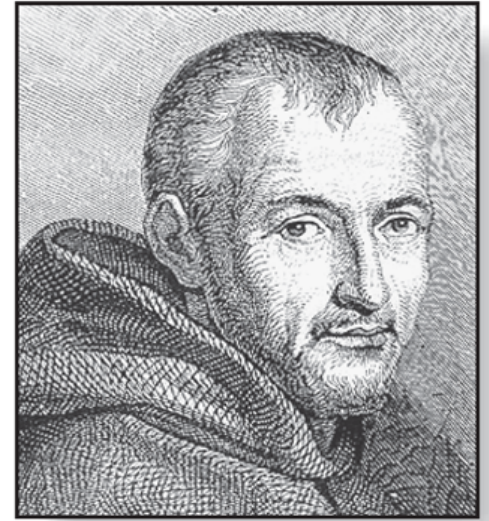
- **Theorem** Let m be a positive integer and let a, b, c be integers. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

- **Theorem** Let m be a positive integer and let a, b, c be integers. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

  Proof. Since $ac \equiv bc \pmod{m}$, we have $m \mid ac - bc = c(a - b)$. Because $\gcd(c, m) = 1$, it follows that $m \mid a - b$.

Mersenne Primes

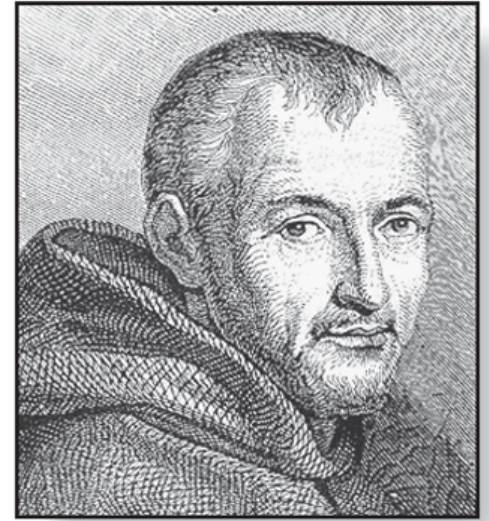- Prime numbers of the form $2^p - 1$, where p is a prime.

Marin Mersenne

- Prime numbers of the form $2^p - 1$, where p is a prime.

  ◇ $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, $2^7 - 1 = 127$ are Mersenne primes.

  ◇ $2^{11} - 1 = 2047 = 23 \cdot 89$ is not a Mersenne prime.

Marin Mersenne

- Prime numbers of the form $2^p - 1$, where p is a prime.

  ◇ $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, $2^7 - 1 = 127$ are Mersenne primes.

  ◇ $2^{11} - 1 = 2047 = 23 \cdot 89$ is not a Mersenne prime.

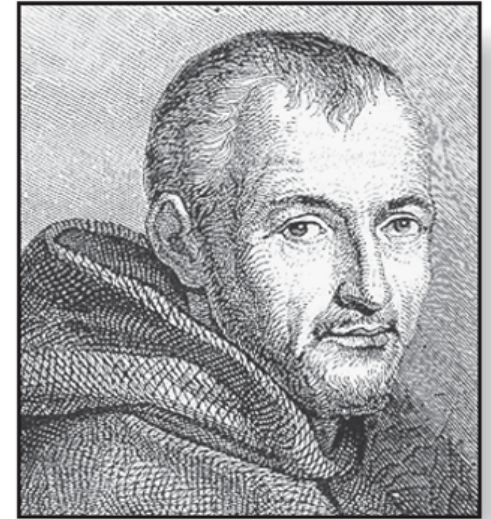 ◇ The largest known prime numbers are Mersenne primes.

Marin Mersenne

Mersenne Primes

- Prime numbers of the form $2^p - 1$, where p is a prime.

  ◇ $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, $2^7 - 1 = 127$ are Mersenne primes.

  ◇ $2^{11} - 1 = 2047 = 23 \cdot 89$ is not a Mersenne prime.

Marin Mersenne

Largest Known Prime, 49th Known Mersenne Prime Found!

January 7, 2016 — GIMPS celebrated its 20th anniversary with the discovery of the largest known prime number, $2^{74,207,281}$-1.

50th Known Mersenne Prime Found!

January 3, 2018 — Persistence pays off. Jonathan Pace, a GIMPS volunteer for over 14 years, discovered the 50th known Mersenne prime, $2^{77,232,917}$-1 on December 26, 2017. The prime number is calculated by multiplying together 77,232,917 twos, and then subtracting one. It weighs in at 23,249,425 digits, becoming the largest prime number known to mankind. It bests the previous record prime, also discovered by GIMPS, by 910,807 digits.

51st Known Mersenne Prime Found!

December 21, 2018 — The Great Internet Mersenne Prime Search (GIMPS) has discovered the largest known prime number, $2^{82,589,933}$-1, having 24,862,048 digits. A computer volunteered by Patrick Laroche from Ocala, Florida made the find on December 7, 2018. The new prime number, also known as M82589933, is calculated by multiplying together 82,589,933 twos and then subtracting one. It is more than one and a half million digits larger than the previous record prime number.
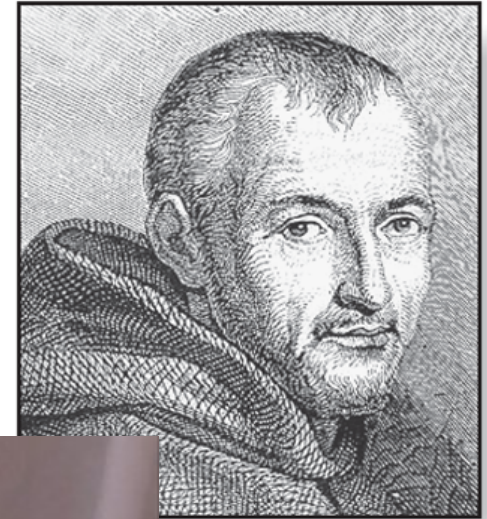
http://www.mersenne.org/

21 - 4

Mersenne Primes

- Prime numbers of the form $2^p - 1$, where p is a prime.

  - $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, $2^7 - 1 = 127$ are Mersenne primes.

Marin Mersenne

Prime Found!

number, $2^{74,207,281}$-1.

Janua

January
December
the largest

prime, $2^{77,232,917}$-1 on
23,249,425 digits, becoming

Decembe
digits. A co
calculated b
number.

$89,933$-1, having 24,862,048
known as M82589933, is
previous record prime

http://www.mersenne.org/

21 - 5

- *Goldbach's Conjecture* $(1 + 1)$: Every even integer $n > 2$, is the sum of two primes.

■ *Goldbach's Conjecture* $(1+1)$: Every even integer $n > 2$, is the sum of two primes.

"$3 + 4$", "$3 + 3$", "$2 + 3$" $-$ Y. Wang, 1956

"$1 + 5$" $-$ C. Pan, 1962

"$1 + 4$" $-$ Y. Wang, 1962

"$1 + 2$" $-$ J. Chen, 1973

- *Goldbach's Conjecture* $(1+1)$: Every even integer $n > 2$, is the sum of two primes.

  "$3 + 4$", "$3 + 3$", "$2 + 3$" $-$ Y. Wang, 1956

  "$1 + 5$" $-$ C. Pan, 1962

  "$1 + 4$" $-$ Y. Wang, 1962

  "$1 + 2$" $-$ J. Chen, 1973

- *Twin-prime Conjecture*: There are infinitely many twin primes.

- A congruence of the form $ax \equiv b \pmod{m}$, where m is a positive integer, a and b are integers, and x is a variable, is called a *linear congruence*.

- A congruence of the form $ax \equiv b \pmod{m}$, where m is a positive integer, a and b are integers, and x is a variable, is called a *linear congruence*.

  The solutions to a linear congruence $ax \equiv b \pmod{m}$ are all integers x that satisfy the congruence.

Linear Congruences

- A congruence of the form $ax \equiv b \pmod{m}$, where m is a positive integer, a and b are integers, and x is a variable, is called a *linear congruence*.

  The solutions to a linear congruence $ax \equiv b \pmod{m}$ are all integers x that satisfy the congruence.

  Systems of linear congruences have been studied since ancient times.

  今有物不知其数 三三数之剩二 五五数之剩三 七七数之剩二 问物几何

  About 1500 years ago, the Chinese mathematician Sun-Tsu asked: "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

23 - 3

- An integer $\bar{a}$ such that $\bar{a}a \equiv 1 \pmod{m}$ is said to be an *inverse* of a modulo m.

- An integer $\bar{a}$ such that $\bar{a}a \equiv 1 \pmod{m}$ is said to be an *inverse* of a modulo m.

  One method of solving linear congruences makes use of an inverse $\bar{a}$ if it exists. From $ax \equiv b \pmod{m}$, it follows that $\bar{a}ax \equiv \bar{a}b \pmod{m}$ and then $x \equiv \bar{a}b \pmod{m}$.

Modular Inverse

- An integer $\bar{a}$ such that $\bar{a}a \equiv 1 \pmod{m}$ is said to be an *inverse* of a modulo m.

  One method of solving linear congruences makes use of an inverse $\bar{a}$ if it exists. From $ax \equiv b \pmod{m}$, it follows that $\bar{a}ax \equiv \bar{a}b \pmod{m}$ and then $x \equiv \bar{a}b \pmod{m}$.

  When does an inverse of a modulo m exist?

- **Theorem** If a and m are relatively prime integers and $m > 1$, then an inverse of a modulo m exists. Furthermore, the inverse is uinque modulo m.

- **Theorem** If a and m are relatively prime integers and $m > 1$, then an inverse of a modulo m exists. Furthermore, the inverse is uinque modulo m.

    Proof. Since $\gcd(a, m) = 1$, there are integers s and t such that $sa + tm = 1$. Hence $sa + tm \equiv 1 \pmod{m}$. Since $tm \equiv 0 \pmod{m}$, it follows that $sa \equiv 1 \pmod{m}$. This means that s is an inverse of a modulo m.

- **Theorem** If a and m are relatively prime integers and $m > 1$, then an inverse of a modulo m exists. Furthermore, the inverse is uinque modulo m.

Proof. Since $\gcd(a, m) = 1$, there are integers s and t such that $sa + tm = 1$. Hence $sa + tm \equiv 1 \pmod{m}$. Since $tm \equiv 0 \pmod{m}$, it follows that $sa \equiv 1 \pmod{m}$. This means that s is an inverse of a modulo m.

How to prove the uniqueness of the inverse?

How to find inverses?

- Using *extended Euclidean algorithm*

- Using *extended Euclidean algorithm*

  Example. Find an inverse of 101 modulo 4620.

How to find inverses?

- Using *extended Euclidean algorithm*

  Example. Find an inverse of 101 modulo 4620.

  $4620 = 45 \cdot 101 + 75$

  $101 = 1 \cdot 75 + 26$

  $75 = 2 \cdot 26 + 23$

  $26 = 1 \cdot 23 + 3$

  $23 = 7 \cdot 3 + 2$

  $3 = 1 \cdot 2 + 1$

  $2 = 2 \cdot 1$

How to find inverses?

- Using *extended Euclidean algorithm*

Example. Find an inverse of 101 modulo 4620.

$$4620 = 45 \cdot 101 + 75$$
$$101 = 1 \cdot 75 + 26$$
$$75 = 2 \cdot 26 + 23$$
$$26 = 1 \cdot 23 + 3$$
$$23 = 7 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$
$$2 = 2 \cdot 1$$

$$1 = 3 - 1 \cdot 2$$
$$1 = 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$$
$$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$$
$$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = 26 \cdot 26 - 9 \cdot 75$$
$$1 = 26 \cdot (101 - 1 \cdot 75) - 9 \cdot 75$$
$$= 26 \cdot 101 - 35 \cdot 75$$
$$1 = 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$$
$$= -35 \cdot 4620 + 1601 \cdot 101$$

- Solve the congruence $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.

- Solve the congruence $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.

  Example. What are the solutions of the congruence $3x \equiv 4 \pmod{7}$?

- Solve the congruence $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.

  Example. What are the solutions of the congruence $3x \equiv 4 \pmod{7}$?

  Solution: We found that -2 is an inverse of 3 modulo 7. Multiply both sides of the congruence by -2, we have $x \equiv -8 \equiv 6 \pmod{7}$.

- **Theorem*** Let $d = \gcd(a, m)$ and $m' = m/d$. The congruence $ax \equiv b \pmod{m}$ has solutions if and only if $d|b$. If $d|b$, then there are exactly d solutions. If x_0 is a solution, then the other solutions are given by $x_0 + m', x_0 + 2m', \ldots, x_0 + (d-1)m'$.

■ **Theorem*** Let $d = \gcd(a, m)$ and $m' = m/d$. The congruence $ax \equiv b \pmod{m}$ has solutions if and only if $d|b$. If $d|b$, then there are exactly d solutions. If x_0 is a solution, then the other solutions are given by $x_0 + m', x_0 + 2m', \ldots, x_0 + (d-1)m'$.

Proof.
1) "only if": If x_0 is a solution, then $ax_0 - b = km$. Thus, $ax_0 - km = b$. Since d divides $ax_0 - km$, we must have $d|b$.

2) "if": Suppose that $d|b$. Let $b = kd$. There exist integers s, t such that $d = as + mt$. Multiply both sides by k. Then $b = ask + mtk$. Let $x_0 = sk$. Then $ax_0 \equiv b \pmod{m}$.

3) "$\# = d$": $ax_0 \equiv b \pmod{m}$ $ax_1 \equiv b \pmod{m}$ imply that $m|a(x_1 - x_0)$ and $m'|a'(x_1 - x_0)$. This implies further that $x_1 = x_0 + km'$, where $k = 0, 1, \ldots, d-1$.

28 - 2

The Chinese Remainder Theorem

- About 1500 years ago, the Chinese mathematician Sun-Tsu asked:

  "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

  今有物不知其数 三三数之剩二 五五数之剩三 七七数之剩二 问物几何

The Chinese Remainder Theorem

- About 1500 years ago, the Chinese mathematician Sun-Tsu asked:

  "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

  今有物不知其数 三三数之剩二 五五数之剩三 七七数之剩二 问物几何

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

- **Theorem** (*The Chinese Remainder Theorem*) Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system

$$x \equiv a_1 \pmod{m_1}$$
$$x \equiv a_2 \pmod{m_2}$$
$$\cdots$$
$$x \equiv a_n \pmod{m_n}$$

  has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k, an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots a_n M_n y_n.$$

It is checked that x is a solution to the n congruences.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k, an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots a_n M_n y_n.$$

  It is checked that x is a solution to the n congruences.

  How to prove the uniqueness of the solution modulo m?

- **Example**

$$x \equiv 2 \quad (\text{mod } 3)$$
$$x \equiv 3 \quad (\text{mod } 5)$$
$$x \equiv 2 \quad (\text{mod } 7)$$

- **Example**

$$x \equiv 2 \quad (\text{mod } 3)$$
$$x \equiv 3 \quad (\text{mod } 5)$$
$$x \equiv 2 \quad (\text{mod } 7)$$

Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

$$35 \cdot 2 \equiv 1 \quad (\text{mod } 3)$$
$$21 \equiv 1 \quad (\text{mod } 5)$$
$$15 \equiv 1 \quad (\text{mod } 7)$$

- **Example**

$$x \equiv 2 \quad (\text{mod } 3)$$
$$x \equiv 3 \quad (\text{mod } 5)$$
$$x \equiv 2 \quad (\text{mod } 7)$$

Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

$$35 \cdot 2 \equiv 1 \quad (\text{mod } 3) \qquad y_1 = 2$$
$$21 \equiv 1 \quad (\text{mod } 5) \qquad y_2 = 1$$
$$15 \equiv 1 \quad (\text{mod } 7) \qquad y_3 = 1$$

- **Example**

$$x \equiv 2 \quad (\text{mod } 3)$$
$$x \equiv 3 \quad (\text{mod } 5)$$
$$x \equiv 2 \quad (\text{mod } 7)$$

Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

$$35 \cdot 2 \equiv 1 \quad (\text{mod } 3) \qquad y_1 = 2$$
$$21 \equiv 1 \quad (\text{mod } 5) \qquad y_2 = 1$$
$$15 \equiv 1 \quad (\text{mod } 7) \qquad y_3 = 1$$

$$x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \quad (\text{mod } 105)$$

- **Example**

$$x \equiv 2 \quad (\text{mod } 3)$$
$$x \equiv 3 \quad (\text{mod } 5)$$
$$x \equiv 2 \quad (\text{mod } 7)$$

三人同行七十稀，五树梅花廿一枝，
七子团圆正月半，除百零五便得知。
-- 程大位 《算法统要》 （1593年）

Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

$$35 \cdot 2 \equiv 1 \quad (\text{mod } 3) \qquad y_1 = 2$$
$$21 \equiv 1 \quad (\text{mod } 5) \qquad y_2 = 1$$
$$15 \equiv 1 \quad (\text{mod } 7) \qquad y_3 = 1$$

$$x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \quad (\text{mod } 105)$$

- We may also solve systems of linear congruences with pairwise relatively prime moduli by *back substitution*.

Back Substitution

- We may also solve systems of linear congruences with pairwise relatively prime moduli by *back substitution*.

Example

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

Back Substitution

- We may also solve systems of linear congruences with pairwise relatively prime moduli by *back substitution*.

Example

$$x \equiv 2 \quad (\text{mod } 3)$$
$$x \equiv 3 \quad (\text{mod } 5)$$
$$x \equiv 2 \quad (\text{mod } 7)$$

$$x \equiv 8 \quad (\text{mod } 15)$$
$$x \equiv 2 \quad (\text{mod } 21)$$

■ Modular arithmetic and congruencies are used in CS:

◇ Pseudorandom number generators

◇ Hash functions

◇ Cryptography

- *Linear congruential method*

  We choose four numbers:

  - ◇ the modulus m
  - ◇ multiplier a
  - ◇ increment c
  - ◇ seed x_0

- *Linear congruential method*

  We choose four numbers:

  - ◇ the modulus m
  - ◇ multiplier a
  - ◇ increment c
  - ◇ seed x_0

  We generate a sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ with $0 \leq x_i < m$ by using the congruence

  $$x_{n+1} = (ax_n + c) \pmod{m}$$

Pseudorandom Number Generators

- *Linear congruential method*

$$x_{n+1} = (ax_n + c) \pmod{m}$$

Pseudorandom Number Generators

- *Linear congruential method*

$$x_{n+1} = (ax_n + c) \pmod{m}$$

Example:

- Assume : m=9,a=7,c=4, $x_0 = 3$

- $x_1 = $ 7*3+4 mod 9=25 mod 9 =7
- $x_2 = 53$ mod 9 = 8
- $x_3 = 60$ mod 9 = 6
- $x_4 = $ 46 mod 9 =1
- $x_5 = 11$ mod 9 =2
- $x_6 = 18$ mod 9 =0

- ....

Hash Functions

- A *hash function* is an algorithm that maps data of arbitrary length to data of a fixed length. The values returned by a hash function are called *hash values* or hash codes.

- A *hash function* is an algorithm that maps data of arbitrary length to data of a fixed length. The values returned by a hash function are called *hash values* or hash codes.

Example:
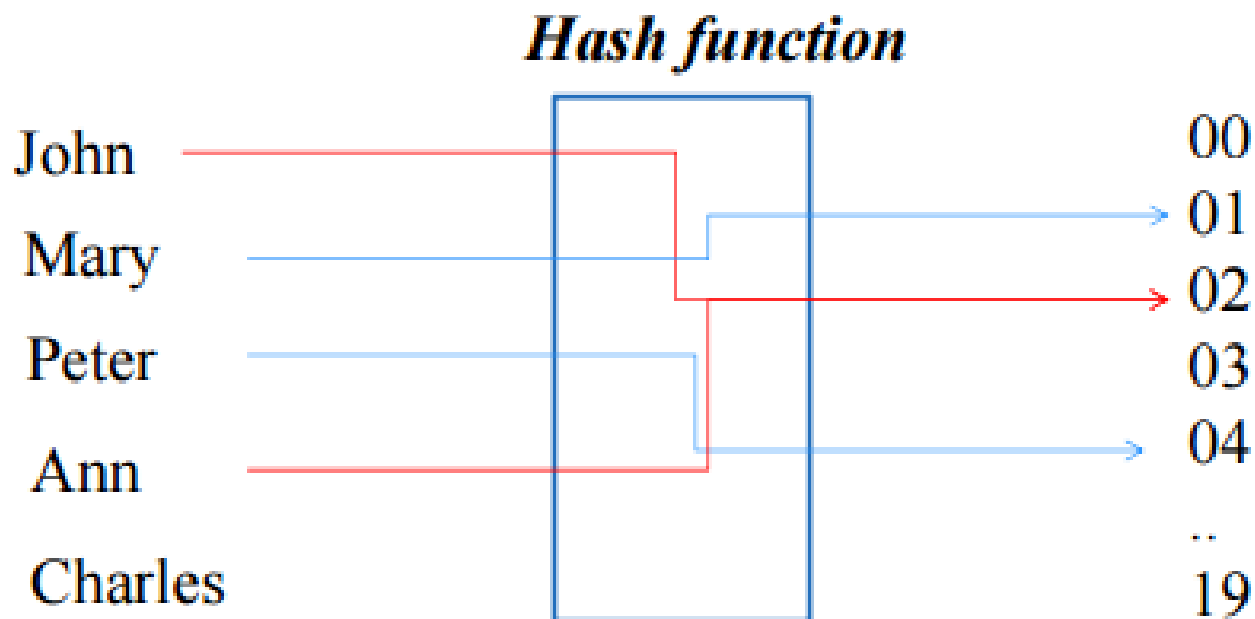
Hash function

John	00
Mary	01
Peter	02
Ann	03
	04
Charles	..
	19

- **Problem**: Given a large collection of records, how can we store and find a record quickly?
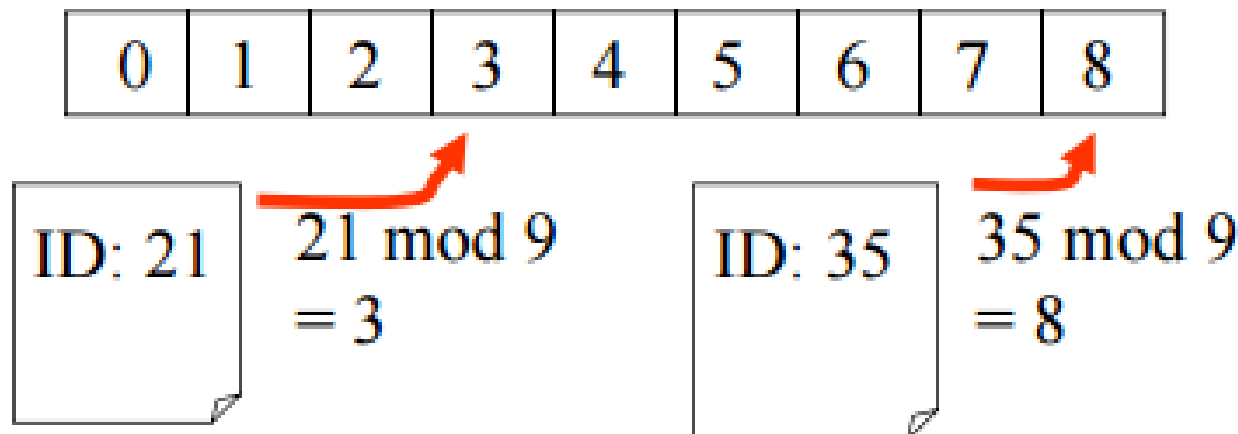
- **Problem**: Given a large collection of records, how can we store and find a record quickly?

  Solution: Use a hash function, calculate the location of the record based on the record's ID.
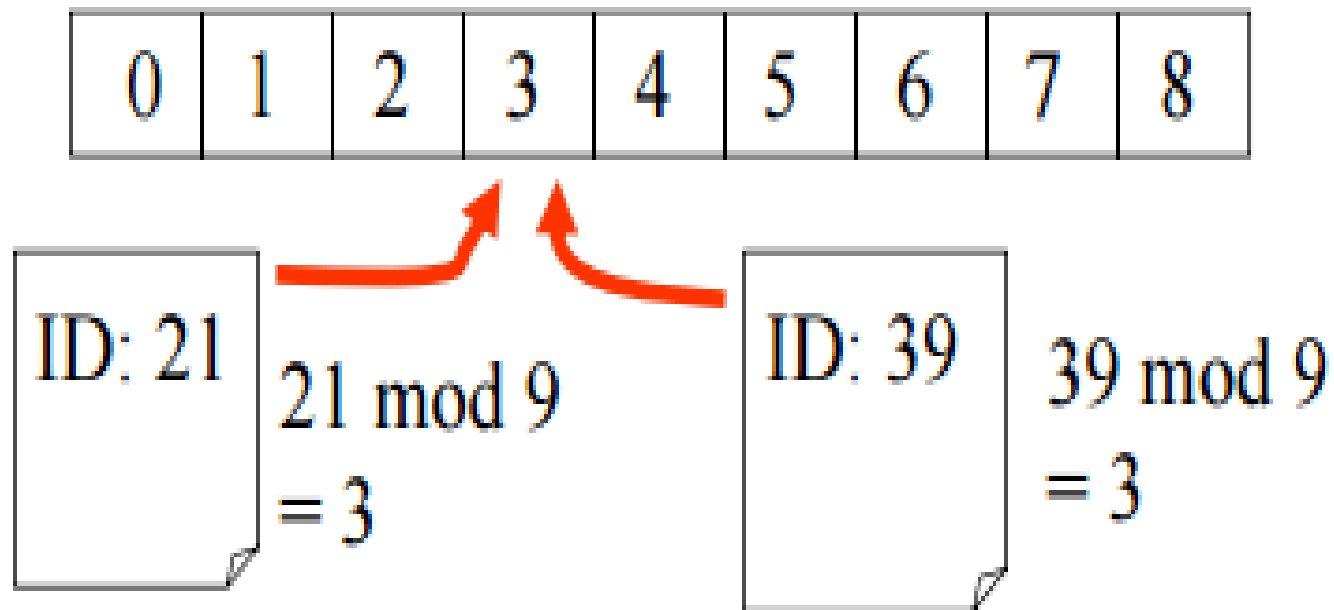
Example: A common hash function is

- $h(k) = k \bmod n,$

where n is the number of available storage locations.

0	1	2	3	4	5	6	7	8

ID: 21   21 mod 9 = 3

ID: 35   35 mod 9 = 8

Hash Functions

- Two records mapped to the same location

- **Solution 1**: move to the next available location

try

$$h_0(k) = k \bmod n$$
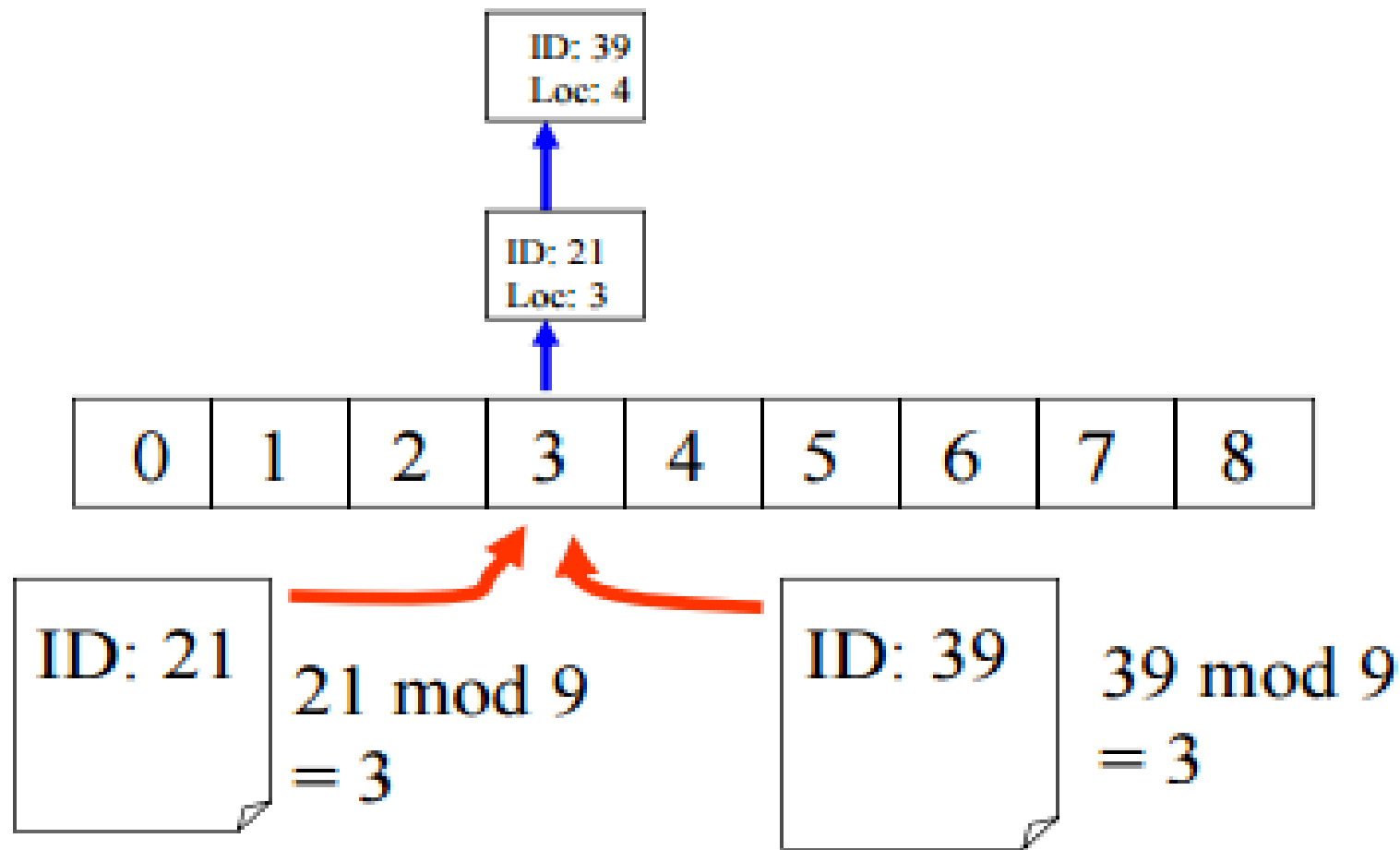$$h_1(k) = (k+1) \bmod n$$
$$...$$
$$h_m(k) = (k+m) \bmod n$$

0	1	2	3	4	5	6	7	8

ID: 21   21 mod 9 = 3

ID: 39   39 mod 9 = 3
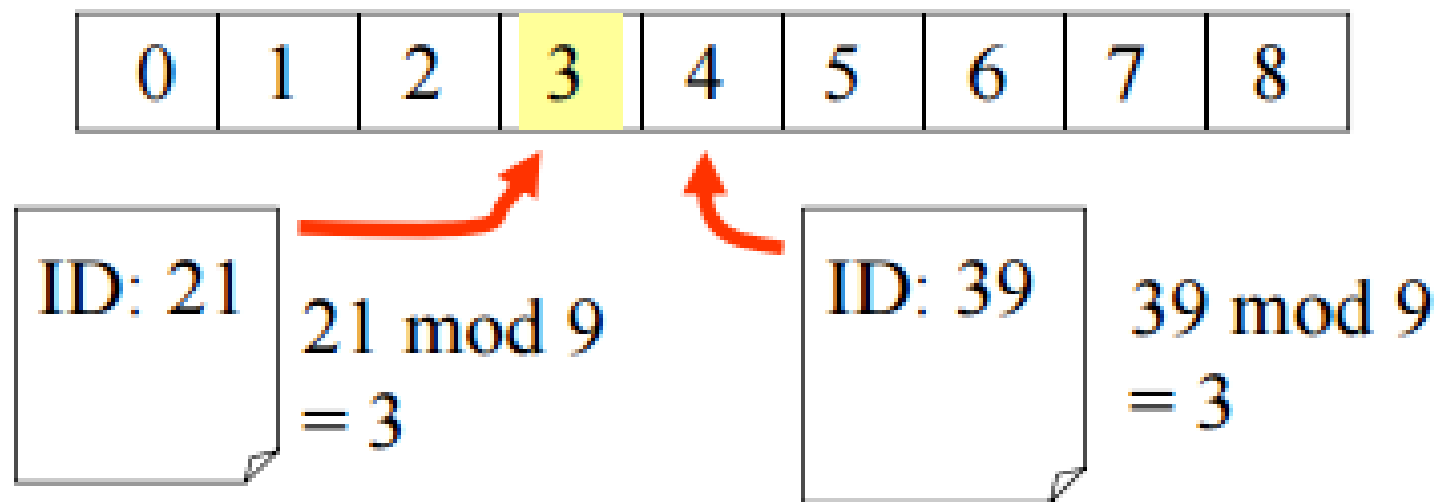
- **Solution 2**: remember the exact location in a secondary structure that is searched sequentially

- cryptography ...