CS215: Discrete Math (H)
2021 Fall Semester   Written Assignment # 5
Due: Dec. 17th, 2021, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b) | a$ and b have no letters in common$\}$

(2) $R_2 = \{(a, b) | a$ and b are not the same length$\}$

(3) $R_3 = \{(a, b) | a$ is longer than $b\}$

Q.2 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Q.3 Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

Q.4 Give an example of a relation R such that its transitive closure R^* satisfies $R^* = R \cup R^2 \cup R^3$, but $R^* \neq R \cup R^2$.

Q.5 Suppose that R_1 and R_2 are both *reflexive* relations on a set A.

(1) Show that $R_1 \oplus R_2$ is *irreflexive*.

(2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.

(3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Q.6 Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$.

(a) Show that R is an equivalence relation.

(b) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation R?

(c) Give an interpretation of the equivalence classes for the equivalence relation R.

Q.7 Show that the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined on $(a, b)\mathbb{R}(c, d)$ if and only if $a + d = b + c$ is an *equivalence* relation.

Q.8 How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?

Q.9 Show that $\{(x, y)|x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Q.10 Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both *equivalence relations*, then T is also an equivalence relation.

Q.11 Let $\sim$ be a relation defined on $\mathbb{N}$ by the rule that $x \sim y$ if $x = 2^k y$ or $y = 2^k x$ for some $k \in \mathbb{N}$. Show that $\sim$ is an equivalence relation.

Q.12 Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

(a) Prove that $\preceq$ is a partial ordering.

(b) Prove or disprove: $\preceq$ is a total ordering.

Q.13 Which of these are posets?

(a) $(\mathbf{R}, =)$

(b) $(\mathbf{R}, <)$

(c) $(\mathbf{R}, \leq)$

(d) $(\mathbf{R}, \neq)$

Q.14 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Q.15 For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3 + 5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Q.16 Answer these questions for the partial order represented by this Hasse diagram.

(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) Find all lower bounds of $\{f, g, h\}$.

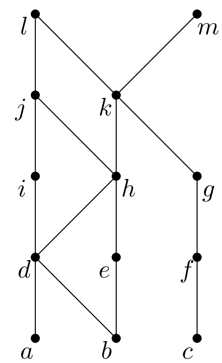

Figure 1: Q.16

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Q.17 We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0, 1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0, 1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.