CS215: Discrete Math
2021 Fall Semester Written Assignment # 6
Due: Dec. 31st, 2021, please submit at the beginning of class

Q.1 Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

Solution:

If uRv, then there is an edge associated with $\{u, v\}$. But $\{u, v\} = \{v, u\}$, so this edge is associated with $\{v, u\}$ and therefore vRu. Thus, by definition, R is a symmetric relation. A simple graph does not allow loops; therefore uRu never holds, and so by definition R is irreflexive.

$\square$

Q.2 The *complementary graph* $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. If the degree sequence of the simple graph G is $4, 3, 3, 2, 2$, what is the degree sequence of $\overline{G}$?

Solution:

The degree sequence can be obtained by subtracting each of these numbers from 4 and reversing the order. We obtain $2, 2, 1, 1, 0$.

$\square$

Q.3 The complementary graph of a simple graph $G = (V, E)$ is the graph $(V, \{(x, y) : \ x, y \in V, x \neq y\} \setminus E)$. A graph is *self-complementary* if it is isomorphic to its complement.

(a) Prove that no simple graph with two or three vertices is self-complementary, without enumerating all isomorphisms of such simple graphs.

(b) Find examples of self-complementary simple graphs with 4 and 5 vertices.

Solution:

(a) Obviously, two isomorphic graphs must have the same number of edges. Thus, for a graph with n vertices to be self-complementary, the total number of possible edges, $\binom{n}{2}$, must be even so that the graph and its complement can have the same number of edges. $\binom{2}{2} = 1$ and $\binom{3}{2} = 3$, so no graph with 2 or 3 vertices can be self-complementary.

(b) see below.

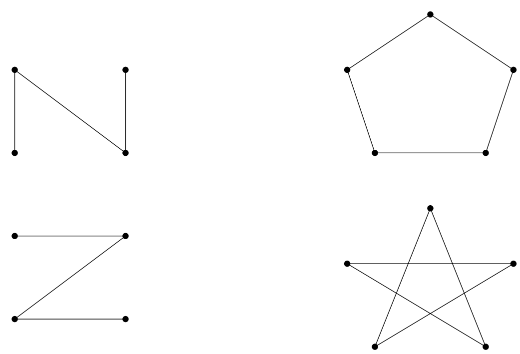

Figure 1: Q.3

$\square$

Q.4 Let G be a *simple* graph with n vertices. Show that if the degree of any vertex of G is $\geq (n-1)/2$, then G must be connected.

Solution: We prove this by contradiction. Suppose that the minimum degree is $(n-1)/2$ and G is not connected. Then G has at least two connected components. In each of the components, the minimum vertex degree is still $(n-1)/2$, and this means that each connected component must have at least $(n-1)/2+1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2}+1) = n+1$ vertices, which is a contradiction.

$\square$

Q.5 Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Solution:

(a) omitted.

(b) The degree of each vertex is $\binom{n-2}{2}$.

$\square$

Q.6 Let G be a simple graph with n vertices and k connected components.

(a) What is the minimum possible number of edges of G?

(b) What is the maximum possible number of edges of G?

Solution:

(a) Let each component i have c_i vertices. If we put a minimum spanning tree to keep it connected, we get $c_i - 1$ edges. So the total number of edges is

$$\sum_{i=1}^{k}(c_i - 1) = \sum_{i=1}^{k} c_i - k = n - k.$$

Thus, it does not matter how the components are selected, we always get this minimum.

(b) Let each component i have c_i vertices. If we put a complete graph for each connected component, we will maximize edges. So the total number of edges is

$$\sum_{i=1}^{k}\binom{c_i}{2} = \sum_{i=1}^{k}\frac{c_i(c_i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{k}(c_i^2 - c_i)\right) = \frac{1}{2}\left(\sum_{i=1}^{k}c_i^2 - n\right).$$

We now need to find some distribution of vertices for each connected component such that we maximize this expression. Consider some sequence $\{c_1, c_2, \ldots, c_k\}$ such that $c_1 \le c_2 \le \cdots \le c_k$. Let's compare the number of edges produced with sequence $\{c_1 - 1, c_2 + 1, c_3, \ldots, c_k\}$. Notice that this sequence is still in increasing order. The additional edges gained from using this new sequence for number of vertices for each of the components:

$$(c_1 - 1)^2 + (c_2 + 1)^2 - c_1^2 - c_2^2 = 2(c_2 - c_1 + 1).$$

So this is a positive increase as long as $c_1 \leq c_2 + 1$. We have assumed that $c_1 \leq c_2$, so this means that decreasing c_1 by 1 and increasing c_2 by 1 results in creation of additional edges. We can apply this argument to any two consecutive c_i and c_j repeatedly, thus resulting in $c_1 = c_2 = \cdots = c_{k-1} = 1$ and $c_k = n - (k-1)$. Therefore, the maximal number of edges that can be created is

$$\frac{1}{2}(k - 1 + (n - (k-1))^2 - n) = \frac{1}{2}\left((n-k)^2 + (n-k)\right).$$

□

Q.7 Suppose that G is a graph on a finite set of n vertices. Prove that if G is disconnected, then its complement is connected.

Solution: Let $\overline{G}$ denote the complement of G. Consider any two vertices u, v in G. If u and v are in different connected components in G, then no edge of G connects them, so there will be an edge $\{u, v\}$ in $\overline{G}$. If u and v are in the same connected component in G, then consider any vertex w in a different connected component (since G is disconnected, there must be at least one other connected component). By our first argument, the edges $\{u, w\}$ and $\{v, w\}$ exist in $\overline{G}$, so u and v are connected by the path (u, w, v). Hence, any two vertices are connected in $\overline{G}$, so the whole graph is connected.

□

Q.8 In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner – there are no ties. The results of such a tournament can then be represented with a *tournament directed graph* where the vertices correspond to players and there is an edge $x \to y$ iff x beats y in their game.

(a) Explain whey a tournament directed graph cannot have cycles of length 1 or 2.

(b) Is the "beats" relation for a tournament graph always/sometimes/never: antisymmetric? reflexive? irreflexive? transitive?

(c) Show that a tournament graph represents a total ordering iff there are no cycles of length 3.

Solution:

(a) There are no self-loops in a tournament graph since no player plays with himself, so no length 1 cycles. Also, it cannot be that x beats y and y beats x for $x \neq y$, since every pair competes exactly once and there are no ties. This means there are no length 2 cycles.

(b) No self-loops implies the relation is irreflexive. It is also antisymmetric since for every pair of distinct players, exactly one game is played and results in a win for one of the players. Some tournament graphs represent transitive relations and others don't.

(c) As observed in (b), the "beats" relation whose graph is a tournament is antisymmetric and irreflexive. Since every pair of players is comparable, the relation is a total ordering iff it is transitive. "Beats" is transitive iff for any players x, y and z, $x \to y$ and $y \to z$ implies that $x \to z$, and consequently that there is no edge $z \to x$. Therefore, "beats" is transitive iff there are no cycles of length 3.

$\square$

Q.9 Let G be a connected simple graph. Show that if an edge in a connected graph is not traversed by any simple cycle, then this edge is a *cut edge*.
Solution: Suppose that the edge u–v is not a cut-edge. We show that it must be traversed by a simple cycle.

Since the edge is not a cut-edge, the graph obtained by removing the edge is connected. So there exists a path from u to v which does not traverse u–v. We proved in class that if there exists a path from u to v, then there exists a simple path from u to v. But this simple path together with u–v is a simple cycle that traverses u–v.

Q.10 Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Let G be a bipartite k-regular graph (the degree of every vertex is k) for $k \geq 2$. Prove that G has no bridge.

Solution: We will prove the result by contradiction. Assume that G has a bridge $e = \{u, v\}$. Let's start with a couple of easy observations. Firstly, note that a bridge affects only the connected component it belongs to. Every connected component of a bipartite k-regular graph is itself bipartite k-regular, so we can assume, without loss of generality, that G is a connected bipartite k-regular graph. Secondly, removal of an edge can split a connected graph into at most two connected component – to see why, observe that if we restore the edge, the graph should be connected, but three or more disjoint components cannot be linked by a single edge.

Now assume that G has classes A and B, where $u \in A$ and $v \in B$. Removal of e splits G into disjoint components G_1 and G_2. Let A' be the set of vertices of A in G_1 and A'' be those in G_2 – both these sets are non-empty. Similarly let B', B'' be the vertices of B in G_1 and G_2, respectively. Observe that the bridge e must be the only edge linking G_1 and G_2, and assume without loss of generality that $u \in A'$ and $v \in B''$.

Now look at G_1, which is a bipartite graph with classes A' and B'. Since e is the only edge linking G_1 and G_2, every other edge of G incident on A' or B' is retained in G_1. So every vertex in A' and B' still has degree k in G_1, except u which has degree $k-1$. Let $a := |A'|$ and $b := |B'|$. Since no edge links two vertices in B' (bipartite property), the number of edges in G_1 is simply kb (every edge is incident to some vertex in B', so we can add up the degrees of the vertices in B'). Similarly, adding up the degrees in A' instead, the number of edges is $k(a-1)+k-1$. Equating the two formulae, we have $k(a-1)+k-1 = kb$, which implies that $k(a-b) = 1$. But this implies that $k = 1$, contradicting the given condition that $k \geq 2$. Hence, the bridge cannot exist.

$\square$

Q.11 Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimal* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(i) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(ii) $dist(u, v) = dist(v, u)$.

(iii) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Solution:

(i) By definition, the $dist(u, v)$ is the minimal length of a path from u to v, and the length if the number of edges in the path. Thus, $dist(u, v)$ cannot be negative. Furthermore, $dist(u, v) = 0$ if and only if there is a path of length 0 from u to v, which means that $u = v$.

(ii) Suppose that P is path from u to v of the minimal length. We reverse all the edges in the path P, and will get a path P' from v to u. Note that P' must be the minimal path from v to u. Otherwise, we reverse P' and will get a shorter path from u to v, which is a contradiction. Thus, $dist(u, v) = dist(v, u)$.

(iii) By definition, $dist(u, v) = \#$ of edges in the path P, where P is the path from u to v with the minimum length. Suppose that P_1 and P_2 are the paths of minimal length from u to w, and from w to v, respectively. Then $u\tilde{w}\tilde{v}$ is a new path P' from u to v. By the minimality of P, we must have $dist(u, v) \leq dist(u, w) + dist(w, v)$.

$\square$

Q.12 Given a connected graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices u, v in G is defined as the length of a shortest path between u and v. The *diameter* $\mathrm{diam}(G)$ of G is defined as the greatest distance among all pairs of vertices in G. That is, $\max_{u,v \in V} d_G(u, v)$. The *eccentricity* $\mathrm{ecc}(v)$ of a vertex v of G is defined as $\max_{u \in V} d_G(u, v)$. Finally, the *radius* $\mathrm{rad}(G)$ of G is defined as the minimal eccentricity of a vertex in G, namely $\min_{v \in V} \mathrm{ecc}(v)$. Prove the following.

(a) $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$.

(b) For every positive integer n, there are connected graphs G_1 and G_2 with $\mathrm{diam}(G_1) = \mathrm{rad}(G_1) = n$ and $\mathrm{diam}(G_2) = 2\mathrm{rad}(G_2) = 2n$.

Solution:

(a) As $\mathrm{rad}(G) = \min_{v \in V}[\max_{u \in V} d_G(u, v)]$, obviously $\mathrm{rad}(G) \leq \mathrm{diam}(G)$.

Now suppose that $\mathrm{diam}(G)$ goes from vertices v_1 to v_2, where $v_1, v_2 \in V$. Recall that $\mathrm{rad}(G) = \min_{v \in V} \mathrm{ecc}(v)$. Let the chosen v for minimal eccentricity be v^*. Note that $\mathrm{diam}(G) \leq d_G(v^*, v_1) + d_G(v^*, v_2)$. Since $\mathrm{diam}(G)$ is the shortest path from v_1 to v_2, any other path from v_1 to v_2 is either as long or longer. Also note that $d_G(v^*, v_1) + d_G(v^*, v_2) \leq \mathrm{rad}(G) + \mathrm{rad}(G) = 2\mathrm{rad}(G)$ since $\mathrm{rad}(G)$ is the maximum distance of any other vertex from v^* in G. Hence, we have

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G).$$

(b) Consider a cycle with $2n$ or $2n+1$ vertices. This will always have $\mathrm{diam}(G_1) = \mathrm{rad}(G_1) = n$. For $\mathrm{diam}(G_2) = 2\mathrm{rad}(G_2) = 2n$, consider a line graph with $2n+1$ vertices.

□

Q.13 Show that isomorphism of simple graphs is an equivalence relation.

Solution:

G is isomorphic to itself by the identity function, so isomorphism is reflexive. Suppose that G is isomorphic to H. Then there exists a one-to-one correspondence f from G to H that preserves adjacency and nonadjacency. It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and nonadjacency. Hence, isomorphism is symmetric. If G is isomorphic to H and H is isomorphic to K, then there are one-to-one correspondences f and g from G to H and from H to K that preserve adjacency and nonadjacency. It follows that $g \circ f$ is a one-to-one correspondence from G to K that preserves adjacency and nonadjacency. Hence, isomorphism is transitive.

□

Q.14 Suppose that G_1 and H_1 are isomorphic and that G_1 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.

Solution: The isomorphism need not hold. For the simplest counterexample, let G_1, G_2 and H_1 each be the graph consisting of the single vertex v, and let H_2 be the graph consisting of the single vertex w. Then of course G_1 and H_1 are isomorphic, as are G_2 and H_2. But $G_1 \cup G_2$ is a graph with one vertex, and $H_1 \cup H_2$ is a graph with two vertices.

□

Q.15 Given a graph G, its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding edges of G share a common endpoint. Prove that if G is regular (all vertices have the same degree) and connected, then $L(G)$ has an Euler circuit.
Solution: If the degree of regular graph G is d, then every edge of G has $2(d-1)$ neighbours in $L(G)$. Since this is even, $L(G)$ has an Euler circuit.

□

Q.16 Show that if G is simple graph with at least 11 vertices, then either G or its complement graph $\overline{G}$, the complement of G, is nonplanar.
Solution: If G is planar, then because $e \leq 3v - 6$, G has at most 27 edges. (If G is not connected it has even fewer edges.) Similarly, $\overline{G}$ has at most 27 edges. However, the union of G and $\overline{G}$ is K_{11}, which has 55 edges, and $55 > 27 + 27$.

□

Q.17 Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.
Solution:

As in the argument in the proof of Corollary 1, we have $2e \geq 5r$ and $r = e - v + 2$. Thus $e - v + 2 \leq 2e/5$, which implies that $e \leq (5/3)v - (10/3)$.

□

Q.18 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

(1) K_6

(2) $K_{4,5}$

(3) Q_3

(4) C_6

Solution:

(1) K_6: The diameter is clearly 1, since the maximum distance between two vertices is 1; the radius is also 1, with any vertex serving as the center.

(2) $K_{4,5}$: The diameter is clearly 2, since vertices in the same part are not adjacent, but no pair of vertices are at a distance greater than 2. Similarly, the radius is 2 with any vertex serving as the center.

(3) Q_3: Vertices at diagonally opposite corners of the cube are a distance of 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3.

(4) C_6: Vertices at opposite corners of the hexagon are a distance 3 from each other, and this it the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3. (Note that despite the appearances in this exercise, it is not always the case that the radius equals the diameter; for example, $K_{1,n}$ has radius 1 and diameter 2.)

$\square$

Q.19 Let G be a graph in which all vertices have degree at least d. Prove that G contains a path of length d.

Solution: Let the longest path have length p. Consider the last vertex in the path. It has degree at least d. Therefore, they must all be in the path otherwise we can make a longer path by adding any of those. Hence, the longest path must include at least $d + 1$ vertices, meaning the longest path must be at least length d, so a path of length d can be found by taking a subpath of the longest path.

□

Q.20 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly* **two** vertices of degree i for each $i = 1, 2, \ldots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Solution:

We draw a bipartite graph in the following way:

The vertex set contains two sets of vertices, V_1 and V_2, with each containing n vertices. For the ith vertex in V_1, it is connected to the first i vertices in the set V_2. In this way, the ith vertex in V_1 has degree i, and the ith vertex in V_2 has degree $n-i+1$, since the previous $i-1$ vertices in $V-1$ are not connected to the ith vertex in V_2 by the adjacency rule. The constructed graph is connected in an obvious way: $(1, 1, 2, 2, 3, 3, 4, 4, \ldots, n, n)$, where the first i denotes the ith vertex in V_1 and the second i denotes the ith vertex in V_2 (see the following figure).

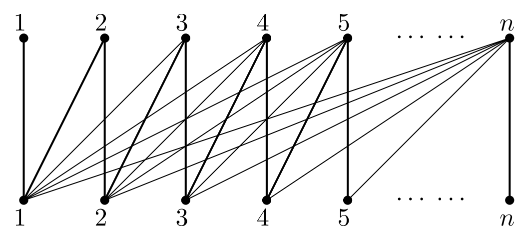

Figure 2: Q.20

□

Q.21 Consider the two graphs G and H. Answer the following three questions, and explain your answers.

(1) Which of the two graphs is/are *bipartite*?

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

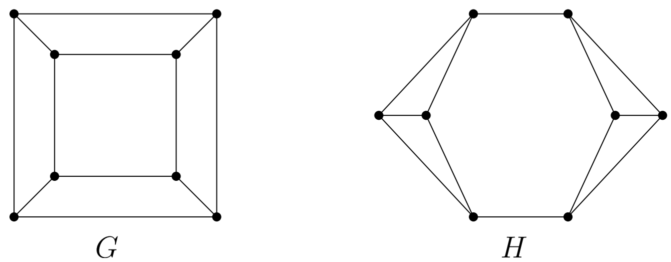

Figure 3: Q.21

Solution:

(1) The graph G is bipartite, and the graph H is not. It is checked below that the graph G can be 2-colored, which H cannot.

(2) The two graphs are not isomorphic, since there exists a circuit of length 3 in H, which does not exist in G.

(3) Since both of the two graphs have only degree-3 vertices, neither of these two graphs has an Euler circuit.

□

Q.22 There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Solution: We use vertices to denote the 17 students and edges to denote the communication among these students. In addition, we use 3 different colors to color the edges to denote the 3 problems they are discussing. For one fixed student A, A communicates with the other 16 students. By the Pigeonhole Principle, at least 6 edges are of the same color, w.l.o.g., we assume that the edges AB, AC, AD, AE, AF, AG are all of color red.

If among the six students B, C, D, E, F, G there is one edge, e.g., BC whose color is also red, then all the three edges of the triangle ABC are red.

If among the six students B, C, D, E, F, G there is no red edge, we consider the edges BC, BD, BE, BF, BG. There are only two colors for these five edges, so at least there are three of these five edges of the same color. W.l.o.g., assume that the three edges BC, BD, BE are of the same color, yellow. We consider the triangle CDE. If there is one yellow edge of the triangle CDE, say, CD is yellow, then the triangle BCD is a triangle with three edges all yellow. If the triangle CDE does not have yellow edge, which means all edges of CDE are blue, we again have a triangle with three edges of the same color.

□

Q.23 Prove that $G = (V, E)$ is a tree if and only if $|V| = |E| + 1$ and G has no cycles.
Solution: We will use induction on $|V|$. For a tree with a single vertex, the claim holds since $|E| + 1 = 0 + 1 = 1$. Now suppose that the claim holds for all n-vertex tress and consider an $(n + 1)$-vertex tree. Let v be a leaf of the tree. Deleting v and its incident edge gives a smaller tree for which the equation $|V| = |E| + 1$ holds by induction. If we add back the vertex v and its incident edge, then the equation still holds because the number of vertices and number of edges both increased by 1. Thus, the claim holds for the $n + 1$-vertex tree and, by induction, for all trees.

□

Q.24 The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?
Solution:

The number of vertices in the tree T_n satisfies the recurrence relation $v_n = v_{n-1} + v_{n-2} + 1$ (the "+1" is for the root), with $v_1 = v_2 = 1$. Thus the sequence begins $1, 1, 3, 5, 9, 15, 25, \ldots$. It is easy to prove by induction that $v_n = 2f_n - 1$, where f_n is the n-th Fibonacci number. The number of leaves satisfies the recurrence relation $l_n = l_{n-1} + l_{n-2}$, with $l_1 = l_2 = 1$, so $l_n = f_n$. Since $i_n = v_n - l_n$, we have $i_n = f_n - 1$. Finally, it is clear that the height of the tree T_n is one more than the height of the tree T_{n-1} for $n \geq 3$, with the height of T_2 being 0. Therefore the height of T_n is $n - 2$ for all $n \geq 2$ (and of course the height of T_1 is 0).

□

Q.25

What is the value of each of these postfix expressions?

(a) $5\ 2\ 1\ -\ -\ 3\ 1\ 4\ +\ +\ *$

(b) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

Solution:

We exhibit the answers by showing with parentheses the operation that is applied next, working from left to right (it always involves the first occurrence of an operator symbol).

(a) $5\ (2\ 1\ -)\ -3\ 1\ 4\ +\ +* = (5\ 1\ -)3\ 1\ 4\ +\ +* = 4\ 3\ (1\ 4\ +)\ +\ * = 4\ (3\ 5\ +)* = (4\ 8\ *) = 32$

(b) $(9\ 3\ /)\ 5 + 7\ 2\ -\ * = (3\ 5\ +)\ 7\ 2\ -\ * = 8(\ 7\ 2\ -)* = (8\ 5\ *) = 40$

□

Q.26

Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

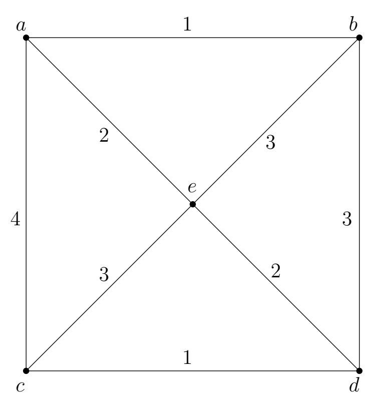

Figure 4: Q.26

Solution:

We start with the minimum weight edge $\{a, b\}$. The least weight edge incident to the tree constructed so far is edge $\{a, e\}$, with weight 2, so we add it to the tree. Next we add edge $\{d, e\}$, and then edge $\{c, d\}$. This completes the tree, whose total weight is 6.

$\square$

Q.27

Use Kruskal's algorithm to find a minimum spanning tree for the weighted graph in Q.26.

Solution:

With Kruskal's algorithm, we add at each step the shortest edge and will not complete a simple circuit. Thus we pick edge $\{a, b\}$ first, and then edge $\{c, d\}$ (alphabetical order breaks ties), followed by $\{a, e\}$ and $\{d, e\}$. The total weight is 6.

$\square$