南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Algorithm Design and Analysis (H)

CS216

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About me

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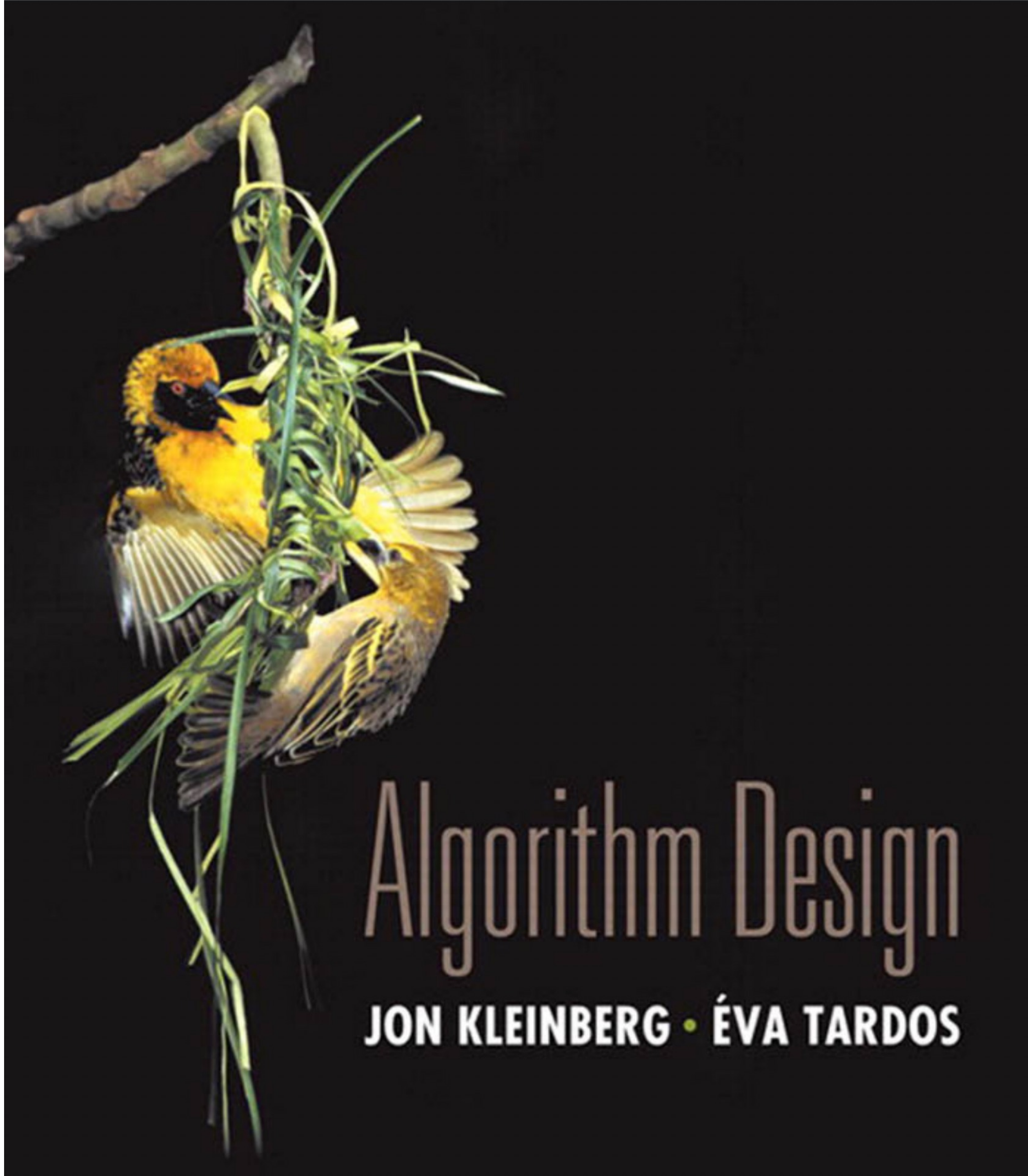
About the Course

Grade Component

- Attendance: 10%
- Assignment: 20%
- Lab: 30% (N-1)
- Exam: 40%

Resources

- **Textbook**
  - **Algorithm Design by Jon Kleinberg and Éva Tardos**
- **Blackboard:**



Introduction

Some Representative Problems

Algorithms

- Example:
  - Postgraduate admission
  - College admission: National College Entrance Exam of China/Gaokao/高考

Algorithms

- Algorithms can help to distribute resources efficiently
  - Qin Dynasty -> Tang Dynasty -> Current

1. Stable Matching

Stable matching

- David Gale and Lloyd Shapley asked a question in 1962
  - Could one design a college admissions process, or job recruiting process, that was self-enforcing?

Formulating the problem

- A "bare-bones" version:
  - n applicants
  - n companies
- Or marriage:
  - n men
  - n women

Some definitions

- Matching
  - M={m1, m2, …, mn}, W={w1, w2, …, wn}
  - All possible pairs: MxW
- Perfect matching: everyone is matched monogamously.
  - Each man gets exactly one woman.
  - Each woman gets exactly one man.

Some definitions

- Stability: no incentive for some pair of participants to undermine assignment by joint action.
  - In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
  - Unstable pair m-w could each improve by eloping.
- Stable matching: perfect matching with no unstable pairs.

Some definitions

- Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.
- Example: n=2

Questions

- Do stable matchings always exist?

Propose-And-Reject Algorithm

- Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

- Observation 1. Men propose to women in decreasing order of preference.
- Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."
- Claim. Algorithm terminates after at most n^2 iterations of while loop.
- Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ■

Proof of Correctness: Perfection

- Claim. All men and women get matched.
- Pf. (by contradiction)
  - Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
  - Then some woman, say Amy, is not matched upon termination.
  - By Observation 2, Amy was never proposed to.
  - But, Zeus proposes to everyone, since he ends up unmatched. ■

Proof of Correctness: Stability

- Claim. No unstable pairs.
- Pf. (by contradiction)
  - Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.
  - Case 1: Z never proposed to B.
    $\Rightarrow$ Z prefers B to A.
    $\Rightarrow$ A-Z is stable.

    men propose in decreasing order of preference

  - Case 2: Z proposed to B.
    $\Rightarrow$ B rejected Z (right away or later)
    $\Rightarrow$ B prefers Y to Z.
    $\Rightarrow$ A-Z is stable.

    women only trade up

  - In either case A-Z is stable, a contradiction. ■

S*
Amy-Yancey
Bertha-Zeus
. . .

Summary

- Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

- Q. How to implement GS algorithm efficiently?

- Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

- Efficient implementation. We describe $O(n^2)$ time implementation.

- Representing men and women.
  - Assume men are named 1, …, n.
  - Assume women are named 1', …, n'.

- Engagements.
  - Maintain a list of free men, e.g., in a queue.
  - Maintain two arrays `wife[m]`, and `husband[w]`.
    - ✓ set entry to `0` if unmatched
    - ✓ if m matched to w then `wife[m]=w` and `husband[w]=m`

- Men proposing.
  - For each man, maintain a list of women, ordered by preference.
  - Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

- Women rejecting/accepting.
  - Does woman `w` prefer man `m` to man `m'`?
  - For each woman, create inverse of preference list of men.
  - Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

- An instance with two stable matchings.
  - A-X, B-Y, C-Z.
  - A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

- Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

- Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

- Man-optimal assignment. Each man receives best valid partner.

- Claim. All executions of GS yield man-optimal assignment, which is a stable matching!
  - No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
  - Simultaneously best for each and every man.

Man Optimality

- Claim. GS matching S* is man-optimal.
- Pf. (by contradiction)
  - ➢ Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference $\Rightarrow$ some man is rejected by valid partner.
  - ➢ Let Y be first such man, and let A be first valid woman that rejects him.
  - ➢ Let S be a stable matching where A and Y are matched.
  - ➢ When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
  - ➢ Let B be Z's partner in S.
  - ➢ Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
  - ➢ But A prefers Z to Y.
  - ➢ Thus A-Z is unstable in S. ■

since this is first rejection by a valid partner

S
Amy-Yancey
Bertha-Zeus
. . .

Stable Matching Summary

- **Stable matching problem**. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with
each other than assigned partner

- **Gale-Shapley algorithm**. Finds a stable matching in $O(n^2)$ time.

- **Man-optimality**. In version of GS where men propose, each man receives best valid partner.
-

w is a valid partner of m if there exist some
stable matching where m and w are paired

- **Q.** Does man-optimality come at the expense of the women?

Woman Pessimality

- Woman-pessimal assignment. Each woman receives worst valid partner.

- Claim. GS finds woman-pessimal stable matching S*.

- Pf.
  - Suppose A-Z matched in S*, but Z is not worst valid partner for A.
  - There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
  - Let B be Z's partner in S.
  - Z prefers A to B. ← man-optimality
  - Thus, A-Z is an unstable in S. ■

S
Amy-Yancey
Bertha-Zeus
. . .

Extensions: Matching Residents to Hospitals

- Ex: Men ≈ hospitals, Women ≈ med school residents.

- Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

- Variant 2. Unequal number of men and women.

- Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

- Def. Matching S unstable if there is a hospital h and resident r such that:
  - h and r are acceptable to each other; and
  - either r is unmatched, or r prefers h to her assigned hospital; and
  - either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Application: Matching Residents to Hospitals

- NRMP. (National Resident Matching Program)
  - Original use just after WWII. predates computer usage
  - Ides of March, 23,000+ residents.

- Rural hospital dilemma.
  - Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
  - Rural hospitals were under-subscribed in NRMP matching.
  - How can we find stable matching that benefits "rural hospitals"?

- Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Lessons Learned

- Powerful ideas learned in course.
  - Isolate underlying structure of problem.
  - Create useful and efficient algorithms.

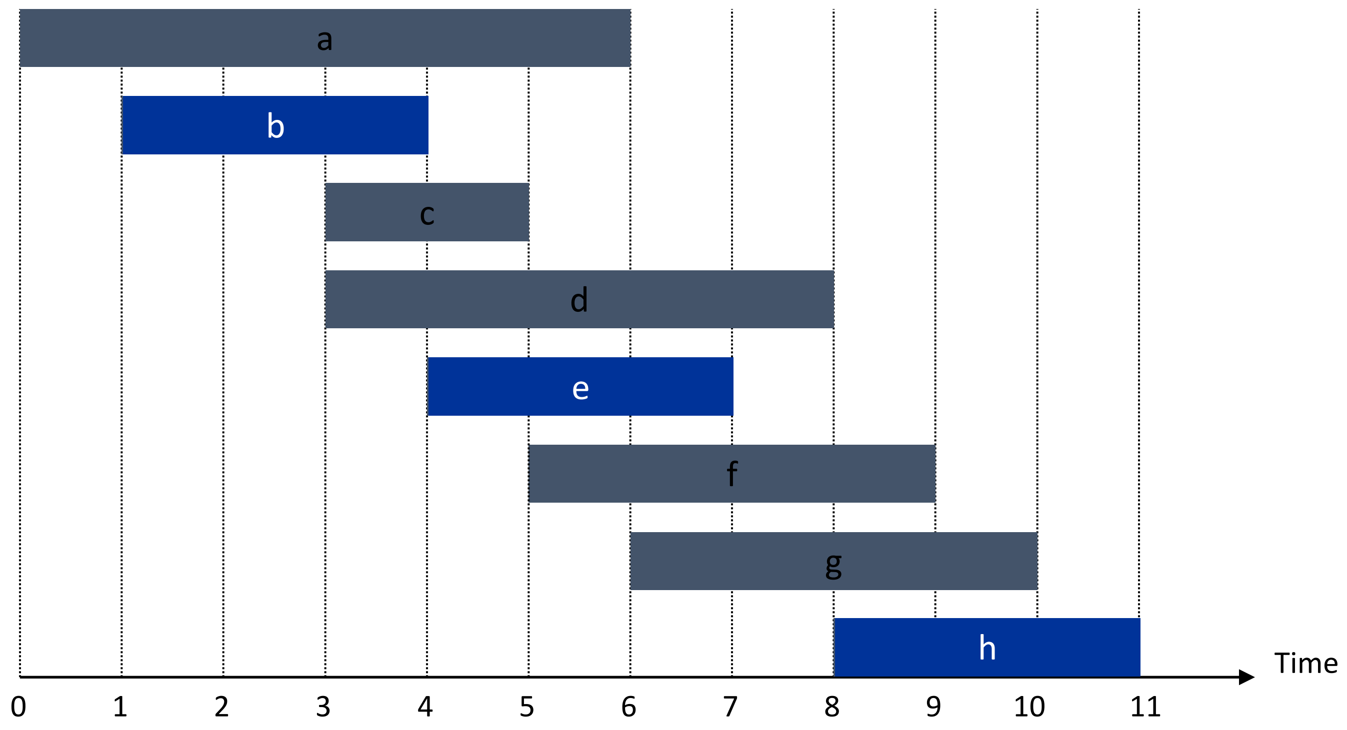
- Potentially deep social ramifications. [legal disclaimer]

2. Five Representative Problems

Interval Scheduling

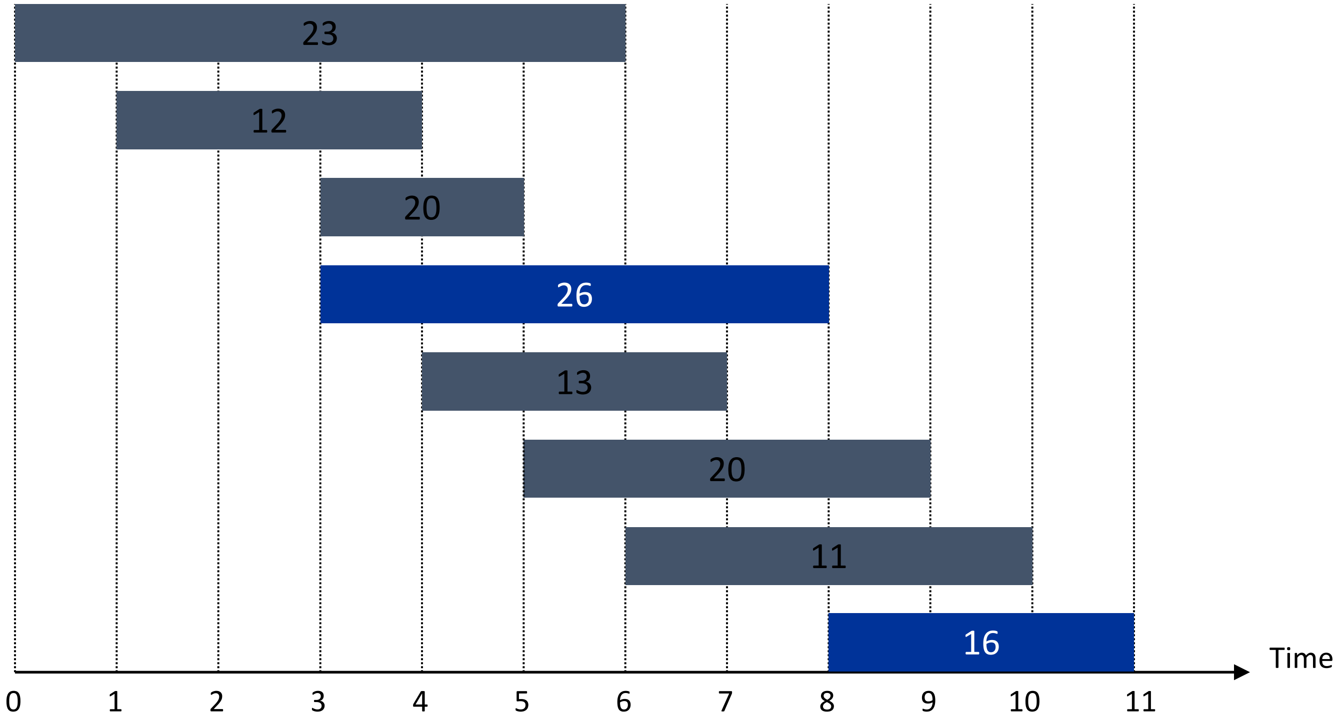
- Input. Set of jobs with start times and finish times.
- Goal. Find maximum cardinality subset of mutually compatible jobs.

jobs don't overlap



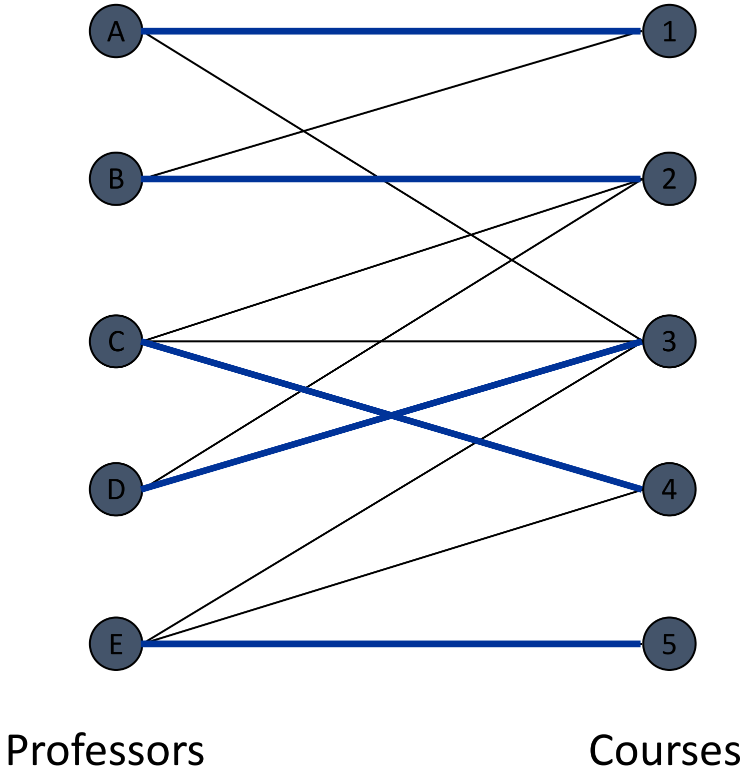
Weighted Interval Scheduling

- Input. Set of jobs with start times, finish times, and weights.
- Goal. Find maximum weight subset of mutually compatible jobs.



Bipartite Matching

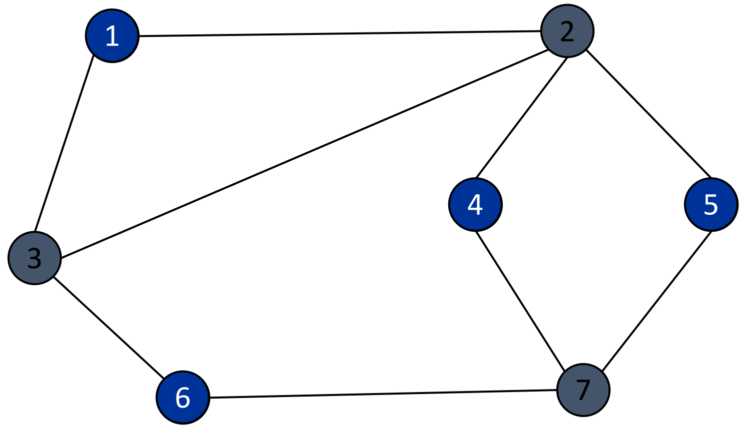
- Input. Bipartite graph.
- Goal. Find maximum cardinality matching.



Independent Set

- Input. Graph.
- Goal. Find maximum cardinality independent set.

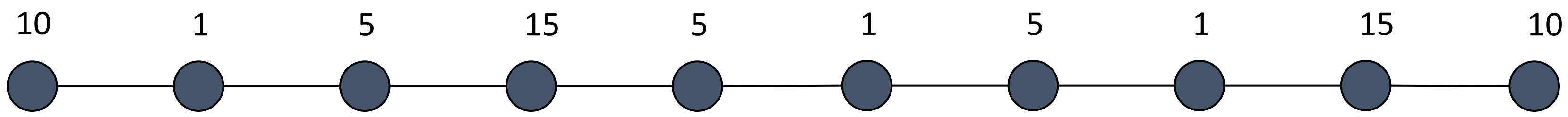
subset of nodes such that no two joined by an edge



Competitive Facility Location

- Input. Graph with weight on each node.
- Game. Two competing players alternate in selecting nodes.
  Not allowed to select a node if any of its neighbors have been selected.

- Goal. Select a maximum weight subset of nodes.



Second player can guarantee 20, but not 25.

Five Representative Problems

- Variations on a theme: independent set.

- Interval scheduling: n log n greedy algorithm.
- Weighted interval scheduling: n log n dynamic programming algorithm.
- Bipartite matching: n^k max-flow based algorithm.
- Independent set: NP-complete.
- Competitive facility location: PSPACE-complete.