CS215: Discrete Math (H)
2021 Fall Semester Written Assignment # 4
Due: Dec. 8th, 2021, please submit at the beginning of class

Q.1 Prove by induction that, for any integer $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\left(1-\frac{1}{4^2}\right)\cdot\cdots\cdot\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Solution:

We begin by showing the **base case** $n=2$ is satisfied. In fact, we have $(1-\frac{1}{2^2}=\frac{3}{4}=\frac{2+1}{2\cdot 2}$. Suppose that the property holds for $n=k$, then we show it must hold for $n=k+1$. We have

$$\begin{aligned}
\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) &= \left(1-\frac{1}{(k+1)^2}\right)\cdot\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right) \\
&= \left(1-\frac{1}{(k+1)^2}\right)\cdot\frac{k+1}{2k} \qquad \text{by i.h} \\
&= \frac{(k+1)^2-1}{(k+1)^2}\cdot\frac{k+1}{2k} \\
&= \frac{(k+1)^2-1}{2k(k+1)} \\
&= \frac{k^2+2k}{2k(k+1)} \\
&= \frac{k+2}{2(k+1)}.
\end{aligned}$$

By mathematical induction, we have proved the conclusion.

□

Q.2 Use induction to prove that 4 divides $2n^2+6n$ whenever n is a positive integer.

Solution:

Base case: $n=1$, $2n^2+6n=8$, which is divisible by 4.

Inductive hypothesis: Suppose that 4 divides $2n^2+6n$.

Inductive step: We now prove that 4 divides $2(n+1)^2+6(n+1)$. We have

$$\begin{aligned} 2(n+1)^2+6(n+1) &= 2n^2+4n+2+6n+6 \\ &= (2n^2+6n)+4(n+2). \end{aligned}$$

Since $2n^2+6n$ is divisible by 4 by i.h., and also $4(n+2)$ is divisible by 4, it then follows that $2(n+1)^2+6(n+1)$ is divisible by 4.

Conclusion: By mathematical induction, we prove the result.

$\square$

Q.3 Let $x \in \mathbb{R}$ and $x \neq 1$. Using mathematical induction, prove that for all integers $n \geq 0$,

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}.$$

Solution:

Base case: $n=0$, we have $\sum_{i=0}^{0} x^i = 1 = \frac{x^{0+1}-1}{x-1} = 1$.

Inductive hypothesis: Suppose that $\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$.

Inductive step: For $n=k+1$, we have

$$\begin{aligned} \sum_{i=0}^{k+1} x^i &= x^{k+1} + \sum_{i=0}^{k} x^i \\ &= x^{k+1} + \frac{x^{k+1}-1}{x-1} \quad \text{by i.h.} \\ &= \frac{x^{k+1}(x-1)}{x-1} + \frac{x^{k+1}-1}{x-1} \\ &= \frac{x^{k+2}-1}{x-1}. \end{aligned}$$

Conclusion: By mathematical induction, we prove the result.

$\square$

Q.4 Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$\begin{aligned} &(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \\ &= (A_1 \cap A_2 \cap \cdots \cap A_n) - B. \end{aligned}$$

Solution:

If $n = 1$, there is nothing to prove, and then $n = 2$, this says that $(A_1 \cap \bar{B}) \cap (A_2 \cap \bar{B}) = (A_1 \cap A_2) \cap \bar{B})$, which is the distributive law. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B;$$

we must show that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = (A_1 \cap A_2 \cap \cdots A_n \cap A_{n+1}) - B.$$

We have

$$\begin{aligned}
&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\
&\quad = ((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)) \cap (A_{n+1} - B) \\
&\quad = ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1} - B) \\
&\quad = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B.
\end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

$\square$

Q.5 Prove that if $h > -1$, then $1 + nh \leq (1 + h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

Solution:

Let $P(n)$ be "$1 + nh \leq (1 + h)^n$, $h > -1$."

Basic step: $P(0)$ is true because $1 + 0 \cdot h = 1 \leq 1 = (1 + h)^0$.

Inductive step: Assume that $1 + kh \leq (1 + h)^k$. *Then because* $(1 + h) > 0$, $(1+h)^{k+1} = (1+h)(1+h)^k \geq (1+h)(1+kh) = 1+(k+1)h+kh^2 \geq 1+(k+1)h$.

Inductive conclusion: By mathematical induction, we have $P(n)$ is true for all nonnegative integers n.

$\square$

Q.6 Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

(a) Show statements $P(18)$, $P(19)$, $P(20)$ and $P(21)$ are true, completing the basis step of the proof.

(b) What is the inductive hypothesis of the proof?

(c) What do you need to prove in the inductive step?

(d) Complete the inductive step for $k \geq 21$.

(e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Solution:

(a) $P(18)$ is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps. $P(19)$ is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp. $P(20)$ is true, because we can form 20 cents of postage with five 4-cent stamps. $P(21)$ is true, because we can form 20 cents of postage with three 7-cent stamps.

(b) The inductive hypothesis is the statement that using just 4-cent and 7-cent stamps we can form j cents postage for all j with $18 \leq j \leq k$, where we assume that $k \geq 21$.

(c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $k + 1$ cents postage using just 4-cent and 7-cent stamps.

(d) We want to form $k + 1$ cents of postage. Since $k \geq 21$, we know that $P(k - 3)$ is true, that is, we can form $k - 3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k + 1$ cents of postage, as desired.

(e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

□

Q.7 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Solution: The strong induction principle clearly implies ordinary induction, for if one has shown that $P(k) \to P(k+1)$, then it automatically follows that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$; in other words, strong induction can always be invoked whenever ordinary induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose that $Q(k)$ is true, i.e., $P(1) \wedge \cdots \wedge P(k)$ is true. By the proof of $\forall n P(n)$ by strong induction, it follows that $P(k+1)$ is true. But $Q(k) \wedge P(k+1)$ is just $Q(k+1)$. Thus, we have proved $\forall n Q(n)$ by ordinary induction.

$\square$

Q.8 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

(a) Find $f(16)$

(b) Find a big-O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Solution:

(a) $f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 2(2 \cdot 1 + 2) + 4 = 12$.

(b) Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + m$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of g we have $g(m) = 2g(m/2) + m$. Theorem 2 (with $a = 2, b = 2, c = 1$, and $d = 1$ now tells us that $g(m)$ is $O(m \log m)$. Since $m = \log n$, this means that our function is $O(\log n \cdot \log \log n)$.

$\square$

Q.9 The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in *closed form* (in terms of b if necessary) and should *not* use summation.

(b) Find a solution for $T(n)$. Your solution should be in *closed form* (in terms of a and c if necessary) and should *not* use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Solution:

(a) By repeated substitution, we get

$$\begin{aligned}
S(n) &= 9S(n/2) + n^2 \\
&= 9\left[9S\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 \\
&= 9^2 S\left(\frac{n}{2^2}\right) + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^2\left[9S\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= 9^3 S\left(\frac{n}{2^3}\right) + \left(\frac{9}{4}\right)^2 n^2 + \left(\frac{9}{4}\right) n^2 + n^2 \\
&= \cdots \\
&= 9^{\log_2 n} S(1) + n^2 \sum_{i=0}^{\log_2 n - 1} \left(\frac{9}{4}\right)^i \\
&= b n^{\log_2 9} + \frac{4}{5} n^{\log_2 9} - \frac{4}{5} n^2 \\
&= \left(b + \frac{4}{5}\right) n^{\log_2 9} - \frac{4}{5} n^2,
\end{aligned}$$

where we are using the fact that

$$\left(\frac{9}{4}\right)^{\log_2 n} = \frac{9^{\log_2 n}}{n^2} = \frac{n^{\log_2 9}}{n^2}.$$

(b) Similar to (a), we get

$$\begin{aligned}
T(n) &= aT\left(\frac{n}{4}\right) + n^2 \\
&= a\left[aT\left(\frac{n}{4^2}\right) + \left(\frac{n}{4}\right)^2\right] + n^2 \\
&= a^2T\left(\frac{n}{4^2}\right) + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^2\left[aT\left(\frac{n}{4^3}\right) + \left(\frac{n}{4^2}\right)^2\right] + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= a^3T\left(\frac{n}{4^3}\right) + \left(\frac{a}{16}\right)^2 n^2 + \left(\frac{a}{16}\right)n^2 + n^2 \\
&= \cdots \\
&= a^{\log_4 n}T(1) + n^2 \sum_{i=0}^{\log_4 n-1} \left(\frac{a}{16}\right)^i \\
&= cn^{\log_4 a} + \frac{16}{a-16}n^{\log_4 a} - \frac{16}{a-16}n^2 \\
&= \left(c + \frac{16}{a-16}\right)n^{\log_4 a} - \frac{16}{a-16}n^2,
\end{aligned}$$

where we are using the fact that

$$\left(\frac{a}{16}\right)^{\log_4 n} = \frac{a^{\log_4 n}}{n^2} = \frac{n^{\log_4 a}}{n^2}.$$

(c) For $T(n) = O(S(n))$, we should have

$$\begin{aligned}
n^{\log_4 a} &\leq n^{\log_2 9} \\
\log_4 a &\leq \log_2 9 \\
a &\leq 9^2 = 81.
\end{aligned}$$

So the range of values is $16 < a \leq 81$.

$\square$

Q.10 Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Solution:

(a) There are 3^n functions.

(b) If $n \leq 3$, there are $P(3, n)$ one-to-one functions. Hence, there are 3 when $n = 1$, 6 when $n = 2$, and 6 when $n = 3$. If $n > 3$, then there are 0 injective functions; there cannot be a one-to-one function from A to B if $|A| > |B|$.

(c) By the Inclusion-Exclusion Principle, we have

$$\begin{aligned} \# &= \#\{f : f(A) \subseteq \{1,2,3\}\} - \#\{f : f(A) \subseteq \{1,2\}\} - \#\{f : f(A) \subseteq \{1,3\}\} \\ &\quad -\#\{f : f(A) \subseteq \{2,3\}\} + \{f : f(A) \subseteq \{1\}\} + \{f : f(A) \subseteq \{2\}\} \\ &\quad \#\{f : f(A) \subseteq \{3\}\} \\ &= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 \\ &= 3^n - 3 \cdot 2^n + 3. \end{aligned}$$

□

Q.11 How many bit strings of length 19 contain at least 9 1's and at least 9 0's? You may leave your answer as an equation.
Solution: This requirement leaves only 1 bit undecided. So, there are 2 cases to deal with: the case with nine 1's and ten 0's, and the case with ten 1's and nine 0's. In each case, we simply choose which 9 or 10 of the 19 places to make 1's, giving $\binom{19}{9} + \binom{19}{10} = 2\binom{19}{9}$.

□

Q.12 Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding n that are relatively prime to n, i.e., the Euler function $\phi(n)$.
Solution: Let P be the set of numbers in $\{1, 2, 3, \ldots, n\}$ that are divisible by p, and similarly define the set Q. We want to count the numbers not divisible by either p or q, so we want $n - |P \cup Q|$. By the principle of

inclusion-exclusion, $|P \cup Q| = |P| + |Q| - |P \cap Q|$. Every pth number is divisible by p, so $|P| = \lfloor n/p \rfloor = q$. Similarly $|Q| = \lfloor n/q \rfloor = q$. Clearly, n is the only positive integer not exceeding n that is divisible by both p and q, so $|P \cap Q| = 1$. Therefore, the number of positive integers not exceeding n that are relatively prime to n is $n - p - q + 1$.

$\square$

Q.13 Alice is going to choose a selection of 12 chocolates. There are 25 different brands of them and she can have as many as she wants of each brand (but can only choose 12 pieces). How many ways can she make this selection?

Solution:

This is equivalent to count the number of solutions $x_1 + x_2 + \cdots + x_{25} = 12$, where all x_i's are nonnegative. The number of solutions is $\binom{36}{12}$.

$\square$

Q.14 16 points are chosen inside a 5×3 rectangle. Prove that two of these points lie within $\sqrt{2}$ of each other.

Solution: The area of the rectangle is 15 square units. By the pigeonhole principle, two of these points must lie in the same 1 by 1 square. Therefore, they are no further apart than the diagonal of the square $\sqrt{2}$.

$\square$

Q.15 Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Solution:

Let $K(x)$ be the number of other people at the party that person x knows. The possible values for $K(x)$ are $0, 1, \ldots, n-1$, where $n \geq 2$ is the number of people at the party. We cannot apply the pigeonhole principle directly, since there are n pigeons and n pigeonholes. However, it is impossible for both 0 and $n-1$ to be in the range of K, since if one person knows everybody else, then nobody can know no one else (we assume that "knowing" is symmetric). Therefore, the range of K has at most $n-1$ elements, whereas the domain has n elements, so K is not one-to-one, precisely what we wanted to prove.

□

Q.16 Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Solution:

(a) $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternatively, suppose that the $(j+1)$st term is the last term equal to 1, so that $n \le j \le n+r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j-n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r} \binom{j}{j-n} = \sum_{k=0}^{r} \binom{n+k}{k}$ ways to this.

(b) Let $P(r)$ be the statement to be proved. The basis step is the equation $\binom{n}{0} = \binom{n+1}{0}$, which is just $1 = 1$. Assume that $P(r)$ is true. Then

$$\begin{aligned} &\sum_{k=0}^{r+1} \binom{n+k}{k} \\ &= \sum_{k=0}^{r} \binom{n+k}{k} + \binom{n+r+1}{r+1} \\ &= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \\ &= \binom{n+r+2}{r+1}, \end{aligned}$$

using the inductive hypothesis and Pascal's identity.

□

Q.17

Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.
Solution: The CE is

$$r^3 - 2r^2 - r + 2 = (r+1)(r-1)(r-2).$$

The roots are $r = -1$, $r = 1$ and $r = 2$. Hence, the solutions to this recurrence are of the form

$$a_n = \alpha_1(-1)^n + \alpha_2 1^n + \alpha_3 2^n.$$

To find the constants α_1, α_2 and α_3, we use the initial conditions. Plugging in $n = 0, n = 1$, and $n = 2$, we have

$$a_0 = 1 = \alpha_1 + \alpha_2 + \alpha_3 a_1 = 0 = -\alpha_1 + \alpha_2 + 2\alpha_3 a_2 = 7 = \alpha_1 + \alpha_2 + 4\alpha_3.$$

We then have $\alpha_1 = 3/2$, $\alpha_2 = -5/2$, and $\alpha_3 = 2$. Hence,

$$a_n = 3/2 \cdot (-1)^n - 5/2 \cdot 1^n + 2 \cdot 2^n = 2^{n+1} + (-1)^n \cdot 3/2 - 5/2.$$

□

Q.18 Solve the recurrence relation

$$a_n = 4a_{n-2},$$

with initial conditions $a_0 = 3$, $a_1 = 2$.
Solution: The CE is $r^2 = 4$, and the two distinct roots are $r = \pm 2$. Thus, $a_n = \alpha_1 2^n + \alpha_2(-2)^n$. Plugging in the two initial conditions, we have

$$a_0 = 3 = \alpha_1 + \alpha_2 a_2 = 2 = 2\alpha_1 - 2\alpha_2.$$

It then follows that $\alpha_1 = 2$ and $\alpha_2 = 1$. Therefore, we have

$$a_n = 2^{n+1} + (-2)^n.$$

□

Q.19

(a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

(b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

Solution:

(a) The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = p_2 n^2 + p_1 n + p_0$. (Note that $s = 1$ here, and 1 is not a root of the characteristic polynomial.) We plug this into our recurrence relation and obtain $p_2 n^2 + p_1 n + p_0 = 2(p_2(n-1)^2 + p_1(n-1) + p_0) + 2n^2$. We rewrite this by grouping terms with equal powers of n, obtaining $(-p_2 - 2)n^2 + (4p_2 - p_1)n + (-2p_2 + 2p_1 - p_0) = 0$. In order for this equation to be true for all n, we must have $p_2 = -2$, $4p_2 = p_1$, and $-2p_2 + 2p_1 - p_0 = 0$. This tells us that $p_1 = -8$ and $p_0 = -12$. Therefore the particular solution we seek is $a_n^{(p)} = -2n^2 - 8n - 12$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n - 2n^2 - 8n - 12$.

(b) We plug the initial condition into our solution from part (a) to obtain $4 = a_1 = 2\alpha - 2 - 8 - 12$. This tells us that $\alpha = 13$. So the solution is $a_n = 13 \cdot 2^n - 2n^2 - 8n - 12$.

□

Q.20 A computer system considers a string of decimal digits $(0, 1, \ldots, 9)$ to be a **valid** code word if and only if it contains an **odd number of zero digits**. For example, 12030 and 11111 are **not** valid, but 29046 is. Let $V(n)$ denote the number of valid n-digit code words. Find a recurrence relation for $V(n)$ with initial cases, and give a closed-form solution to this recurrence relation. Please explain how you find the recurrence relation. (Hint: notice that the number of non-valid code words is equal to $10^n - V(n)$.)
Solution: There are two ways to construct a valid code of length n from a string of $n-1$ digits:

(a) take a valid code of length $n-1$, append a number between 1 and 9: there are $9V(n-1)$ ways;

(b) take a non-valid code of length $n-1$, append a 0: there are $10^{n-1} - V(n-1)$ ways.

In total, we have

$$V(n) = 9V(n-1) + 10^{n-1} - V(n-1) = 10^{n-1} + 8V(n-1),$$

with initial cases $V(1) = 1$.

By iterating this recurrence, we have

$$\begin{aligned}
V(n) &= 8V(n-1) + 10^{n-1} \\
&= 8(8V(n-2) + 10^{n-2}) + 10^{n-1} \\
&= 8^2V(n-2) + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^2(8V(n-3) + 10^{n-3}) + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^3V(n-3) + 8^2 \cdot 10^{n-3} + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= \vdots \\
&= 8^{n-1}V(1) + 8^{n-2}10^1 + 8^{n-3}10^2 + \cdots + 8 \cdot 10^{n-2} + 10^{n-1} \\
&= 8^{n-1}\left(1 + \frac{5}{4} + (\frac{5}{4})^2 + \cdots + (\frac{5}{4})^{n-1}\right) \\
&= 8^{n-1} \cdot \frac{1 - (\frac{5}{4})^n}{1 - \frac{5}{4}} \\
&= 5 \cdot 10^{n-1} - 4 \cdot 8^{n-1}.
\end{aligned}$$

□

Q.21 Use generating functions to prove Pascal's identity: $C(n,r) = C(n-1,r) + C(n-1,r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

Solution:

First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$. Expanding both sides of this equality using the binomial theorem, we have

$$\begin{aligned}\sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r.\end{aligned}$$

Thus,

$$1+\left(\sum_{r=1}^{n-1} C(n,r)x^r\right)+x^n = 1+\left(\sum_{r=1}^{n-1}(C(n-1,r)+C(n-1,r-1))x^r\right)+x^n.$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n,r)$ must equal $C(n-1,r)+C(n-1,r-1)$ for $1 \le r \le n-1$, as desired.

$\square$