南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Properties of Relations

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Yes. If $a|b$ and $b|c$, then $a|c$.

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  No. $(1,2),(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$.

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  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

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  Set operations: union, intersection, difference, etc.

Combining Relations

- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and
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  We may also combine relations by matrix operations.

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$R^k =?$ for $k > 3$

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"only if" part: by induction.

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  How many subsets on $n(n-1)$ elements are there?

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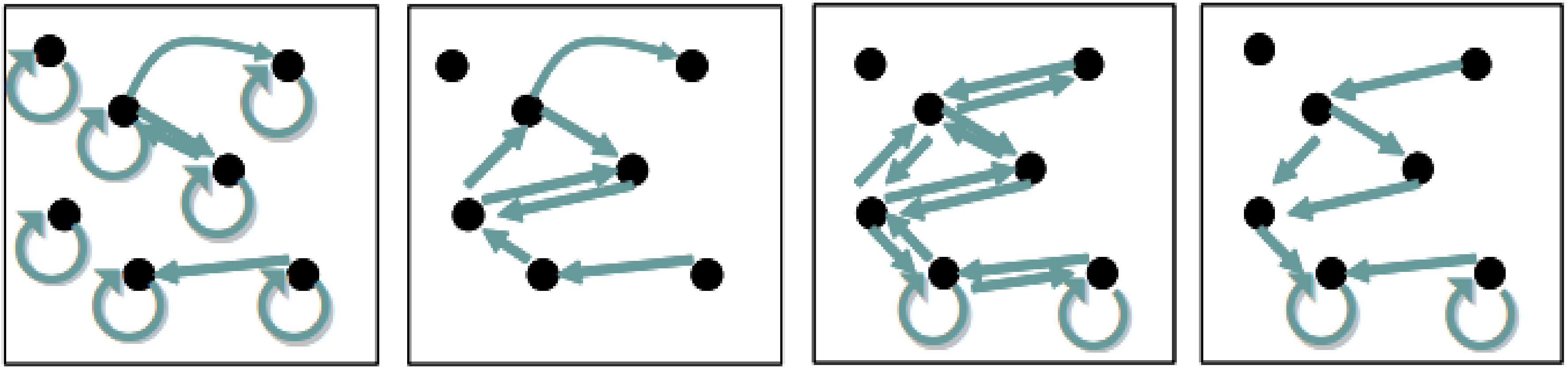

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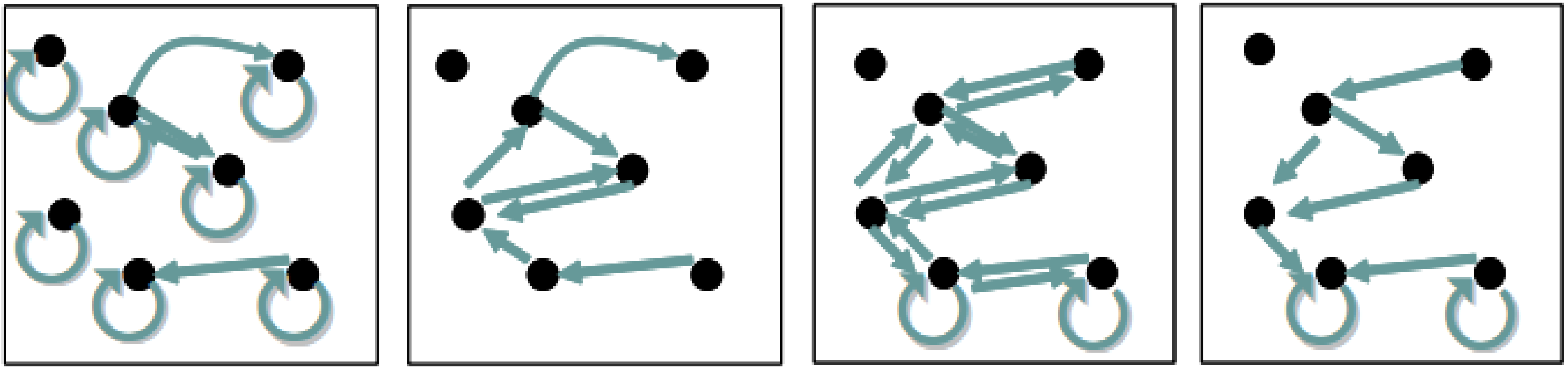

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  The minimal set $S \supseteq R$ is called *the reflexive closure of R*.

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Reflexive Closure

- The set S is called *the reflexive closure of* R if it:

  ◇ contains R
  ◇ is reflexive
  ◇ is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

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  - symmetric
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We define:

- reflexive closures
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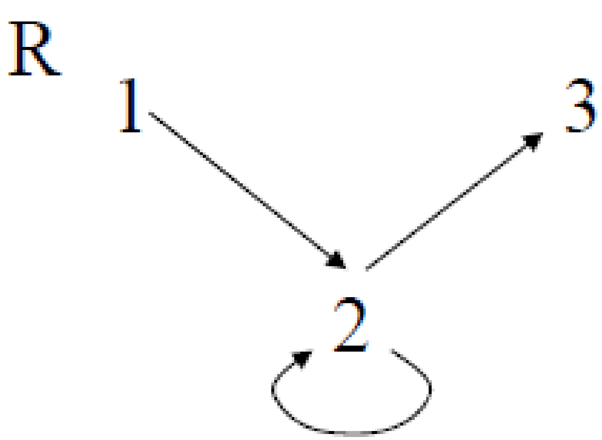

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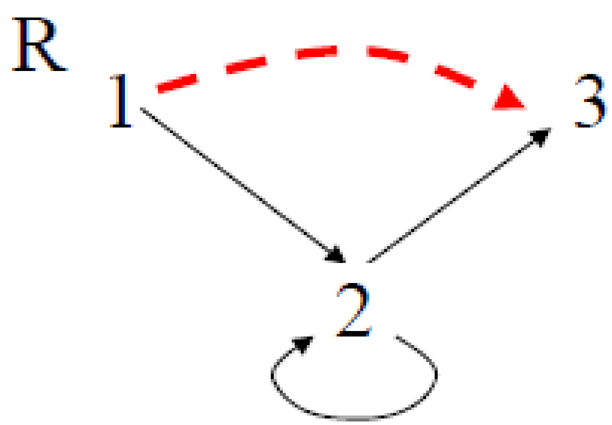

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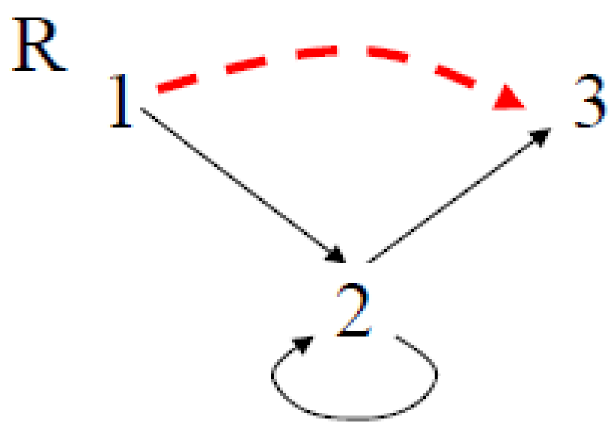

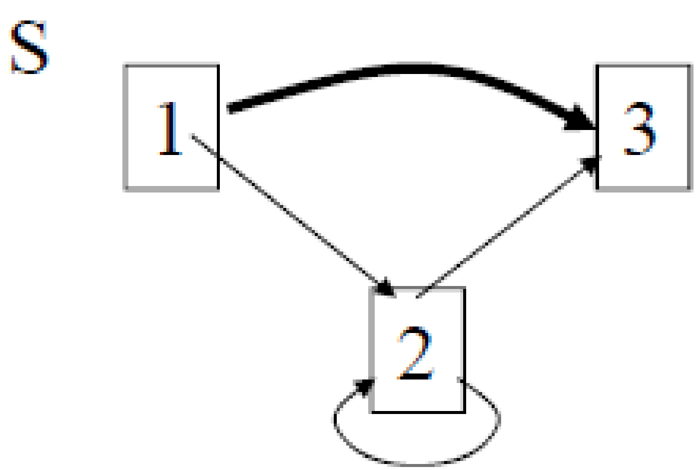

Paths in Directed Graphs

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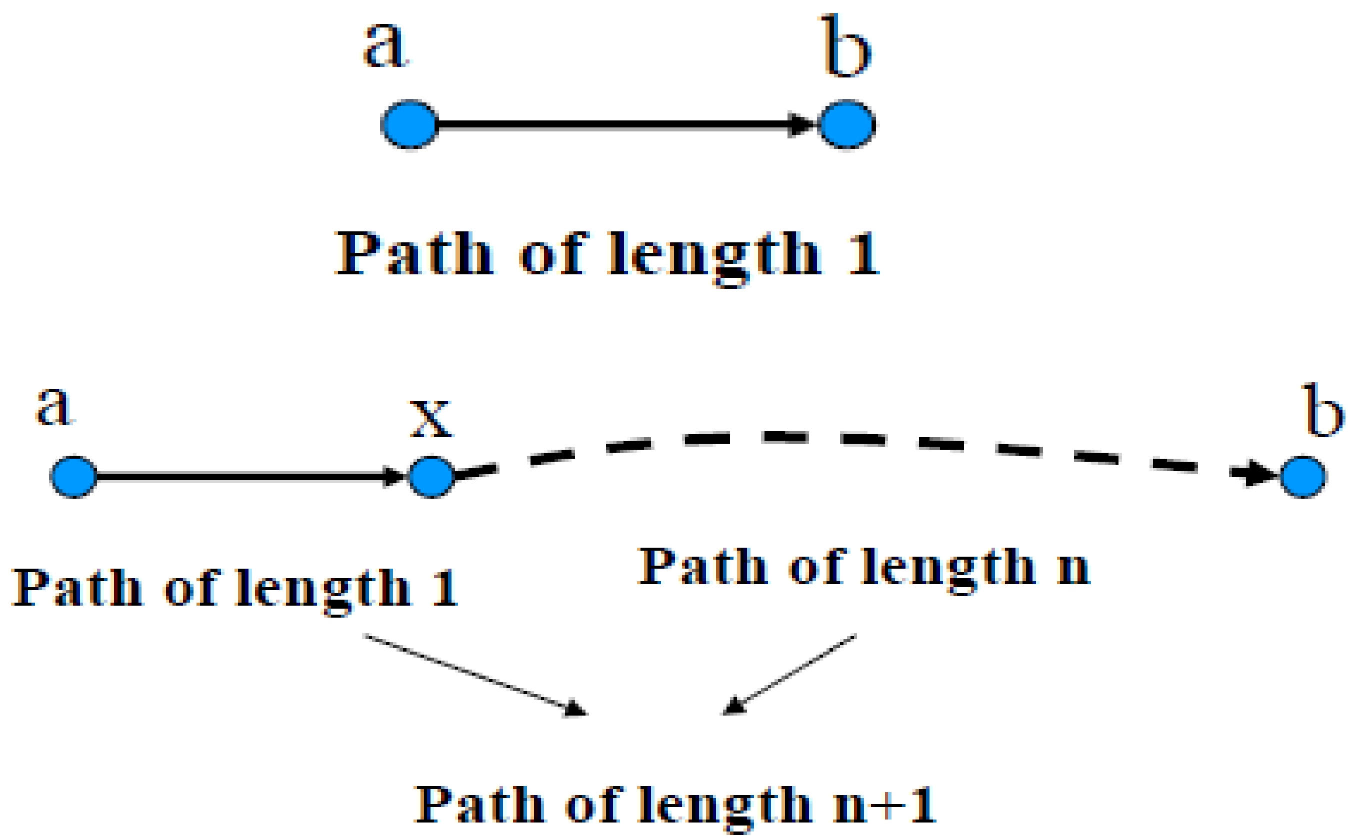

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- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

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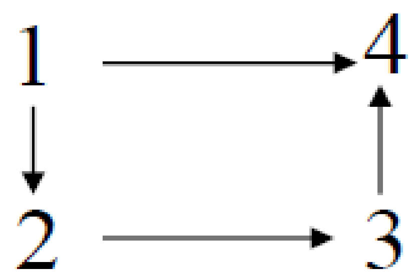

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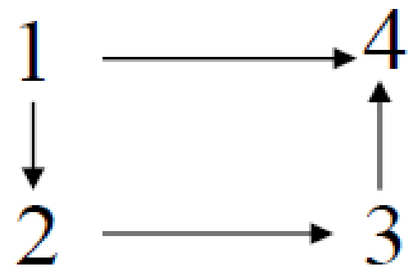

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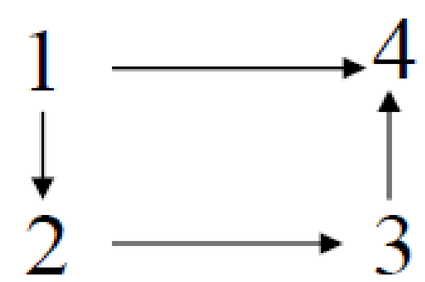

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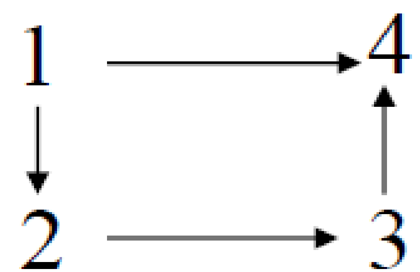

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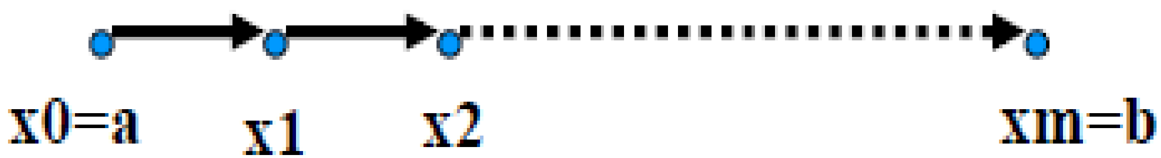

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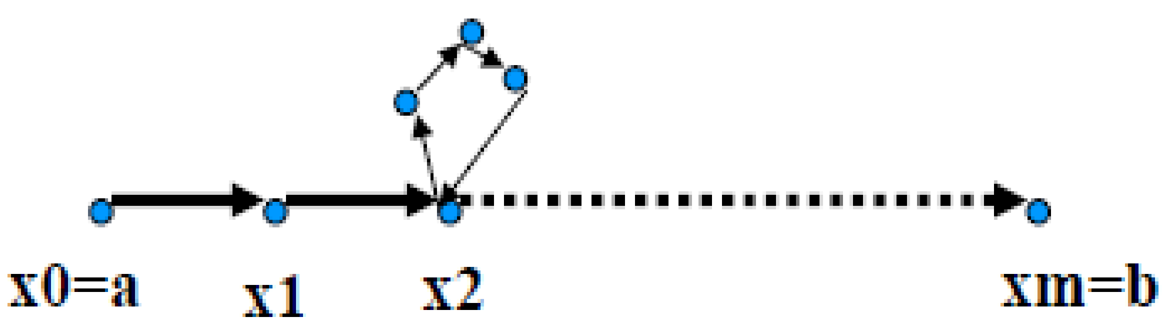

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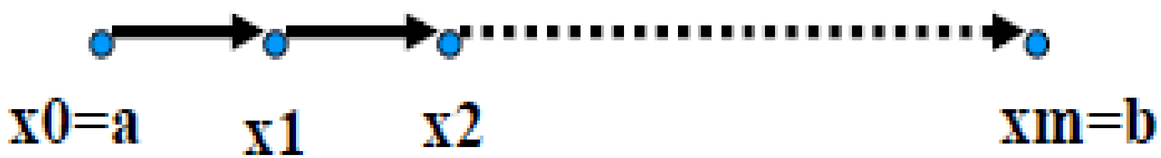

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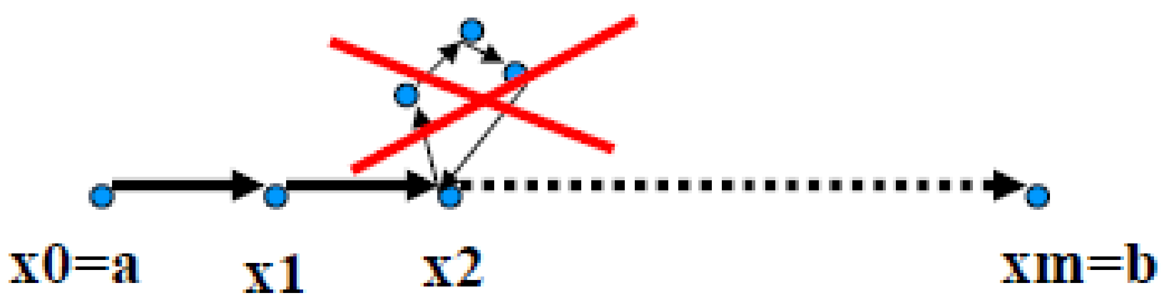

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1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.

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We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

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Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
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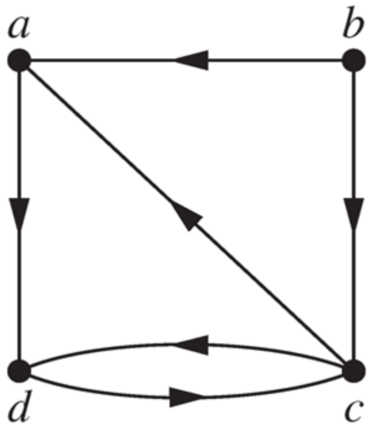
Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



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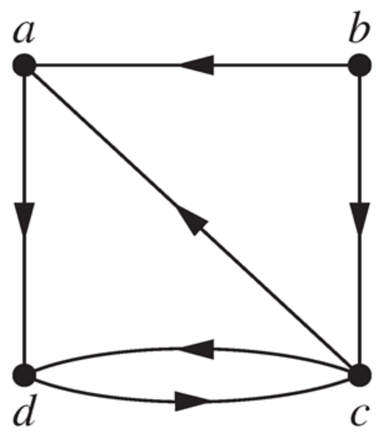

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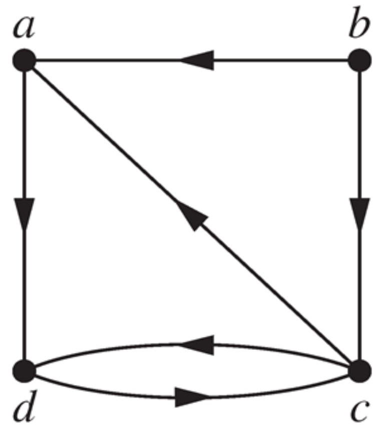

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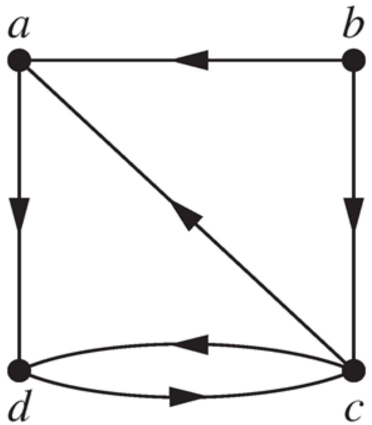

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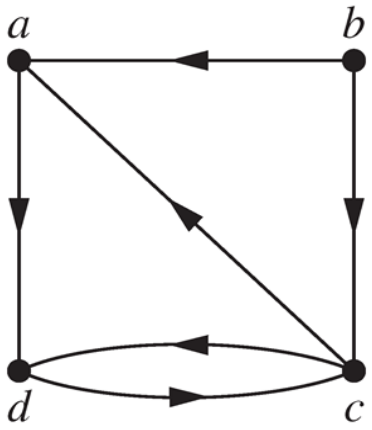

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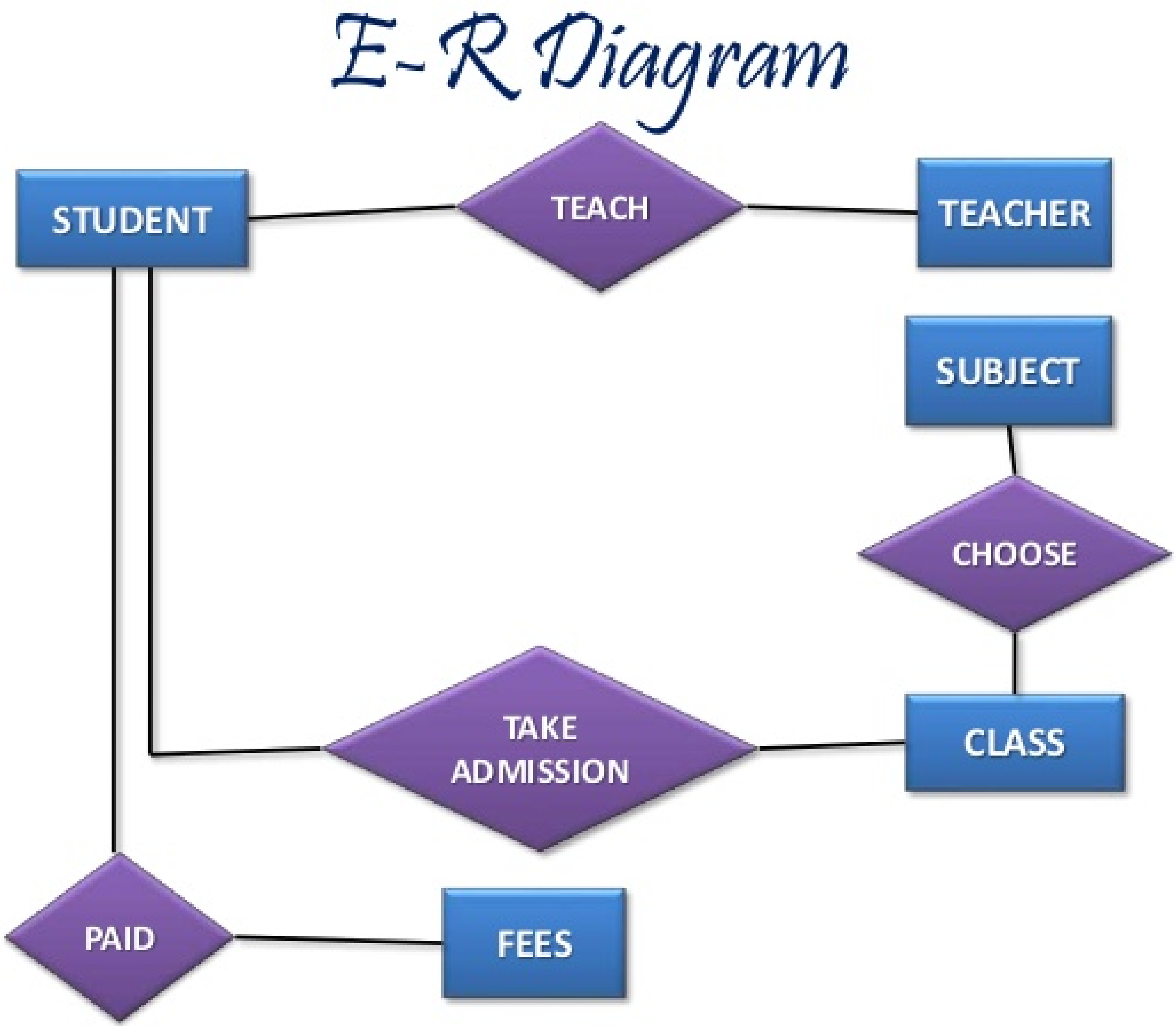
- A *composite key* for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

Relational Databases



Selection Operators

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  – $\forall R \subseteq A$,

$$\begin{aligned} s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\ &= \{a \in R \mid s_C(a) = T\}. \end{aligned}$$

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- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.

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- Then the *projection operator* on n-tuples

  $$P_{\{i_k\}} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

  is defined by

  $$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

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- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of $model/color$ combinations available.

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- A, B, C can also be sequences of elements rather that single elements.

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- Suppose that R_2 is a room assignment table relating *Courses* to *Rooms* and *Times*.

- Then $J(R_1, R_2)$ is like your class schedule, listing $(professor, course, room, time)$.

Next Lecture

- relation II ...