



CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Properties of Relations

■ **Reflexive Relation:** A relation R on a set A is called *reflexive* if $(a, a) \in R$ for **every** element $a \in A$.

Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for **every** element $a \in A$.

Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for **all** $a, b \in A$.

Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for **all** $a, b \in A$.

Transitive Relation: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for **all** $a, b, c \in A$.



Connectivity

- **Lemma:** Let A be a set with n elements, and R a relation on A . If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n - 1$.

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Theorem: The transitive closure of a relation R equals the connectivity relation R^* .

Recall Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path



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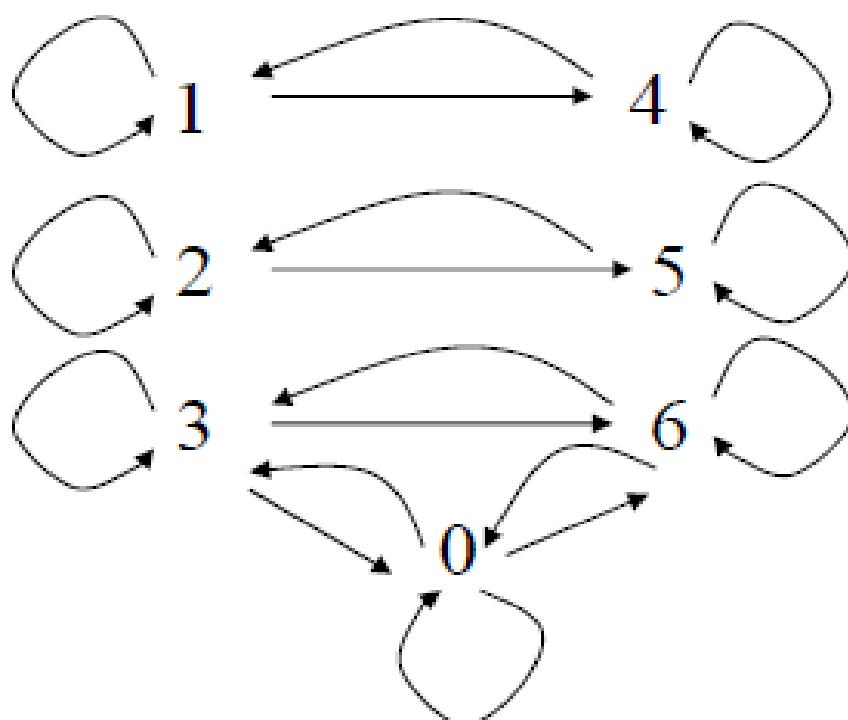
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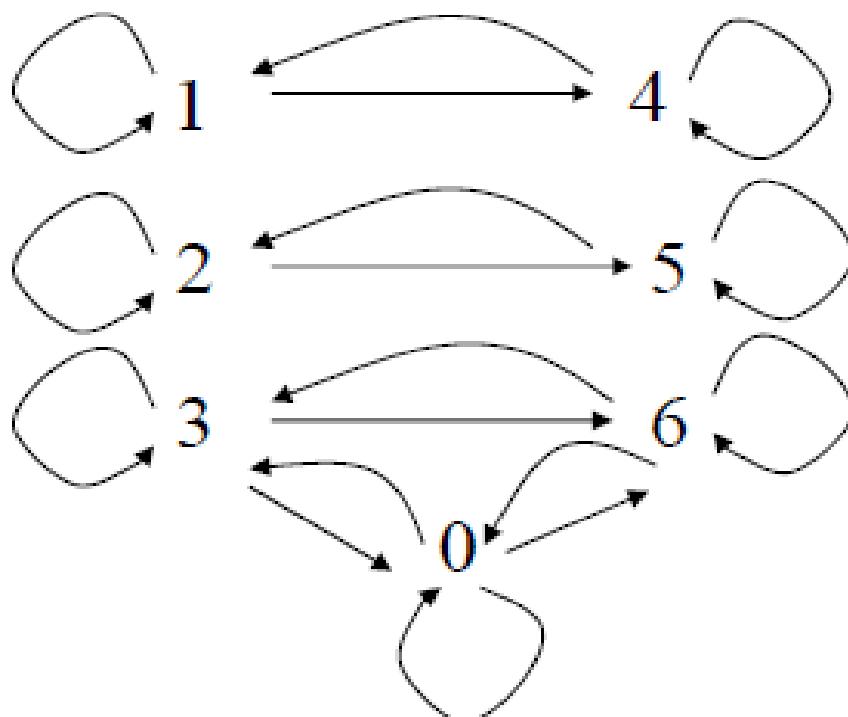
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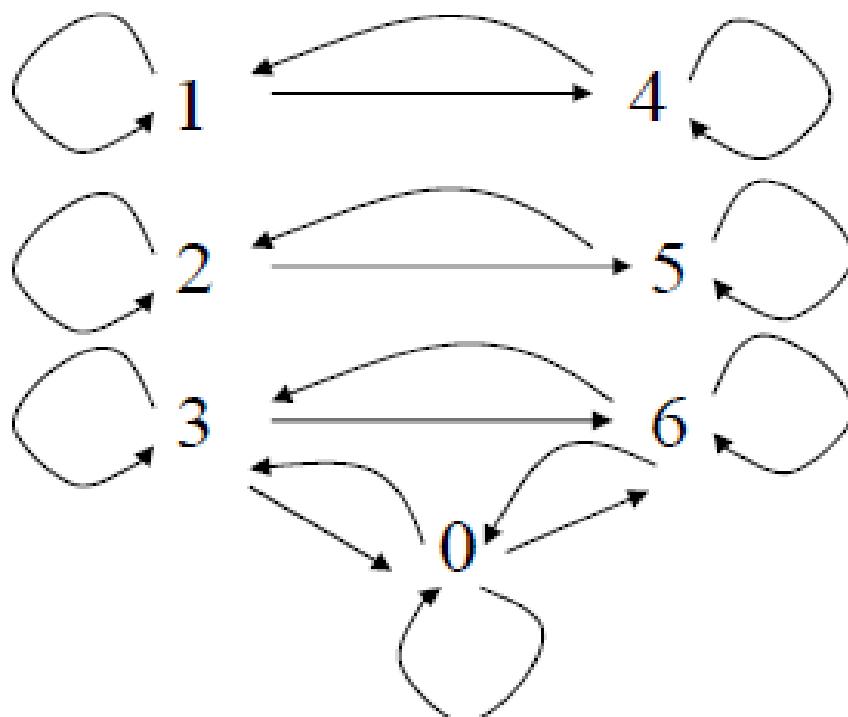


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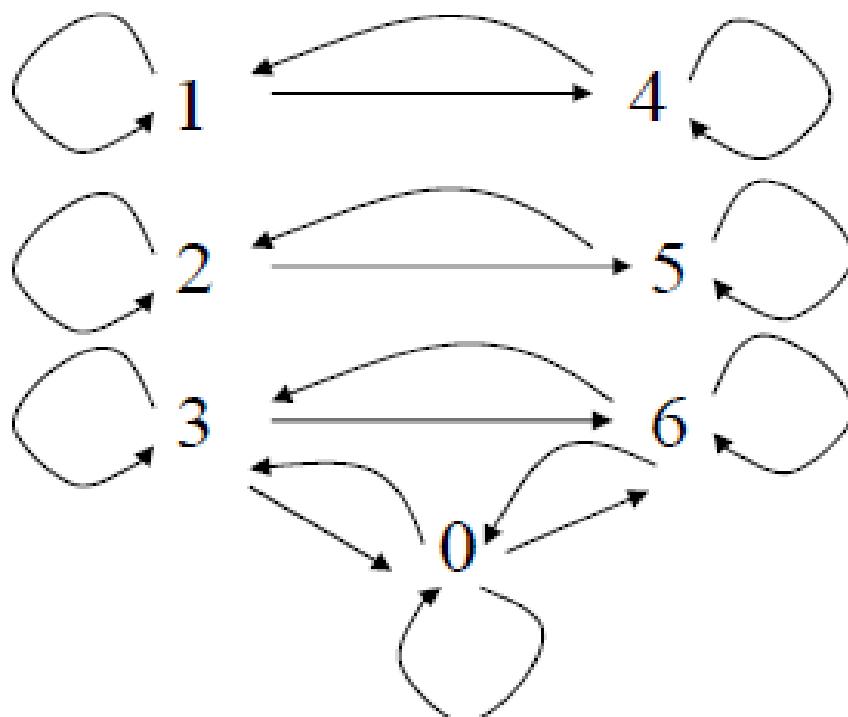


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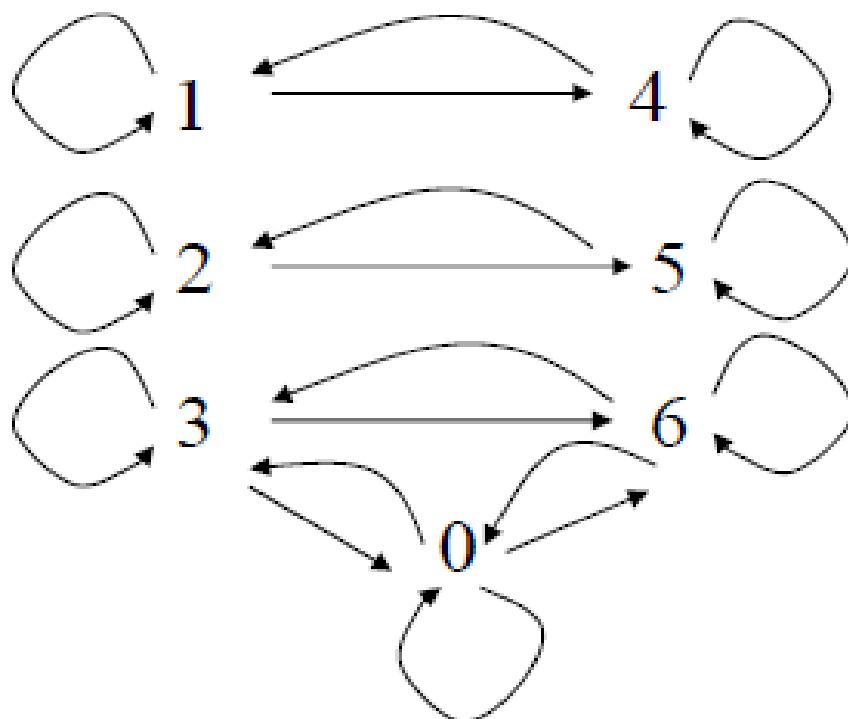
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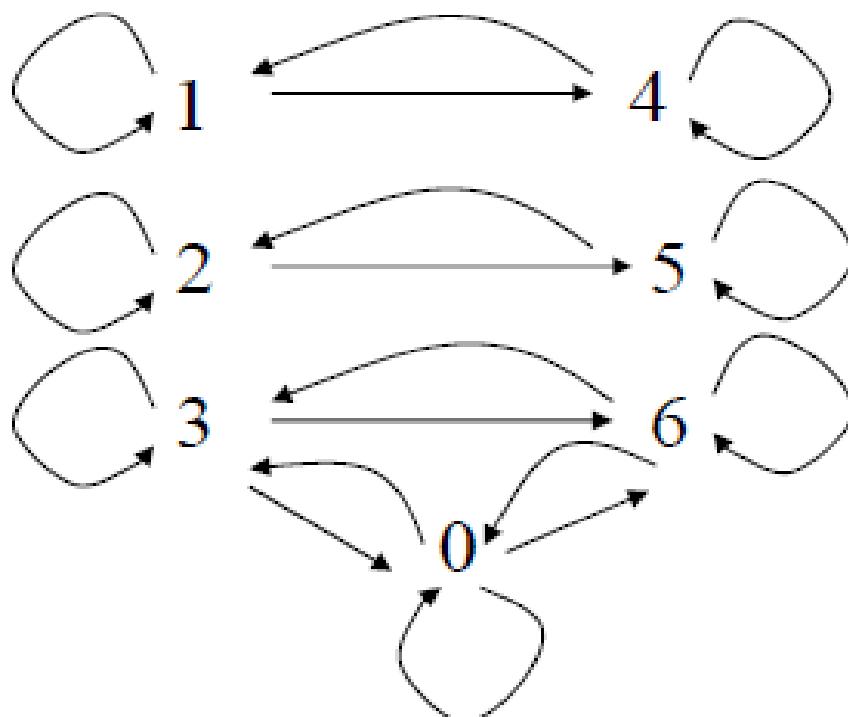
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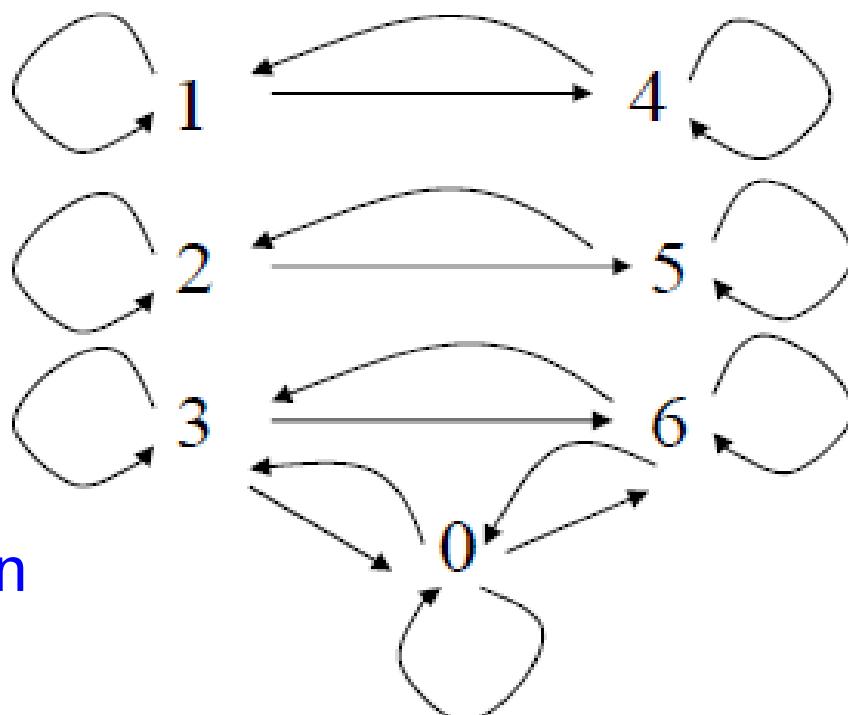
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Is R symmetric? Yes

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R is an equivalence relation



Examples of Equivalence Relations

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“Strings a and b have the same length.”

“Integers a and b have the same absolute value.”

“Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbb{Z}$).”

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“The relation \geq between real numbers.”

“has a common factor greater than 1 between natural numbers.”



Equivalence Class

- **Definition** Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class* of a , denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.



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$$\begin{aligned}[0] &= [3] = [6] = \{0, 3, 6\} \\[1] &= [4] = \{1, 4\} \\[2] &= [5] = \{2, 5\}\end{aligned}$$



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$[a] =$ the set of all strings of the same length as a

“Integers a and b have the same absolute value.”

$[a] =$ the set $\{a, -a\}$

“Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$).”

$[a] =$ the set $\{\dots, a - 2, a - 1, a, a + 1, a + 2, \dots\}$



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- **Theorem** Let R be an equivalence relation on a set A . The following statements are equivalent:

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(iii) \rightarrow (i): there exists a c s.t. $c \in [a]$ and $c \in [b]$



Partition of a Set S

- **Definition** Let S be a set. A collection of nonempty subsets of S A_1, A_2, \dots, A_k is called *a partition of S* if:

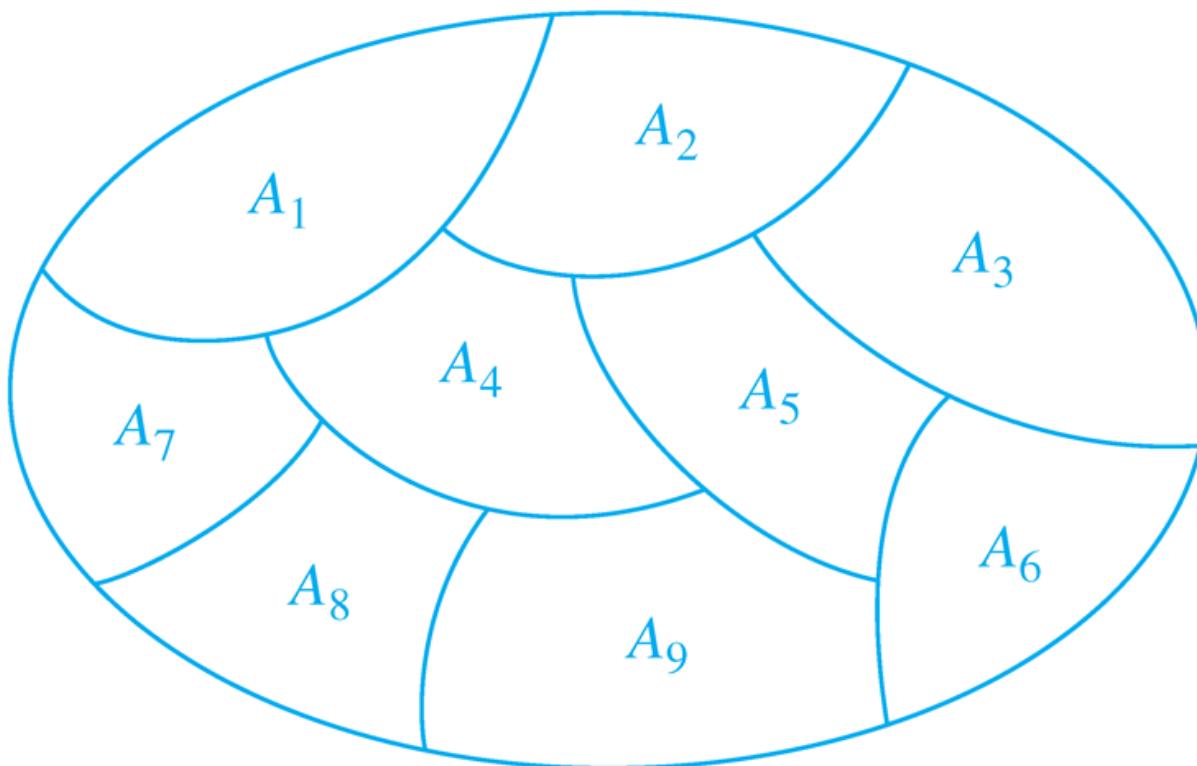
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Example:

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Is A_1, A_2, A_3 a partition of S ?



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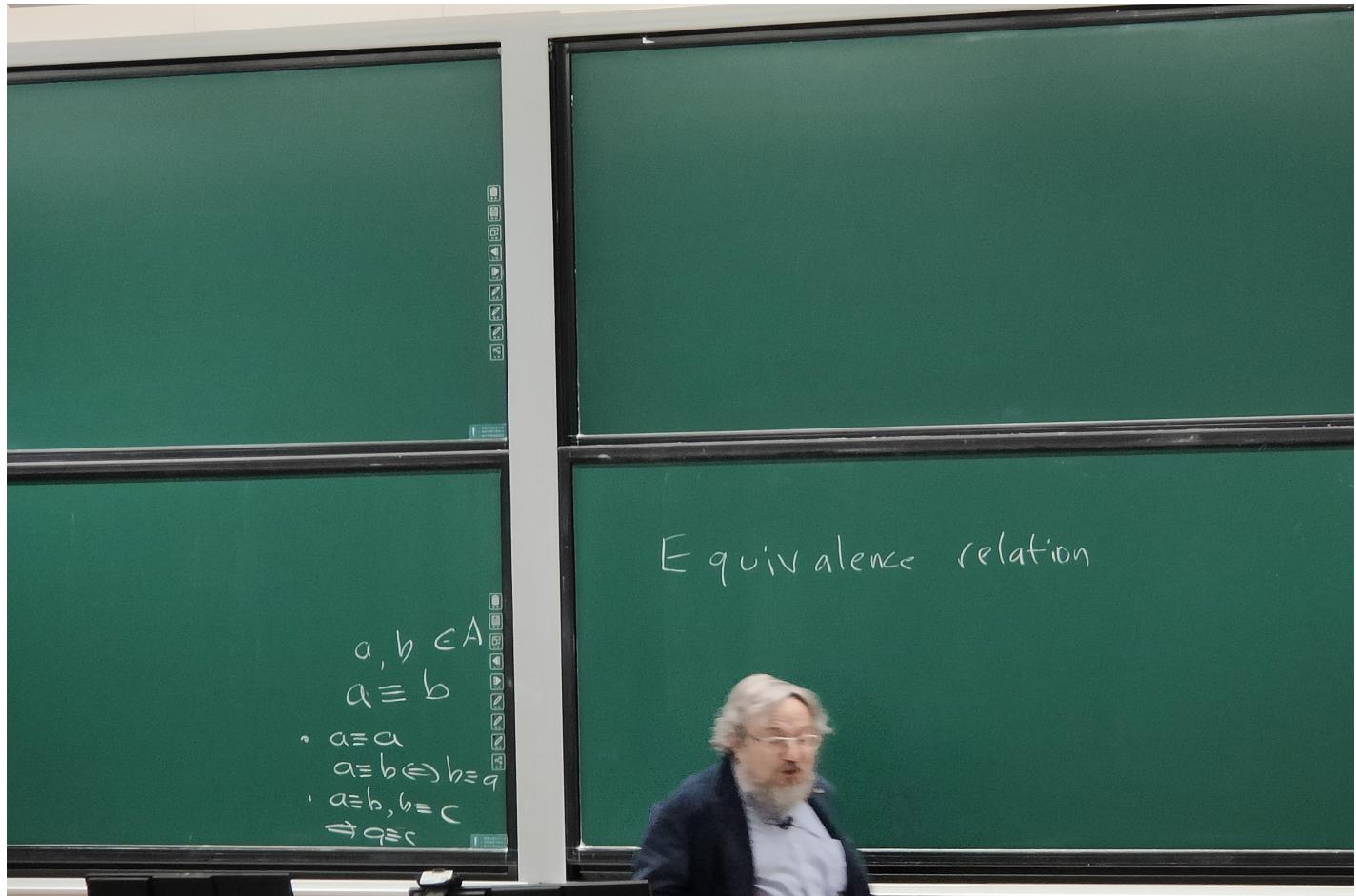
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Theorem Let $\{A_1, A_2, \dots, A_i, \dots\}$ be a partition of S . Then there is an equivalence relation R on S , that has the sets A_i as its equivalence classes.

Equivalence Relation



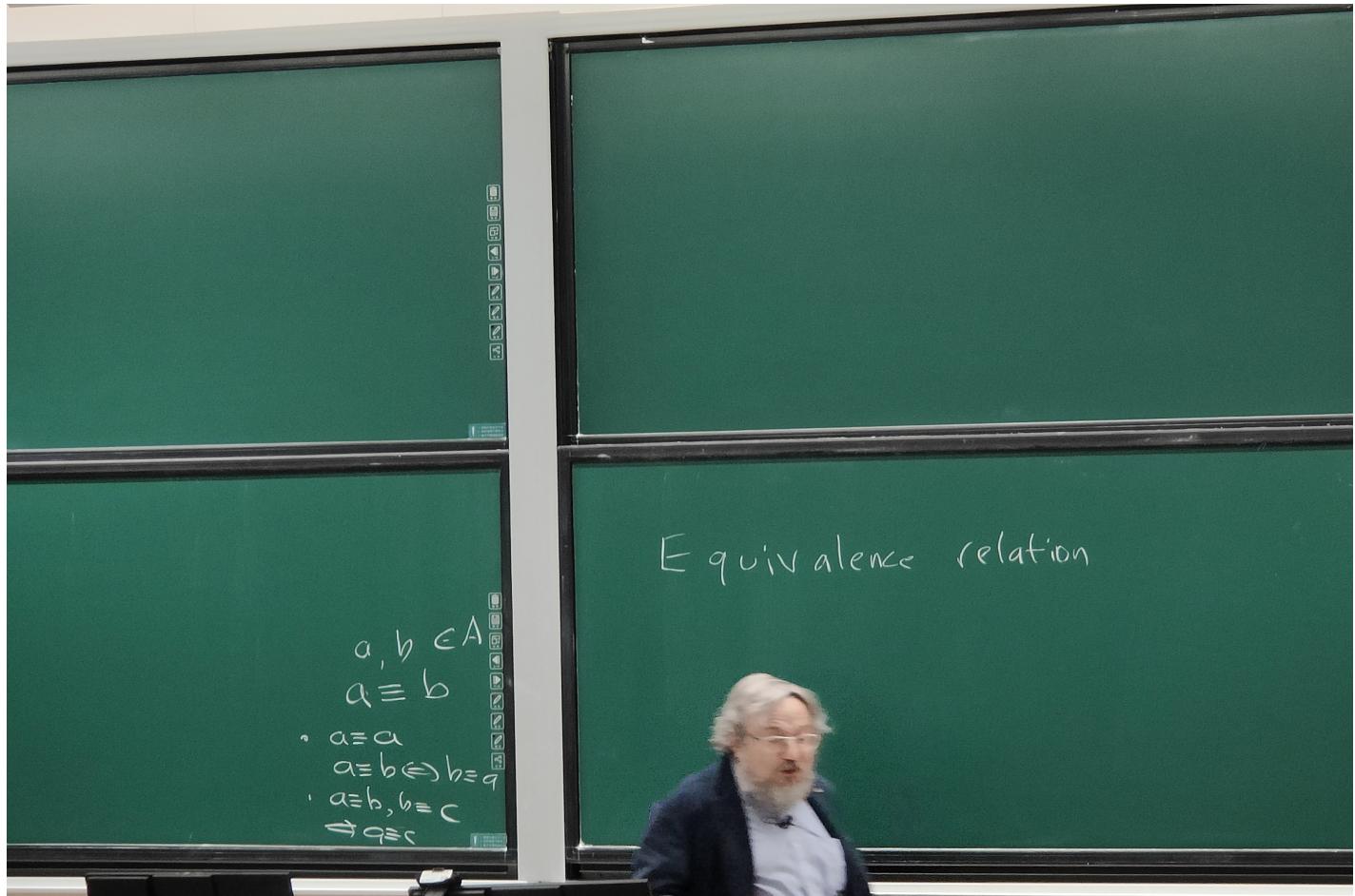
Don Zagier



Equivalence Relation



Don Zagier



他四十年前來中國，在合肥一周就寫了四百中國漢字，說是在日本學的中日通用。他在波恩講演之前，先問聽眾贊成用英語，德語，。。。分別舉手，他用多數人贊成的語言講。張賢科給他起中文名：查吉爾，或查杰爾。他選後者

.....

Partial Ordering

- **Definition** A relation R on a set S is called a *partial ordering*, or *partial order*, if it is **reflexive**, **antisymmetric**, and **transitive**. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, denoted by (S, R) . Members of S are called *elements of the poset*.

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2, 4 are comparable, 3, 5 are incomparable.

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- **Definition** Given two posets (A_1, \preccurlyeq_1) and (A_2, \preccurlyeq_2) , the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is **less than** (b_1, b_2) , i.e., $(a_1, a_2) \preccurlyeq (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \preccurlyeq_2 b_2$.

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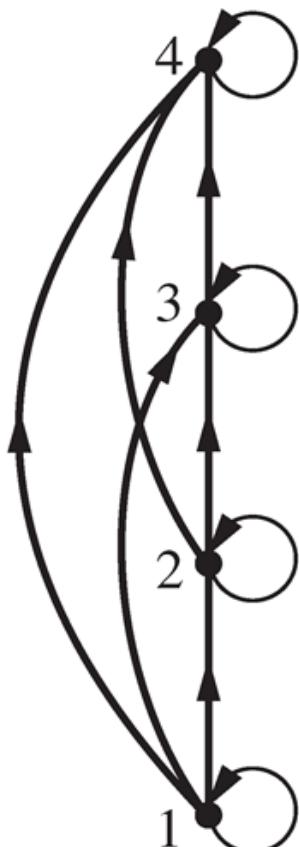
- ◊ *discreet* \prec *discrete*
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Hasse Diagram

- A Hasse diagram is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

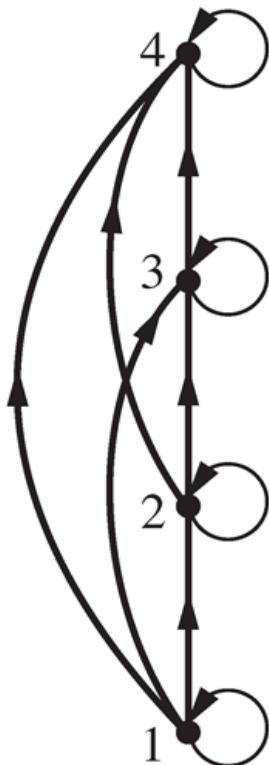
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Hasse Diagram

- (a) A partial ordering. The loops are due to the reflexive property
- (b) The edges that must be present due to the transitive property are deleted
- (c) The Hasse diagram for the partial ordering (a)



Procedure for Constructing Hasse Diagram

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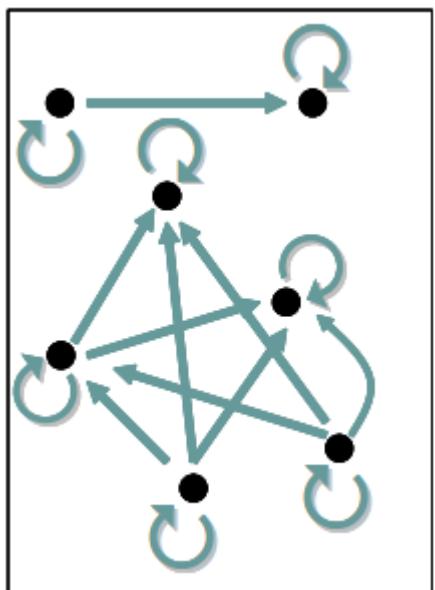
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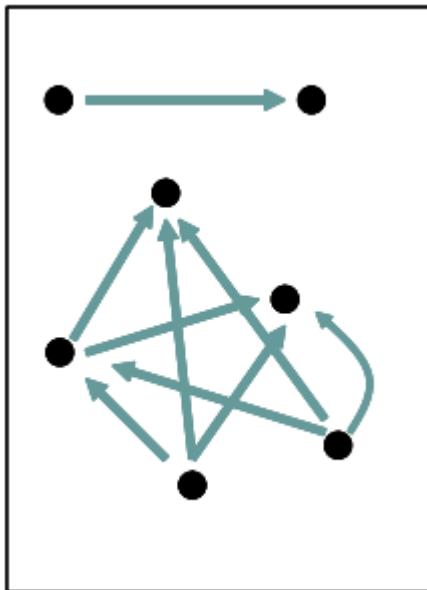
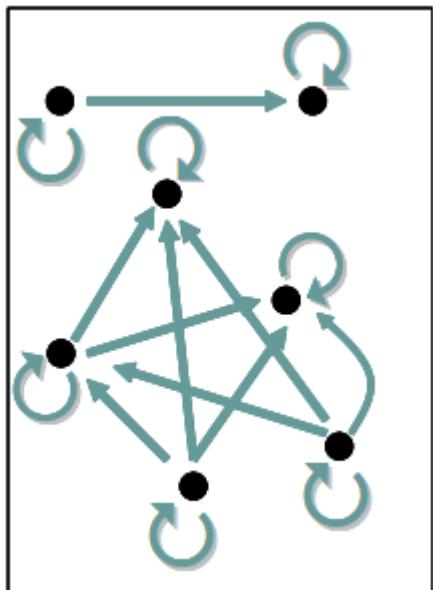
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 - ◊ Arrange each edge so that its initial vertex is **below** the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

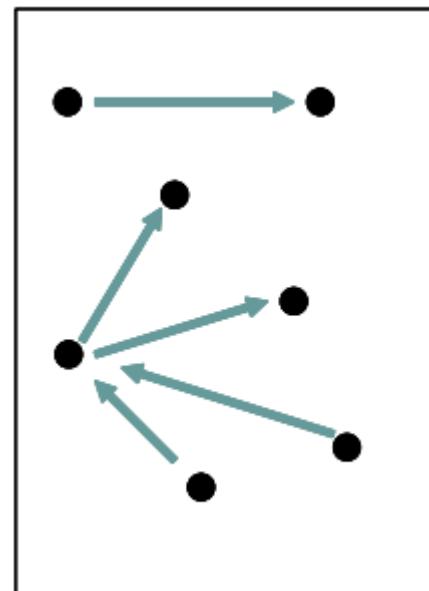
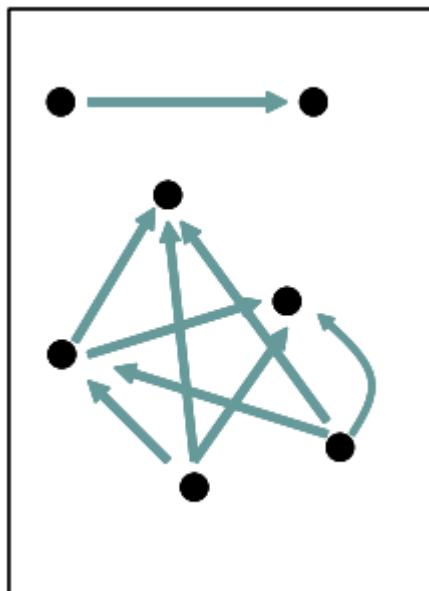
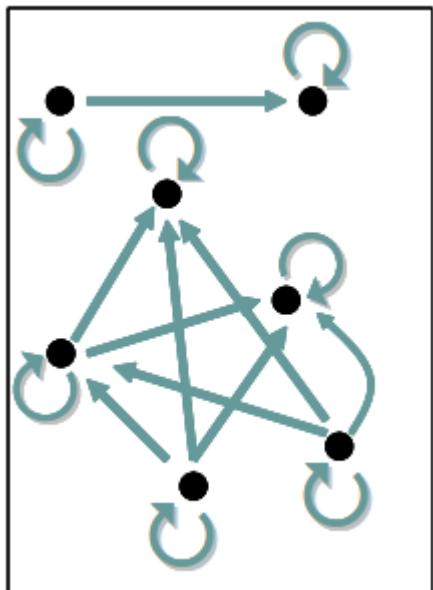
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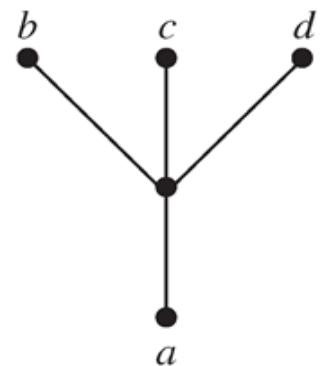
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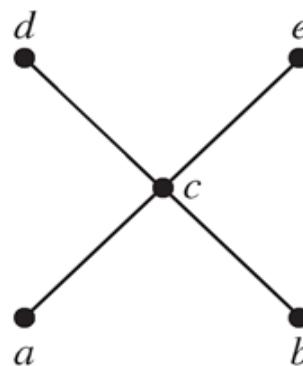
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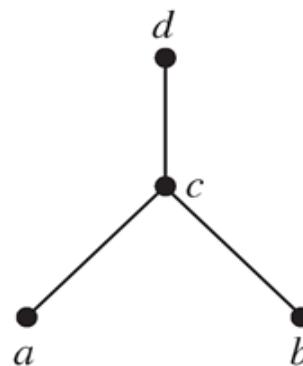
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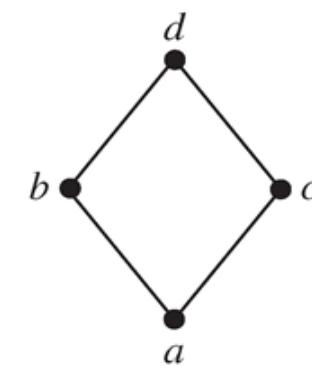
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(d)

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Example Find the *greatest lower bound* and the *least upper bound* of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbb{Z}^+, |)$.



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p. 620, Theorem 1

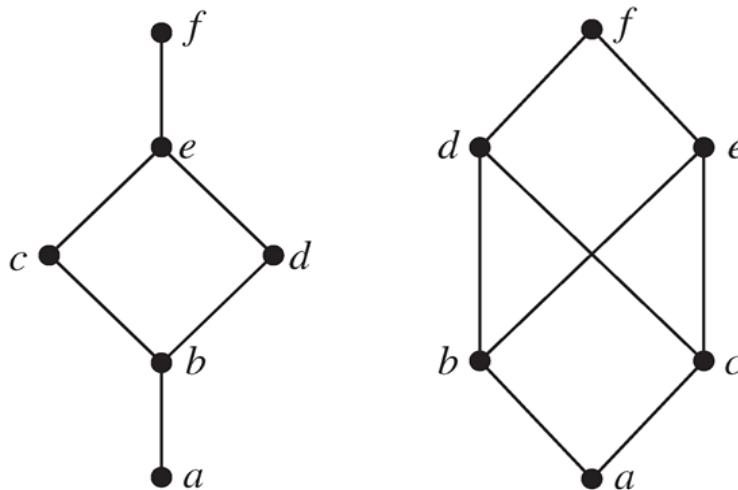


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- **Definition** A partial ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

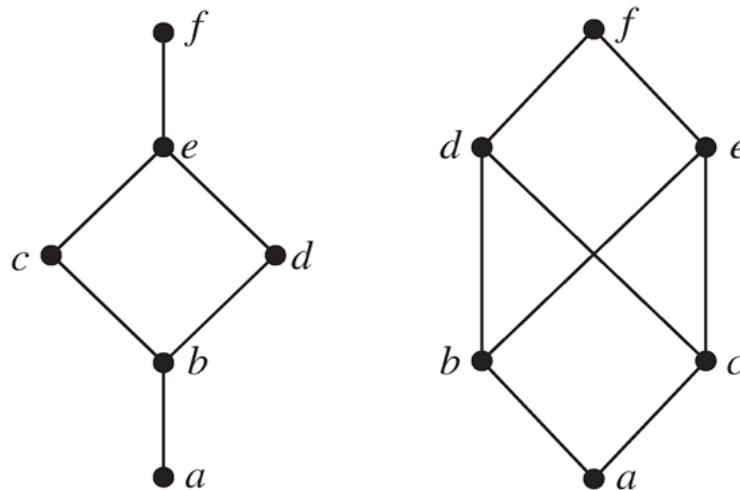
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Example Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Topological Sorting

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Topological sorting: Given a partial ordering R , find a total ordering \preccurlyeq such that $a \preccurlyeq b$ whenever $a R b$. \preccurlyeq is said *compatible with R* .



Topological Sorting for Finite Posets

procedure topological_sort (S : finite poset)

$k := 1;$

while $S \neq \emptyset$

$a_k :=$ a minimal element of S

$S := S \setminus \{a_k\}$

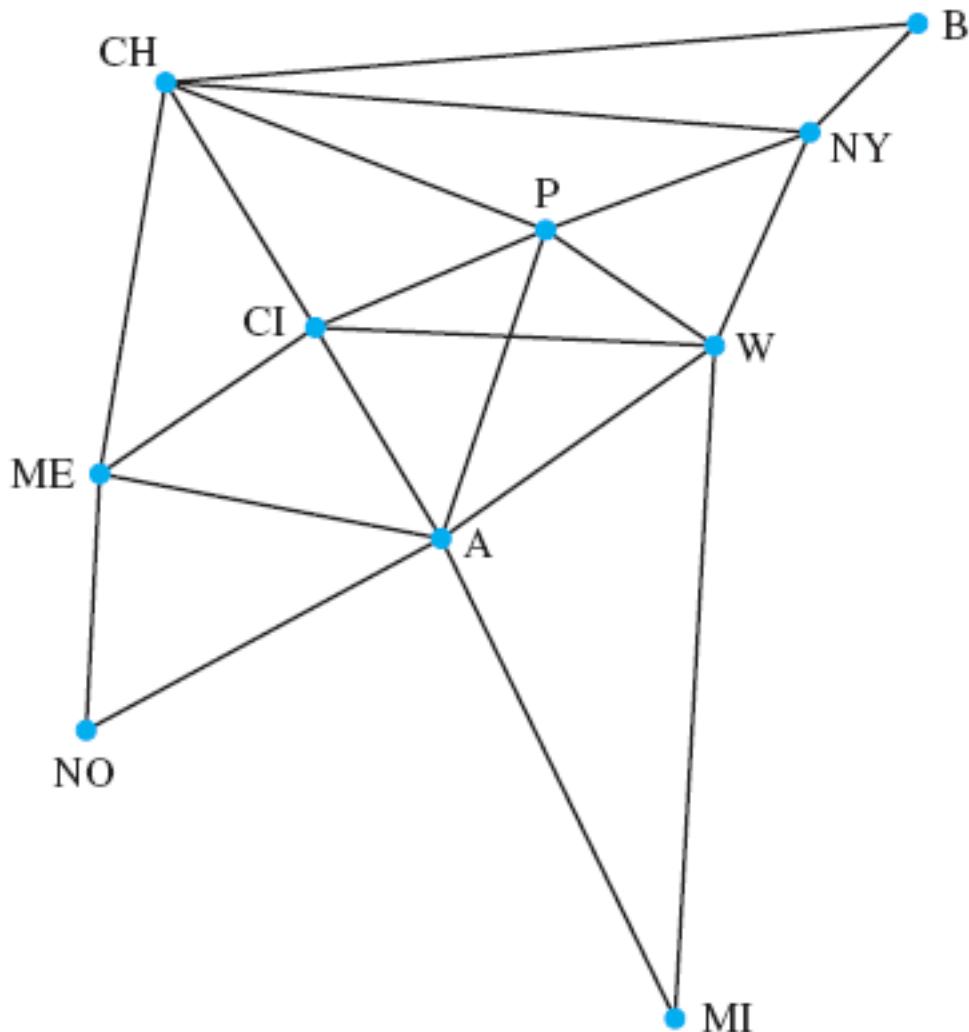
$k := k + 1$

end while

// $\{a_1, a_2, \dots, a_n\}$ is a compatible total ordering of S

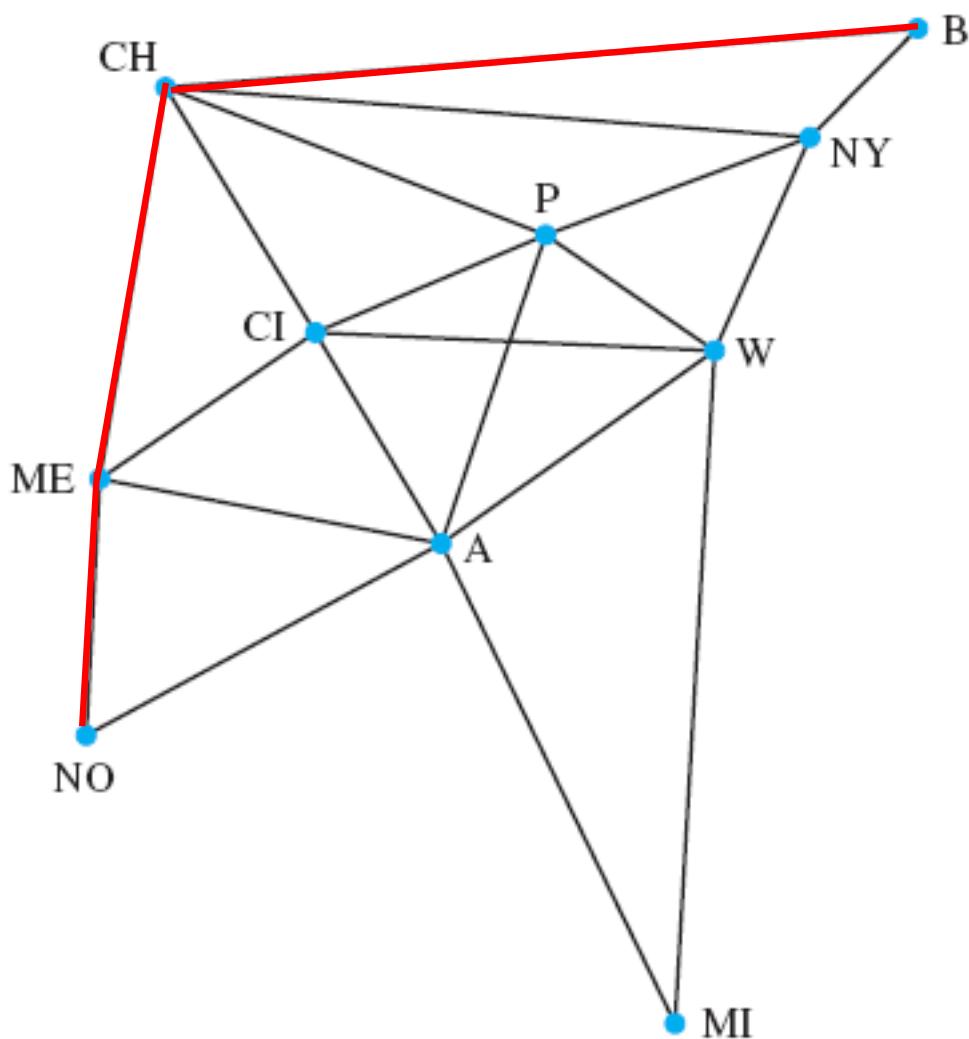


Example



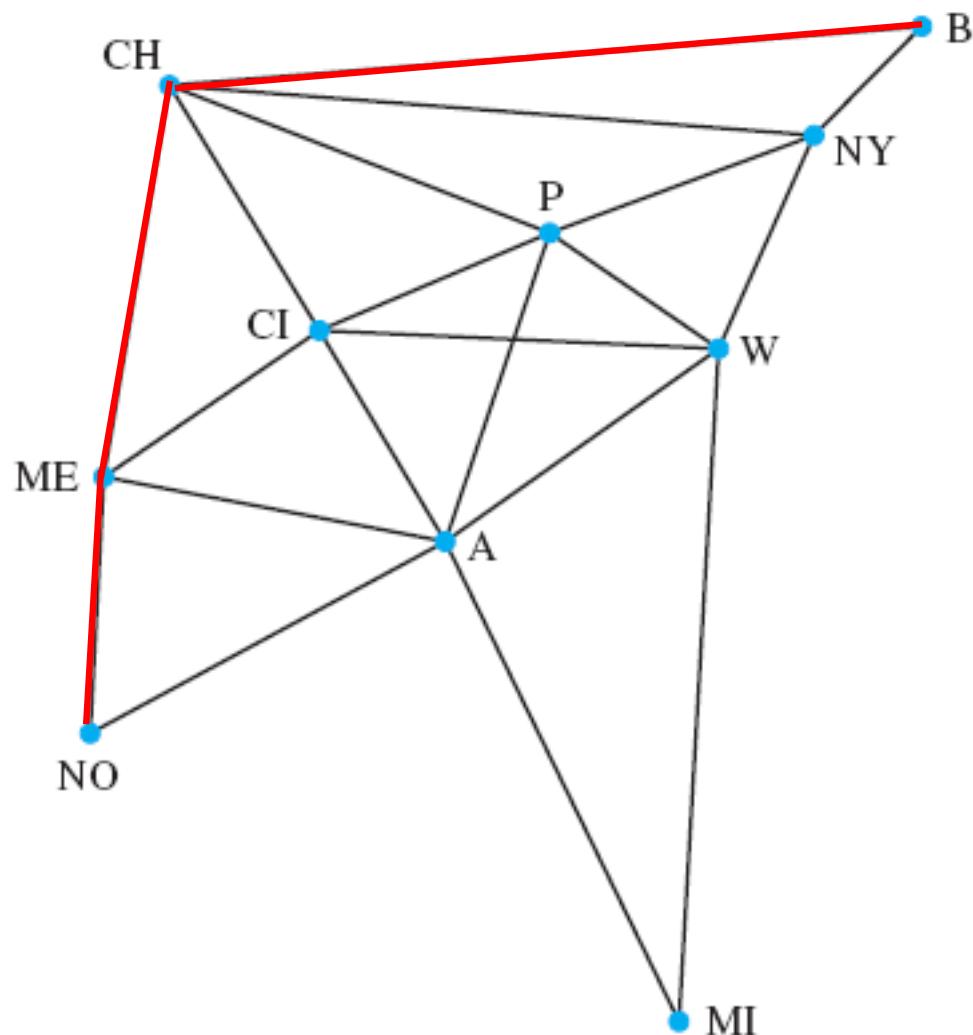
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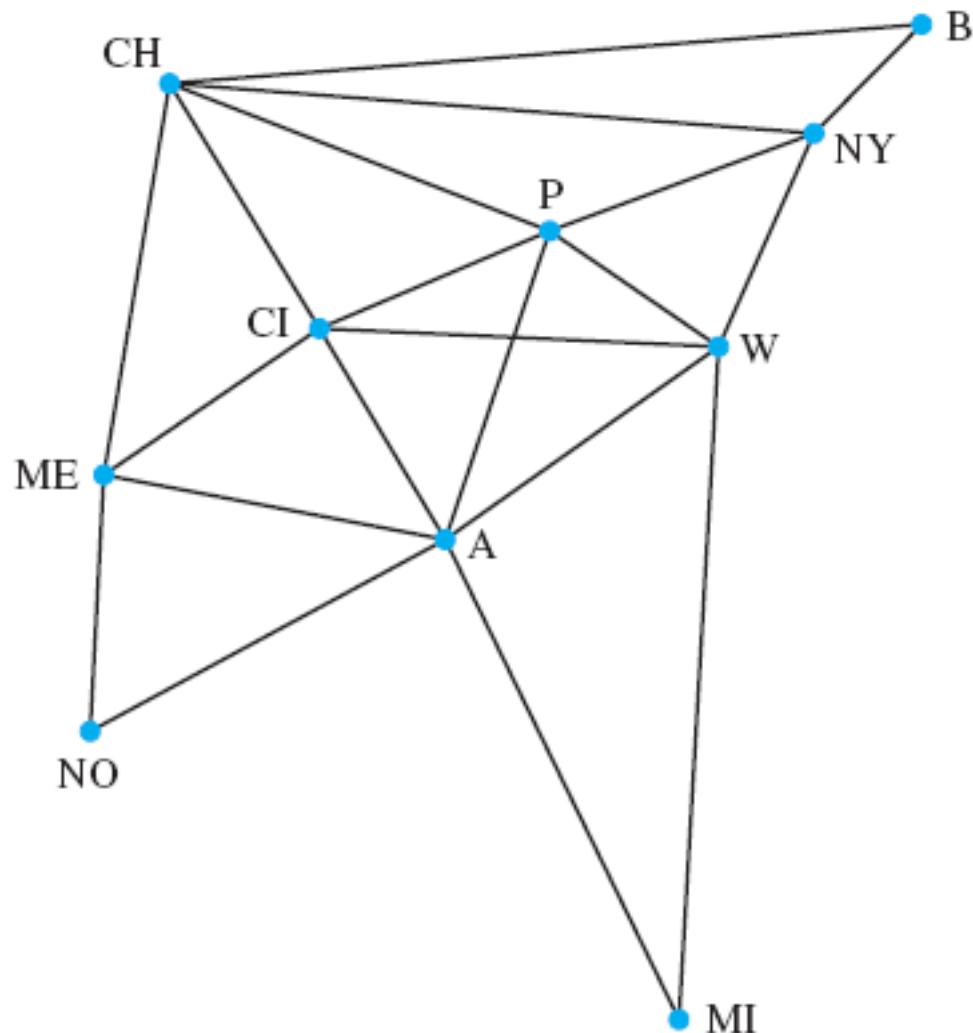
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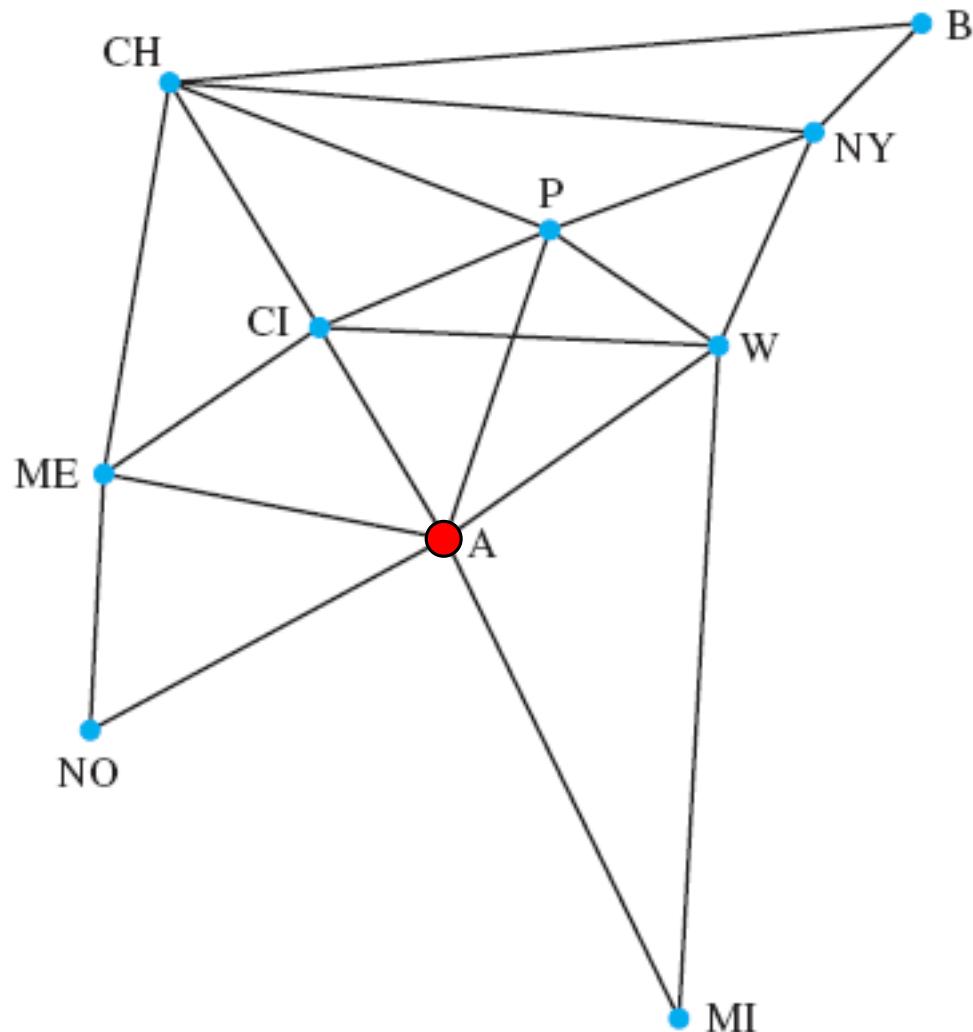


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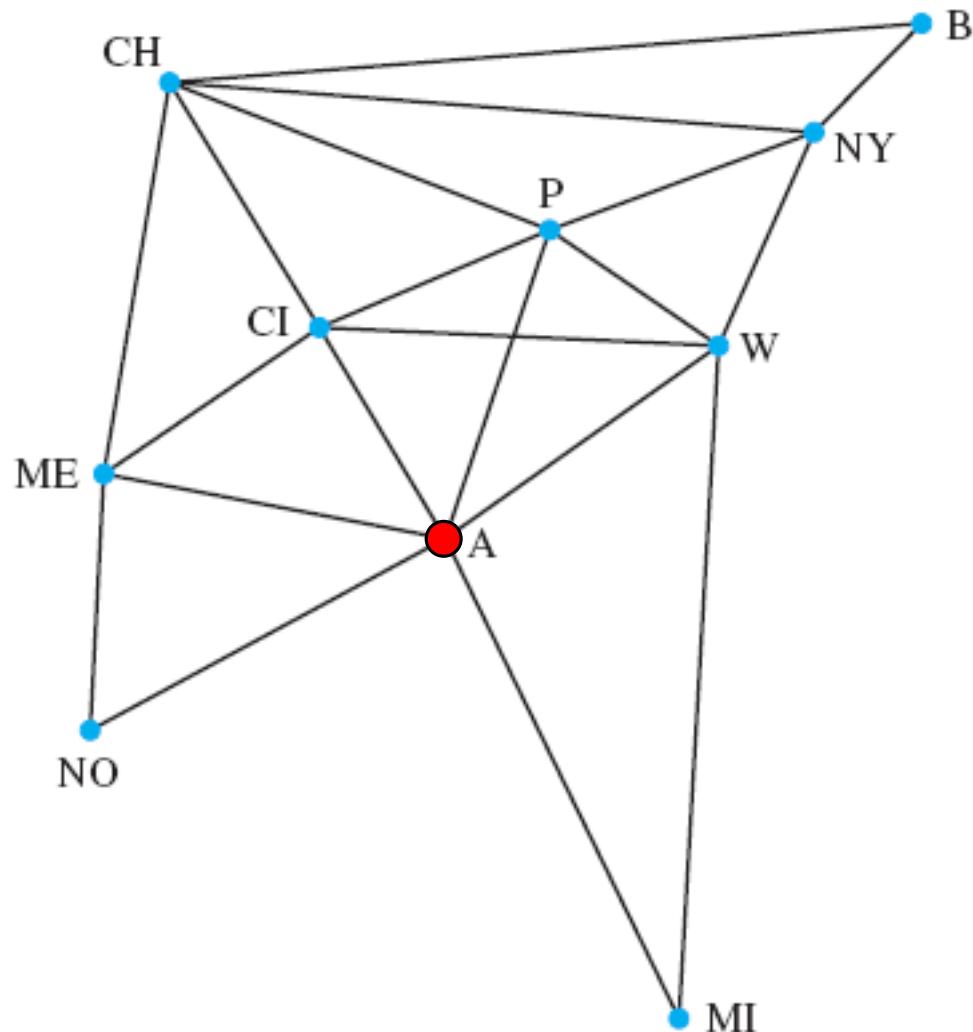


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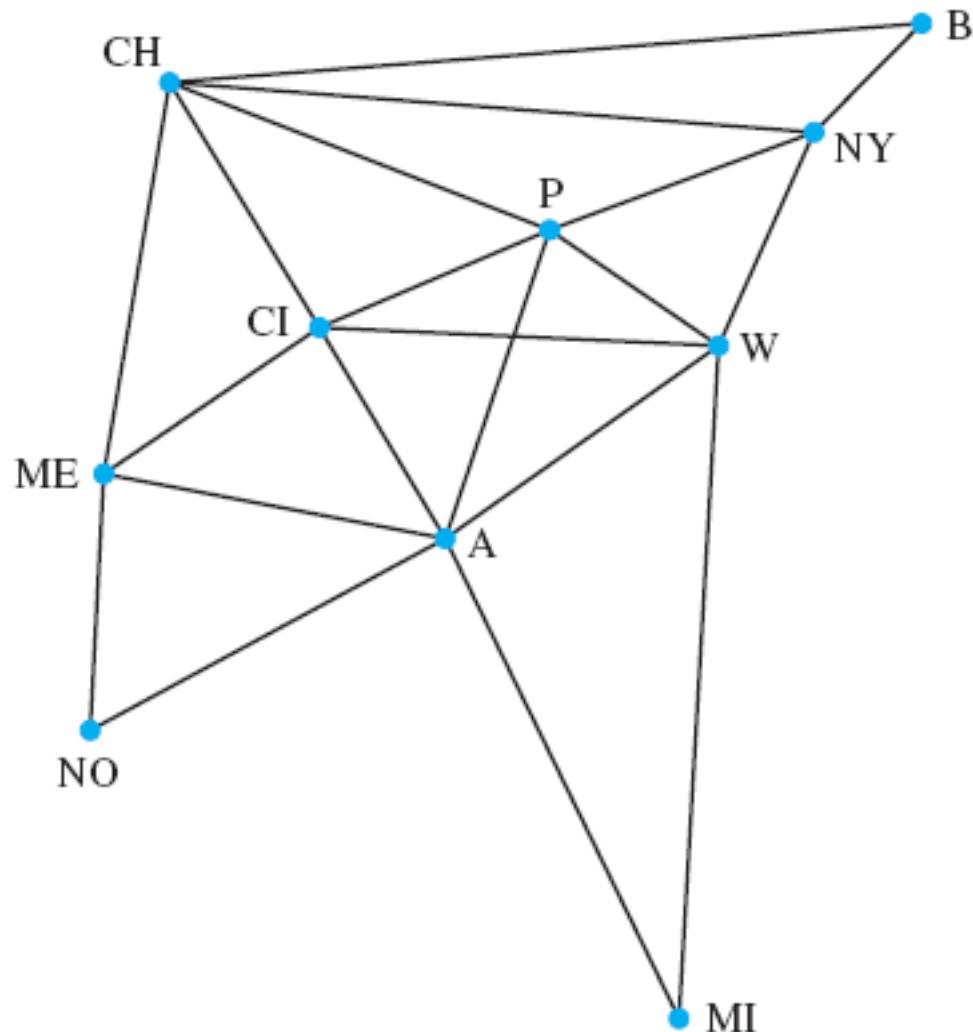
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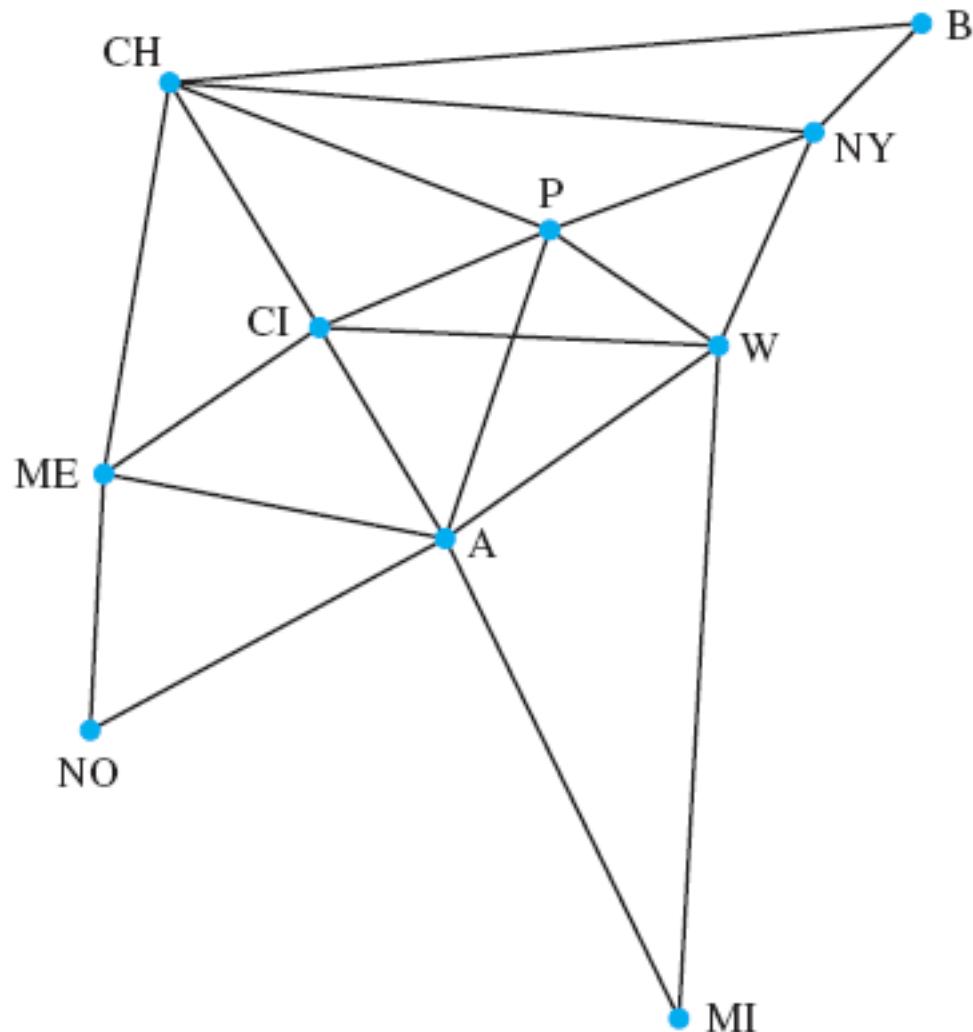
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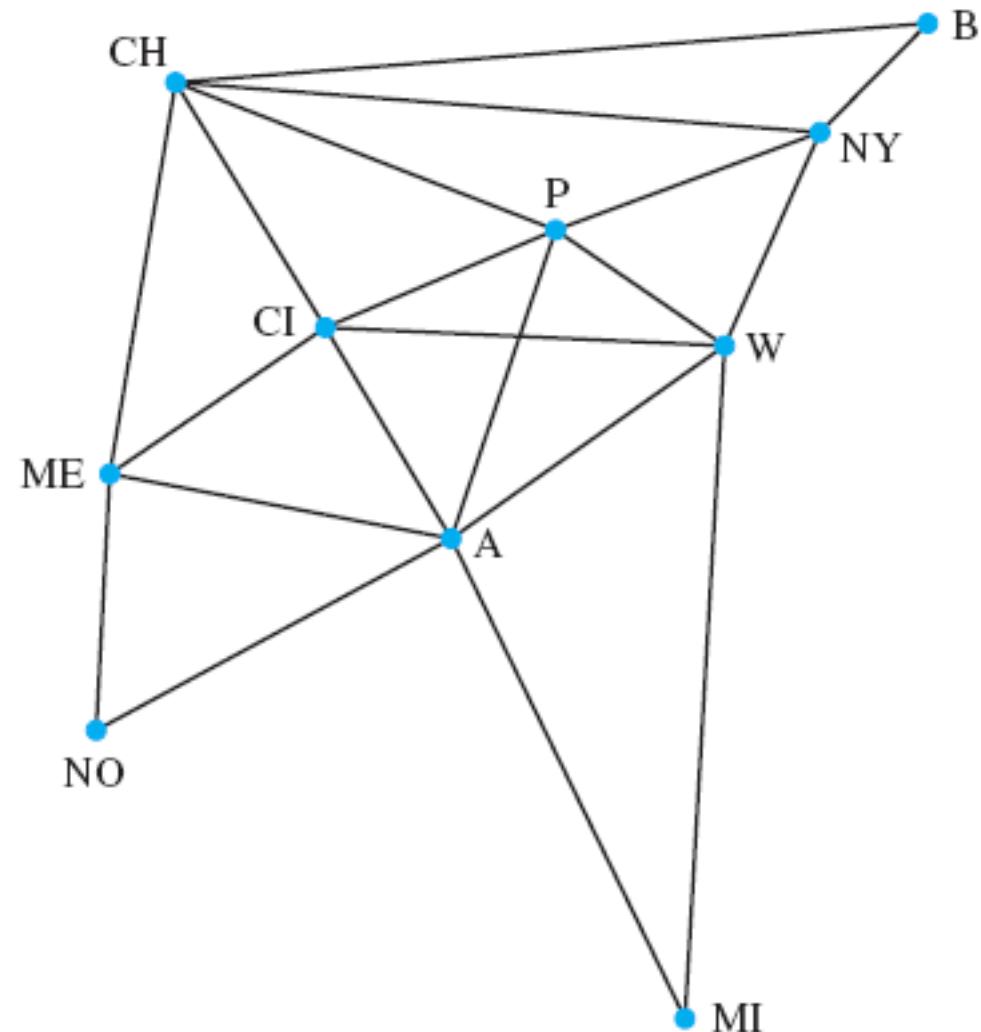
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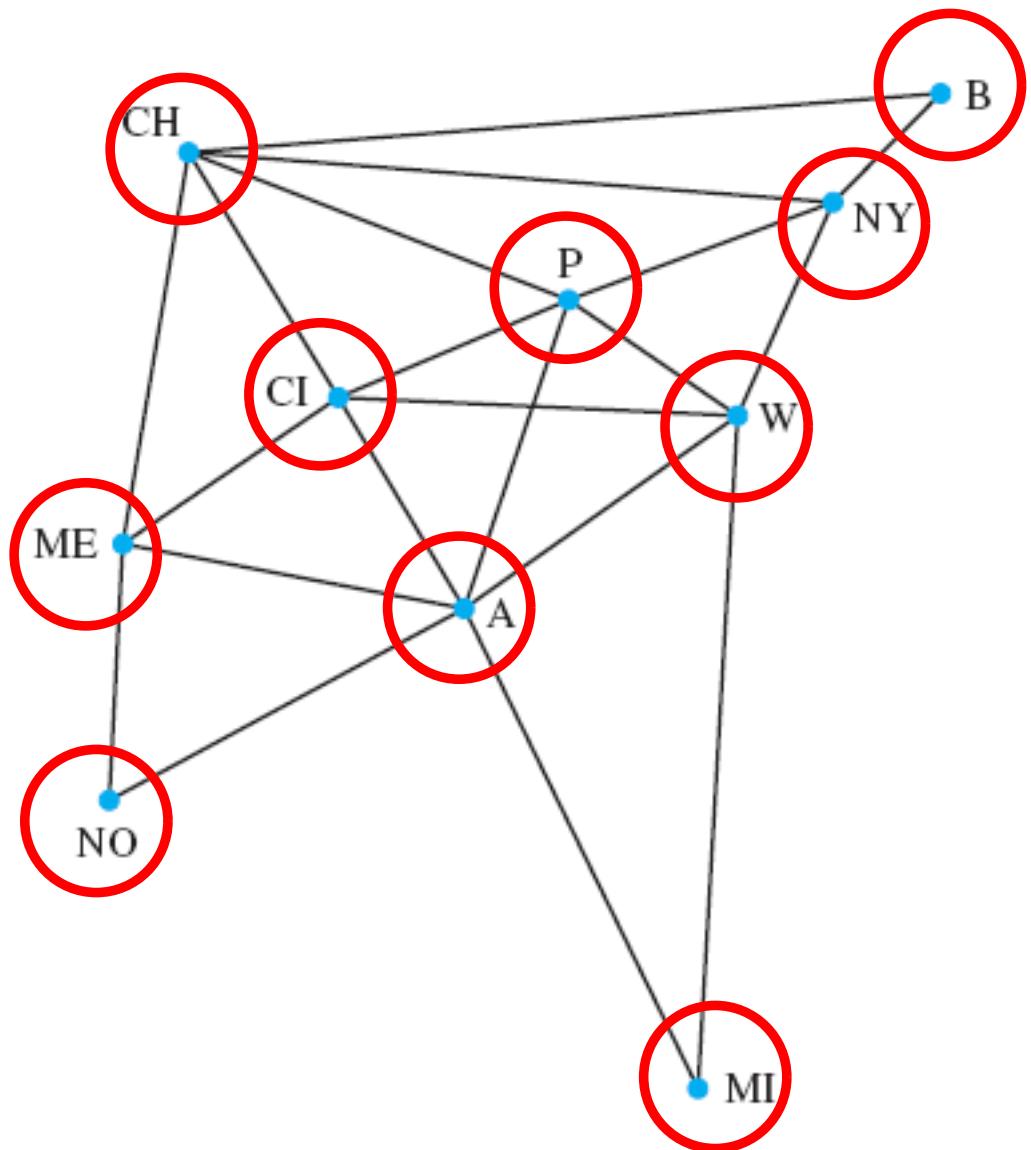
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20 links

Graph G

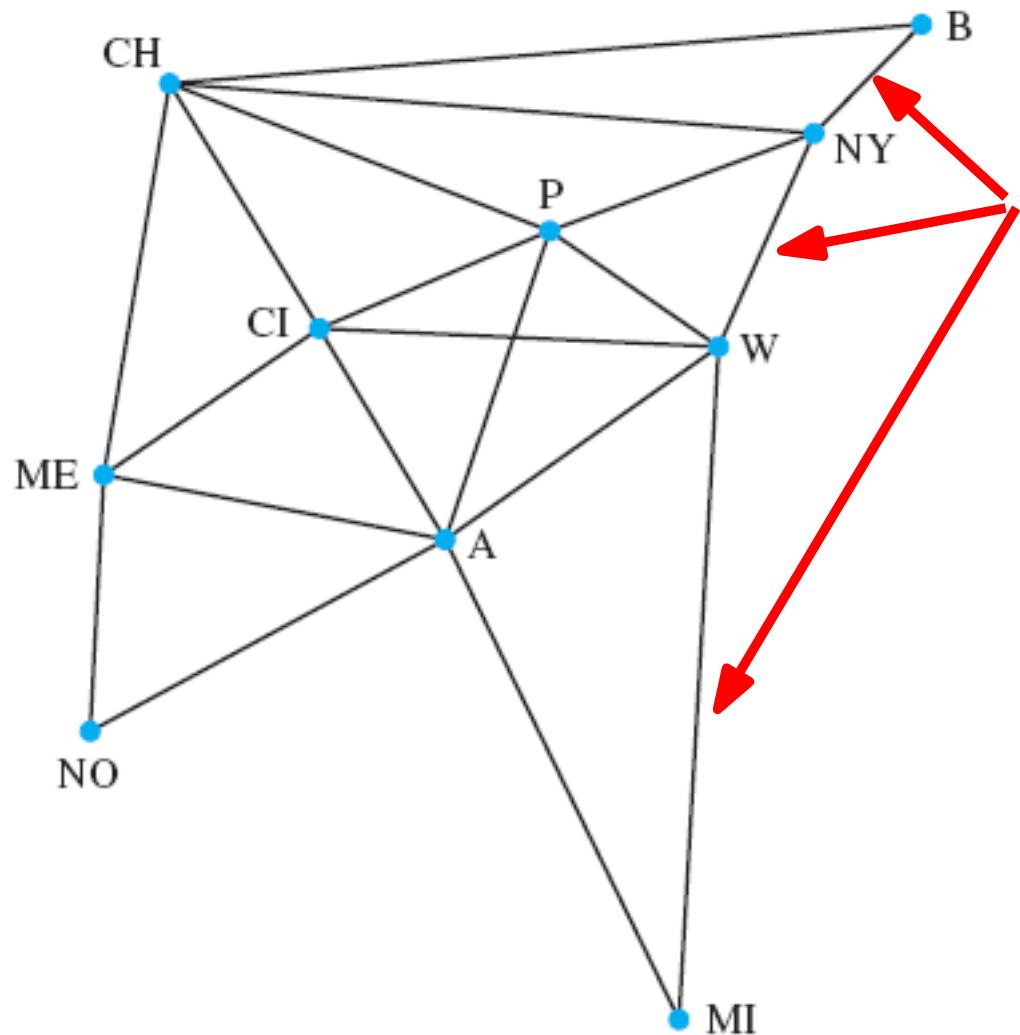


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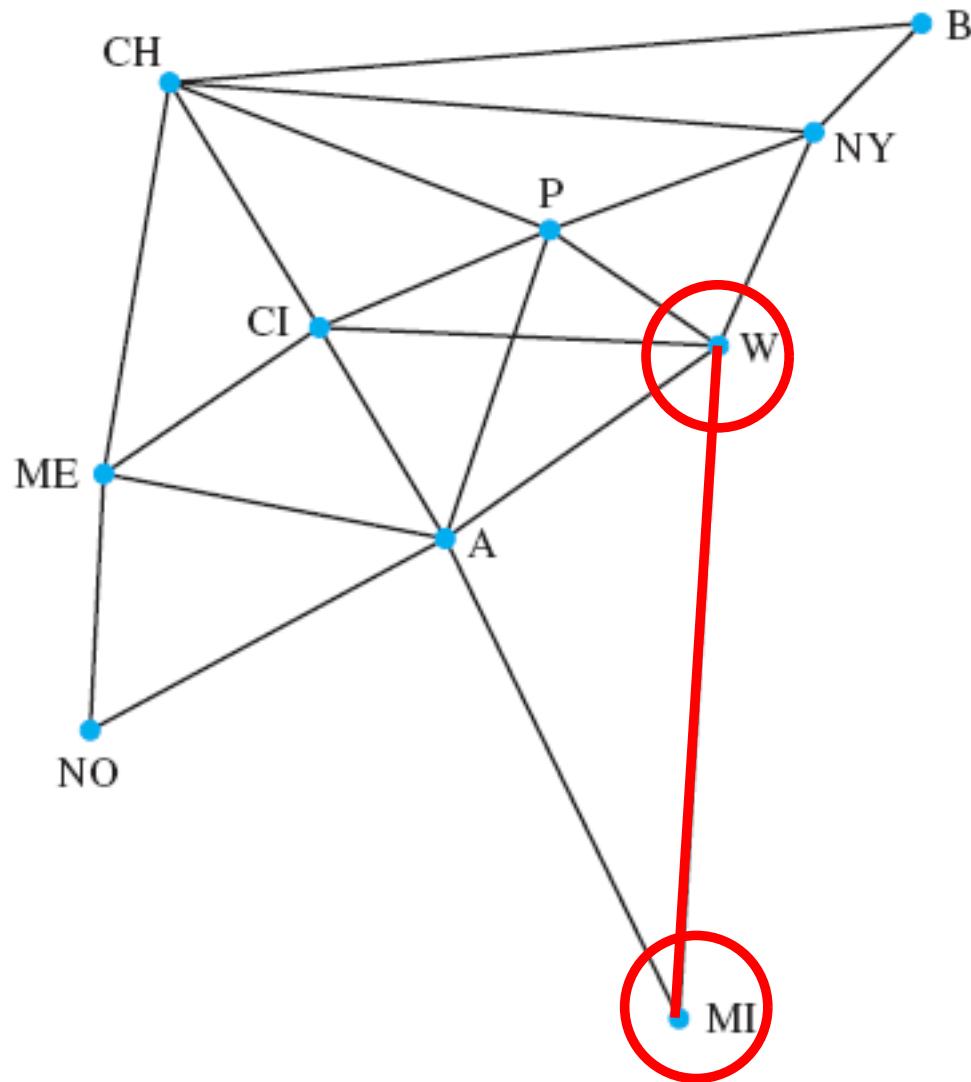
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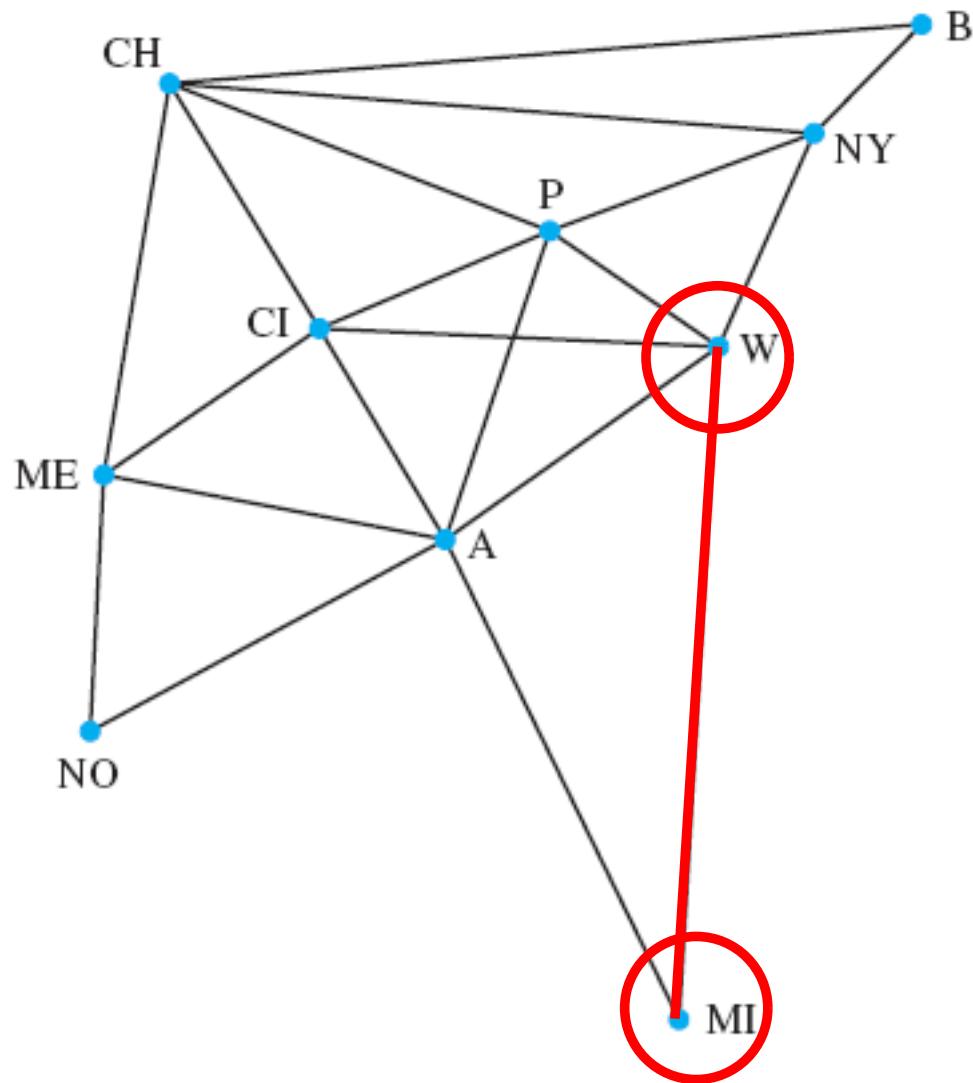


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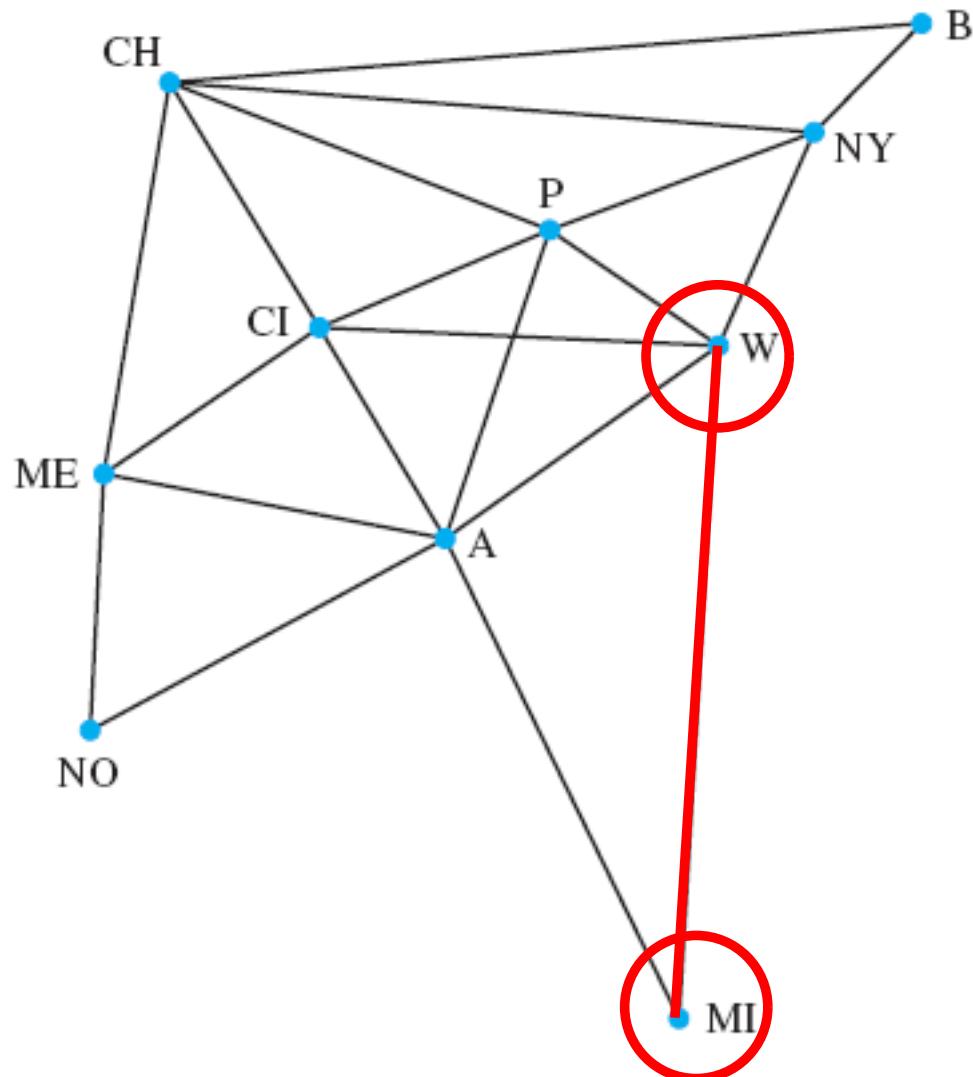
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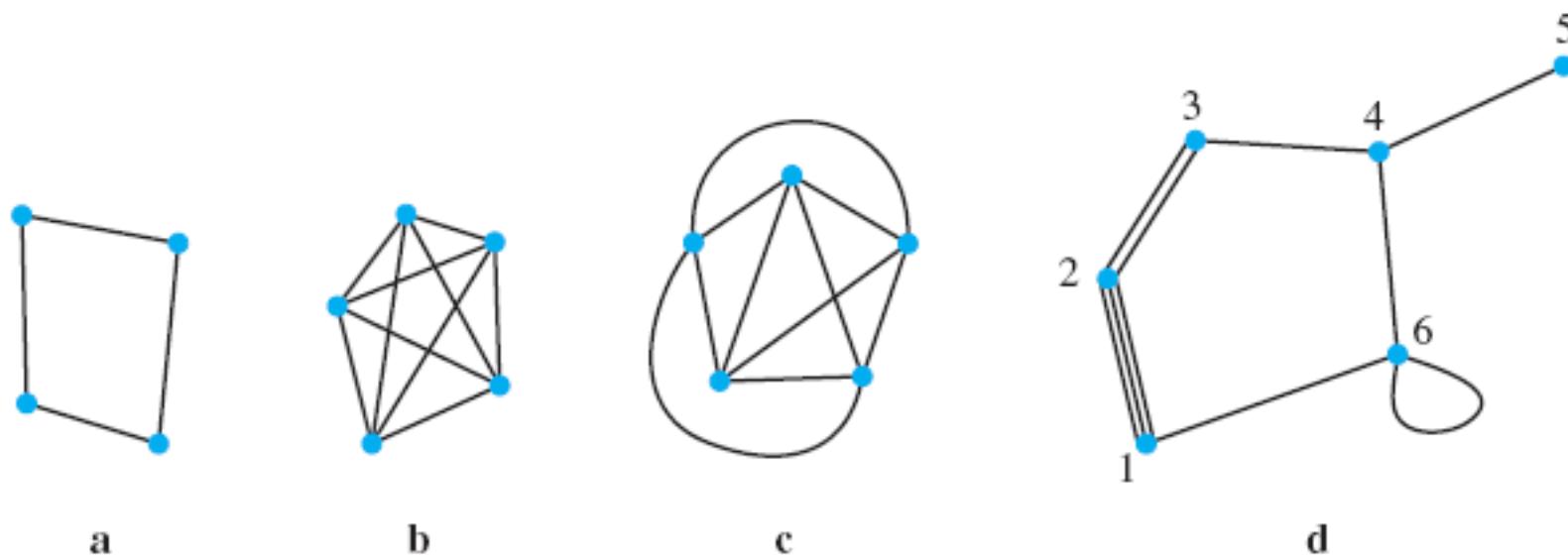
When a vertex is an
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- *Complete graph* K_n

A graph with n vertices that has an edge between **each pair** of vertices

Graphs

- Graphs and graph theory can be used to model:
 - ◊ Computer networks
 - ◊ Social networks
 - ◊ Communication networks
 - ◊ Information networks
 - ◊ Software design
 - ◊ Transportation networks
 - ◊ Biological networks

Graph Models

- Computer Networks

Vertices: computers

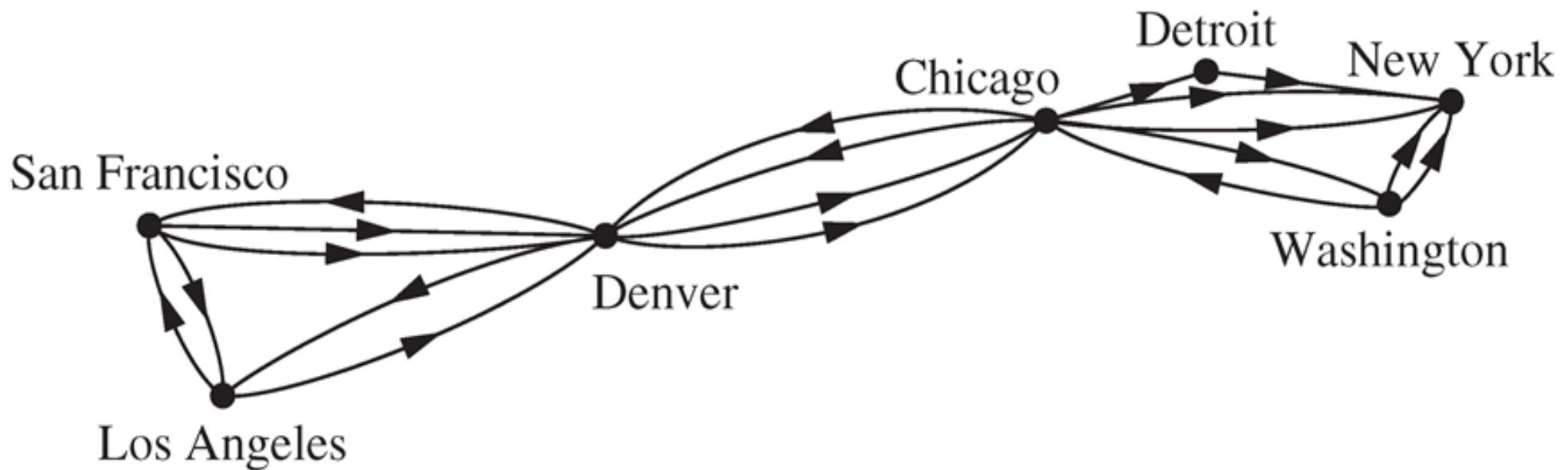
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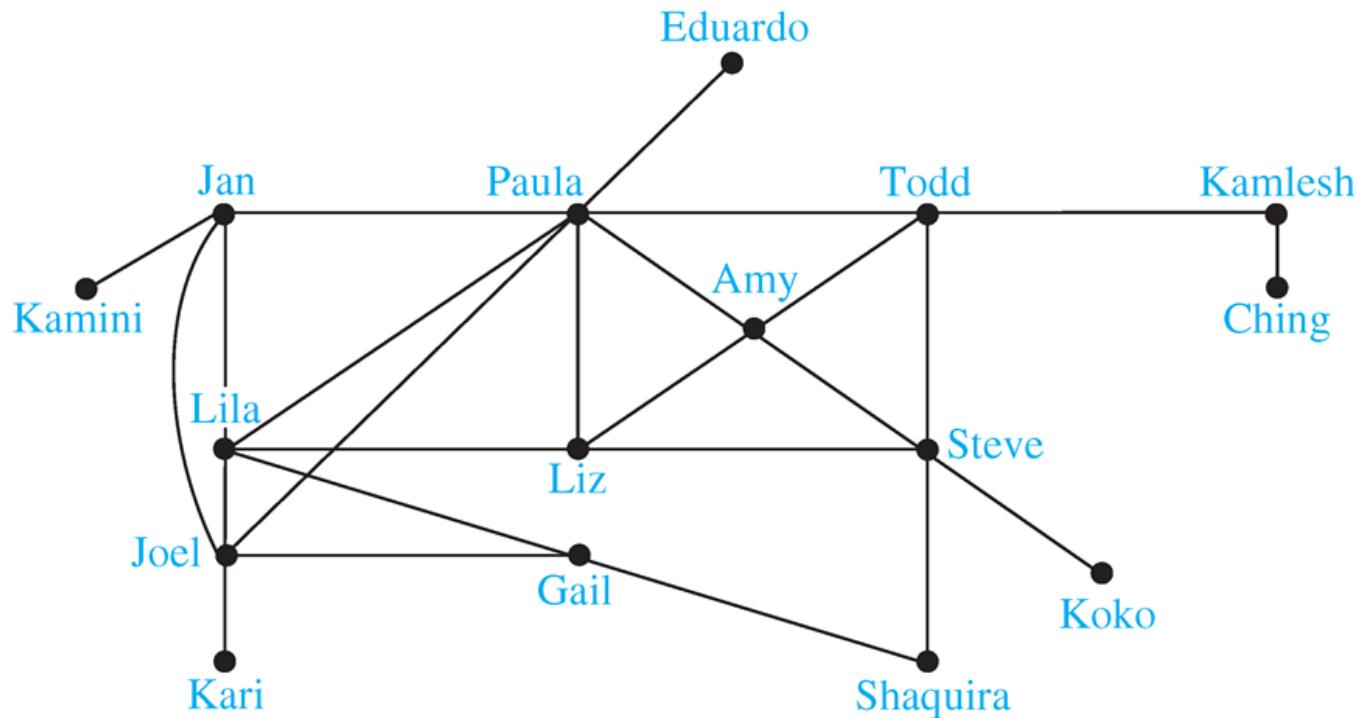
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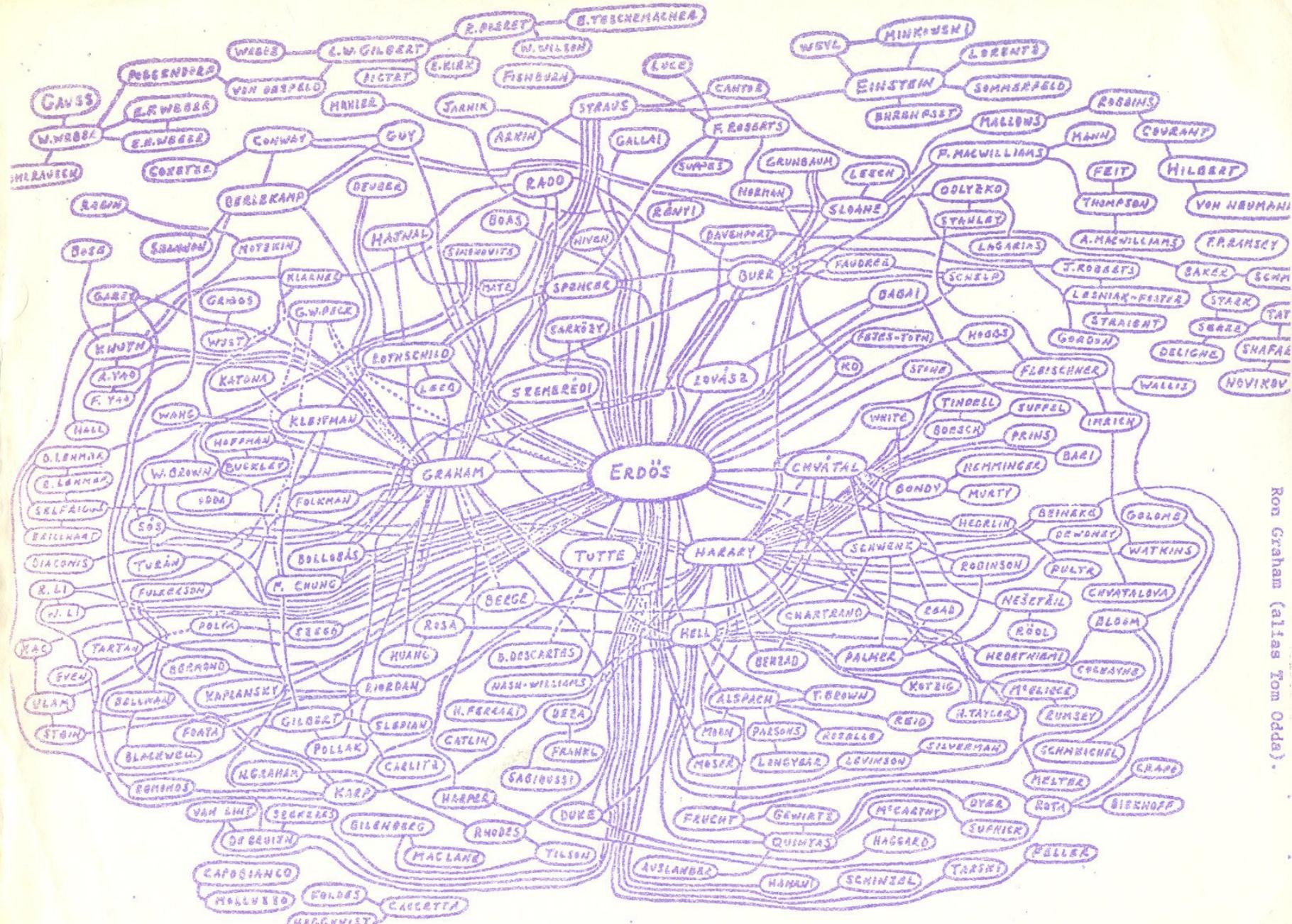
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Example

- the Hollywood graph

- the Erdős number

The Erdős Number

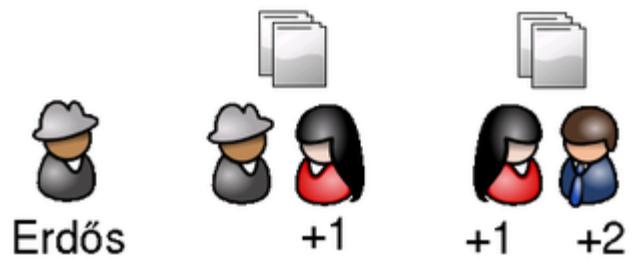


Ron Graham (alias Tom Odda).

Figure 1

To appear in Topics in Graph Theory (F. Harary, ed.), New York Academy of Sciences (1979).

The Erdős Number



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Erdős number 0	---	1 person
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Statistics on Mathematical Collaboration, 1903-2016

◆	#Laureates ◆	#Erdős ◆	%Erdős ◆	Min ◆	Max ◆	Average ◆	Median ◆
Fields Medal	56	56	100.0%	2	6	3.36	3
Nobel Economics	76	47	61.84%	2	8	4.11	4
Nobel Chemistry	172	42	24.42%	3	10	5.48	5
Nobel Medicine	210	58	27.62%	3	12	5.50	5
Nobel Physics	200	159	79.50%	2	12	5.63	5

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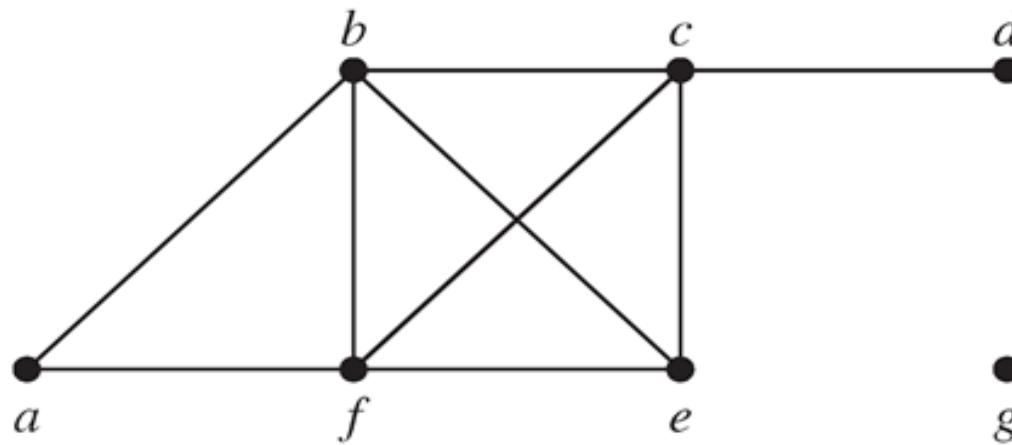
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Definition The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.



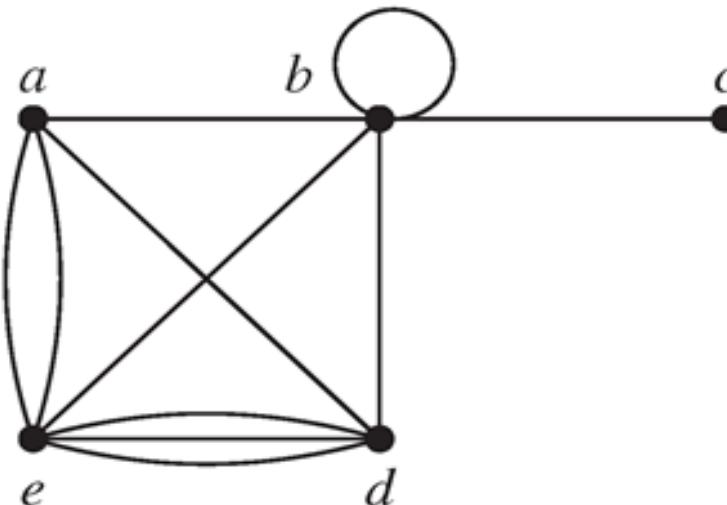
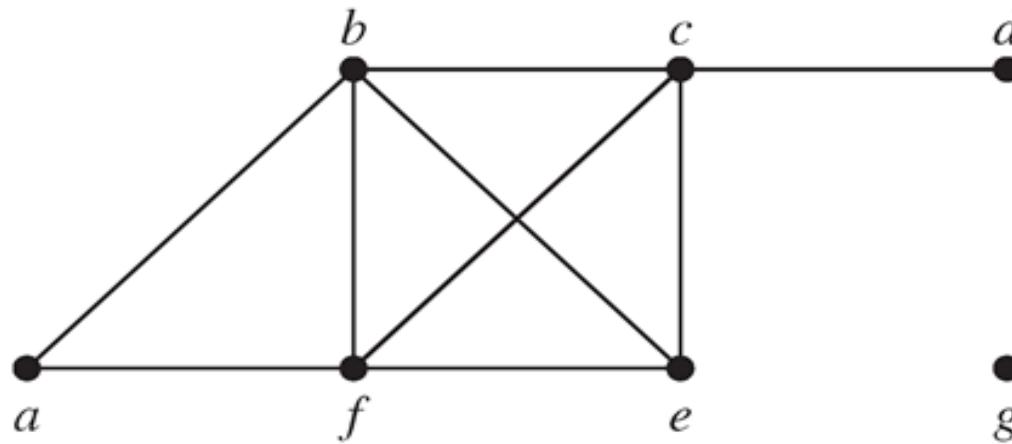
Undirected Graphs

- **Example:** What are the degrees and neighborhoods of the vertices in the graph G ?



Undirected Graphs

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Undirected Graphs

- **Theorem 1 (Handshaking Theorem)** If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof

Undirected Graphs

- **Theorem 2** An undirected graph has an even number of vertices of odd degree.

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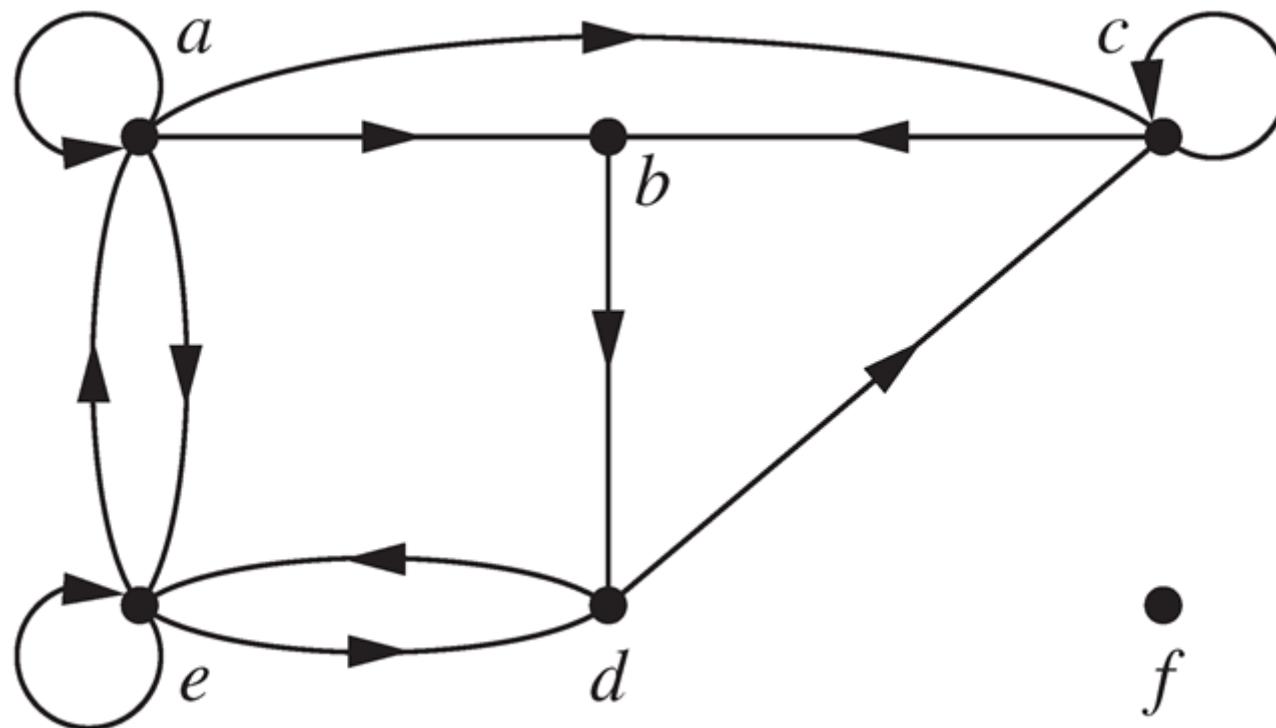
Definition Let (u, v) be an edge in G . Then u is the *initial vertex* of the edge and is *adjacent to v* and v is the *terminal vertex* of this edge and is *adjacent from u* . The initial and terminal vertices of a loop are the same.

Directed Graphs

- **Definition** The *in-degree* of a vertex v , denoted by $\deg^-(v)$, is the number of edges which terminate at v . The *out-degree* of v , denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. Note that a **loop** at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

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Directed Graphs

- **Theorem 3** Let $G = (V, E)$ be a graph with directed edges. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

Proof



Complete Graphs

- A *complete graph* on n vertices, denoted by K_n , is the simple graph that contains exactly one edge between **each pair** of distinct vertices.

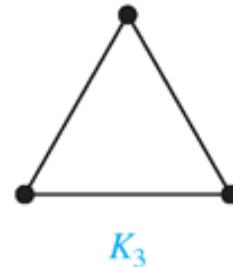
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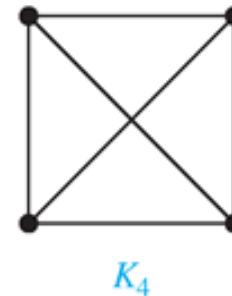
K_1



K_2



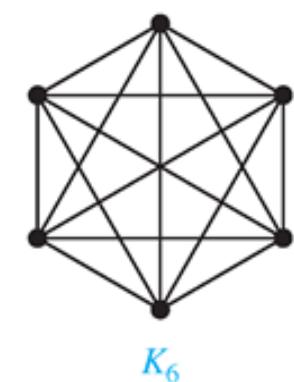
K_3



K_4



K_5



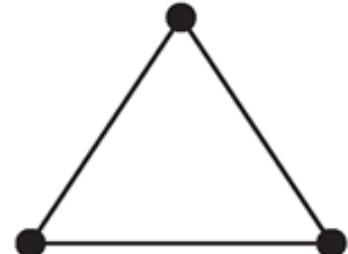
K_6

Cycles

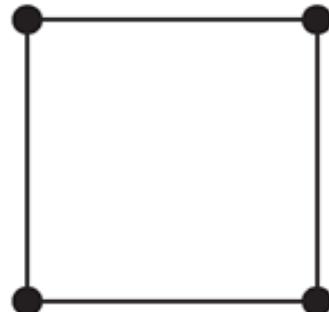
- A *cycle* C_n for $n \geq 3$ consists of n vertices v_1, v_2, \dots, v_n , and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

Cycles

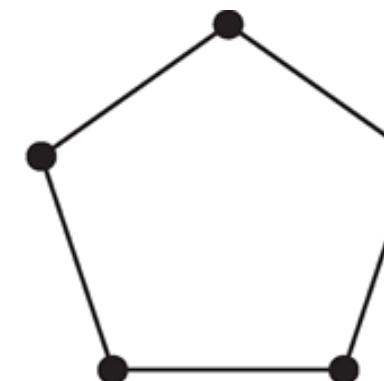
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C_3



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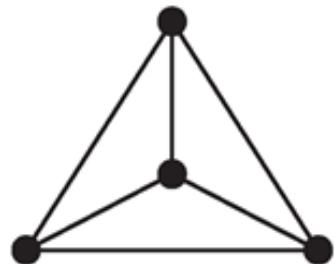
C_6

Wheels

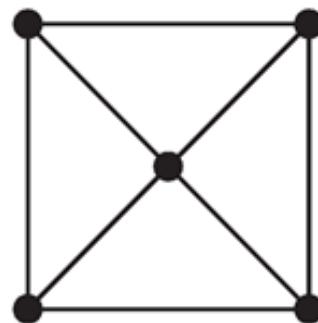
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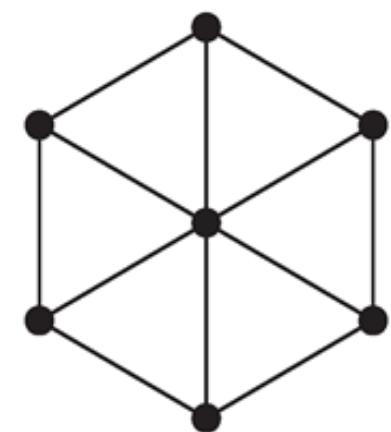
W_3



W_4



W_5



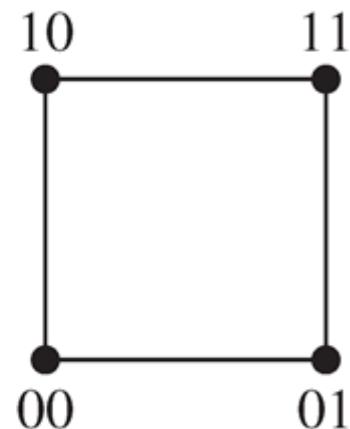
W_6

N -dimensional Hypercube

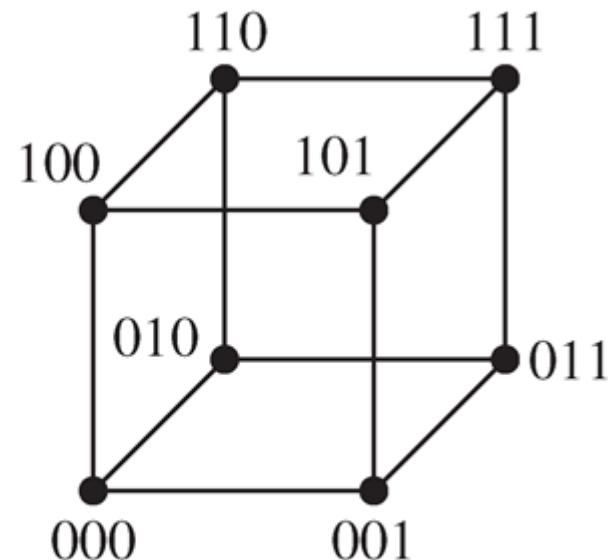
- An *n-dimensional hypercube*, or *n-cube*, Q_n is a graph with 2^n vertices representing all bit strings of length n , where there is an edge between two vertices that differ in exactly one bit position.

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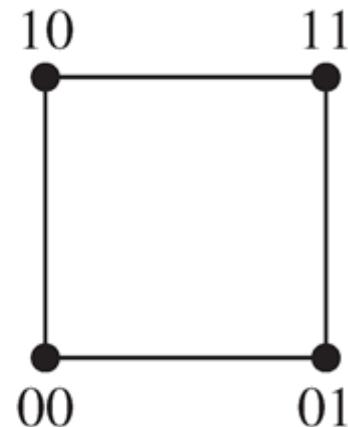
Q_1



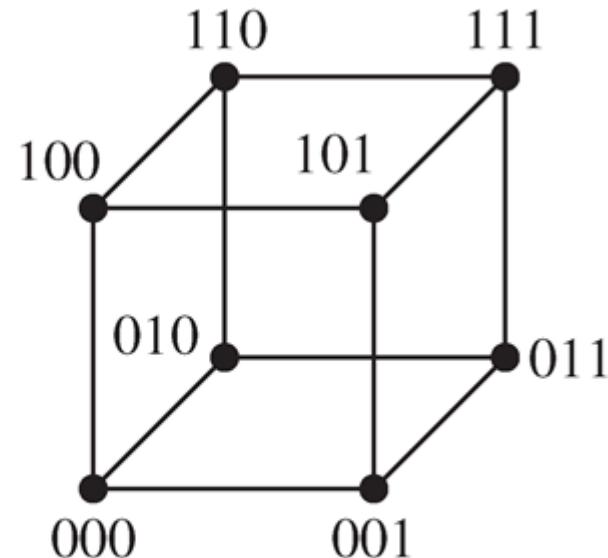
Q_3

N -dimensional Hypercube

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Q_1



Q_3

How many vertices? How many edges?

Bipartite Graphs

- **Definition** A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2 .

Bipartite Graphs

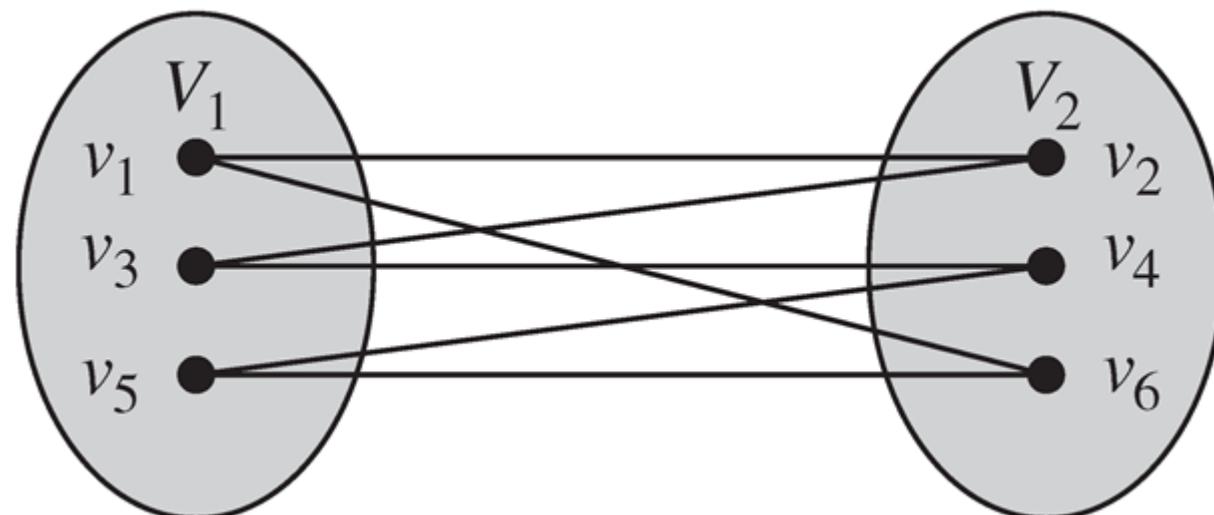
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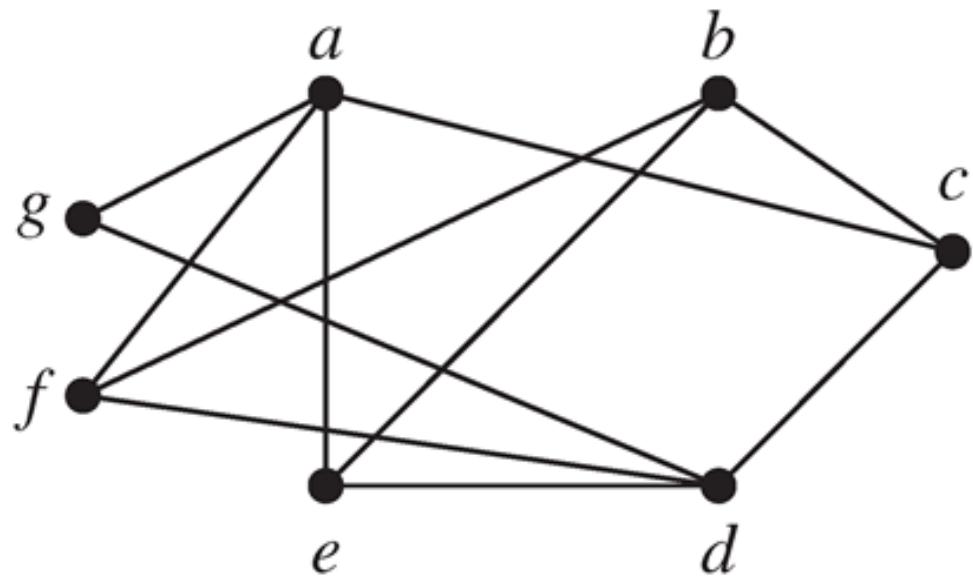
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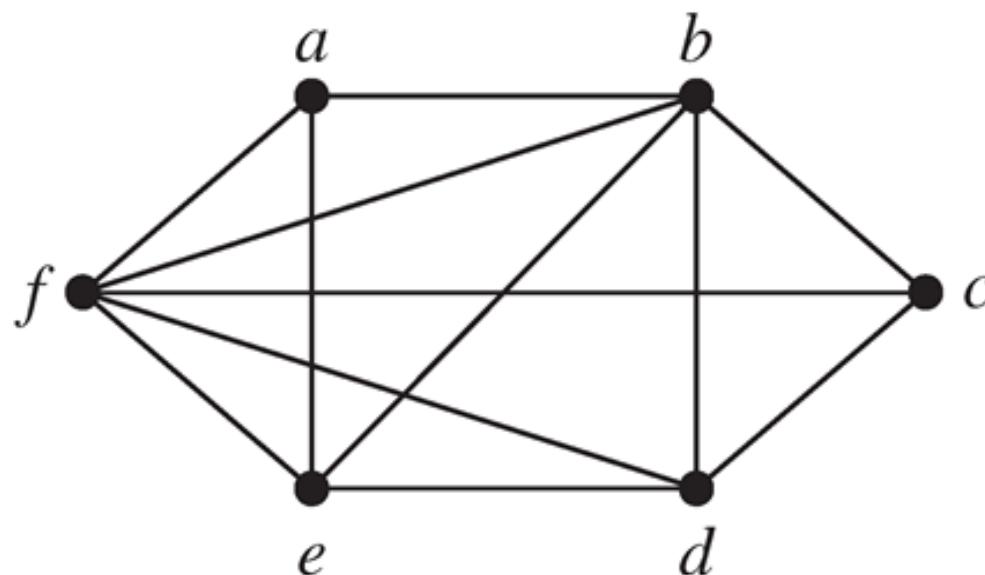
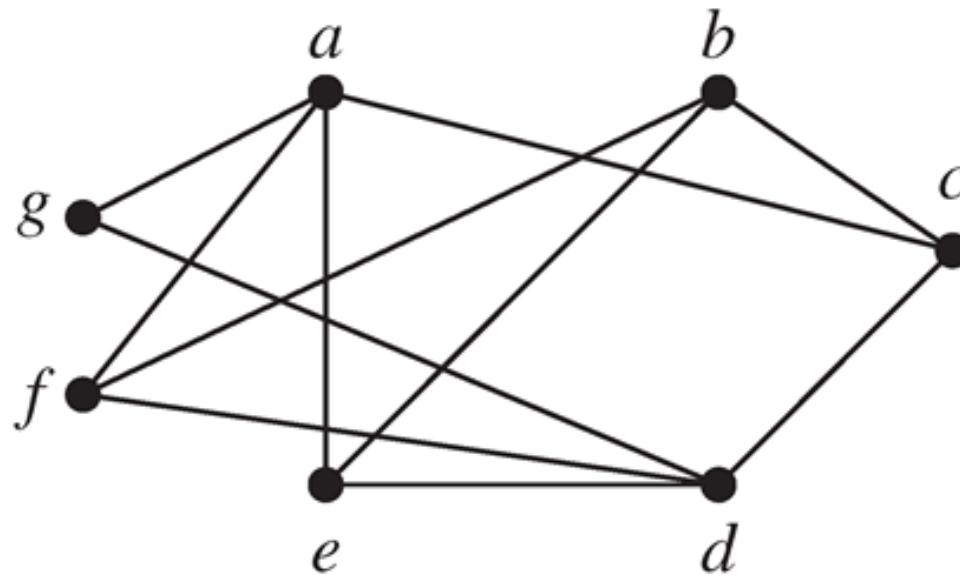
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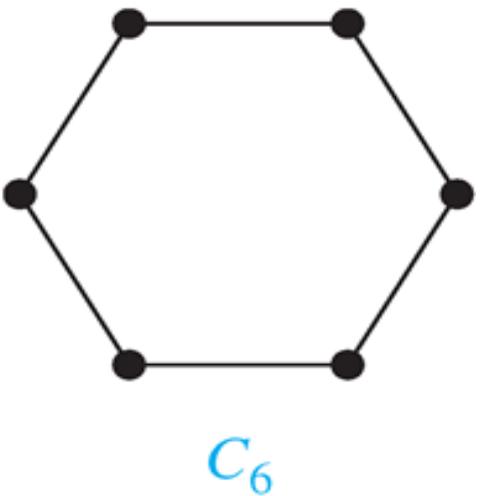


Bipartite Graphs



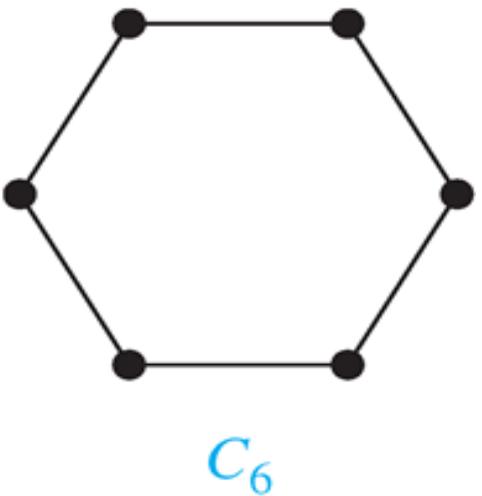
Bipartite Graphs

- **Example** Show that C_6 is bipartite.

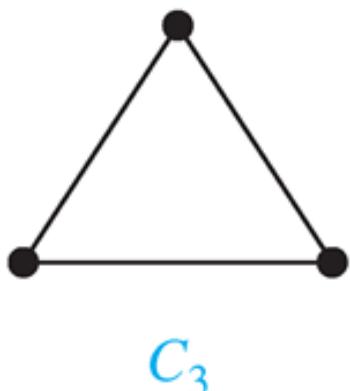


Bipartite Graphs

- **Example** Show that C_6 is bipartite.



- **Example** Show that C_3 is not bipartite.



Complete Bipartite Graphs

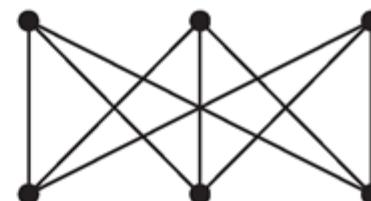
- **Definition** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2 .

Complete Bipartite Graphs

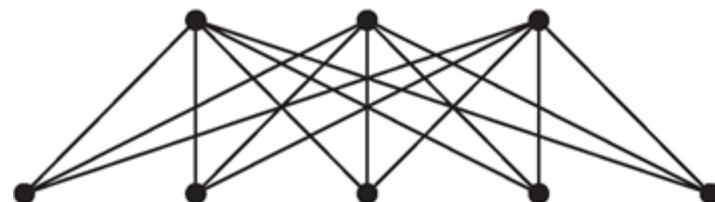
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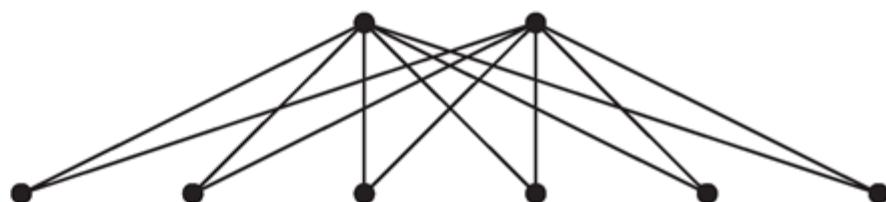
$K_{2,3}$



$K_{3,3}$



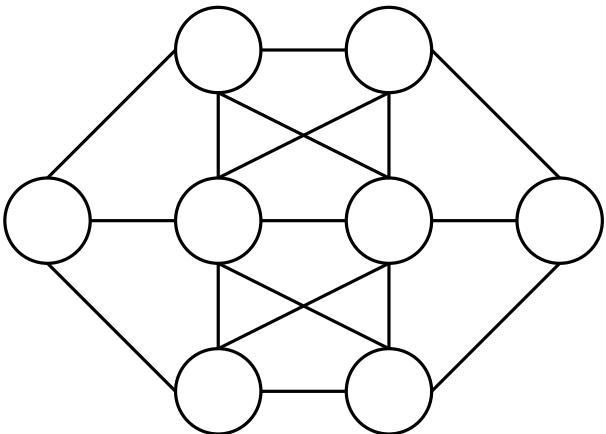
$K_{3,5}$



$K_{2,6}$

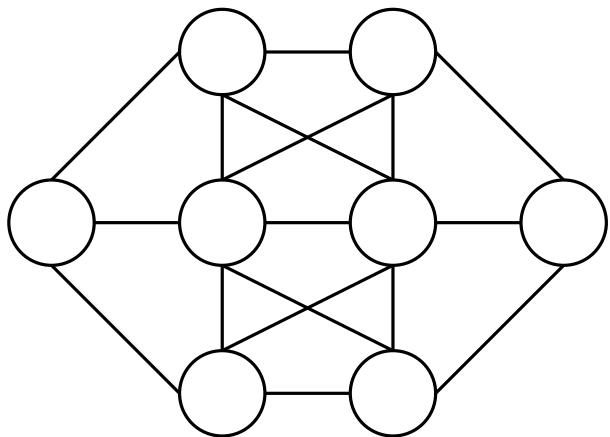
Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that **no** letter is adjacent to a letter that is next to it in the alphabet.



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- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

Next Lecture

- graph ...

