南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS215 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Graph Concepts

- $G = (V, E)$, *simple* graph, *multigraph*, *pseudograph*
- *Undirected*, *directed* graph
- Special graphs

  K_n, C_n, W_n, Q_n, $K_{m,n}$

  Hall's Marriage Theorem on *bipartite* graphs

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  Hall's Marriage Theorem on *bipartite* graphs
- Representation of graphs

  adjacency list, *adjacency matrix*, *incidence matrix*

Isomorphism of Graphs

- **Definition** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an *isomorphism*.

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- Useful graph invariants include the number of vertices, number of edges, degree sequence, etc.

Paths and Isomorphism

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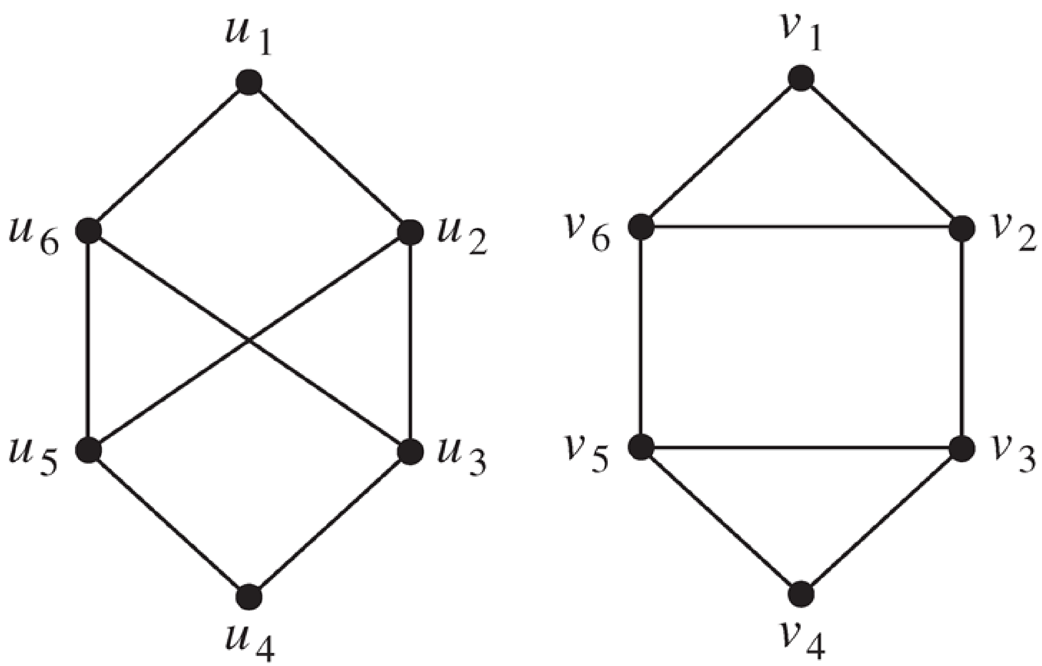

G H

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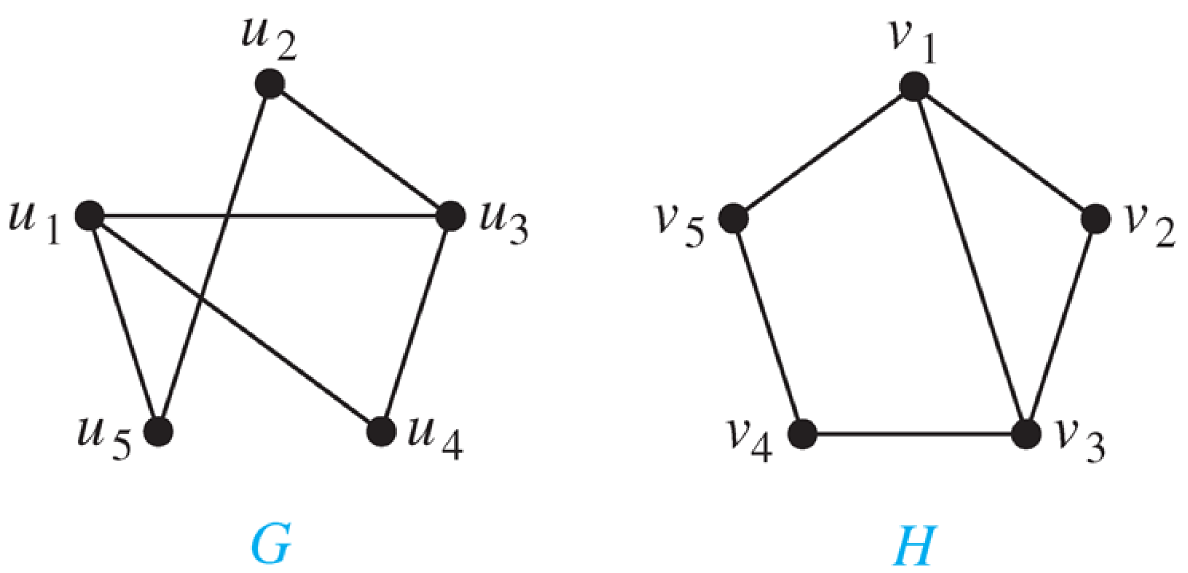

Counting Paths between Vertices

- **Theorem** Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where $r > 0$ is positive, equals the (i,j)-th entry of $\mathbf{A}^r$.

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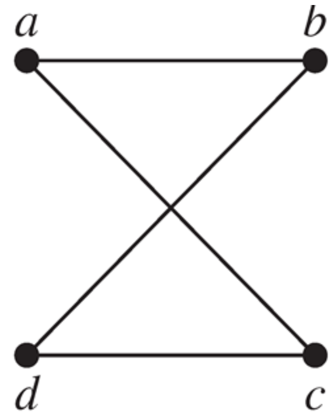
$\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$, the (i, j)-th entry of $\mathbf{A}^{r+1}$ equals $b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj}$, where b_{ik} is the (i, k)-th entry of $\mathbf{A}^r$.

Counting Paths between Vertices

- **Example** How many paths of length 4 are there from a to d in the graph G?



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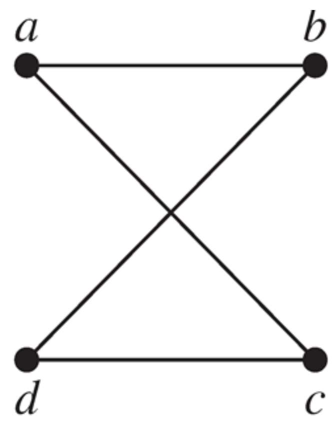

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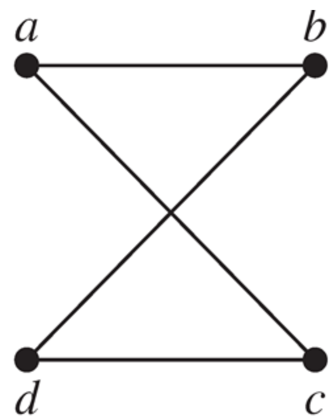

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- **Königsberg seven-bridge problem**

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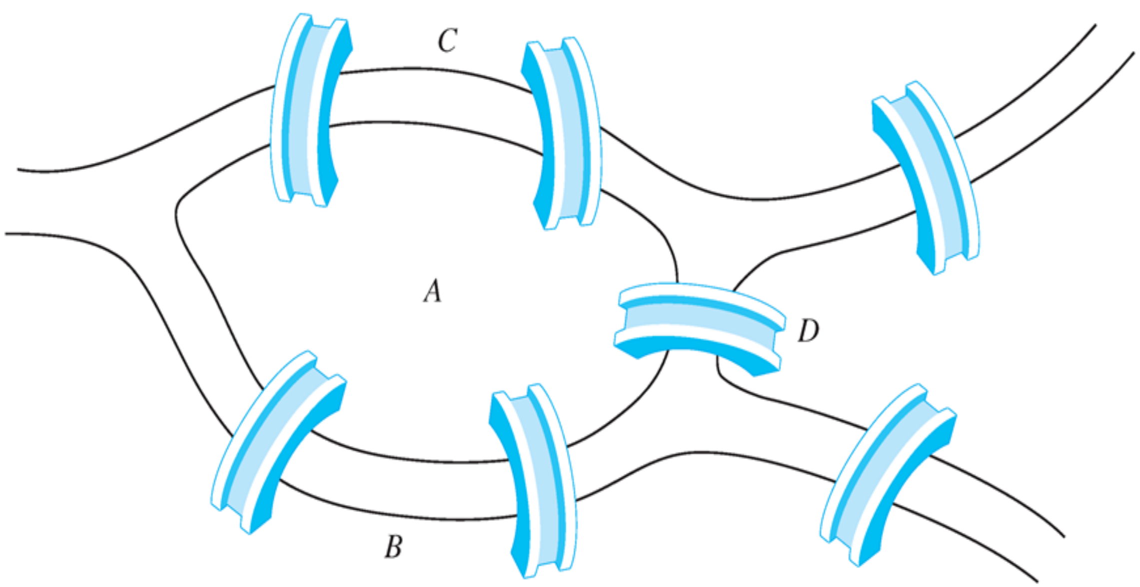

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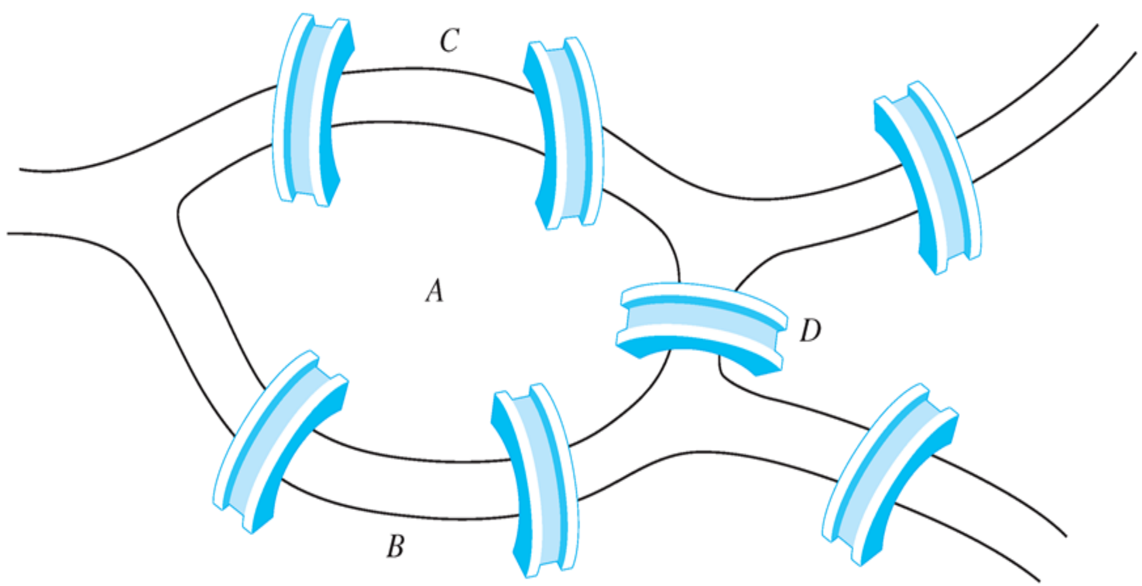

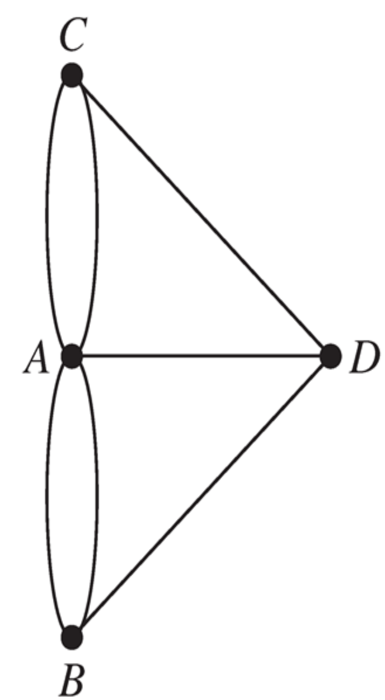

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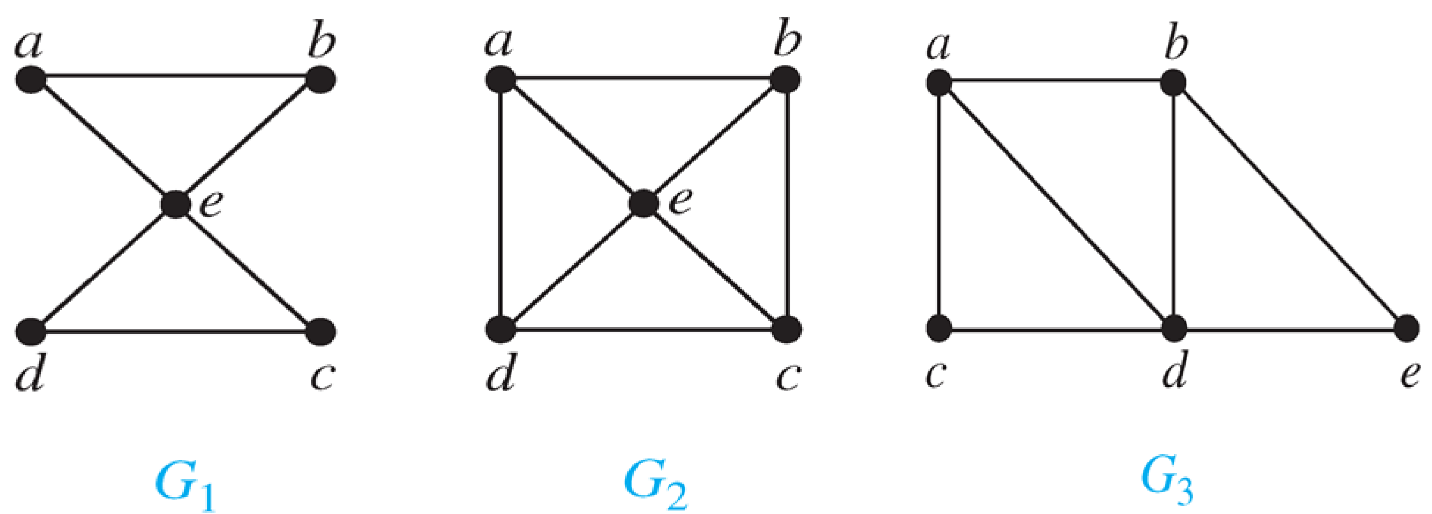

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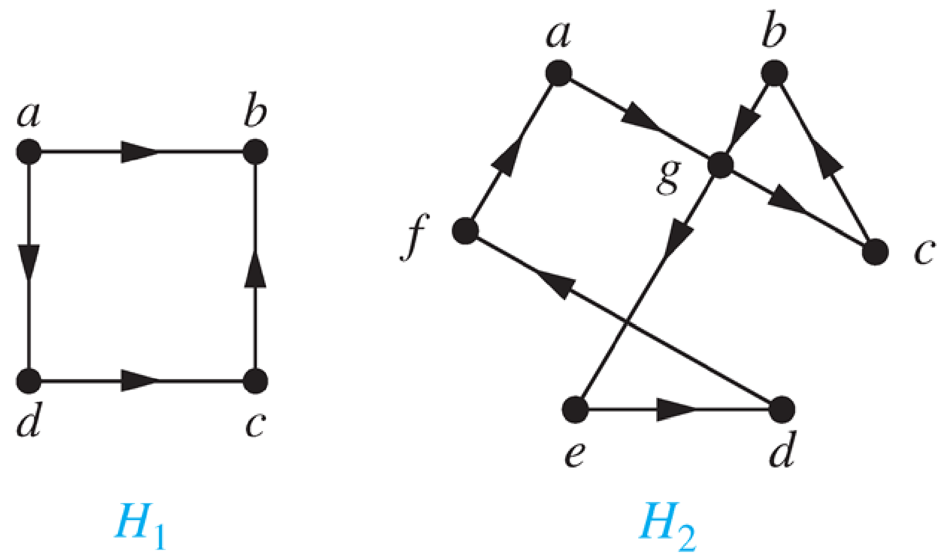

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  - The initial vertex and the final vertex of an Euler path have odd degree.

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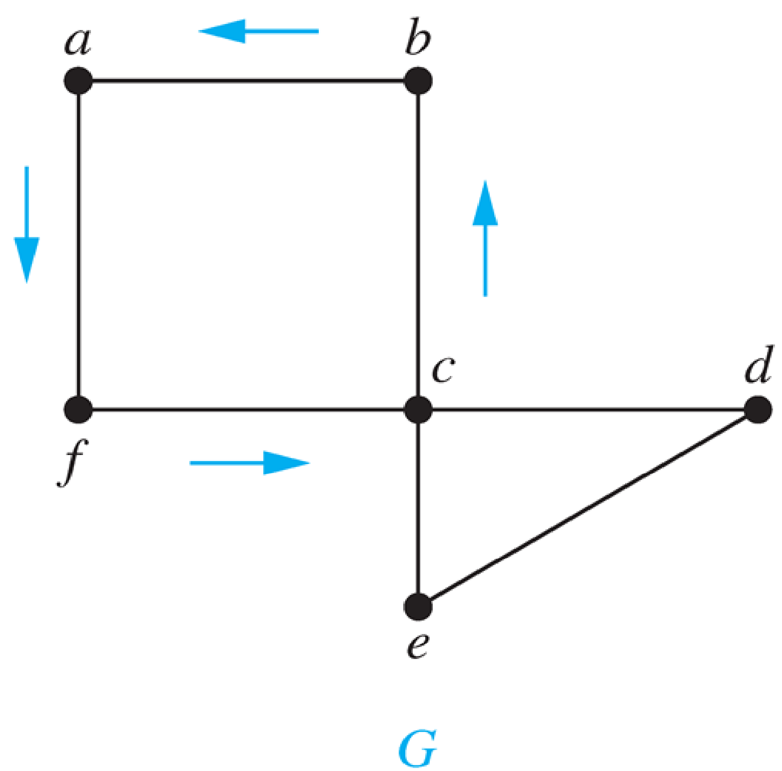

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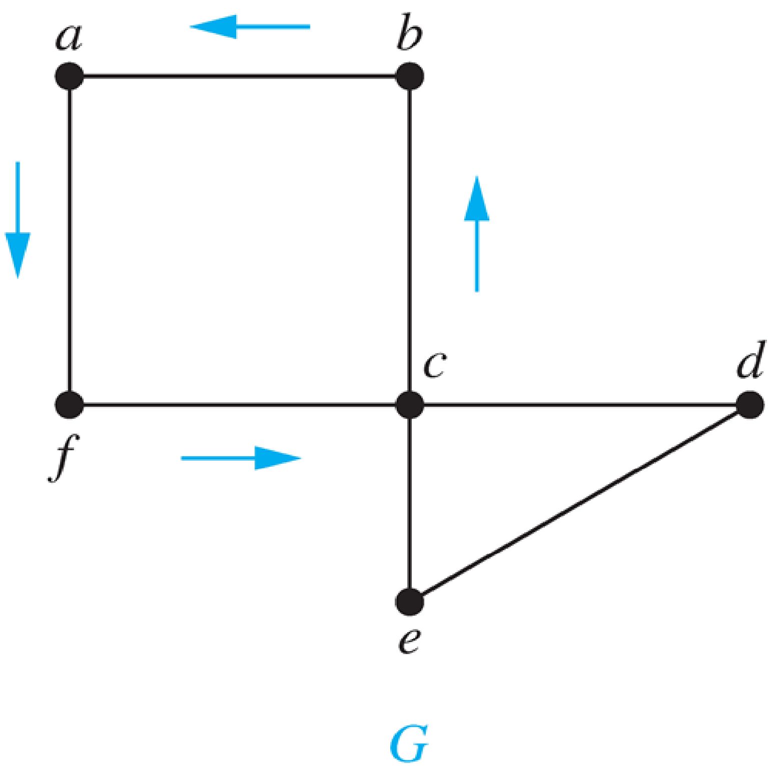

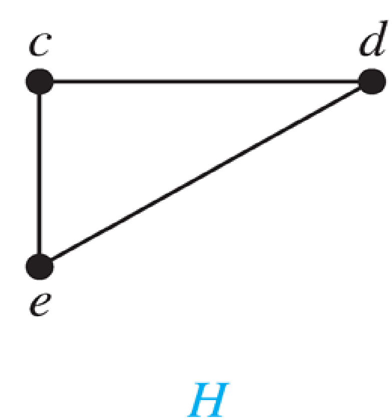

Algorithm for Constructing an Euler Circuit

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex.
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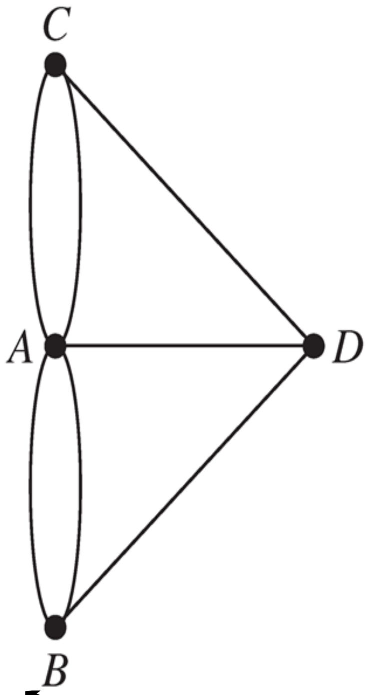

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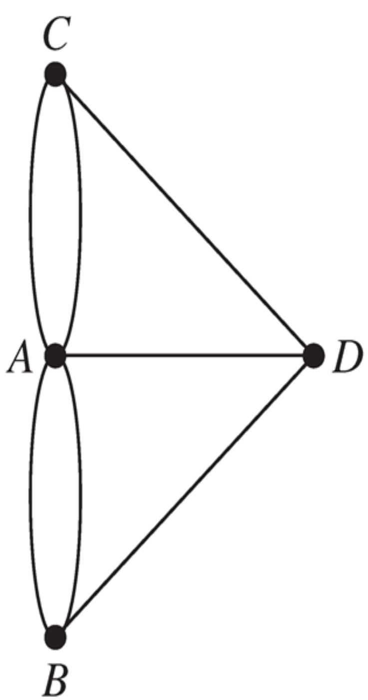

No Euler circuit

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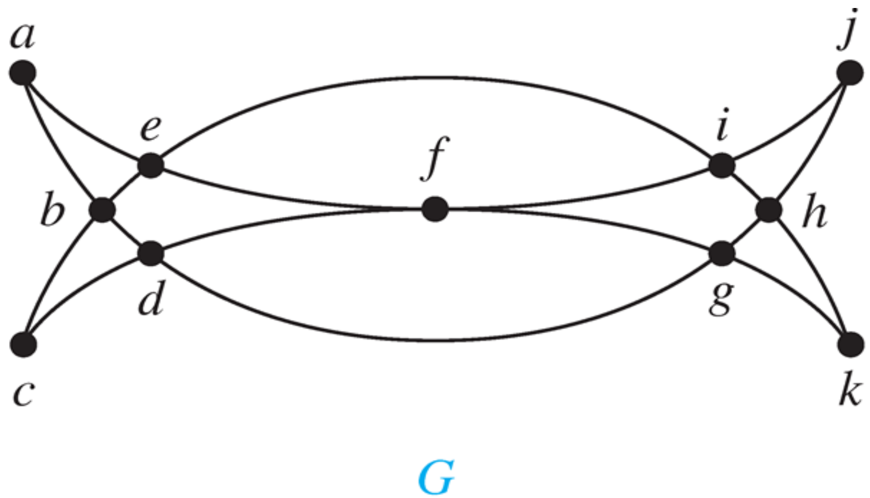

FIGURE 6 **Mohammed's Scimitars.**

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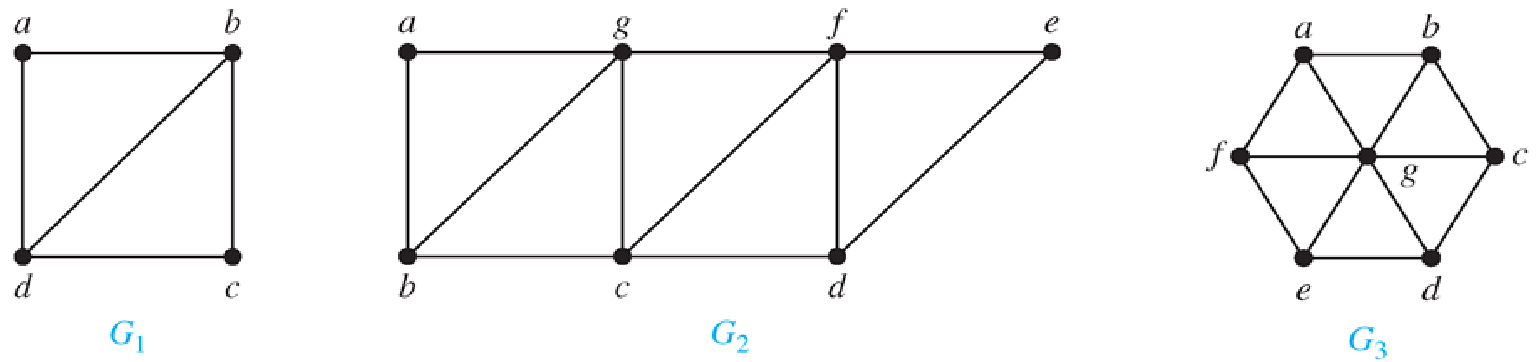

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- Finding a path or circuit that traverses each
  - ◇ street in a neighborhood
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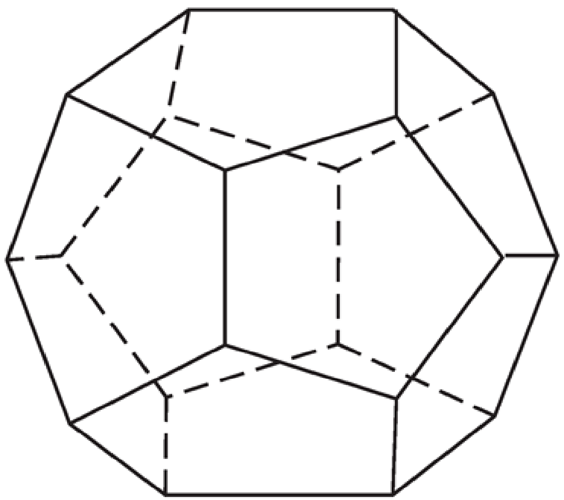

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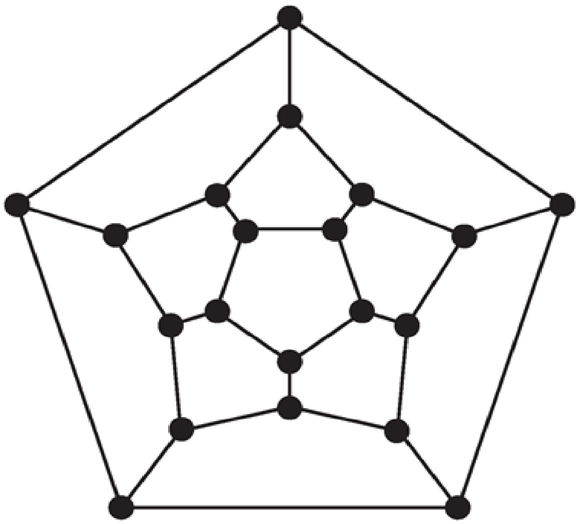

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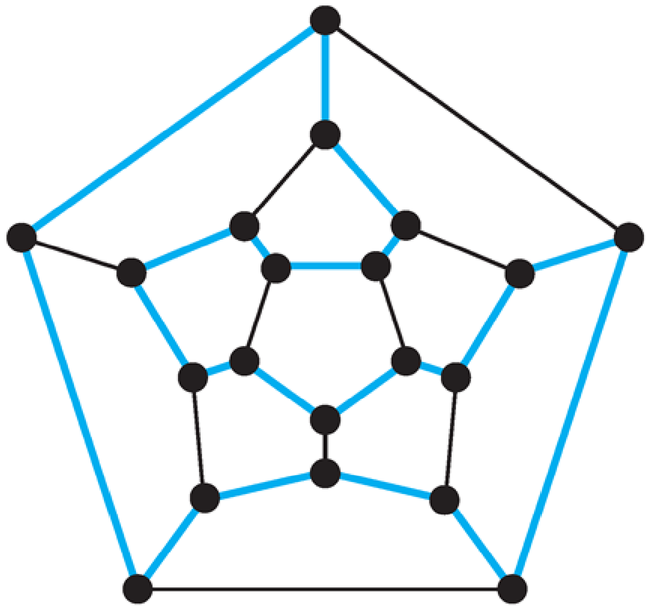

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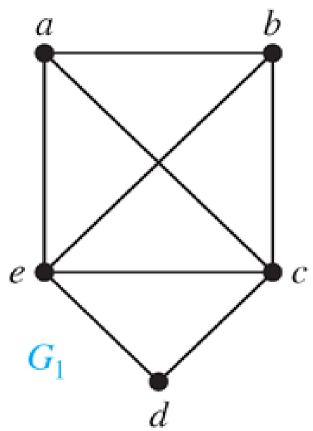
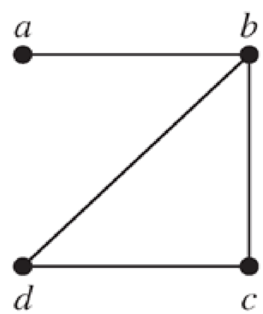


G_1



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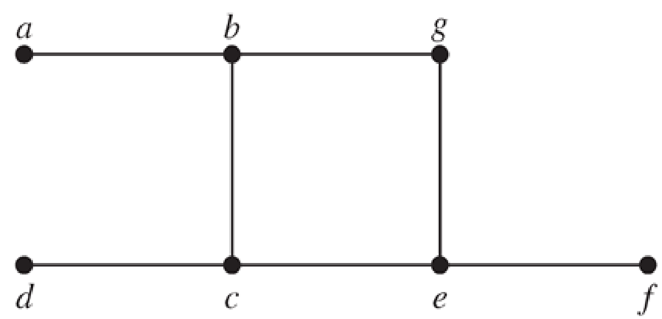

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  Hamilton path problem $\in$ NPC

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  the decision version of the TSP $\in$ NPC

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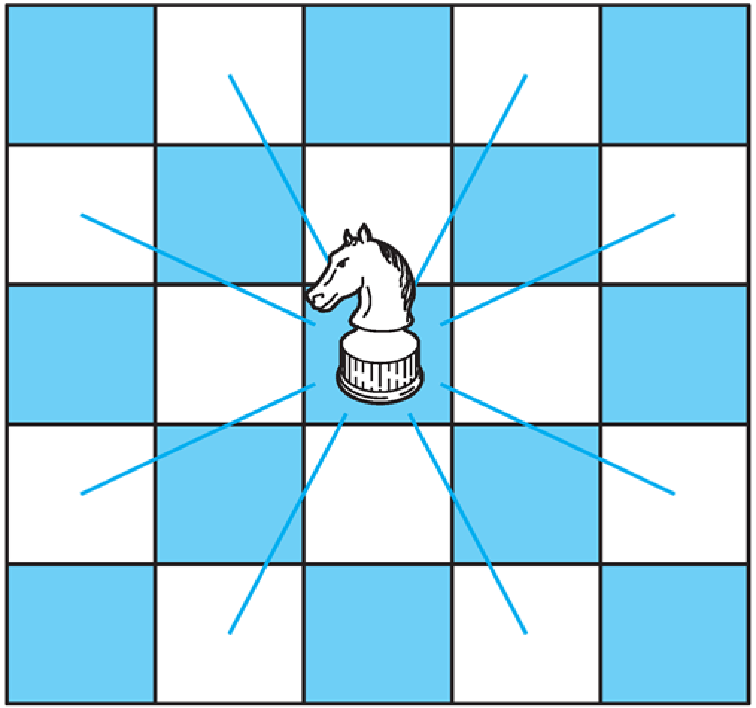

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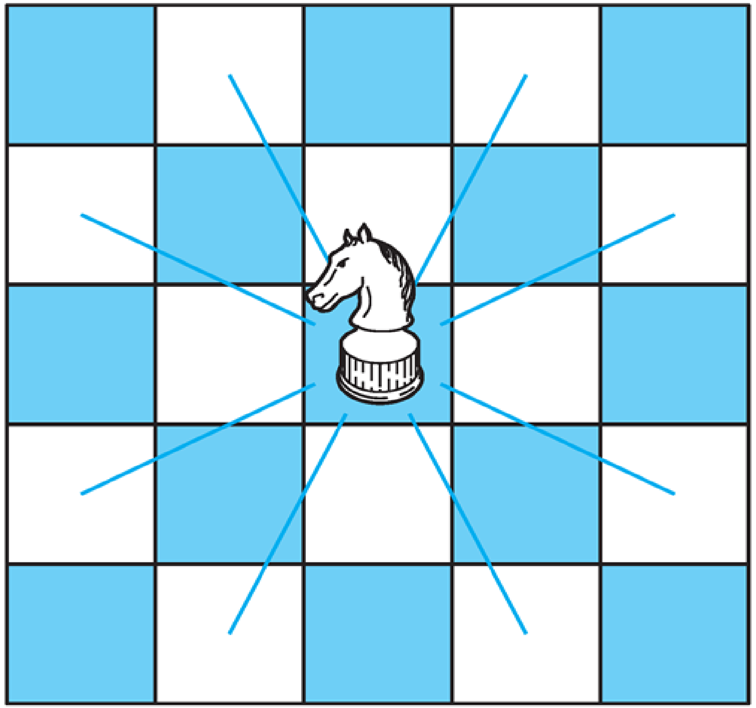

What about in 6×6 chessboard?

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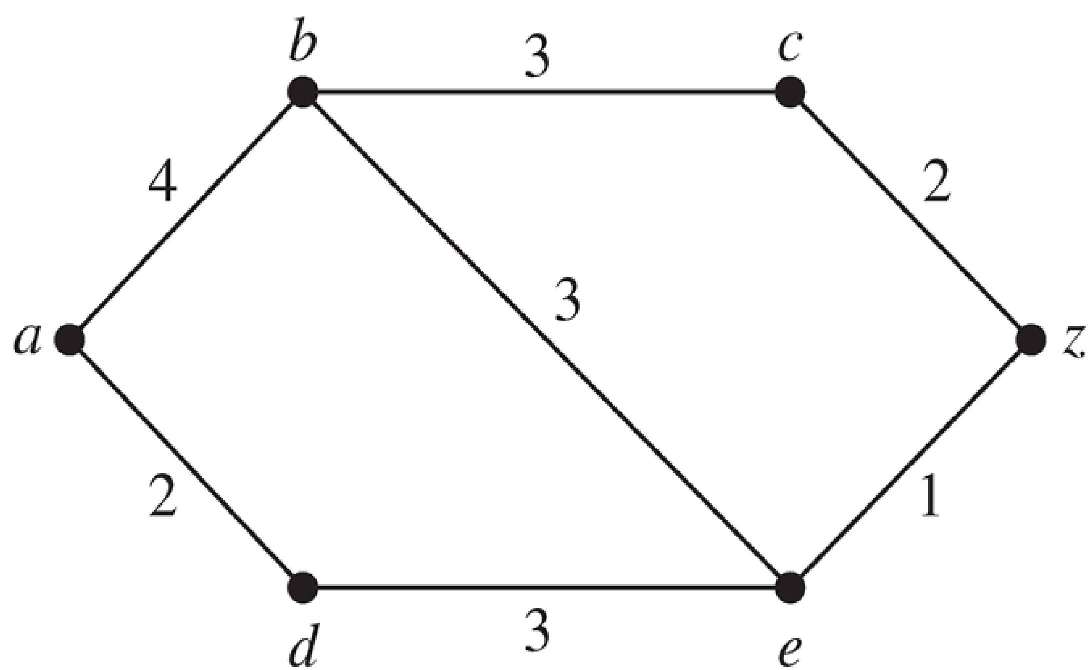

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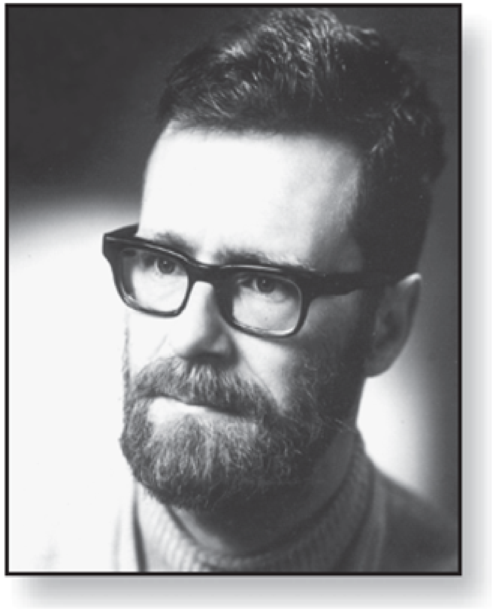

Edsger Wybe Dijkstra

Dijkstra's Algorithm

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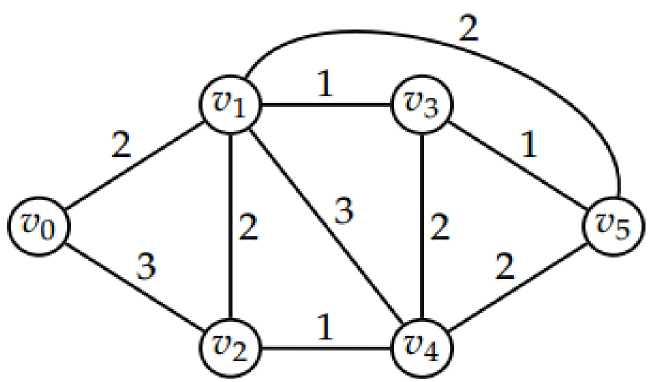

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  (ii) while $S \neq V$

  let $v \notin S$ be the vertex with the least value $d(v)$,

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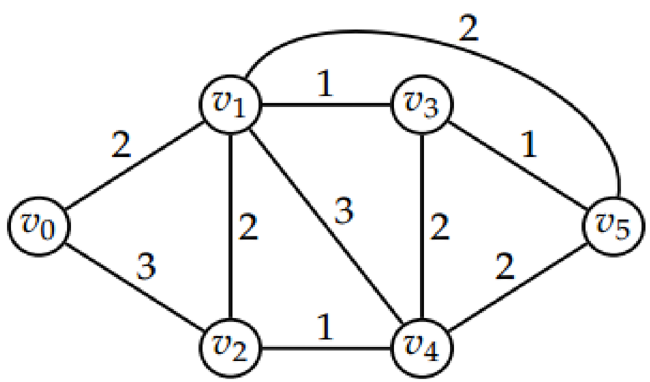
  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

  Example



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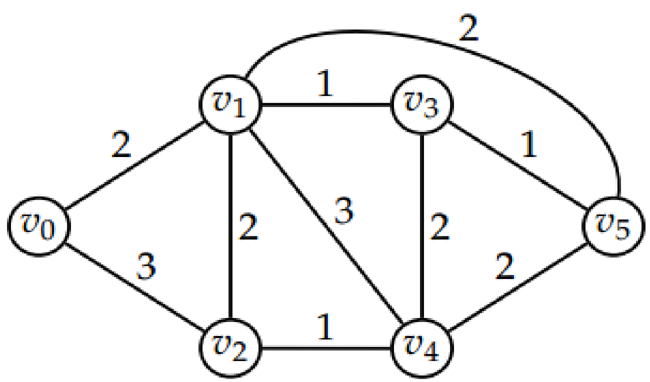

v_0	v_1	v_2	v_3	v_4	v_5
0	∞	∞	∞	∞	∞

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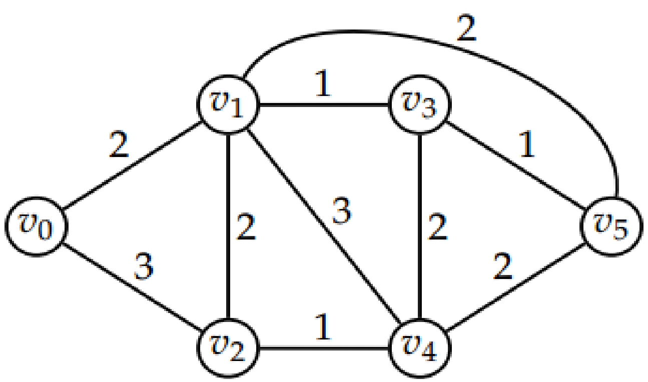

v_0	v_1	v_2	v_3	v_4	v_5
0	∞	∞	∞	∞	∞

$i = 0$

$d(v_1) = \min\{\infty, 2\} = 2$, $d(v_2) = \min\{\infty, 3\} = 3$

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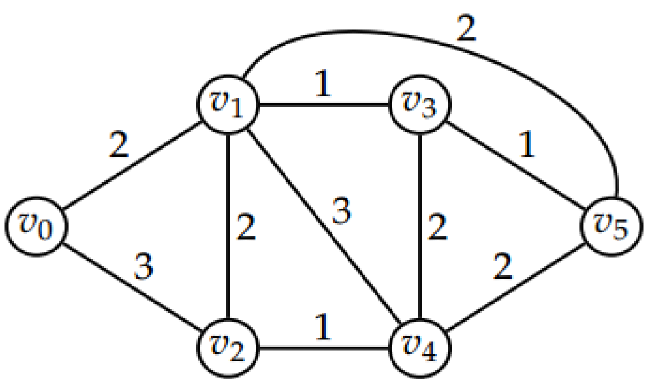

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	∞	∞	∞

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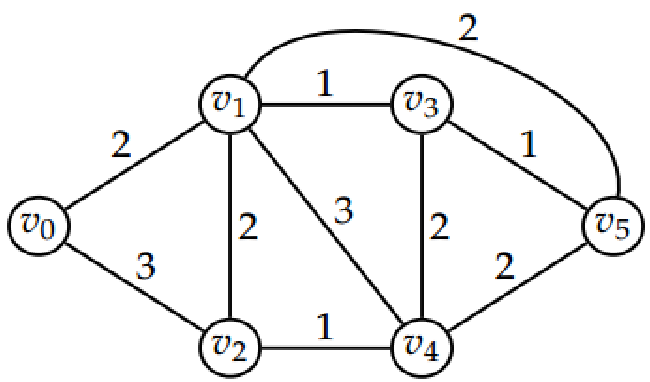

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	∞	∞	∞

$i = 1$

$d(v_2) = \min\{3, d(v_1) + \alpha(v_1 v_2)\} = \min\{3, 4\} = 3,$

$d(v_3) = 2 + 1 = 3,\ d(v_4) = 2 + 3 = 5,$

$d(v_5) = 2 + 2 = 4$

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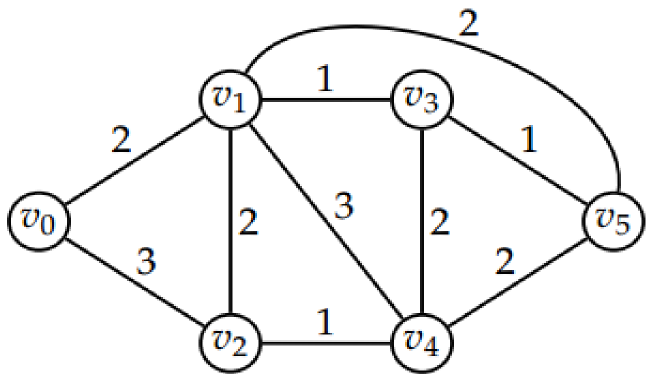

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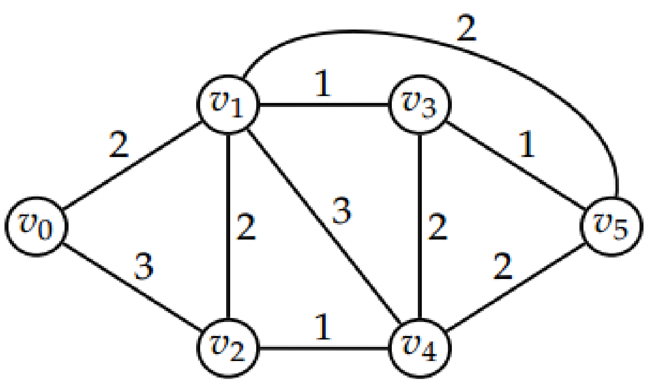

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	5	4

$i = 2$

$d(v_3) = \min\{3, \infty\} = 3,$

$d(v_4) = \min\{5, 3 + 1\} = 4,$

$d(v_5) = \min\{4, \infty\} = 4$

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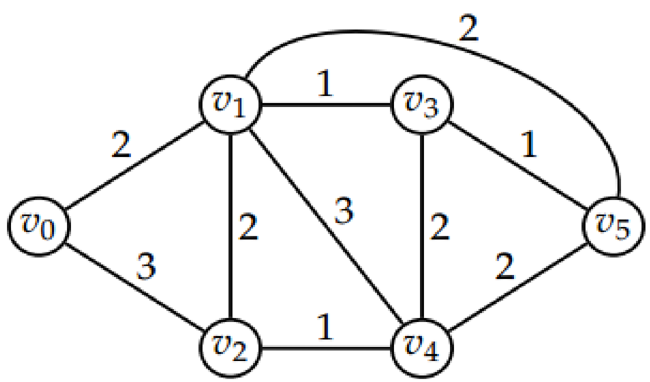

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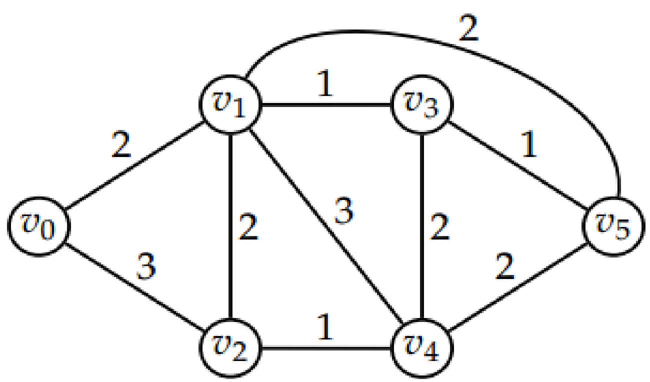

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 3$

$d(v_4) = \min\{4, 3 + 2\} = 4,$

$d(v_5) = \min\{4, 3 + 1\} = 4$

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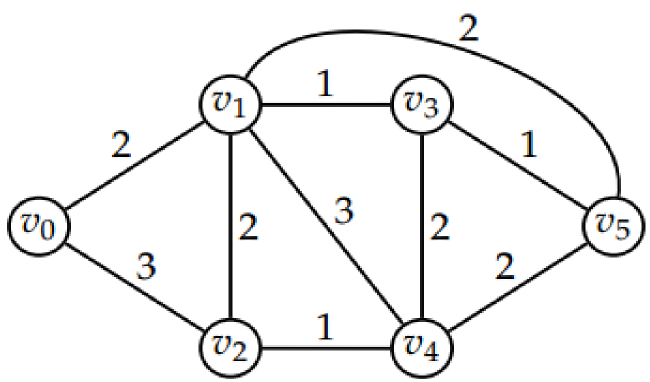
  Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 4$

$d(v_5) = \min\{4, 4 + 2\} = 4$

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- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

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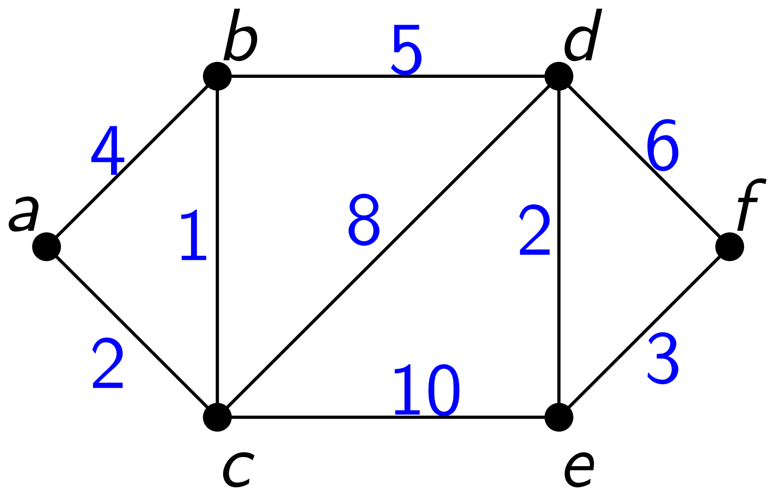
  Complexity

  read the Textbook p.712 – p.714

Another Example



Next Lecture

- Graph theory III ...