CS201H: Discrete Math for Computer Science
2021 Fall Semester Written Assignment #1
Due: Oct. 8th, 2021, please submit at the beginning of class

Q.1 Let p, q and r be the propositions
p: You get an A on the final exam.
q: You do every exercise in this book.
r: You get an A in this class.
Write these propositions using p, q, and r and logical connectives (including negations).

(a) You get an A in this class, but you do not do every exercise in this book.

(b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

(c) To get an A in this class, it is necessary for you to get an A on the final.

(d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

(e) Getting an A on the final and doing every exercise in this book is suffcient for getting an A in this class.

(f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Solution:

(a) $r \wedge \neg q$

(b) $p \wedge q \wedge r$

(c) $r \rightarrow p$

(d) $p \wedge \neg q \wedge r$

(e) $(p \wedge q) \rightarrow r$

(f) $r \leftrightarrow (q \vee p)$

□

Q.2 Construct a truth table for each of these compound propositions.

(a) $p \oplus \neg p$

(b) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

(c) $(p \oplus q) \rightarrow (p \oplus \neg q)$

Solution:

(a)

p	$\neg p$	$(p \oplus \neg p)$
T	F	T
F	T	T

(b)

p	q	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	T
T	F	T
F	T	T
F	F	T

(c)

p	q	$(p \oplus q) \rightarrow (p \oplus \neg q)$
T	T	T
T	F	F
F	T	F
F	F	T

□

Q.3 Use truth tables to decide whether or not the following two propositions are equivalent.

(a) $p \oplus q$ and $\neg p \vee \neg q$

(b) $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$

(c) $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$

(d) $(\neg q \wedge \neg(p \rightarrow q))$ and $\neg p$

Solution:

(a) The combined truth table is:

p	q	$p \oplus q$	$\neg p \vee \neg q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	F	F

By comparing the last two columns, we have that they are not equivalent.

(b) The truth table for $p \to (q \vee r)$ is :

p	q	r	$q \vee r$	$p \to (q \vee r)$
F	F	F	F	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

The truth table for $(p \to q) \vee (p \to r)$ is

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	T	F	T
T	T	T	T	T	T

Since the final columns are the same in both truth tables, we know that these two propositions are equivalent.

(c) The combined truth table is:

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
F	F	F	T	F	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

Since the fifth and last columns are not the same in both truth tables, we know that these two propositions are not equivalent.

(d) The combined truth table is:

p	q	$p \to q$	$\neg q$	$\neg(p \to q)$	$\neg q \wedge \neg(p \to q)$	$\neg p$
F	F	T	T	F	F	T
F	T	T	F	F	F	T
T	F	F	T	T	T	F
T	T	T	F	F	F	F

By comparing the last two columns, we have that they are not equivalent.

□

Q.4 Use logical equivalences to prove the following statements.

(a) $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are equivalent.

(b) $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology.

(c) $\neg(p \to q) \to \neg q$ is a tautology.

Solution:

(a) We have

$$\begin{aligned}
&\neg p \to (q \to r) \\
&\quad \equiv \quad p \vee (q \to r) \quad \text{Useful and double negation} \\
&\quad \equiv \quad p \vee (\neg q \vee r) \quad \text{Useful} \\
&\quad \equiv \quad (p \vee \neg q) \vee r \quad \text{Associative} \\
&\quad \equiv \quad (\neg q \vee p) \vee r \quad \text{Commutative} \\
&\quad \equiv \quad \neg q \vee (p \vee r) \quad \text{Associative} \\
&\quad \equiv \quad q \to (p \vee r) \quad \text{Useful}
\end{aligned}$$

(b) We have

$$\begin{aligned}
&((p \to q) \wedge (q \to r)) \to (p \to r) \\
&\quad \equiv \quad \neg((p \to q) \wedge (q \to r)) \vee (p \to r) \quad \text{Useful} \\
&\quad \equiv \quad \neg(p \to q) \vee \neg(q \to r) \vee (p \to r) \quad \text{De Morgan} \\
&\quad \equiv \quad \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (p \to r) \quad \text{Useful} \\
&\quad \equiv \quad (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg r) \vee (p \to r) \quad \text{De Morgan} \\
&\quad \equiv \quad (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r) \quad \text{Double negation, Useful} \\
&\quad \equiv \quad ((p \wedge \neg q) \vee q) \wedge ((p \wedge \neg q) \vee \neg r) \vee (\neg p \vee r) \quad \text{Distributive} \\
&\quad \equiv \quad ((p \vee q) \wedge (\neg q \vee q)) \wedge ((p \vee \neg r) \wedge (\neg q \vee \neg r)) \vee (\neg p \vee r) \quad \text{Distributive} \\
&\quad \equiv \quad ((p \vee q) \wedge (p \vee \neg r) \wedge (\neg q \vee \neg r)) \vee (\neg p \vee r) \quad \text{Negation} \\
&\quad \equiv \quad ((p \vee q) \vee (\neg p \vee r)) \wedge ((p \vee \neg r) \vee (\neg p \vee r)) \wedge ((\neg q \vee \neg r) \vee (\neg p \vee r)) \quad \text{Distributive} \\
&\quad \equiv \quad (p \vee q \vee \neg p \vee r) \wedge (p \vee \neg r \vee \neg p \vee r) \wedge (\neg q \vee \neg r \vee \neg p \vee r) \quad \text{Associative} \\
&\quad \equiv \quad T \wedge T \wedge T \quad \text{Negation} \\
&\quad \equiv \quad T.
\end{aligned}$$

Thus, it is a tautology.

(c) We have

$$\begin{aligned}
&\neg(p \to q) \to \neg q \\
&\quad \equiv \quad \neg\neg(p \to q) \vee \neg q \quad \text{Useful} \\
&\quad \equiv \quad (p \to q) \vee \neg q \quad \text{Double negation} \\
&\quad \equiv \quad (\neg p \vee q) \vee \neg q \quad \text{Useful} \\
&\quad \equiv \quad \neg p \vee (q \vee \neg q) \quad \text{Associative} \\
&\quad \equiv \quad T \quad \text{Domination}
\end{aligned}$$

Therefore, it is a tautology.

$\square$

Q.5

Explain, without using a truth table, why $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true, when at least one of p, q, and r is true and at least one is false, but is false when all three variables have the same truth value.

Solution: The first clause is true if and only if at least one of p, q and r is true. The second clause is true if and only if at least one of the three variables is false. Therefore the entire statement is true if and only if there is at least one T and one F among the truth values of the variables, in other words, that they don't all have the same truth value.

$\square$

Q.6 Determine whether or not the following two are logically equivalent, and explain your answer.

(a) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(b) $(p \to q) \to r$ and $p \to (q \to r)$

(c) $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$

(d) $(p \vee q) \to r$ and $(p \to r) \vee (q \to r)$.

Solution:

(a) The second statement is false only when p is true and $q \vee r$ is false, which means both q and r are false.

The first statement if false only when both $p \to q$ and $p \to r$ are false. This only happens when p is true, and both q and r are false.

Thus, these two statements are logically equivalent.

(b) It suffices to give a counterexample. When p, q and r are all false, $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true.

(c) What we need only is to find an assignment of truth values such that one of these propositions is true and the other false. Let p be true, and the other two be false. Then the first statement will be $F \to F$, which is true. But, the other will be $F \wedge T$, which is false.

(d) No. What we need only is to find an assignment of truth values such that one of these propositions is true and the other is false. Let p be false, q be true, and r be false. Then the first statement is false. However, the second statement is true.

□

Q.7 Prove that if $p \to q$, $\neg p \to r$, $r \to s$, then $\neg q \to s$.
Solution:

(1)	$p \to q$	Premise
(2)	$\neg q \to \neg p$	Contrapositive of (1)
(3)	$\neg p \to r$	Premise
(4)	$\neg q \to r$	Hypothetical syllogism (2) (3)
(5)	$r \to s$	Premise
(6)	$\neg q \to s$	Hypothetical syllogism (4) (5)

□

Q.8 Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++". Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

(a) There is a student at your school who can speak Russian and who knows C++.

(b) There is a student at your school who can speak Russian but who doesn't know C++.

(c) Every student at your school either can speak Russian or knows C++.

(d) No student at your school can speak Russian or knows C++.

Solution:

(a) $\exists x(P(x) \wedge Q(x))$

(b) $\exists x(P(x) \wedge \neg Q(x))$

(c) $\forall x(P(x) \vee Q(x))$

(d) $\forall x \neg(P(x) \vee Q(x))$

$\square$

Q.9 Let $L(x, y)$ be the statement "x loves y", where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statement.

(a) Everybody loves Jerry.

(b) Everybody loves somebody.

(c) There is somebody whom everybody loves.

(d) Nobody loves everybody.

(e) There is somebody whom Lydia does not love.

(f) There is somebody whom no one loves.

(g) There is exactly one person whom every body loves.

(h) There are exactly two people whom Lynn loves.

(i) Everyone loves himself or herself.

(j) There is someone who loves no one besides himself or herself.

Solution:

(a) $\forall x \ L(x, Jerry)$

(b) $\forall x \exists y \ L(x, y)$

(c) $\exists y \forall x\ L(x, y)$

(d) $\forall x \exists y \neg L(x, y)$

(e) $\exists x \neg L(Lydia, x)$

(f) $\exists x \forall y \neg L(y, x)$

(g) $\exists x(\forall y L(y, x) \wedge \forall z((\forall w L(w, z)) \rightarrow z = x))$

(h) $\exists x \exists y(x \neq y \wedge L(Lynn, x) \wedge L(Lynn, y) \wedge(\forall z(L(Lynn, z) \rightarrow (z = x \vee z = y)))$

(i) $\forall x L(x, x)$

(j) $\exists x \forall y(L(x, y) \leftrightarrow x = y)$

$\square$

Q.10 Suppose that variables x and y represent real numbers, and $L(x, y)$: $x < y$, $Q(x, y)$: $x = y$, $E(x)$: x is even, $I(x)$: x is an integer. Write the following statements using these predicates and any needed quantifiers.

(1) Every integer is even.

(2) If $x < y$, then x is not equal to y.

(3) There is no largest real number.

Solution:

(1) $\forall x(I(x) \rightarrow E(x))$

(2) $\forall x \forall y(L(x, y) \rightarrow \neg Q(x, y))$

(3) $\forall x \exists y L(x, y)$

$\square$

Q.11 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(a) $\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)$

(b) $\forall x \exists y P(x,y) \vee \forall x \exists y Q(x,y)$

(c) $\forall x \exists y (P(x,y) \wedge \exists z R(x,y,z))$

(d) $\forall x \exists y (P(x,y) \rightarrow Q(x,y))$

Solution:

(a)

$$\begin{aligned}
\neg(\exists x \exists y P(x,y) \wedge \forall x \forall y Q(x,y)) &\equiv \neg\exists x \exists y P(x,y) \vee \neg\forall x \forall y Q(x,y) \\
&\equiv \forall x \neg\exists y P(x,y) \vee \exists x \neg\forall y Q(x,y) \\
&\equiv \forall x \forall y \neg P(x,y) \vee \exists x \exists y \neg Q(x,y)
\end{aligned}$$

(b)

$$\begin{aligned}
\neg(\forall x \exists y P(x,y) \vee \forall x \exists y Q(x,y)) &\equiv \exists x \neg\exists y P(x,y) \wedge \exists x \neg\exists y Q(x,y) \\
&\equiv \exists x \forall y \neg P(x,y) \wedge \exists x \forall y \neg Q(x,y).
\end{aligned}$$

(c)

$$\begin{aligned}
\neg\forall x \exists y (P(x,y) \wedge \exists z R(x,y,z)) &\equiv \exists x \neg\exists y (P(x,y) \wedge \exists z R(x,y,z)) \\
&\equiv \exists x \forall y (\neg P(x,y) \vee \forall z \neg R(x,y,z))
\end{aligned}$$

(d)

$$\begin{aligned}
\neg\forall x \exists y (P(x,y) \rightarrow Q(x,y)) &\equiv \exists x \neg\exists y (P(x,y) \rightarrow Q(x,y)) \\
&\equiv \exists x \forall y \neg(\neg P(x,y) \vee Q(x,y)) \\
&\equiv \exists x \forall y (P(x,y) \wedge \neg Q(x,y)).
\end{aligned}$$

$\square$

Q.12

Let $P(x,y)$ be a propositional function. Prove or disprove that $\exists x \forall y P(x,y) \rightarrow \forall y \exists x P(x,y)$ is a tautology.

Solution: We assume that $\exists x \forall y P(x,y)$ holds. This means that there is some x_0 such that $P(x_0,y)$ holds for all y. Then it is certainly that for all y there exists an x such that $P(x,y)$ is true, since in each case we can at least take $x = x_0$. Note that the converse is not always true, since x depends on y in $\forall y \exists x P(x,y)$.

$\square$

Q.13 Each of the two below contains a pair of statements, (i) and (ii). For each pair, say whether (i) is equivalent to (ii), i.e., for all $P(x)$ and $Q(x)$, (i) is true if and only if (ii) is true. Here $\mathbb{R}$ denotes the set of all *real numbers.*

If they are equivalent, *all you have to do is to say that they are equivalent.* If they are not equivalent, give a counterexample. A counterexample should involve a specification of $P(x)$ and $Q(x)$ and an explanation as to why the resulting statement is false.

(1) (i) $(\forall x \in \mathbb{R}\ P(x)) \vee (\forall x \in \mathbb{R}\ Q(x))$
(ii) $\forall x \in \mathbb{R}\,(P(x) \vee Q(x))$

(2) (i) $(\forall x \in \mathbb{R}\ P(x)) \wedge (\forall x \in \mathbb{R}\ Q(x))$
(ii) $\forall x \in \mathbb{R}\,(P(x) \wedge Q(x))$

(3) (i) $(\forall x \in \mathbb{R}\ P(x)) \wedge (\exists y \in \mathbb{R}\ Q(y))$
(ii) $\forall x \in \mathbb{R}\,(\exists y \in \mathbb{R}\ (P(x) \wedge Q(y)))$

Solution:

(1) Not equivalent. Let $P(x)$ be "$x \geq 0$" and $Q(x)$ be "$x < 0$". (i) is false but (ii) is true.

(2) Equivalent.

(3) Equivalent.

$\square$

Q.14 For the following argument, explain which rules of inference are used for each step.

"Each of five roommates, A, B, C, D, and E, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

Solution:

Let $r(x)$ be "x is one of the five roommates listed", let $d(x)$ be "x has taken a course in discrete mathematics", and let $a(x)$ be "x can take a course in algorithms". We are given premises $\forall x(r(x) \rightarrow d(x)$, $\forall x(d(x) \rightarrow a(x)$, and we want to conclude $\forall x(r(x) \wedge a(x))$.

Step	**Reason**
1. $\forall x(r(x) \rightarrow d(x))$	Hypothesis
2. $r(y) \rightarrow d(y)$	Universal Instantiation using 1.
3. $\forall x(d(x) \rightarrow a(x))$	Hypothesis
4. $d(y) \rightarrow a(y)$	Universal instantiation using 3.
5. $r(y) \rightarrow a(y)$	Hypothetical syllogism using 2. and 4.
6. $\forall x(r(x) \rightarrow a(x))$	Universal generalization using 5.

□

Q.15 Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$.
Solution: We prove by four cases.

Case 1: $x \geq 0$ and $y \geq 0$. Then $|x| + |y| = x + y = |x + y|$.

Case 2: $x < 0$ and $y < 0$. Then $|x| + |y| = -x + (-y) = -(x + y) = |x + y|$.

Case 3: $x \geq 0$ and $y < 0$. Then $|x| + |y| = x + (-y)$. If $x \geq -y$, then $|x + y| = x + y$. But because $y < 0$, $-y > y$, so $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$, then $|x + y| = -(x + y)$. But because $x < 0$, $x \geq -x$, so $|x| + |y| = x + (-y) \geq -x + (-y) = |x + y|$.

Case 4: $x < 0$ and $y \geq 0$. Similar to Case 3.

□

Q.16

Prove that between every two rational numbers there is an irrational number.

Solution:

By finding a common denominator, we can assume the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$. In particular, $(a+1)/b \leq c/b$. Thus, $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well, which is wrong.

$\square$

Q.17

Prove that between every rational number and every irrational number there is an irrational number.

Solution: The average of two different numbers is certainly always between the two numbers. Furthermore, the average a of rational number x and irrational number y must be irrational, because the equation $a = (x + y)/2$ leads to $y = 2a - x$, which would be rational if a were rational.

$\square$

Q.18 Prove that $\sqrt[3]{2}$ is irrational.

Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

$\square$

Q.19 Suppose that we have a theorem: "$\sqrt{n}$ is irrational whenever n is a positive integer that is *not* a perfect square." Use this theorem to prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution: We give a proof by contradiction. If $\sqrt{2}+\sqrt{3}$ is a rational number, then its square is also rational, which is $5+2\sqrt{6}$. Subtracting 5 and dividing by 2, we have $\sqrt{6}$ is also rational. However, this contradicts the theorem.

$\square$

Q.20

(a) Let P be a proposition in atomic propositions p and q. If $P = \neg(p \leftrightarrow (q \vee \neg p))$, prove that P is equivalent to $\neg p \vee \neg q$.

(b) If P is of any length, using any of the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, prove that P is logically equivalent to a proposition of the from

$$A \Box B,$$

where $\Box$ is one of $\wedge$, $\vee$, $\leftrightarrow$, and A and B are chosen from $\{p, \neg p, q, \neg q\}$.

Solution:

(a) This can be proved by truth table.

Alternatively, we prove it using logical equivalences as follows.

$$\begin{aligned}
P &= \neg(p \leftrightarrow (q \vee \neg p)) \\
&\equiv \neg((p \rightarrow (q \vee \neg p)) \wedge ((q \vee \neg p) \rightarrow p)) \quad \text{Definition} \\
&\equiv \neg((\neg p \vee (q \vee \neg p)) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Useful} \\
&\equiv \neg((\neg p \vee q) \wedge (\neg(q \vee \neg p) \vee p)) \quad \text{Idempotent}
\end{aligned}$$

For simplicity, let $r = \neg p \vee q$, then we have

$$\begin{aligned}
P &\equiv \neg(r \wedge (\neg r \vee p)) \\
&\equiv \neg r \vee \neg(\neg r \vee p) \quad \text{De Morgan} \\
&\equiv \neg r \vee (r \wedge \neg p) \quad \text{De Morgan and double negation} \\
&\equiv (\neg r \vee r) \wedge (\neg r \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg r \vee \neg p) \quad \text{Negation} \\
&\equiv \neg r \vee \neg p \quad \text{Identity} \\
&\equiv (p \wedge \neg q) \vee \neg p \quad \text{De Morgan} \\
&\equiv (p \vee \neg p) \wedge (\neg q \vee \neg p) \quad \text{Distributive} \\
&\equiv T \wedge (\neg q \vee \neg p) \quad \text{Negation} \\
&\equiv \neg p \vee \neg q \quad \text{Identity.}
\end{aligned}$$

(b) For the proposition P, since the two involved atomic propositions p and q can have at most 4 combinations of truth tables, P has at most 2^4 different forms in terms of truth tables up to logical equivalence. It then suffices to prove that a proposition of the form $A \Box B$ has also 2^4 different forms in terms of truth tables up to logical equivalence.

If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$, and $\Box \in \{\wedge, \vee\}$, then $A \Box B$ has $2 \times 2 \times 2 = 8$ different possible forms. If $A \in \{p, \neg p\}$, $B \in \{q, \neg q\}$ and $\Box = \leftrightarrow$, then there are two extra different possibilities: $p \leftrightarrow q$ and $p \leftrightarrow \neg q$. Together with $p \vee p \equiv p$, $p \vee \neg p \equiv T$, $q \vee q \equiv q$, $p \wedge \neg p \equiv F$ and similarly $\neg p$, $\neg q$, we will have the $2^4 = 16$ different forms by $A \Box B$. This proves the statement.

$\square$