南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Algorithm Design and Analysis (H)

CS216

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Greedy Algorithms

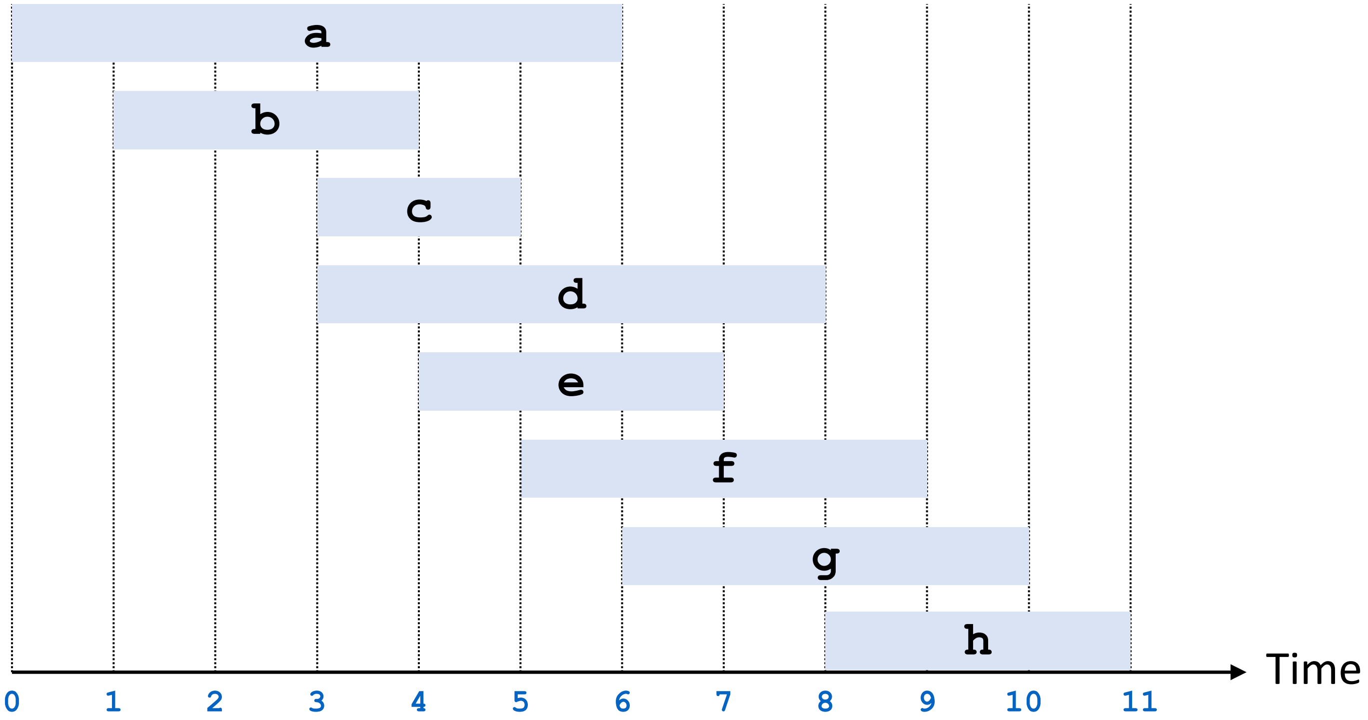
Greedy algorithms

- Build up a solution in small steps,
- Choose a decision at each step myopically to optimize some underlying criterion.
- May not produce an optimal solution,
- But can yield locally optimal solutions that approximate a globally optimal solution in a reasonable amount of time.

1. Interval Scheduling

Interval Scheduling

- **Interval scheduling.**
  - Job j starts at s_j and finishes at f_j.
  - Two jobs compatible if they don't overlap.
  - Goal: find maximum subset of mutually compatible jobs.



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Interval Scheduling: Greedy Algorithms

- **Greedy template.** Consider jobs in some natural order.
  Take each job provided it's compatible with the ones already taken.

  - [Earliest start time] Consider jobs in ascending order of s_j.
  - [Earliest finish time] Consider jobs in ascending order of f_j.
  - [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.
  - [Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

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Interval Scheduling: Greedy Algorithms

- **Greedy template**. Consider jobs in some natural order.
  Take each job provided it's compatible with the ones already taken.

counterexample for earliest start time

counterexample for shortest interval

counterexample for fewest conflicts

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Interval Scheduling: Greedy Algorithm

- **Greedy algorithm**. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

A ← φ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

set of jobs selected

- Implementation. O(n log n).
  - Remember job j* that was added last to A.
  - Job j is compatible with A if $s_j \geq f_{j^*}$.

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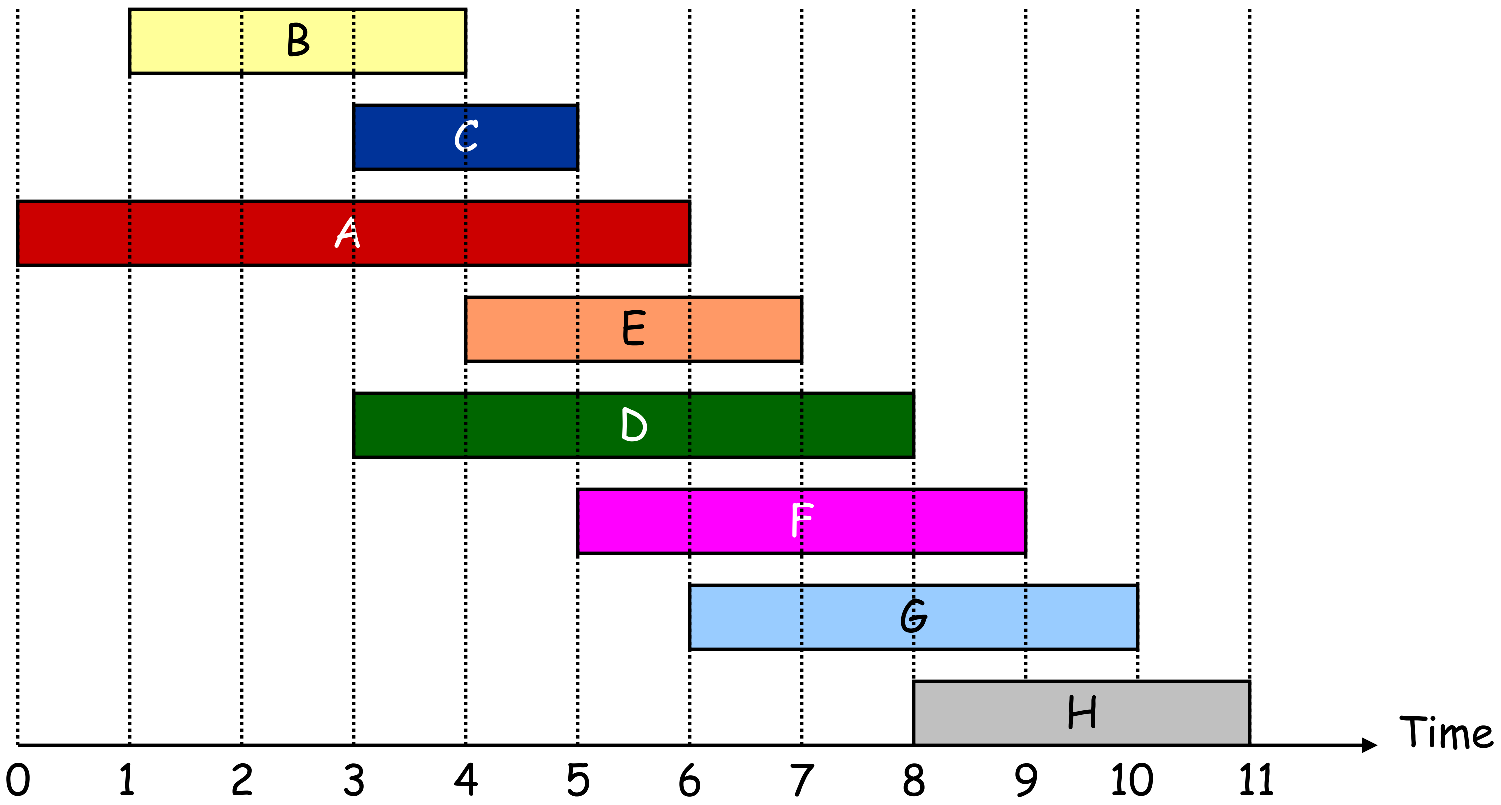
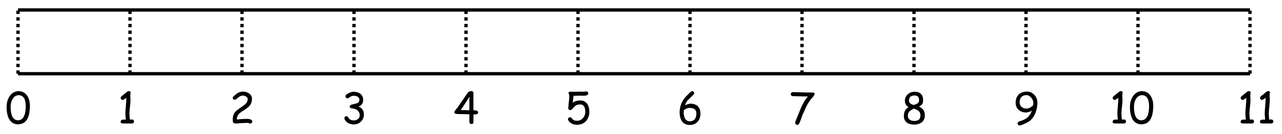
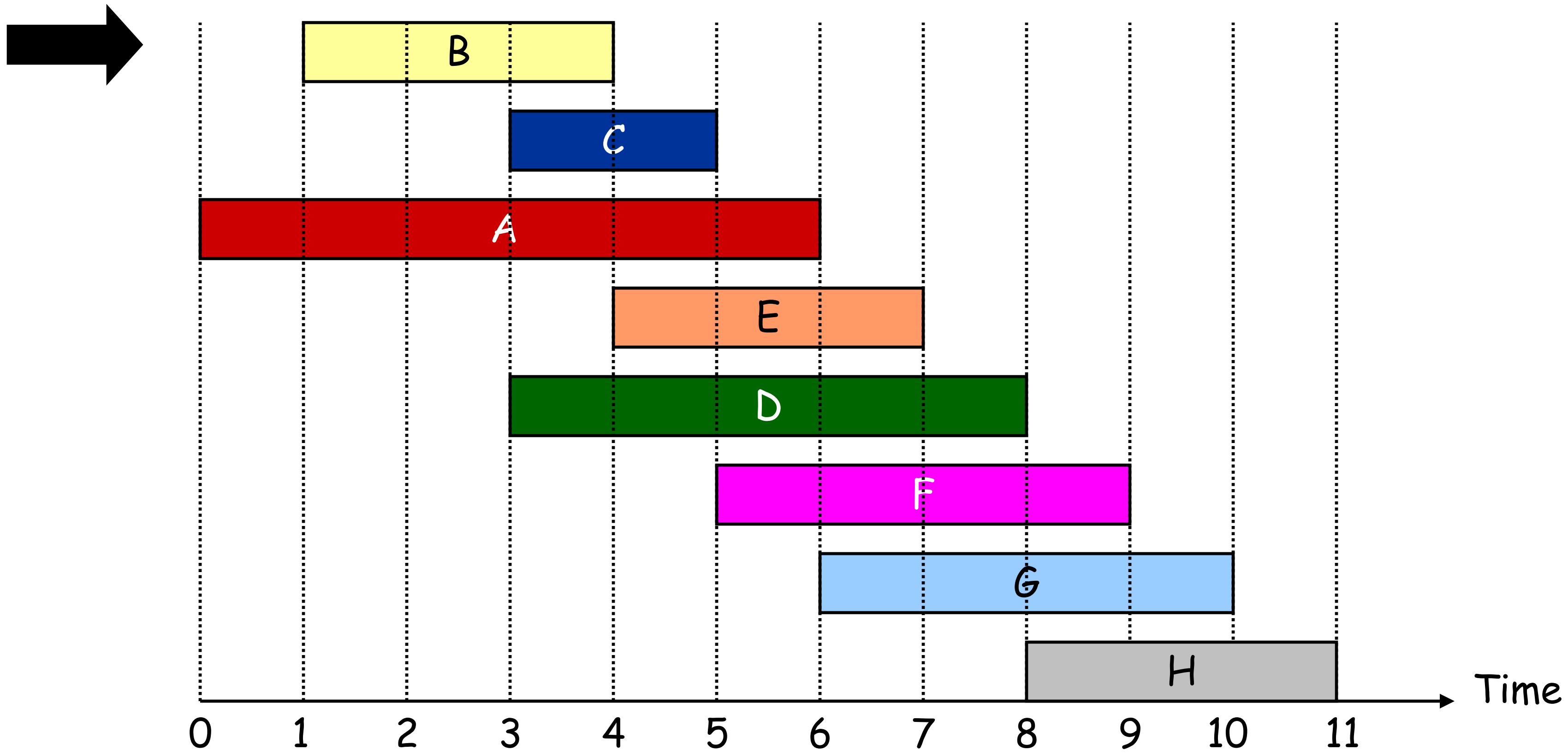
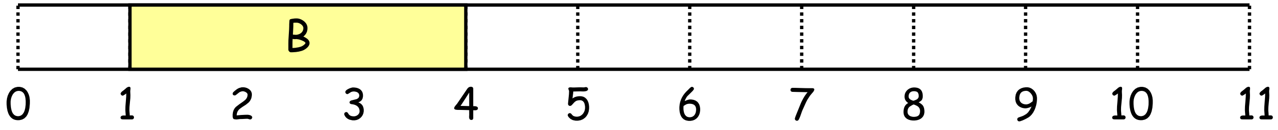
Interval Scheduling



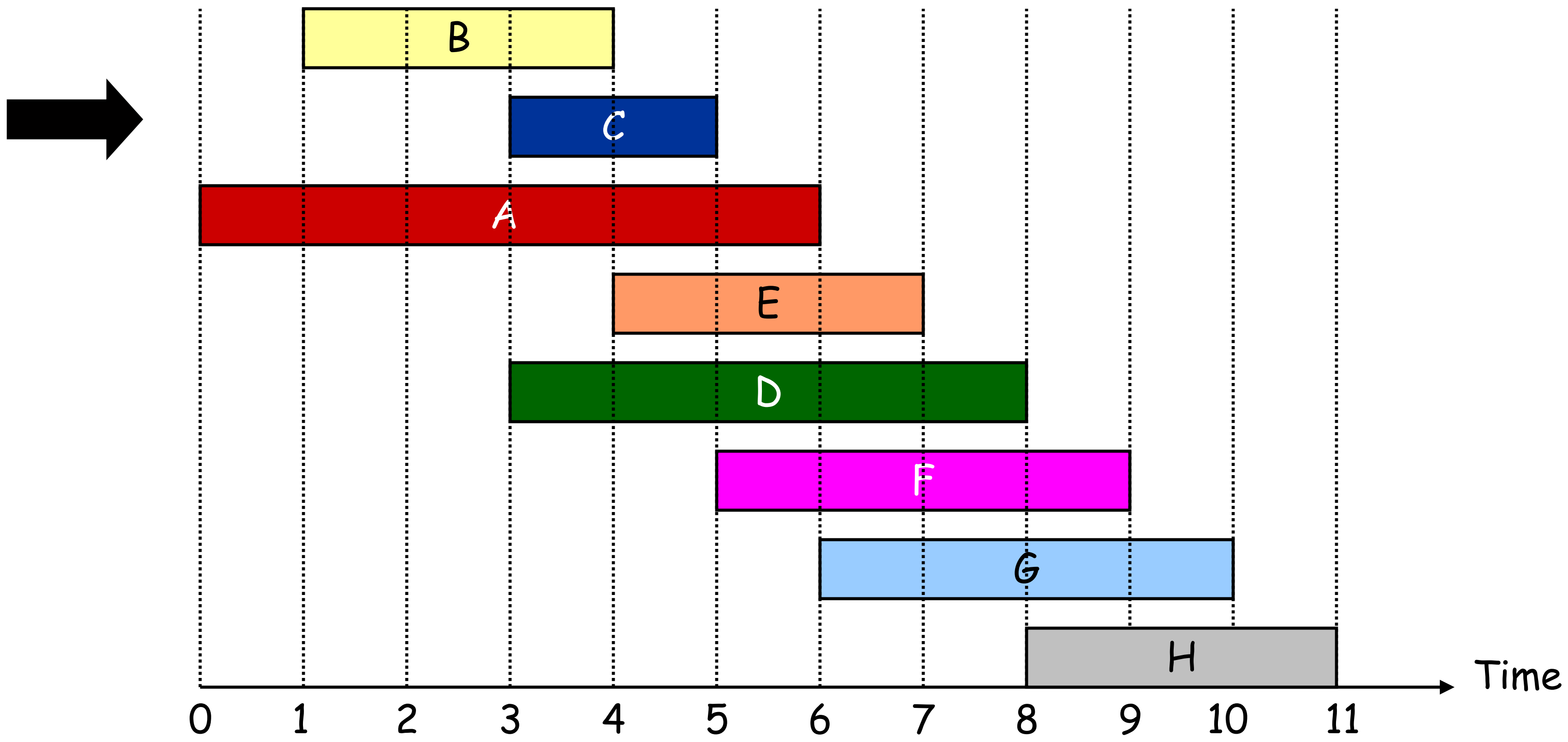


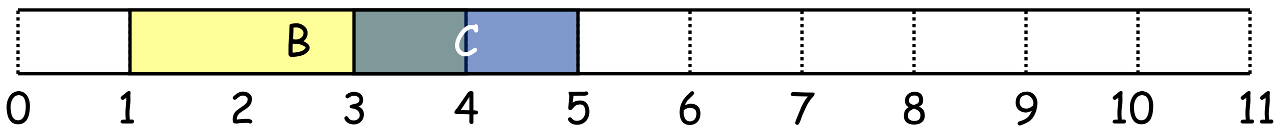
Interval Scheduling



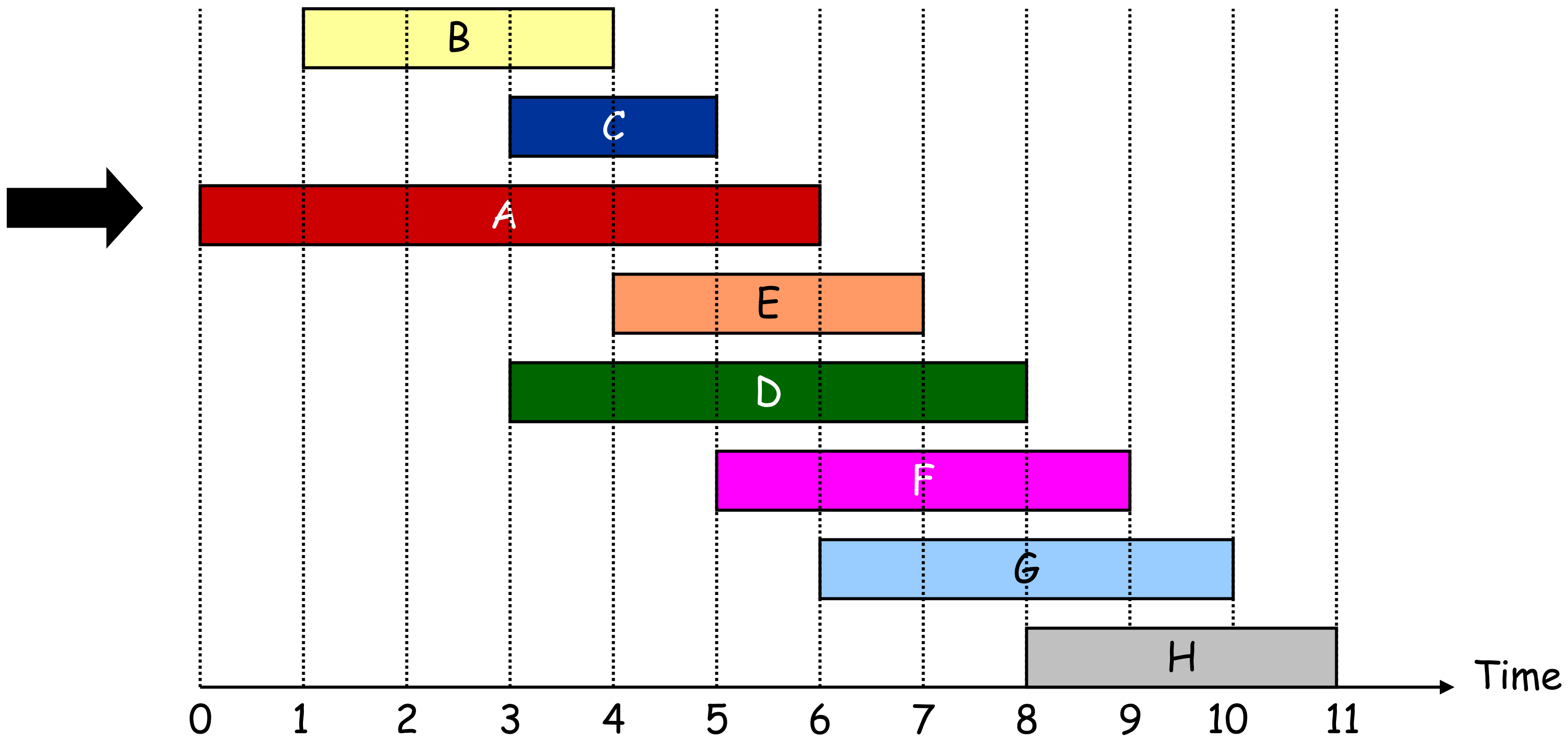


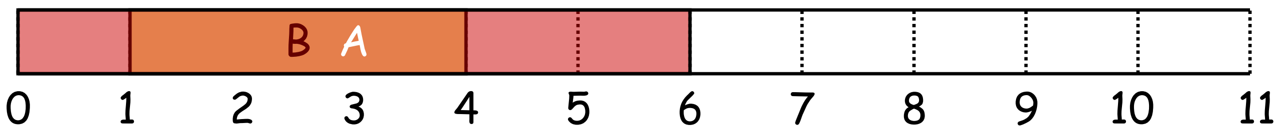
Interval Scheduling



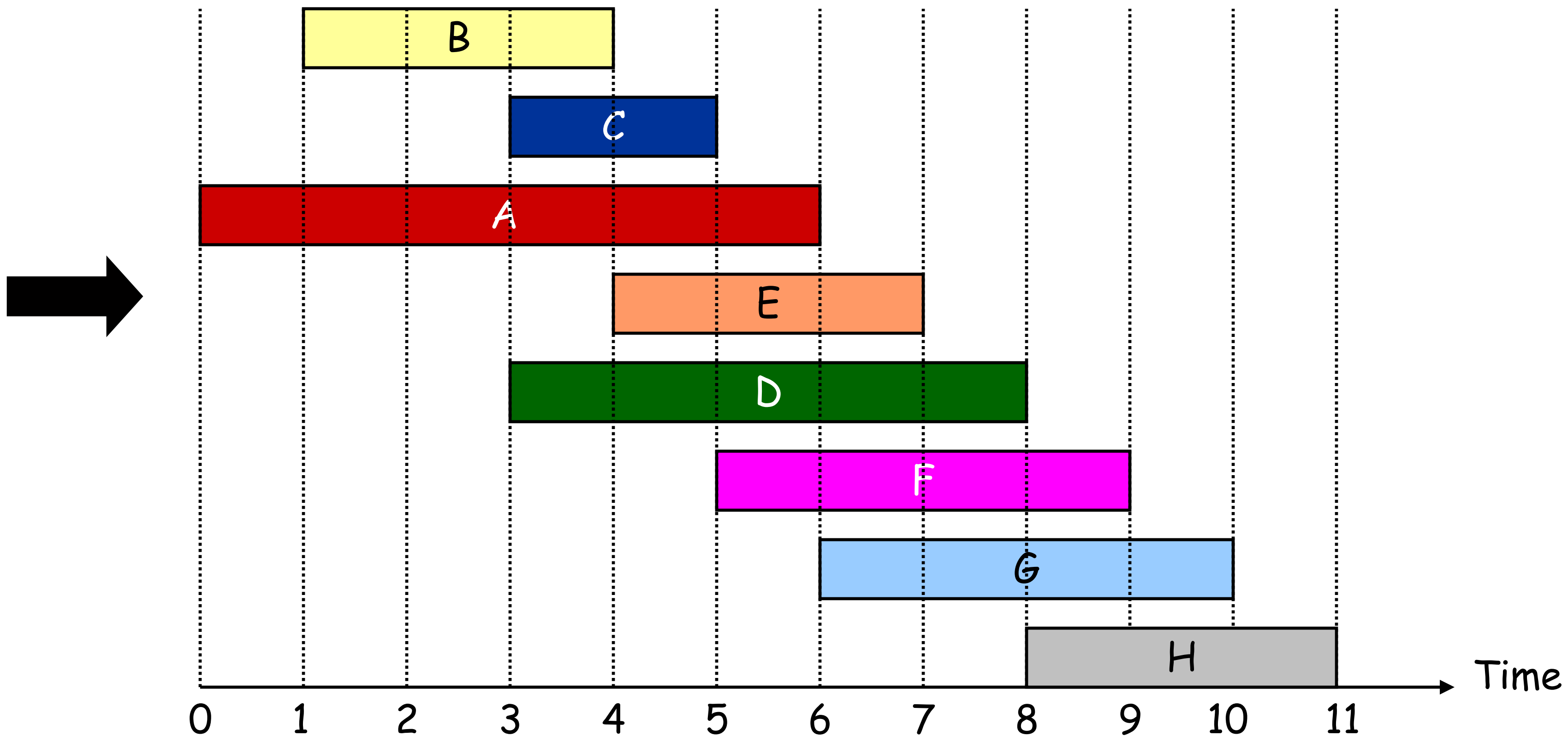


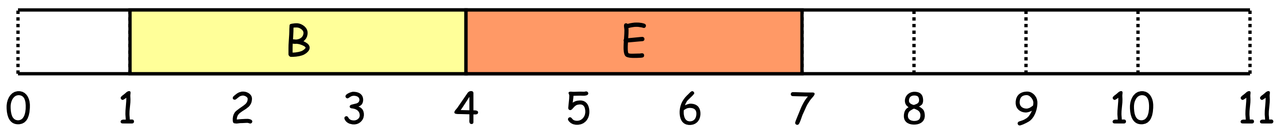
Interval Scheduling



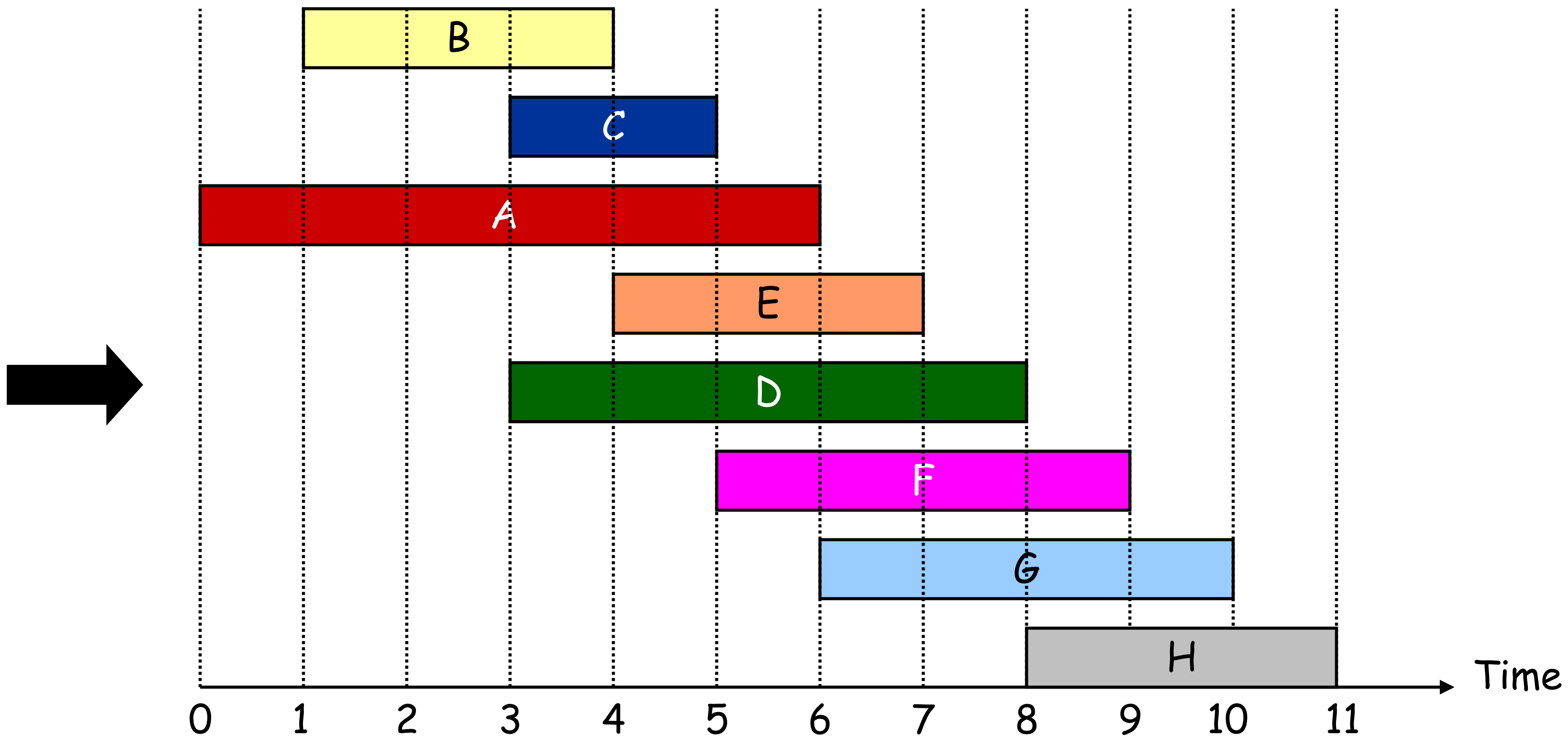


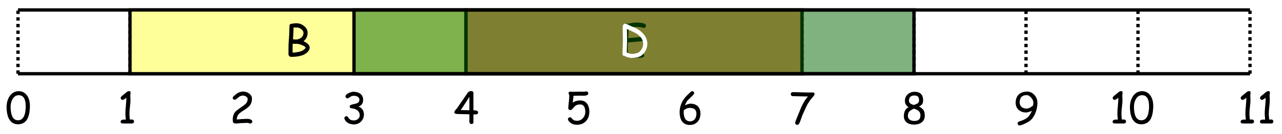
Interval Scheduling



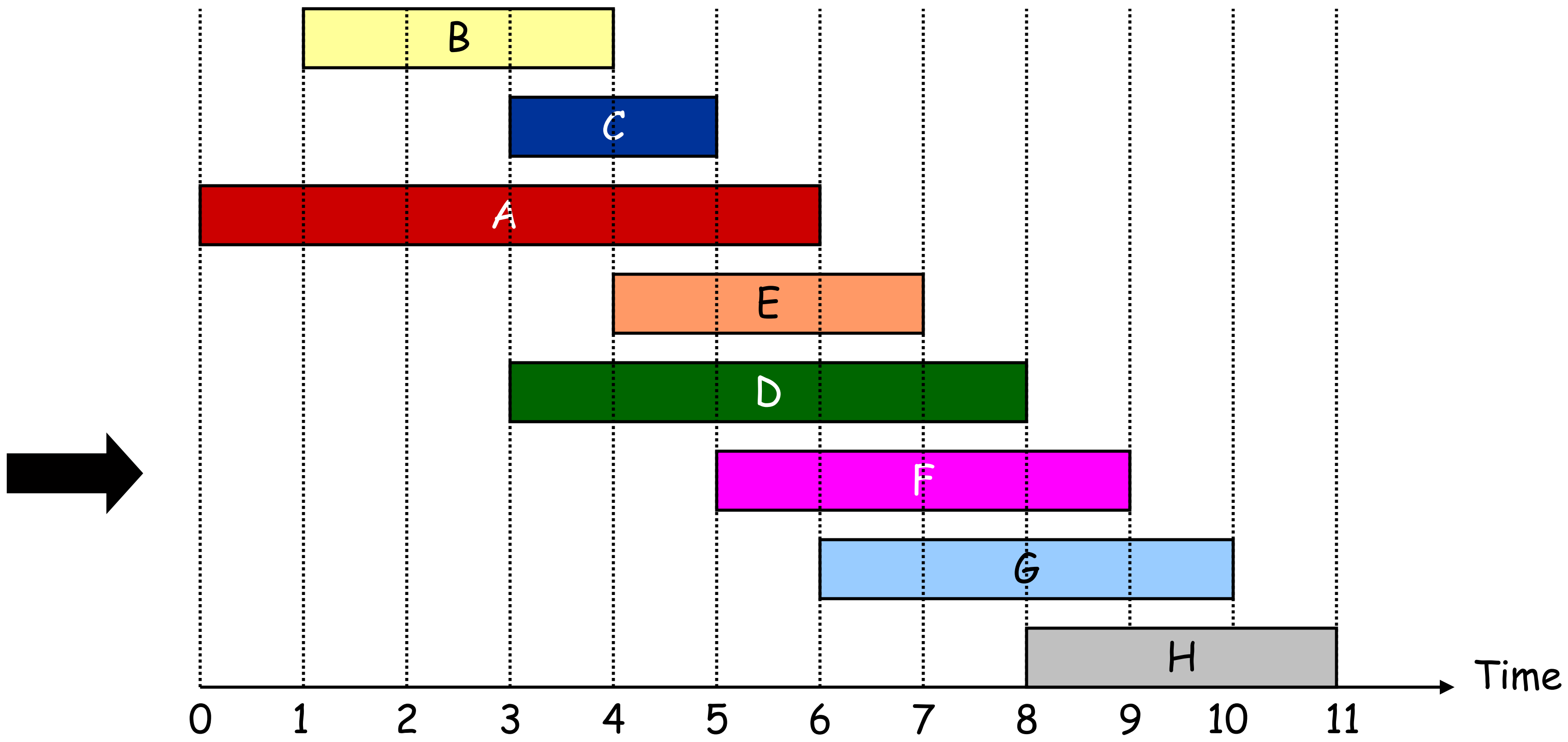


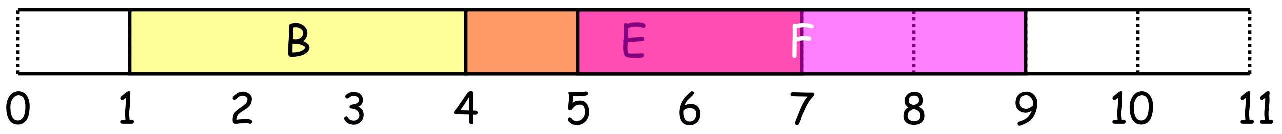
Interval Scheduling



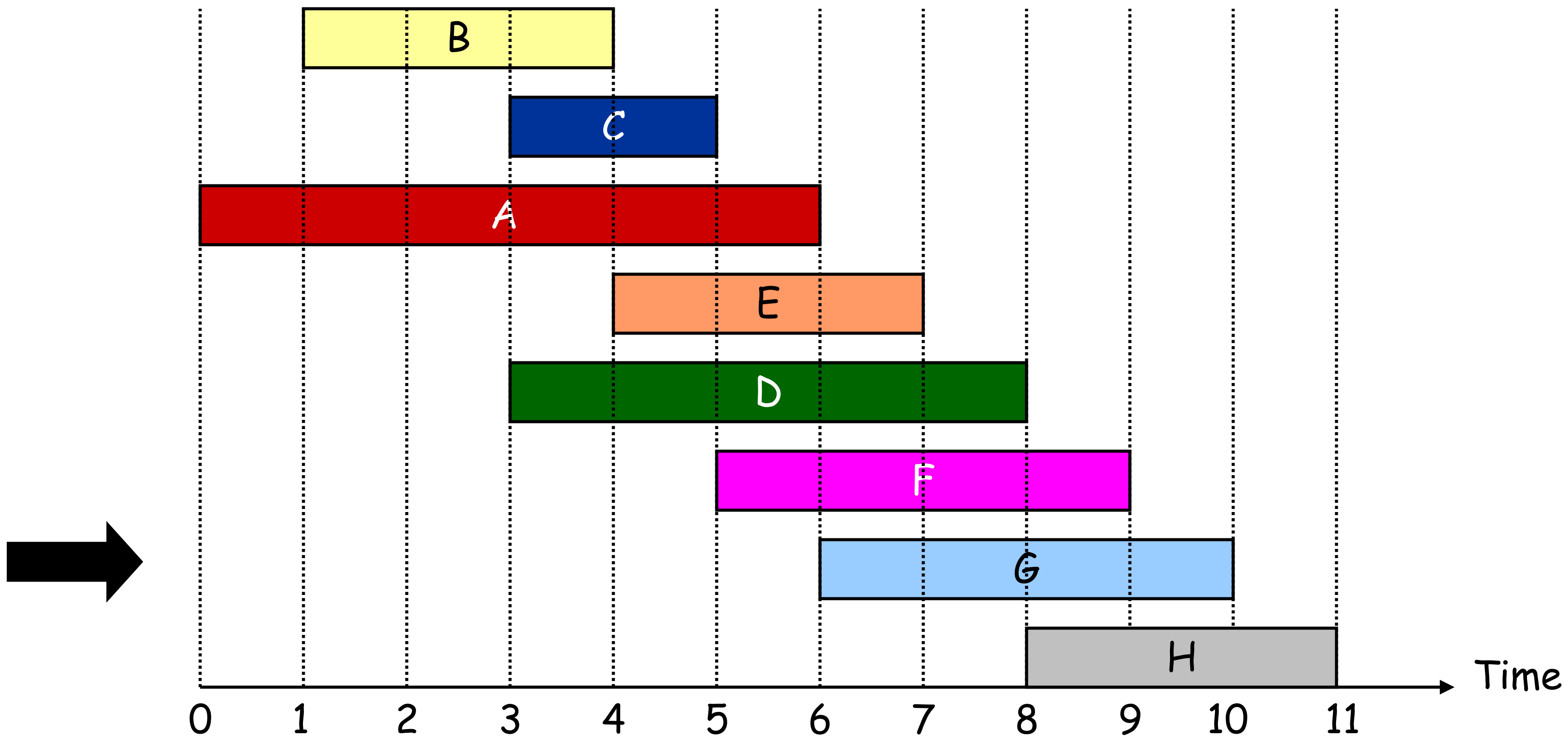


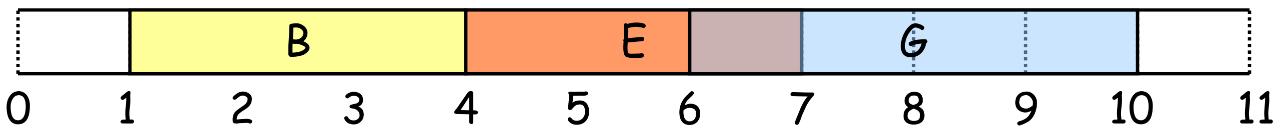
Interval Scheduling



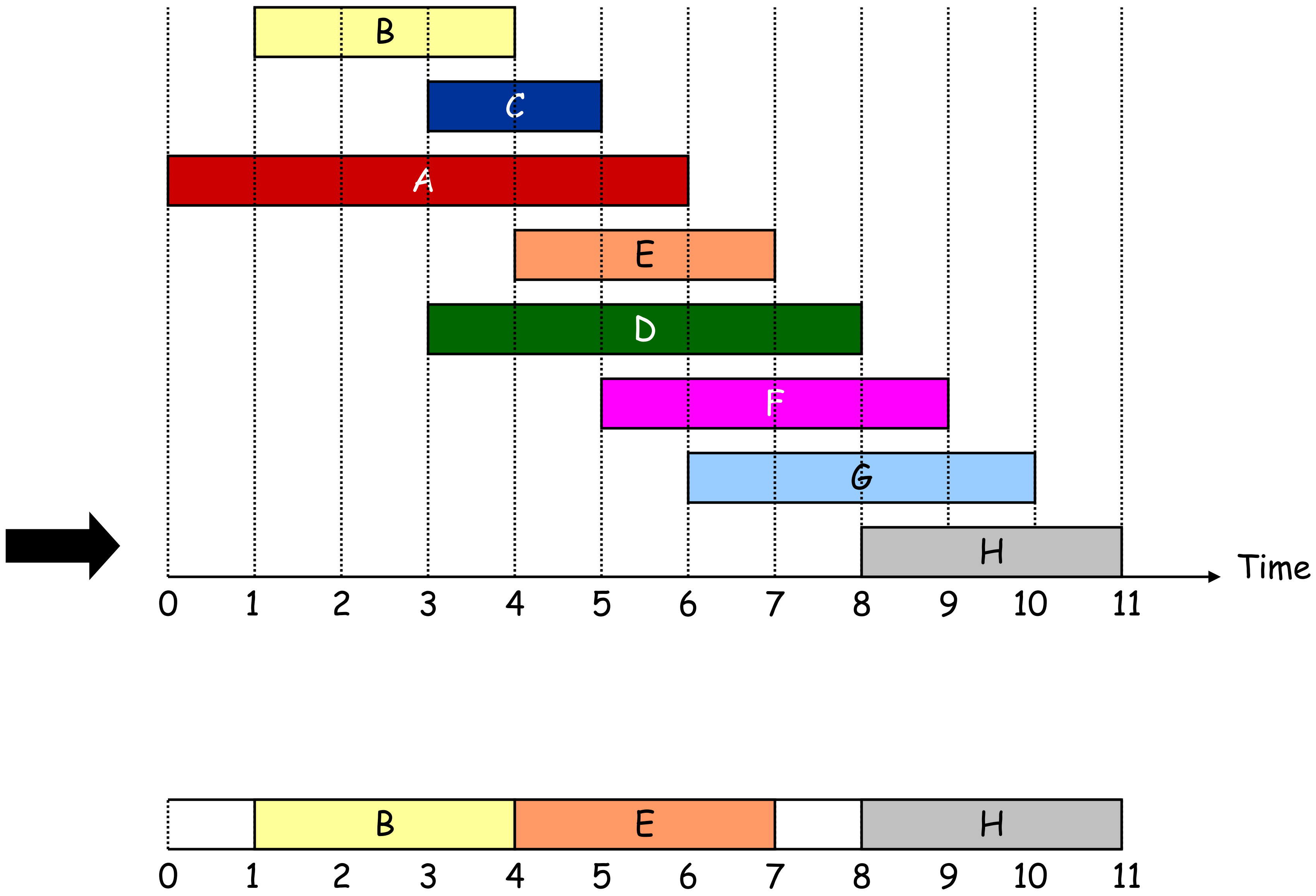


Interval Scheduling



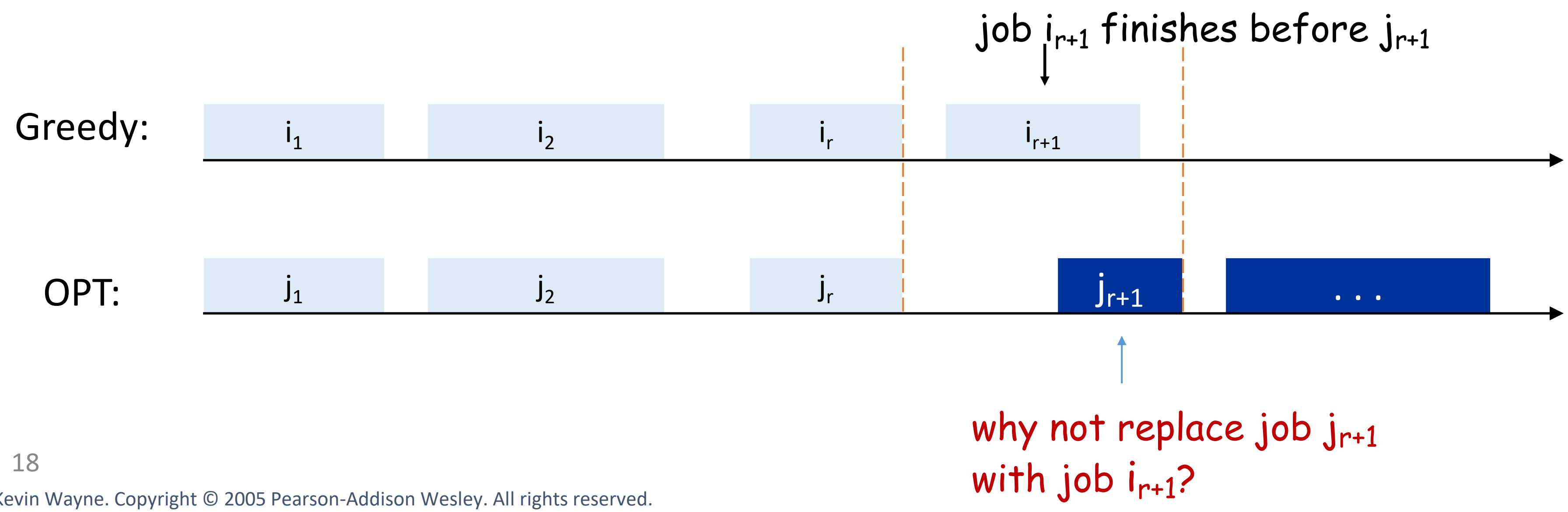


Interval Scheduling



Interval Scheduling: Analysis

- Theorem. Greedy algorithm is optimal.
- Pf. (by contradiction)
  - Assume greedy is not optimal, and let's see what happens.
  - Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
  - Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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Interval Scheduling: Analysis

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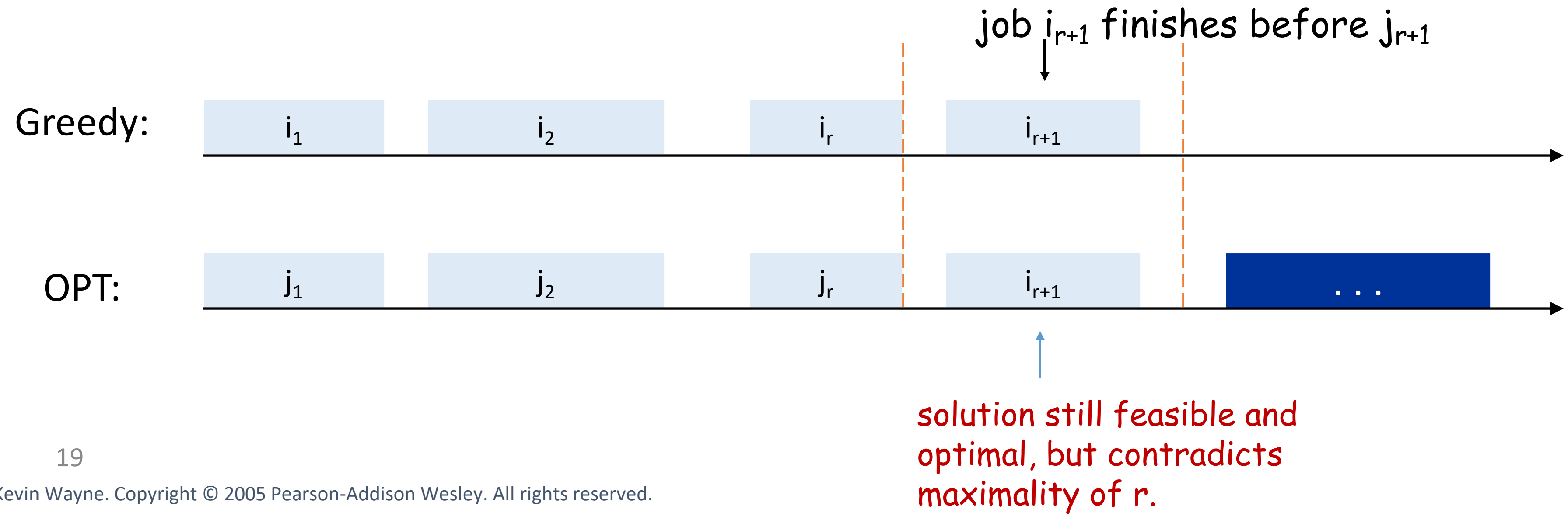

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2. Interval Partitioning

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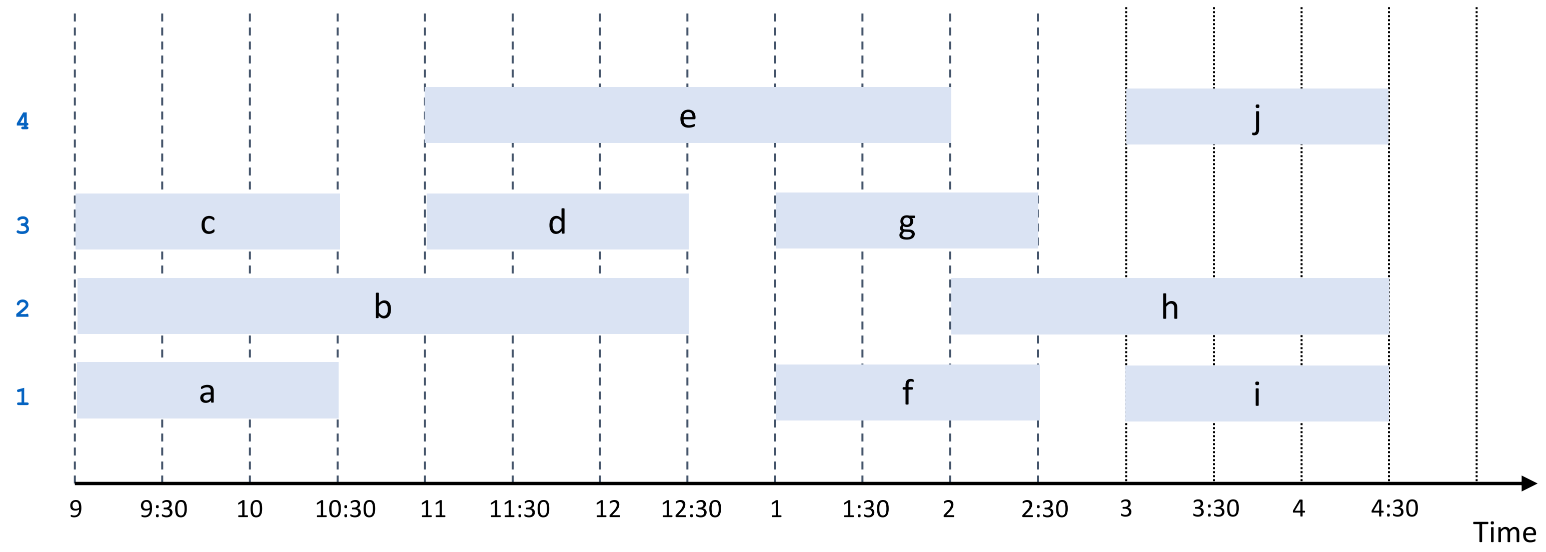
Interval Partitioning

- Interval partitioning.
  - Lecture j starts at s_j and finishes at f_j.
  - Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

- Ex: This schedule uses 4 classrooms to schedule 10 lectures.



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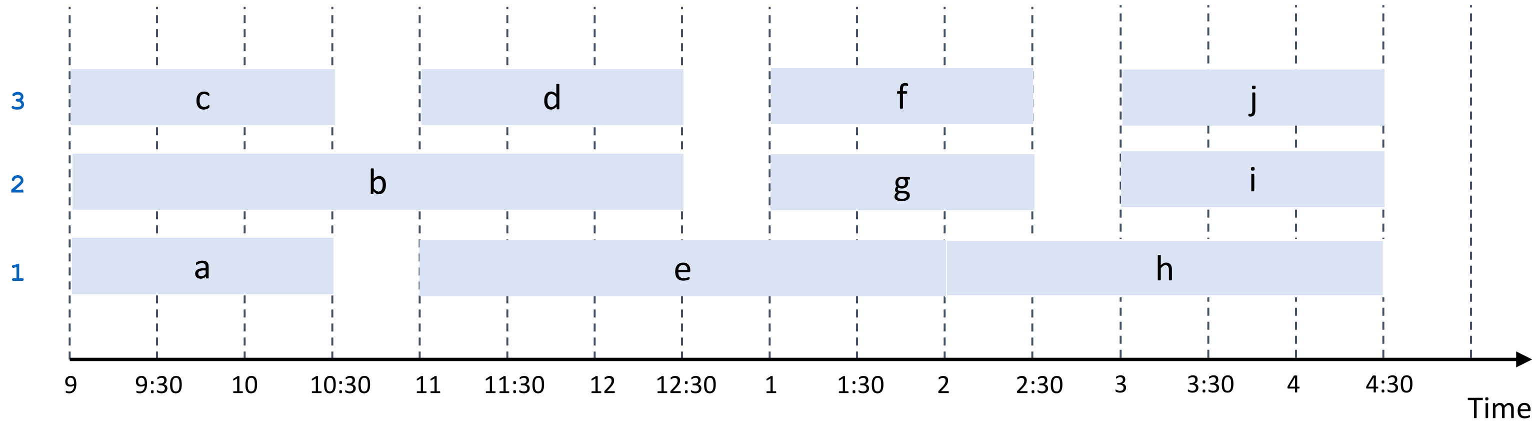
Interval Partitioning

- Interval partitioning.
  - Lecture j starts at s_j and finishes at f_j.
  - Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

- Ex: This schedule uses only 3.



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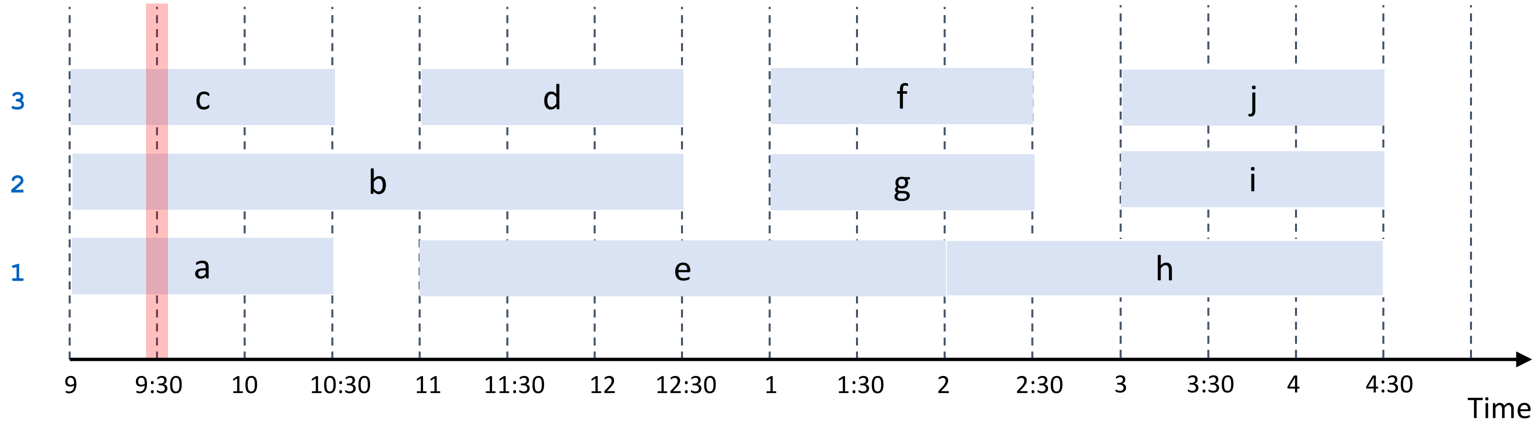
Interval Partitioning: Lower Bound on Optimal Solution

- Def. The depth of a set of open intervals is the maximum number that contain any given time.
- Key observation. Number of classrooms needed $\geq$ depth.
- Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

a, b, c all contain 9:30

- Q. Does there always exist a schedule equal to depth of intervals?



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Interval Partitioning: Greedy Algorithm

- Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0      number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

- Implementation. O(n log n).
  - For each classroom k, maintain the finish time of the last job added.
  - Keep the classrooms in a priority queue.

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Interval Partitioning: Greedy Analysis

- Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

- Theorem. Greedy algorithm is optimal.
- Pf.
  - Let d = number of classrooms that the greedy algorithm allocates.
  - Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
  - These d jobs each end after s_j.
  - Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
  - Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
  - Key observation $\Rightarrow$ all schedules use $\geq$ d classrooms. ▪

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3. Scheduling to Minimize Lateness

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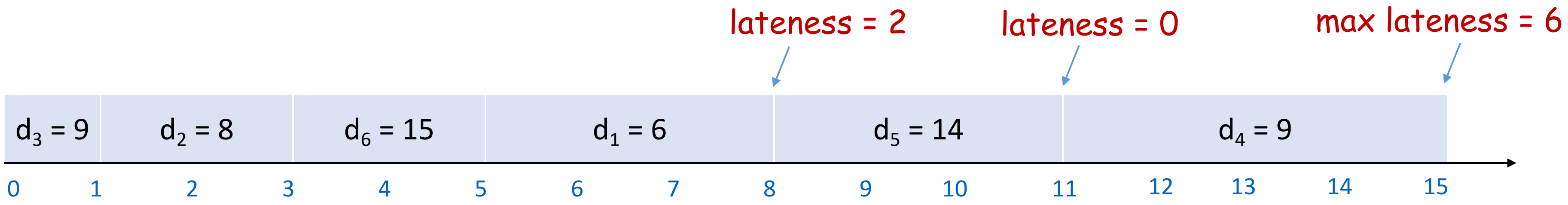
Scheduling to Minimizing Lateness

- Minimizing lateness problem.
  - Single resource processes one job at a time.
  - Job j requires t_j units of processing time and is due at time d_j.
  - If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
  - Lateness: $\ell_j = \max \{ 0, \ f_j - d_j \}$.
  - Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

- Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



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Minimizing Lateness: Greedy Algorithms

- Greedy template. Consider jobs in some order.

  - [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

  - [Earliest deadline first] Consider jobs in ascending order of deadline d_j.

  - [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

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Minimizing Lateness: Greedy Algorithms

- Greedy template. Consider jobs in some order.

  - [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

  - [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

counterexample

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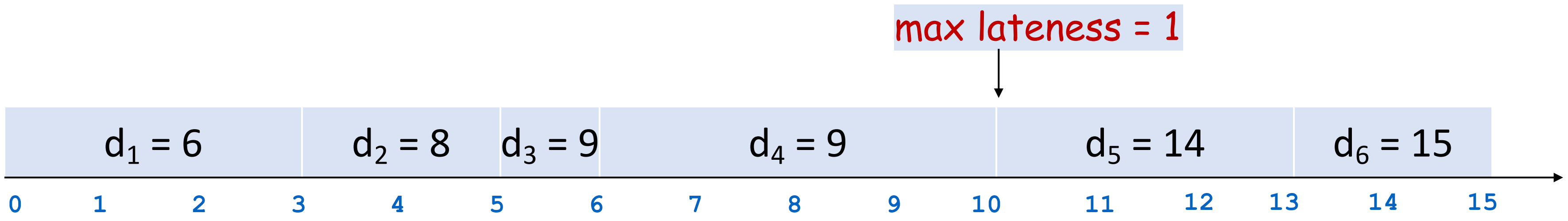
Minimizing Lateness: Greedy Algorithm

- Greedy algorithm. Earliest deadline first.

Sort n jobs by deadline so that $d_1 \le d_2 \le \ldots \le d_n$

$t \leftarrow 0$
for $j = 1$ to n
  Assign job j to interval $[t,\ t + t_j]$
  $s_j \leftarrow t,\ f_j \leftarrow t + t_j$
  $t \leftarrow t + t_j$
output intervals $[s_j,\ f_j]$



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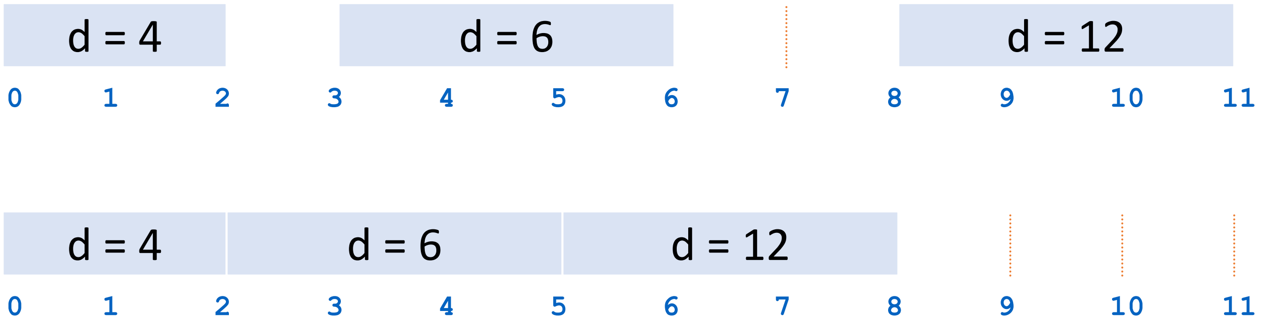
Minimizing Lateness: No Idle Time

- Observation. There exists an optimal schedule with no idle time.



- Observation. The greedy schedule has no idle time.

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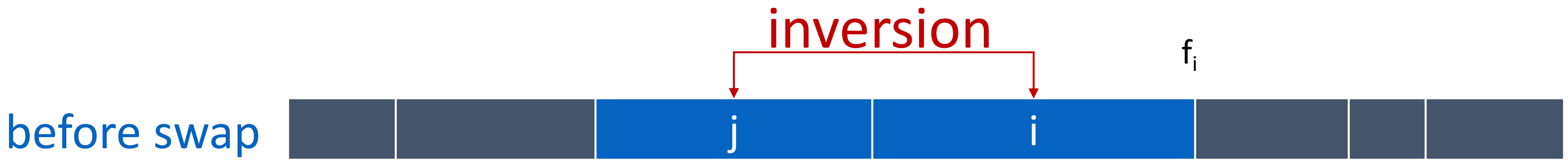
Minimizing Lateness: Inversions

- Def. Given a schedule S, an inversion is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i.



[as before, we assume jobs are numbered so that $d_1 \le d_2 \le \ldots \le d_n$]

- Observation. Greedy schedule has no inversions.

- Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

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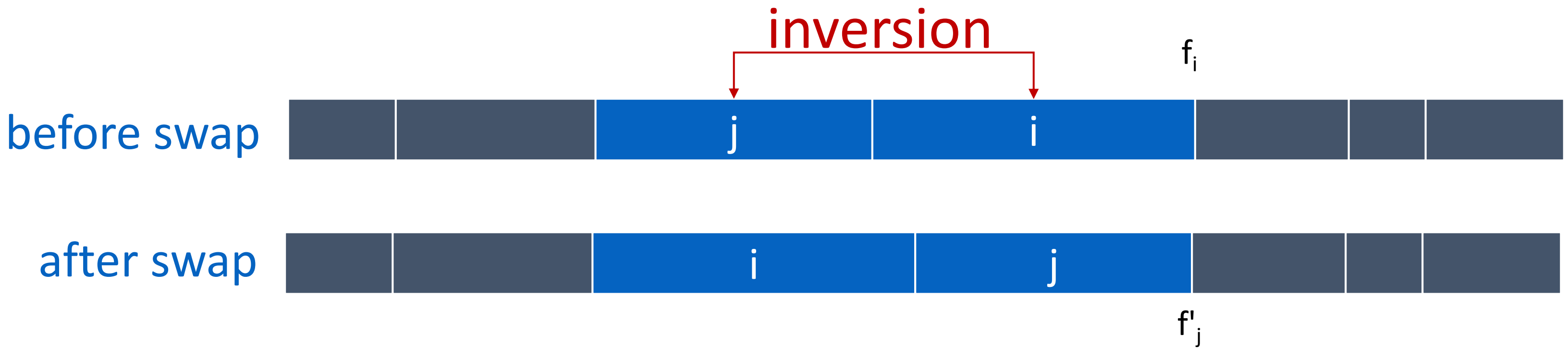
Minimizing Lateness: Inversions

- Def. Given a schedule S, an inversion is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i.



- Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

- Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.
  - $\ell'_k = \ell_k$ for all $k \neq i, j$
  - $\ell'_i \leq \ell_i$
  - $\ell'_j = f'_j - d_j = f_i - d_j < f_i - d_i = \ell_i$

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Minimizing Lateness: Analysis of Greedy Algorithm

- Theorem. Greedy schedule S is optimal.
- Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.
  - Can assume S* has no idle time.
  - If S* has no inversions, then S = S*.
  - If S* has an inversion, let i-j be an adjacent inversion.
    - ✓ swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
    - ✓ this contradicts definition of S* ▪

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Greedy Analysis Strategies

- Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

- Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

- Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

- Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, …

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