

Mechanics - Landau.

Leo.



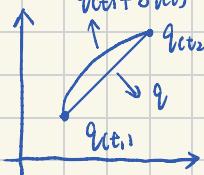
- Chap 1. Least Action Theory [$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$]

81. Define Action : $S = \int L dt$

$$\text{Variational Inference } \delta S = \int \delta L(q, \dot{q}, t) dt = \int (\frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q}) dt = 0$$

- Comments on \dot{q}

$$L \rightarrow \dot{q} \text{ for } q = q_{\min} + \underbrace{\delta q(t)}_{\text{cause change}}$$



As seen in definition

$$q(t_1) + \delta q(t_1) = q(t_1)$$

$$\delta q(t_1) = \delta q(t_2) = 0$$

$$\int \delta q \cdot \frac{\partial L}{\partial q} dt + \delta \dot{q} \cdot \frac{\partial L}{\partial \dot{q}} dt \Big|_{t_1}^{t_2} - \int \delta \dot{q} \left(\frac{\partial^2 L}{\partial \dot{q}^2} \right) dt = 0$$

$$\int \delta q \left(\frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial \dot{q}^2} \right) dt = 0$$

$$\frac{\partial L}{\partial q} = \left(\frac{\partial^2 L}{\partial \dot{q}^2} \right) \frac{d\dot{q}}{dt} \quad \text{Lagrange Equation}$$

82. - Limitation / Variation form on $L(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(x, t)$

$$\text{For } S' = \int_{t_1}^{t_2} [L + \frac{d}{dt} f(x, t)] dt$$

$$= S + f(x, t_2) - f(x, t_1)$$

$$\delta S' = \delta S + \delta \left[f(x, t_2) - f(x, t_1) \right] \quad \text{Since } t_1, t_2 \rightarrow q_1, q_2 \text{ is assigned.} \quad \delta S' = \delta S + 0 \quad \text{both Function Same}$$

\Rightarrow Lagrange isn't change when $\pm \frac{d}{dt} f(x, t)$

83. - Frame of reference

Galilean reference Frame - isotropy in time and position

$$(\frac{\partial L}{\partial \dot{q}})^{\frac{d}{dt}} = \frac{\partial L}{\partial \dot{q}}$$

$$\frac{\partial L}{\partial \dot{q}} = \text{constant} \Rightarrow V \text{ is constant}$$

$$J_{C(x)} = J_{C(y)}$$

$$J_{C(t_1)} = J_{C(t_2)}$$

$\Rightarrow L(q, \dot{q}, t)$ only include $L(\dot{q})$

$$\begin{aligned} &\Rightarrow \text{No vector include} \\ &\downarrow \\ &V^2 = \dot{q} \cdot \dot{q} \\ &= \text{only values} \\ &L(\dot{q}^2) \end{aligned}$$

Newton's 1st Law

- Comments on dif. Frame

L' move V along L

$$r' = r + Vt \quad t = t'$$

84. - Same movement in dif. Frame.

$$L(v'^2) = L(v^2) \quad \text{where } v' = v + \varepsilon$$

due to Frame dif.

$$L(v^2 + 2v\varepsilon + \varepsilon^2) \text{ have to } = L(v^2)$$

$$= L(v^2) + \frac{\partial L}{\partial v^2} \cdot (2v\varepsilon + \varepsilon^2)$$

small term cancel

$$= L(v^2) + \boxed{2 \frac{\partial L}{\partial v^2} (\varepsilon \cdot v)} \quad \text{Must belongs to } \frac{d}{dt} f(x, t) \Rightarrow \frac{\partial L}{\partial v^2} = \text{Constant}$$

$$\text{Separation } \frac{d}{dt} f(x, t) = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t}$$

$\nabla f \cdot v$
Linear relation with v .

$$\text{Assume to } L = kv^2 \quad \text{Loc } v^2$$

- Verification on normal value \underline{v}

$$Lcv + \underline{v}^2 = k(v + \underline{v})^2 = k[v^2 + 2v\underline{v} + \underline{v}^2] = 2cv^2 + \frac{d}{dt}(kbn\underline{v} + kv^2t)$$

- Comments on k

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} kv^2 dt \quad \text{Since } S \text{ always minimum value, if } kv^2 < 0, \text{ No minimum exists}$$

⇒ k always > 0 .

85. Interactions on particles in Same Frame.

$$\text{For } L = kv^2 - V(x_1, x_2, \dots, x_n)$$

$$(\frac{\partial L}{\partial v}) \frac{dv}{dt} = \frac{\partial k}{\partial v} \frac{dv}{dt} = -\frac{\partial v}{\partial x} \quad \text{Since } F = m \frac{dv}{dt} \quad \text{define } k = \frac{1}{m}$$

$$\text{Similar on Dual-Frame } f(q_1, \dots, q_n) \quad f = \sum \frac{\partial f}{\partial q_i} \cdot \dot{q}_i$$

$$L = \sum \alpha_i g_i \dot{q}_i \dot{q}_k - V(q)$$

Chap 2. - Energy // Conservation.

86. Energy

- Homogeneity of time [happens not related with time]

$$= L(q, \dot{q}) \quad \text{Try on } \frac{d}{dt} L(q, \dot{q}) = \underbrace{\frac{\partial L}{\partial q} \cdot \dot{q}}_{\text{value of }} + \underbrace{\frac{\partial L}{\partial \dot{q}} \cdot \ddot{q}}_{\text{not related with t.}} = + \underbrace{\left(\frac{\partial^2}{\partial \dot{q}^2} \right) \frac{d}{dt} \cdot \dot{q}}_{\left(\frac{\partial^2}{\partial \dot{q}^2} \cdot \dot{q} \right)} + \underbrace{\frac{\partial^2}{\partial \dot{q}^2} \cdot \ddot{q}}$$

$$\Rightarrow \frac{d}{dt} (\underline{L}(q, \dot{q}) - \frac{\partial L}{\partial \dot{q}} \cdot \dot{q}) = 0$$

value of \downarrow not related with t . define $E = L - \frac{\partial L}{\partial \dot{q}} \cdot \dot{q}$ as energy of system

$$\text{Since } L = T(q, \dot{q}) - V(q) \quad \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} = \frac{\partial T}{\partial \dot{q}} \cdot \dot{q} = 2T$$

$$\Rightarrow E = T(q, \dot{q}) + V(q)$$

87 - Homogeneity on position [Diff with homo-on time, L always include q_i . Try on compare $L(q)$ with $L(q+\epsilon)$]

$$\delta L = \sum \frac{\partial L}{\partial r_i} \cdot \delta r_i = \epsilon \sum \frac{\partial^2}{\partial r^2} = 0 \quad \delta L = 0 \text{ cause homo of position}$$

$$\epsilon \sum (\frac{\partial^2}{\partial r^2}) \frac{d}{dt} = \epsilon \frac{d}{dt} (\sum \frac{\partial^2}{\partial r^2}) = 0 \quad \text{define } P \text{ as } \sum \frac{\partial^2}{\partial r^2} - \text{Momentum}$$

$$-\text{Comments on } \frac{\partial^2}{\partial r^2} \quad \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} = -F \Rightarrow \sum F = 0$$

88. Center of Mass.

$$P = \sum mv \quad \text{For Another Frame } P = \sum m(v + v') = P' + v \sum m$$

Always exist Frame that $P=0 \Rightarrow V = P/\Sigma m = \Sigma mv/\Sigma m \leftarrow V$ of COM (center of mass)
 Integrate
 $\vec{r} = \Sigma m\vec{r}_i / \Sigma m \leftarrow \text{Position of COM}$

- Energy On different frame

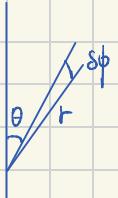
$$\begin{aligned} E_{(v_0 + v)} &= \sum \frac{1}{2} m v_0^2 + \sum m v_0 \cdot v + \sum \frac{1}{2} m v^2 + V \\ &= E' + v \cdot P + \frac{1}{2} m v^2 \quad \text{For Frame of } P=0 \\ &= E' + \frac{1}{2} m v^2 \end{aligned}$$

- Action in dif. Frame $S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$

$$\begin{aligned} L(q, (q+v), \dot{q}, \dot{q}+v) &= \frac{1}{2} m v^2 + v \cdot P + \frac{1}{2} m v_0^2 + V_{(q+v)} \\ &= L' + \frac{1}{2} m v^2 + v \cdot P' \quad \rightarrow S = S' + \frac{1}{2} m v^2 t + m v R' \end{aligned}$$

89. Rotation Motion

$$\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$



$$\begin{aligned} r \sin \theta \cdot \delta \phi &= \delta r = \delta \phi \times r \\ \delta v &= \delta \phi \times v \end{aligned}$$

$$\delta L = (\partial^2 / \partial v \cdot \delta v + \partial^2 / \partial q \cdot \delta q)$$

$$A \times [B \times C] \leftarrow \text{Assume } B \times C = D$$

$$(P \cdot \delta v + q \cdot \delta r) = P \cdot (\delta \phi \times v) + \dot{P}(\delta \phi \times r) = P(\delta \phi \times r) + \dot{P}(\delta \phi \times r)$$

$$= \delta \phi [P \times r + \dot{P} \times r]$$

Define $M = r \times p$ as Momentum - Conservation

$$\leftarrow = \delta \phi \cdot \frac{d}{dt}(r \times p) = 0$$

Similar Method on Momentum. $M' = (r+R) \times p = M + R \times p \rightarrow M$ is independent to coordinates when $P=0$

$$M' = r \times m(v+\epsilon) = M + m \times v \quad \text{if } \epsilon=v \text{ System Steady in Second Frame : } M = \underline{M_0 + R \times p} \quad \text{System Momentum.}$$

Intrinsic Momentum.

Weight of Angular Momentum $M_z = \sum m_a [r \times \dot{r}]_z = \sum m_a (x \dot{y} - y \dot{x}) = \sum m_a r^2 \dot{\phi}$

$$x = r \cos \phi \quad y = r \sin \phi \quad z = z \quad L = \frac{1}{2} m c (r^2 + (r \dot{\phi})^2 + z^2) - V \quad \leftarrow \frac{\partial L}{\partial \phi} = \sum m_a r^2 \dot{\phi}$$

810. Similarity / Ratio Idea.

Expand every term in motion

$$r \rightarrow dr$$

$$t \rightarrow \beta t$$

$$U(r_1, \dots, r_n) = \alpha^k U(r_1, r_2, \dots, r_n) \quad \rightarrow \alpha^k = (\alpha/\beta)^2$$

$$\alpha^{1-k} = \beta$$

$$V = dr/dt = (\alpha/\beta) \cdot v_0 \rightarrow E = (\alpha/\beta)^2 \cdot E_0$$

Since $L = E - V$ if Both have same scale $L = L_0 \cdot \text{Constant} \rightarrow \text{Same Equation}$

$$\Rightarrow \left(\frac{t}{t_0} \right) = \left(\frac{1}{1} \right)^{1-k} \quad \left(\frac{y}{y_0} \right) = \left(\frac{1}{1} \right)^{1-k} \quad \left(\frac{E}{E_0} \right) = \left(\frac{1}{1} \right)^{2k} \quad \frac{M}{M_0} = \left(\frac{1}{1} \right)^{1+1/2}$$

$$\text{Eg. } k=-1 \text{ Such as } V = G \frac{Mm}{r} \rightarrow \left(\frac{1}{1} \right)^{1/2} = \left(\frac{t}{t_0} \right) \rightarrow A^3/T^2 = \text{Constant.}$$

810.5 Average idea on relation of V and E .

$$f(t) = \lim_{t \rightarrow \infty} \frac{1}{t} \cdot t \int_0^t f(t) dt \quad \text{For } 2\bar{T} = \frac{\partial L}{\partial \dot{q}} \cdot \ddot{q} = \sum P \cdot V = \cancel{\frac{\partial L}{\partial t} \sum P \cdot r} - \sum \dot{P} \cdot r$$

For $\frac{\partial L}{\partial t}$ of limited function ($f(t \rightarrow \infty) = 0$) $f(t \rightarrow \infty) = 0$

$$2\bar{T} = \sum \dot{P} \cdot r \quad \text{Since } \dot{P} = \frac{\partial L}{\partial \dot{r}} \quad 2\bar{T} = k\bar{U} \quad \text{where } k \text{ is power of } r \text{ in } U$$

$$\bar{T} + \bar{U} = \bar{E} \quad \bar{U} = \frac{2}{k+2} \bar{E} \quad \bar{T} = \frac{k}{k+2} \bar{E}$$

8.11. 1D Normal Motion's Integration [Derive Time by U]

$$L = \frac{1}{2} m \dot{x}^2 - U(x) = E - 2U$$

$$\frac{1}{2} m \dot{x}^2 = E - U(x) \quad \frac{dx}{dt} = \sqrt{\frac{2(E-U(x))}{m}} \quad t = \int \frac{dx}{\sqrt{\frac{2(E-U(x))}{m}}} + \text{Constant}$$

e.g. SHM.

Only exist In $E > U(x)$ Range

$$\frac{1}{2} T = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \sqrt{\frac{1}{(E-U(x))}} dx \quad \text{where } x_1, x_2 \text{ belongs to } E = U(x_1, x_2)$$

Pendulum Situation: l, mg E in equation is constant *

$$T = \sqrt{\frac{2m}{l}} \int_{-\psi_0}^{+\psi_0} \sqrt{[mg(l(1-\cos\psi) - mg(l(1-\cos\psi_0))] \cdot l]} d\psi \quad \text{where as } \psi_0 \text{ is max angel.}$$

$$= \sqrt{\frac{2m}{l}} \int_{-\psi_0}^{+\psi_0} [mg(l(\cos\psi - \cos\psi_0))]^{-1/2} d\psi \leftarrow \text{Semi-Circle Integration: } (1-x)^{-1/2} = 1 + \frac{1}{2}x \text{ when } x \rightarrow 0$$

$$\text{See } \sin\psi = \sin(\psi_0/2)/\sin(\psi_0/2) \quad \cos\psi - \cos\psi_0 = 2\sin(\psi_0/2)^2 - 2\sin(\psi_0/2)^2$$

$$= \sqrt{\frac{2m}{l}} \cdot (mg/l)^{-1/2} \cdot l \cdot \int_{-\psi_0/2}^{\psi_0/2} [(sin\psi_0/2)^2 - (sin\psi \cdot sin(\psi_0/2))^2]^{1/2} d\psi \cdot \frac{d\psi}{d\psi} \star \text{ Try on derivative on both side}$$

$$\sin\psi = \sin(\psi_0/2)/\sin(\psi_0/2)$$

↓ derivative both side

$$\cos\psi \cdot d\psi = \cos(\psi_0/2) \cdot \frac{1}{2} \cdot \frac{1}{\sin(\psi_0/2)} d\psi$$

$$\Rightarrow d\psi/d\psi = \cos\psi \cdot 2\sin(\psi_0/2) / \cos(\psi_0/2) = \cos\psi \cdot 2\sin(\psi_0/2) / \sqrt{1 - \sin^2(\psi_0/2) \cdot \sin^2(\psi)}$$

$$= 4\sqrt{\frac{l}{g}} \cdot \int_{-\psi_0/2}^{\psi_0/2} [\sin(\psi_0/2)^{-1} \cos^{-1}\psi] \cdot [1 - \sin^2(\psi_0/2) \cdot \sin^2(\psi)]^{-1/2} \cdot \cos\psi \cdot 2\sin(\psi_0/2) d\psi$$

$$= 4\sqrt{\frac{l}{g}} \cdot \int_{-\psi_0/2}^{\psi_0/2} (1 - \sin^2(\psi_0/2) \sin^2(\psi))^{-1/2} d\psi = 4\sqrt{\frac{l}{g}} K(\sin\psi_0/2)$$

where K is semi circle Integration: $K = \int_0^{\pi/2} \frac{ds}{\sqrt{1 - k^2 \sin^2 s}}$

$$= 2\pi \cdot \sqrt{\frac{l}{g}} (1 + \frac{1}{16} \psi_0^2) \dots$$

$$\text{Normal Pendulum Period} = 2\pi \sqrt{\frac{l}{g}}$$

$$\text{For } U(x_1) = k|x_1^n| \quad T = 2\sqrt{2m} \int_0^{\infty} \frac{dx}{\sqrt{E-U(x_1)}} \quad E = k|x_1^n| \quad \dot{x}_{\max} = (E/k)^{1/n} \Rightarrow 2\sqrt{2m} \cdot \int_0^{(E/k)^{1/n}} \frac{dx}{\sqrt{E-kx^n}}$$

Try Simplify

$$T(z) = \int_0^{\infty} t^{z-1} e^{-t} dt = (z-1)! \quad [\text{For } z \in N^+]$$

$$B(Cp, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = T(p) T(q) / T(p+q)$$

$$= 2\sqrt{2m} \cdot E^{1/n - 1/2} / k^{1/n} \cdot \int_0^1 \frac{dy}{\sqrt{1-y^n}} \cdot (E/k)^{1/n}$$

Separate to $x^a (1-x)^b$
For Euler Int.B

$$\downarrow \text{Set } y^n = u \quad n y^{n-1}$$

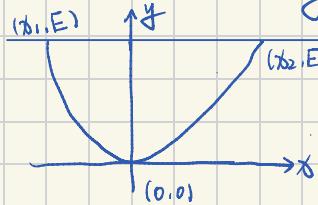
$$= \vartheta \cdot \int_0^1 du / \sqrt{1-u} \cdot \frac{dy}{du} = \vartheta \int_0^1 (1-u)^{-1/2} \cdot \frac{1}{ny^{n-1}} du$$

$$\rightarrow T = (\vartheta/h) \cdot T(c_{12}) T(c_{13}) / T(c_{1/2} + c_{1/3}) \quad \text{where } \vartheta = 2\sqrt{2m} E^{1/n - 1/2} / k^{1/n}$$

values of these equations only able proceed by computers.

$$\delta 12. \text{ Derive } U \text{ by } T, E \quad T = \int_{x_1}^{x_2} \frac{dx}{\sqrt{E-U(x_1)}} \cdot \sqrt{2m}$$

Assume Movement along $U(x_1)$ End up both side



$$T(E) = \int_E^0 \frac{dx_1}{du} \frac{du}{\sqrt{E-U(x_1)}} \cdot \sqrt{2m} + \int_0^E \frac{dx_2}{dv} \frac{dv}{\sqrt{E-U(x_2)}} \cdot \sqrt{2m}$$

$$= \int_0^E \frac{du}{\sqrt{E-U(x_1)}} \cdot \sqrt{2m} \left(\frac{dx_2}{dv} - \frac{dx_1}{du} \right) \quad \star \text{ Inverse Problem}$$

$$B(w) = \int_{\alpha}^U \frac{dE}{\sqrt{(U-E)(E-\alpha)}} = \pi \quad \text{Since } \alpha \text{ is any value - Set } \alpha = V$$

$$\int_0^{\alpha} T(E) / \sqrt{E-\alpha} \cdot dE = \int_0^U \int_0^E \frac{dudv}{\sqrt{E-U(x_1)} \sqrt{E-\alpha}} \cdot \sqrt{2m} \left(\frac{dx_2}{dv} - \frac{dx_1}{du} \right)$$

$$= \pi \cdot \int_0^U \left(\frac{dx_2}{dv} - \frac{dx_1}{du} \right) \cdot \sqrt{2m} = \sqrt{2m} \cdot \pi (B_2(\alpha) - B_1(\alpha))$$

$$\int_0^U T(E) / \sqrt{E-\alpha} \cdot dE = \sqrt{2m} \cdot \pi (B_2(U) - B_1(U))$$

δ13 Reduced Mass

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - U(x_1 - x_2)$$

$$\text{Set } \vec{r} = \vec{r}_2 - \vec{r}_1, \quad m_1 r_1 = m_2 r_2$$

$$L = \frac{1}{2} m \dot{r}^2 - U(r) \quad \text{where } m = \frac{m_1 m_2}{m_1 + m_2}$$

E.g. M and n numbers of m particle

$$M \text{ placed in } \vec{R} \quad m_a \text{ in } \vec{R}_a \quad \text{Define } \vec{r}_a = \vec{R}_a - \vec{R}$$

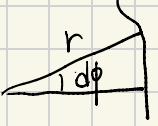
$$M \cdot R + \sum m_a r_a = 0 \quad \text{As } M = m + nm \quad \vec{R} = \frac{m}{M} \cdot \sum \vec{r}_a \quad \vec{R}_a = \vec{R} + \vec{r}_a$$

$$L = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \sum m_a \dot{r}_a^2 - U = m$$

8.14 Movements In Central Field: $F = \frac{\partial U(r)}{\partial r}$

$L = \frac{1}{2}m(c\dot{r}^2 + cr\dot{\phi}^2) - V$ Since L didn't include $\dot{\phi}$ $\cancel{\frac{\partial L}{\partial \dot{\phi}}(\frac{\partial^2}{\partial \dot{\phi}^2})} = \frac{\partial L}{\partial \dot{\phi}} = 0$

For arbitrary movements area $f = \frac{1}{2}r \cdot r \cdot d\phi$ $M = Mr^2\dot{\phi} = 0 \rightarrow$ Same with $M = r \times p$.



$$\dot{f} = (\frac{1}{2}r^2 d\phi)/dt = \frac{d\phi}{dt} \frac{1}{2}r^2 \rightarrow M = 2m\dot{f} = mr^2\dot{\phi}$$

$$E = \frac{1}{2}m(c\dot{r}^2 + r^2\dot{\phi}^2) + V$$

$$\dot{r} = \sqrt{\frac{2(E-V)}{m} - r^2\dot{\phi}^2} = dr/dt$$

$$= \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\phi}^2 + V$$

$$t = \int \left(\frac{(2E-V)/m - r^2\dot{\phi}^2}{M^2/m^2r^2} \right)^{-1/2} dr = \int \left(\frac{(2E-V)/m - r^2\dot{\phi}^2}{M^2/m^2r^2} \right)^{-1/2} dr$$

$$d\phi = \frac{M}{mr^2} dt \Rightarrow$$

$$\dot{\phi} = \int \left(\frac{(2E-V)/m - M^2/m^2r^2}{M^2/m^2r^2} \right)^{-1/2} \frac{M}{mr^2} dr \quad \text{Orbital Equation}$$

Traditional E V with Centrifugal

$$U_{\text{centri}} = \frac{1}{2}mr^2\dot{\phi}^2 = \frac{M^2}{2mr^2}$$

For Function $r = r_{\min} \cup r_{\max}$ $\Delta\phi = \int_{r_{\min}}^{r_{\max}} \left(\frac{(2E-V)/m - M^2/m^2r^2}{M^2/m^2r^2} \right)^{-1/2} \frac{M}{mr^2} dr$

$$f \Delta\phi = 2\pi n \quad n \in \mathbb{Z} \quad \text{path closed}$$

Conditions On path throw center of force $E = \frac{1}{2}mv^2 + \frac{M^2}{2mr^2} + U(r) > 0$ when $r \rightarrow 0$

$$U(r) > -\frac{M^2}{2mr^2} \quad U(r)|_{r \rightarrow 0} < -\frac{M^2}{2m}$$

E.g. Sphere Pendulum $E = \frac{1}{2}m(c\dot{r}^2 + (r\sin\theta\dot{\phi})^2) + mg\cos\theta\dot{\phi}$

$$\dot{\phi} = \sqrt{\frac{2(E-V)}{mr^2}} = d\phi/dt \quad \text{2 only include } \dot{\theta}$$

$$t = \int \left[\frac{(2E-V)/mr^2}{M^2/m^2r^2} \right]^{-1/2} d\phi \quad \text{with } U_{\text{eff}} = \frac{1}{2}mr^2\sin^2\theta\dot{\theta}^2 + mg\cos\theta\dot{\phi}$$

$$d\theta = [M \cdot cmr^2 \sin^2\theta]^{-1} d\theta \quad \leftarrow r^2 \sin^2\theta \cdot \dot{\theta} \cdot m = M$$

$$\theta = \int \left[\frac{(2E-V)/mr^2}{M^2/m^2r^2} \right]^{-1/2} [M \cdot cmr^2 \sin^2\theta]^{-1} d\phi$$

8.15 Kepler's Motion $|F| \propto 1/r^2$, $V = -\alpha/r$

$$U_{\text{eff}} = \frac{M^2}{2mr^2} - \frac{\alpha}{r} \quad \begin{array}{ll} \text{with } r \rightarrow \infty & U_{\text{eff}} = 0 \\ & r \rightarrow 0 \quad U_{\text{eff}} \rightarrow \infty \end{array} \quad U_{\text{eff}}(\min) = -\frac{M\alpha^2}{2M^2}$$



$$\dot{\phi} = \int \left(\frac{(2E-V)/m - M^2/m^2r^2}{M^2/m^2r^2} \right)^{-1/2} \frac{M}{mr^2} dr \quad \text{Set } u = 1/r$$

$$= \int \left(\frac{(2E + \alpha/r)/m - M^2/m^2r^2}{M^2/m^2r^2} \right)^{-1/2} \frac{M}{mr^2} \cdot du \cdot (-r^2)$$

$$= - \int \left(\frac{(2E + 2d(u))/m - M^2/m^2 \cdot u^2}{M^2/m^2 \cdot u^2} \right)^{-1/2} \frac{M}{m} \cdot du - \arccos(x/A)' = \frac{1}{\sqrt{A^2 - x^2}} \quad \star$$

$$M^2(2E/mM^2 + 2du/m^2 - u^2/m^2)^{-1/2} \cdot M/m - \alpha \dot{x}^2 + b \dot{y}^2 \rightarrow (\sqrt{a}\dot{x} + \frac{b}{2}\sqrt{a}\dot{y})^2 - b^2/a$$

$$[-(U/m - \alpha/m^2)^2 + \alpha^2/m^4 + 2E/mn^2]^{-1/2} \cdot dn$$

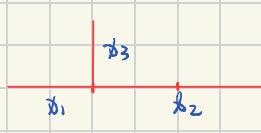
$$= \arccos [(cn - m\alpha/m^2)/A] \text{ with } A = (\alpha^2 m^2/m^4 + 2Em/m^2)^{1/2} \quad n = 1/r$$

$$\Rightarrow \psi = \arccos [cm/r - m\alpha/m] / \sqrt{\alpha^2 m^2/m^2 + 2Em}$$

Goal is $\cos\psi + 1 = r/l$ $\cos\psi = (M/r - M/\alpha^2) / \sqrt{1+2EmM^2/\alpha^2m} \cdot \sqrt{\alpha^2 m^2/m^2}$
 $= M(1/r - 1/\alpha^2) / \sqrt{1+2Em} / \alpha \cdot M/dm$

$\cos\psi + 1 = r/l$ $\Leftrightarrow r = l(1 - e^2)$
with $\alpha = M^2/dm$ $e = \sqrt{1+2Em}/\alpha$

Semi-Circle polar coordinates eq. $a_r = \alpha/(1-e^2) = M^2/dm - 2Em/\alpha = M^2/2mdE = \alpha/2E$



$$r = \alpha/(1-e^2) \Rightarrow x_1 = \alpha/(1+e) \quad x_2 = \alpha/(1-e) \quad c = \alpha$$

$$a = \frac{1}{2}(x_1+x_2) = \frac{\alpha}{2} \cdot \frac{2}{1-e^2} = \alpha/(1-e^2) \quad b = \sqrt{\alpha/(1-e^2)}$$

> Special Point on $e=0$ $2EM^2/\alpha^2m = -1$ $E = -\alpha^2m/2M^2$ Trajectory Change to circle

> Period of Motion $M = 2mf \rightarrow MT = 2mf \quad f = \pi ab$

$$T = 2\pi ab/M = 2\pi \sqrt{a^2 \sqrt{1-e^2}} / M = 2\pi a^{3/2} \sqrt{m/\alpha}$$

> Traj for dif. E, $E \geq 0$, $e > 1$, Traj is open

> $t = \int (2CE - V)/m - r^2\dot{\phi}^2)^{-1/2} dr$ Substitute to related with a, e, m

$$e = \sqrt{1+2Em}/\alpha \quad \alpha = M^2/dm \quad V = M^2/2mr^2 + \alpha/r \quad \dot{\phi} = M/mr^2 \quad a = \alpha/(1-e^2) = \alpha/2E$$

$$1-e^2 = M^2/m\alpha$$

$$t = \int (2E/m + 2\alpha/mr - M^2/m^2r^2)^{-1/2} dr$$

$$= \int (2Er^2/m + 2\alpha r/m - M^2/m^2r^2)^{-1/2} r dr$$

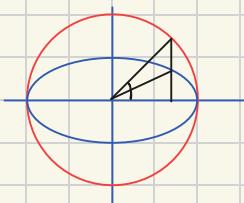
$$= \int dm/a [-r^2 + 2ar - a^2(1-e^2)]^{-1/2} \cdot r dr$$

$$= \int \frac{dm}{a} \cdot \int r dr / \sqrt{a^2e^2 - (r-a)^2} \quad \text{with } r-a = -ae \cos\theta \quad ae=c$$

- Semi Circle

$$r = a \cos\theta \quad \frac{r^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \text{Focal Radius } r = a - e \cos\theta$$

$$\Rightarrow r = a - e \cos\theta$$



$$= \sqrt{\frac{ma}{\alpha}} \int r(c^2 e^2 - e^2 a^2 \cos^2 \xi)^{-\frac{1}{2}} dr = \sqrt{\frac{ma}{\alpha}} \int r dr / a e \sin \xi = \sqrt{\frac{ma}{\alpha}} \int a(1 - e \cos \xi) d\xi = \sqrt{\frac{ma^3}{\alpha}} (\xi - e \sin \xi)$$

* Since we intake ξ as parameters, Goal of \uparrow is derive Eq. of $t \rightarrow \xi$ and $r \rightarrow \xi \Rightarrow r \rightarrow t$.

$$t = \sqrt{\frac{ma^3}{\alpha}} (\xi - e \sin \xi) \quad \text{with } r = a(1 - e \cos \xi)$$

> Similar Situation On $V = d/r \quad d > 0$

$$V = \frac{d}{r} + \frac{M^2}{2mr^2} \quad \text{with } r \text{ from } 0 \rightarrow \infty \quad V \text{ always } > 0$$

\Rightarrow Hyperbola with $P/r = -1 + e \cos \varphi \quad r_{\min} = a(e+1)$

> Int. of Motion $v_x m + \frac{dr}{r} = \text{constant}$

$$\frac{d}{dt}(v_x m + \frac{dr}{r}) = \dot{v}_x m + \frac{d}{dt}(\dot{r} \cdot r + v_r \hat{r}/r^2) = \dot{v}_x m + \frac{d\dot{r}}{r} - (v \cdot r) \hat{r}/r^3$$

$$\text{With } M = mr \times v \Rightarrow mr(v_x \dot{v}_r - v_r \dot{v}_x) + \frac{d\dot{r}}{r} - (v \cdot r) \hat{r}/r^3 = 0.$$

E.g. For $V = -\frac{d}{r}$ with $E=0$ find traj.

* diff with previous : $a = \frac{d}{2|E|} \rightarrow \infty$ when $E \rightarrow 0$

$$t = \int \frac{dr}{\sqrt{2ar/m - M^2/m^2}}$$

* Uniformization Method. : $\int \frac{dr}{\sqrt{Ar-B}}$



To cancel $\sqrt{Ar-B}$, Assume $U(r)^2 = Ar - B \quad r = \sqrt{A(B+U^2)}$

$$2ar/m - M^2/m^2 = C \cdot r^2$$

Convert int. of r to int. of u

To keep same unit. c choose as M^2/m^2

$$2ar/m = \frac{M^2}{2am}(1+J^2) \Rightarrow r = \frac{M^2}{2am}(1+J^2) \quad \frac{dr}{dJ} = \frac{M^2}{am} \cdot J$$

$$t = \int \frac{M^2}{2am}(1+J^2) \cdot \frac{M^2}{am} \cdot \left(\frac{M}{am}\right)^{-1} \cdot dJ = \int \frac{M^3}{2am}(1+J^2) = \frac{M^3}{2am}(J + \frac{J^3}{3})$$

$$\text{with } r = \frac{M^2}{2am}(1+J^2)$$

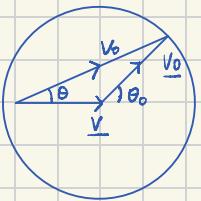
816. Split of Point of Mass.

Circle is assigned to $r = \underline{v_0}$

2 Frame : \underline{v}_{cm}

S Frame : $\underline{v}_0 \underline{v}_0$ As particle speed

$$\underline{v} + \underline{v}_0 = \underline{v}_0$$



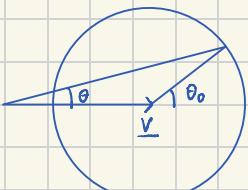
> Energy During Separate.

$$E_{int} = E_{int1} + E_{int2} + P_0^2/2m_1 + P_0^2/2m_2$$

With ϵ as split energy = $E_{int} - E_{int1} - E_{int2}$.

$$\epsilon = P_0^2/2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = P_0^2/2m$$

with $\theta_{max} = \arcsin \frac{v_0}{v_0}$



$$\text{Obv. } \tan \theta = \sin \theta \cdot \frac{v_0}{v} \cdot (v + v_0 \cos \theta)^{-1}$$

> Avg. Energy Distribution.

Azimuth : $dA/4\pi = \frac{1}{2} \sin\theta d\theta = \frac{1}{2} (d\cos\theta)$

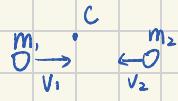
$$V^2 = V_0^2 + V_{\theta}^2 - 2V_0 V_{\theta} \cos\theta \quad \cos\theta = V_0^2 + V_{\theta}^2 - V^2 / 2V_0 V_{\theta} \rightarrow d\cos\theta = d(V_{\theta}^2) / 2V_0 V_{\theta}$$

$$T = \frac{1}{2} m v^2 \quad dT = \frac{1}{2} m d(V^2) \Rightarrow \frac{1}{2} \cdot \frac{dV^2}{2V_0 V_{\theta}} = \frac{dT}{2V_0 V_{\theta} m}$$

$$T \in [\frac{1}{2} m (V_0 + V_{\theta})^2 \sim \frac{1}{2} m (V_0 - V_{\theta})^2]$$

> Max Energy of particle. [M_i ref to all other parts] $T_{max} = \frac{P_0^2/m_i = M - m_i/M (E_{int} - E_{int'} - E)}{E}$

87. Elastic Collision.



V_{10}, V_{20} : C.M. Frame
 V_1, V_2 : Lab. Frame.

$$V = \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2} \Rightarrow V_{10} = V_1 - V = \frac{m_1 V_1 + m_2 V_2 - m_1 V_1 - m_2 V_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} (V_1 - V_2)$$

$$V_{20} = \frac{m_1}{m_1 + m_2} (V_2 - V_1). \text{ define } V = V_1 - V_2$$

$$V_{10} = \frac{m_2}{m_1 + m_2} \cdot V \quad V_{20} = \frac{m_1}{m_1 + m_2} \cdot V$$

↓ After Collision.

$$V_{10}' = \frac{m_2}{m_1 + m_2} \cdot \hat{n}V \quad V_{20}' = \frac{m_1}{m_1 + m_2} \cdot (-\hat{n}V)$$

$$V_1' = \frac{m_2}{m_1 + m_2} \hat{n}V + \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2} \quad \left. \begin{array}{l} \text{(C.M. Frame)} \\ \text{(C.M. Relate to Lab.)} \end{array} \right\}$$

Vice Versa - V_{20}'

Motion in Lab frame

> Try plot Momentum.

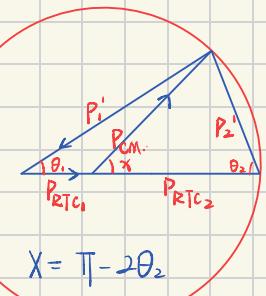
$$P_1' = m \cdot \hat{n}V + \frac{m_1}{m_1 + m_2} (P_1 + P_2)$$

$$P_2' = m \cdot \hat{n}V + \frac{m_2}{m_1 + m_2} (P_1 + P_2)$$

$$\left\{ \begin{array}{l} P_1' = P_{10}' + P_{lab. to 1} \\ P_2' = P_{20}' + P_{lab. to 2} \end{array} \right.$$

$$\tan\Theta_1 = P_2' \sin\Theta_2 / P_{RTC_1} + P_{CM} \cos\Theta_1$$

↔



> Specially for m_2 steady

$$P_{RTC_1} = \frac{m_1 V}{m_1 + m_2}$$

$$P_{RTC_2} = \frac{m_2 m_1 V}{m_1 + m_2}$$

$$P_{CM} = m \hat{n} V_1 = \frac{m_1 m_2 V_1}{m_1 + m_2}$$

Same

* P_{CM}. For two particle is same.

* P_{RTC}. P of Frame refrence to center of Mass.

$$V_1' = P_1'/m_1 = (\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos\chi} / m_1 + m_2) \cdot V$$

$$V_2' = 2m_1 V / m_1 + m_2 \cdot \sin\chi/2$$

88. Scattering of points. [★ Center of forth is in O.]

$$\phi_0 = \int_{r_{min}}^{\infty} (M/r^2) \cdot dr / \sqrt{2m(E-V) - M^2/r^2} \quad (\text{chap 14}).$$

$$\text{With } E = \frac{1}{2} m V_{\infty}^2 \quad M = m P \times V_{\infty}$$

* P is related to x

$$\phi_0 = \int_{r_{min}}^{\infty} (m P V_{\infty} / r^2) \cdot dr / \sqrt{m^2 V_{\infty}^2 - 2mV - m^2 P^2 V_{\infty}^2 / r^2}$$

$$= \int_{r_{min}}^{\infty} (P/r^2) \cdot dr / \sqrt{1 - (P^2/r^2) - (2V/m V_{\infty}^2)} \quad [\text{Relation of } \phi_0 \text{ & } P]$$

$$X = |\pi - 2\phi_0|$$

define $dN \rightarrow$ num. of particle in $x \sim x+dx$
 $n \rightarrow$ num. in unit time and area. $\frac{d\sigma}{d\Omega} = dN/n$
 Num of part. $dN = 2\pi p \cdot dp \cdot n = 2\pi P_{(x)} \cdot dP_{(x)}/dx \cdot dx$

$$\text{Use solid angle } d\Omega = 2\pi \sin x \cdot dx \Rightarrow d\sigma = \frac{P_{(x)}}{\sin x} \cdot \frac{dp}{dx} \cdot dx$$

Prob. 4 $U = -\alpha/r_2$ fall to center effective cross-section $d\sigma$

$$1/2mv^2 > -\alpha/r_2 \quad P_{\max} = \sqrt{2\alpha/mv_\infty^2}$$

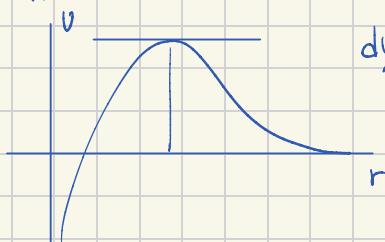
$$\alpha > 1/2mv_\infty^2 P^2 \quad \text{with } \sigma = \pi P_{\max}^2 = 2\pi \alpha/mv_\infty^2$$

$\cdot U_{\text{eff}} < E$ Attach

$$\Rightarrow U_{\text{eff max}} = E$$

Prob. 5 $U = -\alpha/r^n$ ($n > 2, \alpha > 0$) Solve $P = \infty$

$$U_{\text{eff}} = mp^2 V_\infty^2 / 2r^2 - \alpha/r^n$$



$$\frac{dU}{dr} = -mp^2 V_\infty^2 + \alpha nr^{-n+2} = 0$$

$$-mp^2 V_\infty^2 / \alpha n = r^{-n+2}$$

$$r = (-mp^2 V_\infty^2 / \alpha n)^{1/(n-2)}$$

$$U_{\max} = 1/2(n-2) \alpha (mp^2 V_\infty^2 / \alpha n)^{n/(n-2)}$$

$$\text{Similar } U_{\text{eff}} = E = 1/2mv_\infty^2 \Rightarrow P = \infty \quad \text{Area} = \pi P^2 = \infty$$

Prob. 6 m_1 with m_2, R sphere. Area $\star \frac{1}{r} \rightarrow \infty$ when $r \rightarrow 0$ No U_{\max} exist.

$$r_{\min} = R \quad \text{with } V=0$$

$$E = M^2/2mr^2 - \alpha/r = 1/2mv_\infty^2 \quad \text{with} \quad \alpha = 2m_1 m_2$$

$$M = mv_\infty P$$

$$\frac{V_\infty P^2}{r^2} - \frac{2\alpha}{rmv^2} = 1 \quad P^2 = (1 + 2\alpha/rmv^2) \cdot r/V_\infty$$

Prob. 7 ACE, x) derive $U(r)$

$$\underline{\pi P^2} = \int_0^\pi (d\sigma/d\Omega) r \cdot dx$$

Area of collision

819. Rutherford's Formula [effective cross section in $V = \alpha/r$]

$$\text{with } V = \alpha/r \quad \rightarrow \quad \sigma = \int_{r_{\min}}^{\infty} (M/r^2) dr / \sqrt{2m[E - Ur] - M^2/r^2}$$

$$= \int_{r_{\min}}^{\infty} (P/r^2) dr / \sqrt{1 - (P^2/r^2) - (2U/mv_{\infty}^2)}$$

$$= \int_{r_{\min}}^{\infty} (P/r^2) dr \cdot (1 - (P^2/r^2) - 2U/mv_{\infty}^2)^{-1/2} dr \quad u = r^{-1} \quad -r^2$$

$$\cos \phi_0 = \alpha/mv_{\infty} P / (1 + (\alpha/mv_{\infty} P)^2)^{1/2}$$

$$P \nu^2 du (1 - P^2 u^2 - 2\alpha u / mv_{\infty}^2)^{-1/2} \cdot dr / du \cdot -X u^2$$

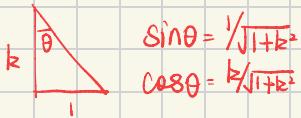
$$\begin{array}{c} \uparrow \\ \downarrow \end{array}$$

$$\phi_0 = \cos^{-1} \left(\frac{\alpha/mv_{\infty}^2 P}{\sqrt{1 + (\alpha/mv_{\infty}^2 P)^2}} \right) \leftarrow \int_u^0 \left\{ -Cu + \frac{\alpha^2}{mv_{\infty}^2 P^2} + \frac{\alpha^2}{m^2 v_{\infty}^4} P^4 + \frac{1}{2} X \right\}^{-1/2} du.$$

* Trigonometric functions

$$\tan \phi = mv_{\infty}^2 P / \alpha \rightarrow P^2 = (\alpha^2 / m^2 v_{\infty}^4) \cdot \tan^2 \phi_0$$

$$= (\alpha^2 / m^2 v_{\infty}^4) \cdot \cot^2 \frac{1}{2} X$$



$$\sin \theta = 1/\sqrt{1+k^2}$$

$$\cos \theta = k/\sqrt{1+k^2}$$

$$\rightarrow \tan \theta = 1/k$$

$$\Rightarrow d\sigma = \pi P^2 \cdot dP/dx = (\alpha/2mv_{\infty}^2)^2 \cdot dx / \sin^4 \frac{1}{2} X$$

[Rutherford's Formula]

> Special Situations

1. Particle m_2 rest initially

$$X = \pi - 2\theta_2$$

$$X = \pi - 2\theta_2 \quad d\sigma_2 = \pi (\alpha/mv_{\infty}^2)^2 \cdot \cos^{1/2} X \cdot dx / \sin^3 \frac{1}{2} X \leftarrow -\sin(\theta_2) 2d\theta_2 / \cos^3(\theta_2)$$

$$= +2\pi (\alpha/mv_{\infty}^2)^2 \cdot \sin \theta_2 \cdot d\theta_2 / \cos^3(\theta_2) \quad \text{with } d\sigma_2 = 2\pi \sin \theta_2 \cdot d\theta_2$$

$$= (\alpha/mv_{\infty}^2)^2 \cdot d\theta_2 / \cos^3(\theta_2)$$

2. Mass Relation

$$M_2 \gg m_1 \quad \theta_1 \approx X \quad m_1 \approx m \quad d\sigma_i = (\alpha/4E_1)^2 \cdot d\theta_i / \sin^4(\frac{1}{2}\theta_1) \quad \text{with } E = \frac{1}{2}m_1 v_{\infty}^2$$

$$m_1 = M_2 \quad m = \frac{1}{2}m \quad X = 2\theta_1 \quad d\sigma_i = (\alpha/E_1)^2 \cos \theta_1 \cdot d\theta_i / \sin^4 \theta_1$$

3. Stick After Collision

$$d\sigma_i = (\alpha/E_1)^2 (1/\sin^4 \theta + 1/\cos^4 \theta) \cos \theta \cdot d\theta$$

> Use split Energy & express σ [In m_2 Steady State]

$$V_2' = \frac{2mv}{m_1+m_2} \cdot \sin \frac{1}{2} X \quad \epsilon = \frac{1}{2} m_2 V_2'^2 = (\frac{2m}{m_2}) V_{\infty}^2 \cdot \sin^2 \frac{1}{2} X$$



$$d\sigma = 2\pi (X^2 / m_2 V_{\infty}^2) \cdot dE / \epsilon^2$$

Prob. 1 $V = \alpha/r_2$ — Find effective cross-section. ($\alpha > 0$)

$$\int_{r_{\min}}^{\infty} (P/r^2) dr / \sqrt{1 - (P^2/r^2) - (2U/mv_{\infty}^2)}$$

$$V = \alpha/r_2$$

[X, m, V_{∞} is known]

$$= \int_{r_{\min}}^{\infty} (P/r^2) dr / \sqrt{1 - (P^2/r^2) - (2\alpha/mr^2v_{\infty}^2)}$$

$$= \int_u^0 P \cdot du / \sqrt{1 - P^2 u^2 - 2\alpha u^2 / m v_{\infty}^2}$$

$$= \int_u^0 \frac{P \cdot du}{\sqrt{1 - P^2 u^2}} \cdot \frac{P}{P} \quad [\arcsin(Ax)' = \frac{Adx}{\sqrt{1-A^2x^2}}]$$

$$= [\arcsin(Pu) \cdot \frac{P}{P}]_u^0 = P/2[\pi/2] \Rightarrow X = \pi [1 - P/2] \quad P = mv$$

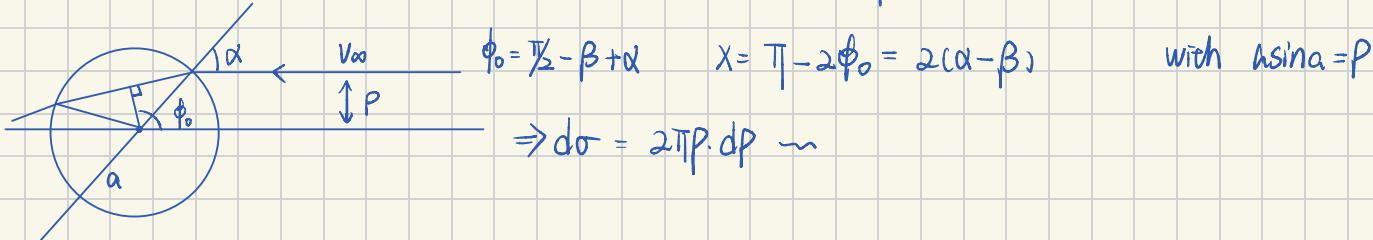
$$\text{with } d\sigma = 2\pi P dp$$

Prob 2. Radius a depth v_0 . Potential Wall

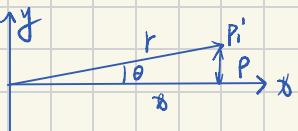
$$\text{Incident angle } v_i \sin \theta_i = v_0 \sin \theta_2 \text{ with } \frac{1}{2}mv_i^2 + v_i = \frac{1}{2}mv_2^2 + v_2$$

$$\sin \theta / \sin \theta_2 = \sqrt{1 + \frac{2}{m}mv_i^2(v_1 - v_2)}$$

$$\sin \alpha / \sin \beta = \sqrt{1 + \frac{2v_0}{m}v_{\infty}^2}$$



8.20. Small-angle Scattering [Small angle deflection. Weak impact on v . Large impact parameters.]



$$\sin \theta_i = P_{iy}' / P_i' \approx \theta_i \quad \text{as } P_i \approx P_i' \quad \theta_i \approx P_{iy}' / m_i v_{\infty}$$

Connect P_{iy}' to (y, r)

$$F_y = -\frac{\partial v}{\partial y} = -(\frac{\partial y}{\partial r}) \cdot \frac{\partial r}{\partial y} \quad \text{Since } r = \sqrt{x^2 + y^2 + z^2} \quad \star \quad \text{Use } r^2 = x^2 + y^2 + z^2$$

Derivative both side

$$= -(\frac{dy}{dr}) \cdot \frac{y}{r} \quad \text{For particle in straight line } y = p \quad 2r \frac{\partial y}{\partial r} = 2y \rightarrow \frac{\partial y}{\partial r} = \frac{y}{r}$$

$$= -(\frac{dy}{dr}) \cdot \frac{P}{r} \quad \Rightarrow \quad P_{iy}' = -P_{V_{\infty}} \cdot \int_r^{\infty} \frac{dy}{dr} \cdot \frac{dr}{r} \quad \text{For } r = \sqrt{r^2 - p^2} \quad 2r \frac{dy}{dr} = 2r$$

$$P_{iy}' = m_i v_{\infty} \sin \theta_i \quad \rightarrow \quad \theta_i = P_{iy}' / m_i v_{\infty} = -2P_{V_{\infty}} \int_r^{\infty} \frac{dy}{dr} \cdot \frac{dr}{r} \cdot \frac{1}{r}$$

$$= -2P_{V_{\infty}} \int_r^{\infty} \frac{dy}{dr} \cdot \frac{dr}{\sqrt{r^2 - p^2}}$$

$$\text{with } d\sigma = |\frac{dP}{d\theta_i}| \cdot P(\theta_i) / \theta_i \cdot d\theta_i$$

Chap V. OSC.

8.21. Free oscillation 1-D.

$$U(q) - U(q_0) \approx \frac{1}{2}k(q - q_0)^2 \quad \text{define } x = q - q_0$$

$$\Rightarrow \ddot{x} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad \text{with Eq. } m\ddot{x} + kx = 0$$

$$E = \frac{1}{2}mv^2 a^2$$

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) \quad \text{Or} \quad \cos(\omega t + \phi)$$

$$x = \text{Re}[A e^{i\omega t}]$$

$$A = \sqrt{C_1^2 + C_2^2}$$

$$\tan \phi = -C_2/C_1 \leftarrow = a[\cos \omega t \cos \phi - \sin \omega t \sin \phi]$$

$A = A \exp(i\omega t)$ with $A =$ complex amplitude.

8.22. Forced Oscillations \square

8.23. More degree oscillations

Set $V(q_1, q_2, \dots)$ minimum when $q_i = q_{i0}$. Set $\dot{q}_i = \dot{q}_i - \dot{q}_{i0}$. As small oscillation close to q_{i0} .

$$V = \sum_k \frac{1}{2} k_{ik} \dot{q}_i \dot{q}_k \quad \text{with } k_{ik} = k_{ki} \text{ obs.} \rightarrow L = \frac{1}{2} \sum_k (m_{ik} \ddot{q}_i \dot{q}_k - k_{ik} \dot{q}_i \ddot{q}_k)$$
$$dL = \frac{1}{2} \sum_k [m_{ik} (\dot{q}_i \ddot{q}_k + \ddot{q}_k \dot{q}_i) - k_{ik} (\dot{q}_i \ddot{q}_k + \ddot{q}_k \dot{q}_i)]$$

Since k_{ik} is symmetry $dL = \sum_k [m_{ik} \dot{q}_i \dot{q}_k - k_{ik} \dot{q}_i \dot{q}_k]$

$$\Rightarrow \frac{d}{dt} (\partial L / \partial \dot{q}_i) - \frac{\partial L}{\partial q_i} = 0 \rightarrow \sum m_{ik} \ddot{q}_k + \sum k_{ik} \dot{q}_k = 0 \quad \text{For } (i=1, 2, \dots, n)$$

Similar Solution Trying - $\dot{q}_k = A_k \exp(i\omega t)$ $\rightarrow \sum_k (-\omega^2 m_{ik} + k_{ik}) A_k = 0$

* Linear Algebra : $B \cdot \text{adj}(CB) = \det(CB) \cdot I \leftarrow \text{adj}(CB) = \Delta^T$ $\sum_k B \Delta_k = 0$ Goal is Solve A_k

$$B \cdot \Delta^T = \det(CB) \cdot I = 0 \quad [W = WA]$$

$\sum_k B \Delta_k = 0$ For Δ_k is Algebraic Residual $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} c & d \\ a & b \end{bmatrix} = a$

$$\Rightarrow \dot{q}_k = \Delta_k a \exp(i\omega t) \leftarrow A = \Delta_k a \quad a \text{ is random complex value.}$$

[Comments on w] : $\dot{q} = \exp(i\omega t)$ If W_A include i , $i\omega t$ include time \rightarrow function unstable.

$$\dot{q}_k = \operatorname{re} \sum_{\alpha=1}^8 \Delta_{k\alpha} C_\alpha \exp(i\omega t) = \sum_{\alpha=1}^8 \Delta_{k\alpha} \Theta_\alpha \quad \text{with } \Theta_\alpha = \operatorname{Re}[C_\alpha \exp(i\omega t)]$$

* Observe Θ_α as new coordinates, \dot{q}_k base on Θ_α exist.

i). Separate Complex vibration to linear compose of SHM.

ii). Since $\Theta_\alpha = \operatorname{re}[C_\alpha \exp(i\omega t)]$ Θ_α satisfy $\ddot{\Theta}_\alpha + \omega^2 \Theta_\alpha = 0$

each Θ_α independent oscillations.

$$L = \sum_k m_k (\dot{\Theta}_\alpha^2 - \omega^2 \Theta_\alpha^2)$$

Also, define $Q_\alpha = \sqrt{m_\alpha} \Theta_\alpha$

$$L = \frac{1}{2} C (\dot{Q}_\alpha^2 - \omega^2 Q_\alpha^2) \quad \text{All } Q_\alpha \text{ oscillation independent.}$$

> For Forced Oscillation $F(t)$.

$$L = L_0 + \sum_k F_k(t) \cdot \dot{q}_k \leftarrow \text{Try use } Q_k = \operatorname{Re}[C_k \exp(i\omega_k t)].$$

$$= \frac{1}{2} (Q_k^2 - \omega^2 Q_k^2) + \sum_k f_{k(t)} \cdot Q_k \quad \text{for } f_{k(t)} = \Delta k / \sqrt{m} \cdot F_{k(t)}$$

With eq. $\ddot{Q}_a + \omega_a^2 Q_a = f_{act}$

$$\text{Prob 1. } L = \frac{1}{2} (x^2 + y^2) - \frac{1}{2} \omega_0^2 (x^2 + y^2) + axy$$

$$\ddot{x} = \omega_0^2 x - ay$$

$$\ddot{y} = \omega_0^2 y - ax$$

$$\text{with } x, y = A_k \exp(j\omega_k t)$$

General Form is $A_x \ddot{Q}_1 + A_y \ddot{Q}_2$.

$$A_x \omega_x^2 = \omega_0^2 A_x - a A_y$$

$$A_y \omega_y^2 = \omega_0^2 A_y - a A_x$$

$$[\Delta = \omega^2 - \omega_0^2]$$

The general solution is $\det(CB) = 0$

$$(\omega^2 - \omega_0^2)^2 = a^2$$

$$B = \begin{vmatrix} -a & \Delta \\ \Delta & -a \end{vmatrix}$$

$$x^2 + y^2 = (Q_1 + Q_2)^2 + (Q_1 - Q_2)^2$$

$$= 2Q_1^2 + 2Q_2^2 \rightarrow \text{Normalize } \frac{1}{L} \Rightarrow \omega^2 = a + \omega_0^2$$

$$\text{or } \omega_0^2 = a$$

$$x = \frac{1}{L}(Q_1 + Q_2) \quad y = \frac{1}{L}(Q_1 - Q_2) \quad \leftarrow A_x = A_y$$

$$A_x = -A_y$$

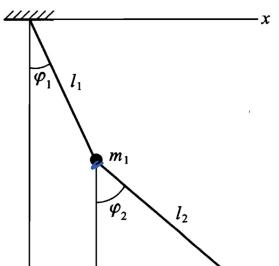


图 1

Prob 2. Solve 1.

$$[\psi_1, \psi_2 \ll 1 \cos \psi = 1 - \frac{1}{2}\psi^2]$$

$$L = \frac{1}{2} (m_1 + m_2) \cdot (l_1 \dot{\psi}_1)^2 + \frac{1}{2} m_2 (l_2 \dot{\psi}_2)^2 + m_2 l_1 l_2 \dot{\psi}_1 \dot{\psi}_2 - \frac{1}{2} (m_1 + m_2) g l_1 \psi_1^2 - \frac{1}{2} m_2 g l_2 \psi_2^2$$

$$m_2: \begin{cases} \ddot{x} = l_1 \sin \psi_1 + l_2 \sin \psi_2 \rightarrow l_1 \cos \psi_1 \cdot \dot{\psi}_1 + l_2 \cos \psi_2 \cdot \dot{\psi}_2 \\ \ddot{y} = l_1 \cos \psi_1 + l_2 \cos \psi_2 \rightarrow -l_1 \sin \psi_1 \cdot \dot{\psi}_1 - l_2 \sin \psi_2 \cdot \dot{\psi}_2 \end{cases}$$

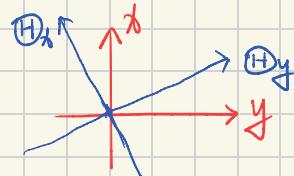
$$V_s^2 = l_1^2 \dot{\psi}_1^2 + l_2^2 \dot{\psi}_2^2 + 2l_1 l_2 \dot{\psi}_1 \dot{\psi}_2 (\sin \psi_1 \sin \psi_2 + \cos \psi_1 \cos \psi_2) \quad \leftarrow \psi_1, \psi_2 \ll 1 = 1$$

$$L \text{ on } \dot{\psi}_1: (m_1 + m_2) l_1^2 \ddot{\psi}_1 + m_2 l_1 l_2 \ddot{\psi}_2 = (m_1 + m_2) g l_1 \psi_1$$

$$L \text{ on } \dot{\psi}_2: m_2 l_2^2 \ddot{\psi}_2 + m_2 l_1 l_2 \ddot{\psi}_1 = m_2 g l_2 \psi_2$$

$$\dot{\phi}_a = A_a \exp(j\omega_a t)$$

* Ideas on coordinates.



Motion Separate to θ_x, θ_y

θ_x, θ_y contribute to x, y

$$-A_1 (m_1 + m_2) l_1^2 \omega_1^2 - m_2 l_1 l_2 \omega_2^2 \cdot A_2 = (m_1 + m_2) g l_1 A_1$$

$$-A_2 m_2 l_2^2 \omega_2^2 - m_2 l_1 l_2 \omega_1^2 A_1 = m_2 g l_2 A_2$$

$$A_1 (m_1 + m_2) (l_1^2 \omega^2 + g l_1) + A_2 m_2 l_1 l_2 \omega^2 = 0$$

→ Convert to $B \cdot A = 0$ form

$$A_2 (g + l_2 \omega^2) + l_1 \omega^2 A_1 = 0$$

$$B = \begin{bmatrix} (m_1 + m_2)(g - l_1 \omega^2) & -m_2 l_2 \omega^2 \\ -l_1 \omega^2 & g - l_2 \omega^2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \det(CB) = 0 \rightarrow (g - l_2 \omega^2)(g - l_1 \omega^2)(m_1 + m_2) = -m_2 l_2 l_1 \omega^4$$

$$W_1 W_2 A_1 A_2 = \dots$$

