Mechanics – Landau.

Leo.

- Chap 1. Least Action Theory [$S=\int_{t_1}^{t_2} \mathcal{L}(q,\dot q,t)\cdot dt$]

§1. Defin Action : $S=\int \mathcal{L}\, dt$

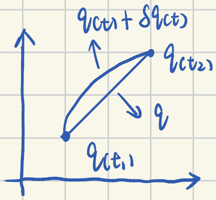
Variational Inference $\delta S=\int \delta \mathcal{L}(q,\dot q,t)\, dt = \int \left(\partial\mathcal{L}/\partial q \cdot \delta q + \underbrace{\partial\mathcal{L}/\partial\dot q\cdot \delta\dot q}\right)\cdot dt = 0$

$\delta\dot q = d/dt\,\delta q$

- Comments on q

$\mathcal{L}\to q$ for $q=q_{min}+\underbrace{\delta q(t)}_{\text{cause change}}$

$$\int \partial\mathcal{L}/\partial q\cdot\delta q + \underbrace{\delta q\cdot\partial\mathcal{L}/\partial\dot q\Big|_{t_1}^{t_2}}_{0} - \int \delta q\,(\partial\mathcal{L}/\partial\dot q)\, d/dt = 0$$



← As seen in defination
$q(t_1)+\delta q(t_1)=q(t_1)$
$\delta q(t_1)=\delta q(t_2)=0$

$$\int \delta q\left(\partial\mathcal{L}/\partial q - (\partial\mathcal{L}/\partial\dot q)\, d/dt\right)=0$$

$\partial\mathcal{L}/\partial q = (\partial\mathcal{L}/\partial\dot q)\, d/dt$ Lagrange Equation

§2. - Limitation / Variation form on $\mathcal{L}'(q,\dot q,t)=\mathcal{L}(q,\dot q,t)+d/dt\, f(x,t)$

For $S'=\int_{t_1}^{t_2}(\mathcal{L}+d/dt\, f(x,t))\,dt$

$= S + f(x,t_2) - f(x,t_1)$

$\delta S'=\delta S+\underbrace{\delta[f(x,t_2)-f(x,t_1)]}_{0}$ Since $t_1 t_2 \to q_1 q_2$ is assigned. $\delta S'=\delta S+0$ both Function Same

$\Rightarrow$ Lagrange isn't change when $\pm d/dt\, f(x,t)$

§3. - Frame of refrence

Galilean refrence Frame - isotropy in time and position. $\Rightarrow \mathcal{L}(q,\dot q,t)$ only include $\mathcal{L}(\dot q)$

$\Omega(x)=\Omega(y)$ $\Omega(t_1)=\Omega(t_2)$ → No vector include $\dot q^2=\dot q\cdot\dot q$ = only values

$\mathcal{L}(\dot q^2)$

$(\partial\mathcal{L}(\dot q^2)/\partial\dot q)\, d/dt = \underbrace{\partial\mathcal{L}/\partial q}_{0}$

⇓

$\partial\mathcal{L}/\partial\dot q$ = constant $\Rightarrow$ V is constant

$\boxed{\text{Neuton's } 1^{st} \text{ Law}}$

- Comments on dif Frame

$\mathcal{L}'$ move v along $\mathcal{L}$ $r'=r+Vt$ $t=t'$

§4. - Same movement in dif Frame

$\mathcal{L}(v'^2)=\mathcal{L}(v^2)$ where $v'=v+\varepsilon$

due to Frame dif.

$\mathcal{L}(v^2+2\varepsilon v+\varepsilon^2)$ have to $=\mathcal{L}(v^2)$

Seperation $d/dt\, f(x,t) = \partial f/\partial t + \underbrace{\sum_i \partial f/\partial x_i\cdot \partial x_i/\partial t}_{\nabla f\cdot v}$

Linear relasion with v.

$=\mathcal{L}(v^2)+\partial\mathcal{L}/\partial v^2\cdot(2\varepsilon v+\underbrace{\varepsilon^2}_{\text{small term cancel}})$

Assume to $\mathcal{L}=kv^2$

$=\mathcal{L}(v^2)+\boxed{2\,\partial\mathcal{L}/\partial v^2\,(\varepsilon\cdot v)}$ Must belongs to $d/dt\, f(x,t)$ $\Rightarrow \partial\mathcal{L}/\partial v^2$ = Constant $\mathcal{L}\propto v^2$

- Verification on normal value $\underline{v}$

$$L(v+\underline{v})^2 = k(v+\underline{v})^2 = k[v^2+2v\cdot\underline{v}+\underline{v}^2] = L(v^2) + d/dt\,(2kr\cdot\underline{v} + k\underline{v}^2 t)$$

- Comments on k

$$S = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} kv^2\cdot dt$$ Since S always minmum value, if kv^2 $k<0$, No minimum exsist.

$\Rightarrow$ k alwayse >0.

§5. Interactions on particles in Same Frame.

For $L = kv^2 - U(x_1, x_2 \cdots x_n)$

$$(\partial L/\partial v)\,d/dt = 2k\,dv/dt = -\partial U/\partial x$$ Since $F = m\cdot dv/dt$ define $k = 1/2\,m$

Similar on Dual-Frame $f(q_1 \cdots q_n)$ $\dot{f} = \Sigma\, \partial f/\partial q_i \cdot \dot{q}_i$

$$L = \Sigma\, a_{ik}(q)\,\dot{q}_i \dot{q}_k - U(q)$$

Chap 2. - Energy // Conservation.

§6. Energy

- Homogenity of time [happens not related with time]

$= L(q, \dot{q})$ Try on $d/dt\, L(q,\dot{q}) = \partial L/\partial q \cdot \dot{q} + \partial L/\partial \dot{q}\cdot \ddot{q} = +(\partial L/\partial \dot{q})\,d/dt \cdot \dot{q} + \partial L/\partial \dot{q}\cdot \ddot{q}$

$= (\partial L/\partial \dot{q}\cdot \dot{q})\, d/dt$

$$\Rightarrow d/dt\,(L(q,\dot{q}) - \partial L/\partial \dot{q} \cdot \dot{q}) = 0$$

value of ____ not related with t. define $E = L - \partial L/\partial \dot{q}\cdot \dot{q}$ as energy of system.

Since $L = T(q,\dot{q}) - U(q)$ $\partial L/\partial \dot{q}\cdot \dot{q} = \partial T/\partial \dot{q} \cdot \dot{q} = 2T$

$$\Rightarrow E = T(q,\dot{q}) + U(q)$$

§7 - Homogenety on position [Dif with homo- on time, L alwayse include q, Try on compare $L(q)$ with $L(q+\varepsilon)$]

$\delta L = \Sigma\, \partial L/\partial r \cdot \delta r_a = \varepsilon\, \Sigma\, \partial L/\partial r = 0$ $\delta L = 0$ cause homo of position

$\Downarrow$

$\varepsilon\, \Sigma(\partial L/\partial \dot{r})\, d/dt = \varepsilon\, d/dt\,(\Sigma\, \partial L/\partial \dot{r}) = 0$ define P as $\Sigma\, \partial L/\partial \dot{r}$ - Momentum

- Comments on $\partial L/\partial r$ $\partial L/\partial r = \partial U/\partial r = -F \Rightarrow \Sigma F = 0$

§8. Center of Mass.

$P = \Sigma mv$ For Another Frame $P = \Sigma m(V + v') = P' + v\Sigma m$

Alwayse exsist Frame that $P=0 \Rightarrow V = P/\Sigma m = \Sigma mv/\Sigma m$ ← V of COM (center of mass)

Integrate ↓

$\vec{r} = \Sigma m x / \Sigma m$ ← Position of COM

– Energy On different frame

$$E(v_a'+v) = \Sigma \tfrac{1}{2} m v_a'^2 + \Sigma m_a v_a' v + \Sigma \tfrac{1}{2} m V^2 + U$$

$$= E' + v\cdot p + \tfrac{1}{2}\mu v^2$$ For Frame of $P=0$

$$= E' + \tfrac{1}{2}\mu v^2$$

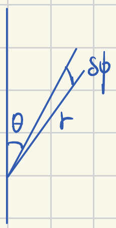
– Action in dif. Frame $S = \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$

$$L(q,(q+q')',t) = \tfrac{1}{2}\mu v^2 + \underline{v\cdot p} + \tfrac{1}{2}\underline{\mu v_a^2} + \underline{U(q+q')}$$

$$= L' + \tfrac{1}{2}\mu v^2 + v\cdot p' \longrightarrow S = S' + \tfrac{1}{2}\mu v^2 t + \mu v R'$$

§9. Rotation Motion

$\varepsilon_{ijk}\cdot\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$



$r\sin\theta\cdot\delta\phi = \delta r = \delta\phi\times r$

$\delta v = \delta\phi\times v$

$A\times[B\times C]$ ← Assume $B\times C = D$ ..

$\varepsilon_{ijk}A_jD_k = \varepsilon_{ijk}A_j(\varepsilon_{kmn}B_mC_n)$

$= \varepsilon_{ijk}\cdot\varepsilon_{kmn}\cdot A_jB_mC_n = \cdots$

$$\delta L = (\partial L/\partial v\cdot\delta v + \partial L/\partial q\cdot\delta q)$$

$$(P\cdot\delta v + \dot{q}\cdot\delta r) = P\cdot(\delta\phi\times v) + \dot{p}(\delta\phi\times r) = P(\delta\phi\times\dot{r}) + \dot{P}(\delta\phi\times r)$$

$$= \delta\phi[P\times\dot{r} + \dot{p}\times r]$$

Define $M = r\times p$ as Momentum – Conservation ← $= \delta\phi\cdot d/dt(r\times p) = 0$.

Similar Method on Momentum. $M' = (r+R)\times p = M + R\times P$ → M is independent to coordinates when $P=0$

$M' = r\times\mu(v+\varepsilon) = M + \mu r\times\varepsilon$ if $\varepsilon = v$ System Steady in Second Frame : $\overbrace{M}^{\text{System Momentum}} = \underbrace{M_0}_{\text{Intrinsic Momentum}} + R\times P$

Weight of Angular Momentum $M_z = \Sigma m_a[r\times\dot{r}]_z = \Sigma m_a(x_a\dot{y}_a - \dot{x}_a y_a) = \Sigma m_a r^2\dot{\varphi}$

$x = r\cos\varphi$ $y = r\sin\varphi$ $z = z$ $L = \tfrac{1}{2}m(\dot{r}^2 + (r\dot{\phi})^2 + \dot{z}^2) - U \longrightarrow \partial L/\partial\dot{\phi} = \Sigma m_a r^2\dot{\varphi}$

§10. Similarity / Ratio Idea.

Expand every term in motion $r\to\alpha r$, $t\to\beta t$

$U(\alpha r_1,\cdots,\alpha r_n) = \alpha^k U(r_1,r_2,\cdots,r_n) \longrightarrow \alpha^k = (\alpha/\beta)^2$ $\boxed{\alpha^{1-\frac{1}{2}k} = \beta}$

$v = dr/dt = (\alpha/\beta)\cdot v_0 \to E = (\alpha/\beta)^2\cdot E_0$

Since $L = E - U$ if Both have same scale $L = L_0\cdot$ Constant → Same Equation

$\Rightarrow (t/t') = (l/l')^{1-\frac{1}{2}k}$ $(v/v') = (l/l')^{\frac{1}{2}k}$ $(E/E') = (l/l')^{k}$. $M/M' = (l/l')^{1+k/2}$.

Eg. $k=-1$ Such as $U = G\frac{Mm}{r} \to (l/l')^{3/2} = (t/t') \to A^3/T^2 =$ Constant.

§10.5 Average idea on relation of U and E.

$\overline{f(t)} = \lim_{t\to\infty} 1/t \cdot \int_0^t f(t)\, dt$ For $2T = \partial L/\partial \dot{q} \cdot \dot{q} = \Sigma p \cdot v = \underbrace{d/dt\, \Sigma p \cdot r}_{=0} - \Sigma \dot{p} \cdot r$

For d/dt of limited function ($f(\infty) = 0$) $\overline{f(t)} = 0$

$2\bar{T} = \overline{\Sigma \dot{p} \cdot r}$ Since $\dot{p} = \partial U/\partial r$ $2\bar{T} = k\bar{U}$ where k is power of r in U

$\bar{T} + \bar{U} = \bar{E}$ $\bar{U} = \frac{2}{k+2}\bar{E}$ $\bar{T} = \frac{k}{k+2}\bar{E}$

§11. 1D Normal Motion's Integration [Derive Time by U]

$L = \frac{1}{2}m\dot{x}^2 - U(x) = E - 2U$

$\frac{1}{2}m\dot{x}^2 = E - U(x)$ $dx/dt = \sqrt{2(E - U(x))/m}$ $t = \int \sqrt{m/2(E - U(x))} \cdot dx + \text{Constant}$

Only exsist In $E > U(x)$ Range

eg. SHM.

$\frac{1}{2}T = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \sqrt{1/(E - U(x))}\, dx$ where $x_{1,2}$ belongs to $E = U(x_1, x_2)$

Pendulam Situation: l, mg E in equation is constant ☆

$T = \sqrt{2m} \int_{-\varphi}^{+\varphi} \sqrt{1/[mgl(1-\cos\varphi_0) - mgl(1-\cos\varphi)]} \cdot d\varphi \cdot l$ where as φ_0 is max angel

$= \sqrt{2m} \int_-^+ [mgl(\cos\varphi - \cos\varphi_0)]^{-1/2}\, d\varphi$ ← Semi-Circle Integration: $(1-x)^{-1/2} = 1 + \frac{1}{2}x$ when $x \to 0$.

Set $\sin\xi = \sin(\varphi/2)/\sin(\varphi_0/2)$ $\cos\varphi - \cos\varphi_0 = 2\sin(\varphi_0/2)^2 - 2\sin(\varphi/2)^2$

$= \sqrt{2m} \cdot (mgl)^{-1/2} l \cdot \int_-^+ [(\sin\varphi_0/2)^2 - (\sin\xi \sin(\varphi_0/2))^2]^{-1/2} d\xi \cdot d\varphi/d\xi$ ☆ Try on derivitive on both side

$\sin\xi = \sin(\varphi/2)\, 1/\sin(\varphi_0/2)$

↓ derivative both side

$\cos\xi \cdot d\xi = \cos(\varphi/2) \cdot \frac{1}{2} \cdot 1/\sin(\varphi_0/2)\, d\varphi$

$\Rightarrow d\varphi/d\xi = \cos\xi \cdot 2\sin(\varphi_0/2)/\cos(\varphi/2) = \cos\xi\, 2\sin(\varphi_0/2)/\sqrt{1 - \sin^2(\varphi_0/2) \cdot \sin^2(\xi)}$

$= 4\sqrt{l/g} \cdot \int_-^+ \cancel{[\sin(\varphi_0/2)^{-1} \cos^{-1}\xi]} \cdot [1 - \sin^2(\varphi_0/2)\sin^2(\xi)]^{-1/2} \cancel{\cos\xi}\, \cancel{2\sin(\varphi_0/2)}$

$= 4\sqrt{l/g} \cdot \int_-^+ (1 - \sin^2(\varphi_0/2)\sin^2(\xi))^{-1/2} \cdot d\xi = 4\sqrt{l/g}\, K(\sin\varphi_0/2)$

where K is semi circle Integration: $K = \int_0^{\pi/2} d\xi/\sqrt{1 - k^2\sin^2\xi}$

$= 2\pi\sqrt{l/g}\,(1 + \frac{1}{16}\varphi_0^2 - \cdots)$

Normal Pendulam Period $= 2\pi\sqrt{l/g}$

For $U(x)=k|x^n|$ $\quad T=2\sqrt{2m}\int_0^{x}dx/\sqrt{E-U(x)}$ $\quad E=k|x^n|$ $\quad x_{max}=(E/k)^{1/n}$ $\quad\Rightarrow 2\sqrt{2m}\int_0^{(E/k)^{1/n}}dx/\sqrt{E-kx^n}$

Try simplify

☆ Eular Integration

$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}\,dt=(z-1)!$ [For $z\in N^+$]

$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx=\Gamma(p)\Gamma(q)/\Gamma(p+q)$

Set $y^n\cdot E=x^n\cdot k$ $\quad y=(k/E)^{1/n}\cdot x$

$2\sqrt{2m}\cdot\int_0^1 dy/\sqrt{E-Ey^n}\cdot(E/k)^{1/n}$

$=2\sqrt{2m}\cdot E^{1/n-1/2}/k^{1/n}\cdot\int_0^1 dy/\sqrt{1-y^n}$ Seperate to $x^a(1-x)^b$ For Eular Int.B

Set $y^n=u$ $\quad ny^{n-1}$

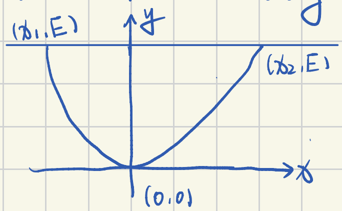
$=\gamma\cdot\int_0^1 du/\sqrt{1-u}\cdot dy/du=\gamma\int_0^1(1-u)^{-1/2}\cdot 1/ny^{n-1}\cdot du$

$T=(\gamma/n)\cdot\Gamma(1/2)\Gamma(1/n)/\Gamma(1/2+1/n)$ where $\gamma=2\sqrt{2m}\,E^{1/n-1/2}\cdot k^{-1/n}$ $\quad\Leftarrow\quad \gamma\int_0^1(1-u)^{-1/2}\,n^{-1}\,u^{1/n-1}\,du$

values of these equations only able proceed by computers.

§12. Derive U by T, E $\quad T=\int_{x_1}^{x_2}dx/\sqrt{E-U(x)}\cdot\sqrt{2m}$

Assume Movement along $U(x)$ End up both side.



$T(E)=\int_E^0 dx_1/dU\cdot dU/\sqrt{E-U(x)}\cdot\sqrt{2m}+\int_0^E dx_2/dU\cdot dU/\sqrt{E-U(x)}\cdot\sqrt{2m}$

$=\int_0^E dU/\sqrt{E-U(x)}\cdot\sqrt{2m}\,(dx_2/dU-dx_1/dU)$ ☆ Inverse Problem

$B(\alpha)=\int_\alpha^U dE/\sqrt{(U-E)(E-\alpha)}=\pi$ $\quad$ Since α is any value - Set $\alpha=U$

$\int_0^\alpha T(E)/\sqrt{E-\alpha}\cdot dE=\int_0^U\int_0^E dU\,dE/\sqrt{E-U(x)}\sqrt{E-\alpha}\cdot\sqrt{2m}\,(dx_2/dU-dx_1/dU)$

$=\pi\cdot\int_0^U(dx_2/dU-dx_1/dU)\sqrt{2m}=\sqrt{2m}\cdot\pi\cdot(x_2(\alpha)-x_1(\alpha))$

$\int_0^U T(E)/\sqrt{E-\alpha}\cdot dE=\sqrt{2m}\cdot\pi\,(x_2(U)-x_1(U))$

§13. Reduced Mass

$L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-U(|x_1-x_2|)$

Set $\vec{r}=\vec{r}_2-\vec{r}_1$, $\quad m_1r_1=m_2r_2$

$L=\frac{1}{2}m\dot{r}^2-U(r)$ where $m=m_1m_2/m_1+m_2$

E.g. M and n numbers of m particle

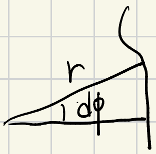
M placed in $\vec{R}$ $\quad m_a$ in $\vec{R}_a$ $\quad$ Define $\vec{r}_a=\vec{R}_a-\vec{R}$

$M\cdot R+\sum m_aR_a=0$ $\quad$ As $\mu=M+nm$ $\quad R=-m/\mu\cdot\sum r_a$ $\quad R_a=R+\vec{r}_a$

$L=\frac{1}{2}M\dot{R}^2+\frac{1}{2}\sum m\cdot\dot{R}_a^2-U=\sim$

§14. Movements In Central Feild: $F = \partial U(x)/\partial x$

> $\mathcal{L} = \frac{1}{2}m(\dot r^2 + (r\dot\varphi)^2) - U$ Since $\mathcal{L}$ didn't include φ $\frac{d}{dt}(\underbrace{\partial \mathcal{L}/\partial\dot\varphi}) = \partial \mathcal{L}/\partial\varphi = 0$



For arbitary movements area $f = \frac{1}{2}\cdot r\cdot r\,d\varphi$ $M = mr^2\dot\varphi = 0 \to$ Same with $M = r\times p$.

$\dot f = (\frac{1}{2}r^2 d\varphi)/dt = d\varphi/dt\cdot\frac{1}{2}r^2$ $\to M = 2m\dot f = mr^2\dot\varphi$

$m^2r^4\dot\varphi^2$

$E = \frac{1}{2}m(\dot r^2 + r^2\dot\varphi^2) + U$

$\dot r = \sqrt{2(E-U)/m - r^2\dot\varphi^2} = dr/dt$ $t = \int(2(E-U)/m - \overbrace{r^2\dot\varphi^2}^{\frac{M^2}{m^2r^2}})^{-1/2}\cdot dr = \int(2(E-U)/m - r^2\dot\varphi^2)^{-1/2}\,dr$ $t(r)$

$= \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\varphi^2 + U$ $d\varphi = M/mr^2\cdot dt \Rightarrow$

> Traditional E   U with Centrifugal $\varphi(r)$ Orbital Equation

$\varphi = \int(2(E-U)/m - M^2/m^2r^2)^{-1/2}\cdot M/mr^2\,dr$

$U_{centri} = \frac{1}{2}mr^2\dot\varphi^2 = M^2/2mr^2$

> For Function $r = r_{min} \sim r_{max}$ $\Delta\varphi = \int_{r_{min}}^{r_{max}}(2(E-U)/m - M^2/m^2r^2)^{-1/2}\cdot M/mr^2\,dr$

If $\Delta\varphi = 2\pi n$ $n\in\mathbb{Z}$ path closed

> Conditions On path throw center of force $E = \frac{1}{2}mv^2 + M^2/2mr^2 + U(r)$ >0 when $r\to0$

$U(r) > -M^2/2mr^2$ $U(r)\cdot r^2|_{r\to0} < -M^2/2m$

E.g. Sphere Pendulam $E = \frac{1}{2}m((r\dot\varphi)^2 + (r\sin\varphi\cdot\dot\theta)^2) + mg\cos\varphi$

$\dot\varphi = \sqrt{2(E-U)/mr^2} = d\varphi/dt$ $\mathcal{L}$ only include $\dot\theta$

$t = \int[2(E-U)/mr^2]^{-1/2}\cdot d\varphi$ with $U_{eff} = \frac{1}{2}mr^2\sin^2\varphi\cdot\dot\theta^2 + mg\cos\varphi$

$d\theta = [M\cdot(mr^2\sin^2\varphi)^{-1}]\cdot dt$ $\leftarrow r^2\sin^2\varphi\cdot\dot\theta\cdot m = M$

$\Downarrow$

$\theta = \int[2(E-U)/mr^2]^{-1/2}\cdot[M(mr^2\sin\varphi)^{-1}]\cdot d\varphi$

§15. Keplar's Motion $[F\propto 1/r^2,\ U = -\alpha/r]$

$U_{eff} = M^2/2mr^2 - \alpha/r$ with $r\to\infty$ $U_{eff} = 0$; $r\to0$ $U_{eff}\to\infty$ $V_{eff}(min) = -m\alpha^2/2M^2$

$\varphi = \int(2(E-U)/m - M^2/m^2r^2)^{-1/2}\cdot M/mr^2\,dr$ Set $u = 1/r$

$= \int(2(E+\alpha/r)/m - M^2/m^2r^2)^{-1/2}\cdot M/mr^2\cdot du\cdot(-r^2)$

$= -\int\underline{(2E+2\alpha u)/m - M^2/m^2\,u^2)}^{-1/2}\cdot M/m\cdot du$ $-\arccos(x/A)' = 1/\sqrt{A^2-x^2}$ ☆

$M^2(2E/mM^2 + 2\alpha u/mM^2 - u^2/m^2)^{-1/2}\cdot M/m$ $-ax^2+bx \to (\sqrt{a}x + b/2\sqrt{a})^2 - b^2/4a$

$[-(u/m - \alpha/M^2)^2 + \alpha^2/M^4 + 2E/mM^2]^{-1/2} \cdot m^{-1} \cdot du$

$= \arccos[(u - m\alpha/M^2)/A]$ with $A = (\alpha^2 m^2/M^4 + 2Em/M^2)^{1/2}$ $u = 1/r$

$\Rightarrow \varphi = \arccos[(M/r - m\alpha/M)/\sqrt{(\alpha^2 m^2/M^2 + 2Em)}]$

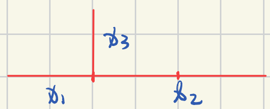
Goal is $e\cos\varphi + 1 = \nu/r$ $\cos\varphi = (M/r - M/\nu)/\sqrt{1+2EmM^2/\alpha^2 m} \cdot \sqrt{\alpha^2 m^2/M^2}$

$= M(1/r - 1/\nu)/\sqrt{1+2E\nu/\alpha} \cdot M/\alpha m$

$e\cos\varphi + 1 = \nu/r \Leftarrow = 1/\nu\,(1/r - 1/\nu)/e$

with $\nu = M^2/\alpha m$ $e = \sqrt{1+2E\nu/\alpha}$

Semi-Circle polar coordinates eq. $a = \nu/1-e^2 = M^2/\alpha m - 2E\nu/\alpha = M^2/-2mE\nu = \alpha/2E$



$r = \nu/1+e\cos\varphi \Rightarrow x_1 = \nu/1+e$ $x_2 = \nu/1-e$ $c = \nu$

$a = \frac{1}{2}(x_1+x_2) = \nu/2 \cdot 2/1-e^2 = \nu/1-e^2$ $b = \nu/\sqrt{1-e^2}$

> Special Point on $e=0$ $2EM^2/\alpha^2 m = -1$ $E = -\alpha^2 m/2M^2$ Trajactory Change to circle

> Period of Motion $M = 2m\dot{f} \rightarrow MT = 2mf$ $f = \pi ab$

$T = 2m\pi ab/M = 2m\pi a^2\sqrt{1-e^2}/M = 2\pi a^{3/2}\sqrt{m/\alpha}$

> Traj for dif. E. $E \geq 0$ $e > 1$, Traj is open

> $t = \int(2(E-U)/m - r^2\dot{\varphi}^2)^{-1/2} dr$ Subsititute to related with a, e, m

$e = \sqrt{1+2E\nu/\alpha}$ $\nu = M^2/m\alpha$ $U = M^2/2mr^2 + \alpha/r$ $\dot{\varphi} = M/mr^2$ $a = \nu/1-e^2 = \alpha/2E$
$1-e^2 = M^2/m\alpha a$

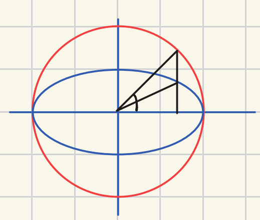
$t = \int(2E/m + 2\alpha/mr - M^2/m^2r^2)^{-1/2} dr$

$= \int(2Er^2/m + 2\alpha r/m - M^2/m^2)^{-1/2} r\,dr$

$= \int \alpha/ma[-r^2 + 2ar - a^2(1-e^2)]^{-1/2} \cdot r\,dr$

$= \sqrt{ma/\alpha} \cdot \int r\,dr/\sqrt{a^2e^2 - (r-a)^2}$ with $r - a = -ae\cos\xi$ $ae = c$

- Semi Circle



$x = a\cos\xi$ $a/x^2 + b/y^2 = 1 \Rightarrow$ Focal Radius $r = a - ex$

$\Rightarrow r = a - ea\cos\xi$

$= \sqrt{\frac{ma}{\alpha}} \int r(a^2e^2 - e^2a^2\cos^2\xi)^{-1/2} \cdot dr = \sqrt{\frac{ma}{\alpha}} \cdot \int r dr / ae\sin\xi = \sqrt{\frac{ma}{\alpha}} \int a(1-e\cos\xi) \cdot d\xi = \sqrt{\frac{ma^3}{\alpha}} (\xi - e\sin\xi)$

☆ Since we intake ξ as parameters, Goal of ↑ is derive Eq. of $t \to \xi$ and $r \to \xi$ ⇒ $r \to t$.

$t = \sqrt{\frac{ma^3}{\alpha}} \cdot (\xi - e\sin\xi)$ with $r = a(1 - e\cos\xi)$

> Similar Situation On $U = \alpha/r$ $\alpha > 0$

$U = \alpha/r + M^2/2mr^2$ with r from $0 \to \infty$ U alwayse >0

⇒ Hyperbola with $P/r = -1 + e\cos\varphi$ $r_{min} = a(e+1)$

> Int. of Motion $v \times M + \alpha\vec{r}/r = \text{Constant}$

$d/dt(v \times M + \alpha\vec{r}/r) = \dot{v} \times M + \alpha \cdot (\dot{v} \cdot r + v \cdot \hat{r}/r^2) = \dot{v} \times M + \alpha\hat{v}/r - (v \cdot r)\vec{r}/r^3$

with $M = mr \times v$ ⇒ $mr(v \cdot \dot{v}) - mv(r \cdot \dot{v}) + \alpha v/r - \alpha r(v \cdot r)/r^3 = 0$

E.g. For $V = -\alpha/r$ with $E=0$ find traj.

☆ dif with previous : $a = \alpha/2|E| \to \infty$ when $E \to 0$

$t = \int r dr / \sqrt{2\alpha r/m - M^2/m^2}$

☆ Uniformization Method. : $\int dr/\sqrt{Ar-B}$ ☆

To cancel $\sqrt{Ar-B}$, Assume $u(r)^2 = Ar - B$ $r = 1/A(B+u^2)$

Convert int. of r to int. of u

$2\alpha r/m - M^2/m^2 = C \cdot \eta^2$

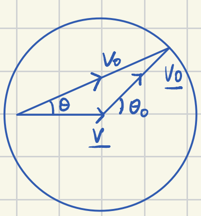
To keep same unit. c choose as M^2/m^2

$2\alpha r/m = M^2/m^2 (1+\eta^2)$ ⇒ $r = M^2/2\alpha m (1+\eta^2)$ $dr/d\eta = M^2/\alpha m \cdot \eta$

$t = \int M^2/2m\alpha (1+\eta^2) \cdot M^2/\alpha m \cdot (M/m)^{-1} \cdot d\eta = \int M^3/2\alpha m (1+\eta^2) = M^3/2\alpha m (\eta + \eta^3/3)$

with $r = M^2/2\alpha m (1+\eta^2)$

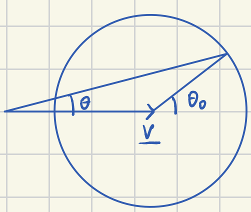
§16. Split of Point of Mass.



λ Frame : $\underline{V}_{cm}$

S Frame : $\underline{V_0}$ v_0 As particle speed.

$\underline{V} + \underline{V_0} = V_0$



with $\theta_{max} = \arcsin \frac{v_0}{V_0}$

Obv. $\tan\theta = \sin\theta_0 \cdot \underline{V_0} \cdot (\underline{V} + \underline{V_0} \cdot \cos\theta_0)^{-1}$

Circle is assigned to $r = \underline{V_0}$

> Energy During Seperate

$E_{int} = E_{int1} + E_{int2} + P_0^2/2m_1 + P_0^2/2m_2$

With ε as split energy $= E_{int} - E_{int1} - E_{int2}$

$\varepsilon = P_0^2/2 (1/m_1 + 1/m_2) = P_0^2/2m$

> Avg. Energy Distribution.

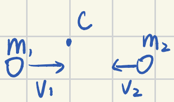
Azimuth : $dA/4\pi = \frac{1}{2}\sin\theta\, d\theta = \frac{1}{2}(d\cos\theta)$

$\underline{V}^2 = V_0^2 + \underline{V_0}^2 - 2V_0\underline{V_0}\cos\theta$ $\quad \cos\theta = V_0^2 + \underline{V_0}^2 - \underline{V}^2 / 2V_0\underline{V_0}$ $\rightarrow$ $d\cos\theta = d(V_0^2)/2V_0\underline{V_0}$

$T = \frac{1}{2}mv^2$ $\quad dT = \frac{1}{2}md(v^2)$ $\Rightarrow \frac{1}{2}\cdot dv^2/2V_0\underline{V_0} = dT/2V_0\underline{V_0}m$ $\quad T \in [\frac{1}{2}m(V_0+\underline{V_0})^2 \sim \frac{1}{2}m(V_0-\underline{V_0})^2]$

> Max Energy of particle [m_1 ref to all other parts] $\quad T_{max} = P_0^2/2m_1 = M-m_1/M\,[\underbrace{E_{int} - E_{1int} - E_{int}'}_{\varepsilon}]$

§17. Elastic Collision.



$\left[\begin{array}{l} V_{10}\ V_{20}:\ \text{C.M. Frame} \\ V_1 V_2:\ \text{Lab. Frame.} \end{array}\right.$

$V = \dfrac{m_1V_1 + m_2V_2}{m_1+m_2}$ $\Rightarrow V_{10} = V_1 - \underline{V} = \dfrac{\cancel{m_1V_1}+m_2V_1-\cancel{m_1V_1}-m_2V_2}{m_1+m_2} = \dfrac{m_2}{m_1+m_2}(V_1-V_2)$

$V_{20} = \dfrac{m_1}{m_1+m_2}(V_2-V_1)$. $\quad$ define $v = V_1 - V_2$

$V_{10} = m_2/m_1+m_2 \cdot v$ $\quad V_{20} = m_1/m_1+m_2 \cdot v$

↓ After Collision.

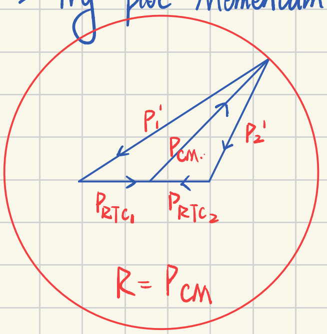
$V_{10}' = m_2/m_1+m_2 \cdot \hat{n}v$ $\quad V_{20}' = m_1/m_1+m_2 \cdot (-\hat{n}v)$

↕

$V_1' = \underbrace{m_2/m_1+m_2\, \hat{n}v}_{\text{(C.M. Frame)}} + \underbrace{m_1V_1+m_2V_2/m_1+m_2}_{\text{(C.M Relate to Lab.)}}$ $\quad$ Vice Versa — V_{20}'

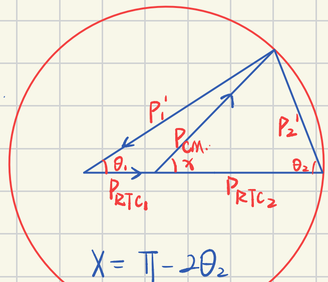
★ θ_1, θ_2 represent angle of Motion in Lab frame.

> Try plot Momentum. $\quad P_1' = m\cdot\hat{n}v + m_1/m_1+m_2\,(P_1+P_2)$ $\quad P_2' = m\hat{n}v + m_2/m_1+m_2\,(P_1+P_2)$



$\left\{\begin{array}{l} P_1' = P_{10}' + P_{Lab.\ to\ 1} \\ P_2' = P_{20}' + P_{Lab.\ to\ 2} \end{array}\right.$

$\tan\theta_1 = P_2'\sin\theta_2 / P_{RTC_1} + P_{CM}\cos\chi$ ⇐



> Specially for m_2 steady

$P_{RTC_1} = m_1^2V_1/m_1+m_2$

$P_{RTC_2} = m_2m_1V_1/m_1+m_2$. Same

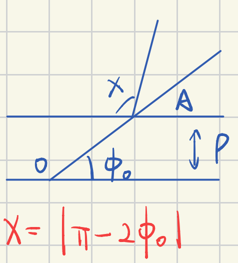
$P_{CM} = m\hat{n}V_1 = m_1m_2V_1/m_1+m_2$

★ P_{CM}. For two particle is same.

★ P_{RTC}: P of Frame refrence to center of Mass.

$\left[\begin{array}{l} V_1' = P_1'/m_1 = \left(\sqrt{m_1^2+m_2^2+2m_1m_2\cos\chi}\,/m_1+m_2\right)\cdot v \\ V_2' = 2m_1v/m_1+m_2 \cdot \sin\chi/2 \end{array}\right.$

§18. Scattering of points. [★ Center of forth is in O.]



$\phi_0 = \int_{r_{min}}^{\infty} (M/r^2)\cdot dr / \sqrt{2m(E-U) - M^2/r^2}$ $\quad$ (chap 14).

With $E = \frac{1}{2}mV_\infty^2$ $\quad M = m\rho \times V_\infty$ $\quad$ ★ P is related to χ

$\phi_0 = \int_{r_{min}}^{\infty} (m\rho V_\infty/r^2)\cdot dr / \sqrt{m^2V_\infty^2 - 2mU - m^2\rho^2V_\infty^2/r^2}$

$= \int_{r_{min}}^{\infty} (\rho/r^2)\cdot dr / \sqrt{1 - (\rho^2/r^2) - (2U/mV_\infty^2)}$ ← [Relation of $\phi_0 \sim P$]

define $dN \to$ num. of particle in $\chi \sim \chi + d\chi$ $\quad d\sigma = dN/n$

$n \to$ num. in unit time and area. $\quad$ Num of part. $dN = 2\pi\rho \cdot d\rho \cdot n = 2\pi\rho(\chi) \cdot d\rho(\chi)/d\chi \cdot d\chi$

Use solid angle $dO = 2\pi \sin\chi \, d\chi \Rightarrow d\sigma = \rho(\chi)/\sin\chi \cdot d\rho/d\chi \cdot dO$

Prob. 4 $\quad U = -\alpha/r^2$ $\quad$ fall to center $\quad$ effective cross-section $d\sigma$

$\frac{1}{2}mv^2 > -\alpha/r^2$ $\qquad \rho_{max} = \sqrt{2\alpha/mv_\infty^2}$

$\alpha > \frac{1}{2}mv_\infty^2\rho^2$ $\qquad$ with $\sigma = \pi\rho_{max}^2 = 2\pi\alpha/mv_\infty^2$

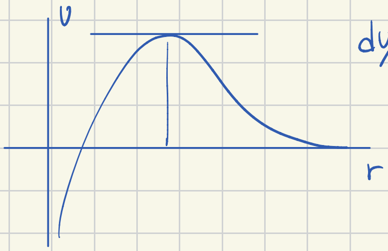
· $V_{eff} < E$ Attach

$\Rightarrow V_{eff\,max} = E$

Solve $\rho = \sim$

Prob. 5 $\quad U = -\alpha/r^n \quad (n>2, \alpha>0)$

$V_{eff} = m\rho^2 v_\infty^2/2r^2 - \alpha/r^n$



$dU/dr = -m\rho^2 v_\infty^2 + \alpha n r^{-n+2} = 0$

$-m\rho^2 v_\infty^2/\alpha n = r^{-n+2}$

$r = (-m\rho^2 v_\infty^2/\alpha n)^{1/(2-n)}$ $\qquad U_{max} = \frac{1}{2}(n-2)\alpha(m\rho^2 v_\infty^2/\alpha n)^{n/(n-2)}$

Similar $V_{eff} = E = \frac{1}{2}mv_\infty^2 \Rightarrow \rho = \sim$ $\qquad$ Area $= \pi\rho^2 = \sim$

Prob. 6 $\quad m_1$ with m_2, R sphere. Area $\quad$ ☆ $U/r \to \infty$ when $r \to 0$ $\quad$ No U_{max} exsist.

$r_{min} = R$ $\quad$ with $V = 0$

$E = M^2/2mr^2 - \alpha/r = \frac{1}{2}mv_\infty^2$ $\qquad$ with $\qquad \alpha = Gm_1m_2$

$M = mv_\infty\rho$

$\frac{v_\infty\rho^2}{r^2} - \frac{2\alpha}{rmv^2} = 1 \qquad \rho^2 = (1 + 2\alpha/rmv^2) \cdot r^2/v_\infty$

Prob. 7 $\quad A(E, \chi)$ derive $U(r)$.

$\pi\rho^2 = \int_\chi^\pi (d\sigma/d\chi) \cdot d\chi$

Area of collision

§19. Rutherford's Formula [effective cross section in $U=\alpha/r$]

with $U=\alpha/r$ $\longrightarrow$ $\varphi = \int_{r_{min}}^{\infty} (M/r^2)\,dr / \sqrt{2m[E-U(r)]-M^2/r^2}$

$E = \frac{1}{2}mv_\infty^2 + M^2/2mr^2$

$= \int_{r_{min}}^{\infty} (\rho/r^2)\,dr/\sqrt{1-(\rho^2/r^2)-(2U/mv_\infty^2)}$

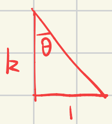
$= \int_{r_{min}}^{\infty} (\rho/r^2)\,dr\cdot(1-(\rho^2/r^2)-2\alpha/mv_\infty^2 r)^{-1/2}\,dr$ $\qquad u=r^{-1}$ $\quad -r^{-2}$ $\quad -r^2$

$\cos\phi_0 = \alpha/mv_\infty\rho \,/\, (1+(\alpha/mv_\infty\rho)^2)^{1/2}$ $\qquad \rho\,u^2\,du\,(1-\rho^2u^2-2\alpha u/mv_\infty^2)^{-1/2} \cdot dr/du \cdot -1/u^2$

$\uparrow$ $\qquad du\,[1/\rho^2-u^2-2\alpha u/mv_\infty^2\rho^2]^{-1/2}$

$\phi_0 = \cos^{-1}\left(\frac{\alpha/mv_\infty^2\rho}{\sqrt{1+(\alpha/mv_\infty^2\rho)^2}}\right) \longleftarrow \int_u^0 \{-(u+\alpha/mv_\infty^2\rho^2)^2+\alpha^2/m^2v_\infty^4\rho^4+1/\rho^2\}^{-1/2}\,du$

★ Trigonometric functions



$\sin\theta = 1/\sqrt{1+k^2}$

$\cos\theta = k/\sqrt{1+k^2}$ $\rightarrow$ $\tan\theta = 1/k$

$\tan\phi = mv_\infty^2\rho/\alpha \rightarrow \rho^2 = (\alpha^2/m^2v_\infty^4)\cdot\tan^2\phi_0$

$= (\alpha^2/m^2v_\infty^4)\cdot\cot^2\tfrac{1}{2}\chi$

$\Rightarrow d\sigma = \pi\rho^2\cdot d\rho/d\chi = (\alpha/2mv_\infty^2)^2\cdot do/\sin^4\tfrac{1}{2}\chi$

[Rutherford's Formula]

> Special Situations

1. Particle m_2 rest initially

$\chi = \pi - 2\theta_2$

$\chi = \pi-2\theta_2$ $\qquad d\sigma_2 = \pi(\alpha/mv_\infty^2)^2\cdot\cos\tfrac{1}{2}\chi\cdot d\chi/\sin^3\tfrac{1}{2}\chi \leftarrow -\sin(\theta_2)\,2d\theta_2/\cos^3(\theta_2)$

$= +2\pi(\alpha/mv_\infty^2)^2\cdot\sin\theta_2\cdot d\theta_2/\cos^3(\theta_2)$ $\qquad$ with $do_2 = 2\pi\sin\theta_2\,d\theta_2$

$= (\alpha/mv_\infty^2)^2\cdot do_2/\cos^3(\theta_2)$

2. Mass Relation

$m_2 \gg m_1$ $\quad \theta_1 \simeq \chi$ $\quad m_1\simeq m$ $\qquad d\sigma_1 = (\alpha/4E_1)^2\cdot do_1/\sin^4(\tfrac{1}{2}\theta_1)$ with $E=\tfrac{1}{2}m_1v_\infty^2$

$m_1 = m_2$ $\quad m=\tfrac{1}{2}m$ $\quad \chi = 2\theta_1$ $\qquad d\sigma_1 = (\alpha/E_1)^2\cos\theta_1\cdot do_1/\sin^4\theta_1$

3. Stick After Collision

$d\sigma_1 = (\alpha/E_1)^2(1/\sin^4\theta + 1/\cos^4\theta)\cos\theta\cdot do$

> Use split Energy & express σ [In m_2 Steady State]

$v_2' = \frac{2m_1v}{m_1+m_2}\cdot\sin\tfrac{1}{2}\chi$ $\qquad \varepsilon = \tfrac{1}{2}m_2v_2'^2 = (2m^2/m_2)\,v_\infty^2\cdot\sin^2\tfrac{1}{2}\chi$

$\Downarrow$

$d\sigma = 2\pi(\chi^2/m_2v_\infty^2)\cdot d\varepsilon/\varepsilon^2$

Prob. 1 $\quad U=\alpha/r^2$ — Find effective cross-section. $\quad (\alpha>0)$

$\int_{r_{min}}^{\infty} (\rho/r^2)\,dr/\sqrt{1-(\rho^2/r^2)-(2U/mv_\infty^2)}$ $\qquad U=\alpha/r^2$ $\qquad$ [χ, m, v_∞ is known]

$= \int_{r_{min}}^{\infty} (P/r^2)\, dr / [\sqrt{1-(P^2/r^2)-(2\alpha/mr^2V_\infty^2)}]$

$= \int_u^0 P\cdot du / [\sqrt{1-P^2u^2-2\alpha u^2/mV_\infty^2}]$

$= \int_u^0 \gamma\cdot du/\sqrt{1-\gamma^2u^2}\cdot P/\gamma$ $\quad [\arcsin(Ax)]' = A\,dx/\sqrt{1-A^2x^2}\,]$

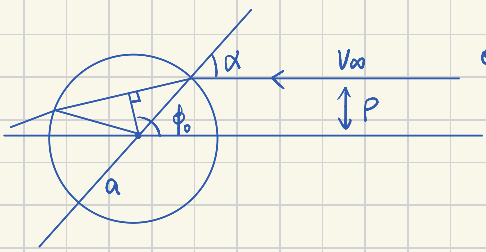
$= [\arcsin(\gamma u)\cdot P/\gamma]_{1/\gamma}^{0} = P/\gamma\,[\pi/2] \Rightarrow \chi = \pi[1-P/\gamma] \quad P = \sim$

with $d\sigma = 2\pi P\, dP$

Prob 2. Radius a depth U_0 Potential Wall

Incident angle $V_1\sin\theta_1 = V_2\sin\theta_2$ with $\frac{1}{2}mv_1^2+U_1 = \frac{1}{2}mv_2^2+U_2$

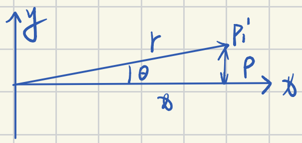
$\sin\theta_1/\sin\theta_2 = \sqrt{1+2/mv_1^2\,(U_1-U_2)}$ $\quad \sin\alpha/\sin\beta = \sqrt{1+2U_0/mV_\infty^2}$



$\phi_0 = \pi/2-\beta+\alpha \quad \chi = \pi - 2\phi_0 = 2(\alpha-\beta)$ with $a\sin\alpha = P$

$\Rightarrow d\sigma = 2\pi P\, dP \sim$

§20. Small-angle Scattering [Small angle deflection, Weak impact on U, Large impact parameters.]



Define P_1' as momentum of particle after collision.

$\sin\theta_1 = P_{1y}'/P_1' \approx \theta_1$ as $P_1 \simeq P_1'$ $\quad \theta_1 \approx P_{1y}'/m_1V_\infty$ $\quad$ Connect P_{1y}' to $(U, r \ldots)$

$F_y = -\partial U/\partial y = -(\partial U/\partial r)\cdot \partial r/\partial y$ Since $r = \sqrt{x^2+y^2+z^2}$ $\quad$ ☆ Use $r^2 = x^2+y^2+z^2$ Derivative both side

$= -(dU/dr)\cdot y/r$ $\quad$ For particle in straight line $y = P$ $\quad 2r\,\partial r/\partial y = 2y \rightarrow \partial r/\partial y = y/r$

$= -(dU/dr)\, P/r \Rightarrow P_{1y}' = -P/V_\infty \cdot \int_-^+ dU/dr\cdot dx/r$ $\quad$ For $x = \sqrt{r^2-P^2}$ $\quad 2x\cdot dx/dr = 2r$

$P_{1y}' = m_1V_\infty\sin\theta_1 \longrightarrow \theta_1 = P_{1y}'/m_1V_\infty = -2P/m_1V_\infty^2 \int_0^+ dU/dr\cdot dr/r\cdot r/x$

$= -2P/m_1V_\infty^2 \int_0^+ dU/dr\cdot dr/\sqrt{r^2-P^2}$

with $d\sigma = |dP/d\theta_1|\cdot P(\theta_1)/\theta_1\cdot d\theta_1$

Chap V. OSC.

§21. Free oscilliation 1-D.

$U(q) - U(q_0) \approx \frac{1}{2}k(q-q_0)^2$ $\quad$ define $x = q-q_0$

$\Rightarrow L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ with Eq. $m\ddot{x} + kx = 0$

$E = \frac{1}{2}m\omega^2a^2$ $\quad x = C_1\cos(\omega t) + C_2\sin(\omega t)$. Or $a\cos(\omega t+\alpha)$.

$x = \mathrm{Re}[A\exp(i\omega t)]$ $\quad a = \sqrt{C_1^2+C_2^2}$ $\quad \tan\alpha = -C_2/C_1 \leftarrow = a[\cos\omega t\cos\alpha - \sin\omega t\sin\alpha]$

$A = a\exp(i\alpha)$ with A = complex amplitude.

§22. Forced Oscilliations

§23. More degree oscilliations

Set $U(q_1, q_2 \cdots)$ minimum when $q_i = q_{i0}$ Set $x_i = q_i - q_{i0}$ As small oscilliation close to q_{i0}.

$U = \sum_{ik} \frac{1}{2} k_{ik} x_i x_k$ with $k_{ik} = k_{ki}$ obv. $\longrightarrow L = \frac{1}{2}\sum_{ik}(m_{ik}\dot{x}_i\dot{x}_k - k_{ik} x_i x_k)$

$$dL = \frac{1}{2}\sum_{ik}[m_{ik}(\dot{x}_i d\dot{x}_k + \dot{x}_k d\dot{x}_i) - k_{ik}(x_i dx_k + x_k dx_i)]$$

Since k_i is symmetry $dL = \sum_{ik}[m_{ik}\dot{x}_i d\dot{x}_k - k_{ik} x_i dx_k]$

$\Rightarrow d/dt(\partial L/\partial \dot{q}) - \partial L/\partial q = 0 \rightarrow \sum m_{ik}\ddot{x}_k + \sum k_{ik} x_k = 0$ For $(i = 1, 2 \cdots n)$

Similar Solution Trying — $x_k = A_k \exp(i\omega t) \longrightarrow \sum_k(\underbrace{-\omega^2 m_{ik} + k_{ik}}_{\text{Set} = B}) A_k = 0$

$\sum_k BA_k = 0$ Goal is Solve A_k

☆ Linear Algebra : $B\,\mathrm{adj}(B) = \det(B)\cdot I \leftarrow \mathrm{adj}(B) = \Delta^T$

For $\omega = \omega_A$ $\det(B(\omega_A)) = 0$

$$B\cdot\Delta^T = \det(B)\cdot I = 0 \quad [\omega = \omega_A]$$

$\sum_k B\Delta_k = 0$ For Δ_k is Algebratic Residual $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $C(d) = a$

$\Rightarrow x_k = \Delta_{k\alpha} C_\alpha \exp(i\omega_\alpha t) \leftarrow A = \Delta_{k\alpha}$ C_α is random complex value.

[Comments on ω]: $x = \exp(i\omega_\alpha t)$ If ω_α include i, $i\omega_\alpha t$ include time $\rightarrow$ function unstab.

$$x_k = re\sum_{\alpha=1}^{s}\Delta_{k\alpha} C_\alpha \exp(i\omega_\alpha t) = \sum_{\alpha=1}^{s}\Delta_{k\alpha}\Theta_\alpha \quad \text{with } \Theta_\alpha = Re[C_\alpha \exp(i\omega_\alpha t)]$$

☆ Observe Θ_α as new coordinates, x_k base on Θ_α exsist.

i). Seperate Complex vibration to linear compose of SHM.

ii). Since $\Theta_\alpha = re[C_\alpha\exp(i\omega_\alpha t)]$ Θ_α satisfy $\ddot{\Theta}_\alpha + \omega^2\Theta_\alpha = 0$

each Θ_α independent oscilliations.

$$L = \frac{1}{2}m_\alpha(\dot{\Theta}_\alpha^2 - \omega^2\Theta_\alpha^2)$$

Also, define $Q_\alpha = \sqrt{m_\alpha}\,\Theta_\alpha$

$L = \frac{1}{2}(\dot{Q}_\alpha^2 - \omega^2 Q_\alpha^2)$ All Q_α oscilliation independent.

> For Forced Oscilliation $F(t)$.

$L = L_0 + \sum_k F_k(t)\cdot x_k \leftarrow$ Try use $Q_k = Re[C_k\exp(i\omega_k t)]$.

$= \frac{1}{2}(\dot{Q}_k^2 - \omega^2 Q_k^2) + \sum_k f_k(t)\cdot Q_k$ for $f_k(t) = \Delta_k/\sqrt{m}\cdot F_k(t)$.

With eq. $\ddot{Q}_a + \omega_a^2 Q_a = f_a(t)$

Prob 1. $\mathcal{L} = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - \frac{1}{2}\omega_0^2(x^2+y^2) + axy$

$\ddot{x} = \omega_0^2 x - ay$ $\qquad \ddot{y} = \omega_0^2 y - ax$

with $x, y = A_k \exp(i\omega_k t)$ $\qquad$ General Form is $A_x Q_1 + A_y Q_2$.

$A_x \omega_x^2 = \omega_0^2 A_x - aA_y$ $\qquad A_y\omega_y^2 = \omega_0^2 A_y - aA_x$ $\qquad [\Delta = \omega^2 - \omega_0^2]$

The general solution is $\det(B) = 0$ $\qquad B = \begin{vmatrix} -a & \Delta \\ \Delta & -a \end{vmatrix}$

$(\omega^2-\omega_0^2)^2 = a^2$

$x^2+y^2 = (Q_1+Q_2)^2 + (Q_1-Q_2)^2$

$= 2Q_1^2 + 2Q_2^2 \to$ Normalize $1/\sqrt{2}$ $\Rightarrow$ $\omega^2 = a + \omega_0^2$ or $\omega_0^2 - a$.

$x = \frac{1}{\sqrt{2}}(Q_1+Q_2)$ $\quad y = \frac{1}{\sqrt{2}}(Q_1-Q_2)$ $\leftarrow$ $A_x = A_y$ $\qquad A_x = -A_y$

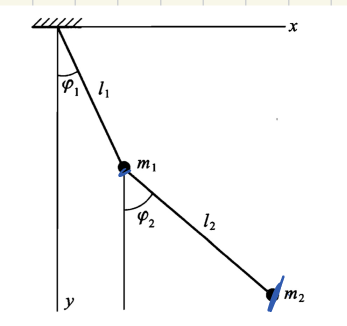


图 1

Prob 2. Solve 图 1.

$[\varphi_1, \varphi_2 \ll 1 \quad \cos\varphi = 1 - \frac{1}{2}\varphi^2]$

$\mathcal{L} = \frac{1}{2}(m_1+m_2)\cdot(l_1\dot{\varphi}_1)^2 + \frac{1}{2}m_2(l_2\dot{\varphi}_2)^2 + m_2 l_1 l_2 \dot{\varphi}_1\dot{\varphi}_2 - \frac{1}{2}(m_1+m_2)gl_1\varphi_1^2 - \frac{1}{2}m_2 g l_2 \varphi_2^2$

m_2: $\begin{cases} x = l_1\sin\varphi_1 + l_2\sin\varphi_2 \\ y = l_1\cos\varphi_1 + l_2\cos\varphi_2 \end{cases}$ $\begin{matrix} \to l_1\cos\varphi_1\cdot\dot{\varphi}_1 + l_2\cos\varphi_2\cdot\dot{\varphi}_2 \\ \to -l_1\sin\varphi_1\cdot\dot{\varphi}_1 - l_2\sin\varphi_2\cdot\dot{\varphi}_2 \end{matrix}$

$\varphi_1\varphi_2 \ll 1 \quad = 1$

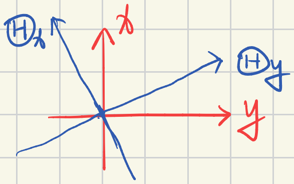
$V_2^2 = l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2l_1l_2\dot{\varphi}_1\dot{\varphi}_2(\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2)$

$\mathcal{L}$ on φ_1: $(m_1+m_2)l_1^2\ddot{\varphi}_1 + m_2 l_1 l_2 \ddot{\varphi}_2 = (m_1+m_2)gl_1\varphi_1$

$\mathcal{L}$ on φ_2: $m_2 l_2^2 \ddot{\varphi}_2 + m_2 l_1 l_2 \ddot{\varphi}_1 = m_2 g l_2 \varphi_2$

$\varphi_\alpha = A_\alpha \exp(i\omega t)$

☆ Ideas on coordinates.



Motion Seprate to Θ_x Θ_y

Θ_x Θ_y contribute to xy

$-A_1(m_1+m_2)l_1^2\omega_1^2 - m_2 l_1 l_2 \omega_2^2\cdot A_2 = (m_1+m_2)gl_1A_1$

$-A_2 m_2 l_2^2\omega_2^2 - m_2 l_1 l_2 \omega_1^2 A_1 = m_2 g l_2 A_2$

$A_1(m_1+m_2)(l_1^2\omega^2 + gl_1) + A_2 m_2 l_1 l_2 \omega^2 = 0$

$\to$ Convert to $B\cdot A = 0$ form.

$A_2(g + l_2\omega^2) + l_1\omega^2 A_1^2 = 0$.

$$\underbrace{\begin{bmatrix} (m_1+m_2)(g-l_1\omega^2) & -m_2l_2\omega^2 \\ -l_1\omega^2 & g-l_2\omega^2 \end{bmatrix}}_{B}\cdot\underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{A} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$\omega_1\omega_2 A_1 A_2 = \sim$

$\uparrow$

$\to \det(B) = 0 \to (g - l_2\omega^2)(g - l_1\omega^2)(m_1+m_2) = -m_2 l_2 l_1 \omega^4$

§25. Damped Oscilliations $[m\ddot{x} = -kx - m\dot{x}]$

Same / Simple process with $x = A\exp(rt)$

$$x = c_1\exp(r_1 t) + c_2\exp(r_2 t)$$

$\xrightarrow{\mathrm{Re}[\ ]}$

$$= a\exp(-\lambda t)\cos(\omega t + \alpha)$$

For Multi DoF System. $f_i = -\sum_k \alpha_{ik}\cdot \dot{x}_k$

> 2. Eqa expansion $[\ d/dt(\partial L/\partial \dot{q}) - \partial L/\partial q = 0\ ]$

$\partial L/\partial q = \partial U/\partial q$ = Format of Force $\rightarrow$ Right Side design for Left Force

For $F = \frac{1}{2}\sum_k \alpha_{ik}\cdot \dot{x}_i \dot{x}_k \xrightarrow{\ast} \partial F/\partial \dot{x}_i = \sum_k \alpha_{ik}\cdot \dot{x}_k \Rightarrow d/dt(\partial L/\partial \dot{q}) = \partial L/\partial q + f_i = \partial L/\partial q + \partial F/\partial \dot{x}_i$

$\ast$: $\partial F/\partial \dot{x}_i = \frac{1}{2}\alpha_{ik}\sum_k(\dot{x}_i\cdot \partial \dot{x}_k/\partial \dot{x}_i + \dot{x}_k\cdot \partial \dot{x}_i/\partial \dot{x}_i) = \frac{1}{2}\alpha_{ik}\sum_k(\dot{x}_i\cdot \delta_{ik} + \dot{x}_k) = \alpha_{ik}\cdot \dot{x}_k$

$dE/dt = d/dt[\dot{x}\,\partial L/\partial \dot{x} - L] = \dot{x}\cdot \partial F/\partial \dot{x}$ Since $F \propto \dot{x}^2 \rightarrow \dot{x}\cdot \partial F/\partial \dot{x} = 2F$