Modern Quantum Mechanics

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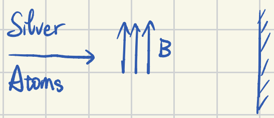
1.1 Experiment

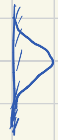
- Magnetic Moment μ $\quad U = -\mu \cdot B \quad F = \mu \times B \quad$ define as $\mu = I \cdot$ Enclosed Area



i). Silver 47 electron $\rightarrow$ 46 Steady | 1 Self Spin

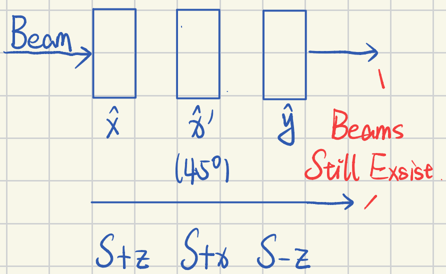
ii). $\mu \propto$ Spin s , $F = \mu_z \cdot \partial B_z / \partial z$

iii). Expected $\mu \in -|\mu| \sim |\mu|$ — Quantum $\mu = \mu_+$ or μ_- with $S_{z\pm} = \pm \hbar/2$

$\Rightarrow$  (p of detected) $\quad \Rightarrow$ 

IV). Cancel $S_z -$ Add B_x $\quad S_{z\pm}$ Both Appeard Again.

- Polarized light experiment.



$S_{\pm z} \leftrightarrow x, y$ polarized
$S_{\pm x} \leftrightarrow x', y'$ polarized
noted as "ket" sign in Dirac notation

$E = E_0 \hat{x} \cos(kz - \omega t) = |S_z+\rangle$
$= E_0 \hat{y} \cos(kz - \omega t) = |S_z-\rangle$

$E_0 \hat{x}' \cos(\omega) = E_0 [\cos 45^\circ \cdot \hat{x} \cos(\omega) + \cos 45^\circ \cdot \hat{y} \cos(\omega)]$

$E_0 \hat{y}' \cos(\omega) = E_0 [-\cos 45^\circ \cdot \hat{x} \cos(\omega) + \cos 45^\circ \cdot \hat{y} \cos(\omega)]$

$\Leftarrow$

$|S_x+\rangle = 1/\sqrt{2}\, |S_z+\rangle + 1/\sqrt{2}\, |S_z-\rangle$

$|S_x-\rangle = -1/\sqrt{2}\, |S_z+\rangle + 1/\sqrt{2}\, |S_z-\rangle$

1.2 Kets Bras Operators

Ket Space

Observe System in Vector Space $\longrightarrow$ State is n-dimention vector.

Define Ket Space as Vector space $\quad |\alpha\rangle + |\beta\rangle = |\gamma\rangle \quad c \cdot |\alpha\rangle = |\alpha\rangle \cdot c$

$A|\alpha\rangle = a|\alpha\rangle$ For A is observable $\quad |\alpha'\rangle, |\alpha''\rangle \cdots$ is eigenket of $|\alpha\rangle$

e.g. $S_z |S_z\pm\rangle = \pm \hbar/2\, |S_z\pm\rangle$

Since Linear Additive $\quad |\alpha\rangle = \Sigma_{a'} C_{a'} |\alpha'\rangle$

Bra Space - Inner Produce.

Define $\langle\alpha|$ corresponding to $|\alpha\rangle$ $\quad$ Dual Correspondence

$|\alpha\rangle \; |\alpha'\rangle \; |\alpha''\rangle \cdots \; |\alpha\rangle + |\beta\rangle \xleftrightarrow{DC} \langle\alpha| \; \langle\alpha'| \; \langle\alpha''| \cdots \; \langle\alpha| + \langle\beta|$

☆ $c|\alpha\rangle \xleftrightarrow{DC} c^*\langle\alpha|$ Define innerproduct $\langle\beta|\alpha\rangle = (\langle\beta|)\cdot(|\alpha\rangle) \xrightarrow{DC} = \langle\alpha|\beta\rangle^*$

- $\langle\alpha|\alpha\rangle \geq 0$, Real Number
- $\langle\alpha|\beta\rangle = 0$ Orthogonal on α β

Operators

$X\cdot(|\alpha\rangle) = X|\alpha\rangle$ $X=Y$ if $X|\alpha\rangle = Y|\alpha\rangle$ if $X|\alpha\rangle = 0$ X is null

☆ $X|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|X^\dagger$ if $X = X^\dagger$ X is Hermitian.

Define outer product $(|\beta\rangle)(\langle\alpha|) = |\beta\rangle\langle\alpha|$

Associative Axiom

- $(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle = |\beta\rangle\cdot(\langle\alpha|\gamma\rangle)$
- if $X = |\beta\rangle\langle\alpha|$ $X^\dagger = |\alpha\rangle\langle\beta|$
- $(\langle\beta|)\cdot(X|\alpha\rangle) = (\langle\beta|X)\cdot(|\alpha\rangle) \rightarrow \langle\beta|X|\alpha\rangle$
- $\langle\beta|X|\alpha\rangle = \langle\beta|\cdot(X|\alpha\rangle) = \{(\langle\alpha|X^\dagger)|\beta\rangle\}^* = \langle\alpha|X^\dagger|\beta\rangle^*$.

1.3 Base kets and matrix representations.

- Eigenvalue of Hermitian operator A is real

  Eigenkets of A to dif. eigenvalues are orthnormal.

Proof. $A|\alpha'\rangle = a'|\alpha'\rangle$ $\langle a''|A = a''^*\langle a''|$

$\Downarrow$

$0 = \langle\alpha''|a'|\alpha'\rangle - a''^*\langle a''|\alpha'\rangle$ $0 = \langle\alpha''|\alpha'\rangle(a' - a''^*)$

if:

i). $a'' = a' \rightarrow a' = a'^*$

ii). $a'' \neq a' \rightarrow \langle a''|a'\rangle = 0$ Each eigenvalues orthnormal

$\langle a''|a'\rangle = \delta_{a'a''}$ if $a' \neq a''$ $= 0$ $\Lambda^2_{a'}$: Project on a'

- Base Kets. $|\alpha\rangle = \sum C_{a'}|\alpha'\rangle$ $\sum \Lambda^2_{a'} = 1$: Project on all a Must $= 1$

$\langle\alpha''|\alpha\rangle = \langle\alpha''|\sum C_a|\alpha'\rangle = \sum_{a'} C_{a'}\delta_{a'a''} \Rightarrow C_{a'} = \langle\alpha'|\alpha\rangle$

$|\alpha\rangle = \sum_{a'}\langle\alpha'|\alpha\rangle\cdot|\alpha'\rangle = \sum_{a'}|\alpha'\rangle\langle\alpha'|\alpha\rangle \Rightarrow \boxed{\sum_{a'}|\alpha'\rangle\langle\alpha'| = 1}$

$\Updownarrow$

$V = \sum_i \hat{e}_i(\hat{e}_i\cdot V)$ ☆ $\langle\alpha'|$ must orthnormal with $|\alpha\rangle$

Since $\sum|\alpha'\rangle\langle a'|=1 \Rightarrow \langle\alpha|\alpha\rangle = \langle\alpha|\sum_{a'}|a'\rangle\langle a'||\alpha\rangle = \sum_{a'}|\langle a'|\alpha\rangle|^2 \Rightarrow \sum_{a'}|C_{a'}|^2 = \sum_{a'}|\langle a'|\alpha\rangle|^2 = 1$

☆ $(|\alpha'\rangle\langle\alpha'|)\cdot|\alpha\rangle = |\alpha'\rangle\langle\alpha'|\alpha\rangle = C_{a'}|\alpha'\rangle \Rightarrow |a'\rangle\langle a'|$ portion of $|\alpha'\rangle$ (for $|\alpha\rangle$ is normalized).

define $\Lambda_{a'} = |a'\rangle\langle a'|$ $\quad \sum\Lambda_{a'} = 1$. $\quad \Lambda_{a'}$ Subtract valv. of propotion of a' on a

Matrix Representation. ☆ $|a'\rangle\ |a''\rangle \cdots$ From eigenket list. No dif.

Define $X = \sum_{a'}\sum_{a''}|a''\rangle\langle a''|\sum|a'\rangle\langle a'|$

Convert to $N\times N$ Matrix where N is dim. of ket space

$$X = \begin{bmatrix} \langle a_1''|X|a_1'\rangle & \langle a_1''|X|a_2'\rangle & \cdots \\ \langle a_2''|X|a_1'\rangle & \ddots & \\ \vdots & & \ddots \end{bmatrix} \Rightarrow \langle a''|X|a'\rangle = \langle a'|X^\dagger|a''\rangle^*$$

Insert Identity Matrix

- For $Z = XY$ $\quad \langle a''|Z|a'\rangle = \sum_{a'''}\langle a''|X|a'''\rangle\langle a'''|Y|a'\rangle$

- For $|\gamma\rangle = X|\alpha\rangle$ Acting on base $\langle a'|$ $\quad \langle a'|\gamma\rangle = \langle a'|X|a\rangle = \sum_{a''}\underbrace{\langle a'|X|a''\rangle}_{\text{Matrix of } X}\underbrace{\langle a''|a\rangle}_{\text{Portion of old vectors.}}$

$$|\alpha\rangle = \begin{pmatrix}\langle a^1|a\rangle \\ \langle a^2|a\rangle \\ \vdots\end{pmatrix} \qquad |\gamma\rangle = \begin{pmatrix}\langle a^1|\gamma\rangle \\ \langle a^2|\gamma\rangle \\ \vdots\end{pmatrix}$$

As column Matrix.

$$|\alpha\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle = \begin{pmatrix}|a_1'\rangle\langle a_1'|\alpha\rangle \\ |a_2'\rangle\langle a_2'|\alpha\rangle \\ \vdots\end{pmatrix} = \begin{pmatrix}\langle a_1'|\alpha\rangle \\ \langle a_2'|\alpha\rangle \\ \vdots\end{pmatrix}$$

☆ $|a_1\rangle\ |a_2\rangle \cdots$ Have no valv. — only directions.

- Similar on $\langle\gamma| = \langle\alpha|X$ $\quad \langle\gamma|a'\rangle = \sum_{a''}\langle\alpha|a''\rangle\langle a''|X|a'\rangle$

$\langle\gamma| = (\langle\gamma|a^1\rangle, \langle\gamma|a^2\rangle \cdots) = (\langle a^1|\gamma\rangle, \langle a^2|\gamma\rangle \cdots)^*$ As row Matrix.

- Column Matrix · Raw Matrix. $\quad \langle\beta|\alpha\rangle = \sum_{a'}\langle\beta|a'\rangle\langle a'|\alpha\rangle$

$$= (\langle\beta|a_1\rangle\ \langle\beta|a_2\rangle \cdots)\cdot\begin{pmatrix}\langle a_1|a\rangle \\ \langle a_2|a\rangle \\ \vdots\end{pmatrix} = \text{Some value}$$

- Raw · Column $\quad |\beta\rangle\langle\alpha| = \sum_{a'}\langle a'|\beta\rangle\langle\alpha|a'\rangle = \begin{bmatrix}\langle a^1|\beta\rangle\langle a^1|\alpha\rangle^* & \langle a^1|\beta\rangle\langle a^2|\alpha\rangle^* & \cdots \\ \langle a^2|\beta\rangle\langle a^1|\alpha\rangle^* & \ddots & \vdots \\ \vdots & & \ddots \end{bmatrix}$

- Eigenkets of A used as baseket. [$\sum_{a'}|a'\rangle\langle a'| = 1$ — a' is baseket]

$A = \sum_{a''}\sum_{a'}|a''\rangle\langle a''|A|a'\rangle\langle a'|$ = if $\langle a'|A|a''\rangle$ where $a' \neq a''$ $=0 \Rightarrow \delta_{a''a'}\langle a'|A|a'\rangle = a'\delta_{a''a'}$

$A = \sum_{a'} a' |a'\rangle\langle a'| = \sum_{a'} a' \Lambda_{a'}$

☆ defin $S_\pm$ as Up down operators

$S_\pm = \hbar |\pm\rangle\langle\mp|$

Set $|-\rangle$ to $\hbar|+\rangle$

Spin ½ Systems

$1 = \sum_{a'} |a'\rangle\langle a'| = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$ $\qquad S_+ |-\rangle = \hbar|+\rangle\langle -|-\rangle = \hbar|+\rangle$

$S_z = \sum_{a'} \underbrace{a'}_{\text{eigen value}} \underbrace{\Lambda_{a'}}_{\text{eigenket}} = (\hbar/2)|\uparrow\rangle\langle\uparrow| - (\hbar/2)|\downarrow\rangle\langle\downarrow| = \hbar/2\,(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$ $\qquad S_z|\updownarrow\rangle = \pm(\hbar/2)|\updownarrow\rangle$

as $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $\quad S_z = \hbar/2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

1.4 Measurements, Observables, Uncertainty

$|\alpha\rangle = \sum \underset{i}{\langle a'|}\underset{j}{\alpha\rangle} \cdot \underset{k}{|a'\rangle} = \sum_{a'} \underset{k}{|a'\rangle}\underset{i}{\langle a'|}\underset{j}{\alpha\rangle}$

$|\alpha\rangle$ [Series of possiple eigenvalue] $\xrightarrow{\text{Observe}}$ $|a'\rangle$ [Singele co. value]

$|a'\rangle \xrightarrow{\text{Observe}} |\alpha'\rangle$

Prob. of a' observed $= |\Lambda_{a'}\cdot\alpha|^2 = |\underbrace{|a'\rangle}_{\text{Veetor}}\underbrace{\langle a'|\alpha\rangle}_{\text{Constant}}|^2$

$|a'\rangle^2 = \langle a'|a'\rangle$

$= |\text{Constant}|^2 \cdot |\text{Veetor}|^2 = |\langle a'|\alpha\rangle|^2 \cdot \underbrace{\langle a'|a'\rangle}_{=1}$

$\langle A\rangle = \sum_{a'} \underbrace{a'}_{\text{valv.}} \underbrace{|\langle a'|\alpha\rangle|^2}_{P.}$

$P = |\langle a'|\alpha\rangle|^2$

$= \sum_{a'}\sum_{a''} \langle\alpha|a''\rangle\underbrace{\langle a''|A|a'\rangle}_{\delta_{a''a'}a'}\langle a'|\alpha\rangle = \sum_{a'}\sum_{a''} \langle a'|\alpha\rangle\langle a''|\alpha\rangle^* \delta_{a'a''}\, a'$

$= \sum_{a'} |\langle a'|\alpha\rangle|^2\, a'$

$|S_x:+\rangle$ on S_z direction $= \vec{A}|+\rangle + \vec{B}|-\rangle$

with $|A| = |B| = 1/\sqrt{2}$ and direction unknown

Spin ½ System, again

$P = \sum|\Lambda_{a'}\cdot\alpha|^2 = 1$ with Same P. on $S_{z\pm}$ for $S_x +$ $\qquad$ $1/\sqrt{2}|+\rangle + 1/\sqrt{2}\exp(i\delta_1)|-\rangle \Leftarrow 1/\sqrt{2}\exp(i\alpha)|+\rangle + 1/\sqrt{2}\exp(i\beta)|-\rangle$

$|\langle \underset{S_z+}{+}|\underset{S_x+}{S_x:+}\rangle| = |\langle\underset{S_z-}{-}|\underset{S_x+}{S_x:+}\rangle| = 1/\sqrt{2}$ $\quad\Rightarrow\quad |S_x:+\rangle = 1/\sqrt{2}|+\rangle + 1/\sqrt{2}\underbrace{\exp(i\delta_1)}_{\text{phase difference in } |+\rangle\,|-\rangle}|-\rangle$

Since $|S_x:+\rangle$ orthnormal with $|S_x:-\rangle$. $\langle +|-\rangle = 0$ $\Rightarrow |S_x:-\rangle = 1/\sqrt{2}|+\rangle - 1/\sqrt{2}\exp(i\delta_1)|-\rangle$

$S_x = \sum_{a'} a'\Lambda_{a'} = \hbar/2\,[|S_x+\rangle\langle S_x:+| - |S_x:-\rangle\langle S_x:-|] = \hbar/2\,[e^{-i\delta_1}|+\rangle\langle -| + e^{i\delta_1}|-\rangle\langle +|]$

Similar
$|S_y:\pm\rangle = 1/\sqrt{2}|+\rangle \pm 1/\sqrt{2}\exp(i\delta_2)|-\rangle$ $\qquad \Rightarrow S_x = \begin{pmatrix} 0 & \exp(-i\delta_1)\\ \exp(i\delta_1) & 0\end{pmatrix}$

$S_y = \sum_{a'} \Lambda_{a'} a' = \hbar/2\,[\exp(-i\delta_2)|+\rangle\langle -| + \exp(i\delta_2)|-\rangle\langle +|]$

To determine $\delta_2\ \delta_1$ $\qquad |\langle S_y:\pm|S_x:+\rangle| = |\langle S_y:\pm|S_x:-\rangle| = 1/\sqrt{2}$ $\qquad \Rightarrow S_y = \begin{pmatrix} 0 & \exp(-i\delta_2)\\ \exp(i\delta_2) & 0\end{pmatrix}$

$= [1/\sqrt{2}\langle +| + 1/\sqrt{2}\exp(i\delta_2)\langle -|]\,[1/\sqrt{2}|+\rangle - 1/\sqrt{2}\exp(i\delta_1)|-\rangle]$

$= \frac{1}{2} - \frac{1}{2}\exp[i(\delta_2 - \delta_1)] = \frac{1}{\sqrt{2}}$ $(\langle +|+\rangle = 1 \quad \langle -|+\rangle = 0)$

$\Rightarrow \delta_2 - \delta_1 = \pm \pi/2$. ☆ $\exp(i\theta) \in \mathrm{Re}$ when $\theta \in 0, \pi, 2\pi \cdots$

$\delta_2 - \delta_1 = \pi/2 \rightarrow$ Must have one imagine. Assume $x \in \mathrm{Re}$. $y \in \mathrm{Im}$.

$S_x = \hbar/2\,[|+\rangle\langle -| + |-\rangle\langle +|]$ $\quad |S_x : \pm\rangle = 1/\sqrt{2}\,|+\rangle \pm 1/\sqrt{2}\,|-\rangle$

$S_y = \hbar/2\,[-i|+\rangle\langle -| + i|-\rangle\langle +|]$ $\quad |S_y : \pm\rangle = 1/\sqrt{2}\,|+\rangle \pm i/\sqrt{2}\,|-\rangle$.

☆ $S_\pm = \hbar|\pm\rangle\langle \mp| \rightarrow S_\pm = S_x \pm iS_y = \hbar|\pm\rangle\langle\mp|$ [Up / Down degree operators].

with $S_x S_y S_z$ $\quad [S_x, S_y] = \hbar^2/4\,[i|+\rangle\langle +| - i|-\rangle\langle -| + i|+\rangle\langle +| - i|-\rangle\langle -|]$

$[S_z = (\hbar/2)\,[|+\rangle\langle +| - |-\rangle\langle -|]]$

$= i\hbar^2/2\,[|+\rangle\langle +| - |-\rangle\langle -|] = S_z \cdot i\hbar$ $\quad \Rightarrow [S_i, S_j] = \varepsilon_{ijk} \cdot S_k \cdot i\hbar$

Similar Anti - Commutation $\{S_x S_y\} = \frac{1}{2}\hbar^2 \delta_{ij}$ $\quad [\ \{A, B\} = AB + BA\ ]$

> Define $S^2 = S_x^2 + S_y^2 + S_z^2 = 3 \cdot (1/4\,\hbar^2)$ $\quad [S^2, S_i] = 0$

Compatible Observables.

define $[A, B] = 0$ compatible vise versa.

☆ degenerate : two dif. eigen state $\rightarrow$ same correspond eigen value $\Rightarrow |a'\rangle$ isn't complete.

> Theory : A, B compatible eigenvalue of A nonedegenerate. $\langle a''|B|a'\rangle$ all diagonal

$\langle a''|[A, B]|a'\rangle = 0$

$\langle a''|AB - BA|a'\rangle = \langle a''|AB|a'\rangle - \langle a''|BA|a'\rangle = \langle a''|a''\rangle B|a'\rangle - \langle a''|B(a'|a'\rangle = \langle a''|B|a'\rangle(a'' - a') = 0$

Obv. $a' = a''$ or $\langle a''|B|a'\rangle = 0$. $\Rightarrow$ $\langle a''|B|a'\rangle = \delta_{a'a''}\langle a'|B|a''\rangle$. $\rightarrow$ Diagnal Matrix.

> Simutaneous Eigenkits.

$B = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$ adding on ket $|a'\rangle$ = $\sum_{a''} |a''\rangle\langle a''|B|a''\rangle\underbrace{\langle a''|a'\rangle}_{\delta_{a'a''}} = (\langle a'|B|a'\rangle)\,|a'\rangle$

Simotanious Eigenkets

define $|K'\rangle$ collective index $\leftarrow$ $\quad A|a', b'\rangle = a'|a', b'\rangle$ $\quad \leftarrow$ $\quad$ ☆ $B|a'\rangle = b'|a'\rangle$ with $b' = \langle a'|B|a'\rangle$

$|K'\rangle = |a', b'\rangle$ $\quad B|a', b'\rangle = b'|a', b'\rangle$ $\quad A|a'\rangle = a'|a'\rangle$

AB Use Same eigenstate $|a', b'\rangle$

> Degeneracy Notations.

$|K'\rangle = |a', b', c' \cdots\rangle$ (Assume compatible)

obv. $\sum_{k'} |K'\rangle\langle K'| = 1$ $\quad \langle K''|K'\rangle = \delta_{k''k'}$

$|\alpha\rangle \xrightarrow{\text{Measure A}} |a', b'\rangle$ with same value a' $\xrightarrow{\text{Mea. B}} |a', b'\rangle$ with dif. b'

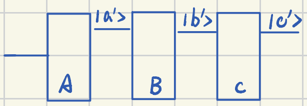
degeneracy in a   dif. in b'

☆ degeneracy → Linear combination of kets. $= \sum_{i}^{n} c_{a'}^{i} |a', b^{i}\rangle$ [For n degeneracy A state, measure gives a'
B state, gives $b', b^2 \dots b^n$]

> Incompatible // Collective Index.

For $[A, B] \neq 0$ $AB|a', b'\rangle = a'b'|a', b'\rangle$, $BA|a', b'\rangle = b'a'|a', b'\rangle$ → $AB = BA$ $[A, B]$ must $= 0$. (x)

For $[A, B] \neq 0$ $|a', b'\rangle$ not exsist.

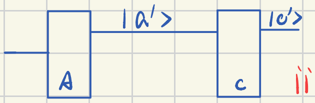


i)

P of output $= \underbrace{|\langle c'|b'\rangle|^2}_{c \text{ in } b} \underbrace{|\langle b'|a'\rangle|^2}_{b \text{ in } a}$

As B have eigen states. summing over b' for all possible routes.

$$\sum_{b'} |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2 = \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle \quad (\text{Eq. 1})$$



ii).

Difference in b' and b'' value.

$$P_{out} = |\langle c'|a'\rangle|^2 = \sum_{b'} |\langle c'|b'\rangle\langle b'|a'\rangle|^2 = \sum_{b'}\sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle \quad (\text{Eq. 2})$$

State in i). $|b'\rangle$ is detected. just left normal P. adding together.

State in ii). $\sum_{b'}\sum_{b''}$ cause interference part when $b' \neq b''$

For $[A, B] = 0$:   S-M. eigenkets $|a\rangle$ have $B|a'\rangle = b'|a\rangle$ $\langle b|a\rangle = \delta_{bb'}$

☆ $\text{Eq.1} = |\langle c|b'\rangle\langle b'|a\rangle|^2 = \text{Eq.2} = |\langle c|b'\rangle|^2 |\langle b'|a\rangle|^2$ [only $b = b'$ exsist].

Interference $\sum_{b}\sum_{b''}$ cause by differente possible path. $[A, B] = 0$ // Test on $|b\rangle$

will left only one path of $|b\rangle$ → kill intef.

> Uncertainty Relation.

Define $\Delta A = A - \langle A\rangle$ As operators $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$ as dispersion.

$\langle(\Delta A)^2\rangle$ Assign to "sharpness" of value. $\langle(\Delta S_x)^2\rangle = \hbar^2/4 - 0 = \hbar^2/4$ → Fuzzy
$\langle(\Delta S_{z+})^2\rangle = \hbar^2/4 - \hbar^2/4 = 0$ → Sharp.

Theory : $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2$

Proof on this theory require three Lemmas.

> Lemma 1. In Hib. Space $|\nu|^2 \geq 0$ For $|\nu\rangle = |\alpha\rangle + \nu|\beta\rangle$

$$(\langle\alpha| + \nu^*\langle\beta|)\cdot(|\alpha\rangle + \nu|\beta\rangle) \geq 0$$

$$\langle\alpha|\alpha\rangle + \underline{\nu\langle\alpha|\beta\rangle} + \nu^*\langle\alpha|\beta\rangle^* + \underline{|\nu|^2\langle\beta|\beta\rangle} \geq 0$$

Since ν is free, Set $\nu = -\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle$ To cancle terms.

$$\langle\alpha|\alpha\rangle - \underline{|\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle} + \nu^*\langle\alpha|\beta\rangle + \underline{|\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle} \Rightarrow \langle\alpha|\alpha\rangle - |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle \geq 0 \quad \square$$

> Lemma 2. Expection Value of Hermiton operators must real.

$$\langle\alpha'|H|\alpha''\rangle = \langle\alpha''|H^\dagger|\alpha'\rangle^* = \langle H\rangle \rightarrow \text{Real} \quad \square$$

> Lemma 3. Vice Versa on Lemma 2. $\square$

> Proof. Lm. 1 $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

Define operator $|\alpha\rangle = \Delta A|\alpha\rangle$, $|\beta\rangle = \Delta B|\beta\rangle$

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\,\Delta B\rangle|^2$$

$$\Delta A\Delta B = \tfrac{1}{2}[\Delta A\Delta B - \Delta B\Delta A + \Delta A\Delta B + \Delta B\Delta A] = \tfrac{1}{2}[\Delta A, \Delta B] + \tfrac{1}{2}\{\Delta A, \Delta B\}$$

$$\langle\Delta A\Delta B\rangle = \tfrac{1}{2}\langle[\Delta A, \Delta B]\rangle + \tfrac{1}{2}\langle\{\Delta A, \Delta B\}\rangle$$

$[\Delta A, \Delta B]$: Since Commutation Relation on $[A, \text{Constant Value}] = 0$

$$= [A, B] - [\langle A\rangle, B] - [A, \langle B\rangle] + [\langle A\rangle, \langle B\rangle] = [A, B]$$

Lemma 2/3 $[\Delta A, \Delta B]^\dagger = [A, B]^\dagger = BA - AB = -[B, A] \longrightarrow \langle[A, B]\rangle$ Pure Im.

$\{\Delta A, \Delta B\}^\dagger = \Delta B\Delta A + \Delta A\Delta B = \{\Delta A, \Delta B\} \longrightarrow \{\Delta A, \Delta B\}$ Pure Re.

$$\langle(\Delta A)^2\rangle + \langle(\Delta B)^2\rangle = \tfrac{1}{4}|\langle[A, B]\rangle|^2 + \tfrac{1}{4}|\langle\{\Delta A, \Delta B\}\rangle|^2 \geq \tfrac{1}{4}|\langle[A, B]\rangle|^2 \quad \square$$

1.5 Change of basis.

Transforming Operators. [Define $|b^1\rangle = U|a^1\rangle$, $|b^2\rangle = U|a^2\rangle$, -- with U is unitary operators]

$U^\dagger U = UU^\dagger = 1$

Assume For $U = \sum_k |b^k\rangle\langle a^k|$ $\quad U \cdot U^\dagger = \sum_k \sum_l |b^k\rangle\langle a^k|a^l\rangle\langle b^l|$

$$= \sum_k |b^k\rangle\,\delta_{kl}\,\langle b^k| = 1$$

Transforming Matrix. [Transforming $\{|a'\rangle\}$ to $\{|b'\rangle\}$]

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$ Transform to $\langle b^k|\alpha\rangle = \sum_l \langle b^k|a^l\rangle\langle a^l|\alpha\rangle$ with $\langle b^k| = \langle a^k|U^\dagger\rangle$

(new) $= U^\dagger$ (old) $\quad = \sum_l \langle a^k|U^\dagger|a^l\rangle\langle a^l|\alpha\rangle$

$$\langle b^k|X|b^l\rangle = \sum_m\sum_n \langle b^k|a^n\rangle\langle a^n|X|a^m\rangle\langle a^m|b^l\rangle$$

with $|b\rangle = U|\alpha\rangle$ $\quad = \sum_{m,n} \langle a^k|U^\dagger|a^n\rangle\langle a^n|X|a^m\rangle\langle a^m|U|b^l\rangle$

$\Rightarrow \quad X' = U^\dagger X U$

> define Trace $tr(X) = \sum_{a'} \langle a'|X|a'\rangle$

$tr(X)$ isn't depends on basis. $= \sum_{a'b'b''} \underline{\langle a'|b'\rangle\langle b'|X|b''\rangle\langle b''|a'\rangle}$ Three scalors → change position freely $= \sum_{b'b''} \underbrace{\langle b''|b'\rangle}_{\delta_{b'b''}}\langle b'|X|b''\rangle$

$= \sum_{b'} \langle b'|X|b'\rangle = Tr(X)$

Diagonalization

For $B|b'\rangle = b'|b'\rangle$

$\sum_{a''} \langle a'|B|a''\rangle\langle a''|b'\rangle = b'\langle a'|b'\rangle$. as $\langle a'|$ represent "direction"

$$\begin{pmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} = b'(b'=\lambda) \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} \rightarrow \det(B-\lambda I) = 0$$ For I is unit vector.

For $B_{ij} = \langle a_i|B|a_j\rangle \qquad C_i = \langle a^i|b'\rangle$

↓ N dimention function with $\lambda_1 \cdots \lambda_N \in b$

↓ one col. of trs. Matrix

For Transforming Matrix define as $|b\rangle = U|\alpha\rangle \qquad U = \sum_{b'} \langle a^i|b'\rangle = \sum_{b'} C_i'$

Unitary Equivalent Observables. [Unitary Define as $UU^\dagger = I \quad U^\dagger = U^{-1}$]

For sets $\{|a'\rangle\}$ $\{|b'\rangle\}$ Conneet by U. Unitary transformation on A is UAU^{-1}

$A|a^l\rangle = a^l|a^l\rangle$

↓ Transformation

both $|a^l\rangle\, |b^l\rangle$ are eigenkets of UAU^{-1} with same eigenvalue a^l

$UAU^{-1}U|a^l\rangle = a^l U|a^l\rangle \longrightarrow (UAU^{-1})|b^l\rangle = a^l|b^l\rangle$

as $B|b^l\rangle = b^l|b^l\rangle$) UAU^{-1} Similar to B.

1.6 Position Momentum Translation

Continuous Spectra. [e.g. P_z is avilable in any value $(-\infty \sim +\infty)$ - Continues]

For Continues spectrum $\underset{\text{operators}}{\xi}|\xi'\rangle = \underset{\text{eigenvalues}}{\xi'}|\xi'\rangle$

$\langle a'|a''\rangle = \delta_{a'a''} \rightarrow \langle \xi'|\xi''\rangle = \delta(\xi'-\xi'')$

$\sum_{a'} |a'\rangle\langle a'| = 1 \longrightarrow \int d\xi |\xi'\rangle\langle\xi'| = 1$

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \longrightarrow \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle$

$\sum_{a} |\langle a'|\alpha\rangle|^2 = 1 \longrightarrow \int d\xi' |\langle\xi'|\alpha\rangle|^2 = 1$

$\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle \rightarrow \langle\beta|\alpha\rangle = \int d\xi' \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle$

$\langle a''|A|a'\rangle = a'\delta_{a'a''} \longrightarrow \xi'\delta(\xi''-\xi')$

Position Eigenkets. — Position Measurement.

$x|x'\rangle = x'|x'\rangle$

with $|\alpha\rangle = \int_-^+ dx' |x'\rangle\langle x'|\alpha\rangle \xrightarrow{\text{Measure close to } (x'-\Delta/2,\, x'+\Delta/2)} \int_{x'-\Delta/2}^{x'+\Delta/2} dx'' |x''\rangle\langle x''|\alpha\rangle$

$P(\text{on position } x') = |\langle x'|\alpha\rangle|^2 \cdot dx'$ with $\int_-^+ |\langle x'|\alpha\rangle|^2 \cdot dx' = 1$ $[\langle\alpha|\alpha\rangle = 1]$

> Coordinates Operators.

$|\alpha\rangle = \int d^3x' |\underline{x}'\rangle\langle\underline{x}'|\alpha\rangle$ with $|\underline{x}'\rangle = |x', y', z'\rangle$ (Simultanious Ket) Restrict by $[x_i, x_j] = 0$.

Translation [Change state define in $|\underline{x}'\rangle \rightarrow |\underline{x}'+d\underline{x}'\rangle$] $[\mathcal{J}(dx') \rightarrow \mathcal{T}(dx')]$

Define $\mathcal{J}(d\underline{x}')|\underline{x}'\rangle = |\underline{x}'+d\underline{x}'\rangle$ Particle move dx' but system keep same.

$|\alpha\rangle \rightarrow \mathcal{J}(d\underline{x}')|\alpha\rangle = \mathcal{J}(dx')\int d^3x' |\underline{x}'\rangle\langle\underline{x}'|\alpha\rangle = \int d^3x' |\underline{x}'+d\underline{x}'\rangle\langle\underline{x}'|\alpha\rangle$ ☆ $= \int d^3x' |\underline{x}'\rangle\langle\underline{x}'-d\underline{x}'|\alpha\rangle$

i). Unitary: $\langle\alpha|\alpha\rangle = \langle\alpha|\mathcal{J}^\dagger(dx')\mathcal{J}(dx')|\alpha\rangle \longrightarrow \mathcal{J}\cdot\mathcal{J}^\dagger = 1$

ii). Additive: $\mathcal{J}(dx'')\mathcal{J}(dx') = \mathcal{J}(dx'+dx'')$ For $x'x''$ dif. direction.

iii). Inverse: $\mathcal{J}(-dx') = \mathcal{J}^{-1}(dx')$

iv). $\lim_{dx'\to 0} \mathcal{J}(dx') = 1$

Assume $\mathcal{J}(dx') = 1 - iK\cdot dx'$ Satisfy all requirements. $[K \Rightarrow K_x K_Y K_Z \quad K^\dagger = K]$

> Commutation of $[X', \mathcal{J}(dx')]$

second deriv of x'

$[X, \mathcal{J}(dx')]|x'\rangle = (x'+dx')|x'+dx'\rangle - x'|x'+dx'\rangle = dx'|x'+dx'\rangle \simeq dx'|x'\rangle$

$[X, \mathcal{J}(dx')] = dx'$ Take $\mathcal{J}(dx') = 1 - iK\cdot dx'$ $-iXK\cdot dx' + iK\cdot dx' X = dx'$ $[x_i, K_j] = i\delta_{ij}$

Momentum as Generator of Translation.

☆ Classical Ideas of momentum. [P is Generation Function]

State 1 $(x_1 P_1) \rightarrow (x_2 P_2)$

$\mathcal{L}_1 = \dot{x}_1 P_1 - U_1$ $\quad \mathcal{L}_2 = \dot{x}_2 P_2 - U_2 + dF(t)/dt$ $\quad$ [$\mathcal{L}$ Unchange for $dF(t)/dt$]

dF/dt = Generation Function $\quad$ For $\mathcal{L}_1 = \mathcal{L}_2$ $\quad \dot{x}_1 P_1 - U_1 dt = \dot{x}_2 P_2 - U_2 dt + dF$

$\Rightarrow dF = dx_1 P_1 - dx_2 P_2 + (U_2 - U_1) dt$ $\quad$ dt term vanish $F(x_1 x_2)$

Type I - Generation Function $\left[\begin{array}{l} \text{For } F(x_1, x_2) \quad dF = \partial F/\partial x_1 \, dx_1 + \partial F/\partial x_2 \, dx_2 \\ P_1 = \partial F/\partial x_1 \quad P_2 = -\partial F/\partial x_2 \end{array} \right.$

Type II G.F. $\left[\begin{array}{l} ☆ \text{ Set } F_2(x_1 p) = F_1(x_1 x_2) + X_2 \cdot P_2 = P_1 dx_1 - \cancel{P_2 dx_2} + \cancel{P_2 dx_2} + x_2 dP_2 . \\ \partial F_2/\partial x_1 = P_1 \quad \partial F_2/\partial p_2 = x_2 \quad \leftarrow = P_1 dx_1 + x_2 dP_2 \end{array} \right.$

[Comments on momentum]: Consider small deviation of $F_2 = x_1 \cdot P_2 + \underbrace{\boxed{\varepsilon G(x_1, P_2)}}$ Generator $(\varepsilon \rightarrow 0)$

$X_2 = \partial F_2/\partial p_2 = x_1 + \varepsilon \, \partial G(x_1 p_2)/\partial p_2 + \partial x_1/\partial p_2$

$\delta x = x_2 - x_1 = \varepsilon \cdot \partial G(x_1 p_2)/\partial p_2$ — For δx Translation
ε is value . $\partial G/\partial p_2 = C$.
Momentum P_2 is Generator. G. $\leftarrow$ G is linear to P_2

As $\mathcal{J}(dx') = 1 - ik\,dx'$ Try connect $\mathcal{J}(dx')$ with Generator function G.

Since K is dimentional P is simply value. Assume $K = P/$dim. of action [Universal Constant] $\rightarrow$ dim. $= \hbar$

[Ref. From Gold Stein : Classical Mechanics]

Chap 9. Canonical Transformation.

$\mathcal{L}(q, \dot{q}, t) \longrightarrow H(q, p, t)$ use momentum p instead of $\dot{q}$

$\dot{q} = \partial H/\partial p \quad \dot{p} = -\partial H/\partial q \quad \delta \int (p\dot{q} - H) \cdot dt = 0$

9.1 Canonical Transformation Equations.

For motion with Hamiltonian is C [$\dot{H} = 0$]. Coordinates q_i cyclic [$\partial H/\partial q_i = 0$]

$\Rightarrow \dot{p} = \partial H/\partial q = 0$ $\quad P_i = \alpha_i$ As constants.

$H = H(\alpha_1 \alpha_2 \cdots \alpha_n) \Rightarrow \dot{q}_i = \partial H/\partial \alpha_i = W_i$ $\quad$ For $W_i = W_i(\alpha_i) \Rightarrow q_i = W_i t + \beta$

☆ Point Transformations [Eg. Use $Q(q, t)$ as new coordinates of motion. Q described by (q, t)]

New coord. define as $Q_i = Q_i(q, p, t)$ $\quad P_i = P_i(q, p, t)$ $\quad$ [$H(q, p, t)$ with p, q same level]

Point. Transf. of Configuration Space : $Q(q,t)$ Only position in time t

Point. Transf. of Phase Space : $Q(q,p,t)$ Position / State in time t

In new coordinates, $\delta\int_{t_1}^{t_2}(p\dot{q}-H)\cdot dt = \int_{t_1}^{t_2}\delta(P_i\dot{Q}_i - K(Q,P,t))\,dt = 0$ with $\dot{Q} = \partial K/\partial p$ $\dot{P} = -\partial K/\partial Q$

Since $\mathcal{L} = \mathcal{L} + dF/dt$ $\boxed{\lambda}(p\dot{q}-H) = P_i\dot{Q}_i - K(Q,P,t) + dF/dt$

Simply for scale transformation.

$\lambda = \mu\nu$

$K'(Q',P') = \mu\nu H(q,p)$

For simple scale change $Q_i' = \mu q_i$ $P_i' = \nu P_i$

$\mu\nu(P_i\dot{q}_i - H) = P_i'\dot{Q}_i' - K'$

> Focusing On $\lambda = 1$ Unscaled Canonical Transformation.

$p\dot{q} - H = P\dot{Q} - K + dF/dt$ define F = Generation function.

Since dF/dt discribe change from old coordinates to new. F may include both side coordinates.

For $F = F_1(q,Q,t)$ $p\dot{q} - H = P\dot{Q} - K + \partial F/\partial t + \partial F/\partial q\cdot\dot{q} + \partial F/\partial Q\cdot\dot{Q}$

☆ vanished with other terms since $\dot{q}\dot{Q}$ Independent

$\Rightarrow K = H + \partial F/\partial t$

$p_i = \partial F_1/\partial q_i$ $P_i = -\partial F_1/\partial Q_i$

☆ Legendar Transformation [In $F_1(q,Q,t)$ If $Q=q$ System break down]

$dF_1 = p\cdot dq - P\cdot dQ$ Try adding PQ to F_1

$dF_2 = p\,dq - \cancel{P\cdot dQ} + \cancel{P\,dQ} + Q\cdot dP = p\,dq + Q\cdot dP \Rightarrow F_2(q,P,t) = F_1 + QP$

with same restriction: $p\dot{q} - H = P\dot{Q} - K + dF/dt$ $\Rightarrow p = \partial F_2/\partial q_i$ $Q = \partial F_2/\partial P_i$

Type	Restrict		Simple example		
$F_1 = F_1(q,Q,t)$	$p = \partial F_1/\partial q$	$P = -\partial F_1/\partial Q$	$F_1 = qQ$	$Q = p$	$P = -q$
$F_2 = F_1 + QP$	$p = \partial F_2/\partial q$	$Q = \partial F_2/\partial P$	$F_2 = qP$	$Q = q$	$P = p$
$F_3 = F_1 - qp$	~		~		
$F_4 = F_1 + QP - qp$	~		~		

> For DoF. more than 1 System. F is able to choose freely in each Dimention.

$$\begin{array}{ccc} x & \xrightarrow{F_1} & X \\ y & \xrightarrow{F_2} & Y \\ z & \xrightarrow{F_3} & Z \end{array}$$

Represent by $F'(q_1 p_2 P_1 Q_2 t)$ ☆ Obv. maximum coord in $F' = 2\times$ DoF.

9.2 Examples on Canonical Transformation.

$F_2 = q_i P_i$ As simplest form $p_i = \partial F_2/\partial q_i$ $Q_i = \partial F_2/\partial P_i$

$\Rightarrow F_2 = f_i(q_1, q_2 \cdots q_n; t) P_i$ [For $f_i(\cdots)$ represent any function pick in $(\cdots)$]

$Q_i = \partial F_2 / \partial P_i = f_i(q_1, q_2 \cdots q_n, t)$ ☆ Only old position include $-Q(q,t)$ Point transformation $\Rightarrow$ All Points trans. are canonical

☆ For any $F_2(q, P)$ Have $Q = \partial F_2 / \partial P$ $\quad p = \partial F_2 / \partial q$ As canonical coordinate transf. ☆

> Generation Unit [Adding $g(m)$ as generation unit]

$F_2 = f_i(m) P_i + g(m)$ $\quad p_j = \partial F_2 / \partial q_j = \partial f_i / \partial q_j \cdot P_i + \partial g / \partial q_j \longrightarrow p = \partial f / \partial q \cdot P + \partial g / \partial q$

In matrix notation Eg. 2D. $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial q_1 & \partial f_1/\partial q_2 \\ \partial f_2/\partial q_1 & \partial f_2/\partial q_2 \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \partial g/\partial q_1 \\ \partial g/\partial q_2 \end{bmatrix}$

$P = (\partial f / \partial q)^{-1} (p - \partial g / \partial q)$ $\quad$ with $Q = f_i(m)$.

9.3 Harmonic Oscillator [$H = p^2/2m + \frac{1}{2} k q^2$]

$H = \frac{1}{2m}(p^2 + m^2 \omega^2 q^2)$ [Try on convert to $\sin^2\theta + \cos^2\theta = 1$ Build cyclic coordinates]

$p = f(P) \cdot \cos Q \quad q = f(P)/m\omega \sin Q \longrightarrow H = f(P)^2/2m \qquad \partial H / \partial Q = 0.$

Try Find $f(p)$ make transf. system canonical

$p/q = m\omega \cot Q \rightarrow f(p) = m\omega q / \sin Q \qquad p = q m \omega \cot Q$ $\qquad p = p(q, Q) \rightarrow F_1$ or F_2.

For $F_2(q, Q, t)$ $\begin{cases} p = \partial F_2 / \partial q_i = q m \omega \cot Q \rightarrow F_2 = \frac{1}{2} q^2 m \omega \cot Q \\ P = m\omega q^2 / 2\sin^2 Q \quad \Rightarrow q = \sqrt{2P/m\omega} \cdot \sin Q \quad f(P) = \sqrt{2 P m \omega} \end{cases}$ ☆ Reason on $F_1(q, Q)$ include Q

According to H. Canonical Function $\begin{aligned} \partial H / \partial Q &= \dot{P} \rightarrow P \text{ is const.} = E/\omega \\ \partial H / \partial P &= \dot{Q} \rightarrow Q = \omega t + \alpha \end{aligned}$ New Coord. P.Q Very Simple

$\Rightarrow \quad q = \sim \qquad p = \sim$

$H = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2 = \frac{1}{2}(p^2/m + k q^2) = \frac{1}{2m}(p^2 + m^2 \omega^2 q^2) = f(P)^2 / 2m$

Set new coord. $p = f(P) \sin Q \qquad q = f(P) \cos Q / m\omega$

Solve $f(P)$, Q $\quad p = q m \omega \tan Q = p(q, Q) \longrightarrow F = F_1(q, Q)$

$\partial F / \partial q = p = q m \omega \tan Q \longrightarrow F_1 = \frac{1}{2} q^2 m \omega \tan Q \qquad -\partial F / \partial Q = q^2 m \omega / 2\cos^2 Q = P$

$f(P) = p / \sin Q = q m \omega / \cos Q = \sqrt{2 m \omega P} \qquad H = f(P)^2 / 2m = \omega P$

For Cano. H. $\partial H / \partial P = \dot{Q} = \omega \qquad \partial H / \partial Q = \dot{P} = 0 \quad \rightarrow Q = \omega t + \alpha \qquad P = E/\omega$

$q = \sqrt{2E / m\omega^2} \sin(\omega t + \alpha) \qquad p = \sqrt{2Em} \cos(\omega t + \alpha)$

9.4 Symplectic Aproach

For $Q_i = Q(q,p)$ $\quad P_i = P(q,p)$

$\dot{Q}_i = \partial Q/\partial q \cdot \dot{q} + \partial Q/\partial p \cdot \dot{p} = \partial Q/\partial q\, \partial H/\partial p - \partial Q/\partial p\, \partial H/\partial q = \partial H/\partial P$

Similar On $P = P(Q,P)$ $\quad q = q(Q,P)$ $\quad \partial H/\partial P = \partial H/\partial q \cdot \partial q/\partial P + \partial H/\partial p\, \partial p/\partial P$

$$\Rightarrow \begin{cases} (\partial Q/\partial q)_{q,p} = (\partial P/\partial P)_{Q,P} & (\partial Q/\partial p)_{q,p} = -(\partial q/\partial P)_{QP} \\ (\partial P/\partial q)_{qp} = -(\partial P/\partial Q)_{QP} & (\partial P/\partial p)_{qp} = (\partial q/\partial Q)_{QP} \end{cases}$$

> Symplectic Notation $[\eta, \xi]$

η = Column Matrix of $\begin{bmatrix} \text{old } q \\ \text{old } p \end{bmatrix}$

ξ = 〃 of $\begin{bmatrix} \text{new} Q \\ \text{new} P \end{bmatrix}$

$J = \begin{bmatrix} & 1 \\ & & 1 \\ -1 \\ & -1 \end{bmatrix}$ ← add -1 for deriv. on p

$\dot{\eta} \begin{cases} \dot{p} = -\partial H/\partial q \\ \dot{q} = \partial H/\partial p \end{cases} = J \cdot \partial H/\partial \eta$

Obv. $\xi = \xi(\eta)$ $\quad$ for $\dot{\xi}(\eta) = \partial \xi_i/\partial \eta_j \cdot \dot{\eta}_j$ $[i,j \in 1,2,3 \cdots n]$ ☆ new/old not perfect match

> define Jacobian Matrix $\quad M_{ij} = \partial \xi_i/\partial \eta_j$ [Recall Gausses Integral]

$\dot{\xi} = MJ \cdot \partial H/\partial \eta$ ☆ Reason on M^T : $\partial H/\partial \eta_i = \sum_j \partial H/\partial \xi_j \cdot \partial \xi_j/\partial \eta_i = \sum_j M_{ji}\, \partial H/\partial \xi_j = \sum_j M^T \cdot \partial H/\partial \xi_j$

η_i only column matrix M_{ji} Unable match η_i

$\dot{\xi} = MJM^T \cdot \partial H/\partial \xi$ ↔ Since $\dot{\xi} = \partial \xi/\partial \eta \cdot \partial \eta/\partial t = M \cdot J \cdot \partial H/\partial \eta = M J \cdot \partial H/\partial \xi\, (\partial \xi/\partial \eta)^{M^{-1}} = J\, \partial H/\partial \xi$

$MJM^T = J$ ☆ if System (η, ξ) $MJM^{-1} = J$ Transformation is Canonical

Focusing On $\xi = \xi(\eta, t)$ $\quad \xi$ change with t.

if $\eta \to \xi(t)$ $\Rightarrow$ $\eta \to \xi(t_0)$ $\Rightarrow$ $\xi(t_0) \to \xi(t)$ Canonical

☆ Consider $\xi(t+dt)$ Compare to $\xi(t)$? Canonical Trans.

Set Small Change δ : $\begin{cases} Q_i = q_i + \delta q_i \\ P_i = P_i + \delta P_i \\ \xi = \eta + \delta \eta \end{cases}$ $\Rightarrow F_2 = q_i P_i + \varepsilon G(q, P, t)$ $\begin{cases} p_i = P_i + \varepsilon \cdot \partial G/\partial q_i \to \delta P = -\varepsilon\, \partial G/\partial q_i \\ Q = q_i + \varepsilon \cdot \partial G/\partial P_i \to \delta Q = \varepsilon\, \partial G/\partial P_i \end{cases}$

$\delta \eta = \varepsilon J\, \partial G/\partial \eta$

To proof if it's canonical. $\quad MJM^T = J \quad M = \partial \xi/\partial \eta = 1 + \partial \delta \eta/\partial \eta = 1 + \varepsilon J (\partial G^2/\partial \eta \partial \eta)$ ← ☆

$MJM^T = (1+\varepsilon J \cdot \partial^2 G/\partial\eta_i\partial\eta_j) J (1-\varepsilon \partial^2 G/\partial\eta_i\partial\eta_j \cdot J) \longleftarrow$ All operation is on matrix. $(\partial^2 G/\partial\eta_i\partial\eta_j)_{ij}$ Built Matrix.

$= [1-\varepsilon \cdot \partial^2 G/\partial\eta\partial\eta \cdot J + \varepsilon J \cdot \partial^2 G/\partial\eta\partial\eta - M \cdot M^T] \cdot J$

$= J + \varepsilon J \cancel{\partial^2 G/\partial\eta\partial\eta} J - J\varepsilon \cdot \cancel{\partial^2 G/\partial\eta\partial\eta} J = J \Rightarrow$ For small deviations, Trans. Alwayse Canonical.

9.5 Poisson Brackets + Other Canonical Invariants

$[u,v]_{q,p} = \partial u/\partial q \cdot \partial v/\partial p - \partial u/\partial p \, \partial v/\partial q \longrightarrow [u,v]_\eta = \partial u/\partial\eta_i \, \partial v/\partial\eta_j - \partial u/\partial\eta_j \cdot \partial v/\partial\eta_i = \partial u \partial v/\partial\eta\partial\eta \, J = (\partial u/\partial\eta)^T J (\partial v/\partial\eta)$

> Momentum As Generator of Translation [Ski. Quantum] Assume $J(dx') = 1 - iK \cdot dx'$

Use II Generation Function $X_{new} = X + dx$ $P = P'$ $F(x,p) = x \cdot p + p \cdot dx$ K belongs to 1/L dimention

For $F_2(x,p) = \Sigma q_i P_i + \varepsilon G(q,p,t)$ System move $x \to x+\varepsilon$ Generation Unit $G = P_x$

$[\delta P = 0 \quad \delta q = \varepsilon]$

$\delta q = Q - q = \varepsilon \cdot \partial G/\partial P_i \qquad \delta P_i = P - p = -\varepsilon \cdot \partial G/\partial q_i \qquad \Rightarrow G = P$

☆ Axiom in Quantum $[x, p] = i\hbar$ Use $[x, J(dx)]$ Replace $[x, K]$.

For $[x, J(dx')]|x'\rangle = x|x'+dx'\rangle - J(dx') \, x'|x'\rangle = (x'+dx')|x'+dx'\rangle - x'|x'+dx'\rangle = dx'|x'+dx'\rangle \simeq dx'|x'\rangle$

$\Rightarrow x(1-iKdx') - (1-iKdx') \cdot x = -xiKdx' + iKdx'x = dx'$ ☆ $= idx'[K, x] \Rightarrow [x, K] = i$

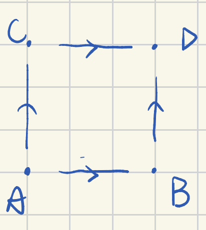
$xK - Kx = i$ with $xp - Px = i\hbar \Rightarrow K$ must $= P/\hbar$

> Consider Finitive Change $|x + \Delta x\rangle$ [Seperate to $J(dx)^N$]

$J(\Delta x'\hat{x})|x'\rangle = |x' + \Delta x'\hat{x}\rangle \rightarrow J(\Delta x \hat{x}) = J(dx')^N$ with $dx = \Delta x/N$ $N \to \infty$

$= \lim_{n\to\infty} (1 - ip\Delta x'/n\hbar)^n$ ☆ e^x defined as $\lim_{N\to\infty} (1 + x/N)^N = e^x$

$= \exp(-ip\Delta x'/\hbar)$



Obviously, $J(\Delta y'\hat{y}) J(\Delta x'\hat{x}) = J(\Delta x'\hat{x} + \Delta y'\hat{y}) = J(\Delta x'\hat{x}) J(\Delta y'\hat{y}) \rightarrow [J(y_n), J(x_n)] = 0$

> define Abelian = Commutable = $([A,B]=0)$ $\Rightarrow [P_x, P_y] = 0$

> define Simuta. Eigenket $|P'\rangle = |P_x' P_y' P_z'\rangle$ $J(dx')|P'\rangle = (1 - ip'dx'/\hbar)|P'\rangle$ $[p, J(dx')] = 0$

> Canonical Commutation Relations

Basic Axioms $[x, x] = [p, p] = 0$ $[x, p] = i\hbar$

Notice Poisson Brackets $[A, B]_{(q,p)} = \sum_s \{ \partial A/\partial q \, \partial B/\partial p - \partial A/\partial p \, \partial B/\partial q \}$

$[x_i, p_j]$ classical $= \delta_{ij}$

Quantum $[\,,\,] = i\hbar$ Classical $[\,\,]$ With Same Operation Laws.

Chap 1.7 Wave Function in x and p space

> Position – Space wave function [$\langle\beta|f(x)|\alpha\rangle = ?$]

For $x|x'\rangle = x'|x'\rangle$ In Continues Spectra $|\alpha\rangle = \int dx'|x'\rangle\langle x'|\alpha\rangle$ with $\psi_\alpha(x')$ define as $\langle x'|\alpha\rangle$ for appear probability of x' in α

$\langle\beta|\alpha\rangle = \int dx'\langle\beta|x'\rangle\langle x'|\alpha\rangle = \int dx'\,\psi_\beta^*(x')\psi_\alpha(x')$ → Prob. of State β found in State α

Just Simply Define ψ

> Try Calculate $\langle\beta|f(x)|\alpha\rangle$

As $|\alpha\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle$ $\quad\langle x'|\alpha\rangle = \int\langle x|a'\rangle\langle a'|\alpha\rangle \Rightarrow \psi(x')_\alpha = \sum C_{a'}u_{a'}(x')$ with $u_{a'}(x') = \langle x'|a'\rangle$

$\langle\beta|A|\alpha\rangle = \int dx'\int dx''\langle\beta|x''\rangle\langle x''|A|x'\rangle\langle x'|\alpha\rangle = \int dx'\int dx''\,\psi_\beta(x'')\langle x''|A|x'\rangle\psi_\alpha(x') \leftarrow$ by having value of $\langle x'|A|x''\rangle$ □

> Momentum Operator In Position Basis. [$\hat p = -i\hbar\,\partial/\partial x\cdot\psi$]

In Classical Mechanics P represent by Generation Unit of $F(\mathrm{II})$

$J(\Delta x)|\alpha\rangle = (1 - ip\Delta x/\hbar)|\alpha\rangle = \int dx'\,J(\Delta x')|x'\rangle\langle x'|\alpha\rangle = \int dx'\,|x'+\Delta x'\rangle\langle x'|\alpha\rangle$

☆ $-\Delta x'$ on all x'

☆ $\Delta x'$ Term not include in x → Only subtract Δx Term

$= \int dx'\,|x'\rangle\langle x'-\Delta x'|\alpha\rangle = \int dx'\,|x'\rangle(\langle x'|\alpha\rangle - \Delta x'\,\partial/\partial x'\langle x'|\alpha\rangle)$

$-ip\Delta x'/\hbar\,|\alpha\rangle = -\Delta x'\,\partial/\partial x'\langle x'|\alpha\rangle$

$p|\alpha\rangle = \int dx'|x'\rangle(-i\hbar\,\partial/\partial x'\langle x'|\alpha\rangle) \longrightarrow \langle x'|p|\alpha\rangle = -i\hbar\,\partial/\partial x'\langle x'|\alpha\rangle$

Replace α by x'' $\quad\langle x'|p|x''\rangle = -i\hbar\,\partial/\partial x'\,\delta(x'-x'')$

$\langle\beta|p|\alpha\rangle = \int dx'\langle\beta|x'\rangle(-i\hbar\cdot\partial/\partial x'\langle x'|\alpha\rangle) = \int dx'\,\psi_\beta^*(x')\cdot(-i\hbar\,\partial/\partial x')\,\psi_\alpha(x')$

> Momentum – Space Wave function

$P|p'\rangle = P'|p'\rangle \qquad \langle p'|p''\rangle = \delta(p'-p'') \qquad |\alpha\rangle = \int dp'|p'\rangle\underline{\langle p'|\alpha\rangle}$

$|\langle p'|\alpha\rangle|^2$ represent prob. of measure get p' define as $\phi_\alpha(p')$

$\int dp'\langle\alpha|p'\rangle\langle p'|\alpha\rangle = 1$ For normalize

> Connections on x and p representations [Transform Bases Matrix $U_{ij} = \langle a_i|b_j\rangle$]

$\langle x'|p|\alpha\rangle$ replace α with p' → $\langle x'|p|p'\rangle = -i\hbar\,\partial/\partial x'\langle x'|p'\rangle \longrightarrow \underbrace{\langle x'|p'\rangle}_{U} = -\frac{i\hbar}{p'}\cdot\partial/\partial x'\underbrace{\langle x'|p'\rangle}_{U}$

$-\ln U = (-ip'/\hbar) \rightarrow U = N\exp(ip'x'/\hbar)$

$U = \langle x'|p'\rangle = N\exp(ip'x'/\hbar)$ $\qquad |\langle x'|p'\rangle|^2 = \langle x'|p'\rangle\cdot\langle p'|x'\rangle$]

Left N for normalize $\quad$ Use $\langle x'|x''\rangle = \int p'\langle x'|p'\rangle\langle p'|x''\rangle$

$\delta(x'x'') = \int p'\,N\exp(ip'x'/\hbar)\cdot N\exp(-ip'x''/\hbar)$

$\delta(x'-x'') = \int p'\,N^2\exp[i/\hbar\,(p'x'-p'x'')]\cdot dp' = |N|^2\int dp'\exp[ip'/\hbar\,(x'-x'')]\cdot dp$

$\star$ Fourier's Transformation [$f(x) = \sum_n C_n \exp(ik_n x) = \int_{-\infty}^{+\infty} g(x)\cdot \exp(-ikx)\, dx$]

Transform — $f(x) = 1/\sqrt{2\pi} \int_-^+ g(x) \exp(+ikx)$

Inverseform — $g(x) = 1/\sqrt{2\pi} \int_-^+ f(x) \exp(-ikx)$

$f(x) = (1/\sqrt{2\pi})^2 \int_-^+ \exp(ikx)\cdot \int_-^+ f(x') \exp(-ikx')\, dx'\, dx$

$f(x) = 1/2\pi$ $\boxed{\text{Sth. Take } f(x') \text{ out of } f(x)}$ $= 1/2\pi\, \delta(x'-x)$

$\delta(x-x') = 1/2\pi \int \exp(ik(x-x'))\cdot dk$

$\star$ Proofments On Why Left Pure $\delta(x-x')$ without any constant.

Original Eqs. $= |N|^2 2\pi\, \delta(x'-x'')\hbar = \delta(x'-x'') \Rightarrow N = 1/\sqrt{2\pi\hbar}$

$\langle x'|p'\rangle = 1/\sqrt{2\pi\hbar}\, \exp(ip'x'/\hbar)$.

Similar form.

> Symboles of $\psi_\alpha(x')$ and $\phi_\alpha(p')$ Finally

$$\begin{cases} \psi_\alpha(x') = [1/\sqrt{2\pi\hbar}]\cdot \int dp' \exp(ip'x'/\hbar)\, \phi_\alpha(p') \\ \phi_\alpha(p') = [1/\sqrt{2\pi\hbar}]\cdot \int dx' \exp(-ip'x'/\hbar)\, \psi_\alpha(p') \end{cases} \leftrightarrow \quad \begin{matrix} Q = f(p) \\ P = f(q) \end{matrix}$$

II Generate Unit.

> Gaussian Wave Packets. [$\psi_\alpha(x') = [1/\pi^{1/4}\sqrt{d}] \exp(ikx - x'^2/2d^2)$] Use as example illustrates formalism.

$\langle x\rangle = \int_-^+ dx' \langle\alpha|x'\rangle x \langle x|\alpha\rangle = \int_-^+ dx\, |\psi_\alpha(x')|^2\cdot x = \frac{1}{d\sqrt{\pi}}\cdot \int_- \exp(-x'^2/d^2)\cdot x\cdot dx = 0$

$\langle x^2\rangle = \frac{1}{\sqrt{\pi}\cdot d}\cdot \int_-^+ dx' x'^2 \exp(-x'^2/d^2) = d^2/2$

$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = d^2/2$ Similar $\langle(\Delta p)^2\rangle = \hbar^2/2d^2$ $\quad \langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$.

$\langle p|\alpha\rangle$ also occupied by $\phi_\alpha = [\sim] \int \exp(-ipx/\hbar)\cdot \psi_\alpha(x') = \cdots$

> Generalize to 3D.

$q \ni (q_x\, q_y\, q_z)$ $\quad \langle q'|q''\rangle = \delta^3(q'-q'')$ $\quad$ Same as P

$\langle\beta|p|\alpha\rangle \longrightarrow \int d^3x'\, \psi_\beta^*(x')(-i\hbar\nabla)\psi_\alpha(x')$ $\quad$ with $\psi_\alpha(x)$ $\phi_\alpha(x)$ Transform by coef. $1/(2\pi\hbar)^{3/2}$