

Modern Quantum Mechanics

Leo

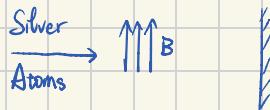
Liu, C.



1.1 Experiment.

- Magnetic Moment μ $U = -\mu \cdot B$ $F = \mu \times B$ define as $\mu = I \cdot \text{Enclosed Area}$

i). Silver 47 electron \rightarrow 46 Steady 1 Self Spin



ii). $\mu \propto$ Spin s . $F = \mu_z \frac{\partial B_z}{\partial z}$

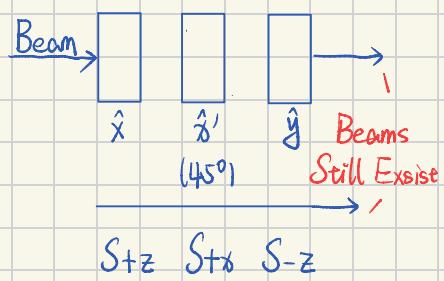
iii). Expected $\mu \in -|u| \sim |u|$ — Quantum $\mu = \mu_+$ or μ_- with $S_{z\pm} = \pm \frac{\hbar}{2}$

$$\Rightarrow \begin{cases} \uparrow \\ \downarrow \end{cases} \text{(P of detected)}$$

$$\Rightarrow \begin{cases} \uparrow \\ \downarrow \end{cases}$$

IV. Cancel $S_z -$ Add B_z $S_{z\pm}$ Both Appear Again.

- Polarized light experiment.



$S_{z\pm} \leftrightarrow x, y$ polarized

$S_{x\pm} \leftrightarrow x', y'$ polarized

noted as "ket" sign in Dirac notation

$$E = E_0 \hat{x} \cos(\omega t) = |S_z + \rangle$$

$$= E_0 \hat{y} \cos(\omega t) = |S_z - \rangle$$

$$|S_x + \rangle = \frac{1}{\sqrt{2}} |S_z + \rangle + \frac{1}{\sqrt{2}} |S_z - \rangle$$

$$E_0 \hat{x}' \cos(\omega t) = E_0 [\cos 45^\circ \hat{x} \cos(\omega t) + \cos 45^\circ \hat{y} \cos(\omega t)]$$

$$|S_x - \rangle = -\frac{1}{\sqrt{2}} |S_z + \rangle + \frac{1}{\sqrt{2}} |S_z - \rangle$$

$$E_0 \hat{y}' \cos(\omega t) = E_0 [-\cos 45^\circ \hat{x} \cos(\omega t) + \cos 45^\circ \hat{y} \cos(\omega t)]$$

1.2 Kets Bras Operators

Ket Space

Observe System in Vector Space \rightarrow State is n -dimensional vector.

Define Ket Space as Vector space $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$ $c|\alpha\rangle = |\alpha\rangle \cdot c$

$A|\alpha\rangle = a|\alpha\rangle$ For A is observable $|\alpha'\rangle, |\alpha''\rangle, \dots$ is eigenket of $|\alpha\rangle$

$$\text{e.g. } S_z |S_z \pm \rangle = \pm \frac{\hbar}{2} |S_z \pm \rangle$$

Since Linear Additive $|\alpha\rangle = \sum \alpha'_i |\alpha'_i\rangle$

Bra Space - Inner Product.

Define $\langle \alpha |$ corresponding to $|\alpha\rangle$ Dual Correspondence

$$|\alpha\rangle \langle \alpha'| \langle \alpha''| \dots \langle \alpha''| + |\beta\rangle \langle \beta| \xrightarrow{\text{DC}} \langle \alpha | \langle \alpha' | \langle \alpha'' | \dots \langle \alpha'' | + \langle \beta |$$

$\star C|\alpha\rangle \xleftrightarrow{DC} C^*|\alpha\rangle$ Define innerproduct $\langle\beta|\alpha\rangle = (\langle\beta|)\cdot(|\alpha\rangle) \xrightarrow{DC} = \langle\alpha|\beta\rangle^*$

- $\langle\alpha|\alpha\rangle \geq 0$, Real Number

- $\langle\alpha|\beta\rangle = 0$ Orthogonal on α, β

Operators

$$X \cdot (|\alpha\rangle) = X|\alpha\rangle \quad X=Y \text{ if } X|\alpha\rangle = Y|\alpha\rangle \quad \text{if } X|\alpha\rangle = 0 \quad X \text{ is null}$$

$\star X|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|X^\dagger$ if $X=X^\dagger$ X is Hermitian.

Define outer product $(|\beta\rangle)(\langle\alpha|) = |\beta\rangle\langle\alpha|$

Associative Axiom

$$- (|\beta\rangle\langle\alpha|) \cdot |\gamma\rangle = |\beta\rangle \cdot (\langle\alpha|\gamma\rangle)$$

$$- \text{ if } X=|\beta\rangle\langle\alpha| \quad X^\dagger = |\alpha\rangle\langle\beta|$$

$$- (\langle\beta|) \cdot (X|\alpha\rangle) = (\langle\beta|X) \cdot (|\alpha\rangle) \rightarrow \langle\beta|X|\alpha\rangle$$

$$- \langle\beta|X|\alpha\rangle = \langle\beta|\cdot(X|\alpha\rangle) = \{(\langle\alpha|X^\dagger)|\beta\rangle\}^* = \langle\alpha|X^\dagger|\beta\rangle^*.$$

1.3 Base kets and matrix representations.

- Eigenvalue of Hermitian operator A is real

Eigenkets of A to diff. eigenvalues are orthonormal

$$\text{Proof, } A|\alpha'\rangle = \alpha'|\alpha'\rangle \quad \langle\alpha''|A = \alpha''^*\langle\alpha''|$$

\downarrow

$$0 = \langle\alpha''|A|\alpha'\rangle - \alpha''^*\langle\alpha''|\alpha'\rangle \quad 0 = \cancel{\langle\alpha''|\alpha'\rangle} (\alpha' - \alpha''^*)$$

If:

$$\text{i). } \alpha'' = \alpha' \rightarrow \alpha' = \alpha'^*$$

$$\text{ii). } \alpha'' \neq \alpha' \rightarrow \langle\alpha''|\alpha'\rangle = 0 \quad \text{Each eigenvalues orthonormal}$$

$$\langle\alpha''|\alpha'\rangle = \delta_{\alpha''\alpha'} \quad \text{if } \alpha' \neq \alpha'' = 0$$

$\Lambda_{\alpha'}^2$: Project on α'

$$- \text{Base Kets. } |\alpha\rangle = \sum C_{\alpha'} |\alpha'\rangle \quad \sum \Lambda_{\alpha'}^2 = 1 : \text{Project on all } \alpha \text{ Must} = 1$$

$$\langle\alpha''|\alpha\rangle = \langle\alpha''|\sum C_{\alpha'} |\alpha'\rangle = \sum_{\alpha'} C_{\alpha'} \delta_{\alpha''\alpha'} \Rightarrow C_{\alpha'} = \langle\alpha'|\alpha\rangle$$

$$|\alpha\rangle = \sum_{\alpha'} \langle\alpha'|\alpha\rangle \cdot |\alpha'\rangle = \sum_{\alpha'} |\alpha'\rangle \langle\alpha'|\alpha\rangle \quad \Rightarrow \quad \boxed{\sum_{\alpha'} |\alpha'\rangle \langle\alpha'| = 1}$$

$$V = \sum_i e_i^\dagger (e_i \cdot V) \quad \star \langle\alpha'\rangle \text{ must orthonormal with } |\alpha\rangle$$

Since $\sum |\alpha'| \langle \alpha' | = 1 \Rightarrow \langle \alpha | \alpha' = \langle \alpha | \sum_{\alpha'} |\alpha'| \langle \alpha' | \alpha \rangle = \sum_{\alpha'} |\alpha'| \langle \alpha' | \alpha \rangle|^2 \Rightarrow \sum_{\alpha'} |\alpha'|^2 = \sum_{\alpha'} |\langle \alpha' | \alpha \rangle|^2 = 1$

$\star (|\alpha'| \langle \alpha'|) \cdot |\alpha\rangle = |\alpha'\rangle \langle \alpha'| \alpha \rangle = C_{\alpha'} |\alpha'\rangle \Rightarrow |\alpha'\rangle \langle \alpha'| \text{ portion of } |\alpha\rangle \text{ (for } |\alpha\rangle \text{ is normalized)}$

define $\Lambda_{\alpha'} = |\alpha'\rangle \langle \alpha'| \quad \sum \Lambda_{\alpha'} = 1 \quad \Lambda_{\alpha'} \text{ Subtract valv. of proportion of } \alpha' \text{ on } \alpha$

Matrix Representation. $\star |\alpha'\rangle \langle \alpha'| \dots$ From eigenket list. No df.

Define $X = \sum_{\alpha''} \sum_{\alpha'} |\alpha''\rangle \langle \alpha''| \sum |\alpha'| \langle \alpha'|$

Convert to $N \times N$ Matrix where N is dim. of ket space

$$X = \begin{bmatrix} \langle \alpha''_1 | X | \alpha'_1 \rangle & \langle \alpha''_1 | X | \alpha'_2 \rangle & \dots \\ \langle \alpha''_2 | X | \alpha'_1 \rangle & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix} \Rightarrow \langle \alpha''_i | X | \alpha'_j \rangle = \langle \alpha'_j | X + |\alpha''_i\rangle^*$$

Insert Identity Matrix

- For $Z = XY \quad \langle \alpha'' | Z | \alpha' \rangle = \sum_{\alpha'''} \langle \alpha'' | X | \alpha''' \rangle \langle \alpha''' | Y | \alpha' \rangle$

- For $|\beta\rangle = X|\alpha\rangle \quad \text{Acting on base } \langle \alpha' | \quad \langle \alpha' | \beta \rangle = \langle \alpha' | X | \alpha \rangle = \sum_{\alpha''} \langle \alpha' | X | \alpha'' \rangle \langle \alpha'' | \alpha \rangle$

$$|\alpha\rangle = \begin{pmatrix} \langle \alpha' | \alpha \rangle \\ \langle \alpha^2 | \alpha \rangle \\ \vdots \end{pmatrix} \quad |\beta\rangle = \begin{pmatrix} \langle \alpha' | \beta \rangle \\ \langle \alpha^2 | \beta \rangle \\ \vdots \end{pmatrix} \quad \text{As column Matrix.}$$

$$|\alpha\rangle = \sum_{\alpha'} |\alpha'\rangle \langle \alpha' | \alpha \rangle = \begin{pmatrix} |\alpha'_1\rangle \langle \alpha'_1 | \alpha \rangle \\ |\alpha'_2\rangle \langle \alpha'_2 | \alpha \rangle \\ \vdots \end{pmatrix} \quad \star |\alpha_1\rangle \langle \alpha_2 | \dots \text{ Have no valv. - only directions.}$$

- Similar on $\langle \beta | = \langle \alpha | X \quad \langle \beta | \alpha' \rangle = \sum_{\alpha''} \langle \alpha | \alpha'' \rangle \langle \alpha'' | X | \alpha' \rangle$

$$\langle \beta | = (\langle \beta | \alpha'_1 \rangle, \langle \beta | \alpha^2 \rangle, \dots)^* \quad (\langle \alpha' | \beta \rangle, \langle \alpha^2 | \beta \rangle, \dots)^* \quad \text{As row Matrix.}$$

- Column Matrix \cdot Row Matrix $\langle \beta | \alpha \rangle = \sum_{\alpha'} \langle \beta | \alpha' \rangle \langle \alpha' | \alpha \rangle$

$$= (\langle \beta | \alpha'_1 \rangle \langle \beta | \alpha_2 \rangle \dots) \cdot \begin{pmatrix} \langle \alpha_1 | \alpha \rangle \\ \langle \alpha_2 | \alpha \rangle \\ \vdots \end{pmatrix} = \text{some value}$$

$$- \text{Row} \cdot \text{Column} \quad |\beta\rangle \langle \alpha| = \sum_{\alpha'} \langle \alpha' | \beta \rangle \langle \alpha | \alpha' \rangle = \begin{pmatrix} \langle \alpha'_1 | \beta \rangle \langle \alpha'_1 | \alpha \rangle^* & \langle \alpha'_1 | \beta \rangle \langle \alpha^2 | \alpha \rangle^* & \dots \\ \langle \alpha^2 | \beta \rangle \langle \alpha'_1 | \alpha \rangle^* & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

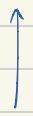
- Eigenkets of A used as baseket. [$\sum_{\alpha'} |\alpha'\rangle \langle \alpha'| = 1 - \alpha'$ is baseket]

$$A = \sum_{\alpha''} \sum_{\alpha'} |\alpha''\rangle \langle \alpha''| A |\alpha' \rangle \langle \alpha'| = \text{if } \langle \alpha' | A \alpha'' \rangle \text{ where } \alpha' + \alpha'' = 0 \Rightarrow \delta_{\alpha'' \alpha'} \langle \alpha' | A | \alpha' \rangle = \alpha' \delta_{\alpha'' \alpha'}$$

$$A = \sum_{\alpha'} \alpha' |\alpha' \rangle \langle \alpha'| = \sum_{\alpha'} \alpha' \Lambda_{\alpha'}$$

* defin S_{\pm} as Up down operators

$$S_{\pm} = \hbar |\pm\rangle \langle \mp|$$



Set $|-\rangle$ to $\hbar |-\rangle$

$$S_{+} |-\rangle = \hbar |+\rangle \langle -| = \hbar |+\rangle$$

$$I = \sum_{\alpha'} |\alpha' \rangle \langle \alpha'| = |\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|$$

$$S_z = \sum_{\alpha'} \alpha' \Lambda_{\alpha'} = (\frac{\hbar}{2}) |\uparrow\rangle \langle \uparrow| - (\frac{\hbar}{2}) |\downarrow\rangle \langle \downarrow| = \frac{\hbar}{2} (|\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|) \quad S_z |\pm\rangle = \pm (\frac{\hbar}{2}) |\pm\rangle$$

\downarrow eigenket.

eigen value as $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

1.4 Measurements, Observables, Uncertainty

$$|\alpha\rangle = \sum_{i,j,k} \langle \alpha' | \alpha \rangle \cdot |\alpha'\rangle = \sum_{\alpha'} |\alpha' \rangle \langle \alpha' | \alpha \rangle$$

$|\alpha\rangle$ [Series of possible eigenvalue] $\xrightarrow{\text{Observe}}$ $|\alpha'\rangle$ [Single CD. value]

$|\alpha'\rangle$ $\xrightarrow{\text{Observe}}$ $|\alpha'\rangle$

$$\text{Prob. of } \alpha' \text{ observed} = |\Lambda_{\alpha'} \cdot \alpha|^2 = \left| \sum_{\alpha'} \langle \alpha' | \alpha \rangle \right|^2$$

$$\text{Vector} \downarrow \quad = |\text{Constant}|^2 \cdot |\text{Vector}|^2 = |\langle \alpha' | \alpha \rangle|^2 \cdot \underbrace{\langle \alpha' | \alpha' \rangle}_{=1}$$

$$\langle A \rangle = \sum_{\alpha'} \underbrace{\alpha'}_{\text{valv.}} \underbrace{\frac{|\langle \alpha' | \alpha \rangle|^2}{P}}$$

$$= \sum_{\alpha'} \sum_{\alpha''} \langle \alpha | \alpha' \rangle \langle \alpha'' | A | \alpha' \rangle \langle \alpha' | \alpha \rangle = \sum_{\alpha'} \sum_{\alpha''} \langle \alpha' | \alpha \rangle \langle \alpha'' | \alpha \rangle^* \delta_{\alpha'' \alpha'} \alpha'$$

$$= \sum_{\alpha'} |\langle \alpha' | \alpha \rangle|^2 \alpha' \quad |S_{\theta} + \rangle \text{ on } S_z \text{ direction} = \vec{A} |+\rangle + \vec{B} |-\rangle$$

Spin 1/2 System, again

with $|A| = |B| = \frac{1}{\sqrt{2}}$ and direction unknown.

$$P = \sum |\Lambda_{\alpha'} \cdot \alpha|^2 = 1 \text{ with Same P. on } S_{\theta} \pm \text{ for } S_{\theta} + \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} \exp(i\delta_1) |-\rangle \Leftarrow \frac{1}{\sqrt{2}} \exp(i\omega_1) |+\rangle + \frac{1}{\sqrt{2}} \exp(i\beta_1) |-\rangle$$

$$|\underbrace{\langle + | S_{\theta} + \rangle}_{S_{\theta}+} | = |\underbrace{\langle - | S_{\theta} + \rangle}_{S_{\theta}-} | = \frac{1}{\sqrt{2}} \Rightarrow |S_{\theta} + \rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} \exp(i\delta_1) |-\rangle$$

phase difference in $|+\rangle |-\rangle$.

$$\text{Since } |S_{\theta} + \rangle \text{ orthonormal with } |S_{\theta} - \rangle \langle + | - \rangle = 0 \Rightarrow |S_{\theta} - \rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} \exp(i\delta_1) |-\rangle$$

$$S_{\theta} = \sum_{\alpha'} \alpha' \Lambda_{\alpha'} = \frac{\hbar}{2} [|S_{\theta} + \rangle \langle S_{\theta} + | - |S_{\theta} - \rangle \langle S_{\theta} - |] = \frac{\hbar}{2} [e^{-i\delta_1} |+\rangle \langle -| + e^{i\delta_1} |-\rangle \langle +|]$$

$$\text{Similar } |S_y \pm \rangle = \frac{1}{\sqrt{2}} |+\rangle \pm \frac{1}{\sqrt{2}} \exp(i\delta_2) |-\rangle \Rightarrow S_y = \begin{pmatrix} 0 & \exp(i\delta_2) \\ \exp(i\delta_1) & 0 \end{pmatrix}$$

$$\text{To determine } \delta_1, \delta_2 \quad |\langle S_y \pm | S_{\theta} + \rangle| = |\langle S_y \pm | S_{\theta} - \rangle| = \frac{1}{\sqrt{2}} \Rightarrow S_y = \begin{pmatrix} 0 & \exp(i\delta_2) \\ \exp(i\delta_1) & 0 \end{pmatrix}$$

$$= [\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \exp(i\delta_2) \langle - |] [\frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} \exp(i\delta_1) |-\rangle]$$

$$= \frac{1}{2} - \frac{1}{2} \exp[i(\delta_2 - \delta_1)] = \frac{1}{2} \quad (\langle +|+ \rangle = 1 \quad \langle -|- \rangle = 0)$$

$$\Rightarrow \delta_2 - \delta_1 = \pm \frac{\pi}{2}. \quad \star \exp(i\theta) \in \text{Re} \text{ when } \theta \in [0, \pi, 2\pi \dots]$$

$\delta_2 - \delta_1 = \frac{\pi}{2} \rightarrow$ Must have one imagine. Assume $x \in \text{Re}, y \in \text{Im}$

$$S_x = \frac{i}{2} [|+\rangle\langle -| + |-\rangle\langle +|] \quad |S_x|_{\pm} = \frac{1}{2} |+\rangle \pm \frac{1}{2} |-\rangle$$

$$S_y = \frac{i}{2} [-i|+\rangle\langle -| + i|-\rangle\langle +|] \quad |S_y|_{\pm} = \frac{1}{2} |+\rangle \pm \frac{i}{2} |-\rangle$$

$$\star S_{\pm} = \hbar | \pm \rangle \langle \mp | \rightarrow S_{\pm} = S_x \pm iS_y = \hbar | \pm \rangle \langle \mp | \quad [\text{Up/Down degree operators}]$$

$$\text{with } S_x S_y S_z \quad [S_x, S_y] = \frac{i\hbar}{4} [i|+\rangle\langle +| - i|-\rangle\langle -| + i|+\rangle\langle +| - i|-\rangle\langle -|] \\ = i\hbar \frac{1}{2} [|+\rangle\langle +| - |-\rangle\langle -|] = S_z \cdot i\hbar \quad \Rightarrow [S_z, S_y] = E_y b \cdot S_z \cdot i\hbar$$

$$\text{Similar Anti-Commutation } \{S_x, S_y\} = \frac{1}{2}\hbar^2 \delta_{xy} \quad [\{A, B\} = AB - BA]$$

$$> \text{Define } S^2 = S_x^2 + S_y^2 + S_z^2 = 3 \cdot (\frac{1}{4}\hbar^2) \quad [S^2, S_z] = 0$$

Compatible Observables.

define $[A, B] = 0$ compatible vice versa.

\star degenerate : two diff. eigen state \rightarrow same correspond eigen value $\Rightarrow |a'\rangle$ isn't complete.

> Theory: A, B compatible eigenvalue of A nondegenerate. $\langle a''|B|a'\rangle$ all diagonal

$$\langle a''|[A, B]|a'\rangle = 0$$

$$\langle a''|AB - BA|a'\rangle = \langle a''|AB|a'\rangle - \langle a''|BA|a'\rangle = \langle a''|a''\rangle B|a'\rangle - \langle a''|B|a'\rangle a''|a'\rangle = \langle a''|B|a'\rangle (a'' - a'') = 0$$

Obv. $a' = a''$ or $\langle a''|B|a'\rangle = 0 \Rightarrow \langle a''|B|a'\rangle = \delta_{a''a'} \langle a'|B|a''\rangle \rightarrow$ Diagonal Matrix.

Simultaneous Eigenkets.

$$B = \sum_{a''} |a''\rangle \langle a''| B |a''\rangle \langle a''| \quad \text{adding on ket } |a'\rangle = \sum_{a''} |a''\rangle \langle a''| B |a''\rangle \langle a''| a'\rangle = (\langle a'|B|a'\rangle) |a'\rangle$$

$\delta_{a''a'}$ Simultaneous Eigenkets

define $|K'\rangle$ collective index \leftarrow

$$|K'\rangle = |a', b'\rangle$$

$$A|a', b'\rangle = a'|a', b'\rangle$$

$$B|a', b'\rangle = b'|a', b'\rangle$$

$$\star B|a'\rangle = b'|a'\rangle \text{ with } b' = \langle a'|B|a'\rangle$$

$$A|a'\rangle = a'|a'\rangle$$

AB Use Same eigenstate $|a', b'\rangle$

Degeneracy Notations.

$$|K'\rangle = |a', b', c' \dots \rangle \quad (\text{Assume compatible})$$

$$\text{obv. } \sum_{K'} |K'\rangle \langle K'| = 1 \quad \langle K''|K'\rangle = \delta_{K''K'}$$

$|a\rangle$ Measure A $\rightarrow |a', b'\rangle$ with same value a' degeneracy in a Meas. B $\rightarrow |a', b'\rangle$ with diff. b' diff. in b'

* degeneracy \rightarrow linear combination of kets. $= \sum_i^n |a^i|a', b^i\rangle$ [For n degeneracy A state, measure gives a'
B state, gives $b^1, b^2 \dots b^n$]

> Incompatible // Collective Index

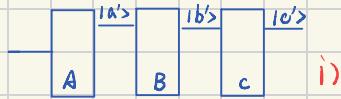
For $[A, B] \neq 0$

$$AB|a', b'\rangle = a'b' |a', b'\rangle \Rightarrow AB = BA \quad [A, B] \text{ must } = 0. \quad (\times)$$

$$BA|a', b'> = b'a'|a', b'>$$

$\Rightarrow AB = BA$ $[A, B]$ must = 0. (X)

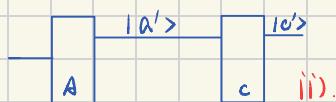
For $[A, B] \neq 0$, $|a', b'\rangle$ not exist



c in b b in a

$$\text{P of output} = |\langle 0' | b' \rangle|^2 |\langle b' | a' \rangle|^2$$

As B have eigen states. summing over b' for all possible routes.



$$\sum_{B'} |C' \langle B' |^2 | \langle B' | A' \rangle |^2 = \sum_{B'} \cancel{C' \langle B' |} \cancel{\langle B' | A' \rangle} \cancel{\langle A' | B' \rangle} \cancel{\langle B' | C' \rangle} \quad (\text{Eq. 1})$$

Difference in b' and b'' value.

$$P_{\text{out}} = |\langle c' | a' \rangle|^2 = \sum_{b'} |\langle c' | b' \rangle \langle b' | a' \rangle|^2 = \sum_{b'} \sum_{b''} \langle c' | b' \rangle \langle b' | a' \rangle \langle a' | b'' \rangle \langle b'' | c' \rangle \quad (\text{Eq. 2})$$

State in i). $|b\rangle$ is detected, just left normal P. adding together.

State in ii). $\sum_b \sum_{b''}$ cause interference pair when $b \neq b''$

For $[A, B] = 0$: S-m. eigenkets $|a\rangle$ have $B|a'\rangle = b'|a\rangle$ $\langle b|a\rangle = \delta_{bb'}$

$$\text{Eq.1} = |\langle c|b'\rangle \langle b'|a\rangle|^2 \quad = \quad \text{Eq.2} = |\langle c|b\rangle|^2 |\langle b'|a\rangle|^2 \quad [\text{only } b=b' \text{ exist}]$$

Interference $\sum_B \sum_{B'}$ cause by different possible path. $[A, B] = 0$ // Test on $|b\rangle$

will leave only one path of $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ → kill inf.

> Uncertainty Relation

Define $\Delta A = A - \langle A \rangle$ As operators $\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$ as dispersion

$\langle (\Delta A)^2 \rangle$ Assign to "sharpness" of value

$$\begin{aligned} \nabla & \langle (\Delta S_x)^2 \rangle = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4} & \rightarrow \text{Fuzzy} \\ & \langle (\Delta S_x + i)^2 \rangle = \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = 0 & \rightarrow \text{Sharp} \end{aligned}$$

$$\text{Theory : } \langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} [\langle [A, B] \rangle]^2$$

Proof on this theory require three Lemmas.

> Lemma 1. In Hilb. Space $|n|^2 \geq 0$ For $|n\rangle = |\alpha\rangle + n|\beta\rangle$

$$(\langle\alpha| + n^* \langle\beta|) \cdot (|\alpha\rangle + n|\beta\rangle) \geq 0$$

$$\langle\alpha|\alpha\rangle + \underline{n\langle\alpha|\beta\rangle} + n^* \langle\alpha|\beta\rangle^* + \underline{|n|^2 \langle\beta|\beta\rangle} \geq 0$$

Since n is free, Set $n = -\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle$ To cancel terms.

$$\underbrace{\langle\alpha|\alpha\rangle - |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle}_{+ n^* \langle\alpha|\beta\rangle + |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle} \Rightarrow \langle\alpha|\alpha\rangle - |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle \geq 0 \quad \square$$

> Lemma 2. Expectation Value of Hermitian Operators must real.

$$\langle\alpha'|\hat{H}|\alpha''\rangle = \langle\alpha''|\hat{H}^\dagger|\alpha'\rangle^* = \langle\hat{H}\rangle \rightarrow \text{Real.} \quad \square$$

> Lemma 3. Vice Versa on Lemma 2. \square

> Proof. Lm. 1 $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle >= |\langle\alpha|\beta\rangle|^2$

$$\begin{aligned} \text{Define Operator } |\alpha\rangle &= \Delta A |\alpha\rangle \\ |\beta\rangle &= \Delta B |\beta\rangle \end{aligned}$$

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq |\langle \Delta A \Delta B \rangle|^2$$

$$\Delta A \Delta B = \frac{1}{2} [\Delta A \Delta B - \Delta B \Delta A + \Delta A \Delta B + \Delta B \Delta A] = \frac{1}{2} [\Delta A, \Delta B] + \frac{1}{2} \{ \Delta A, \Delta B \}$$

$$\langle \Delta A \Delta B \rangle = \frac{1}{2} \langle [\Delta A, \Delta B] \rangle + \frac{1}{2} \langle \{ \Delta A, \Delta B \} \rangle$$

$[\Delta A, \Delta B]$: Since Commutation Relation on $[\Delta A, \text{Constant Value}] = 0$

$$= [\Delta A, \Delta B] - \langle \Delta A \rangle \langle \Delta B \rangle - \langle \Delta A \rangle \langle \Delta B \rangle + \langle \Delta A \rangle \langle \Delta B \rangle = [\Delta A, \Delta B]$$

Lm. 2 / 3 $[\Delta A, \Delta B]^\dagger = [\Delta A, \Delta B]^\dagger = \Delta B \Delta A - \Delta A \Delta B = -[\Delta B, \Delta A] \rightarrow \langle [\Delta A, \Delta B] \rangle \text{ Pure Im.}$

$$\{ \Delta A, \Delta B \}^\dagger = \Delta B \Delta A + \Delta A \Delta B = \{ \Delta A, \Delta B \} \rightarrow \{ \Delta A, \Delta B \} \text{ Pure Re.}$$

$$\langle (\Delta A)^2 \rangle + \langle (\Delta B)^2 \rangle = \frac{1}{4} \langle [\Delta A, \Delta B] \rangle^2 + \frac{1}{4} \langle \{ \Delta A, \Delta B \} \rangle^2 \geq \frac{1}{4} \langle [\Delta A, \Delta B] \rangle^2 \quad \square$$

1.5 Change of basis.

Transforming Operators. [Define $|b^1\rangle = U|a^1\rangle$, $|b^2\rangle = U|a^2\rangle$, ... with U is unitary operators.]

$$\begin{aligned} \text{Assume For } U &= \sum_k |b^k\rangle \langle a^k| \quad U \cdot U^\dagger = \sum_k \sum_l |b^k\rangle \langle a^k| a^l \rangle \langle b^l| \\ &= \sum_k |b^k\rangle \delta_{kl} \langle b^k| = I \end{aligned}$$

$$U^\dagger U = U U^\dagger = I$$

Transforming Matrix. [Transforming $\{ |a^1\rangle \}$ to $\{ |b^1\rangle \}$]

$$|\alpha\rangle = \sum_{\alpha'} |\alpha'\rangle \langle \alpha'| \alpha\rangle \quad \text{Transform to} \quad \langle b^k | \alpha\rangle = \sum_i \langle b^k | \alpha^i \rangle \langle \alpha^i | \alpha\rangle \quad \text{with} \quad \langle b^k | = \langle \alpha^k | U^\dagger \rangle$$

$$(\text{new}) = U^\dagger (\text{old}) \quad = \sum_i \langle \alpha^k | U^\dagger | \alpha^i \rangle \langle \alpha^i | \alpha\rangle$$

$$\langle b^k | X | b^l \rangle = \sum_m \sum_n \langle b^k | \alpha^n \rangle \langle \alpha^n | X | \alpha^m \rangle \langle \alpha^m | b^l \rangle$$

$$\text{with} \quad |b\rangle = U|\alpha\rangle \quad = \sum_{m,n} \langle \alpha^k | U^\dagger | \alpha^n \rangle \langle \alpha^n | X | \alpha^m \rangle \langle \alpha^m | U | b^l \rangle$$

$$\Rightarrow X' = U^\dagger X U$$

$$> \text{define Trace} \quad \text{tr}(X) = \sum_{\alpha'} \langle \alpha' | X | \alpha' \rangle$$

$$\text{tr}(X) \text{ isn't depends on basis.} \quad = \sum_{\alpha' \beta' \beta''} \langle \alpha' | b' \rangle \langle b' | X | b'' \rangle \langle b'' | \alpha' \rangle \quad = \sum_{b' b''} \frac{\langle b'' | b' \rangle \langle b' | X | b'' \rangle}{\delta_{b' b''}}$$

Three scalars \rightarrow change position freely

$$= \sum_{b'} \langle b' | \alpha | b' \rangle = \text{Tr}(x)$$

Diagonalization

$$\text{For } B|b\rangle = b'|b\rangle$$

$$\left(\sum_{\alpha''} \langle \alpha' | B | \alpha'' \rangle \langle \alpha'' | b' \rangle \right) = b' \langle \alpha' | b' \rangle \quad \text{as } \langle \alpha' | \text{ represent "direction"}$$

$$\begin{pmatrix} B_{11} & B_{12} & \dots \\ B_{21} & \dots & \dots \\ \vdots & \ddots & \dots \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} = b' (b' = n) \cdot \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} \quad \rightarrow \det(CB - nI) = 0. \quad \text{For } i \text{ is unit vector.}$$

$$\text{For } B_{ij} = \langle \alpha_i | B | \alpha_j \rangle \quad C_i = \langle \alpha^i | b' \rangle \quad N \text{ dimension function with } n_1 \dots n_N \in b$$

\downarrow one col. of trs Matrix

$$\text{For Transforming Matrix define as} \quad |b\rangle = U|\alpha\rangle \quad U = \sum_{b'} \langle \alpha^i | b' \rangle = \sum_{b'} C_i$$

Unitary Equivalent Observables [Unitary Define as $U^\dagger U = I \quad U^\dagger = U^{-1}$]

For sets $\{|\alpha'\rangle\} \{|\beta'\rangle\}$ Connect by U . Unitary transformation on A is UAV^{-1}

$$A|\alpha^i\rangle = \alpha^i |\alpha^i\rangle$$

\downarrow Transformation

$$UAV^{-1}U|\alpha^i\rangle = \alpha^i U|\alpha^i\rangle \quad \rightarrow \quad (UAV^{-1})|b^i\rangle = \alpha^i |b^i\rangle$$

both $|\alpha^i\rangle$ $|b^i\rangle$ are eigenvectors of UAV^{-1} with same eigenvalue α^i

as $B|b^i\rangle = b^i |b^i\rangle$) UAV^{-1} Similar to B .

1.6 Position Momentum Translation

Continuous Spectra. [e.g. P_z is available in any value ($-\infty \dots +\infty$) - Continuous].

For Continuous spectrum $\underline{s}|s\rangle = s'|s'\rangle$
operators eigenvalues

$$\langle \alpha' | \alpha'' \rangle = \delta_{\alpha' \alpha''} \rightarrow \langle \xi' | \xi'' \rangle = \delta(\xi' - \xi'')$$

$$\sum_{\alpha'} |\alpha'\rangle \langle \alpha'| = 1 \rightarrow \int d\xi' |\xi'\rangle \langle \xi'| = 1$$

$$|\alpha\rangle = \sum_{\alpha'} |\alpha'\rangle \langle \alpha'| \alpha \rangle \rightarrow \int d\xi' |\xi'\rangle \langle \xi'| \alpha \rangle$$

$$\sum_{\alpha} |\langle \alpha' | \alpha \rangle|^2 = 1 \rightarrow \int d\xi' |\langle \xi' | \alpha \rangle|^2 = 1$$

$$\langle \beta | \alpha \rangle = \sum_{\alpha'} \langle \beta | \alpha' \rangle \langle \alpha' | \alpha \rangle \rightarrow \langle \beta | \alpha \rangle = \int d\xi' \langle \beta | \xi' \rangle \langle \xi' | \alpha \rangle$$

$$\langle \alpha'' | A | \alpha' \rangle = \alpha' \delta_{\alpha'' \alpha''} \rightarrow \xi' \delta(\xi'' - \xi')$$

Position Eigenkets. — Position Measurement.

$$|x|x'\rangle = |x'|x\rangle$$

with $|\alpha\rangle = \int_{-\infty}^{\infty} dx' |x'\rangle \langle x'| \alpha \rangle$ Measure close to $(x' - \Delta/2, x' + \Delta/2)$. $\int_{x'-\Delta/2}^{x'+\Delta/2} dx'' |x''\rangle \langle x''| \alpha \rangle$

$$P(\text{on position } x') = |\langle x' | \alpha \rangle|^2 \cdot dx' \quad \text{with} \quad \int_{-\infty}^{\infty} |\langle x' | \alpha \rangle|^2 \cdot dx' = 1 \quad [\langle \alpha | \alpha \rangle = 1]$$

> Coordinates Operators.

(Simultaneous Ket)

$$|\alpha\rangle = \int d^3x' |x'\rangle \langle x' | \alpha \rangle \quad \text{with} \quad |\underline{x}'\rangle = |x', y', z'\rangle \quad \text{Restrict by} \quad [x_i, x_j] = 0.$$

Translation [Change state define in $|x'\rangle \rightarrow |x'+dx'\rangle$] $[J(dx')] \rightarrow J(dx')$.

$$\text{Define } J(dx') |x'\rangle = |x'+dx'\rangle$$

Particle move dx' but system keep same

$$|\alpha\rangle \rightarrow J(dx') |\alpha\rangle = J(dx') \int d^3x' |x'\rangle \langle x' | \alpha \rangle = \int d^3x' |x'+dx'\rangle \langle x' | \alpha \rangle \quad \star = \int d^3x' |x'\rangle \langle x'-dx' | \alpha \rangle$$

i). Unitary: $\langle \alpha | \alpha \rangle = \langle \alpha | J^\dagger(dx') J(dx') | \alpha \rangle \rightarrow J \cdot J^\dagger = 1$

ii). Additive: $J(dx'') J(dx') = J(dx'+dx'')$ For $x' x''$ dif. direction.

iii). Inverse: $J(-dx') = J^{-1}(dx')$

iv). $\lim_{dx' \rightarrow 0} J(dx') = 1$

Assume $J(dx') = 1 - ik \cdot dx'$ Satisfy all requirements. $[K \rightarrow k_x, k_y, k_z \quad K^\dagger = K]$

> Commutation of $[x', J(dx')]$

Second deriv of x'

$$[x', J(dx')] |x'\rangle = (x'+dx') |x'+dx'\rangle - x' |x'+dx'\rangle = dx' |x'+dx'\rangle \approx dx' |x'\rangle$$

$$[x, J(dx')] = dx' \quad \text{Take } J(dx') = 1 - ik \cdot dx'$$

$$-ixk \cdot dx' + ik \cdot dx' x = dx' \quad [x_i, k_j] = i\delta_{ij}$$

Momentum as Generator of Translation

* Classical Ideas of momentum [P is Generation Function]

State I $(x_1, p_1) \rightarrow (x_2, p_2)$

$$\dot{x}_1 = \dot{x}_1 P_1 - U_1 \quad \dot{x}_2 = \dot{x}_2 P_2 - U_2 + dF(x)/dt \quad [L \text{ Unchange for } dF(x)/dt]$$

dF/dt = Generation Function For $\dot{x}_1 = \dot{x}_2$ $\dot{x}_1 P_1 - U_1 dt = \dot{x}_2 P_2 - U_2 dt + dF$

$$\Rightarrow dF = d\dot{x}_1 P_1 - d\dot{x}_2 P_2 + (U_2 - U_1) dt \quad \text{dt term vanish}$$

$$\left[\begin{array}{l} \text{For } F(x_1, x_2) \quad dF = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 \\ P_1 = \frac{\partial F}{\partial x_1} \quad P_2 = -\frac{\partial F}{\partial x_2} \end{array} \right]$$

Type I - Generation Function

Type II ★ Set $F_2(x_1, p_1) = F_1(x_1, x_2) + x_2 \cdot P_2 = P_1 dx_1 - P_2 dx_2 + P_2 dx_2 + x_2 dp_2$.

$$\frac{\partial F_2}{\partial x_1} = P_1 \quad \frac{\partial F_2}{\partial p_2} = x_2 \quad \leftarrow = P_1 dx_1 + x_2 dp_2$$

Generator ($\varepsilon \rightarrow 0$)

[Comments on momentum]: Consider small deviation of $F_2 = x_1 \cdot P_2 + \boxed{\varepsilon G(x_1, P_2)}$

$$x_2 = \frac{\partial F_2}{\partial P_2} = x_1 + \varepsilon \frac{\partial G(x_1, P_2)}{\partial P_2} + \frac{\partial \varepsilon G}{\partial P_2}$$

$$\delta x = x_2 - x_1 = \varepsilon \frac{\partial G(x_1, P_2)}{\partial P_2} \quad \text{- For } \delta x \text{ Translation}$$

ε is value, $\frac{\partial G}{\partial P_2} = C$.

Momentum P_2 is Generator G. $\leftarrow G$ is linear to P_2

As $J(dx') = -ik dx'$ Try connect $J(dx')$ with Generator function G.

Since K is dimensional P is simply value. Assume $K = P/\text{dim. of action [Universal Constant]} \rightarrow \text{dim.} = \hbar$

[Ref. From Gold Stein : Classical Mechanics]

Chap 9. Canonical Transformation.

$L(q, \dot{q}, t) \rightarrow H(q, p, t)$ use momentum p instead of \dot{q}

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \delta \int (p_i \dot{q}_i - H) dt = 0$$

9.1 Canonical Transformation Equations.

For motion with Hamiltonian is 0 [$H=0$], Coordinates q_i cyclic [$\frac{\partial H}{\partial q_i} = 0$]

$$\Rightarrow \dot{p}_i = \frac{\partial H}{\partial q_i} = 0 \quad p_i = \text{const.}$$

$$H = H(x_1, x_2, \dots, x_n) \Rightarrow \dot{q}_i = \frac{\partial H}{\partial x_i} = w_i \quad \text{For } w_i = w_i(\text{const.}) \Rightarrow q_i = w_i t + \beta$$

* Point Transformations [Eg. Use $Q_i(q, t)$ as new coordinates of motion, Q described by (q, t)]

New coord. define as $Q_i = Q_i(q, p, t)$ $P_i = P_i(q, p, t)$ [$H(q, p, t)$ with p, q same level]

Point. Transf. of Configuration Space : $Q(q, t)$ Only position in time t

Point. Transf. of Phase Space : $Q(q, p, t)$ Position / State in time t

In new coordinates, $\int_{t_1}^{t_2} (pq - H) dt = \int_{t_1}^{t_2} (\dot{P}_i Q_i - K(Q, P, t)) dt = 0$ with $\dot{Q} = \frac{\partial K}{\partial P}$ $\dot{P} = -\frac{\partial K}{\partial Q}$

Since $L = L + dF/dt$ $\boxed{[n]}(pq - H) = P_i \dot{Q}_i - K(Q, P, t) + dF/dt$

simply for scale transformation.

$$K'(Q', P') = M^2 H(Q, P)$$

For simple scale change $Q'_i = M q_i$ $P'_i = \lambda P_i$

$$\lambda^2 (P_i \dot{Q}_i - H) = P'_i \dot{Q}'_i - K'$$

> Focusing On $n=1$ Unscaled Canonical Transformation.

$pq - H = P \dot{Q} - K + dF/dt$ define F = Generation function.

Since dF/dt describe change from old coordinates to new. F may include both side coordinates.

For $F = F(q, Q, t)$ $pq - H = P \dot{Q} - K + \frac{\partial F}{\partial t} + \underbrace{\frac{\partial F}{\partial q} \cdot \dot{q}}_{\text{I}} + \underbrace{\frac{\partial F}{\partial Q} \cdot \dot{Q}}$

$$\Rightarrow K = H + \frac{\partial F}{\partial t}$$

★ vanished with other terms since \dot{Q} Independent

$$P_{ii} = \frac{\partial F}{\partial q_i} \quad P_i = -\frac{\partial F}{\partial Q_i}$$

* Legendre Transformation [In $F_1(q, Q, t)$ If $Q=q$ System break down]

$dF = P dq - P_i dQ$ Try adding PQ to F ,

$$dF_2 = P dq - P_i dQ + P_i dQ + Q dP = P dq + Q dP \Rightarrow F_2(q, P, t) = F_1 + QP$$

$$\text{with same restriction: } pq - H = P \dot{Q} - K + dF/dt \Rightarrow P = \frac{\partial F_2}{\partial q_i} \quad Q = \frac{\partial F_2}{\partial P_i}$$

Type

$$F_1 = F_1(q, Q, t)$$

Restrict.

$$P = \frac{\partial F_1}{\partial q} \quad P_i = -\frac{\partial F_1}{\partial Q}$$

Simple example.

$$F_1 = qQ \quad Q = P \quad P = -q$$

$$F_2 = F_1 + QP$$

$$P = \frac{\partial F_2}{\partial q} \quad Q = \frac{\partial F_2}{\partial P}$$

$$F_2 = qP \quad Q = P \quad P = P$$

$$F_3 = F_1 - qP$$

~

~

$$F_4 = F_1 + QP - qP$$

~

~

> For DoF. more than 1 System. F is able to choose freely in each Dimension.



Represent by $F'(q_1, P_1, Q_2, t)$ ★ Obv. maximum coord in $F' = 2 \times \text{DoF}$.

9.2 Examples on Canonical Transformation.

$$F_2 = q_i P_i \quad \text{As simplest form} \quad P_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$\Rightarrow F_2 = f_2(q_1, q_2, \dots, q_n; t) P_i \quad [\text{For } f_2 \text{ can represent any function pick in } c \dots]$$

$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_1, q_2, \dots, q_n; t)$ \star Only odd position include $-Q(q_i, t)$ Point transformation \Rightarrow All Points trans. are canonical.

\star For any $F_2(q, P)$ Have $Q = \frac{\partial F_2}{\partial P}$ $P = \frac{\partial F_2}{\partial q}$ As canonical coordinate transf. \star

> Generation Unit [Adding $g(m)$ as generation unit]

$$F_2 = f_2(m) P_i + g(m) \quad P_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_2}{\partial q_j} \cdot P_i + \frac{\partial g}{\partial q_j} \rightarrow P = \frac{\partial f}{\partial q} \cdot P + \frac{\partial g}{\partial q}$$

$$\text{In matrix notation Eg. 2D. } \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial q_1} & \frac{\partial f}{\partial q_2} \\ \frac{\partial g}{\partial q_1} & \frac{\partial g}{\partial q_2} \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial q_2} \end{bmatrix}$$

$$P = (\frac{\partial f}{\partial q})^{-1} (P - \frac{\partial g}{\partial q}) \quad \text{with } Q = f_2(m).$$

9.3 Harmonic Oscillator [$H = P^2/2m + \frac{1}{2}kq^2$]

$$H = \frac{1}{2}m(CP^2 + m^2\omega^2q^2) \quad [\text{Try to convert to } \sin^2\theta + \cos^2\theta = 1 \text{ Build cyclic coordinates}]$$

$$P = f_{CP} \cdot \cos Q \quad q = f_{CP}/m\omega \sin Q \rightarrow H = \frac{f_{CP}^2}{2m} \quad \frac{\partial H}{\partial Q} = 0.$$

Try Find f_{CP} make transf. system canonical

$$P = P(q, Q) \rightarrow F_1 \text{ or } F_2.$$

$$\frac{P}{q} = m\omega \cot Q \rightarrow f_{CP} = \frac{m\omega q}{\sin Q} \quad P = qm\omega \cot Q$$

\star Reason on $F_1(q, Q)$

$$\text{For } F_2(q, Q, t) \quad \left\{ \begin{array}{l} P = \frac{\partial F_2}{\partial q_i} = qm\omega \cot Q \rightarrow F_2 = \frac{1}{2}q^2 m\omega \cot Q \xrightarrow{\text{include } Q} \\ P = \frac{m\omega q^2}{2\sin^2 Q} \rightarrow q = \sqrt{\frac{2P}{m\omega}} \cdot \sin Q \quad f_{CP} = \sqrt{\frac{2P}{m\omega}} \end{array} \right.$$

$$\text{According to H. Canonical Function} \quad \begin{aligned} \frac{\partial H}{\partial Q} &= \dot{P} \rightarrow P \text{ is const.} = E/\omega \\ \frac{\partial H}{\partial P} &= \dot{Q} \rightarrow Q = \omega t + \alpha \end{aligned} \quad \text{New coord. P, Q Very Simple}$$

$$\Rightarrow q = \omega \quad P = E/\omega$$

$$H = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 = \frac{1}{2}(P^2/m + kq^2) = \frac{1}{2}m(P^2/m + \underline{m^2\omega^2q^2}) = \frac{f_{CP}^2}{2m}$$

$$\text{Set new coord. } P = f_{CP} \sin Q \quad q = f_{CP} \cos Q / m\omega$$

$$\text{Solve } f_{CP}, Q \quad P = qm\omega \tan Q = P(q, Q) \rightarrow F = F_1(q, Q)$$

$$\frac{\partial F}{\partial q} = P = qm\omega \tan Q \rightarrow F_1 = \frac{1}{2}q^2 m\omega \tan Q \quad -\frac{\partial F}{\partial Q} = q^2 m\omega / 2\cos^2 Q = P$$

$$f_{CP} = P/\sin Q = qm\omega / \cos Q = \sqrt{2mE/\omega}$$

$$H = f_{CP}^2/2m = E/\omega$$

$$\text{For Cano. H. } \frac{\partial H}{\partial P} = \dot{Q} = \omega \quad \frac{\partial H}{\partial Q} = \dot{P} = 0 \rightarrow Q = \omega t + \alpha \quad P = E/\omega$$

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

$$P = \sqrt{2E/m} \cos(\omega t + \alpha)$$

9.4 Symplectic Approach.

For $Q_i = Q(q, p)$ $P_i = P(q, p)$

$$\dot{Q}_i = \frac{\partial Q}{\partial q} \dot{q} + \frac{\partial Q}{\partial p} \dot{p} = \frac{\partial Q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial H}{\partial q} = \frac{\partial H}{\partial p}$$

$$\text{Similar On } P = P(Q, p) \quad q = q(Q, p) \quad \frac{\partial H}{\partial p} = \frac{\partial H}{\partial q} \cdot \frac{\partial q}{\partial p} + \frac{\partial H}{\partial p} \frac{\partial}{\partial p}$$

$$\Rightarrow \begin{cases} (\frac{\partial Q}{\partial q})_{Q,p} = (\frac{\partial P}{\partial p})_{Q,p} & (\frac{\partial Q}{\partial p})_{Q,p} = -(\frac{\partial P}{\partial p})_{Q,p} \\ (\frac{\partial P}{\partial q})_{Q,p} = -(\frac{\partial P}{\partial Q})_{Q,p} & (\frac{\partial P}{\partial p})_{Q,p} = (\frac{\partial Q}{\partial Q})_{Q,p} \end{cases}$$

> Symplectic Notation. [J, S]

J = Column Matrix of $\begin{bmatrix} \text{old } q \\ \text{old } p \end{bmatrix}$

S = \sim of $\begin{bmatrix} \text{new } q \\ \text{new } p \end{bmatrix}$

$$J = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \leftarrow \text{add } -1 \text{ for deriv. on } p$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} = J \cdot \frac{\partial H}{\partial p}$$

Obv. $S = S(cg)$ for $\dot{S}_{ij} = \frac{\partial \dot{S}_i}{\partial q_j} \cdot \dot{q}_j$ $[i, j \in 1, 2, 3, \dots, n]$ * new/old not perfect match

> define Jacobian Matrix $M_{ij} = \frac{\partial S_i}{\partial q_j}$ [Recall Gausses Integral]

$$S = MJ \cdot \frac{\partial H}{\partial p} \quad * \text{Reason on MT: } \frac{\partial H}{\partial p} = \sum_j \frac{\partial H}{\partial q_j} \cdot \frac{\partial S_j}{\partial q_i} = \sum_j M_{ji} \frac{\partial H}{\partial q_j} = \sum_j M_{ji} \frac{\partial H}{\partial S_j}$$

\downarrow J is only column matrix M_{ji} unable match \dot{q}_i

$$\dot{S} = MJMT \cdot \frac{\partial H}{\partial S} \leftrightarrow \text{Since } \dot{S} = \frac{\partial S}{\partial t} \cdot \frac{\partial q}{\partial t} = M \cdot J \cdot \frac{\partial H}{\partial p} = M \cdot J \cdot \frac{\partial H}{\partial S} \cdot \frac{\partial S}{\partial t}^{M^{-1}} = J \cdot \frac{\partial H}{\partial S}$$

$MJMT = J$ * if System (J, S) $MJMT = J$ Transformation is Canonical.

Focusing On $S = S(cg, t)$ S change with t .

if $J \rightarrow S(t)$ $\Rightarrow J \rightarrow S(t_0)$ $\Rightarrow S(t_0) \rightarrow S(t)$ Canonical.

* Consider $S(t+t_0)$ Compare to $S(t)$ ② Canonical Trans.

Set Small Change δ :

$$\begin{cases} Q_i = q_i + \delta q_i \\ P_i = p_i + \delta p_i \end{cases} \Rightarrow F_i = q_i P_i + \epsilon G(q, P, t) \quad \begin{cases} P_i = p_i + \epsilon \cdot \frac{\partial S}{\partial q_i} \rightarrow \delta P = -\epsilon \frac{\partial S}{\partial q_i} \\ Q = q + \epsilon \cdot \frac{\partial S}{\partial P_i} \rightarrow \delta Q = \epsilon \frac{\partial S}{\partial P_i} \end{cases}$$

$$\delta J = \epsilon J \frac{\partial S}{\partial q}$$

To proof if it's canonical. $MJMT = J$ $M = \frac{\partial S}{\partial q} = 1 + \frac{\partial S}{\partial q} = 1 + \epsilon J \frac{\partial S}{\partial q} \rightarrow *$

$$M^* J M = (1 + \epsilon \frac{\partial G}{\partial q_j} \frac{\partial G}{\partial p_j}) \hat{J} (1 - \epsilon \frac{\partial G}{\partial q_j} \frac{\partial G}{\partial p_j}) \hat{J} \leftarrow \text{All operation is on matrix. } (\frac{\partial G}{\partial q_j} \frac{\partial G}{\partial p_j}) \text{ by Built Matrix.}$$

$$= [1 - \epsilon \frac{\partial G}{\partial q_j} \frac{\partial G}{\partial p_j} \hat{J} + \epsilon \frac{\partial G}{\partial q_j} \frac{\partial G}{\partial p_j} - M^* M] \cdot \hat{J}$$

$$= \hat{J} + \epsilon \frac{\partial G}{\partial q_j} \frac{\partial G}{\partial p_j} \hat{J} - \epsilon \frac{\partial G}{\partial q_j} \frac{\partial G}{\partial p_j} \hat{J} = \hat{J} \Rightarrow \text{For small deviations, Trans. Always Canonical.}$$

9.5 Poisson Brackets + Other Canonical Invariants

$$[u, v]_{q, p} = \frac{\partial u}{\partial q_i} \cdot \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \cdot \frac{\partial v}{\partial q_i} \rightarrow [u, v]_J = \frac{\partial u}{\partial q_j} \cdot \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \cdot \frac{\partial v}{\partial q_j} = \text{average } J = (\frac{\partial u}{\partial q_j} J) (\frac{\partial v}{\partial p_j})$$

> Momentum As Generator of Translation [Ski. Quantum] Assume $J(c dx') = 1 - ik \cdot dx'$

Use II Generation Function $x_{\text{new}} = x + dx$ $P = P$ $F(x, p) = x \cdot P + p \cdot dx$ K belongs to $\frac{1}{2}$ dimention

For $F_2(x, p) = \sum q_i p_i + \epsilon G(q, p, t)$ System move $x \rightarrow x + \epsilon$ Generation Unit $G = P_x$

$$\delta q = Q - q = \epsilon \frac{\partial G}{\partial p_x}, \quad \delta p = P - p = -\epsilon \frac{\partial G}{\partial q_x} \quad [\delta p = 0, \delta q = \epsilon] \Rightarrow G = P$$

* Axiom in Quantum $[x, p] = i\hbar$ Use $[x, J(c dx)]$ Replace $[x, K]$.

$$\text{For } [x, J(c dx')]|_{x'} = x|x' + dx\rangle - J(dx') \cdot x'|x'\rangle = (x' + dx')|x' + dx\rangle - x'|x' + dx'\rangle = dx'|x' + dx'\rangle \approx dx|x'\rangle$$

$$\Rightarrow x(1 - ik dx') - (1 - ik dx') \cdot x = -xik dx + ik dx' x = dx' \star = i dx'[k, x] \Rightarrow [x, k] = \star$$

$$\star K - K \star = i \text{ with } \star p - P \star = i\hbar \Rightarrow K \text{ must } = P/\hbar$$

> Consider Finitive Change $|x + \Delta x\rangle$ [Separate to $J(c dx)^N$]

$$J(c \Delta x' \hat{x})|x'\rangle = |x' + \Delta x' \hat{x}\rangle \rightarrow J(c \Delta x \hat{x}) = J(c dx')^N \text{ with } dx = \frac{\Delta x}{N}, N \rightarrow \infty$$

$$= \lim_{N \rightarrow \infty} (1 - i \frac{p \Delta x}{\hbar N})^N \quad \star e^x \text{ defined as } \lim_{N \rightarrow \infty} (1 + \frac{x}{N})^N = e^x$$

$$= \exp(-i p \Delta x / \hbar)$$

C. \rightarrow . \triangleright Obviously, $J(c y' \hat{y}) J(c \Delta x' \hat{x}) = J(c \Delta x' \hat{x} + c y' \hat{y}) = J(c \Delta x \hat{x}) J(c y \hat{y}) \rightarrow [J(c y), J(c x)] = 0$

\uparrow \uparrow > define Abelian = Commutable = $[A, B] = 0$ $\Rightarrow [P_x, P_y] = 0$

A. \rightarrow . B. > define Simulta. Eigenket $|P'\rangle = |P_x P_y P_z'\rangle$ $J(c dx)|P'\rangle = (1 - i p' dx / \hbar) |P'\rangle \quad (p, J(c dx)) = 0$

> Canonical Commutation Relations

Basic Axioms $[x, x] = [p, p] = 0$ $[x, p] = i\hbar$

Notice Poisson Brackets $[A, B]_{(q, p)} = \sum_i \{ \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \}$

$$[x_i, p_j]_{\text{classical}} = \delta_{ij}$$

Quantum $[,] = i\hbar$ Classical $[]$ With Same Operation Laws.

Chap 1.1 Wave Function in \mathbf{x} and \mathbf{p} space

> Position - Space wave function [$\langle \beta | f(x) | \alpha \rangle = ?$]

For $|x| \langle x' | = x' | x \rangle$ In Continuous Spectra. $|\alpha\rangle = \int dx' |x'\rangle \langle x' | \alpha \rangle$ with $\Psi_\alpha(x')$ define as $\langle x' | \alpha \rangle$ for appear probability of x' in α

$$\langle \beta | \alpha \rangle = \int dx' \langle \beta | x' \rangle \langle x' | \alpha \rangle = \int dx' \Psi_\beta^*(x') \Psi_\alpha(x') \rightarrow \text{Prob. of State } \beta \text{ found in State } \alpha$$

Just Simply Define Ψ

As $|\alpha\rangle = \sum_a |a\rangle \langle a' | \alpha \rangle \quad \langle x' | \alpha \rangle = \int \langle x' | a' \rangle \langle a' | \alpha \rangle \Rightarrow \Psi(x') \alpha = \sum a' U_{a'}(x') \quad \text{with } U_{a'}(x') = \langle x' | a' \rangle$

$$\langle \beta | A | \alpha \rangle = \int dx' \int dx'' \langle \beta | x'' \rangle \langle x'' | A | x' \rangle \langle x' | \alpha \rangle = \int dx' \int dx'' \Psi_\beta^*(x'') \langle x'' | A | x' \rangle \Psi_\alpha(x') \leftarrow \text{by having value of } \langle x'' | A | x'' \rangle \quad \square$$

> Momentum Operator In Position Basis. [$\hat{P} = -i\hbar \frac{\partial}{\partial x} \Psi$]

In Classical Mechanics P represent by Generation Unit of $F(\mathbb{II})$

$$J(\Delta x) |\alpha\rangle = (1 - i\hbar \frac{\partial}{\partial x}/\hbar) |\alpha\rangle = \int dx' J(\Delta x) |x'\rangle \langle x' | \alpha \rangle = \int dx' |x'+\Delta x'\rangle \langle x' | \alpha \rangle$$

* $\Delta x'$ Term not include in x → Only subtract Δx Term

$$-i\hbar \frac{\partial}{\partial x} |\alpha\rangle = -\Delta x' \frac{\partial}{\partial x} \langle x' | \alpha \rangle$$

$$P|\alpha\rangle = \int dx' |x'\rangle (-i\hbar \frac{\partial}{\partial x} \langle x' | \alpha \rangle) \rightarrow \langle x' | P | \alpha \rangle = -i\hbar \frac{\partial}{\partial x} \langle x' | \alpha \rangle$$

$$\text{Replace } \alpha \text{ by } x'' \quad \langle x' | P | x'' \rangle = -i\hbar \frac{\partial}{\partial x} \delta(x' - x'')$$

$$\langle \beta | P | \alpha \rangle = \int dx' \langle \beta | x' \rangle (-i\hbar \frac{\partial}{\partial x} \langle x' | \alpha \rangle) = \int dx' \Psi_\beta^*(x') \cdot (-i\hbar \frac{\partial}{\partial x}) \Psi_\alpha(x')$$

> Momentum - Space Wave function

$$P|p'\rangle = p'|p'\rangle \quad \langle p' | p'' \rangle = \delta(p' - p'') \quad |\alpha\rangle = \int dp' |p'\rangle \langle p' | \alpha \rangle$$

$$\int dp' \langle \alpha | p' \rangle \langle p' | \alpha \rangle = 1 \quad \text{For normalize} \quad \langle p' | \alpha \rangle \text{ represent prob of measure get } p' \text{ define as } \phi_\alpha(p')$$

> Connections on x and p representations [Transform Bases Matrix $U_{ij} = \langle a_i | b_j \rangle$]

$$\langle x' | p | \alpha \rangle \text{ replace } \alpha \text{ with } p' \rightarrow \langle x' | p | p' \rangle = -i\hbar \frac{\partial}{\partial x'} \langle x' | p' \rangle \rightarrow \underbrace{\langle x' | p' \rangle}_{U} = -\frac{i\hbar}{p'} \frac{\partial}{\partial x'} \underbrace{\langle x' | p' \rangle}_{U}$$

$$U = \langle x' | p' \rangle = N \exp(i p' x'/\hbar)$$

$$|\langle x' | p' \rangle|^2 = \langle x' | p' \rangle \cdot \langle p' | x' \rangle$$

$$- (n_U = -i p'/\hbar) \rightarrow U = N \exp(i p' x'/\hbar)$$

Left N for normalize. Use $\langle x' | x'' \rangle = \int p' \langle x' | p' \rangle \langle p' | x'' \rangle$

$$\delta(x - x'') = \int p' N^2 \exp(i p' x'/\hbar) \cdot N \exp(-i p' x''/\hbar)$$

$$\delta(x - x'') = \int p' N^2 \exp[i \hbar (p' x' - p' x'')] \cdot dp' = |N|^2 \int dp' \exp[i \hbar (p' x' - p' x'')] \cdot dp'$$

★ Fourier's Transformation [$f(x) = \sum_n C_n \exp(ik_n x) = \int_{-\infty}^{+\infty} g(k) \cdot \exp(-ikx) dk$]

Transform — $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(k) \exp(ikx) dk$

Inverseform — $g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \exp(-ikx) dx$

$\delta(x-x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(ik(x-x')) dk$

$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{+\infty} \exp(ikx) \cdot \int_{-\infty}^{+\infty} f(x') \exp(-ikx') dx' dk$

$f(x) = \frac{1}{2\pi} \boxed{\text{Sch. Take } f(x') \text{ out of } f(x)} = \frac{1}{2\pi} \delta(x-x')$

* Proofs On Why Left Pure $\delta(x-x')$ without any constant.

Original Eqs. $= |N|^2 \frac{1}{2\pi} \delta(x-x') \hbar = \delta(x-x') \Rightarrow N = \frac{1}{\sqrt{2\pi}\hbar}$

$\langle x' | p' \rangle = \frac{1}{\sqrt{2\pi}\hbar} \exp(ip'x'/\hbar)$.

Similar form.

> Symbols of $\Psi_{\alpha}(x')$ and $\Phi_{\alpha}(p')$. Finally

$$[\Psi_{\alpha}(x')] = \left[\frac{1}{\sqrt{2\pi}\hbar} \right] \cdot \int dp' \exp(ip'x'/\hbar) \Phi_{\alpha}(p')$$

$$[\Phi_{\alpha}(p')] = \left[\frac{1}{\sqrt{2\pi}\hbar} \right] \cdot \int dx' \exp(-ip'x'/\hbar) \Psi_{\alpha}(x')$$

$Q = f(p)$

$P = f(q)$

II Generate Unit.

> Gaussian Wave Packets. [$\Psi_{\alpha}(x) = \left[\frac{1}{\sqrt{\pi}\hbar\sqrt{d}} \right] \exp(ikx - x^2/d^2)$] Use as example illustrates formalism.

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx' \langle \alpha | x' \rangle \delta(x-x') | \alpha \rangle = \int_{-\infty}^{+\infty} dx' |\Psi_{\alpha}(x')|^2 \cdot x = \frac{1}{d\sqrt{\pi}} \cdot \int_{-\infty}^{+\infty} \exp(-x'^2/d^2) \cdot x \cdot dx = 0$$

$$\langle x^2 \rangle = \frac{1}{d\sqrt{\pi}} \cdot \int_{-\infty}^{+\infty} dx' x'^2 \exp(-x'^2/d^2) = d^2/2$$

$$\langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = d^2/2$$

Similar $\langle (\Delta p)^2 \rangle = \hbar^2/2d^2$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = \hbar^2/4$$

$\langle p | \alpha \rangle$ also occupied by $\Phi_{\alpha} = [\sim] \int \exp(-ipx/\hbar) \cdot \Psi_{\alpha}(x) = \dots$

> Generalize to 3D.

$q \rightarrow (q_x, q_y, q_z) \quad \langle q' | q'' \rangle = \delta^3(q' - q'')$ Same as P

$\langle p | p | \alpha \rangle \rightarrow \int d^3x' \Psi_B^*(x') (-i\hbar\nabla) \Psi_{\alpha}(x')$ with $\Psi_{\alpha}(x)$, $\Phi_{\alpha}(p)$ Transform by coef. $1/(2\pi\hbar)^{3/2}$

