Modern Quantum Mechanics

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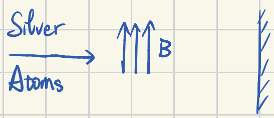
1.1 Experiment

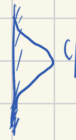
– Magnetic Moment μ $U = -\mu \cdot B$ $F = \mu \times B$ define as $\mu = I \cdot$ Enclosed Area



i). Silver 47 electron → 46 Steady 1 Self Spin

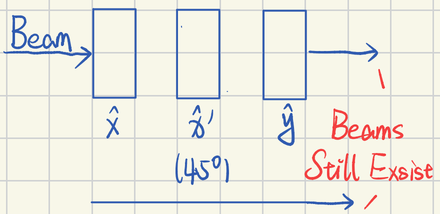
ii). $\mu \propto$ Spin s . $F = \mu_z \cdot \partial B_z / \partial z$

iii). Expected $\mu \in -|\mu| \sim |\mu|$ — Quantum $\mu = \mu_+$ or μ_- with $S_{z\pm} = \pm \hbar/2$

⇒  (p of detected) ⇒ 

IV). Cancel S_z- Add B_x $S_{z\pm}$ Both Appeard Again

– Polarized light experiment



S_{+z} S_{+x} S_{-z}

$S_{\pm z} \leftrightarrow x, y$ polarized

$S_{\pm x} \leftrightarrow x', y'$ polarized

noted as "ket" sign in Dirac notation

$E = E_0 \hat{x} \cos(kz - wt) = |S_z+\rangle$

$= E_0 \hat{y} \cos(kz - wt) = |S_z-\rangle$

$E_0 \hat{x}' \cos(\omega) = E_0 [\cos 45° \cdot \hat{x} \cos(\omega) + \cos 45° \cdot \hat{y} \cos(\omega)]$

$E_0 \hat{y}' \cos(\omega) = E_0 [-\cos 45° \cdot \hat{x} \cos(\omega) + \cos 45° \cdot \hat{y} \cos(\omega)]$

⇐

$|S_x+\rangle = 1/\sqrt{2}\, |S_z+\rangle + 1/\sqrt{2}\, |S_z-\rangle$

$|S_x-\rangle = -1/\sqrt{2}\, |S_z+\rangle + 1/\sqrt{2}\, |S_z-\rangle$

1.2 Kets Bras Operators

Ket Space

Observe System in Vector Space ⟶ State is n-dimention vector.

Define Ket Space as Vector space $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$ $c \cdot |\alpha\rangle = |\alpha\rangle \cdot c$

$A|\alpha\rangle = a|\alpha\rangle$ For A is observable $|\alpha'\rangle, |\alpha''\rangle \cdots$ is eigenket of $|\alpha\rangle$

e.g. $S_z |S_{z\pm}\rangle = \pm \hbar/2 \, |S_{z\pm}\rangle$

Since Linear Additive $|\alpha\rangle = \Sigma_{a'} C_{a'} |\alpha'\rangle$

Bra Space – Inner Produce

Define $\langle\alpha|$ corresponding to $|\alpha\rangle$ Dual Correspondence

$|\alpha\rangle \; |\alpha'\rangle \; |\alpha''\rangle \cdots \; |\alpha\rangle + |\beta\rangle \xleftrightarrow{DC} \langle\alpha| \; \langle\alpha'| \; \langle\alpha''| \cdots \; \langle\alpha| + \langle\beta|$

☆ $c|\alpha\rangle \xleftrightarrow{DC} c^*\langle\alpha|$ Define innerproduct $\langle\beta|\alpha\rangle = (\langle\beta|)\cdot(|\alpha\rangle) \xrightarrow{DC} = \langle\alpha|\beta\rangle^*$

- $\langle\alpha|\alpha\rangle \geq 0$, Real Number
- $\langle\alpha|\beta\rangle = 0$ Orthogonal on α β

Operators

$X\cdot(|\alpha\rangle) = X|\alpha\rangle$ $X=Y$ if $X|\alpha\rangle = Y|\alpha\rangle$ if $X|\alpha\rangle = 0$ X is null

☆ $X|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|X^\dagger$ if $X = X^\dagger$ X is Hermitian.

Define outer product $(|\beta\rangle)(\langle\alpha|) = |\beta\rangle\langle\alpha|$

Associative Axiom

- $(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle = |\beta\rangle\cdot(\langle\alpha|\gamma\rangle)$
- if $X = |\beta\rangle\langle\alpha|$ $X^\dagger = |\alpha\rangle\langle\beta|$
- $(\langle\beta|)\cdot(X|\alpha\rangle) = (\langle\beta|X)\cdot(|\alpha\rangle) \rightarrow \langle\beta|X|\alpha\rangle$
- $\langle\beta|X|\alpha\rangle = \langle\beta|\cdot(X|\alpha\rangle) = \{(\langle\alpha|X^\dagger)|\beta\rangle\}^* = \langle\alpha|X^\dagger|\beta\rangle^*$.

1.3 Base kets and matrix representations.

- Eigenvalue of Hermitian operator A is real

  Eigenkets of A to dif. eigenvalues are orthnormal.

Proof. $A|a'\rangle = a'|a'\rangle$ $\langle a''|A = a''^*\langle a''|$

$$\Downarrow$$

$$0 = \langle a''|a'|a'\rangle - a''^*\langle a''|a'\rangle \qquad 0 = \underline{\langle a''|a'\rangle}(a' - a''^*)$$

If:

i). $a'' = a' \rightarrow a' = a'^*$

ii). $a'' \neq a' \rightarrow \langle a''|a'\rangle = 0$ Each eigenvalues orthnormal

$\langle a''|a'\rangle = \delta_{a'a''}$ if $a' \neq a''$ $= 0$ $\Lambda^2_{a'}$: Project on a'

- Base Kets. $|\alpha\rangle = \sum C_{a'}|a'\rangle$ $\sum \Lambda^2_{a'} = 1$: Project on all a Must $= 1$

$$\langle a''|\alpha\rangle = \langle a''|\sum C_{a}|a'\rangle = \sum_{a'} C_{a'}\delta_{a'a''} \Rightarrow C_{a'} = \langle a'|\alpha\rangle$$

$$|\alpha\rangle = \sum_{a'}\langle a'|\alpha\rangle\cdot|a'\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle \Rightarrow \boxed{\sum_{a'}|a'\rangle\langle a'| = 1}$$

$$\Updownarrow$$

$V = \sum_i \hat{e}_i(\hat{e}_i\cdot V)$ ☆ $\langle a'|$ must orthnormal with $|\alpha\rangle$

Since $\sum|\alpha'\rangle\langle\alpha'|=1 \Rightarrow \langle\alpha|\alpha\rangle = \langle\alpha|\sum_{a'}|a'\rangle\langle a'||\alpha\rangle = \sum_{a'}|\langle a'|\alpha\rangle|^2 \Rightarrow \sum_{a'}|c_{a'}|^2 = \sum_{a'}|\langle a'|\alpha\rangle|^2 = 1$

☆ $(|\alpha'\rangle\langle\alpha'|)\cdot|\alpha\rangle = |\alpha'\rangle\langle\alpha'|\alpha\rangle = C_{a'}|\alpha'\rangle \Rightarrow |a'\rangle\langle a'|$ portion of $|\alpha'\rangle$ (for $|\alpha\rangle$ is normalized).

define $\Lambda_{a'} = |a'\rangle\langle a'|$ $\quad \sum\Lambda_{a'}=1$. $\quad \Lambda_{a'}$ Subtract valu. of proportion of a' on a

Matrix Representation.

☆ $|a'\rangle\ |a''\rangle$ ... From eigenket list. No dif.

Define $X = \sum_{a'}\sum_{a''}|a''\rangle\langle a''|\sum|a'\rangle\langle a'|$

Convert to $N\times N$ Matrix where N is dim. of ket space

$$X = \begin{bmatrix} \langle a_1''|X|a_1'\rangle & \langle a_1''|X|a_2'\rangle & \cdots \\ \langle a_2''|X|a_1'\rangle & \ddots & \\ \vdots & & \ddots \end{bmatrix} \Rightarrow \langle a''|X|a'\rangle = \langle a'|X^\dagger|a''\rangle^*$$

Insert Identity Matrix

- For $Z = XY$ $\quad \langle a''|Z|a'\rangle = \sum_{a'''}\langle a''|X|a'''\rangle\langle a'''|Y|a'\rangle$

- For $|\gamma\rangle = X|\alpha\rangle$ Acting on base $\langle a'|$ $\quad \langle a'|\gamma\rangle = \langle a'|X|\alpha\rangle = \sum_{a''}\underbrace{\langle a'|X|a''\rangle}_{\text{Matrix of } X}\underbrace{\langle a''|a\rangle}_{\text{Portion of old vectors.}}$

$$|\alpha\rangle = \begin{pmatrix}\langle a^1|a\rangle \\ \langle a^2|a\rangle \\ \vdots\end{pmatrix} \qquad |\gamma\rangle = \begin{pmatrix}\langle a^1|\gamma\rangle \\ \langle a^2|\gamma\rangle \\ \vdots\end{pmatrix}$$

As column Matrix.

$$|\alpha\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle = \begin{pmatrix}|a_1'\rangle\langle a_1'|\alpha\rangle \\ |a_2'\rangle\langle a_2'|\alpha\rangle \\ \vdots\end{pmatrix} = \begin{pmatrix}\langle a_1'|\alpha\rangle \\ \langle a_2'|\alpha\rangle \\ \vdots\end{pmatrix}$$

☆ $|a_1\rangle\ |a_2\rangle$ ... Have no valu. – only directions.

- Similar on $\langle\gamma| = \langle\alpha|X$ $\quad \langle\gamma|a'\rangle = \sum_{a''}\langle\alpha|a''\rangle\langle a''|X|a'\rangle$

$\langle\gamma| = (\langle\gamma|a^1\rangle, \langle\gamma|a^2\rangle \cdots) = (\langle a^1|\gamma\rangle, \langle a^2|\gamma\rangle \cdots)^*$ As row Matrix.

- Column Matrix · Raw Matrix. $\quad \langle\beta|\alpha\rangle = \sum_{a'}\langle\beta|a'\rangle\langle a'|\alpha\rangle$

$$= (\langle\beta|a_1\rangle\ \langle\beta|a_2\rangle \cdots)\cdot\begin{pmatrix}\langle a_1|a\rangle \\ \langle a_2|a\rangle \\ \vdots\end{pmatrix} = \text{Some value}$$

- Raw · Column $\quad |\beta\rangle\langle\alpha| = \sum_{a'}\langle a'|\beta\rangle\langle\alpha|a'\rangle = \begin{pmatrix}\langle a^1|\beta\rangle\langle a^1|\alpha\rangle^* & \langle a^1|\beta\rangle\langle a^2|\alpha\rangle^* & \cdots \\ \langle a^2|\beta\rangle\langle a^1|\alpha\rangle^* & \ddots & \vdots \\ \vdots & & \ddots \end{pmatrix}$

- Eigenkets of A used as baseket. [$\sum_{a'}|a'\rangle\langle a'| = 1$ — a' is baseket]

$A = \sum_{a''}\sum_{a'}|a''\rangle\langle a''|A|a'\rangle\langle a'| = $ if $\langle a'|A|a''\rangle$ where $a'\neq a''$ $=0 \Rightarrow \delta_{a''a'}\langle a'|A|a'\rangle = a'\delta_{a''a'}$

$$A = \sum_{a'} a'|a'\rangle\langle a'| = \sum_{a'} a'\Lambda_{a'}$$

☆ defin $S_\pm$ as Up down operators

$S_\pm = \hbar|\pm\rangle\langle\mp|$

Set $|-\rangle$ to $\hbar|+\rangle$

$S_+|-\rangle = \hbar|+\rangle\langle -|-\rangle = \hbar|+\rangle$

Spin 1/2 Systems

$$1 = \sum_{a'}|a'\rangle\langle a'| = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$$

$$S_z = \sum_{a'} \underbrace{a'}_{\text{eigen value}}\underbrace{\Lambda_{a'}}_{\text{eigenket}} = (\hbar/2)|\uparrow\rangle\langle\uparrow| - (\hbar/2)|\downarrow\rangle\langle\downarrow| = \hbar/2(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$$

$S_z|\updownarrow\rangle = \pm(\hbar/2)|\updownarrow\rangle$

as $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $\quad S_z = \hbar/2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

1.4 Measurements, Observables, Uncertainty

$$|\alpha\rangle = \sum \underset{i}{\langle a'|}\underset{j}{\alpha\rangle}\cdot\underset{k}{|a'\rangle} = \sum_{a'} \underset{k}{|a'\rangle}\underset{i}{\langle a'|}\underset{j}{\alpha\rangle}$$

$|\alpha\rangle$ [Series of possiple eigenvalue] $\xrightarrow{\text{Observe}}$ $|a'\rangle$ [Singele OD. value]

$|a'\rangle \xrightarrow{\text{Observe}} |\alpha'\rangle$

Prob. of a' observed $= |\Lambda_{a'}\cdot\alpha|^2 = |\underbrace{|a'\rangle}_{\text{Vector}}\underbrace{\langle a'|\alpha\rangle}_{\text{Constant}}|^2$.

$|a'\rangle^2 = \langle a'|a'\rangle$

$= |\text{Constant}|^2\cdot|\text{Vector}|^2 = |\langle a'|\alpha\rangle|^2\cdot\underbrace{\langle a'|a'\rangle}_{=1}$

$P = |\langle a'|\alpha\rangle|^2$

$$\langle A\rangle = \sum_{a'} \underbrace{a'}_{\text{valv.}}\underbrace{|\langle a'|\alpha\rangle|^2}_{P.}$$

$$= \sum_{a'}\sum_{a''}\langle\alpha|a''\rangle\underbrace{\langle a''|A|a'\rangle}_{\delta_{a''a'}a'}\langle a'|\alpha\rangle = \sum_{a'}\sum_{a''}\langle a'|\alpha\rangle\langle a''|\alpha\rangle^*\,\delta_{a'a''}\,a'$$

$$= \sum_{a'}|\langle a'|\alpha\rangle|^2\,a'.$$

$|S_x:+\rangle$ on S_z direction $= \vec{A}|+\rangle + \vec{B}|-\rangle$

with $|A| = |B| = 1/\sqrt{2}$ and direction unknown.

Spin 1/2 System, again.

$P = \sum|\Lambda_{a'}\cdot\alpha|^2 = 1$ with Same P. on $S_z\pm$ for S_x+

$\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}\exp(i\delta_1)|-\rangle \Leftarrow \frac{1}{\sqrt{2}}\exp(i\alpha)|+\rangle + \frac{1}{\sqrt{2}}\exp(i\beta)|-\rangle$

$$|\langle \underbrace{+}_{S_z+}|\underbrace{S_x:+}_{S_x+}\rangle| = |\langle\underbrace{-}_{S_z-}|\underbrace{S_x:+}_{S_x+}\rangle| = 1/\sqrt{2}. \Rightarrow |S_x:+\rangle = 1/\sqrt{2}|+\rangle + 1/\sqrt{2}\underbrace{\exp(i\delta_1)}_{\text{phase difference in } |+\rangle |-\rangle}|-\rangle$$

Since $|S_x:+\rangle$ orthnormal with $|S_x:-\rangle$. $\langle +|-\rangle = 0 \Rightarrow |S_x:-\rangle = 1/\sqrt{2}|+\rangle - 1/\sqrt{2}\exp(i\delta_1)|-\rangle$

$$S_x = \sum_{a'}a'\Lambda_{a'} = \hbar/2[|S_x+\rangle\langle S_x:+| - |S_x:-\rangle\langle S_x:-|] = \hbar/2[e^{-i\delta_1}|+\rangle\langle -| + e^{i\delta_1}|-\rangle\langle +|]$$

Similar $\left[\begin{array}{l} |S_y:\pm\rangle = 1/\sqrt{2}|+\rangle \pm 1/\sqrt{2}\exp(i\delta_2)|-\rangle \\ S_y = \sum_{a'}\Lambda_{a'}a' = \hbar/2[\exp(-i\delta_2)|+\rangle\langle -| + \exp(i\delta_2)|-\rangle\langle +|]\end{array}\right.$

$\Rightarrow S_x = \begin{pmatrix}0 & \exp(-i\delta_1)\\ \exp(i\delta_1) & 0\end{pmatrix}$

To determine $\delta_2\ \delta_1$ $\quad |\langle S_y:\pm|S_x:+\rangle| = |\langle S_y:\pm|S_x:-\rangle| = 1/\sqrt{2}$

$\Rightarrow S_y = \begin{pmatrix}0 & \exp(-i\delta_2)\\ \exp(i\delta_2) & 0\end{pmatrix}$

$$= [1/\sqrt{2}\langle +| + 1/\sqrt{2}\exp(i\delta_2)\langle -|][1/\sqrt{2}|+\rangle - 1/\sqrt{2}\exp(i\delta_1)|-\rangle]$$

$= \frac{1}{2} - \frac{1}{2}\exp[i(\delta_2-\delta_1)] = 1/\sqrt{2}$ ($\langle +|+\rangle = 1$ $\langle -|+\rangle = 0$)

$\Rightarrow \delta_2 - \delta_1 = \pm \pi/2$ ☆ $\exp(i\theta) \in \mathrm{Re}$ when $\theta \in 0, \pi, 2\pi \cdots$

$\delta_2 - \delta_1 = \pi/2 \rightarrow$ Must have one imagine. Assume $x \in \mathrm{Re}$. $y \in \mathrm{Im}$.

$S_x = \hbar/2\,[|+\rangle\langle -| + |-\rangle\langle +|]$ $|S_x:\pm\rangle = 1/\sqrt{2}|+\rangle \pm 1/\sqrt{2}|-\rangle$

$S_y = \hbar/2\,[-i|+\rangle\langle -| + i|-\rangle\langle +|]$ $|S_y:\pm\rangle = 1/\sqrt{2}|+\rangle \pm i/\sqrt{2}|-\rangle$

☆ $S_\pm = \hbar|\pm\rangle\langle\mp| \rightarrow S_\pm = S_x \pm iS_y = \hbar|\pm\rangle\langle\mp|$ [Up/ Down degree operators].

with $S_x S_y S_z$ $[S_x, S_y] = \hbar^2/4\,[i|+\rangle\langle +| - i|-\rangle\langle -| + i|+\rangle\langle +| - i|-\rangle\langle -|]$

[$S_z = (\hbar/2)\,[|+\rangle\langle +| - |-\rangle\langle -|]$]

$= i\hbar/2\,[|+\rangle\langle +| - |-\rangle\langle -|] = S_z \cdot i\hbar$ $\Rightarrow [S_i, S_j] = \varepsilon_{ijk} \cdot S_k \cdot i\hbar$

Similar Anti-Commutation $\{S_x\, S_y\} = \frac{1}{2}\hbar^2 \delta_{ij}$ [$\{A, B\} = AB + BA$]

> Define $S^2 = S_x^2 + S_y^2 + S_z^2 = 3 \cdot (1/4\,\hbar^2)$ $[S^2, S_i] = 0$

Compatible Observables.

define $[A, B] = 0$ compatible vise versa.

☆ degenerate : two dif. eigen state $\rightarrow$ same correspond eigen value $\Rightarrow |a'\rangle$ isn't complete.

> Theory : A, B compatible eigenvalue of A nonedegenerate. $\langle a''|B|a'\rangle$ all diagonal

$\langle a''|[A, B]|a'\rangle = 0$

$\langle a''|AB - BA|a'\rangle = \langle a''|AB|a'\rangle - \langle a''|BA|a'\rangle = \langle a''|a'')B|a'\rangle - \langle a''|B(a'|a'\rangle = \langle a''|B|a'\rangle(a''-a') = 0$

Obv. $a' = a''$ or $\langle a''|B|a'\rangle = 0$. $\Rightarrow$ $\langle a''|B|a'\rangle = \delta_{a'a''}\langle a'|B|a''\rangle$. $\rightarrow$ Diagnal Matrix.

> Simutaneous Eigenkits.

$B = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$ adding on ket $|a'\rangle$ = $\sum_{a''} |a''\rangle\langle a''|B|a''\rangle\underbrace{\langle a''|a'\rangle}_{\delta_{a'a''}} = (\langle a'|B|a'\rangle)\,|a'\rangle$ Simotanious Eigenkets

☆ $B|a'\rangle = b'|a'\rangle$ with $b' = \langle a'|B|a'\rangle$

$A|a'\rangle = a'|a'\rangle$

AB Use Same eigenstate $|a', b'\rangle$

$A|a', b'\rangle = a'|a', b'\rangle$

$B|a', b'\rangle = b'|a', b'\rangle$

define $|K'\rangle$ collective index $\leftarrow$

$|K'\rangle = |a', b'\rangle$

> Degeneracy Notations.

$|K'\rangle = |a', b', c' \cdots\rangle$ (Assume compatible)

obv. $\sum_{k'} |K'\rangle\langle K'| = 1$ $\langle K''|K'\rangle = \delta_{k''k'}$

$|\alpha\rangle \xrightarrow{\text{Measure } A} |a', b'\rangle$ with same value a' $\xrightarrow{\text{Mea. } B} |a', b'\rangle$ with dif. b'

degeneracy in a   dif. in b'

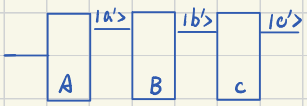
☆ degeneracy → Linear combination of kets. $= \sum_i^n c_{a'}^i |a', b^i\rangle$ [For n degeneracy A state, measure gives a'
B state, gives $b', b^2 \dots b^n$]

> Incompatible // Collective Index.

For $[A,B] \neq 0$

$$AB|a',b'\rangle = a'b'|a',b'\rangle$$
$$BA|a',b'\rangle = b'a'|a',b'\rangle$$
↗ $AB = BA$ $[A,B]$ must $= 0$. (x)

For $[A,B] \neq 0$ $|a',b'\rangle$ not exsist.

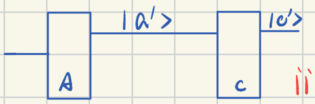


i)

P of output $= \underbrace{|\langle c'|b'\rangle|^2}_{c \text{ in } b} \underbrace{|\langle b'|a'\rangle|^2}_{b \text{ in } a}$

As B have eigen states. summing over b' for all possible routes.

$$\sum_{b'} |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2 = \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle \quad (\text{Eq. } 1)$$



ii).

Difference in b' and b'' value.

$$P_{out} = |\langle c'|a'\rangle|^2 = \sum_{b'} |\langle c'|b'\rangle\langle b'|a'\rangle|^2 = \sum_{b'}\sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle \quad (\text{Eq. } 2)$$

State in i). $|b'\rangle$ is detected. just left normal P. adding together.

State in ii). $\sum_{b'}\sum_{b''}$ cause interference pare when $b' \neq b''$

For $[A,B] = 0$: S-M. eigenkets $|a\rangle$ have $B|a'\rangle = b'|a\rangle$ $\langle b|a\rangle = \delta_{bb'}$

☆ $\text{Eq.}1 = |\langle c|b'\rangle\langle b'|a\rangle|^2 = \text{Eq.}2 = |\langle c|b'\rangle|^2 |\langle b'|a\rangle|^2$ [only $b = b'$ exsist].

Interference $\sum_b \sum_{b'}$ cause by differente possible path. $[A,B]=0$ // Test on $|b\rangle$
will left only one path of $|b\rangle$ → kill inter.

> Uncertainty Relation.

Define $\Delta A = A - \langle A\rangle$ As operators $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$ as dispersion.

$\langle(\Delta A)^2\rangle$ Assign to "sharpness" of value.
$\langle(\Delta S_x)^2\rangle = \hbar^2/4 - 0 = \hbar^2/4$ → Fuzzy
$\langle(\Delta S_z)^2\rangle = \hbar^2/4 - \hbar^2/4 = 0$ → Sharp.

Theory : $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$

Proof on this theory require three Lemmas.

> Lemma 1. In Hib. Space $|\nu|^2 \geq 0$ For $|\nu\rangle = |\alpha\rangle + \nu|\beta\rangle$

$$(\langle\alpha| + \nu^*\langle\beta|)\cdot(|\alpha\rangle + \nu|\beta\rangle) \geq 0$$

$$\langle\alpha|\alpha\rangle + \underline{\nu\langle\alpha|\beta\rangle} + \nu^*\langle\alpha|\beta\rangle^* + \underline{|\nu|^2\langle\beta|\beta\rangle} \geq 0$$

Since ν is free, Set $\nu = -\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle$ To cancle terms.

$$\langle\alpha|\alpha\rangle - \underbrace{|\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle} + \nu^*\langle\alpha|\beta\rangle + \underbrace{|\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle} \Rightarrow \langle\alpha|\alpha\rangle - |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle \geq 0 \quad \square$$

> Lemma 2. Expection Value of Hermiton Operators must real.

$$\langle\alpha'|H|\alpha''\rangle = \langle\alpha''|H^\dagger|\alpha'\rangle^* = \langle H\rangle \rightarrow \text{Real} \quad \square$$

> Lemma 3. Vice Versa on Lemma 2. $\square$

> Proof. Lm. 1 $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

Define operator $|\alpha\rangle = \Delta A|\alpha\rangle$, $|\beta\rangle = \Delta B|\beta\rangle$

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\,\Delta B\rangle|^2$$

$$\Delta A\Delta B = \tfrac{1}{2}[\Delta A\Delta B - \Delta B\Delta A + \Delta A\Delta B + \Delta B\Delta A] = \tfrac{1}{2}[\Delta A, \Delta B] + \tfrac{1}{2}\{\Delta A, \Delta B\}$$

$$\langle\Delta A\Delta B\rangle = \tfrac{1}{2}\langle[\Delta A, \Delta B]\rangle + \tfrac{1}{2}\langle\{\Delta A, \Delta B\}\rangle$$

$[\Delta A, \Delta B]$: Since Commutation Relation on $[A, \text{Constant Value}] = 0$

$$= [A, B] - [\langle A\rangle, B] - [A, \langle B\rangle] + [\langle A\rangle, \langle B\rangle] = [A, B]$$

Lemma 2/3 $[\Delta A, \Delta B]^\dagger = [A, B]^\dagger = BA - AB = -[B, A] \longrightarrow \langle[A, B]\rangle$ Pure Im.

$\{\Delta A, \Delta B\}^\dagger = \Delta B\Delta A + \Delta A\Delta B = \{\Delta A, \Delta B\} \longrightarrow \{\Delta A, \Delta B\}$ Pure Re.

$$\langle(\Delta A)^2\rangle + \langle(\Delta B)^2\rangle = \tfrac{1}{4}|\langle[A, B]\rangle|^2 + \tfrac{1}{4}|\langle\{\Delta A, \Delta B\}\rangle|^2 \geq \tfrac{1}{4}|\langle[A, B]\rangle|^2 \quad \square$$

1.5 Change of basis

Transforming Operators. [Define $|b^1\rangle = U|a^1\rangle$, $|b^2\rangle = U|a^2\rangle$, -- with U is unitary operators]

$U^\dagger U = UU^\dagger = 1$

Assume For $U = \sum_k |b^k\rangle\langle a^k|$

$$U\cdot U^\dagger = \sum_k\sum_l |b^k\rangle\langle a^k|a^l\rangle\langle b^l|$$

$$= \sum_k |b^k\rangle \delta_{kl} \langle b^k| = 1$$

Transforming Matrix. [Transforming $\{|a'\rangle\}$ to $\{|b'\rangle\}$]

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$ Transform to $\langle b^k|\alpha\rangle = \sum_l \langle b^k|a^l\rangle\langle a^l|\alpha\rangle$ with $\langle b^k| = \langle a^k|U^\dagger\rangle$

(new) $= U^\dagger$ (old) $= \sum_l \langle a^k|U^\dagger|a^l\rangle\langle a^l|\alpha\rangle$

$$\langle b^k|X|b^l\rangle = \sum_m \sum_n \langle b^k|a^n\rangle\langle a^n|X|a^m\rangle\langle a^m|b^l\rangle$$

with $|b\rangle = U|\alpha\rangle$ $= \sum_{m,n} \langle a^k|U^\dagger|a^n\rangle\langle a^n|X|a^m\rangle\langle a^m|U|b^l\rangle$

$\Rightarrow$ $X' = U^\dagger X U$

> define Trace $tr(X) = \sum_{a'} \langle a'|X|a'\rangle$

$tr(X)$ isn't depends on basis. $= \sum_{a'b'b''} \underline{\langle a'|b'\rangle\langle b'|X|b''\rangle\langle b''|a'\rangle}$ Three scalors $\rightarrow$ change position freely $= \sum_{b'b''} \underbrace{\langle b''|b'\rangle}_{\delta_{b'b''}}\langle b'|X|b''\rangle$

$= \sum_{b'} \langle b'|X|b'\rangle = Tr(X)$

Diagonalization

For $B|b'\rangle = b'|b'\rangle$

$\sum_{a''} \langle a'|B|a''\rangle\langle a''|b'\rangle = b'\langle a'|b'\rangle$. as $\langle a'|$ represent "direction"

$$\begin{pmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} = b'(b'=\lambda) \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} \rightarrow \det(B-\lambda I)=0.$$ For I is unit vector.

For $B_{ij} = \langle a_i|B|a_j\rangle$ $C_i = \langle a^i|b'\rangle$

N dimention function with $\lambda_1 \cdots \lambda_N \in b$

one col. of trs. Matrix

For Transforming Matrix define as $|b\rangle = U|\alpha\rangle$ $U = \sum_{b'} \langle a^i|b'\rangle = \sum_{b'} C_i'$

Unitary Equivalent Observables. [Unitary Define as $UU^\dagger = I$ $U^\dagger = U^{-1}$]

For sets $\{|a'\rangle\}$ $\{|b'\rangle\}$ Conneet by U. Unitary transformation on A is UAU^{-1}

$A|a^l\rangle = a^l|a^l\rangle$

Transformation

$UAU^{-1}U|a^l\rangle = a^l U|a^l\rangle \longrightarrow (UAU^{-1})|b^l\rangle = a^l|b^l\rangle$

both $|a^l\rangle$ $|b^l\rangle$ are eigenkets of UAU^{-1} with same eigenvalue a^l

as $B|b^l\rangle = b^l|b^l\rangle$) UAU^{-1} Similar to B.

1.6 Position Momentum Translation

Continuous Spectra. [e.g. P_z is avilable in any value $(-\infty \sim +\infty)$ - Continues]

For Continues spectrum $\underset{\text{operators}}{\xi}|\xi'\rangle = \underset{\text{eigenvalues}}{\xi'}|\xi'\rangle$

$\langle a'|a''\rangle = \delta_{a'a''} \rightarrow \langle \xi'|\xi''\rangle = \delta(\xi'-\xi'')$

$\sum_{a'} |a'\rangle\langle a'| = 1 \longrightarrow \int d\xi |\xi'\rangle\langle\xi'| = 1$

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \rightarrow \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle$

$\sum_{a} |\langle a'|\alpha\rangle|^2 = 1 \rightarrow \int d\xi' |\langle\xi'|\alpha\rangle|^2 = 1$

$\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle \rightarrow \langle\beta|\alpha\rangle = \int d\xi' \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle$

$\langle a''|A|a'\rangle = a'\delta_{a'a''} \longrightarrow \xi'\delta(\xi''-\xi')$

Position Eigenkets. — Position Measurement.

$x|x'\rangle = x'|x'\rangle$

with $|\alpha\rangle = \int_-^+ dx' |x'\rangle\langle x'|\alpha\rangle \xrightarrow{\text{Measure close to } (x'-\Delta/2,\, x'+\Delta/2)} \int_{x'-\Delta/2}^{x'+\Delta/2} dx'' |x''\rangle\langle x''|\alpha\rangle$

P(on position x') $= |\langle x'|\alpha\rangle|^2 \cdot dx'$ with $\int_-^+ |\langle x'|\alpha\rangle|^2 \cdot dx' = 1$ [$\langle\alpha|\alpha\rangle = 1$]

> Coordinates Operators.

$|\alpha\rangle = \int d^3x' |\underline{x}'\rangle\langle\underline{x}'|\alpha\rangle$ with $|\underline{x}'\rangle = |x', y', z'\rangle$ (Simultanious Ket) Restrict by $[x_i, x_j] = 0$.

Translation [Change state define in $|\underline{x}'\rangle \rightarrow |\underline{x}'+d\underline{x}'\rangle$] [$J(dx') \rightarrow \mathcal{J}(dx')$]

Define $J(d\underline{x}')|\underline{x}'\rangle = |\underline{x}'+d\underline{x}'\rangle$ Particle move dx' but system keep same.

$|\alpha\rangle \rightarrow J(d\underline{x}')|\alpha\rangle = J(d\underline{x}')\int d^3x' |\underline{x}'\rangle\langle\underline{x}'|\alpha\rangle = \int d^3x' |\underline{x}'+d\underline{x}'\rangle\langle\underline{x}'|\alpha\rangle$ ☆ $= \int d^3x' |\underline{x}'\rangle\langle\underline{x}'-d\underline{x}'|\alpha\rangle$

i). Unitary: $\langle\alpha|\alpha\rangle = \langle\alpha|J^\dagger(dx')J(dx')|\alpha\rangle \longrightarrow J\cdot J^\dagger = 1$

ii). Additive: $J(dx'')J(dx') = J(dx'+dx'')$ For $x'\,x''$ dif. direction.

iii). Inverse: $J(-dx') = J^{-1}(dx')$

iv). $\lim_{dx'\to 0} J(dx') = 1$

Assume $J(dx') = 1 - iK\cdot dx'$ Satisfy all requirements. [$K \to K_x K_y K_z$ $K^\dagger = K$]

> Commutation of $[X', J(dx')]$

$[X, J(dx')]|x'\rangle = (x'+dx')|x'+dx'\rangle - x'|x'+dx'\rangle = dx'|x'+dx'\rangle \simeq dx'|x'\rangle$ Second deriv of x'

$[x, J(dx')] = dx'$ Take $J(dx') = 1 - iK\cdot dx'$ $-ixK\cdot dx' + iK\cdot dx' x = dx'$ $[x_i, K_j] = i\delta_{ij}$

Momentum as Generator of Translation.

☆ Classical Ideas of momentum. [P is Generation Function]

State 1 $(x_1 P_1) \rightarrow (x_2 P_2)$

$\mathcal{L}_1 = \dot{x}_1 P_1 - U_1$ $\quad$ $\mathcal{L}_2 = \dot{x}_2 P_2 - U_2 + dF(t)/dt$ $\quad$ [$\mathcal{L}$ Unchange for $dF(t)/dt$]

dF/dt = Generation Function $\quad$ For $\mathcal{L}_1 = \mathcal{L}_2$ $\quad$ $\dot{x}_1 P_1 - U_1 dt = \dot{x}_2 P_2 - U_2 dt + dF$

$\Rightarrow dF = dx_1 P_1 - dx_2 P_2 + (U_2 - U_1) dt$ $\quad$ dt term vanish $F(x_1 x_2)$

Type I - Generation Function $\quad$ For $F(x_1, x_2)$ $\quad$ $dF = \partial F/\partial x_1 \cdot dx_1 + \partial F/\partial x_2 \cdot dx_2$

$P_1 = \partial F/\partial x_1$ $\quad$ $P_2 = -\partial F/\partial x_2$

Type II G.F. ☆ Set $F_2(x_1 p) = F_1(x_1 x_2) + X_2 \cdot P_2 = P_1 dx_1 - \cancel{P_2 dx_2} + \cancel{P_2 dx_2} + x_2 dP_2$.

$\partial F_2/\partial x_1 = P_1$ $\quad$ $\partial F_2/\partial p_2 = x_2$ $\quad \leftarrow = P_1 dx_1 + x_2 dP_2$

[Comments on momentum]: Consider small deviation of $F_2 = x_1 \cdot P_2 + \boxed{\varepsilon G(x_1, P_2)}$ Generator $(\varepsilon \rightarrow 0)$

$X_2 = \partial F_2/\partial p_2 = x_1 + \varepsilon \, \partial G(x_1 p_2)/\partial p_2 + \partial x_1/\partial p_2$

$\delta x = x_2 - x_1 = \varepsilon \, \partial G(x_1 p_2)/\partial p_2$ — For δx Translation

ε is value. $\partial G/\partial p_2 = C$.

Momentum P_2 is Generator. G. $\leftarrow$ G is linear to P_2

As $\mathcal{J}(dx') = 1 - ik\, dx'$ Try connect $\mathcal{J}(dx')$ with Generator function G.

Since K is dimentional P is simply value. Assume $K = P/\text{dim. of action}$ [Universal Constant] $\rightarrow$ dim. $= \hbar$

[Ref. From Gold Stein: Classical Mechanics]

Chap 9. Canonical Transformation.

$\mathcal{L}(q, \dot{q}, t) \longrightarrow H(q, p, t)$ use momentum p instead of $\dot{q}$

$\dot{q} = \partial H/\partial p$ $\quad$ $\dot{p} = -\partial H/\partial q$ $\quad$ $\delta \int (p\dot{q} - H) \cdot dt = 0$

9.1 Canonical Transformation Equations.

For motion with Hamiltonian is C [$\dot{H} = 0$]. Coordinates q_i cyclic [$\partial H/\partial q_i = 0$]

$\Rightarrow \dot{p} = \partial H/\partial q = 0$ $\quad$ $P_i = \alpha_i$ As constants.

$H = H(\alpha_1 \alpha_2 \cdots \alpha_n)$ $\Rightarrow$ $\dot{q}_i = \partial H/\partial \alpha_i = W_i$ $\quad$ For $W_i = W_i(\alpha_i)$ $\Rightarrow$ $q_i = W_i t + \beta$

☆ Point Transformations [Eg. Use $Q(q,t)$ as new coordinates of motion. Q described by (q,t)]

New coord. define as $Q_i = Q_i(q, p, t)$ $\quad$ $P_i = P_i(q, p, t)$ $\quad$ [$H(q, p, t)$ with p, q same level]

$\Big\langle$ Point. Transf. of Configuration Space : $Q(q,t)$ Only position in time t

Point. Transf. of Phase Space : $Q(q,p,t)$ Position / State in time t

In new coordinates, $\delta\int_{t_1}^{t_2}(p\dot{q}-H)\,dt = \int_{t_1}^{t_2}\delta(P_i\dot{Q}_i - K(Q,P,t))\,dt = 0$ with $\dot{Q}=\partial K/\partial P$ $\dot{P}=-\partial K/\partial Q$

Since $\mathcal{L}' = \mathcal{L} + dF/dt$ $\boxed{\lambda}(p\dot{q}-H) = P_i\dot{Q}_i - K(Q,P,t) + dF/dt$

Simply for scale transformation.

$K'(Q',P') = \mu\nu H(q,p)$

$\lambda = \mu\nu$

For simple scale change $Q_i' = \mu q_i$ $P_i' = \nu P_i$

$\mu\nu(P_i\dot{q}_i - H) = P_i'\dot{Q}_i' - K'$

> Focusing On $\lambda = 1$ Unscaled Canonical Transformation.

$p\dot{q} - H = P\dot{Q} - K + dF/dt$ define F = Generation function.

Since dF/dt discribe change from old coordinates to new. F may include both side coordinates.

For $F = F_1(q,Q,t)$ $p\dot{q} - H = P\dot{Q} - K + \partial F/\partial t + \partial F/\partial q\cdot\dot{q} + \partial F/\partial Q\cdot\dot{Q}$

☆ vanished with other terms since $\dot{q}\dot{Q}$ Independent

$\Rightarrow K = H + \partial F/\partial t$

$p_i = \partial F_1/\partial q_i$ $P_i = -\partial F_1/\partial Q_i$

☆ Legendar Transformation [In $F_1(q,Q,t)$ If $Q=q$ System break down]

$dF_1 = p\cdot dq - P\cdot dQ$ Try adding PQ to F_1

$dF_2 = p\,dq - P\cdot\cancel{dQ} + P\,\cancel{dQ} + Q\cdot dP = p\,dq + Q\cdot dP \Rightarrow F_2(q,P,t) = F_1 + QP$

with same restriction: $p\dot{q} - H = P\dot{Q} - K + dF/dt$ $\Rightarrow p = \partial F_2/\partial q_i$ $Q = \partial F_2/\partial P_i$

Type	Restrict		Simple example		
$F_1 = F_1(q,Q,t)$	$p = \partial F_1/\partial q$	$P = -\partial F_1/\partial Q$	$F_1 = qQ$	$Q = p$	$P = -q$
$F_2 = F_1 + QP$	$p = \partial F_2/\partial q$	$Q = \partial F_2/\partial P$	$F_2 = qP$	$Q = q$	$P = p$
$F_3 = F_1 - qp$	~		~		
$F_4 = F_1 + QP - qp$	~		~		

> For DoF. more than 1 System. F is able to choose freely in each Dimention.

$$\begin{matrix} x & \xrightarrow{F_1} & X \\ y & \xrightarrow{F_2} & Y \\ z & \xrightarrow{F_3} & Z \end{matrix}$$

Represent by $F'(q_1 p_2 P_1 Q_2 t)$ ☆ Obv. maximum Coord in $F' = 2\times$ DoF.

9.2 Examples on Canonical Transformation.

$F_2 = q_i P_i$ As simplest form $p_i = \partial F_2/\partial q_i$ $Q_i = \partial F_2/\partial P_i$

$\Rightarrow F_2 = f_i(q_1, q_2 \cdots q_n; t) P_i$ [For $f_i(\cdots)$ represent any function pick in $(\cdots)$]

$Q_i = \partial F_2/\partial P_i = f_i(q_1, q_2 \cdots q_n, t)$ ☆ Only old position include — $Q(q,t)$ Point transformation $\Rightarrow$ All Points trans. are canonical

☆ For any $F_2(q, P)$ Have $Q = \partial F_2/\partial P$ $\quad P = \partial F_2/\partial q$ As canonical coordinate transf. ☆

> Generation Unit [Adding $g(m)$ as generation unit]

$F_2 = f_i(m) P_i + g(m)$ $\quad p_j = \partial F_2/\partial q_j = \partial f_i/\partial q_j \cdot P_i + \partial g/\partial q_j \longrightarrow p = \partial f/\partial q \cdot P + \partial g/\partial q$

In matrix notation Eg. 2D. $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial q_1 & \partial f_1/\partial q_2 \\ \partial f_2/\partial q_1 & \partial f_2/\partial q_2 \end{bmatrix} \cdot \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \partial g/\partial q_1 \\ \partial g/\partial q_2 \end{bmatrix}$

$P = (\partial f/\partial q)^{-1} (p - \partial g/\partial q)$ with $Q = f_i(m)$

9.3 Harmonic Oscilliator [$H = P^2/2m + \frac{1}{2}kq^2$]

$H = \frac{1}{2m}(p^2 + m^2 w^2 q^2)$ [Try on convert to $\sin^2\theta + \cos^2\theta = 1$ Build cyclic coordinates]

$p = f(P) \cdot \cos Q$ $\quad q = f(P)/mw \sin Q \longrightarrow H = f(P)^2/2m$ $\quad \partial H/\partial Q = 0$

Try Find $f(p)$ make transf. system canonical

$P = P(q, Q) \rightarrow F_1$ or F_2

$p/q = mw \cot Q \rightarrow f(p) = mwq/\sin Q$ $\quad p = qmw\cot Q$

☆ Reason on $F_1(q, Q)$ include Q

For $F_2(q, Q, t)$ $\begin{cases} p = \partial F_2/\partial q_i = qmw\cot Q \rightarrow F_1 = \frac{1}{2}q^2 mw \cot Q \\ P = mwq^2/2\sin^2 Q \Rightarrow q = \sqrt{2P/mw} \cdot \sin Q \quad f(P) = \sqrt{2Pmw} \end{cases}$

According to H. Canonical Function $\begin{matrix} \partial H/\partial Q = \dot{P} \rightarrow P \text{ is const.} = E/w \\ \partial H/\partial p = \dot{Q} \rightarrow Q = wt + \alpha \end{matrix}$ New Coord. P.Q Very Simple

$\Rightarrow q = \sim \quad p = \sim$

$H = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 = \frac{1}{2}(p^2/m + kq^2) = \frac{1}{2m}(p^2 + m^2 w^2 q^2) = f(P)^2/2m$

Set new coord. $p = f(P)\sin Q$ $\quad q = f(P)\cos Q/mw$

Solve $f(P)$, Q $\quad p = qmw\tan Q = P(q, Q) \longrightarrow F = F_1(q, Q)$

$\partial F/\partial q = p = qmw\tan Q \longrightarrow F_1 = \frac{1}{2}q^2 mw \tan Q$ $\quad -\partial F/\partial Q = q^2 mw/2\cos^2 Q = P$

$f(P) = p/\sin Q = qmw/\cos Q = \sqrt{2mwP}$ $\quad H = f(P)^2/2m = wP$

For Cano. H. $\partial H/\partial P = \dot{Q} = w$ $\quad \partial H/\partial Q = \dot{P} = 0 \rightarrow Q = wt + \alpha$ $\quad P = E/w$

$q = \sqrt{2E/mw^2} \sin(wt + \alpha)$ $\quad p = \sqrt{2Em} \cos(wt + \alpha)$

9.4 Symplectic Aproach

For $Q_i = Q(q,p)$ $\quad P_i = P(q,p)$

$\dot{Q}_i = \partial Q/\partial q \cdot \dot{q} + \partial Q/\partial p \cdot \dot{p} = \partial Q/\partial q \, \partial H/\partial p - \partial Q/\partial p \, \partial H/\partial q = \partial H/\partial P$

Similar On $P = P(Q,P)$ $\quad q = q(Q,P)$ $\quad \partial H/\partial P = \partial H/\partial q \cdot \partial q/\partial P + \partial H/\partial p \, \partial p/\partial P$

$$\Rightarrow \left[\begin{array}{ll} (\partial Q/\partial q)_{q,p} = (\partial p/\partial P)_{Q,P} & (\partial Q/\partial p)_{q,p} = -(\partial q/\partial P)_{Q,P} \\ (\partial P/\partial q)_{q,p} = -(\partial p/\partial Q)_{Q,P} & (\partial P/\partial p)_{q,p} = (\partial q/\partial Q)_{Q,P} \end{array}\right.$$

> Symplectic Notation $[\eta, \xi]$

η = Column Matrix of $\begin{bmatrix} \text{old } q \\ \text{old } p \end{bmatrix}$

ξ = 〃 of $\begin{bmatrix} \text{new} Q \\ \text{new} P \end{bmatrix}$

$J = \begin{bmatrix} & 1 \\ & & 1 \\ -1 & & \\ & -1 & \end{bmatrix}$ ← add -1 for deriv. on p

$\dot{\eta} \left\langle \begin{array}{l} \dot{p} = -\partial H/\partial q \\ \dot{q} = \partial H/\partial p \end{array} \right. = J \cdot \partial H/\partial \eta$

Obv. $\xi = \xi(\eta)$ $\quad$ for $\dot{\xi}(\eta) = \partial \xi_i/\partial \eta_j \cdot \dot{\eta}_j$ $[i,j \in 1,2,3 \cdots n]$ ✩ new/old not perfect match

> define Jacobian Matrix $\quad M_{ij} = \partial \xi_i/\partial \eta_j$ [Recall Gausses Integral]

$\dot{\xi} = MJ \cdot \partial H/\partial \eta$ ✩ Reason on M^T: $\quad \partial H/\partial \eta_i = \sum_j \partial H/\partial \xi_j \cdot \partial \xi_j/\partial \eta_i = \sum_j M_{ji} \, \partial H/\partial \xi_j = \sum_j M^T \cdot \partial H/\partial \xi_j$

η_i only column matrix M_{ji} Unable match η_i

$\dot{\xi} = MJM^T \cdot \partial H/\partial \xi \leftrightarrow$ Since $\dot{\xi} = \partial \xi/\partial \eta \cdot \partial \eta/\partial t = M \cdot J \cdot \partial H/\partial \eta = M J \cdot \partial H/\partial \xi \, \underbrace{(\partial \xi/\partial \eta)}_{}^{(M^{-1})} = J \, \partial H/\partial \xi$

$MJM^T = J$ ✩ if System (η, ξ) $MJM^{-1} = J$ Transformation is Canonical

Focusing On $\xi = \xi(\eta, t)$ $\quad \xi$ change with t.

if $\eta \to \xi(t)$ $\Rightarrow$ $\eta \to \xi(t_0)$ $\Rightarrow$ $\xi(t_0) \to \xi(t)$ Canonical

✩ Consider $\xi(t+dt)$ Compare to $\xi(t)$? Canonical Trans.

Set Small Change δ: $\left[\begin{array}{l} Q_i = q_i + \delta q_i \\ P_i = P_i + \delta P_i \\ \xi = \eta + \delta \eta \end{array}\right.$ $\Rightarrow F_2 = q_i P_i + \varepsilon G(q, P, t) \left\langle \begin{array}{l} p_i = P_i + \varepsilon \cdot \partial G/\partial q_i \to \delta P = -\varepsilon \, \partial G/\partial q_i \\ Q = q_i + \varepsilon \cdot \partial G/\partial P_i \to \delta Q = \varepsilon \, \partial G/\partial P_i \end{array}\right.$

$\delta \eta = \varepsilon J \, \partial G/\partial \eta$

To proof if it's canonical. $\quad MJM^T = J \quad M = \partial \xi/\partial \eta = 1 + \partial \delta\eta/\partial \eta = 1 + \varepsilon J (\partial G^2/\partial \eta \partial \eta)$ ← ✩

$$MJM^T = (1+\varepsilon J\cdot \frac{\partial^2 G}{\partial\eta_i\partial\eta_j}) J (1-\varepsilon \frac{\partial^2 G}{\partial\eta_i\partial\eta_j}\cdot J) \longleftarrow$$

All operation is on matrix. $(\partial^2 G/\partial\eta_i\partial\eta_j)_{ij}$ Built Matrix.

$$= [1-\varepsilon\cdot \frac{\partial^2 G}{\partial\eta\partial\eta}\cdot J + \varepsilon J \frac{\partial^2 G}{\partial\eta\partial\eta} - M\cdot M^T]\cdot J$$

$$= J + \varepsilon J \cancel{\frac{\partial^2 G}{\partial\eta\partial\eta}} J - J\varepsilon \cancel{\frac{\partial^2 G}{\partial\eta\partial\eta}} J = J$$

$\Rightarrow$ For small deviations, Trans. Alwayse Canonical.

9.5 Poisson Brackets + Other Canonical Invariants

$$[u,v]_{q,p} = \frac{\partial u}{\partial q}\cdot\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial q} \longrightarrow [u,v]_\eta = \frac{\partial u}{\partial \eta_i}\frac{\partial v}{\partial \eta_j} - \frac{\partial u}{\partial \eta_j}\cdot\frac{\partial v}{\partial \eta_i} = \frac{\partial u\,\partial v}{\partial\eta\,\partial\eta} J = \left(\frac{\partial u}{\partial\eta}\right)^T J \left(\frac{\partial v}{\partial\eta}\right)$$