

Modern Quantum Problems

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Chap. 1 Formalism

$$1. [AB, CD] = ABCD - CDAB$$

$$= -ACDB - ACBD + ACBD + ABCD - CDAB - CADB + CABD + ACDB \quad \square.$$

$$2. X = a_0 + \sigma \cdot a \quad \sigma \text{ represent pauli matrix with } \text{tr}(\sigma) = \sum \text{diagonals.} = 0$$

$$a_i. a_0 a_k c_k (k=1,2,3) \rightarrow \text{tr}(X) \quad \text{tr}(\sigma \cdot X)$$

$$\text{tr}(X) = \text{tr}(a_0 + \sigma \cdot (a_1 a_2 a_3)) = \sigma^2 a_2 + \sigma a_0$$

[Pauli Matrix σ]:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X = a_0 + \sigma \cdot a = \begin{pmatrix} a_0 + a_3 & a_0 + a_1 - ia_2 \\ a_0 + a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

$$a_0 = (x_{11} + x_{22})/2, \quad a_1 = (x_{12} + x_{21} - x_{11} - x_{22})/2$$

$$a_3 = (x_{11} - x_{22})/2, \quad a_2 = (x_{21} - x_{12})/2i.$$

$$3. \sigma \cdot a' \rightarrow \sigma \cdot a \quad \text{by } = \exp(i\theta \hat{n}\phi/2) \sigma \cdot a \exp(-i\theta \hat{n}\phi/2)$$

$$\sigma \cdot a = \sum_i \sigma_i a_i = \begin{pmatrix} a_2 & a_x - ia_y \\ a_x + ia_y & -a_2 \end{pmatrix} \quad \text{Separate } \exp(-i\theta \hat{n}\phi/2) \text{ term by}$$

$$\star e^{i\theta n \cdot \sigma} = \cos \theta I + i \sin \theta (n \cdot \sigma) \quad \text{Prof. for } (n \cdot \sigma)^0 = I$$

$$(n \cdot \sigma)^2 = (n \cdot \sigma)(n \cdot \sigma) = (n \cdot n)I + (n \times n)i\sigma = I$$

$$e^{\lambda t} = 1 + \frac{x^n}{n!} \dots \leftarrow$$

$$(n \cdot \sigma)^1 = n \cdot \sigma$$

$$= 1 + \frac{-\theta(n \cdot \sigma)^2}{2} + \frac{-i\theta(n \cdot \sigma)^3}{3!} + \dots$$

$$= 1 + -\frac{\theta^2 I}{2} + -\frac{i\theta(n \cdot \sigma)}{6} + \dots = \underbrace{\text{Re} : [1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} \dots]}_{\cos \theta} I + \underbrace{\text{Im} : i(n \cdot \sigma) (\frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots)}_{\sin \theta}$$

$$= \cos \theta I + i \sin \theta \cdot (n \cdot \sigma)$$

$$\Rightarrow \exp(i\theta \hat{n}\phi/2) \text{ with } \theta = \phi/2 = \cos \phi/2 I + i \sin \phi/2 \cdot (n \cdot \sigma)$$

$$\text{Vice } \exp(-u) = \cos u I - i \sin u \cdot (n \cdot \sigma)$$

$$\sigma \cdot a \text{ in } z+ = (\cos \phi/2 + i \sin \phi/2 \cdot \sigma_z) \begin{pmatrix} a_2 & a_x - ia_y \\ a_x + ia_y & -a_2 \end{pmatrix} (\cos \phi/2 - i \sin \phi/2 \cdot \sigma_z)$$

$$\star \text{ Since } \vec{a} = a_x \sigma_x + a_y \sigma_y + a_z \sigma_z \rightarrow \text{Calculate each } \sigma \text{ Separately}$$

$$= (\cos \phi/2 \cdot \sigma_x + i \sin \phi/2 \cdot \sigma_z \sigma_x) (\cos \phi/2 - i \sin \phi/2 \cdot \sigma_z)$$

$$= \cos^2 \sigma_x - i \sin \cos \sigma_z \sigma_x + i \sin \cos \sigma_z \sigma_x + \sin^2 \sigma_z^2 \sigma_x \leftarrow \text{For } \sigma_z \sigma_x = -i \sigma_y \quad \sigma_y \sigma_z = i \sigma_x$$

$$= \cos^2 \sigma_x - \sin^2 \sigma_x \quad \star \sigma_x \sigma_z \neq \sigma_z \sigma_x \quad [\sigma_x \sigma_z] \neq 0$$

$$= \cos^2 \sigma_x - \sin^2 \sigma_x + 2 \sin \cos \sigma_y = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \rightarrow \text{Rotation matrix}$$

$$4. a) XY = z|x\rangle\langle xy| + \langle xy|x\rangle = z|x\rangle$$

$$\text{Tr}(XY) = \sum_a \langle a | XY | a \rangle = \sum_{ab} \langle a | X | b \rangle \langle b | Y | a \rangle = \langle a | Y | b \rangle \langle b | X | a \rangle = \sum_a \langle Yx | = \text{Tr}(YX).$$

$$b). (XY)^+ = Y^+ X^+ \quad \langle a | (XY)^+ | b \rangle = \langle b | XY | a \rangle^* = (\sum_c \langle b | X | c \rangle^* \langle c | Y | a \rangle^*) = \sum_c \langle c | X^+ | b \rangle \langle a | Y^+ | c \rangle = \sum_c \langle a | Y^+ | c \rangle \langle c | X^+ | b \rangle = Y^+ X^+$$

c). $\exp[i f(A)] A = A^T$ with $A|a\rangle = a|a\rangle$

$$\exp[i f(\sum a)] = \sum \frac{\text{coeff}}{N!} \sum_a \exp[i f(a)] |a\rangle \langle a| = \sum_a \exp[i f(a)] |a\rangle \langle a|$$

$$d). \sum_{a'} \Psi_{a'}^*(x') \Psi_{a'}(x'') = \sum_{a'} \langle x' | a' \rangle^* \langle x'' | a' \rangle = \sum_{a'} \langle x'' | a' \rangle \langle a' | x' \rangle = \langle x'' | x' \rangle = \delta(x'' - x')$$

	*	T	+ (= * + T)
$\langle a x b \rangle$	$\langle b x^+ a \rangle$	Same	$\langle b x^+ a \rangle$
$\langle a b \rangle$	$\langle b a \rangle$	Same	$\langle b a \rangle$
$ b\rangle$	\sim	$\langle b $	$\langle b $

$$5. a. |\alpha\rangle |\beta\rangle \quad |\alpha\rangle \langle \beta| \quad \langle \alpha_m | \alpha \rangle \langle \beta | \alpha_n \rangle = \sum_{mn} \langle \alpha_m | \alpha \rangle \cdot \langle \alpha_n | \beta \rangle^*$$

b. Spin $\frac{1}{2}$ Intrinsic Angular Momentum: Spin $\pm \frac{1}{2} \rightarrow \frac{\hbar}{2}$

$$\text{In } z\text{-basis } |S_z = \frac{\hbar}{2}\rangle = |\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \star |S_z = -\frac{\hbar}{2}\rangle \text{ in } z\text{-basis} = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$$

$$|S_x = \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$



$$|\alpha\rangle \langle \beta| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{pmatrix}$$

$$b. |\psi\rangle = |i\rangle + |j\rangle \rightarrow A|\psi\rangle = \psi|\psi\rangle \quad \text{Also} = \alpha_i|i\rangle + \alpha_j|j\rangle = \alpha_i \text{ or } j [|i\rangle + |j\rangle]$$

$\Rightarrow \alpha_i = \alpha_j$ degeneracy

$$7. a. \sum |a'\rangle A \prod_{a'} (A - a') |a'\rangle = 0$$

$$(A - a') \underbrace{(A - a'') \dots}_{(A - a'')/a'} |a'\rangle = a' - 0 = A' - a' = 0$$

$$b. \prod_{a=b} (A - a)/a - b |a'\rangle \quad f(a') = \prod_{a'' \neq a'} \frac{(A - a'')}{(a'' - a')} \quad \text{For } f(a'') |a'\rangle \quad \text{others} = 0 \quad \text{Filter Operator}$$

$$c. A = S_z \text{ Spin } \frac{1}{2} \quad (S_z - \frac{\hbar}{2}) \cdot (S_z + \frac{\hbar}{2}) = S_z^2 - \frac{\hbar^2}{4} = 0 \quad \frac{(S_z - \frac{\hbar}{2})}{-\hbar} |-\frac{\hbar}{2}\rangle = |-\frac{\hbar}{2}\rangle$$

$$8. |+\rangle |-\rangle \quad \langle + | - \rangle = 0 \quad \frac{\hbar^2}{4} (\sim).$$

$$|+\rangle \langle - | + \rangle \langle + | - |+\rangle \langle - | - \rangle \langle - | + |-\rangle \langle + | + |-\rangle \langle + | - |-\rangle \langle - | \quad \begin{matrix} x \rightarrow y \\ \uparrow z \end{matrix} \quad \square$$

$$= - |+\rangle \langle - | + |-\rangle \langle + | = i \epsilon_{xyz} S_z$$

9. $|S \cdot \hat{n}\rangle + \rangle$ * Separate term of $a \cdot b$ as $a_x b_x + a_y b_y \dots$

$$S \cdot \hat{n} |S \cdot \hat{n}\rangle + \rangle = (\hbar/2) |S \cdot \hat{n}\rangle + \rangle \quad \cos(\beta/2) = \hbar/2$$

$$S_x \hat{n}_x + S_y \hat{n}_y + S_z \hat{n}_z = \hbar/2 (\sim) \quad * \text{ why } S_{xyz} \text{ must belongs to Pauli Matrix}$$

$$\begin{aligned} N_x &= \sin\beta \cos\alpha \\ N_y &= \sin\beta \sin\alpha \\ N_z &= \cos\beta \end{aligned}$$

Constraints - $[S_x, S_y] = i\hbar S_z$ (Derived From $[\hat{x}, \hat{p}] = i\hbar$)
 - Only $\pm \hbar/2$ is possible value.

On S_z -base

$$\begin{matrix} \uparrow & 0 \\ 0 & \downarrow \end{matrix} = \begin{matrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{matrix} = \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[S_z, S_x] = i\hbar S_y \quad \text{define } |+\rangle = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad |-\rangle = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\text{Only exist these three} \leftarrow \Rightarrow S_x = S_+ + S_- = \hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = S_+ - S_- = \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

10. * Observe Series of $|1\rangle \langle 1| \quad |1\rangle \langle 2| \dots$ As position in matrix.

$$\begin{pmatrix} a & a \\ a & -a \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix} \quad [\hat{H}\psi = E\psi] \quad E \text{ is defined as eigenvalues of } \hat{H}$$

$$\text{with } \det(C\hat{H} - E\mathbb{I}) = 0 \quad N = E = m \quad u, v = m$$

11. [Observe Pauli Matrix as base of H]

$$\begin{pmatrix} \varepsilon_1 & \delta \\ \delta & \varepsilon_2 \end{pmatrix} \text{ is Separable to } E_{\text{avg}} \cdot I + \sigma_z \cdot E_{\text{avg}} + \sigma_x \cdot \delta$$

Unit Matrix + Values on σ_z + Values on σ_x
 * Constants

$$\text{Eigenvalue of } \hat{H} = E = E_{\text{avg}} + \sqrt{\delta^2 + \varepsilon^2} = m$$

Observe eigenstate as angle operator: $\tan\theta = \text{Val. of } \sigma_z / \text{Val. of } \sigma_x = m$

$$\text{with } |\hat{n}\rangle = \cos(\theta/2)|+\rangle + \exp(i\phi/2)\sin(\theta/2)|-\rangle$$

12. \hat{z} plane Spin $1/2$ γ angle to \hat{z}

* S_{xyz} Operators = Equipments on ask Self Spin on xyz direction. $S_{xyz} = \hbar/2 \sigma_{xyz}$

Operating by $|\pm x\rangle, |\pm y\rangle, |\pm z\rangle$ eigenkets to find proportions.

$S_x |\psi\rangle = \text{Useless Steps} \quad \langle \pm x | \psi \rangle = P. \text{ of having } \pm \hbar/2 \text{ on } x \text{ direction.}$

13. $N_z = \cos(\gamma/2)|+z\rangle + \exp(i\phi)\sin(\gamma/2)|-z\rangle$ As General Spin Operator On z -base with γ Angle to \hat{z} ϕ Angle to \hat{y}

$$\langle +z | N_z \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\gamma/2) \\ \sin(\gamma/2) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\gamma/2) \\ \sin(\gamma/2) \end{pmatrix} \Rightarrow P = |\frac{1}{\sqrt{2}} (\cos(\gamma/2) + \sin(\gamma/2))|^2 = \frac{1}{2} (1 + 2\sin(\gamma/2)\cos(\gamma/2)) = \frac{1}{2} (1 + \sin 2\gamma)$$

$$b). \langle (\Delta S_z)^2 \rangle = \langle S_z^2 \rangle - \langle S_z \rangle^2$$

$$\langle S_z \rangle^2 = |\psi| S_z |\psi\rangle|^2 \quad \text{or} = \frac{1}{2}(1+\sin\varphi) \frac{\hbar}{2} + \frac{1}{2}(1-\sin\varphi)(-\frac{\hbar}{2}) = \frac{\hbar}{2}\sin\varphi$$

13 Method 1 - Basic Formalism

State : $\begin{pmatrix} a \\ b \end{pmatrix}$ Eigenkets : $\left| \sum S_i \cdot \hat{n} \pm \right\rangle$ Test $\langle \sum S_i \cdot \hat{n} \pm | \text{State} \rangle$

\hat{z} -base

$$i). \langle S_z \cdot \hat{n} + | \psi \rangle = (0.8(\frac{\hbar}{2}) \sin(\frac{\varphi}{2})) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\frac{\varphi}{2} \quad \psi \rightarrow \text{Eigenket} = \begin{pmatrix} 0.8\frac{\hbar}{2} \\ \sin\frac{\varphi}{2} \end{pmatrix}$$

$$ii). \langle S_z \cdot \hat{n} - | \psi_{\text{new}} \rangle = (0 \ 1) \begin{pmatrix} 0.8\frac{\hbar}{2} \\ \sin\frac{\varphi}{2} \end{pmatrix} = \sin\frac{\varphi}{2}$$

$$P = P_1 \times P_2 = \cos^2\frac{\varphi}{2} \sin^2\frac{\varphi}{2} = \frac{1}{4}\sin^2\varphi$$

Method 2 - Formalism

$$\begin{aligned} P = |\langle + | - \rangle|^2 &\rightarrow |\langle + | S_n + \rangle \langle S_n + | - \rangle|^2 \leftarrow \text{For } |S_n + \rangle = [0.8\frac{\hbar}{2}|+ \rangle + \sin\frac{\varphi}{2}|-\rangle] \\ &= 0.8^2 \frac{\hbar^2}{2} \sin^2\frac{\varphi}{2} \\ &= \frac{1}{4} \sin^2\varphi \end{aligned}$$

$$\hat{n} = \sin\varphi n \cdot \hat{x} + 0 \cdot \hat{y} + \cos\varphi n \cdot \hat{z}$$

— Vector ★ Logic Chain of Vector

$$S \cdot \hat{n} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \cdot \frac{\hbar}{2}$$

— Operator Eigenkets \leftarrow Operator

$$|S \cdot \hat{n}, \pm \rangle = \frac{\hbar}{2} \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \quad \leftarrow \text{Eigenkets of } S \cdot \hat{n} \quad \text{— Eigenkets}$$

Only ratio needed

$$\Rightarrow \cos\varphi a + \sin\varphi b = a$$

Set $a = \cos\frac{\varphi}{2}$ $b = \sin\frac{\varphi}{2}$

$$a = \frac{\sin\varphi}{1 - \cos\varphi} \cdot b = \frac{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}{1 - 1 + 2\sin^2\frac{\varphi}{2}} b = \frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} \cdot b$$



$$|S \cdot \hat{n}, + \rangle = \begin{pmatrix} \cos\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} \end{pmatrix}$$

Similar

$$|S \cdot \hat{n}, - \rangle = \begin{pmatrix} \sin\frac{\varphi}{2} \\ \cos\frac{\varphi}{2} \end{pmatrix}$$

$$14. a. \det(A - NI) = 0$$

$$(-N^3) - 2(-N)(1/\sqrt{2})^2 = 0 \quad N = \pm 1, 0.$$

$$(1, 1, 1) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{2} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad b = \sqrt{2}a \quad a+c = \sqrt{2}b \quad a=c$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$15. \hat{A}\hat{B} \quad \{ |a', b'\rangle \} \quad [A, B] = 0$$

$$[A, B]|ab\rangle = AB|ab\rangle - BA|ab\rangle = AB|a'b'\rangle - BA|a'b'\rangle$$

$$= b'a'|a'b'\rangle - a'b'|a'b'\rangle = 0$$

$$[A, B]\psi = \sum C[A, B]|ab\rangle = 0.$$

$$16. \{ A, B \} = AB + BA = 0 \quad |a, b\rangle$$

$$2ab|a, b\rangle = 0 \quad a \text{ or } b \text{ or } a, b = 0.$$

$$17. A_1, A_2 \quad [A_1, A_2] \neq 0 \quad [A, H] = [A_2 H] = 0$$

$$H|E\rangle = E|E\rangle \rightarrow A_1 H|E\rangle = H A_1 |E\rangle$$
$$\downarrow$$
$$E_2 A_1 |E\rangle \neq E_1 A_2 |E\rangle \leftarrow \# A_1 |E\rangle = \# A_2 |E\rangle$$
$$E_2 = E_1$$