Modern Quantum Problems

Leo

Cheng Xu-Liu.

Chap. 1 Formalism

1. $[AB, CD] = ABCD - CDAB$

$$= \cancel{-ACDB} \cancel{-ACBD} + \cancel{ACBD} + ABCD - CDAB \cancel{- CADB} + \cancel{CADB} + \cancel{ACDB} \quad \square.$$

2. $X = a_0 + \sigma\cdot a$ $\quad$ σ represent pauli matrix with $tr(\sigma) = \Sigma$ diagnals $= 0$

a. a_0 $\quad a_k\ (k=1\,2\,3) \rightarrow tr(X) \quad tr(\sigma_k X)$

$tr(X) = tr(a_0 + \sigma\cdot(a_1 a_2 a_3)) \quad = \sigma^2 a_2 + \sigma a_0$

[Pauli Matrix σ]:

$$X = a_0 + \sigma\cdot a = \begin{pmatrix} a_0 + a_3 & a_0 + a_1 - i a_2 \\ a_0 + a_1 + i a_2 & a_0 - a_3 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$a_0 = (X_{11} + X_{22})/2 \qquad a_1 = (X_{12} + X_{21} - X_{11} - X_{22})/2$$

$$a_3 = (X_{11} - X_{22})/2 \qquad a_2 = (X_{21} - X_{12})/2i.$$

3. $\sigma\cdot a' \rightarrow \sigma\cdot a$ by $= \exp(i\sigma\cdot\hat{n}\phi/2)\ \sigma\cdot a\ \exp(-i\sigma\hat{n}\phi/2)$

$$\sigma\cdot a = \sum_i \sigma_i a_i = \begin{vmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{vmatrix}$$

Seperate $\exp(-i\sigma\hat{n}\phi/2)$ term by

☆ $e^{i\theta n\cdot\sigma} = \cos\theta I + i\sin\theta(n\cdot\sigma)$ $\quad$ Prof. for $(n\cdot\sigma)^0 = 1$

$$(n\cdot\sigma)^2 = (n\cdot\sigma)(n\cdot\sigma) = (n\cdot n)I + (n\times n)i\sigma = I$$

$$(n\cdot\sigma)^1 = n\cdot\sigma$$

$e^x = 1 + x^n/n!\ ---- \quad \leftarrow$

$$= 1 + \frac{-\theta(n\cdot\sigma)^2}{2} + \frac{-i\theta(n\cdot\sigma)^3}{3!} + \cdots$$

$$= 1 + -\theta^2 I/2 + -i\theta(n\cdot\sigma)/6 + \cdots = Re:[\underbrace{1 - \theta^2/2 + \theta^4/4! \cdots}_{\cos\theta}] I + Im: i(n\cdot\sigma)(\underbrace{\theta - \theta^3/3! + \theta^5/5! - \cdots}_{\sin\theta})$$

$$= \cos\theta I + i\sin\theta\cdot(n\sigma)$$

$\Rightarrow \exp(i\sigma\cdot\hat{n}\phi/2)$ with $\theta = \phi/2 = \cos\phi/2\, I + i\sin\phi/2\cdot(n\cdot\sigma)$

Vice $\exp(-u) \quad = \cos\phi/2\, I - i\sin\phi/2\cdot(n\cdot\sigma)$

$$\sigma' a \text{ in } z+ = (\cos\phi/2 + i\sin\phi/2\cdot\sigma_z)\begin{vmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{vmatrix}(\cos\phi/2 - i\sin\phi/2\cdot\sigma_z)$$

☆ Since $x = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z \rightarrow$ Calculate each σ Seperately

$$= (\cos\phi/2\cdot\sigma_x + i\sin\phi/2\,\sigma_z\sigma_x)(\cos\phi/2 - i\sin\phi/2\,\sigma_z)$$

$\cancel{= \cos^2\sigma_x - i\sin\cos\sigma_z\sigma_x + i\sin\cos\sigma_z\sigma_x + \sin^2\sigma_z^2\sigma_x}$ $\leftarrow$ For $\sigma_z\sigma_x = -i\sigma_y \qquad \sigma_y\sigma_z = i\sigma_x$

$= \cos^2\sigma_x - \sin^2\sigma_x$ $\qquad$ ☆ $\sigma_x\sigma_z \neq \sigma_z\sigma_x \quad [\sigma_x\ \sigma_z] \neq 0.$

$$= \cos^2\sigma_x - \sin^2\sigma_x + 2\sin\cos\sigma_y = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \rightarrow \text{Rotation matrix}$$

4. a). $XY = z|XY\rangle \cdot \langle XY|x\rangle = z|x\rangle$

$\mathrm{Tr}(XY) = \sum_a \langle a| XY |a\rangle = \sum_{ab} \langle a|X|b\rangle\langle b|Y|a\rangle = \langle a|Y|b\rangle\langle b|X|a\rangle = \sum_a \langle YX\rangle = \mathrm{Tr}(YX)$.

b). $(XY)^\dagger = Y^\dagger X^\dagger \quad \langle a|(XY)^\dagger|b\rangle = \langle b|XY|a\rangle^* = \left(\sum_c \langle b|X|c\rangle^* \langle c|Y|a\rangle^*\right) = \sum_c \langle c|X^\dagger|b\rangle\langle a|Y^\dagger|c\rangle$

$= \sum_c \langle a|Y^\dagger|c\rangle\langle c|X^\dagger|b\rangle$.

$= Y^\dagger X^\dagger$

c). $\exp[i \cdot f(A)] \quad A = A^\dagger$ with $A|a\rangle = a|a\rangle$

$\exp(i f(\Sigma a)) = \Sigma \frac{[i f(A)]^N}{N!} \qquad \sum_a \exp[i f(A)|a\rangle\langle a|] = \sum_a \exp[i f(a)]|a\rangle\langle a|$

d). $\sum_{a'} \psi_{a'}^*(x')\psi_{a'}(x'') = \sum_{a'} \langle x'|a'\rangle^* \langle x''|a'\rangle = \sum_{a'} \langle x''|a'\rangle\langle a'|x'\rangle = \langle x''|x'\rangle = \delta(x''-x')$

	$*$	T	$\dagger (= * + T)$
$\langle a\|x\|b\rangle$	$\langle b\|x^\dagger\|a\rangle$	Same.	$\langle b\|x^\dagger\|a\rangle$
$\langle a\|b\rangle$	$\langle b\|a\rangle$	Same	$\langle b\|a\rangle$
$\|b\rangle$	$\backsim$	$\langle b\|$	$\langle b\|$

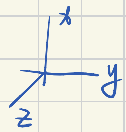
5. a. $|\alpha\rangle \quad |\beta\rangle \qquad |\alpha\rangle\langle\beta| \qquad \langle a_m|\alpha\rangle\langle\beta|a_n\rangle = \sum_{mn} \langle a_m|\alpha\rangle \cdot \langle a_n|\beta\rangle^*$.

b. Spin ½ Intrinsic Angular Momentum : Spin $\pm\frac{1}{2} \to \frac{\hbar}{2}$.

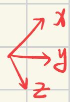
In z - basis $|S_z = \hbar/2\rangle = |\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

☆ $|S_x = \hbar/2\rangle$ in z - basis $= \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$

$|S_x = \hbar/2\rangle = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z) = \begin{vmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{vmatrix}$



Classical Ideas $\longrightarrow$



$|x, y, z\rangle$ only Canonical with them selves.

$|\alpha\rangle\langle\beta| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}^\dagger = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{pmatrix}$

6. $|\psi\rangle = |i\rangle + |j\rangle \longrightarrow A|\psi\rangle = \psi|\psi\rangle$ Also $= a_i|i\rangle + a_j|j\rangle = a_i \text{ or } j[|i\rangle + |j\rangle]$

$\Rightarrow a_i = a_j$ degeneracy.

7. a. $\Sigma|a'\rangle \; A \qquad \prod_{a'} (A - a')|a'\rangle = 0$

$(A - a')(A - a'') \cdots |a'\rangle = 0$

$(A - a'')|a'\rangle = a' - 0 = A' - a' = 0$.

b. $\prod_{a=b} (A-a)/(a-b) \; |a'\rangle \qquad f(a'') = \prod_{a'' \neq a'} \frac{(A - a'')}{(a' - a'')}$ For $f(a'')|a'\rangle$ others $= 0$ Filter Operator

$= a'^n / a'^n = 1$

c. $A = S_z$ Spin ½ $\qquad (S_z - \hbar/2) \cdot (S_z + \hbar/2) = S_z^2 - \hbar^2/4 = 0 \qquad \frac{(S_z - \hbar/2)}{-\hbar} \left| -\hbar/2 \right\rangle = \left| -\hbar/2 \right\rangle$

8. $|+\rangle \; |-\rangle \qquad \langle +|-\rangle = 0 \qquad \hbar^2/4(\backsim)$.

$|+\rangle\langle -|\cancel{+}\rangle\langle +| - |+\rangle\langle -|-\rangle\langle -| + |-\rangle\langle +|+\rangle\langle +| - |-\rangle\langle \cancel{+}|-\rangle\langle -|$

$= -|+\rangle\langle -| + |-\rangle\langle +| = i\varepsilon_{xyz}\hbar S_z \qquad \leftarrow \qquad x \to y, \; \uparrow z \swarrow \qquad \square$

9. $|S\cdot\hat{n};+\rangle$ ☆ Seperate term of $a\cdot b$ as $a_xb_x + a_yb_y \cdots$

$S\cdot\hat{n}\,|S\cdot\hat{n};+\rangle = (\hbar/2)\,|S\cdot\hat{n};+\rangle$ $\quad \cos(\beta/2) = \hbar/2$.

$S_x\hat{n}_x + S_y\hat{n}_y + S_z\hat{n}_z = \hbar/2\,(\sim)$ ☆ why S_{xyz} must belongs to Pauli Matrix

$n_x = \sin\beta\cos\alpha$
$n_y = \sin\beta\sin\alpha$
$n_z = \cos\beta$

Constraints – $[S_i, S_j] = i\hbar S_z$ (Derived From $[x, p] = i\hbar$)
– Only $\pm\hbar/2$ is possible value.

On S_z – base $\begin{matrix}\uparrow & 0 \\ 0 & \downarrow\end{matrix} = \begin{matrix}\hbar/2 & 0 \\ 0 & -\hbar/2\end{matrix} = \hbar/2\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$

$[S_z, S_x] = i\hbar S_y$ $\quad$ define $|+\rangle = \hbar\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$ $\quad |-\rangle = \hbar\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}$

Only exsist these three possible matrix. $\leftarrow$ $\Rightarrow S_x = S_+ + S_-/2 = \hbar/2\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$

$S_y = S_+ - S_-/2 = \hbar/2\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}$

10. ☆ Observe Series of $|1\rangle\langle 1|$ $|1\rangle\langle 2| \cdots$ As position in matrix.

$\begin{pmatrix}a & a \\ a & -a\end{pmatrix}\cdot\begin{pmatrix}u \\ v\end{pmatrix} = E\begin{pmatrix}u \\ v\end{pmatrix}$ $[\hat{H}\psi = E\psi]$ E is defined as eigenvalues of $\hat{H}$

with $\det(H - \lambda I) = 0$ $\quad \lambda = E = \sim$ $\quad u, v = \sim$

11. [Observe Paulie Matrix as base of H]

$\begin{pmatrix}\varepsilon_1 & \delta \\ \delta & \varepsilon_2\end{pmatrix}$ is Seperable to $E_{avg} \cdot I + \sigma_z \cdot E_{avg} + \sigma_x \cdot \delta$

Unit Matrix · Constants + Values on σ_z + Values on σ_x

Eigenvalue of $\hat{H} = E$
$= E_{avg} + \sqrt{\Delta^2 + \delta^2} = \sim$

Observe eigenstate as angle operator : $\tan\theta = \text{Val. of } \sigma_z / \text{Val. of } \sigma_x = \sim$

with $|\hat{n};+\rangle = \cos\frac{\beta}{2}|+\rangle + \exp(i\alpha)\sin\frac{\beta}{2}|-\rangle$.