

PHAS0008 - Wave - Week 1

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Simple Harmonic Motion $F = kx$

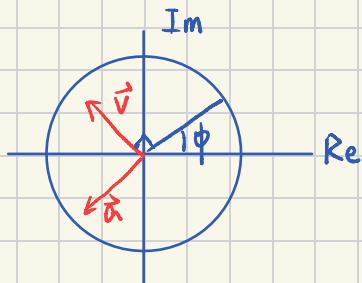
$$F = kx = -m\ddot{x}$$

$$\Rightarrow \ddot{x} = A \exp(\sqrt{k/m}t) = A \cos(\omega t + \phi_0) = B \cos(\omega t) + C \sin(\omega t)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

with $\omega = \sqrt{k/m}$, $B = A \cos \phi$, $C = A \sin \phi$

- Phasor Diagram.



* Motion is exactly $\text{Re}[]$

$$\text{For } \psi = A \exp(i\omega t + i\phi)$$

$$v = iA\omega \exp(i\omega t + i\phi)$$

* v have $\frac{\pi}{2}$ phase different $a = -A\omega^2 \exp(i\omega t + i\phi)$

a have π

- Damped Motion - $F = m\ddot{x} + j\dot{x} + kx = 0$

$$\text{Assume Solution } \psi = A \exp(\alpha t)$$

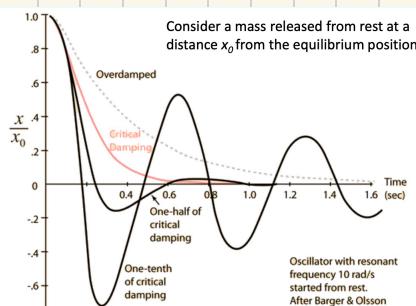
$$m\alpha^2 + j\alpha + k = 0$$

$$\Rightarrow \alpha = \frac{-j \pm \sqrt{j^2 - 4mk}}{2m} \quad \text{To make it looks better} \rightarrow \gamma = j/\omega_m$$

$$= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \quad \text{For } \omega_0 = \sqrt{k/m}$$

$$\Rightarrow \psi = A \exp(-\gamma t + \sqrt{\gamma^2 - \omega_0^2} t) + B \exp(-\gamma t - \sqrt{\gamma^2 - \omega_0^2} t)$$

i) $\gamma > \omega$ $\sqrt{\gamma^2 - \omega_0^2} > 0$ Overdamped
 $\psi = A \exp(\gamma t) + B \exp(-\gamma t) \leftarrow \gamma = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$



* Graph Validation: $A \exp(\gamma t) = -B \exp(-\gamma t)$

$$A \exp(t \sqrt{\gamma^2 - \omega_0^2}) = -B \exp(-t \sqrt{\gamma^2 - \omega_0^2})$$

$$\exp(2t \sqrt{\gamma^2 - \omega_0^2}) = -B/A < 0 \quad \text{None exist}$$

ii) $\gamma = \omega$ $\psi = (A + B) \exp(-\omega t)$ Critical Damp

* $\psi = \text{Constant} \cdot \exp(-\omega t) \leftarrow \text{Lost one solution}$

Approach by overdamped solution

$$\Psi = A \exp(\alpha_+ t) + B \exp(\alpha_- t) \quad \text{assume } \alpha_+ = \alpha_- + \varepsilon$$

$$= A \exp(\alpha_- t) + B \exp(\alpha_- t) \cdot \exp(\varepsilon t) = \exp(\alpha_- t) (A + B \exp(\varepsilon t))$$

$$= \exp(\alpha_- t) (-\frac{1}{\varepsilon} + \frac{1}{\varepsilon} \exp(\varepsilon t)) \quad \text{Assume } A = -\frac{1}{\varepsilon}, B = \frac{1}{\varepsilon}$$

$$\lim_{\varepsilon \rightarrow 0} \Psi(t) = \exp(\alpha_- t) \left(\frac{\exp(\varepsilon t) - 1}{\varepsilon} \right) = t \exp(\alpha_- t) \xrightarrow{\star} B \rightarrow B \cdot t \quad \Psi(t) = (A + Bt) \cdot e^{\alpha_- t}$$

$$\text{with Limitation } \Psi(0) = A \quad \Psi'(0) = 0 \Rightarrow \Psi = A(1 + \omega_0 t) \cdot \exp(-\omega_0 t)$$

iii). $\nu < \omega$ $\Psi(t) = A \exp(-\nu t) + i(\omega_0 t + \phi)$ with $\omega = \sqrt{\omega_0^2 - \nu^2}$

Energy of Damped Oscillation $W = E_k + E_p$

$$\frac{dW}{dt} = m \dot{\Psi} \ddot{\Psi} + k \Psi \dot{\Psi} = \dot{\Psi} (k \Psi + m \ddot{\Psi}) \Rightarrow \frac{dW}{dt} = -b \dot{\Psi}^2$$

Damping Force = $b \dot{\Psi}$

- Forced Motion (ref. From L.D. Landau - Mechanics).

$$U(x,t) = U_{co,t} + \cancel{x} \cdot \frac{\partial U}{\partial \cancel{x}} \Big|_{\cancel{x}=0}$$

$$\text{with } \ddot{x} = \frac{1}{m} \ddot{\Psi}^2 - \frac{1}{m} b \dot{\Psi}^2 - F(t) \cdot x \Rightarrow (\ddot{x}/\cancel{x}) \frac{d}{dt} - \frac{\partial^2}{\partial \cancel{x}^2} = m \ddot{\Psi} - b \dot{\Psi} - F(t) = 0$$

For special state on $F(t) = b \cos(\nu t + \phi)$
↓ Special Solution term

$$\text{Take } x = A \sin(\omega_0 t + \phi) + B \cos(\nu t + \phi)$$

$$-Bm \cos(\nu t + \phi) \nu^2 + k B \cos(\nu t + \phi) = b \cos(\nu t + \phi)$$

Two oscillation composed

Q.

$$x = A \cos(\omega_0 t + \phi) + \frac{b}{m(\omega_0^2 - \nu^2)} \cos(\nu t + \phi) \quad \leftarrow \quad -Bm \nu^2 + k B = b \Rightarrow B = -\frac{b}{m \nu^2 - k}$$

blow up

For special case $\rightarrow \omega = \nu$ → Solution of ODE able to add ordinary solution

$$x = A \cos(\omega_0 t + \phi) + \frac{b}{m(\omega_0^2 - \nu^2)} \cdot [\cos(\nu t + \phi) - \cos(\omega_0 t + \phi)]$$

$$\text{By } \nu \rightarrow \omega \quad x = A \cos(\omega_0 t + \phi) + \frac{\sin(\omega_0 t + \phi) \omega_0 t}{2m \omega_0}$$

Properties on ν closed to ω : $\nu = \omega + \varepsilon$

With $A, B = \exp(i\phi), \exp(i\phi)$

$$\text{Assume as } C = A + B \exp(i\nu t) = A e^{i\phi} + B e^{i(\nu t + \phi)}$$

$$C^2 = A^2 + B^2 + 2AB \cos(\nu t + \phi - \omega_0 t)$$

↓ with frequency ε

$$|A - B| \leq C \leq A + B$$

- General Solution On Driven Force motion.

$$\ddot{x} + \omega^2 x = \frac{1}{m} F(t)$$

* Operator Algebra Method.

$$(\hat{D}^2 + \omega^2)x \text{ where } \hat{D} \text{ is } \frac{d}{dt} \text{ operator}$$

$$\text{Split by } (D - i\omega)(D + i\omega)$$

$$(D^2 + i\omega D - i\omega D + \omega^2)x = \ddot{x} + i\omega \dot{x} + \omega^2 x - i\omega \dot{x}$$

$$\frac{d}{dt} (\underline{\ddot{x} + i\omega \dot{x}}) - i\omega (\underline{\ddot{x} + i\omega \dot{x}}) = \frac{1}{m} F(t).$$

Set as ξ

$$\xi = A(t) \exp(i\omega t) \Leftrightarrow \frac{d}{dt} \xi - i\omega \xi = \frac{1}{m} F(t) \quad * \text{ 2 order} \rightarrow \text{1 order ODE.}$$

$$A(t) \exp(i\omega t) + i\omega \exp(i\omega t) A(t) - i\omega A(t) \exp(i\omega t) = \frac{1}{m} F(t)$$

$$A(t) = \frac{F(t)}{m} \exp(-i\omega t) \Rightarrow \xi = \exp(i\omega t) \left[\int_0^t \frac{1}{m} F(t') \exp(-i\omega t') dt' + \xi_0 \right]$$

\downarrow associate with specific $F(t)$

$$\xi = i\omega x + \dot{x} \Rightarrow x = \xi_0 t + \dots$$