Week 6 - PHAS0004 ASU - Note

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Chap 3.2 Scale of Magnititude

Apparent Magnititude: 1~5 represent 1 ~ 100 x light intensity

Flux $F = L/4\pi r^2$

Absolute Magnititude: M : $F_2/F_1 = 100^{(m_2-m_1)/5}$

$m_1 - m_2 = -2.5\log_{10}(F_1/F_2)$

Define as M at 10pc distance : $M = m - 5\log_{10}(d/10pc)$

Ex. $m_{sun} = -26.83$ $R = 1AU$ $M = m - 5\log_{10}(\frac{1AU}{10pc}) = -31.57$

Chap 3.4 Black Body Radiation.

$\lambda_{max} \cdot T =$ Constant.

(λ when intensity is max).

Proof : Chap 23. Thermal Concept – Photons

$E = \hbar\omega$ where $\omega = 2\pi f$

$P = \hbar k$ where k is wave vector.

$E/p = \omega/k = \nu \times \lambda = c.$

$\omega = 2\pi/T$ Radius moved per Seconds.

$k = 2\pi/\lambda$ Radius moved per length.

23.1 Classical Approach.

$\boxed{u} = U/V = n\hbar\omega$ where define $U =$ Total E $V =$ Volumn

Ener Densi

$u = \hbar\omega$ = Singel photon energy $= mc^2$

Pressure $p = \frac{1}{3} nm\langle v^2\rangle = \frac{1}{3} nmc^2$

☆ $P = \frac{1}{3} u$

$\Phi = \frac{num}{area} = \frac{\langle V\rangle}{Aree} = \frac{1}{4} nc$

→ Power incident per unit area $\underline{P} = \hbar\omega\Phi = \frac{1}{4} uc$

☆ Relate $u = n\hbar\omega$ to Tempreture.

$dU = T \cdot ds - P \cdot dv$

$$\underbrace{(\partial U/\partial v)}_{u} = T \cdot \underbrace{(\partial S/\partial v)}_{T\cdot(\partial P/\partial T)_v} - P$$

$\Rightarrow u = T \cdot \frac{1}{3} \cdot (\partial u/\partial T)_v - u/3 \Rightarrow u = AT^4$

Similar $\underline{P} = \frac{1}{4} uc = \frac{1}{4} ACT^4$

23.2 Spectral energy density – seperate u on ν

$$u = \int_{-\infty}^{+\infty} u_\nu \cdot d\nu$$

23.3 Kirchoff's Law. – emission = absorb.

$\underline{P}_{(\nu)} \cdot \alpha_{(\nu)}$ (absorb energy α coeficient $\cdot$ $\alpha = (\frac{1}{4} u_\nu d\nu c)\, \alpha_\nu$

Emissinon $= e_\nu d\nu \Rightarrow$ Emission $=$ Absorb

$e_\lambda / d\lambda = \frac{1}{4} c u_\lambda$

no relation with λ and T

23.4 Radiation Pressure.

$$\begin{cases} \underline{P} = \frac{1}{4} uc = \sigma T^4 \\ u = (4\sigma / c) \cdot T^4 \\ p = u/3 = 4\sigma T^4 / 3c \end{cases}$$

$$\begin{cases} \underline{P} = uc = \sigma T^4 \\ u = (\sigma / c) \cdot T^4 \\ P = u = \sigma T^4 / c \end{cases}$$

Free photon // all directions. Single direction beam light.

23.5 Statistical mechenics of photon gas. (Cal. number of σ)

density of state for photon gas

: ☆ dif. way from $g(E) \propto \exp(-E/k_B T)$

density of states · Approach by quantum idea

· Numbers of possible state in $k \sim k + dk$

Maxwell distribution · Approach by energy state.

· Number of particles in range $v \sim v + dv$

21.1 Density State.

Begin with box L^3. each wavevector $k = P/\hbar$

$$\psi(x,y,z) = 1/\sqrt{3}\,\sin(k_x x)\sin(k_y y)\sin(k_z z)$$

$\downarrow$ Normalize : $k_x = n_x\pi/L$ $\quad k_y = n_y\pi/L$ $\quad k_z = n_z\pi/L$

☆ Possible State Number $= \dfrac{\text{Volumn Total}}{\text{Volumn each part}}$

$$= \frac{4\pi k^2 dk \cdot 1/8}{(\pi/L)^3}$$

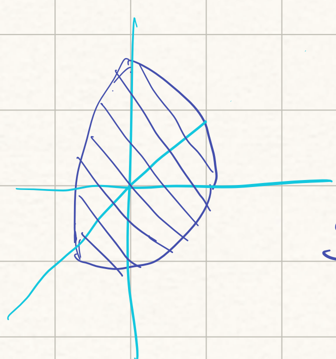
☆ why 1/8 sphere shell ?

$$= Vk^2dk/2\pi^2$$ (V is volumn here).

Only consider k_{xyz} both >0

↑

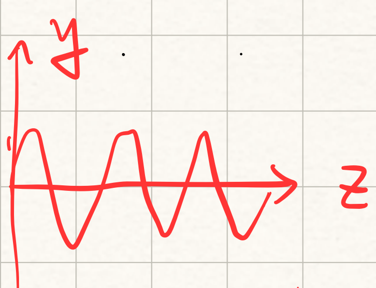
Normalized Condition.



Surface Area for k both >0.

Back to photon density of state $= \dfrac{4\pi k^2 dk}{8(\pi/L)^3} \times 2$

$$= \underbrace{\frac{4\pi k^2 dk}{(2\pi/L)^3}}_{\text{possible state num.}} \times \underbrace{2}_{\text{2 polarization direction}}$$



also

2 "degree of freedom"

$$g(k)\,dk = \frac{k^2 \cdot dk\, L^3}{\pi^2}$$

$w/k = c$

$k = w/c$

$$g(w)\cdot dw = g(k)\, dk$$

$$g(w) = g(k)\cdot dk/dw$$

$$= \frac{w^2 \,{}^1\!/\!c^2 L^3}{\pi^2} \cdot 1/c = \frac{w^2 L^3 dw}{c^3 \pi^2}.$$

☆ why $g(k) \to g(w)$?

$U = \int g(w)\cdot dw \cdot$ energy on each particle.
[☆]

$$= \int_0^\infty w^2 L^3 \cdot dw / c^3 \pi^2 \cdot dw \cdot \left(1/2 + 1/\exp(\beta\hbar w) - 1\right) \cdot \hbar w$$

[☆] Simple harmonic motion particle.

$$= \int_0^\infty \frac{w^3 L^3 \cdot dw}{c^3 \pi^2 (\exp(\beta\hbar w) - 1)}$$

Set $x = \beta\hbar w$.

dw/dx

$$= \frac{L^3 \hbar}{c^3 \pi^2} \int_0^\infty \frac{w^3 dw}{\exp(\beta\hbar w) - 1}$$

$$\frac{(x/\beta\hbar)^3 \cdot dx \cdot 1/\beta\hbar}{\exp(x) - 1}$$

$$= \frac{x^3 \cdot dx}{(\exp(x) - 1)(\beta\hbar)^4}$$

☆ Math Skill, Review

$$\int \frac{x^3}{\exp(x) - 1} dx = \zeta(4)\,\Gamma(4)$$

$$= \pi^4/15$$

$\Rightarrow U = AT^4$ As $A = \frac{\pi^2 k_B^4}{15 c^3 \hbar^3}$

$\sigma = Ac/4 = \pi^2 k_B^4 / 60 c^2 \hbar^3$

23.6 Black Body Distribution / Convert to focuse on $u(\omega)$ $u(T)$ ...

$$u = U/V = \int u(\omega) \cdot d\omega = \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3 \, d\omega}{\exp(\beta\hbar\omega) - 1}$$

$\Downarrow$

Energy Density Distribution on (ω, ν, λ

$$u(\omega) = \frac{\hbar}{\pi^2 c^3} \cdot \frac{\omega^3}{\exp(\beta\hbar\omega) - 1}$$

$u(\nu)$ by $\omega = 2\pi\nu$ $d\omega/d\nu = 2\pi$

$u(\lambda)$ by $c = \lambda\nu$ $d\nu/d\lambda = 1/c$

$\Rightarrow u$ = Light Intensity Distribution.

Wien's Law : λ (when u is max) $\cdot T$ = Constant.

$du/d\lambda = 0$

$\beta hc/\lambda_{max}$ = Constant.

Logic Conclusion ①. $g(\omega) \leftarrow$ density of state in quantum approach.

②. Total Energy U by $\int^\infty g(\omega) \cdot d\omega$ (energy each particle)

③. $U/V = u(\omega)$ → Intensity distribution.

④. $du(\omega)/d\omega$ → $\lambda_{max} \cdot T$ = Constant.

Stefan - Josef Law: $L = A\sigma T^4$ [$\underline{P}$ priviously]

For ball shape → $L = 4\pi R^2 \sigma T_e^4$. flux = $F = \sigma T_e^4$

$F = L_\odot / 4\pi R_\odot^2$

Example: effective T for sun $T_\odot = (L_\odot / 4\pi R_\odot^2 \sigma)^{1/4} = 5777k$

Week 6 Core Slides.

$\lambda_{peak} = \text{Const}/T$

Wien's Law □

☆ Power incident total.

$L = 4\pi R^2 \sigma T_e^4$

infact $\underline{P} = \sigma T^4$

$\underline{P} \cdot$ surf. area $= A\sigma T^4$ □

Rayleigh - Jeans Law — Plank distribution.

$\bar{E} = k_B T$ ——— $\bar{E} = h\nu / \exp(\beta h\nu) - 1$ [☆] □

Different is plank use quantumdize Energy distribusion.

$\bar{E} = h\nu / \exp(\beta h\nu) - 1$

↕ same.

[☆]: SHM. particle: $\frac{1}{2}\hbar\omega + \hbar\omega / \exp(\beta\hbar\omega) - 1$

exsist because dif. basement point.

Bohr model of atom (Energy decrease → Light emitted)

Energy Include $= \frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{r} + \frac{1}{2}mv^2$

↓ require r , v

$mv^2 = \frac{1}{4\pi\varepsilon_0} e^2/r$

$\frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{r^2} = m\frac{v^2}{r}$

$\frac{1}{4\pi\varepsilon_0} e^2 = mv^2r$

$mvr = L = n\hbar$

$\frac{1}{4\pi\varepsilon_0} e^2 = n\hbar \cdot \frac{n\hbar}{mr}$

$\Rightarrow r = \frac{n\hbar^2}{me^2} \cdot 4\pi\varepsilon_0$

$V = \frac{n\hbar}{mr}$

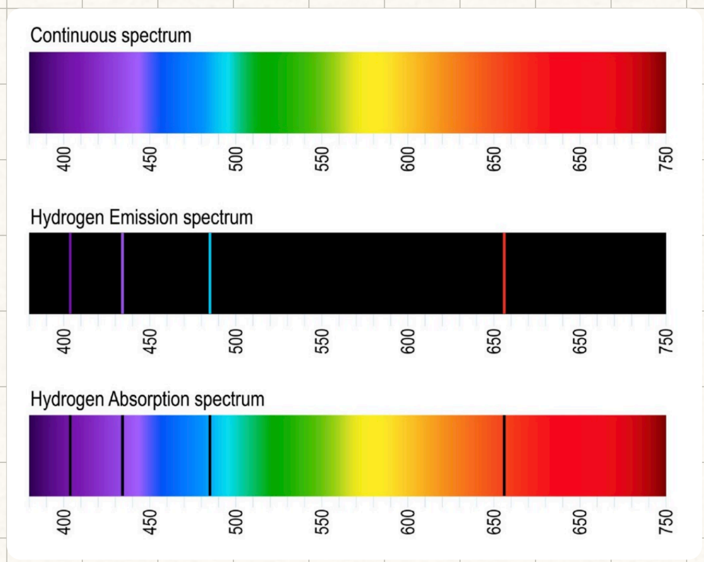
$E = \frac{1}{4\pi\varepsilon_0} \cdot e^2 \cdot \frac{me^2}{4\pi\varepsilon_0 n\hbar^2} + \frac{1}{2}m\left(\frac{n\hbar}{mr}\right)^2 = -13.6\,eV \cdot \frac{1}{n^2}$

↓

$\Delta E = h\nu = h\frac{c}{\lambda} = -13.6eV\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \Rightarrow \frac{1}{\lambda} = R_H\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$ □

m, n → Initial State / Final State.

Emission spectra // Absorb spectra.



Line broadening

Natural : Life time related.

Thermal : ↑ Speed → Dopler Effect.

Pressure : ↑ Collision ↓ life time ↑ emitted

PHAS0004 - Atoms, Stars and The Universe
Problem Solving Tutorial Sheet 2 - 2025

Questions are made available approximately one week before the PST. Please attempt the problem sheet in advance of the PST class. A solution sheet will be made available after all PSTs have taken place. **Please bring an electronic copy of this question sheet to the PST.**

Objectives:

1. Perform a dimensional analysis on Planck's constant.
2. Practise normalising wavefunctions and calculating their properties, including expectation values.
3. Practise with integration and differentiation which arises very often in problems of this kind.
4. Practise using the TISE to calculate energies for electrons in the hydrogen atom.

Useful definitions

Planck's constant h is 6.6×10^{-34} Js (2 s.f.).

The time-independent Schrödinger equation (TISE) for a particle in a one-dimensional potential $V(x)$ with mass m, energy E with wavefunction $\psi(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

1.a) $E = hf = \frac{1}{2}mv^2 = [kg]\cdot[m^2/s^2]$

$h = E/f = [kg][m^2/s^2]/[1/s] = [kg\ m^2/s]$

$\hbar = h/2\pi$ Totally same in dimention.

1.b) $L = mv \times r$

$= [kg][m/s][m]$

$= [kg\ m^2/s]$

1: The dimensions of h

In Physics, studying the dimensions of quantities can be a very important way of discovering relationships (or potential relationships) between them. Here we will consider the dimensions arising in Planck-Einstein's photon energy law and Bohr's atomic model.

☆ dimentions：单位//量纲

1.a) Write down the dimensions of Planck's constant h in terms of the fundamental quantities of mass M, length L and time T, taking the definition of h to be through the Planck-Einstein relationship $E = hf$. Why does $\hbar$ have the same dimensions as h?

1.b) Find the dimensions (in terms of mass M, length L and time T) of angular momentum and show that these are the same as those of Planck's constant h. Hence show that Bohr's quantisation assumption $L = n\hbar$ (where L is the angular momentum and n is an integer) is dimensionally consistent.

$2d.\ d|\psi(x)|^2/dx = 0$

$-8e^{-8x}+20e^{-10x}-12e^{-12x}=0$

$-2e^{4x}+5e^{2x}-12=0$

$2x^2-5x+3=0$

$x=\frac{\ln 3/2}{2}$

☆ wavefunction: $\psi(x) \rightarrow$ probability density $|\psi(x)|^2$

2: Calculating properties of wavefunctions

Consider a particle with the wavefunction $\psi(x)$

☆ Normalize $|\psi^2(x)|$ instead $|\psi(x)|$

$A^2 \int \exp^{-8x}(1-e^{-2x})^2 \cdot dx$

$(1-2e^{-2x}+e^{-4x})$

$\int_0^\infty \exp -8x - 2e-10x + e-12x \cdot dx$

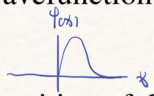
$-1/8(-1)+2\cdot 1/10\cdot(-1)+1/12\cdot(-1)$ $\quad A=\sqrt{120}$

$1/8-1/5-1/12=1/120$

$$\psi(x) = \begin{cases} Ae^{-4x}\left(1-e^{-2x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

2a) Find the value of A which normalises the wavefunction $\psi(x)$.

2b) Sketch the wavefunction $\psi(x)$.



2c) Calculate the expectation value $\langle x\rangle$ for the position of the particle. $\langle x\rangle = \int_{-\infty}^{+\infty} x\cdot f(x)\,dx = 0.31$

2d) Use the probability density function to determine the most probable position of the particle.

2e) Explain the difference between the most probable position and the expectation value.

peak  average

3: Which functions are valid wavefunctions?

To represent a physical particle, wavefunctions must be continuous and normalisable. Consider the following functions. Can they represent a wavefunction for a physical system? If not, explain why.

3a)

$$\int_{-\infty}^{\infty} \cos^2(px/\hbar) \quad f_1(x) = \cos(px/\hbar) \quad \times$$

3b)

$$\int_0^1 \cos^2(4\pi x) - 2\cos(4\pi x)\cos(2\pi x) + \cos(2\pi x)^2\, dx \quad f_2(x) = \begin{cases} \cos(4\pi x) - \cos(2\pi x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \checkmark$$

3c)

$$f_3(x) = \begin{cases} \cos(\pi x) & x \leq 0 \\ \sin(\pi x) & x > 0 \end{cases}$$

× (due none continues).

3d)

$$f_4(x) = e^{-2x}$$

× left part blow

3e)

$$f_5(x) = e^{-x^2}$$

✓

r²a·exp(br)·b

a exp(br)

r²·a·exp(br)·b·

$2b + rb^2$

4: The hydrogen atom

$-\frac{\hbar^2}{2mr^2}(2r\cancel{a\exp(br)}\cdot b + r^2 b^2\cdot\cancel{a\exp(br)})$
$+ V(r)\cdot\cancel{a\exp(br)} = E\cdot\cancel{a\exp(br)}$

The quantum mechanical model of the hydrogen atom can be calculated in the Born-Oppenheimer approximation, in which the nucleus is treated as stationary and taken as the origin of a spherical polar coordinate system. The radial component of the time-independent Schrödinger equation (TISE) for spherically symmetric electron wavefunctions can be written as,

$$-\frac{\hbar^2}{2mr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\psi(r)}{\mathrm{d}r}\right) + V(r)\psi(r) = E\psi(r)$$

where r is the radial electron position relative to the nucleus, m is the mass of the electron, E is the electron energy, $\psi(r)$ is the radial wavefunction and $V(r)$ is the Coulomb potential given by

$$V(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r},$$

where e is the magnitude of the electron charge. The radial probability density function in this situation is given by $\rho = 4\pi r^2 |\psi|^2$ with normalisation $\int_0^\infty \rho(r)dr = 1$.

4a) Show that $\psi(r) = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$ is a solution to the TISE for the hydrogen atom and show that the energy of this state is $E = -\frac{e^2}{8\pi\epsilon_0 a_0}$.
[Hint: What expressions for a_0 and E do you obtain by equating the coefficients of powers of r?]

4b) Determine the most probable value of r for this wavefunction in terms of a_0. What is the quantity a_0?

$d\rho/dr = 0$ $r = \sim$ $\rho = \sim$

4c) Another solution to the hydrogen atom TISE is

$$\psi(r) = \frac{1}{4\sqrt{2\pi a_0^3}}\left(2 - \frac{r}{a_0}\right)e^{-r/2a_0}.$$

For this state show that the energy is given by

$$E = -\frac{e^2}{8\pi\epsilon_0 a_0}\frac{1}{n^2}$$

and determine the value of the quantum number n.

X.

Ryan Nichol, October 2025