Modern Quantum Mechanics

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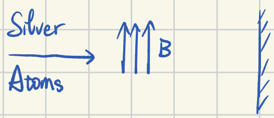
1.1 Experiment

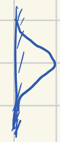
- Magnetic Moment μ $\quad U = -\mu \cdot B \quad F = \mu \times B \quad$ define as $\mu = I \cdot$ Enclosed Area



i). Silver 47 electron $\rightarrow$ 46 Steady | 1 Self Spin

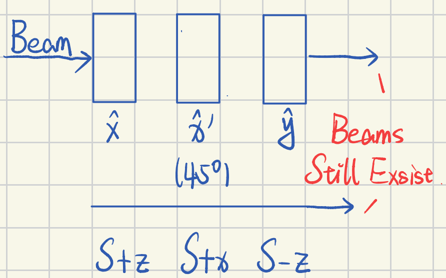
ii). $\mu \propto$ Spin s , $F = \mu_z \cdot \partial B_z / \partial z$

iii). Expected $\mu \in -|\mu| \sim |\mu|$ — Quantum $\mu = \mu_+$ or μ_- with $S_{z\pm} = \pm \hbar/2$.

$\Rightarrow$  (p of detected). $\quad \Rightarrow$ 

Iv). Cancel $S_z -$ Add B_x $S_{z\pm}$ Both Appeard Again.

- Polarized light experiment.



$S_{\pm z} \leftrightarrow x, y$ polarized

$S_{\pm x} \leftrightarrow x', y'$ polarized

noted as "ket" sign in Dirac notation

$E = E_0 \hat{x} \cos(kz - \omega t) = |S_z +\rangle$

$= E_0 \hat{y} \cos(kz - \omega t) = |S_z -\rangle$

$E_0 \hat{x}' \cos(\omega) = E_0 [\cos 45^\circ \cdot \hat{x} \cos(\omega) + \cos 45^\circ \cdot \hat{y} \cos(\omega)]$

$E_0 \hat{y}' \cos(\omega) = E_0 [-\cos 45^\circ \cdot \hat{x} \cos(\omega) + \cos 45^\circ \cdot \hat{y} \cos(\omega)]$.

$\Leftarrow$

$|S_x +\rangle = 1/\sqrt{2}\, |S_z +\rangle + 1/\sqrt{2}\, |S_z -\rangle$

$|S_x -\rangle = -1/\sqrt{2}\, |S_z +\rangle + 1/\sqrt{2}\, |S_z -\rangle$

1.2 Kets Bras Operators

Ket Space

Observe System in Vector Space $\longrightarrow$ State is n-dimention vector.

Define Ket Space as Vector space $\quad |\alpha\rangle + |\beta\rangle = |\gamma\rangle \quad c \cdot |\alpha\rangle = |\alpha\rangle \cdot c$

$A|\alpha\rangle = a|\alpha\rangle$ For A is observable $\quad |\alpha'\rangle, |\alpha''\rangle \cdots$ is eigenket of $|\alpha\rangle$

e.g. $S_z |S_z \pm\rangle = \pm \hbar/2\, |S_z \pm\rangle$

Since Linear Additive $\quad |\alpha\rangle = \Sigma_{\alpha'} C_{\alpha'} |\alpha'\rangle$

Bra Space - Inner Produce.

Define $\langle\alpha|$ corresponding to $|\alpha\rangle$ $\quad$ Dual Correspondence

$|\alpha\rangle \; |\alpha'\rangle \; |\alpha''\rangle \cdots \; |\alpha\rangle + |\beta\rangle \xleftrightarrow{DC} \langle\alpha| \; \langle\alpha'| \; \langle\alpha''| \cdots \; \langle\alpha| + \langle\beta|$

☆ $c|\alpha\rangle \xleftrightarrow{DC} c^*\langle\alpha|$ Define innerproduct $\langle\beta|\alpha\rangle = (\langle\beta|)\cdot(|\alpha\rangle) \xrightarrow{DC} = \langle\alpha|\beta\rangle^*$

- $\langle\alpha|\alpha\rangle \geq 0$, Real Number

- $\langle\alpha|\beta\rangle = 0$ Orthogonal on α β

Operators

$X\cdot(|\alpha\rangle) = X|\alpha\rangle$ $X=Y$ if $X|\alpha\rangle = Y|\alpha\rangle$ if $X|\alpha\rangle = 0$ X is null

☆ $X|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|X^\dagger$ if $X = X^\dagger$ X is Hermitian.

Define outer product $(|\beta\rangle)(\langle\alpha|) = |\beta\rangle\langle\alpha|$

Associative Axiom

- $(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle = |\beta\rangle\cdot(\langle\alpha|\gamma\rangle)$

- if $X = |\beta\rangle\langle\alpha|$ $X^\dagger = |\alpha\rangle\langle\beta|$

- $(\langle\beta|)\cdot(X|\alpha\rangle) = (\langle\beta|X)\cdot(|\alpha\rangle) \rightarrow \langle\beta|X|\alpha\rangle$

- $\langle\beta|X|\alpha\rangle = \langle\beta|\cdot(X|\alpha\rangle) = \{(\langle\alpha|X^\dagger)|\beta\rangle\}^* = \langle\alpha|X^\dagger|\beta\rangle^*$

1.3 Base kets and matrix representations

- Eigenvalue of Hermitian operator A is real

  Eigenkets of A to dif. eigenvalues are orthnormal

Proof. $A|a'\rangle = a'|a'\rangle$ $\langle a''|A = a''^*\langle a''|$

$\Downarrow$

$0 = \langle a''|a'|a'\rangle - a''^*\langle a''|a'\rangle$ $0 = \underline{\langle a''|a'\rangle}(a' - a''^*)$

If:

i). $a'' = a' \rightarrow a' = a'^*$

ii). $a'' \neq a' \rightarrow \langle a''|a'\rangle = 0$ Each eigenvalues orthnormal

$\langle a''|a'\rangle = \delta_{a'a''}$ if $a' \neq a''$ $= 0$ $\Lambda^2_{a'}$: Project on a'

- Base Kets. $|\alpha\rangle = \sum C_{a'}|a'\rangle$ $\sum \Lambda^2_{a'} = 1$: Project on all a Must $= 1$

$\langle a''|\alpha\rangle = \langle a''|\sum C_a|a'\rangle = \sum_{a'} C_{a'}\delta_{a'a''} \Rightarrow C_{a'} = \langle a'|\alpha\rangle$

$|\alpha\rangle = \sum_{a'}\langle a'|\alpha\rangle\cdot|a'\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle \Rightarrow \boxed{\sum_{a'}|a'\rangle\langle a'| = 1}$

$\Updownarrow$

$V = \sum_i \hat{e}_i(\hat{e}_i\cdot V)$ ☆ $\langle a'|$ must orthnormal with $|\alpha\rangle$

Since $\sum |a'\rangle\langle a'| = 1 \Rightarrow \langle\alpha|\alpha\rangle = \langle\alpha| \sum_{a'} |a'\rangle\langle a'| |\alpha\rangle = \sum_{a'} |\langle a'|\alpha\rangle|^2 \Rightarrow \sum_{a'} |c_{a'}|^2 = \sum_{a'} |\langle a'|\alpha\rangle|^2 = 1$

☆ $(|\alpha'\rangle\langle\alpha'|) \cdot |\alpha\rangle = |\alpha'\rangle\langle\alpha'|\alpha\rangle = C_{a'}|\alpha'\rangle \Rightarrow |a'\rangle\langle a'|$ porcion of $|\alpha'\rangle$ (for $|\alpha\rangle$ is normalized).

define $\Lambda_{a'} = |a'\rangle\langle a'|$ $\quad \sum \Lambda_{a'} = 1$. $\Lambda_{a'}$ Subtract valv. of proportion of a' on a

Matrix Representation. ☆ $|a'\rangle\ |a''\rangle \cdots$ From eigenket list. No dif.

Define $X = \sum_{a'}\sum_{a''} |a''\rangle\langle a''| \sum |a'\rangle\langle a'|$

Convert to $N \times N$ Matrix where N is dim. of ket space

$$X = \begin{bmatrix} \langle a''_1|X|a'_1\rangle & \langle a''_1|X|a'_2\rangle & \cdots \\ \langle a''_2|X|a'_1\rangle & \ddots & \\ \vdots & & \ddots \end{bmatrix} \Rightarrow \langle a''|X|a'\rangle = \langle a'|X^\dagger|a''\rangle^*$$

Insert Identity Matrix

- For $Z = XY$ $\quad \langle a''|Z|a'\rangle = \sum_{a'''} \langle a''|X|a'''\rangle\langle a'''|Y|a'\rangle$

- For $|\gamma\rangle = X|\alpha\rangle$ Acting on base $\langle a'|$ $\quad \langle a'|\gamma\rangle = \langle a'|X|a\rangle = \sum_{a''} \underbrace{\langle a'|X|a''\rangle}_{\text{Matrix of } X}\underbrace{\langle a''|a\rangle}_{\text{Porcion of old vectors.}}$

$$|\alpha\rangle = \begin{pmatrix} \langle a^1|a\rangle \\ \langle a^2|a\rangle \\ \vdots \end{pmatrix} \qquad |\gamma\rangle = \begin{pmatrix} \langle a^1|\gamma\rangle \\ \langle a^2|\gamma\rangle \\ \vdots \end{pmatrix}$$

As column Matrix.

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle = \begin{bmatrix} |a'_1\rangle\langle a'_1|\alpha\rangle \\ |a'_2\rangle\langle a'_2|\alpha\rangle \\ \vdots \end{bmatrix} = \begin{pmatrix} \langle a'_1|\alpha\rangle \\ \langle a'_2|\alpha\rangle \\ \vdots \end{pmatrix}$$

☆ $|a_1\rangle\ |a_2\rangle \cdots$ Have no valv. — only directions.

- Similar on $\langle\gamma| = \langle\alpha|X$ $\quad \langle\gamma|a'\rangle = \sum_{a''} \langle\alpha|a''\rangle\langle a''|X|a'\rangle$

$\langle\gamma| = (\langle\gamma|a^1\rangle, \langle\gamma|a^2\rangle \cdots) = (\langle a^1|\gamma\rangle, \langle a^2|\gamma\rangle \cdots)^*$ As row Matrix.

- Column Matrix · Raw Matrix. $\quad \langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle$

$$= (\langle\beta|a_1\rangle\ \langle\beta|a_2\rangle \cdots) \cdot \begin{pmatrix} \langle a_1|a\rangle \\ \langle a_2|a\rangle \\ \vdots \end{pmatrix} = \text{Some value}$$

- Raw · Column $\quad |\beta\rangle\langle\alpha| = \sum_{a'} \langle a'|\beta\rangle\langle\alpha|a'\rangle = \begin{bmatrix} \langle a^1|\beta\rangle\langle a^1|\alpha\rangle^* & \langle a^1|\beta\rangle\langle a^2|\alpha\rangle^* & \cdots \\ \langle a^2|\beta\rangle\langle a^1|\alpha\rangle^* & \ddots & \vdots \\ \vdots & & \ddots \end{bmatrix}$

- Eigenkets of A used as baseket. [$\sum_{a'} |a'\rangle\langle a'| = 1$ — a' is baseket]

$A = \sum_{a''}\sum_{a'} |a''\rangle\langle a''|A|a'\rangle\langle a'| = $ If $\langle a'|A|a''\rangle$ where $a' \neq a''$ $= 0 \Rightarrow \delta_{a''a'}\langle a'|A|a'\rangle = a'\delta_{a''a'}$

$A = \sum_{a'} a'|a'\rangle\langle a'| = \sum_{a'} a'\Lambda_{a'}$

☆ defin $S_\pm$ as Up down operators

Spin ½ Systems

$S_\pm = \hbar|\pm\rangle\langle\mp|$

Set $|-\rangle$ to $\hbar|+\rangle$

$1 = \sum_{a'}|a'\rangle\langle a'| = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$ $\quad S_+|-\rangle = \hbar|+\rangle\langle -|-\rangle = \hbar|+\rangle$

$S_z = \sum_{a'} \underbrace{a'}_{\text{eigen value}}\underbrace{\Lambda_{a'}}_{\text{eigenket}} = (\hbar/2)|\uparrow\rangle\langle\uparrow| - (\hbar/2)|\downarrow\rangle\langle\downarrow| = \hbar/2(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$ $\quad S_z|\updownarrow\rangle = \pm(\hbar/2)|\updownarrow\rangle$

as $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $\quad S_z = \hbar/2\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

1.4 Measurements, Observables, Uncertainty

$|\alpha\rangle = \sum \underset{i}{\langle a'|}\underset{j}{\alpha\rangle}\cdot\underset{k}{|a'\rangle} = \sum_{a'} \underset{k}{|a'\rangle}\underset{i}{\langle a'|}\underset{j}{\alpha\rangle}$

$|\alpha\rangle$ [Series of possiple eigenvalue] $\xrightarrow{\text{Observe}}$ $|a'\rangle$ [Singele ob. value]

$|a'\rangle \xrightarrow{\text{Observe}} |\alpha'\rangle$

Prob. of a' observed $= |\Lambda_{a'}\cdot\alpha|^2 = |\underbrace{|a'\rangle}_{\text{Vector}}\underbrace{\langle a'|\alpha\rangle}_{\text{Constant}}|^2$.

$|a'\rangle^2 = \langle a'|a'\rangle$

$= |\text{Constant}|^2\cdot|\text{Vector}|^2 = |\langle a'|\alpha\rangle|^2\cdot\underbrace{\langle a'|a'\rangle}_{=1}$

$\langle A\rangle = \sum_{a'} \underbrace{a'}_{\text{valv.}}\underbrace{|\langle a'|\alpha\rangle|^2}_{P.}$

$P = |\langle a'|\alpha\rangle|^2$

$= \sum_{a'}\sum_{a''}\langle\alpha|a''\rangle\underbrace{\langle a''|A|a'\rangle}_{\delta_{a''a'}a'}\langle a'|\alpha\rangle = \sum_{a'}\sum_{a''}\langle a'|\alpha\rangle\langle a''|\alpha\rangle^*\,\delta_{a'a''}\,a'$

$= \sum_{a'}|\langle a'|\alpha\rangle|^2\, a'$.

Spin ½ System, again.

$|S_x:+\rangle$ on S_z direction $= \vec{A}|+\rangle + \vec{B}|-\rangle$

with $|A| = |B| = 1/\sqrt{2}$ and direction unknown.

$P = \sum|\Lambda_{a'}\cdot\alpha|^2 = 1$ with Same P. on $S_z\pm$ for S_x+ $\quad 1/\sqrt{2}|+\rangle + 1/\sqrt{2}\exp(i\delta_1)|-\rangle \Leftarrow 1/\sqrt{2}\exp(i\alpha)|+\rangle + 1/\sqrt{2}\exp(i\beta)|-\rangle$

$|\underbrace{\langle +|}_{S_z+}\underbrace{S_x:+\rangle}_{S_x+}| = |\underbrace{\langle -|}_{S_z-}\underbrace{S_x:+\rangle}_{S_x+}| = 1/\sqrt{2}$ $\quad \Rightarrow |S_x:+\rangle = 1/\sqrt{2}|+\rangle + 1/\sqrt{2}\underbrace{\exp(i\delta_1)}_{\text{phase difference in } |+\rangle\,|-\rangle}|-\rangle$

Since $|S_x:+\rangle$ orthnormal with $|S_x:-\rangle$. $\langle +|-\rangle = 0$ $\Rightarrow |S_x:-\rangle = 1/\sqrt{2}|+\rangle - 1/\sqrt{2}\exp(i\delta_1)|-\rangle$

$S_x = \sum_{a'} a'\Lambda_{a'} = \hbar/2[|S_x+\rangle\langle S_x:+| - |S_x:-\rangle\langle S_x:-|] = \hbar/2[e^{-i\delta_1}|+\rangle\langle -| + e^{i\delta_1}|-\rangle\langle +|]$

Similar
$$\left[\begin{array}{l} |S_y:\pm\rangle = 1/\sqrt{2}|+\rangle \pm 1/\sqrt{2}\exp(i\delta_2)|-\rangle \\ S_y = \sum_{a'}\Lambda_{a'}a' = \hbar/2[\exp(-i\delta_2)|+\rangle\langle -| + \exp(i\delta_2)|-\rangle\langle +|] \end{array}\right.$$

$\Rightarrow S_x = \begin{pmatrix} 0 & \exp(-i\delta_1)\\ \exp(i\delta_1) & 0\end{pmatrix}$

To determine $\delta_2\ \delta_1$ $\quad |\langle S_y:\pm|S_x:+\rangle| = |\langle S_y:\pm|S_x:-\rangle| = 1/\sqrt{2}$ $\quad \Rightarrow S_y = \begin{pmatrix} 0 & \exp(-i\delta_2)\\ \exp(i\delta_2) & 0\end{pmatrix}$

$= [1/\sqrt{2}\langle +| + 1/\sqrt{2}\exp(i\delta_2)\langle -|][1/\sqrt{2}|+\rangle - 1/\sqrt{2}\exp(i\delta_1)|-\rangle]$

$= \frac{1}{2} - \frac{1}{2}\exp[i(\delta_2-\delta_1)] = 1/\sqrt{2}$ $(\langle +|+\rangle = 1 \quad \langle -|+\rangle = 0)$

$\Rightarrow \delta_2 - \delta_1 = \pm\pi/2$ ☆ $\exp(i\theta) \in \mathrm{Re}$ when $\theta \in 0, \pi, 2\pi \cdots$

$\delta_2 - \delta_1 = \pi/2 \rightarrow$ Must have one imagine. Assume $x \in \mathrm{Re}$. $y \in \mathrm{Im}$.

$S_x = \hbar/2 [|+\rangle\langle -| + |-\rangle\langle +|]$ $|S_x; \pm\rangle = 1/\sqrt{2}|+\rangle \pm 1/\sqrt{2}|-\rangle$

$S_y = \hbar/2 [-i|+\rangle\langle -| + i|-\rangle\langle +|]$ $|S_y; \pm\rangle = 1/\sqrt{2}|+\rangle \pm i/\sqrt{2}|-\rangle$

☆ $S_\pm = \hbar|\pm\rangle\langle\mp| \rightarrow S_\pm = S_x \pm iS_y = \hbar|\pm\rangle\langle\mp|$ [Up / Down degree operators].

with $S_x S_y S_z$ $[S_x, S_y] = \hbar^2/4 [i|+\rangle\langle +| - i|-\rangle\langle -| + i|+\rangle\langle +| - i|-\rangle\langle -|]$

$= i\hbar^2/2 [|+\rangle\langle +| - |-\rangle\langle -|] = S_z \cdot i\hbar$ $[S_z = (\hbar/2)[|+\rangle\langle +| - |-\rangle\langle -|]]$ $\Rightarrow [S_i, S_j] = \varepsilon_{ijk} \cdot S_k \cdot i\hbar$

Similar Anti - Commutation $\{S_x S_y\} = \frac{1}{2}\hbar^2 \delta_{ij}$ $[\{A, B\} = AB + BA]$

> Define $S^2 = S_x^2 + S_y^2 + S_z^2 = 3 \cdot (1/4\hbar^2)$ $[S^2, S_i] = 0$

Compatible Observables.

define $[A, B] = 0$ compatible vise versa.

☆ degenerate : two dif. eigen state → same correspond eigen value ⇒ $|a'\rangle$ isn't complete.

> Theory : A, B compatible eigenvalue of A nonedegenerate. $\langle a''|B|a'\rangle$ all diagonal

$\langle a''|[A, B]|a'\rangle = 0$

$\langle a''|AB - BA|a'\rangle = \langle a''|AB|a'\rangle - \langle a''|BA|a'\rangle = \langle a''|a'')B|a'\rangle - \langle a''|B(a'|a'\rangle = \langle a''|B|a'\rangle(a'' - a') = 0$

Obv. $a' = a''$ or $\langle a''|B|a'\rangle = 0$. $\Rightarrow$ $\langle a''|B|a'\rangle = \delta_{a'a''}\langle a'|B|a''\rangle$. → Diagnal Matrix.

> Simutaneous Eigenkits.

$B = \sum_{a''} |a''\rangle\langle a''|B|a''\rangle\langle a''|$ adding on ket $|a'\rangle$ = $\sum_{a''} |a''\rangle\langle a''|B|a''\rangle\underbrace{\langle a''|a'\rangle}_{\delta_{a'a''}} = (\langle a'|B|a'\rangle)\,|a'\rangle$ Simotanious Eigenkets

$A|a', b'\rangle = a'|a', b'\rangle$

define $|K'\rangle$ collective index ← ⟵ ☆ $B|a'\rangle = b'|a'\rangle$ with $b' = \langle a'|B|a'\rangle$

$A|a'\rangle = a'|a'\rangle$

$|K'\rangle = |a', b'\rangle$ $B|a', b'\rangle = b'|a', b'\rangle$ AB Use Same eigenstate $|a', b'\rangle$

> Degeneracy Notations.

$|K'\rangle = |a', b', c' \cdots\rangle$ (Assume compatible)

obv. $\sum_{k'} |K'\rangle\langle K'| = 1$ $\langle K''|K'\rangle = \delta_{k''k'}$

$|\alpha\rangle \xrightarrow{\text{Measure } A} |a', b'\rangle$ with same value a' $\xrightarrow{\text{Mea. } B}$ $|a', b'\rangle$ with dif. b'

degeneracy in a — dif. in b'

☆ degeneracy → Linear combination of kets. $= \sum_i^n c_{a'}^i |a', b^i\rangle$ [For n degeneracy A state, measure gives a'
B state, gives $b^1, b^2 \cdots b^n$]

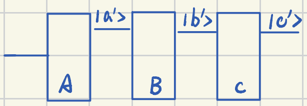
> Incompatible // Collective Index

For $[A, B] \neq 0$

$$AB|a', b'\rangle = a'b'|a', b'\rangle$$
$$BA|a', b'\rangle = b'a'|a', b'\rangle$$

$\nearrow$ $AB = BA$ $[A, B]$ must $= 0$. (x)

For $[A, B] \neq 0$ $|a', b'\rangle$ not exsist.



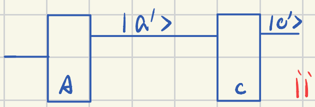
i)

P of output $= \underbrace{|\langle c'|b'\rangle|^2}_{c \text{ in } b} \underbrace{|\langle b'|a'\rangle|^2}_{b \text{ in } a}$

As B have eigen states. summny over b' for all possible routes.

$$\sum_{b'} |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2 = \sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle \quad (\text{Eq. 1})$$

Difference in b' and b'' value.



ii).

$$P_{out} = |\langle c'|a'\rangle|^2 = \sum_{b'} |\langle c'|b'\rangle\langle b'|a'\rangle|^2 = \sum_{b'}\sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle \quad (\text{Eq. 2})$$

State in i). $|b'\rangle$ is detected. just left normal P. adding together.

State in ii). $\sum_{b'}\sum_{b''}$ cause interference pare when $b' \neq b''$

For $[A, B] = 0$: S-M. eigenkets $|a\rangle$ have $B|a'\rangle = b'|a\rangle$ $\langle b|a\rangle = \delta_{bb'}$

☆ $\text{Eq. 1} = |\langle c|b'\rangle\langle b'|a\rangle|^2 = \text{Eq. 2} = |\langle c|b'\rangle|^2 |\langle b'|a\rangle|^2$ [Only $b = b'$ exsist].

Interference $\sum_b \sum_{b'}$ cause by diffenent possible path. $[A, B] = 0$ // Test on $|b\rangle$

will left only one path of $|b\rangle$ → kill intef.

> Uncertainty Relation.

Define $\Delta A = A - \langle A\rangle$ As operators $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$ as dispersion.

$\langle(\Delta A)^2\rangle$ Assign to "sharpness" of value.

$$\langle(\Delta S_x)^2\rangle = \hbar^2/4 - 0 = \hbar^2/4 \quad \rightarrow \text{Fuzzy}$$
$$\langle(\Delta S_{z+})^2\rangle = \hbar^2/4 - \hbar^2/4 = 0 \quad \rightarrow \text{Sharp}$$

Theory : $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2$

Proof on this theory require three Lemmas.

> Lemma 1. In Hib. Space $|\lambda|^2 \geq 0$ For $|\lambda\rangle = |\alpha\rangle + \lambda|\beta\rangle$

$$(\langle\alpha| + \lambda^*\langle\beta|)\cdot(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

$$\langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\alpha|\beta\rangle^* + |\lambda|^2\langle\beta|\beta\rangle \geq 0$$

Since λ is free, Set $\lambda = -\langle\beta|\alpha\rangle/\langle\beta|\beta\rangle$ To cancle terms.

$$\langle\alpha|\alpha\rangle - |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle + \lambda^*\langle\alpha|\beta\rangle + |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle \Rightarrow \langle\alpha|\alpha\rangle - |\langle\alpha|\beta\rangle|^2/\langle\beta|\beta\rangle \geq 0 \quad \square$$

> Lemma 2. Expection Value of Hermiton operators must real.

$$\langle\alpha'|H|\alpha''\rangle = \langle\alpha''|H^\dagger|\alpha'\rangle^* = \langle H\rangle \rightarrow \text{Real} \quad \square$$

> Lemma 3. Vice Versa on Lemma 2. $\square$

> Proof. Lm. 1 $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

Define operator $|\alpha\rangle = \Delta A|\alpha\rangle$, $|\beta\rangle = \Delta B|\beta\rangle$

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\,\Delta B\rangle|^2$$

$$\Delta A\Delta B = \tfrac{1}{2}[\Delta A\Delta B - \Delta B\Delta A + \Delta A\Delta B + \Delta B\Delta A] = \tfrac{1}{2}[\Delta A, \Delta B] + \tfrac{1}{2}\{\Delta A, \Delta B\}$$

$$\langle\Delta A\Delta B\rangle = \tfrac{1}{2}\langle[\Delta A, \Delta B]\rangle + \tfrac{1}{2}\langle\{\Delta A, \Delta B\}\rangle$$

$[\Delta A, \Delta B]$: Since Commutation Relation on $[A, \text{Constant Value}] = 0$

$$= [A, B] - [\langle A\rangle, B] - [A, \langle B\rangle] + [\langle A\rangle, \langle B\rangle] = [A, B]$$

Lemma 2/3 $[\Delta A, \Delta B]^\dagger = [A, B]^\dagger = BA - AB = -[B, A] \longrightarrow \langle[A, B]\rangle$ Pure Im.

$\{\Delta A, \Delta B\}^\dagger = \Delta B\Delta A + \Delta A\Delta B = \{\Delta A, \Delta B\} \longrightarrow \{\Delta A, \Delta B\}$ Pure Re.

$$\langle(\Delta A)^2\rangle + \langle(\Delta B)^2\rangle = \tfrac{1}{4}|\langle[A, B]\rangle|^2 + \tfrac{1}{4}|\langle\{\Delta A, \Delta B\}\rangle|^2 \geq \tfrac{1}{4}|\langle[A, B]\rangle|^2 \quad \square$$

1.5 Change of basis.

Transforming Operators. [Define $|b^1\rangle = U|a^1\rangle$, $|b^2\rangle = U|a^2\rangle$, ... with U is unitary operators]

$U^\dagger U = UU^\dagger = 1$

Assume For $U = \sum_k |b^k\rangle\langle a^k|$ $\quad U\cdot U^\dagger = \sum_k\sum_l |b^k\rangle\langle a^k|a^l\rangle\langle b^l|$

$$= \sum_k |b^k\rangle\delta_{kl}\langle b^k| = 1$$

Transforming Matrix. [Transforming $\{|a'\rangle\}$ to $\{|b'\rangle\}$]

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$ Transform to $\langle b^k|\alpha\rangle = \sum_l \langle b^k|a^l\rangle\langle a^l|\alpha\rangle$ with $\langle b^k| = \langle a^k|U^\dagger\rangle$

(new) $= U^\dagger$ (old) $= \sum_l \langle a^k|U^\dagger|a^l\rangle\langle a^l|\alpha\rangle$

$$\langle b^k|X|b^l\rangle = \sum_m\sum_n \langle b^k|a^n\rangle\langle a^n|X|a^m\rangle\langle a^m|b^l\rangle$$

with $|b\rangle = U|\alpha\rangle$ $= \sum_{m,n} \langle a^k|U^\dagger|a^n\rangle\langle a^n|X|a^m\rangle\langle a^m|U|b^l\rangle$

$\Rightarrow X' = U^\dagger X U$

> define Trace $tr(X) = \sum_{a'} \langle a'|X|a'\rangle$

$tr(X)$ isn't depends on basis. $= \sum_{a'b'b''} \underbrace{\langle a'|b'\rangle\langle b'|X|b''\rangle\langle b''|a'\rangle}_{\text{Three scalors} \to \text{change position freely}} = \sum_{b'b''} \underbrace{\langle b''|b'\rangle}_{\delta_{b'b''}}\langle b'|X|b''\rangle$

$= \sum_{b'} \langle b'|X|b'\rangle = Tr(X)$

Diagonalization

For $B|b'\rangle = b'|b'\rangle$

$\sum_{a''} \langle a'|B|a''\rangle\langle a''|b'\rangle = b'\langle a'|b'\rangle$. as $\langle a'|$ represent "direction"

$$\begin{pmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} = b'(b'=\lambda) \begin{pmatrix} C_1 \\ C_2 \\ \vdots \end{pmatrix} \to \det(B - \lambda 1) = 0 \quad \text{For } 1 \text{ is unit vector.}$$

For $B_{ij} = \langle a_i|B|a_j\rangle$ $\quad C_i = \langle a^i|b'\rangle$ $\quad$ N dimention function with $\lambda_1 \cdots \lambda_N \in b$

$\downarrow$ one col. of trs. Matrix

For Transforming Matrix define as $|b\rangle = U|\alpha\rangle$ $\quad U = \sum_{b'} \langle a^i|b'\rangle = \sum_{b'} C_i'$

Unitary Equivalent Observables. [Unitary Define as $UU^\dagger = I$ $\quad U^\dagger = U^{-1}$]

For sets $\{|a'\rangle\}$ $\{|b'\rangle\}$ Conneet by U. Unitary transformation on A is UAU^{-1}

$A|a^l\rangle = a^l|a^l\rangle$

$\downarrow$ Transformation

both $|a^l\rangle\, |b^l\rangle$ are eigenkets of UAU^{-1} with same eigenvalue a^l

$UAU^{-1}U|a^l\rangle = a^l U|a^l\rangle \longrightarrow (UAU^{-1})|b^l\rangle = a^l|b^l\rangle$

as $B|b^l\rangle = b^l|b^l\rangle$) UAU^{-1} Similar to B.

1.6 Position Momentum Translation

Continuous Spectra. [e.g. P_z is avilable in any value $(-\infty \sim +\infty)$ - Continues].

For Continues spectrum $\underbrace{\xi}_{\text{operators}}|\xi'\rangle = \underbrace{\xi'}_{\text{eigenvalues}}|\xi'\rangle$

$\langle a'|a''\rangle = \delta_{a'a''} \longrightarrow \langle \xi'|\xi''\rangle = \delta(\xi'-\xi'')$

$\sum_{a'} |a'\rangle\langle a'| = 1 \longrightarrow \int d\xi |\xi'\rangle\langle\xi'| = 1$

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \longrightarrow \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle$

$\sum_{a} |\langle a'|\alpha\rangle|^2 = 1 \longrightarrow \int d\xi' |\langle\xi'|\alpha\rangle|^2 = 1.$

$\langle\beta|\alpha\rangle = \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle \longrightarrow \langle\beta|\alpha\rangle = \int d\xi' \langle\beta|\xi'\rangle\langle\xi'|\alpha\rangle$

$\langle a''|A|a'\rangle = a'\delta_{a'a''} \longrightarrow \xi'\,\delta(\xi''-\xi').$

Position Eigenkets. — Position Measurement.

$x|x'\rangle = x'|x'\rangle.$

with $|\alpha\rangle = \int_{-}^{+} dx' |x'\rangle\langle x'|\alpha\rangle \xrightarrow{\text{Measure close to } (x'-\Delta/2,\, x'+\Delta/2).} \int_{x'-\Delta/2}^{x'+\Delta/2} dx'' |x''\rangle\langle x''|\alpha\rangle.$

$P(\text{on position } x') = |\langle x'|\alpha\rangle|^2 \cdot dx'$ with $\int_{-}^{+} |\langle x'|\alpha\rangle|^2 \cdot dx' = 1$ $[\langle\alpha|\alpha\rangle = 1]$

> Coordinates Operators.

$|\alpha\rangle = \int d^3x' |x'\rangle\langle x'|\alpha\rangle$ with $|x'\rangle = |x', y', z'\rangle$ Restrict by $[x_i, x_j] = 0.$