Math Method II

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Week 2

- Int. on dif coordinates

$x-y$: $dA = dx\,dy$

polar : $dA = r \cdot dr\,d\theta$

cylider
- Area : $dA = r \cdot d\theta\,dz$
- Volumn $dV = r\,d\theta\,dr\,dz$

sphere :
- Area : $r\sin\theta \cdot d\theta\,d\phi$
- Volumn $r^2 \sin\theta\,dr\,d\theta\,d\phi$

- Method on derive dA

$dr = \hat{e_x} \cdot dx + \hat{e_y} dy + \hat{e_z} dz$

e.g. Sphere $\quad d\vec{r} = d\vec{r}/dr \cdot dr + d\vec{r}/d\theta \cdot d\theta + d\vec{r}/d\phi \cdot d\phi$

$x = r\sin\phi\cos\theta \quad = \hat{e_r} \cdot dr + \hat{e_\theta} \cdot r \cdot d\theta + r\sin\phi \cdot d\phi$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

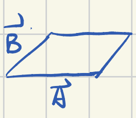
$$dr = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{matrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{matrix} \cdot dr + \begin{matrix} -\sin\phi\sin r \\ \sin\phi\cos r \\ 0 \end{matrix} \cdot r d\theta + \begin{matrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ -r\sin\phi \end{matrix} \cdot r\sin\phi \cdot dr$$

- Int. on scalar field.

eg. $f(x)$ on polar coordinates find int.

$f(x,y)$ with $x = r\cos\theta$ $y = r\sin\theta$ $\Rightarrow f(r,\theta)$ $\quad$ Area $= \int f(r,\theta) \cdot dA = \sim$

- Unit Area



$S = \vec{A} \times \vec{B}$ $\quad \xrightarrow{\text{Arbitany Surface}}$ $\quad \begin{matrix} \partial r/\partial p \cdot dP \text{ As } \vec{A} \\ \partial r/\partial \phi \cdot d\phi \text{ As } \vec{B} \end{matrix}$ $\quad ds = |\partial r/\partial p \times \partial r/\partial \phi|\, dP\, d\phi$

eg. Area Unit For sphere

$$r_\theta = \partial r/\partial\theta = \begin{vmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{vmatrix} \qquad r_\phi = \partial r/\partial\phi = \begin{vmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{vmatrix}$$

$$N = \hat{i}(R^2\sin^2\theta\cos\theta) + \hat{j}(R^2\sin^2\theta\sin\phi) + \hat{k}(R^2\cos^2\phi\sin\theta\cos\theta + R^2\sin^2\phi\cos\theta\sin\theta)$$

$$\downarrow$$

$$+ \hat{k}(R^2\sin\theta\cos\theta)$$

$A = |N| = R^2|\sin\theta| \rightarrow dA = R^2 \sin\theta \, d\theta \, d\phi$

- ∇, ∇^2 in polar coordinates

$\nabla = \partial/\partial x + \partial/\partial y + \partial/\partial z = \partial/\partial r(\partial r/\partial x + \partial r/\partial y + \partial r/\partial z) = \partial/\partial r(1/\sin\phi(1/\sin\theta + 1/\cos\theta) + 1/\cos\phi)$
$+ \partial/\partial\theta(\partial\theta/\partial x + \partial\theta/\partial y) \qquad \partial/\partial\theta(1/r\sin\phi(1 - - -)$
$+ \partial/\partial\phi(\cdots) \qquad + \quad - - -$

$$\begin{cases} x = r\sin\phi\sin\theta \\ y = r\sin\phi\cos\theta \\ z = r\cos\phi \end{cases}$$

X Obsolete Method

☆ General Laplacien + Scale factor method

dr in sph. coor. $= r \cdot d\theta \Rightarrow (\nabla\psi)_i = 1/h_i \cdot \partial\psi/\partial q_i$ as h_i is scale factor

$dr = h \, dq$

For $\psi(ijk)$ $\nabla\psi = \sum_{ijk} 1/h_\alpha \cdot \partial\psi/\partial q_\alpha$

$\nabla \cdot A = \lim_{V\to 0} \oint A \, ds / V$ with $V = (h_1 dq_1)(h_2 dq_2)(h_3 dq_3) = h_1 h_2 h_3 \, dq_1 dq_2 dq_3$

$\oint A \, ds$ = Flux 1. Left Side Flux $= A_{1(q_1)}(h_2 dq_2 \cdot h_3 dq_3)$

Right Side $(q_1 + dq_1)$ Flux $= (A_1 h_2 h_3)_{(q_1+dq_1)} \cdot dq_2 dq_3$

☆ Taylor

$= [(A_1 h_2 h_3)_{q_1} + \partial/\partial q_1 (A_1 h_2 h_3) \cdot dq_1] \cdot dq_2 dq_3$

Net Flux $= \partial/\partial q_1 (A_1 h_2 h_3) \cdot dq_1 dq_2 dq_3$

Add on Three sides : $= \underline{[\partial/\partial q_1 (A_1 h_2 h_3) + \partial/\partial q_2 (A_2 h_1 h_3) + \partial/\partial q_3 (A_3 h_1 h_2)]} \cdot dq_1 dq_2 dq_3$

$\nabla^2\psi = \nabla \cdot A = 1/h_1 h_2 h_3 \cdot [\qquad]$ with $A = 1/h \; \partial\psi/\partial q$

$= 1/h_1 h_2 h_3 \cdot [\partial/\partial q_1 \cdot (h_2 h_3/h_1 \cdot \partial\psi/\partial q_1) + \partial/\partial q_2 (h_1 h_3/h_2 \cdot \partial\psi/\partial q_2) + \partial/\partial q_3 \cdot (h_1 h_2/h_3 \cdot \partial\psi/\partial q_3)]$

eg. Sphere coord.

$q_1 q_2 q_3 = r, \theta, \phi$

$\Rightarrow \nabla^2\psi = 1/r^2 \; \partial/\partial r(r^2 \, \partial\psi/\partial r) + 1/r^2\sin\phi \cdot \partial/\partial\phi(\sin\phi \, \partial\psi/\partial\phi) + 1/r^2\sin^2\phi \; \partial^2\psi/\partial\theta^2$

with $h_1 = 1$ $\quad h_2 = r$ $\quad h_3 = r\sin\theta$

$dr = dr \quad dr = r d\theta \quad dr = r\sin\theta d\phi$