PHAS0005 - Wave - Week 1

ChengXuLin
Leo

Simple Harmonic Motion $F = kx$

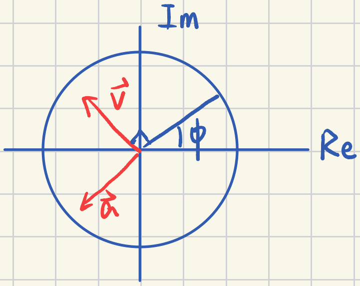
$$F = kx = -m\ddot{x}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\Rightarrow x = A\exp(\sqrt{k/m}\,t) = A\cos(\omega t + \varphi_0) = B\cos(\omega t) + C\sin(\omega t)$$

with $\omega = \sqrt{k/m}$, $B = A\cos\varphi$ $C = A\sin\varphi$

- Phasor Diagram



☆ Motion is exactly Re[ ]

For $\psi = A\exp(i\omega t + i\phi)$

$V = iA\omega\exp(i\omega t + i\phi)$

☆ V have $\pi/2$ phase different $a = -A\omega^2\exp(i\omega t + i\phi)$

a have π ~

- Damped Motion — $F = m\ddot{x} + j\dot{x} + kx = 0$

Assume Solution $\psi = A\exp(\alpha t)$

$$m\alpha^2 + j\alpha + k = 0$$

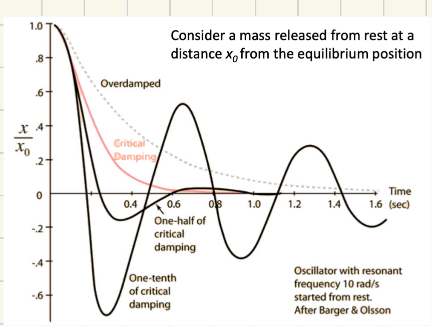
$$\Rightarrow \alpha = \frac{-j \pm \sqrt{j^2 - 4mk}}{2m}$$ To make it looks better → $\gamma = j/2m$

$$= -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$ For $\omega_0 = \sqrt{k/m}$

$$\Rightarrow \psi = A\exp((-\gamma + \sqrt{\gamma^2 - \omega_0^2})t) + B\exp((-\gamma - \sqrt{\gamma^2 - \omega_0^2})t)$$

i) $\gamma > \omega$ $\sqrt{\gamma^2 - \omega_0^2} > 0$ Overdamped

$\psi = A\exp(u_+ t) + B\exp(u_- t)$ ← $u_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$



☆ Graph Valiation: $A\exp(u_+ t) = -B\exp(u_- t)$

$A\exp(t\sqrt{\gamma^2 - \omega_0^2}) = -B\exp(-t\sqrt{\gamma^2 - \omega_0^2})$

$\exp(2t\sqrt{\gamma^2 - \omega_0^2}) = -B/A < 0$ None exsist

ii) $\gamma = \omega$ $\psi = (A+B)\exp(-\omega t)$ Crtical Damp

☆ $\psi = \text{Constant} \cdot \exp(-\omega t)$ ← Lost one solution

Approach by overdamped solution

$\psi = A\exp(u_- t) + B\exp(u_+ t)$ assume $u_+ = u_- + \varepsilon$

$= A\exp(u_- t) + B\exp(u_- t)\cdot\exp(\varepsilon t) = \exp(u_- t)(A + B\exp(\varepsilon t))$

$= \exp(u_- t)(-1/\varepsilon + 1/\varepsilon\,\exp(\varepsilon t))$ Assume $A = -1/\varepsilon$ $B = 1/\varepsilon$

$\lim_{\varepsilon\to 0}\psi(t) = \exp(u_- t)\left(\frac{\exp(\varepsilon t)-1}{\varepsilon}\right) = t\exp(u_- t)$ ← ☆ $B \to B\cdot t$ $\psi(x) = (A+Bt)\cdot e$

with Limitation $\psi(0) = A$ $\psi(0)' = 0$ $\Rightarrow \psi = A(1+\omega_0 t)\cdot\exp(-\omega_0 t)$

iii) $\nu < \omega$ $\psi(t) = A\exp(-\nu + i(\omega t + \phi))$ with $\omega = \sqrt{\omega_0^2 - \nu}$

Energy of Damped Oscilliation $W = E_k + E_p$

$dW/dt = m\dot\psi\ddot\psi + k\psi\dot\psi = \dot\psi(k\psi + m\ddot\psi)$ $\Rightarrow dW/dt = -b\dot\psi^2$

Damping Force $= b\dot\psi$

- Forced Motion (ref From L.D. Landau — Mechanics).

$U(x,t) \simeq U(0,t) + x\cdot \partial U/\partial x\big|_{x=0}$

with $L = \frac{1}{2}m\dot x^2 - \frac{1}{2}kx^2 - F(t)\cdot x$ $\Rightarrow$ $(\partial L/\partial\dot x)\,d/dt - \partial L/\partial q = m\ddot x - kx - F(t) = 0$

For special state on $F(t) = b\cos(\nu t + \phi)$

↕ Special Solution term

Take $x = A\sin(\omega t + \alpha) + B\cos(\nu t + \phi)$

$-Bm\cancel{\cos(\nu t+\phi)}\nu^2 + kB\cancel{\cos(\nu t+\phi)} = b\cancel{\cos(\nu t+\phi)}$

$-Bm\nu^2 + kB = b \Rightarrow B = -b/(m\nu^2 - k)$

$= b/(m(\omega^2 - \nu^2))$

Two oscilliation composed

$\boxed{Q.}$

$x = a\cos(\omega t + \phi) + \frac{b}{m(\omega^2-\nu^2)}\cos(\nu t + \psi)$ ⇐

blow up

For special case $\rightarrow \omega = \nu$

Solution of ODE able to add ordinary solution

$x = a\cos(\omega t + \phi) + \frac{b}{m(\omega^2-\nu^2)}\cdot[\cos(\nu t + \psi) - \cos(\omega t + \phi)]$

By $\nu \to \omega$ $x = a\cos(\omega t + \phi) + \frac{\sin(\omega t + \psi)\,bt}{2m\omega}$

Properties on ν closed to ω : $\nu = \omega + \varepsilon$

With $A, B = \exp(i\phi), \exp(i\psi)$

Focusing on $x = A\exp(i\omega t) + B\exp(i(\omega+\varepsilon)t) = \exp(i\omega t)[A + B\exp(i\varepsilon t)]$

Assume as $C = A + B\exp(i\omega t) = ae^{i\phi} + be^{i(\psi+\omega t)}$

$C^2 = a^2 + b^2 + 2ab\cos(\varepsilon t + \beta - \alpha)$

⇓ with frequency ε

$|a-b| \le C \le a+b$

- General Solution On Driven Force motion

$$\ddot{x} + \omega^2 x = \frac{1}{m} F(t)$$

☆ Operator Algebra Method

$(\hat{D}^2 + \omega^2)x$ where $\hat{D}$ is d/dt operator

Split by $(D - i\omega)(D + i\omega)$

$$(D^2 + i\omega D - i\omega D + \omega^2)x = \ddot{x} + i\omega\dot{x} + \omega^2 x - i\omega\dot{x}$$

$$\frac{d}{dt}\underbrace{(\dot{x} + i\omega x)}_{\text{set as } \xi} - i\omega\underbrace{(\dot{x} + i\omega x)} = \frac{1}{m} F(t)$$

$\xi = A(t)\exp(i\omega t) \Leftarrow \frac{d}{dt}\xi - i\omega\xi = \frac{1}{m}F(t)$ ☆ 2 order → 1 order ODE.

$$\dot{A}(t)\exp(i\omega t) + \cancel{i\omega \exp(i\omega t) A(t)} - \cancel{i\omega A(t)\exp(i\omega t)} = \frac{1}{m}F(t)$$

$$\dot{A}(t) = \frac{F(t)}{m}\exp(-i\omega t) \Rightarrow \xi = \exp(i\omega t)\left[\int_0^t \frac{1}{m} F(t)\cdot \exp(-i\omega t)\cdot dt + \xi_0\right]$$

associate with specific $F(t)$

↓ $\xi = i\omega x + \dot{x} \Rightarrow x = $ sth.