Math Method II

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Week 1 Directional derivative

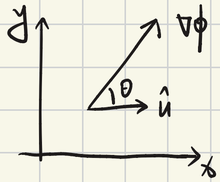
$$f(x)' = \lim_{s\to 0}[f(x+s) - f(x)]/s$$

$$\nabla_{\hat{u}} f = \lim_{s\to 0}[f(r+s\hat{u}) - f(r)]/s$$

As $f(r+s\hat{u}) = f(x,y,z) + s\cdot\left[\partial f/\partial x \cdot u_x + \partial f/\partial y\, u_y + \partial f/\partial z \cdot u_z\right] = f(x,y,z) + \nabla f\cdot\hat{u}\cdot s$

$$\nabla_{\hat{u}} f = \nabla f\cdot\hat{u}$$

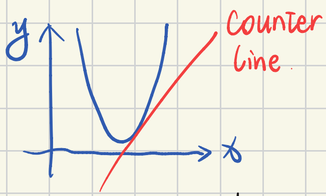
- Comments on directions



Obviously $\theta = 0$ Max
$\theta = \pi$ $= 0$

- Directions of ∇

Similar to basic derivitave.



$\nabla\phi$ points along max change

$\nabla\phi$ perpendicular to counter line.

eg. $y = x^2 \rightarrow \phi(x,y) = y - x^2$ $\qquad \nabla\phi = (\partial\phi/\partial x, \partial\phi/\partial y) = (-2x)i + (1)j$

$\xrightarrow{\text{Counter}}$ $y = 2x$ $(1,2) \longleftarrow$ perpendicular $\longrightarrow (-2,1)$

eg. $x^2 + 2y^2 = 2$. with $P(1, 1/\sqrt{2})$ on counter line.

$$\nabla\phi = \begin{pmatrix}2x\\4y\end{pmatrix} \Rightarrow \nabla\phi|_p = \begin{pmatrix}2\\4/\sqrt{2}\end{pmatrix}$$

Normline of counter line $=$ $y = mx + b$ $\qquad m = y/x = 2/\sqrt{2}$

$y = \sqrt{2}x + b$ with $b = -1/\sqrt{2}$ from $(1, 1/\sqrt{2})$

$$\vec{r} = \begin{pmatrix}1\\1/\sqrt{2}\end{pmatrix} + t\begin{pmatrix}2\\4/\sqrt{2}\end{pmatrix}$$

$y = \sqrt{2}x - 1/\sqrt{2}$ [Normal to counter line]

Tangent line $\quad n_p\cdot\nabla_p\phi = 0 \Rightarrow n_p = \begin{pmatrix}-4/\sqrt{2}\\2\end{pmatrix}$ $\qquad y = -\frac{2}{4/\sqrt{2}}x + b$

$= -1/\sqrt{2}\,x + \sqrt{2}$

eg. 3D Situation $\phi(r) = x^2 + 2y^2 + 3z^2$ Counter line with $P = (1,1,2)$

Normal : $\nabla\phi = \begin{pmatrix}2x\\4y\\6z\end{pmatrix} \Rightarrow \nabla\phi|_p = \begin{pmatrix}2\\4\\12\end{pmatrix} \rightarrow r = \begin{pmatrix}1\\1\\2\end{pmatrix} + t\begin{pmatrix}2\\4\\12\end{pmatrix}$

Tangential : $(r - p)\cdot\nabla\phi = 0 \Rightarrow x + 2y + 6z = 15$

- Divergence / Curl [Already well define in official note, This part focusing on Use of them]

- Electric

$\nabla \cdot E = Q/\varepsilon_0$ with $E = -\nabla V$ $\Rightarrow$ $\nabla^2 V = Q/\varepsilon_0$ if no q include $= 0$.

Laplace Equation.

Example 3.5. An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$, as indicated in Fig. 3.22. Find the potential inside the pipe.

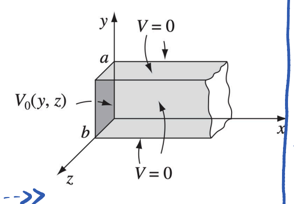


FIGURE 3.22

Griffith « Intro -->>

$$\begin{cases} x=0 & V=V_0 \\ x=\infty & V=0 \\ Y=0 & V=0 \\ Y=a & V=0 \\ Z=0 & V=0 \\ Z=b & V=0 \end{cases}$$

$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$

Guess $V = XYZ$

$\underbrace{\partial X^2/\partial x^2 \cdot 1/X}_{=l^2} + \underbrace{\partial Y^2/\partial y^2 \cdot 1/Y}_{=-m^2} + \underbrace{\partial Z^2/\partial z^2 \cdot 1/Z}_{=-n^2} = 0$

$X \to 0$ at ∞ $\Rightarrow$ $X = A\exp(lx) + B\exp(-lx)$

$\Rightarrow$ $Y = C\sin(mY) + D\cos(mY)$

$\Rightarrow$ $Z = E\sin(nZ) + F\cos(nZ)$

☆ Here assume $l^2 = m^2 + n^2$

i). $X = B\exp(-lx)$ $\Rightarrow$ $B = V_0$ $= V_0\exp(-lx)$

ii). $Y = C\sin(mY)$ $\Rightarrow$ $0 = C\sin(ma)$ $ma = n\pi$ $n = 1, 2, 3 \cdots$
with $C = a$

$Y = C \cdot \sin(m\pi y/a)$

iii). $Z = E\sin(nZ)$ $\Rightarrow$ $0 = E\sin(na)$ $nb = n\pi$ $n = 1, 2, 3 \cdots$

$Z = E \cdot \sin(n\pi z/b)$

$V(x,y,z) = C\sin(m\pi y/a)\cdot\sin(n\pi z/b)\cdot\exp(-lx) = \sum_1^\infty \sum_1^\infty C_{nm}\sin(m\pi y/a)\sin(n\pi z/b)\cdot\exp\left(-\pi\sqrt{(m/a)^2+(n/b)^2}\right)$

Focusing on $F(0,y,z) = V_0 = \sum\sum \sin(m\pi y/a)\sin(n\pi z/b)\cdot C_{nm}$

☆ Furiers Trick [In week 4 PHAS 004 chap 2 orthnormal part]

$\int_0^a\int_0^b V_0(y,z)\sin(m'\pi y/a)\sin(n'\pi z/b) = \int_0^a\int_0^b \sin(m'\pi y/a)\sin(n'\pi z/b)\sum\sum\sin(m\pi y/a)\sin(n\pi z/b)\cdot C_{nm}$

$\Downarrow$ $= ab/4 \cdot C_{nm}$

$C_{n,m} = 4V_0/ab \int_0^a\int_0^b \sin(m'\pi y/a)\sin(n'\pi z/b)\,dy\,dz$ $\to$ if n, m even $= 0$; $\wedge$ odd $= 16V_0/\pi^2 nm$

Finally $V(x,y,z) = 16V_0/\pi^2 \sum_{1,3,5\cdots}^\infty 1/nm \exp\left(-\pi\sqrt{(n/a)^2+(m/b)^2}\,x\right)\sin(m\pi y/a)\sin(n\pi z/b)$

- Line Integrals

$I = \int_C G(r) \cdot dr$ For $G(r)$ vector Field with C is path from $r_A \to r_B$

$C = r(t)$ where $r(a) = r_A$ $r(b) = r_B$ $r(t) = x(t)\hat{e}_x + y(t)\hat{e}_y + z(t)\hat{e}_z$

$I = \int_C G(r(t)) \cdot dt \cdot dr/dt$ ⟵ $dr/dt = dx(t)/dt\,\hat{e}_x + dy(t)/dt\,\hat{e}_y + dz(t)/dt\,\hat{e}_z$

$= \int_C G(r(t)) \cdot r'(t) \cdot dt$

- Comments On Conservative fields. [$G(r) = \nabla\phi(r)$ - Conservative]

$$\int_C G(r) \cdot dr = \int_C \nabla\phi(r) \cdot dr = \int_A^B (\partial\phi/\partial x\,\hat{e}_x \cdots)(dx/dt \cdot \hat{e}_x + \cdots) \cdot dt$$

$$= \int_A^B d\phi/dt \cdot dt = \phi \Big|_{r=r_A}^{r=r_B}$$

☆ No relation with path.