Week 4 – PHAS0004 ASU – Note

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Reading: « Intro --- Quantum » Griffith Chap 1~2

Chap1 Raw Form of SCHD.

$$i\hbar\,\partial\psi/\partial t = -\hbar^2/2m\,\partial^2\psi/\partial x^2 + V\psi$$

— Limitations:

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 = 0$$

☆ If Normalize on t ⇒ $t+dt$ Normalize

$$d/dt \int_{-\infty}^{+\infty} |\psi(x,t)|^2 = 0$$

$$= \int_{-}^{+} d/dt\,\psi^*\psi = d\psi^*/dt\,\psi + d\psi/dt\cdot\psi^*$$

$$\partial\psi/\partial t = +\,i\hbar/2m\,\partial\psi^2/\partial x^2 - i/\hbar\,V\psi$$

$$\Rightarrow (i\hbar/2m\,\partial\psi^2/\partial x^2 - iV/\hbar\,\psi)\,\psi^* + (i\hbar/2m\,\partial\psi^{*2}/\partial x^2 - iV/\hbar\,\psi^*)\,\psi$$

$$= i\hbar/2m(\psi^*\,\partial^2\psi/\partial x^2 - \partial^2\psi^*/\partial x^2\cdot\psi) = \partial/\partial x\left[i\hbar/2m\left(\psi^*\,\partial\psi/\partial x - \psi\,\partial\psi^*/\partial x\right)\right]$$

ψ and ψ^* must $=0$ for $x\to\pm\infty$

$$d/dt\int_{-\infty}^{+\infty} |\psi(x,t)|^2 = 0 \Rightarrow$$ function Automatically Normalize with t.

P.1.7 $A^2\left[\int_0^a x/a\,dx + \int_a^b (b-x)^2/(b-a)^2\,dx + 0\right] = 1$

$A^2\left(\frac{a^2}{2a} + 1/3(b-a)\right) = 1$

$A = \pm\sqrt{a+2b/6}$

Spc. Case. $\partial|\psi|^2/\partial t = i\hbar/2m(\psi^*\,\partial\psi/\partial x - \partial\psi^*/\partial x\,\psi)\cdot dx$

P1.8 $\psi(x,t) = A\exp(-\lambda|x|)\exp(-i\omega t)$

$$\int_{-}^{+} (\psi(x,0))^2\,dx = 1$$

$A^2\exp(-2\lambda|x|)\,1/2\lambda\,\Big|_{-}^{+} = 1$

$2\int_0^+ A^2\exp(-2\lambda|x|) = 1$ $\quad A = \sqrt{\lambda}$

P1.9. $P_{a,b}(t) =$ P of Find in $x \in a\sim b$

$\int_a^b \psi^2(x,t)\,dx = P_{ab}(t)$ $\quad \psi\cdot\psi^*$

$dP_{ab}(t)/dt = \int_a^b d\psi^2/dt\,dx = \int_a^b 2\psi(x,t)\,d\psi/dt$ $\quad J(x,t) = i\hbar/2m(\psi\,\partial\psi^*/\partial x - \psi^*\,\partial\psi/\partial x)$

$\partial\psi/\partial t = +\,i\hbar/2m\,\partial\psi^2/\partial x^2 - i/\hbar\,V\psi$ $\quad = \int_a^b d\psi/dt\cdot\psi^* + d\psi^*/dt\,\psi = J(a,t) - J(b,t)$

P1.10 Assume $P(t) = \int_-^+ |\psi(x,t)|^2 \cdot dx = \exp(-t/\tau)$

$$dP/dt = \int_-^+ dP/dt\, |\psi^* \psi| \cdot dx$$
$$= \int_-^+ \psi(-\partial\psi/\partial t) + \psi^*(\partial\psi/\partial t) \cdot dx = -\exp(-t/\tau)$$

— Momentum. By SCHD.

Momentum — P

$\langle P \rangle$: $P = m \cdot d\langle x\rangle/dt$ $\qquad \langle x\rangle = \int_-^+ x\psi(x,t)^2 \cdot dx$

$$d\langle x\rangle/dt = \int x \cdot \partial/\partial t |\psi|^2 \cdot dx = i\hbar/2m \int x \cdot \partial/\partial x \underbrace{(\psi^* \partial\psi/\partial x - \partial\psi^*/\partial x \cdot \psi)}_{=J} \cdot dx$$
$$= i\hbar/2m \left(\underbrace{Jx\Big|_-^+}_{\text{Must} = 0} - \int_-^+ J \cdot dx \right)$$
$$\Rightarrow -i\hbar/2m \int_-^+ J \, dx$$

$$\langle P\rangle = -i\hbar \int (\psi^* \partial\psi/\partial x) \cdot dx \Leftarrow \quad = -i\hbar/2m \int (\psi^* \partial\psi/\partial x - \partial\psi^*/\partial x \cdot \psi)\, dx$$
$$= \Big|_-^+ \psi\psi^* - \int \partial\psi^*/\partial x \cdot \psi \, dx \qquad - \int \partial\psi^*/\partial x \cdot \psi \, dx$$

★ only Seperate half

$$= -i\hbar/m \int \psi^* \cdot \partial\psi/\partial x \cdot dx$$

$$= \int \psi^* \underbrace{(\hbar/i \cdot \partial/\partial x)}_{\hat{P}} \psi \, dx$$

$$\hat{P} = -i\hbar\, \partial/\partial x \quad (\text{Operators})$$

— Uncertainty Principles $\quad \sigma_p \cdot \sigma_x \geq \hbar/2$

Commutation relation $[A, B] = AB - BA$

★ For two operators $[A, B] = 0$ means able to access both time.

Eg. $[\hat{x}, \hat{p}]\psi = \hat{x}\, i\hbar \cdot \partial/\partial x\, \psi - i\hbar \cdot \partial/\partial x \cdot \hat{x}\psi$

$$= \cancel{\hat{x}\, i\hbar\, \partial\psi/\partial x} - i\hbar\psi - \cancel{i\hbar\, \partial\psi/\partial x \cdot \hat{x}}$$
$$= -i\hbar\psi$$

$[\hat{x}, \hat{p}] \neq 0$ — not accessible same time.

Cauchy - Schwarz : $\quad (\Delta A)^2 (\Delta B)^2 \geq 1/4 |\langle[\hat{A}, \hat{B}]\rangle|^2$

For x, p $\qquad \sigma_x \sigma_p \geq 1/2 |[\hat{x}, \hat{p}]|$
$$\geq 1/2\, \hbar$$

Chap 2. TISE // Seperation Term // Solutions

— Separation Term

$i\hbar \cdot \partial\Psi/\partial t = -\hbar^2/2m \cdot \partial^2\Psi/\partial x^2 + V\Psi$ ← original form

Assume $\Psi(x,t) = \psi(x) \cdot \varphi(t)$

$i\hbar\psi(x) \cdot \partial\varphi(t)/\partial t = -\hbar^2/2m \cdot \partial\psi(x)^2/\partial x^2 \cdot \varphi(t) + V\psi(x)\varphi(t)$

$\underbrace{i\hbar \cdot \partial\varphi(t)/\partial t \cdot 1/\varphi(t)}_{\text{only } t} = \underbrace{-\hbar^2/2m \cdot \partial\psi^2(x)/\partial x^2 \cdot 1/\psi(x) + V}_{\text{only } x}$ ⇒ Constant each side.

$i\hbar \cdot \dot{\varphi} - E\varphi = 0$ $\qquad -\ddot{\psi} \cdot \hbar^2/2m + V\psi = $ Sch Constant $\cdot \psi = E\psi$

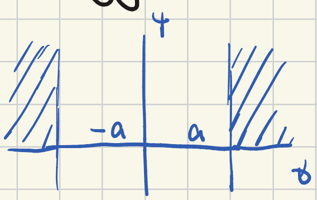
$\varphi(t) = \exp(-iEt/\hbar)$ $\qquad$ TISE

$\Psi(x,t) = \psi(x) \exp(-iEt/\hbar)$

↓ Linear Combination

$\Psi(x,t) = \Sigma_n C_n\psi_n(x) \exp(-iE_nt/\hbar)$

— Energy Solutions - Infinity Square Wall, Harmonic Oscilliator.



$\psi(x)$: $-\ddot{\psi} \cdot \hbar^2/2m + V\psi = E\psi$

Assume $\psi = A\sin(\omega x + \phi_1) + B\cos(\omega t + \phi_2)$

i) $-a < x < a$ $\qquad -\ddot{\psi}\, \hbar^2/2m = E\psi$

$\psi = A\sin kx + B\cos kx$ for $k = \frac{\sqrt{2mE}}{\hbar}$

$\psi(-a) = \psi(a) = 0$

$-A\sin(ak) + B\cos(ak) = A\sin ak + B\cos ak = 0$

$A = 0$ $\qquad \cos(ak) = 0$

$\psi = B\cos(\frac{n\pi}{2a} \cdot x)$ $n \in 1, 3, \cdots$ ← $k \in (\pi/2a, 3\pi/2a, \cdots)$

↓ Normalize

☆ only exsit in $(-a \sim +a)$ → $\int_{-a}^{+a} B^2\cos^2(n\pi x/2a) \cdot dx = 1$

$1/2(1 + \cos(n\pi x/a))$

→ $B^2[1/2x|_-^+ + \sin(n\pi x/a) \cdot a/n\pi |_-^+] = 1$

$\psi = \sqrt{1/a}\cos(n\pi x/2a)$ $\qquad n \in 1, 3, 5 \cdots$

☆ Special Properties — Orthnormal

- Independently distribute
- Use orthnomal derive Furier's Trick

EelectroMagnetism : $V(x,y) = \sum C_n \sin(k_n x) \sinh(k_n y)$

$(\nabla^2 = 0)$ ↓ Solve C_n

$$\sum_m V(x,y) \sin(k_m x) = \sum_{n,m} C_n \sin(k_n x) \sin(k_m x) \sinh(k_n y)$$

☆ ↓ $\sum_{n,m} \sin(n\pi x/a) \sin(m\pi x/a)$

orthrormal keep $n=m$ value
cancel $n \neq m$ term

$[\cos(n+m) \sim \cdot \cos(n-m) \sim]$
two complete cos wave when $n \neq m$

For $n=m \Rightarrow \sum_{n,m} \sim = \frac{1}{2}a \Rightarrow C_n = \sim$

Proof on orthgonal : $\int \psi_m(x)^* \psi_n(x) \cdot dx = 0$

$$= 1/a \int_-^+ \cos(n\pi x/2a) \cos(m\pi x/2a) \cdot dx$$

↓ $2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \cos\alpha + \cos\beta$

$$= \frac{1}{2a} \int_-^+ \cos((n+m)\pi x/4a) + \cos((n-m)\pi x/4a) \cdot dx$$

$$= \frac{1}{2a} \Big|_-^+ [\sin((n+m)\pi x/4a) \cdot 4a/(n+m)\pi + \sin((n-m)\pi x/4a) \cdot 4a/(n-m)\pi]$$

$= 0 \longrightarrow \quad = \delta_{nm}.$

— Harmonic Oscilliator : $V = \frac{1}{2} m w^2 x^2$

$$-\frac{\hbar^2}{2m} d^2\psi/dx^2 + \frac{1}{2} m w^2 x^2 \psi = E\psi$$

$$\frac{1}{2m} (-\hbar^2 \cdot d^2/dx^2 + (mwx)^2) \psi = E\psi$$

A. Algebratic Method — derive $a_+ a_-$ as $E \pm \hbar w$ operators.

define $a_+ a_-$ as $a_+ \cdot a_-$ close to $\frac{1}{2m}(\cdots)\psi$

$$a_{+-} = 1/\sqrt{2m} (\hbar/i \, d/dx \pm (mwx))$$

$a_+ \cdot a_-$ ☆ Cal. by adding on $f(x)$ $\quad a_+ a_- f(x) = \frac{1}{2m} [\hbar/i \, d/dx + (mwx)] [\hbar/i \, d/dx - (mwx)] \cdot f(x)$

$$= \frac{1}{2m} [\hbar/i \, d/dx (\hbar/i \, df/dx) - \hbar/i \, d/dx (mwxf) + (mwx) \hbar/i \, df/dx - (mwx)^2 f]$$

$a_+ a_- = \frac{1}{2m} [(\hbar/i \, d/dx)^2 + (mwx)^2] - \frac{1}{2}\hbar w \Longleftarrow \; = \frac{1}{2m} [-\hbar^2 (d/dx)^2 + i\hbar mw + \cancel{i\hbar \cdot mwx \cdot d/dx} - \cancel{i\hbar mwx \cdot d/dx} - (mwx)^2] f$

||

$= \frac{1}{2m}[-\hbar^2(d/dx)^2 + i\hbar m\omega - (m\omega x)^2] f(x)$

$\Downarrow$

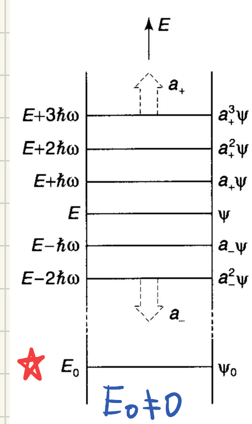
$(a_+a_- + \frac{1}{2}\hbar\omega)\psi = E\psi$

☆ Try use a_+ as upper operator.

$(a_+a_- + \frac{1}{2}\hbar\omega)(\psi a_+) = (a_+a_-a_+ + \frac{1}{2}\hbar\omega a_+)\psi$
$= a_+(a_-a_+ + \frac{1}{2}\hbar\omega)\psi$
$= a_+[(\underbrace{a_-a_+ - \frac{1}{2}\hbar\omega}_{\text{Same with } a_+a_- + \frac{1}{2}\hbar\omega})\psi + \hbar\omega\psi]$
$= a_+[E\psi + \hbar\omega\psi]$
$= a_+[E + \hbar\omega]\psi$
$\Downarrow$
$E(\psi a_+) = (E + \hbar\omega)\psi$ — Vice Versa

$(a_+a_- + \frac{1}{2}\hbar\omega)(\psi a_-) = ---$
$= a_-[E - \hbar\omega]\psi$

$E(\psi a_-) = (E - \hbar\omega)\psi$



☆ base State. $E_0 \neq 0$

☆ Discussion On Base State: $\psi_0 a_-$

1. why E alwayse >0

i). $E = T + V = P^2/2m + \frac{1}{2}m\omega^2x^2$
>0 Mathmaticlly

ii). $E = 0 \rightarrow$ Particle must in Lowest point with $x = 0$ $p = 0$

$\sigma_x\sigma_p = 0$ where as $\sigma_x\sigma_p \geq \hbar/2$ $x.p$ unable measure same time.
$\Downarrow$
$E > 0$

2. Meaning on $\psi_0 E_0$

$E\psi a_- = (E - \hbar\omega)\psi$
Minimum State
$\downarrow$
$E\psi a_-^n = \underline{(E - n\hbar\omega)}\psi$
ψa_-^n $\quad$ $\hbar\omega >$ $\quad$ >0
Lowest Energy State $\quad$ break

Logic

1. Set a_+a_- approach
2. Obseve a_+a_- as operators
3. find base state

define Lowest State as $\psi_0 \rightarrow \psi_0 a_-$ is none exsist $\rightarrow \psi_0 a_-$ define as $= 0$

$\Rightarrow \frac{1}{\sqrt{2m}}[(\frac{\hbar}{i} d\psi_0/dx) - im\omega x\psi_0] = 0$

$\psi_0 = A_0 \exp(-x^2 m\omega/2\hbar)$ with $(a_+a_- + \frac{1}{2}\hbar\omega)\psi_0 = E_0\psi_0 \Rightarrow E_0 = \frac{1}{2}\hbar\omega$ Base State.

Examples: $\psi_1 = \psi_0 \cdot a_+ = A_1 \frac{1}{\sqrt{2m}}(-i\hbar \exp(-m\omega x^2/2\hbar)\cdot(-m\omega x/\hbar) + im\omega x \cdot \exp(-m\omega x^2/2\hbar))$

$= A_1 i\omega x \cdot \sqrt{2m} \cdot \psi_0$

B. Analatic Method

$\frac{1}{2m}(-\hbar^2 d^2/dx^2 + (m\omega x)^2)\psi = E\psi$ [Similar to week 6]

$-\frac{1}{2m}\hbar^2 d^2\psi/dx^2 + \frac{1}{2}m\omega^2x^2\psi = E\psi \rightarrow d^2\psi/d\xi^2 = (sth)\psi$

ODE Methods!

☆ Set $x = c\xi$ $\qquad -\frac{1}{2m}\hbar^2\, d\psi^2/d\xi^2\,(1/c)^2 + \frac{1}{2}m\omega^2(c\xi)^2\psi = E\psi$

valu. before $d\psi^2$ and ψ Same.

$$\frac{\hbar^2}{2mc^2} = \frac{m\omega^2c^2}{2} \qquad c = \sqrt{\frac{\hbar}{m\omega}} \qquad \xi = x/c = x\cdot\sqrt{\frac{m\omega}{\hbar}}$$

$$-\hbar^2/2m\cdot d\psi^2/d\xi^2\cdot m\omega/\hbar + \frac{1}{2}m\omega^2\xi^2\,\hbar/m\omega\,\psi = E\psi$$

$$d\psi^2/d\xi^2 = \psi(\tfrac{1}{2}\hbar\omega\xi^2 - E) = \psi(\xi^2 - \underbrace{2E/\hbar\omega}_{\text{set } = K})$$

$\Rightarrow d\psi^2/d\xi^2 = \psi(\xi^2 - K)$ Focus on $\xi \to \infty$ K is ingnored.

$$d\psi^2/d\xi^2 - \psi\xi^2 = 0 \qquad \psi = Ae^{-\frac{1}{2}\xi^2} + Be^{\frac{1}{2}\xi^2}$$

Obviously B explode $=0$

$$\boxed{K = 2E/\hbar\omega \qquad \xi = x\cdot\sqrt{\frac{\hbar\omega}{\hbar}}}$$

$$\psi = A\exp(-\tfrac{1}{2}\xi^2)$$

$$d\psi^2/d\xi^2 = \cancel{A\exp(-\tfrac{1}{2}\xi^2)}(\xi^2 - K)$$

$\xi^2 - 1 = \xi^2 - K$ ☆ If A used as constant

↓ function only worked when $K=1$ (base state only)

Use $h(\xi)$ $\qquad \psi(\xi) = h(\xi)\cdot\exp(-\tfrac{1}{2}\xi^2)$

$$d\psi(\xi)/d\xi = dh(\xi)/d\xi\,\exp(-\tfrac{1}{2}\xi^2) - \exp(-\tfrac{1}{2}\xi^2)\cdot\xi\cdot h(\xi)$$

$$= \exp(-\tfrac{1}{2}\xi^2)(dh/d\xi - h\xi)$$

$$d\psi(\xi)^2/d\xi^2 = \exp(-\tfrac{1}{2}\xi^2)\cdot -\xi\,(dh/d\xi - h\xi)$$

$$+ (dh^2/d\xi^2 - h)\cdot\exp(-\tfrac{1}{2}\xi^2)$$

$$= \exp(-\tfrac{1}{2}\xi^2)(-2\xi\, dh/d\xi + h\xi^2 + dh^2/d\xi^2 - h)$$

$$d\psi^2/d\xi^2 = (\xi^2 - K)\psi$$

$$\Downarrow$$

$$\cancel{\exp(-\tfrac{1}{2}\xi^2)}[h(\xi^2-1) + dh^2/d\xi^2 - \xi\,\underbrace{dh/d\xi}\,)] = (\xi^2 - K)h\cancel{\exp(-\tfrac{1}{2}\xi^2)}$$

$$dh^2/d\xi^2 - 2\xi\, dh/d\xi + h(K-1) = 0$$

↓ ☆ Try on split $h(\xi) = C_1\xi + C_2\xi^2 + C_3\xi^3 + \cdots = \sum_0^\infty C_j\xi^j$

$$dh^2/d^2\xi = 2C_2 + 6C_3\xi + 24C_4\xi^2 \cdots = \sum_2^\infty n(n-1)\cdot C_n\xi^n$$

Same $\sum_1^\infty$ and $\sum_0^\infty(\cdots)$

$$\sum_2^\infty n(n-1)C_n\xi^{n-2} - \sum_1^\infty n\cdot C_n\cdot\xi^n\cdot 2 + h(K-1) = 0$$

$$\sum_0^\infty (n+1)(n+2)C_{n+2}\xi^n - 2\sum_0^\infty nC_n\xi^n \quad + h(K-1)$$

$$\sum_0^\infty [(n+1)(n+2)C_{n+2}\xi^n - 2nC_n\xi^n] = h(1-K)$$

$$\sum_{0}^{\infty} \underbrace{[(n+1)(n+2)C_{n+2} - 2nC_n + (K-1)C_n]}_{\text{Must } = 0} \xi^n = 0$$

cause ξ is unlimit.

$$(n+1)(n+2)C_{n+2} - 2nC_n + (K-1)C_n = 0$$

$$C_{n+2} = \frac{C_n(2n-K+1)}{(n+1)(n+2)}$$

★ Try $n \to \infty$ $\quad C_{n+2} = C_n \cdot 2/n \Rightarrow C_n = C/(n/2)!$

$$h(\xi) = C\sum 1/(j/2)! \cdot \xi^j = C \cdot \underbrace{\sum 1/k! \cdot \xi^{2k}}_{\text{Taylor Form of exp(x)}} = Ce^{2\xi}$$ ← Unable to Normalize

★ Term of h end in j term

j is defind as ending term

Last term $\rightarrow C_{j+2} = C_j \cdot (2j-K+1)/(j+1)(j+2) = 0 \Rightarrow K = 2j+1$ whereas $E = \hbar\omega(j + 1/2)$

K is constant.

$\Rightarrow C_{n+2} = C_n \cdot -(2n-2j)/(n+1)(n+2)$

★ j is the term number when function end

eg. $j = 4$ $\quad$ Terms end up at $j=4$

$h_0 = a_0 \quad h_2 = -4a_0\xi^2 \quad h_4 = 4/3\, a_0\xi^4 \quad h_6 = 0$

↓ Hermite polynormials

$$\psi_n = (m\omega/\pi\hbar)^{1/4} \cdot 1/\sqrt{2^n n!}\, H_n(\xi)\exp(-\xi^2/2)$$

Week 4 Course Note.

$L = mvr = \hbar n = h/2\pi \cdot n$

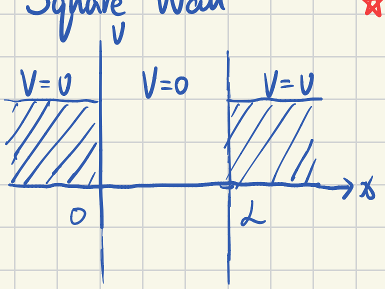
$E_n = R_H(1/m^2 - 1/n^2)$

TISE $\quad -\hbar^2/2m\,(d^2\psi(x)/dx^2) + V(x)\psi(x) = E\psi(x)$ □

Infinitive Square Wall $\quad \psi = \sqrt{2/L}\sin(n\pi x/L)$ □

TISE − operator Form $\quad \hat{H}\psi = E\psi$ □

Square Wall $\quad$ ★ For Function decay at $\pm\infty$ Try on $\psi = A\exp(\omega t) + B\exp(\omega t)$



$$\begin{cases} -\infty \sim 0 & \hbar^2/2m \cdot \partial\psi^2/\partial x^2 = (E-U)\psi \rightarrow \psi_1 = A\exp(kx) + B\exp(-kx) \\ 0 \sim L & \hbar^2/2m \cdot \partial\psi^2/\partial x^2 = E\psi \rightarrow \psi_2 = C\sin(k'x) + D\cos(k'x) \\ L \sim \infty & \hbar^2/2m \cdot \partial\psi^2/\partial x^2 = (E-U)\psi \rightarrow \psi_3 = E\exp(kx) + F\exp(-kx) \end{cases}$$

$k = \sqrt{2m(E-U)}/\hbar$

$k' = \sqrt{2mE}/\hbar$

i) $x \to -\infty \rightarrow B = 0$

ii) $x =$ $\quad \psi_1 = A \quad \psi_2 = C \Rightarrow A = C$

iii) $x = L$ $\quad \psi_2 = C\sin(k'L) + D\cos(k'L) \quad \psi_3 = E\exp(kL) + F\exp(-kL)$

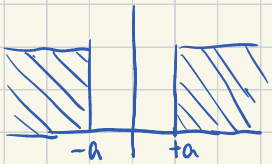
IV) $x \to \infty \quad \to E = 0$

$$\begin{cases} A\exp(-ika) + B\exp(ika) = -C\sin(la) + D\cos(la) \\ ik[A\exp(-ika) - B\exp(ika)] = l\cdot[C\cos(la) + D\sin(la)] \\ C\sin(la) + D\cos(la) = F\exp(ika) \\ l[C\cos(la) - D\sin(la)] = ikF\exp(ika) \end{cases}$$

$\Downarrow$

$A = \cdots \quad F = \cdots$ Since $B, E = 0$ $\qquad$ Transmition Rate $= |F^2| / |A^2|$

☆ Allowed Energy State // Simpler Form Solution



← Function is even (symetry on both side)

$$\to \quad \psi(x) = \begin{cases} F\exp(-kx) & x > a \\ D\cos(k'x) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases} \qquad \Rightarrow \qquad \begin{aligned} & F\exp(-ka) = D\cos(ka) \\ & -kF\exp(-ka) = -kD\sin(k'a) \\ & \qquad \downarrow \\ & k = k' \cdot \tan(k'a) \end{aligned}$$