Topics in Linear Algebra: Homework 3

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Solution 1.3.1.

1.

$$A : \begin{bmatrix} \Re(x) \\ \Im(x) \\ \Re(y) \\ \Im(y) \end{bmatrix} \mapsto \begin{bmatrix} \Re(x) - \Re(y) \\ -\Im(x) \\ \Im(x) - \Re(y) \\ \Im(x) - \Im(y) \end{bmatrix},$$

So

$$A = \begin{bmatrix} 1 & & -1 & \\ & -1 & & \\ & 1 & -1 & \\ & 1 & & -1 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 1-\lambda & & -1 & \\ & -1-\lambda & & \\ & 1 & -1-\lambda & \\ & 1 & & -1-\lambda \end{bmatrix},$$

Then $\lambda = 1$ (multiplicity 1), $\lambda = -1$ (multiplicity 3).

W.R.T. $\lambda = 1$, eigenvector $v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

W.R.T. $\lambda = -1$,

$$A + I = \begin{bmatrix} 2 & & -1 & \\ & 0 & & \\ & 1 & 0 & \\ & 1 & & 0 \end{bmatrix}, (A+I)^2 = \begin{bmatrix} 4 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By observation, $\mathrm{Dim}(\mathrm{Ker}(A+I)) = 2, \mathrm{Dim}(\mathrm{Ker}(A+I)^2) = 3$, so

$$\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

are the basis of the generalized eigenspace associated with $\lambda = -1$. If we let

$$B = \begin{bmatrix} 1 & & & 1 \\ & 2 & & \\ 2 & -1 & & \\ 2 & & 1 & \end{bmatrix}, J = \begin{bmatrix} -1 & 1 & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$$

Then,

$$A = \begin{bmatrix} 1 & & & 1 \\ & 2 & & \\ 2 & -1 & & \\ 2 & & 1 & \end{bmatrix} \begin{bmatrix} -1 & 1 & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} & 1 & 1 & \\ & 2 & & \\ & -2 & -4 & 4 \\ 4 & -1 & -2 & \end{bmatrix}$$

The column vectors of the leftmost matrix is the basis in which the mapping is in its JCF.

2.

$$A : \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \mapsto \begin{bmatrix} p_0 + p_1 \\ 2p_2 \\ p_1 + 3p_3 \\ 4p_4 \\ 0 \end{bmatrix},$$

So

$$A = \begin{bmatrix} 1 & 1 & & & \\ & & 2 & & \\ & 1 & & 3 & \\ & & & & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & & & \\ & -\lambda & 2 & & \\ & 1 & -\lambda & 3 & \\ & & & -\lambda & 4 \\ & & & & -\lambda \end{bmatrix},$$

$$\det(A - \lambda I) = (1-\lambda)(\lambda^2)(\lambda^2 - 2),$$

so $\lambda = 1$, $\lambda = \pm\sqrt{2}$, $\lambda = 0$(multiplicity 2). W.R.T. $\lambda = 1$, eigenvector $v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

W.R.T. $\lambda = \sqrt{2}$, eigenvector $v = \begin{bmatrix} 2+\sqrt{2} \\ \sqrt{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

W.R.T. $\lambda = -\sqrt{2}$, eigenvector $v = \begin{bmatrix} 2+\sqrt{2} \\ -\sqrt{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

W.R.T. $\lambda = 0$, observe that

$$A^2 = \begin{bmatrix} 1 & 1 & 2 & & \\ & 2 & & 6 & \\ & & 2 & & 12 \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} \in \mathrm{Ker}(A^2), \text{ but } A \begin{bmatrix} 12 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -12 \\ 0 \\ 4 \\ 0 \end{bmatrix} \notin \mathrm{Ker}(A),$$

so

$$\left\{ \begin{bmatrix} 12 \\ -12 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 12 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2+\sqrt{2} \\ \sqrt{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2+\sqrt{2} \\ -\sqrt{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is the basis in which the transformation would be expressed in JCF, in which

$$J = \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & \sqrt{2} & & \\ & & & 1 & \\ & & & & -\sqrt{2} \end{bmatrix}$$

3. Since

$$\begin{bmatrix} & & & a_1 \\ & & a_2 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{bmatrix} = \begin{bmatrix} & a_1 & & \\ & & & a_2 \\ & & a_3 & \\ a_4 & & & \end{bmatrix},$$

$$\begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{bmatrix} \begin{bmatrix} & a_1 & & \\ a_4 & & & \\ & & & a_2 \\ & & a_3 & \end{bmatrix} = \begin{bmatrix} & a_1 & & \\ & & & a_2 \\ & & a_3 & \\ a_4 & & & \end{bmatrix},$$

$\begin{bmatrix} & a_1 & & \\ a_4 & & & \\ & & & a_2 \\ & & a_3 & \end{bmatrix}$ and $\begin{bmatrix} & & & a_1 \\ & & a_2 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix}$ shares the same JCF.

Further decompose $\begin{bmatrix} & a_1 & & \\ a_4 & & & \\ & & & a_2 \\ & & a_3 & \end{bmatrix}$ into two parts: $\begin{bmatrix} & a_1 \\ a_4 & \end{bmatrix}$ and $\begin{bmatrix} & a_2 \\ a_3 & \end{bmatrix}$.

Then the problem boils down to finding the Jordan decomposition of the two blocks, in which both in the form of $\begin{bmatrix} & a \\ b & \end{bmatrix}$.

(a) $a = b = 0$, then the basis in this block relative to the matrix we had been decomposed are standard basis: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the Jordan block is zero matrix.

(b) $a = 0, b \neq 0$, then it is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, corresponds to the Jordan block of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(c) $a \neq 0, b = 0$, then it is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, corresponds to the Jordan block of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(d) Otherwise, the block is diagonalizable, and $\begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix}$ is the eigenvector of $\lambda = -\sqrt{ab}$, $\begin{bmatrix} \sqrt{a} \\ -\sqrt{b} \end{bmatrix}$ is the eigenvector of $\lambda = \sqrt{ab}$

Use the basis generated above to form a matrix that preserves the order of decomposing the matrix, and use $\begin{bmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{bmatrix}$ to multiply that matrix, then the column vectors of the product would be the basis, and putting the Jordan blocks correspond to each block back in order will be the Jordan block of that anti-diagonal matrix.

Solution 1.3.2.

1.