LINEAR ALGEBRA-MIDTERM EXAMINATION

Question 1 (10pts=3+3+4)
Let A be a $m \times n$ matrix of rank r.

(1) If the equation $A\vec{x} = \vec{b}$ has solution for all $\vec{b}$, what is the column space of A? What is the relationship between m, n, r for this to hold?
(2) Suppose that the augmented matrix $[A\ \vec{b}]$ can be row reduced to $[C\ \vec{d}]$, is it true that the equation $A\vec{x} = \vec{b}$ and $C\vec{x} = \vec{d}$ has the same solution? Why?
(3) Give an example of a 3×2 matrix A of rank 1 with first row being $(2\ -5)$. Describe completely the column space and null space of A.

Question 2 (16pts=4+4+4+4)
It is known that, with the matrix addition and scalar multiplication, the set of all the matrices over $\mathbf{R}$ of size $n \times n$ form a vector space of dimension n^2 over $\mathbf{R}$. Please briefly but clearly explain your answer to the following question:

(1) Are the set of all $n \times n$ invertible matrices form a sub vector space?
(2) A matrix is said to be *singular*, if it is not invertible. Are the set of all the singular $n \times n$ matrices form a sub vector space?
(3) Consider the set of all the $n \times n$ matrices whose null space contains $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Is it a sub vector space?
(4) Consider the set of all the $n \times n$ matrices whose column space contains $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Is it a sub vector space?

Question 3 (17pts=4+8+5)
Let A be an $m \times n$ matrix over $\mathbf{R}$.

(1) Let $B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$. Is it true that $\mathrm{rk}(A) = \mathrm{rk}(B)$? Why?
(2) Let A be an $n \times n$ invertible matrix, let $B = \begin{bmatrix} A & X \\ X^t & 1 \end{bmatrix}$ with X an $n \times 1$ matrix. What is the condition on X for B to be invertible? When B is invertible, calculate B^{-1}.

(3) Let A be a 3×5 matrix, suppose that the set of solutions to the equation $A^t \vec{y} = 0$ is spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix},$$

what is the rank of A?

Question 4 (12pts=4+8)

Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

(1) Find the LU-factorization of A.

(2) Find the inverse A^{-1}, and use it to solve the equation $AX = \begin{bmatrix} 1 & 3 \\ 2 & -5 \\ 0 & 1 \\ 4 & 2 \end{bmatrix}$.

Question 5 (20pts=6+4+10)

(1) Let $A = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{bmatrix}$. Show that $A^4 = 0$ and use it to calculate $(I - A)(I + A + A^2 + A^3)$.

(2) Let $B = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{bmatrix}$. Show that $B^4 = I$.

(3) Let $A_n = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$ and $B_n = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$ be $n \times n$ matrices. Show that $(A_n)^n = 0$ and $(B_n)^n = I_n$.

Question 6 (25pts=7+18)

Let $A = \begin{bmatrix} -2 & 4 & -2 & -4 & 2 \\ 2 & -6 & -3 & 1 & 2 \\ 3 & -6 & -5 & 19 & -2 \\ -3 & 8 & 2 & -3 & 5 \end{bmatrix}$.

(1) Find the LU-factorization of A.

(2) Find bases for the row space, column space and the null space of A.