作业 7 简答

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习题 1. *Compute* $\det(B^4)$ *for* $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

解答.

$$\det\left(B^4\right) = \det(B)^4 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}^4 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix}^4 = (-2)^4 = 16.$$

习题 2. *Explain briefly the following assertion.*

1. *If* A *is invertible, then* $\det(A^{-1}) = \det(A)^{-1}$.
2. *Let* A *and* B *be square matrices. Even though* AB *and* BA *may not be equal, it is always true that* $\det(AB) = \det(BA)$.
3. *Let* A *and* P *be square matrices with* P *invertible, then* $\det(PAP^{-1}) = \det(A)$.
4. *Let* U *be a square matrix such that* $U^T U = I_n$, *then* $\det(U) = \pm 1$.

解答.

1. 对 $A^{-1}A = I_n$ 取行列式，得到 $\det\left(A^{-1}\right)\det(A) = 1$.
2. $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.
3. $\det\left(PAP^{-1}\right) = \det(P)\det(A)\det\left(P^{-1}\right) = \det(A)$.
4. 对 $U^T U = I_n$ 取行列式，利用 $\det\left(U^T\right) = \det(U)$ 立刻得到.

习题 3. *Compute the adjugate of the matrix* $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$*, and use the inversion formula to calculate its inverse.*

解答. 直接计算

$$\operatorname{adj}\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}, \quad \begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/6 & -1/6 & 5/6 \\ 1/6 & -5/6 & 1/6 \\ 1/6 & 7/6 & -5/6 \end{bmatrix}.$$

习题 4. *Suppose that all the entries of A are integers and $\det(A) = 1$. Explain why all the entries of A^{-1} are integers.*

解答. A^{-1} 的元素是 $adj(A)$ 的元素除去 $\det(A)$，由于 A 的元素都是整数，所以 $adj(A)$ 的元素也都是整数，结合 $\det(A) = 1$ 得到答案.

习题 5. *Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$ and $(5, 1, 0)$.*

解答. 体积 $= \begin{vmatrix} 1 & 1 & 5 \\ 0 & 2 & 1 \\ -3 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 7 & 15 \end{vmatrix} = 23.$

习题 6. *Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3). Show that the area of R equals the absolute value of*

$$\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

解答. 注意 $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$，
右端即是向量 $(x_2 - x_1, y_2 - y_1)$ 与 $(x_3 - x_1, y_3 - y_1)$ 张成的三角形的面积.

习题 7. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$, with a, b, c positive. Let S be the unit ball, bounded by the surface $x_1^2 + x_2^2 + x_3^2 = 1$.*

1. *Show that $T(S)$ is bounded by the ellipsoid* $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$.

2. *It is known that the volume of the unit ball is* $\frac{4}{3}\pi$. *Calculate the volume of* $T(S)$.

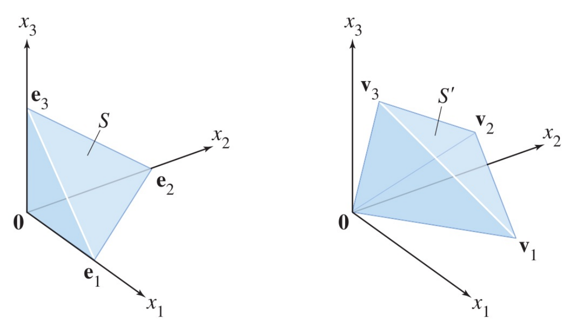
解答. 1. 设 $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ 是 S 中的任意点，则 $T(\vec{x})=\begin{bmatrix} ax_1 \\ bx_2 \\ cx_3 \end{bmatrix}$，注意

$$\frac{(ax_1)^2}{a^2}+\frac{(bx_2)^2}{b^2}+\frac{(cx_3)^2}{c^2}=x_1^2+x_2^2+x_3^2\le 1,$$

因而 $T(\vec{x})$ 是 $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$ 中的点.

2. 题中椭球与单位球的体积比恰为 A 的行列式，从而 $T(S)=\det(A)\cdot\frac{4\pi}{3}=\frac{4}{3}\pi abc$.

习题 8. *Let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors* $0,\vec{e}_1,\vec{e}_2,\vec{e}_3$. *Let S' be the tetrahedron with vertices at the vectors* $0,\vec{v}_1,\vec{v}_2,\vec{v}_3$. *See the figure.*



1. *Describe a linear transformation that maps S onto S'.*

2. *Find a formula for the volume of S', using the fact that the volume of S equals*

$$\frac{1}{3}\cdot\{\textit{area of the base}\}\cdot\{\textit{height}\}.$$

解答. 1. $\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}$.

2. S 的体积 $=\frac{1}{2}\frac{1}{3}=\frac{1}{6}$，从而 S' 的体积 $=|\vec{v_1}\quad\vec{v_2}\quad\vec{v_3}|\cdot S$的体积 $=\frac{1}{6}|\vec{v_1}\quad\vec{v_2}\quad\vec{v_3}|$.