Linear Algebra: Homework 3

October 17, 2021

Question 1.

Let W be the union of the first and the third quadrants in the xy-plane. That is, let

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| xy \geq 0 \right\}.$$

1. If $\vec{u}$ is in W and c any any scalar, is $c\vec{u}$ in W? Why?
2. Find specific vectors $\vec{u}$ and $\vec{v}$ in W such that $\vec{u} + \vec{v}$ is not in W. This is enough to show that W is not a vector space.

Solution 1.

1. Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in W \Rightarrow xy \geq 0.$

$$c\vec{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}, (cx) \cdot (cy) = c^2 \cdot xy \geq 0$$

   Therefore, $c\vec{u} \in W$.

2. Let

$$\vec{u} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}, \vec{v} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\vec{u}, \vec{v} \in W, but\, \vec{u} + \vec{v} = \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{\sqrt{3}-1}{2} \end{bmatrix} \notin W$$

   Thus W is not a vector space.

Question 2.

Determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate n, justify your answer.

1. All polynomials of the form $P(t) = at^2$, with $a \in \mathbb{R}$.
2. All polynomials of the form $P(t) = a + t^2$, with $a \in \mathbb{R}$.
3. All polynomials of degree at most 3, with integers as coefficients.
4. All polynomials in $\mathbb{P}_n$ such that $P(0) = 0$.

Solution 2.

1. **True.**
   (a) Let $a = 0$. Then it is the zero element.
   (b) Let $a, b \in \mathbb{R}$. Then, as $\mathbb{R}$ is a field, $(a + b) \in \mathbb{R}$.

$$P_1(t) + P_2(t) = at^2 + bt^2 = (a + b)t^2 \in \mathbb{P}_2$$

   Thus, the set is closed under addition.

(c) Let $a, c \in \mathbb{R}, (ac) \in \mathbb{R}$

$$c \cdot P(t) = c \cdot at^2 = (ca)t^2 \in \mathbb{P}_2$$

Thus, the set is closed under scalar multiplication.

2. **False.** Zero element does not exist, even though $a = 0$.

3. **False.** The set is not closed under scalar multiplication. That is, for a real number c, if it is not an integer, then the product of the polynomial and c does not belong to the set.

4. **True.**All polynomials in the set has the form of

$$P(t) = \sum_{i=1}^{n} a_i x^i.$$

(a) Let $a_1 = a_2 = \cdots = a_n = 0$. Then it is the zero element.

(b) Let $P_1(t), P_2(t) \in \mathbb{P}_n$. Also let $A, B \in \mathbb{R}$.Then,

$$A \cdot P_1(t) + B \cdot P_2(t) = A \cdot \sum_{i=1}^{n} a_{1i} x^i + B \cdot \sum_{i=1}^{n} a_{2i} x^i = \sum_{i=1}^{n} (A \cdot a_{1i} + B \cdot a_{2i}) x^i \in \mathbb{P}_n$$

Thus, the set is closed under addition and scalar multiplication.

Question 3.

Let H and K be subspaces of a vector space V. The intersection of H and K is defined to be

$$H \cap K = \{\vec{v} \in V | \vec{v} \in H \, and \, \vec{v} \in K\}.$$

Show that $H \cap K$ is a subspace of V. Give and example in $\mathbb{R}^2$ to show that the union of two subspaces is, in general, not a subspace.

Solution 3.

With regards to the intersection of subspaces, H and K are both subspaces of V, thus

1. $\vec{0} \in H \wedge \vec{0} \in K \Rightarrow \vec{0} \in H \cap V$.

2. Let $\vec{u}, \vec{v} \in H \cap V$.
Then, $\vec{u}, \vec{v} \in H \Rightarrow \vec{u} + \vec{v} \in H$.(1)
Similarly, $\vec{u} + \vec{v} \in K$.(2)
With (1) and (2), $\vec{u} + \vec{v} \in H \cap V$, thus $U \cap V$ is closed under addition.

3. Let $\vec{v} \in H \cap V, c \in \mathbb{R}$.
Then, $\vec{v} \in H \Rightarrow c \cdot \vec{v} \in H$.(1)
Similarly, $c \cdot \vec{v} \in K$.(2)
With (1) and (2), $c \cdot \vec{v} \in H \cap V$, thus $U \cap V$ is closed under scalar multiplication.

Q.E.D.
With regards to the union of two subspaces U and V, let $U = \{c_1 \cdot \vec{e_1} | c_1 \in \mathbb{R}\}, V = \{c_2 \cdot \vec{e_2} | c_2 \in \mathbb{R}\}$.
However, $\vec{e_1} + \vec{e_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin (H \cup V)$. Thus, $H \cup V$ is not a subspace.

Question 4.

Consider the polynomials $P_1(t) = 1 + t^2$ and $P_2(t) = 1 - t^2$. Is $\{P_1, P_2\}$ a linear independent set in $\mathbb{P}_3$? Why or why not?

Solution 4.

The set of polynomials is linearly independent $\Leftrightarrow A \cdot P_1(t) + B \cdot P_2(t) = 0$ yields $A = B = 0$.

$$A \cdot P_1(t) + B \cdot P_2(t) = (A + B) + (A - B)t^2 = 0$$

Since t is arbitrary, $A = B \wedge A = -B \Rightarrow A = B = 0$, thus $\{P_1, P_2\}$ is a linear independent set in $\mathbb{P}_3$.

Question 5.

Use coordinate vectors to test the linear independence of the set of polynomials. Explain your work.

1. $1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3$
2. $1 - 2t^2 - t^3, t + 2t^3, 1 + t - 2t^2$

Solution 5.

W.L.O.G., assume $1 \mapsto \vec{e_1}, t \mapsto \vec{e_2}, t^2 \mapsto \vec{e_3}, t^3 \mapsto \vec{e_4}$

1. The images of the polynomials are (1,0,0,2), (2,1,-3,0) and (0,-1,2,-1) respectively.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & -4 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   Thus, all columns are pivot columns, hence the polynomials are linearly independent.

2. The images of the polynomials are (1,0,-2,-1), (0,1,0,2) and (1,1,-2,0) respectively.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Thus, all columns are pivot columns, hence the polynomials are linearly independent.

Question 6.

The first four Laguerre polynomials are $1, 1 - t, 2 - 4t + t^2$ and $6 - 18t + 9t - t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

Solution 6.

The vector representations of these polynomials,relative to $\{1, t, t^2, t^3\}$, are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix}$ respectively.

If $\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \vec{x} = \vec{0}$ has trivial solution only, then these vectors (these polynomials) are linearly independent. Clearly, it is true.As there are 4 vectors, and $dim(\mathbb{P}_3) = 4$, thus these polynomials form a basis of $\mathbb{P}_3$.

Question 7.

Show that the space $C(\mathbb{R})$ of all continuous functions on the real line is an infinite dimensional vector space.

Solution 7.

$\mathbb{P}_n$ is a subspace of $C(\mathbb{R})$.
Suppose $dim(\mathbb{P}) = l < \infty$. $\forall n$, $\mathbb{P}_n \leq \mathbb{P}$. Thus, for $\mathbb{P}_{l-1}$,its dimension is l. Thus, that implies $\mathbb{P}_{L-1} = \mathbb{P}$, arising a contradiction.
In 8 and 9, dim(V) $< \infty$, and the vectors listed belong to V.

Question 8.

1. If there exists a set $\{\vec{v_1}, \cdots, \vec{v_p}\}$ that spans V, then $dim(V) \leq p$.
2. If there exists a linearly independent set $\{\vec{v_1}, \cdots, \vec{v_p}\}$ in V, then $dim(V) \geq p$.
3. If $dim(V) = p$, then there exists a spanning set of $p + 1$ vectors in V.

Solution 8.

1. **True.** Let $\{\vec{v_1}, \cdots, \vec{v_r}\}$ be a linearly independent set by removing p-r vectors. The set spans a space of $r \leq p$, thus $dim(V) \leq p$
2. **True.** Since the set is linearly independent, it spans a space which is the subspace of V with dimension p. Thus, dim(V) must not less than p.
3. **True.** Insert an arbitary vector other than the vectors into the set of basis to form the new spanning set.

Question 9.

1. If there exists a linearly dependent set $\{\vec{v_1}, \cdots, \vec{v_p}\}$ in V, then $dim(V) \leq p$.
2. If every set of p elements in V fails to span V, then $dim(V) > p$.
3. If $p \geq 2$ and $dim(V) = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

Solution 9.

1. **False.** A counter example is by letting the space be $\mathbb{R}^3$ and the set be $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$
2. **True.** The proposition is equivalent to "If $dim(V) \leq p$, then there exists a set of p elements that span V. For a vector subspace V, $dim(V) \leq p$,basis for V has no more than p vectors, which is the spanning set for V.
3. **False.** 9.1 demonstrates a counter-example.

Question 10.

Let $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$.

1. Solve the equation $A\vec{x} = 0$.
2. Let $\vec{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, is $\vec{b}$ in the range of the linear transformation $\vec{x} \mapsto A\vec{x}$? Why or why not?

Solution 10.

1. The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 5 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, $x_4 = 0$, and Let $x_3 = t$ as a parameter, where $t \in \mathbb{R}$.

Thus, $x_2 = -2t$, and $x_1 = -3t$. $\vec{x} = t \cdot \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}$

2. Let $\vec{x} = \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, where $\vec{x} \in Ker(A)$, the kernel of A. If $\vec{b} \in C(A)$, the range of A, then $\vec{b}$ is orthogonal to $\vec{x}$. However,

$$\vec{b} \cdot \vec{x} = 3 - 6 - 1 = -4,$$

thus b is outside the range of transformation of A.

Question 11.

An affine transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\vec{x}) = A\vec{x} + \vec{b}$, with A an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$. Show that T is not a linear transformation if $\vec{b} \neq 0$.

Solution 11.

If T is a linear transformation, then $\forall \vec{v_i} \in \mathbb{R}^n, c_i \in \mathbb{R}, i = \{1, 2, 3, \cdots, k\}$

$$T(\sum_{i=1}^{k} c_i \vec{v_i}) = \sum_{i=1}^{k} c_i T(\vec{v_i})$$

Given $T(\vec{x}) = A\vec{x} + \vec{b}$, so for linear transformation T,

$$\Rightarrow Left = A(\sum_{i=1}^{k} c_i \vec{v_i}) + \vec{b}$$

$$= \sum_{i=1}^{k} c_i A\vec{v_i} + \vec{b},$$

$$Right = \sum_{i=1}^{k} c_i (A\vec{v_i} + \vec{b})$$

$$= \sum_{i=1}^{k} c_i A\vec{v_i} + \sum_{i=1}^{k} c_i \vec{b}$$

Thus,

$$\vec{b} = \sum_{i=1}^{k} c_i \vec{b}$$

As c_i is arbitary, $\vec{b} = 0$.

Question 12.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why $\{T(\vec{v_1}), T(\vec{v_2}), T(\vec{v_3})\}$ is linearly dependent.

Solution 12.

Let $\vec{v_3} = a\vec{v_1} + b\vec{v_2}$. As T is a linear transformation, $T(\vec{v_3}) = T(a\vec{v_1} + b\vec{v_2}) = aT(\vec{v_1}) + bT(\vec{v_2})$

Question 13.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects points through the horizontal x_1-axis and then reflects points through the line $x_1 = x_2$. What is the standard matrix of T?

Solution 13.

W.L.O.G., assume $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,where $\vec{e_1}$ and $\vec{e_2}$ lie on x_1- and x_2- axes respectively.

The two transformations are then represented by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ respectively.

Thus, the standard matrix of T is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

Question 14.

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of the rotation?

Solution 14.

W.L.O.G., assume $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,where $\vec{e_1}$ and $\vec{e_2}$ lie on x_1- and x_2- axes respectively.

The matrices correspond to reflections along x_1- and x_2- axes are $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ respectively.

The matrix that corresponds to this composite transformation is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

Which is a rotation of π rad along the origin.

Question 15.

1. Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.
2. The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.
3. The standard matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that reflects points through the horizontal axis, the vertical axis, or the origin has the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where a and d are ± 1.
4. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.
5. If A is a 3×2 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ can not map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

Solution 15.

1. **False.** Suppose $\{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$ are te basis vectors of $\mathbb{R}^n$, and let the transformation be T. Then, T maps the basis vectors to $T(\vec{e_1}), T(\vec{e_2}), \cdots, T(\vec{e_n})$ respectively.
   Every vector $\vec{u} \in \mathbb{R}^n$ can be expressed as a linear combination of the basis vectors, so linear transformation
   $$T(\vec{u}) = T(\sum_{i=1}^{n} u_i \cdot \vec{e_i}) = \sum_{i=1}^{n} u_i \cdot T(\vec{e_i}) = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$
   $\begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix}$ is the standard matrix corresponds to the linear transformation.
2. **True.** Identity matrix, $I_n = \begin{bmatrix} \vec{e_1} & \vec{e_2} & \cdots & \vec{e_n} \end{bmatrix}$. Hence, from solution 15.1, the columns of the standard matrix are the images of columns of I_n.
3. **False.**
   Unless n=m=2, $\mathbb{R}^n, \mathbb{R}^m$ does neither match the domain and range of the transformation for $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.
4. **True.** Mapping uniquely is the definition of one-to-one.
5. **True.** Since A is 3×2, $dim(C(A)) \leq 2$, so $\mathbb{R}^3 \setminus C(A) \neq \emptyset$. Thus, the range of the transformation is a proper subset of $\mathbb{R}^3$, so the transformation is not onto.