Linear Algebra: Homework 9

December 11, 2021

Question 1.

Determine which pair of vectors are orthogonal:

(1) $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix};$

(2) $\mathbf{u} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}.$

Solution 1.

(1)

$$\langle \mathbf{u}, \mathbf{v} \rangle = 12 \times 2 + 3 \times (-3) + (-5) \times 3 = 24 - 9 - 15 = 0$$

Hence $\mathbf{u} \perp \mathbf{v}$.

(2)

$$\langle \mathbf{u}, \mathbf{v} \rangle = (-3) \times 1 + 7 \times (-8) + 4 \times 15 + 0 \times (-7) = -3 - 56 + 60 + 0 = 1$$

Hence $\mathbf{u} \not\perp \mathbf{v}$.

Question 2.

Mark each statement true or false, and justify your answer.

(1) $\langle \mathbf{v}, \mathbf{v} \rangle = ||\mathbf{v}||^2$.

(2) For any scalar c, $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

(3) If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(4) For a square matrix A, vectors in $\text{Col}(A)$ are orthogonal to vectors in $\text{Nul}(A)$.

(5) If vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p$ span a subspace W, and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_i$ for $i = 1, \cdots, p$, then $\mathbf{x}$ is in $W^\perp$.

Solution 2.

The following assumes $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$.

(1) True.

$$\langle \mathbf{v}, \mathbf{v} \rangle = \mathbf{v}^T\mathbf{v} = \sum_{i=1}^{n} v_i^2 = ||\mathbf{v}||^2 \ \blacksquare$$

(2) True.

$$\langle \mathbf{u}, c\mathbf{v} \rangle = \mathbf{u}^T(c\mathbf{v}) = c\mathbf{u}^T\mathbf{v} = c\langle \mathbf{u}, \mathbf{v} \rangle \ \blacksquare$$

(3) True. The statement can be rewritten as

$$||\mathbf{u}-\mathbf{v}|| = ||\mathbf{u}-(-\mathbf{v})|| \Leftrightarrow ||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}+\mathbf{v}||^2$$

By distributive property of inner product over vector addition,

$$||\mathbf{u}-\mathbf{v}||^2 - ||\mathbf{u}+\mathbf{v}||^2 = \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle - \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle$$

$$= \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u} - \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u} - \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle - \langle \mathbf{v}, \mathbf{v}\rangle$$

$$= -2\langle \mathbf{u}, \mathbf{v}\rangle - 2\langle \mathbf{v}, \mathbf{u}\rangle = -4\langle \mathbf{u}, \mathbf{v}\rangle = 0$$

$$\Leftrightarrow \langle \mathbf{u}, \mathbf{v}\rangle = 0 \blacksquare$$

(4) False. In fact, Col(A^T)=Nul$(A)^\perp$ Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{Nul}(A)$. However, referring to the second column of A,

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \neq 0,$$

which disproved the statement.

(5) True. $W^\perp$ is defined as

$$W^\perp = \{\mathbf{v} \in \mathbf{R}^n : (\forall \mathbf{w} \in W)\langle \mathbf{v}, \mathbf{w}\rangle = 0\}$$

$$W^\perp = \{\mathbf{x} \in \mathbf{R}^n : \langle \mathbf{x}, \sum_{i=1}^{p} c_i \mathbf{v}_i\rangle = 0, c_i \in \mathbf{R}\}$$

$$= \{\mathbf{x} \in \mathbf{R}^n : \sum_{i=1}^{p} c_i \langle \mathbf{x}, \mathbf{v}_i\rangle = 0, c_i \in \mathbf{R}\}$$

Hence $\forall \mathbf{x} \in \mathbf{R}^n$ such that $\langle \mathbf{x}, \mathbf{v}_i\rangle = 0$ for $i \in \{1, 2, \cdots, p\}$,

$$\sum_{i=1}^{p} c_i \langle \mathbf{x}, \mathbf{v}_i\rangle = 0 \Rightarrow \mathbf{x} \in W^\perp$$

Question 3.

Verify the *parallelogram law* for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$.

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2).$$

Solution 3.

By distributive property of inner product over vector addition,

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle$$

$$= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u} - \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle$$

$$= 2\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{v}, \mathbf{v}\rangle = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2) \blacksquare$$

Question 4.

Show that if $\mathbf{x}$ is in both W and $W^\perp$, then $\mathbf{x} = 0$.

Solution 4.

Let $W \subset \mathbf{R}^n$.
$W^\perp$ is defined as

$$W^\perp = \{\mathbf{v} \in \mathbf{R}^n : (\forall \mathbf{w} \in W)\langle \mathbf{v}, \mathbf{w}\rangle = 0\}$$

Hence,

$$\forall \mathbf{x} \in (W \cap W^\perp), \langle \mathbf{x}, \mathbf{x}\rangle = 0 \Rightarrow \mathbf{x} = 0$$

Question 5.

Determine which set of vectors are orthogonal:

(1) $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$

(2) $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}.$

Solution 5.

(1)

$$\begin{bmatrix} -6 & -3 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = -18 - 3 - 9 = -30 \neq 0$$

Hence this set of vectors is not orthogonal.

(2)

$$\begin{bmatrix} 3 & -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} = -3 - 6 - 3 + 12 = 0,$$

$$\begin{bmatrix} 3 & -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = 9 - 16 + 7 + 0 = 0,$$

$$\begin{bmatrix} -1 & 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = -3 + 24 - 21 + 0 = 0$$

Hence this set of vectors is orthogonal.

Question 6.

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbf{R}^2$ or $\mathbf{R}^3$ respectively. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}_i$'s.

(1) $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$.

(2) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$.

Solution 6.

(1) Orthogonality:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 12 - 12 = 0$$

Let $\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2$, then

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ -7 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 4 & -6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix},$$

So

$$\mathbf{x} = 3\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$$

(2) Orthognality:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = -1 + 1 = 0$$

$$\langle \mathbf{u}_1, \mathbf{u}_3 \rangle = 2 - 2 = 0$$

$$\langle \mathbf{u}_2, \mathbf{u}_3 \rangle = -2 + 4 - 2 = 0$$

Let $\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$, then find a, b, c by Gauss-Jordan:

$$\begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 4 & 1 & -4 \\ 1 & 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 4 & 1 & -4 \\ 0 & 2 & -4 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 4 & 1 & -4 \\ 0 & 0 & -9/2 & -9 \end{bmatrix},$$

So $c = 2, b = (-4-2)/4 = -3/2, a = 8 - 4 - 3/2 = 5/2$,

$$\mathbf{x} = \frac{5}{2}\mathbf{u}_1 - \frac{3}{2}\mathbf{u}_2 + 2\mathbf{u}_3$$

Question 7.

(1) Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span$\{\mathbf{u}\}$ and the other orthogonal to $\mathbf{u}$.

(2) Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

Solution 7.

(1) Let $\mathbf{v} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = 28 - 28 = 0 \Rightarrow \mathbf{u} \perp \mathbf{v}.$$

Let $\mathbf{y} = a\mathbf{u} + b\mathbf{v}$, then

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -7 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{65} \begin{bmatrix} 4 & -7 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix},$$

So

$$\mathbf{y} = -\frac{1}{5}\mathbf{u} + \frac{2}{5}\mathbf{v}, \text{ where } \mathbf{v} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

(2) Distance

$$= ||\mathbf{y} - \frac{\langle \mathbf{u}, \mathbf{y} \rangle}{||\mathbf{u}||^2}\mathbf{u}||$$

$$= \left|\left| \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{30}{100}\mathbf{u} \right|\right| = \sqrt{(3-2.4)^2 + (1-1.8)^2} = 1$$

Question 8.

(1) Let U and V be $n \times n$ orthogonal matrices. Explain why UV remains an orthogonal matrix.

(2) Let U be an orthogonal matrix, and construct V by interchanging some of the columns of U. Explain why V is an orthogonal matrix.

Solution 8.

Let $V \in \mathbf{R}^{n \times n}$. V is an orthogonal matrix iff $V^T V = I_n$.

(1) The proof follows by the fact that

$$(UV)^T(UV) = V^T U^T U V = V^T I_n V = V^T V = I_n \square$$

(2) Let $\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ represented by

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

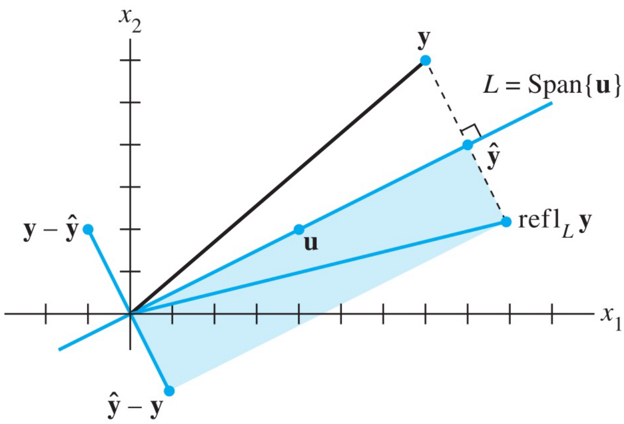
be a permutation of the columns of U that constructs V. Then,

$$P = \begin{bmatrix} \mathbf{e}_{\pi(1)} & \mathbf{e}_{\pi(2)} & \cdots & \mathbf{e}_{\pi(n)} \end{bmatrix}$$

P is an orthogonal matrix, since the columns of P are $\{\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(2)}, \cdots, \mathbf{e}_{\pi(n)}\}$ is just the rearrangement of the standard basis, which is orthogonal.
Then we have $V = UP$. U and P are orthogonal matrices, by (1) V is also an orthogonal matrix. $\blacksquare$

Question 9.

Given $\mathbf{u} \neq 0$ in $\mathbf{R}^n$, let L =Span$\{\mathbf{u}\}$. For $\mathbf{y} \in \mathbf{R}^n$, let $\text{Refl}_L(\mathbf{y})$ be the reflection of $\mathbf{y}$ with respect to L as shown in the figure.



Show that

$$\text{Refl}_L(\mathbf{y}) = 2 \cdot \text{proj}_L(\mathbf{y}) - \mathbf{y},$$

and that $\mathbf{y} \mapsto \text{Refl}_L(\mathbf{y})$ defines a linear transformation.

Solution 9.

By definition,

$$\text{proj}_L(\mathbf{y}) = \mathbf{u}(\mathbf{u}^T\mathbf{u})^{-1}\mathbf{u}^T\mathbf{y},$$

So, if $\text{Refl}_L(\mathbf{y})$ represents the image of y formed by reflection w.r.t. L,

$$\mathbf{y} - \text{proj}_L(\mathbf{y}) = -(\text{Refl}_L(\mathbf{y}) - \text{proj}_L(\mathbf{y}))$$

$$\Leftrightarrow \text{Refl}_L(\mathbf{y}) = 2 \cdot \text{proj}_L(\mathbf{y}) - \mathbf{y}, \square$$

Linearity of reflection can be shown by

$$\text{Refl}_L(a\mathbf{x} + b\mathbf{y}) = 2\mathbf{u}(\mathbf{u}^T\mathbf{u})^{-1}\mathbf{u}^T(a\mathbf{x} + b\mathbf{y}) - (a\mathbf{x} + b\mathbf{y})$$

$$= a(2\mathbf{u}(\mathbf{u}^T\mathbf{u})^{-1}\mathbf{u}^T\mathbf{x} - \mathbf{x}) + b(2\mathbf{u}(\mathbf{u}^T\mathbf{u})^{-1}\mathbf{u}^T\mathbf{y} - \mathbf{y})$$

$$= a \cdot \text{Refl}_L(\mathbf{x}) + b \cdot \text{Refl}_L(\mathbf{y})\blacksquare$$