Linear Algebra: Homework 10

December 14, 2021

Question 1.

Let $\mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. Find the distance from $\mathbf{y}$ to the plane in $\mathbf{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution 1.

Let $A = \begin{bmatrix} -3 & -3 \\ -5 & 2 \\ 1 & 1 \end{bmatrix}$. Then, distance

$$= ||\mathbf{y} - A(A^TA)^{-1}A^T\mathbf{y}||$$

$$= \left|\left| \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} -3 & -3 \\ -5 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 35 & 0 \\ 0 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 35 \\ -28 \end{bmatrix} \right|\right|$$

$$= \left|\left| \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} -3 & -3 \\ -5 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} \right|\right| = 2\sqrt{10}$$

Question 2.

Mark each statement true or false, and justify your answer.

(1) If W is a subspace of $\mathbf{R}^n$ and if $\mathbf{v}$ is both in W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

(2) In the orthogonal decomposition theorem, each term in the fomula for $\text{Proj}_W(\mathbf{y})$ is itself an orthogonal projection of $\mathbf{y}$ onto a subspace of W.

(3) If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, with $\mathbf{z}_1 \in W$ and $\mathbf{z}_2 \in W^\perp$, then $\mathbf{z}_1$ must be orthogonal projection of $\mathbf{y}$ onto W.

(4) The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \text{Proj}_W(\mathbf{y})$.

(5) if an $n \times p$ matrix U has orthonormal columns, then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^n$.

Solution 2.

(1) **True.**
$W^\perp$ is defined as $W^\perp = \{\mathbf{v} \in \mathbf{R}^n : (\forall \mathbf{w} \in W)\langle \mathbf{v}, \mathbf{w} \rangle = 0\}$
Hence, $\forall \mathbf{x} \in (W \cap W^\perp)$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Rightarrow \mathbf{x} = 0$

(2) **True.**
Let $\mathbf{u} \in W$. Then, projection of $\mathbf{y}$ on $\mathbf{u}$ is

$$\frac{\langle \mathbf{u}, \mathbf{y} \rangle}{||\mathbf{u}||^2}\mathbf{u},$$

its projection is

$$\frac{\langle \mathbf{u}, \frac{\langle \mathbf{u}, \mathbf{y} \rangle}{||\mathbf{u}||^2}\mathbf{u} \rangle}{||\mathbf{u}||^2}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{y} \rangle}{||\mathbf{u}||^2}\frac{\langle \mathbf{u}, \mathbf{u} \rangle}{||\mathbf{u}||^2}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{y} \rangle}{||\mathbf{u}||^2}\mathbf{u}$$

(3) **True.**
i). $\mathbf{z}_1 \in W$, so $\text{Proj}_W(\mathbf{z}_1) = \mathbf{z}_1$.
ii). $\mathbf{z}_2 \in W^{\perp}$, so $\text{Proj}_W(\mathbf{z}_2) = 0$.
By i) ii) and the linearity of projection,
$\text{Proj}_W(\mathbf{y}) = \text{Proj}_W(\mathbf{z}_1 + \mathbf{z}_2) = \mathbf{z}_1$

(4) **False.** If $\mathbf{y} \notin W$, then $(\mathbf{y} - \text{Proj}_W(\mathbf{y})) \notin W$. In fact, it should be $\text{Proj}_W(\mathbf{y})$, that gives the best approximation.

(5) **False.** Let $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $UU^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not an identity matrix.

Question 3.

Find an orthogonal basis for the column space of the matrix

$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$$

and a QR-factorization of it.

Solution 3.

Let $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$, $\mathbf{a}_1 = [1\ \text{-}1\ 0\ 1\ 1]^T$, $\mathbf{a}_2 = [3\ \text{-}3\ 2\ 5\ 5]^T$, $\mathbf{a}_3 = [5\ 1\ 3\ 2\ 8]^T$

$$\mathbf{q}_1 = \mathbf{a}_1$$

$$\mathbf{q}_2 = \mathbf{a}_2 - \mathbf{q}_1 \frac{\mathbf{q}_1^T \mathbf{a}_2}{||\mathbf{q}_1||^2} = \mathbf{a}_2 - 4\mathbf{q}_1 = [\begin{matrix} -1 & 1 & 2 & 1 & 1 \end{matrix}]^T$$

$$\mathbf{q}_3 = \mathbf{a}_3 - \mathbf{q}_1 \frac{\mathbf{q}_1^T \mathbf{a}_3}{||\mathbf{q}_1||^2} - \mathbf{q}_2 \frac{\mathbf{q}_2^T \mathbf{a}_3}{||\mathbf{q}_2||^2} = \mathbf{a}_3 - \frac{7}{2}\mathbf{q}_1 - \frac{3}{2}\mathbf{q}_2 = \mathbf{a}_3 - \frac{1}{2}[\begin{matrix} 4 & -4 & 6 & 10 & 10 \end{matrix}]^T = [\begin{matrix} 3 & 3 & 0 & -3 & 3 \end{matrix}]^T$$

So we can set

$$Q = \begin{bmatrix} 1/2 & -\sqrt{2}/4 & 1/2 \\ -1/2 & \sqrt{2}/4 & 1/2 \\ 0 & \sqrt{2}/2 & 0 \\ 1/2 & \sqrt{2}/4 & -1/2 \\ 1/2 & \sqrt{2}/4 & 1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/2 & -1/2 & 0 & 1/2 & 1/2 \\ -\sqrt{2}/4 & \sqrt{2}/4 & \sqrt{2}/2 & \sqrt{2}/4 & \sqrt{2}/4 \\ 1/2 & 1/2 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 6 \end{bmatrix}$$

Question 4.

Mark each statement true or false, and justify your answer.

(1) If $A = QR$ and Q has orthonormal columns, then $R = Q^T A$.

(2) Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ linearly independent, let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

(3) If $\mathbf{x}$ is not in the subspace W, then $\mathbf{x}-\text{Proj}_W(\mathbf{x})$ is non-zero.

(4) In a QR-factorization $A = QR$, where A has linearly independent columns, the column vectors of Q form an orthonormal basis for $\text{Col}(A)$.

Solution 4.

(1) **True.**
Columns of Q are orthonormal, hence $Q^TQ = I$.

$$A = QR \Rightarrow Q^TA = Q^TQR = R$$

(2) **True.**
Each of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, by definition, is a specific linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. So, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in W$. Dim W=3, and orthogonal vectors are linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

(3) **True.**
Assume $\mathbf{x} \in \mathbf{R}^n$, $W \subset \mathbf{R}^n$. Let $A \in \mathbf{R}^{n\times p}$, where p =dim W be a matrix with columns as the basis of W. Since $x \notin W$, $\forall \mathbf{c} \in \mathbf{R}^p$, $\exists \epsilon \in \mathbf{R}^n, \epsilon \neq 0$ s.t.

$$A\mathbf{c} + \epsilon = \mathbf{x}$$

(4) **True.**
The columns of Q are calculated by Gram-Schmidt orthogonalization of the basis of A, in which each of the vector in the basis is still the linear combination of the columns of A, so the basis formed by these orthonormal vectors still spans the same space.

Question 5.

(1) Let $A = QR$, where Q is $m \times n$ and R is $n \times n$. Show if that the columns of A are linearly independent, then R must be invertible.

(2) Let $A = QR$ with R invertible, show that A and Q have the same columns space.

Solution 5.

(1) Assume R is singular, then $\exists \mathbf{x} \in \mathbf{R}^n, \mathbf{x} \neq 0$ s.t. $R\mathbf{x} = 0$

$$\Rightarrow Q(R\mathbf{x}) = Q \cdot 0 = 0 \Rightarrow (QR)\mathbf{x} = A\mathbf{x} = 0,$$

Hence, the columns of A are linearly dependent. $\square$

(2) Let $A, Q \in \mathbf{R}^{n\times p}$. $R \in \mathbf{R}^{p\times p}$.
For each column $\mathbf{a}_j$ of A, $j \in \{1, 2, \cdots, p\}$,

$$\mathbf{a}_j = \sum_{i=1}^{p} \mathbf{q}_i (R)_{ij}$$

Hence every element in $\text{Col}(A)$ is also a specific linear combination of $\mathbf{q}_i$, $i \in \{1, 2, \cdots, p\}$

$$\Rightarrow \text{Col}(A) \subseteq \text{Col}(Q)$$

R is invertible, so $Q = AR^{-1}$.
Hence every element in $\text{Col}(Q)$ is also a specific linear combination of $\mathbf{a}_i$, $i \in \{1, 2, \cdots, p\}$

$$\Rightarrow \text{Col}(Q) \subseteq \text{Col}(A)$$

$$\Rightarrow \text{Col}(A) = \text{Col}(Q) \blacksquare$$

Question 6.

Let A be a QR factorization of an $m \times n$ matrix A with linearly independent columns. Partition A as $[A_1\ A_2]$, where A_1 has p columns. Show how to obtain a QR-factorization of A_1 and explain the reason.

Solution 6.

For each column $\mathbf{a}_j$ of A,

$$\mathbf{a}_j = \sum_{i=1}^{n} \mathbf{q}_i (R)_{ij}$$

R is upper-triangular, so $(R)_{ij} = 0$ for $i > j$. Specially, for $j = p$,

$$\mathbf{a}_p = \sum_{i=1}^{p} \mathbf{q}_i (R)_{ip}$$

Then, $A_1 = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_p \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ & R_{22} & \cdots & R_{2p} \\ & & \ddots & \vdots \\ & & & R_{pp} \end{bmatrix}$.

Partition Q as $[Q_1\ Q_2]$, R as as $[R_1\ R_2]$, where both of Q_1 and R_1 have p columns.
Since the $p+1$-th to the $m-$th rows of R_1 are zeros, pick the first p rows of R_1 to form R'. Then, $A_1 = Q_1 R'$.

Question 7.

Find the least-square solution of $A\mathbf{x} = \mathbf{b}$ by solving the corresponding normal equation:

$$(1)\ A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix};\ (2)\ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

Solution 7.

The following assumes $A\hat{\mathbf{x}} + \epsilon = \mathbf{b}$, where $A^T\epsilon = 0$, so it becomes finding $\hat{\mathbf{x}}$, for $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$.

(1)

$$A^TA = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}, A^T\mathbf{b} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 & 6 \\ 6 & 42 & -6 \end{bmatrix} \sim \begin{bmatrix} 6 & 6 & 6 \\ 0 & 36 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \end{bmatrix}$$

$$\hat{\mathbf{x}} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

(2)

$$A^TA = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}, A^T\mathbf{b} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 1/2 & 7/2 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1/2 & -1/2 & -3/2 \\ 0 & -1/2 & 1/2 & 3/2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1/2 & -1/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Treating x_3 as free variable, $x_2 = -3 + x_3$, $x_1 = 2 - x_2 = 5 - x_3$

$$\hat{\mathbf{x}} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Question 8.

Find the orthognal projection of $\mathbf{b}$ onto Col(A) and the least square solution of $A\mathbf{x} = \mathbf{b}$:

$$(1)\ A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix};\ (2)\ A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

Solution 8.

Let $\hat{\mathbf{b}}$=Proj$_{\mathrm{Col}(A)}(\mathbf{b})$

$$(1)\ \hat{\mathbf{b}} = \tfrac{9}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \tfrac{12}{24}\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 4 & -1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$$

$$(2)\ \hat{\mathbf{b}} = \tfrac{36}{54}\begin{bmatrix} 4 \\ 1 \\ 6 \\ 1 \end{bmatrix} + \tfrac{0}{27}\begin{bmatrix} 0 \\ -5 \\ 1 \\ -1 \end{bmatrix} + \tfrac{9}{27}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 & 3 \\ 1 & -5 & 1 & 1 \\ 6 & 1 & 0 & 4 \\ 1 & -1 & -5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/4 & 3/4 \\ 0 & -5 & 3/4 & 1/4 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & -1 & -21/4 & -7/4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/4 & 3/4 \\ 0 & 1 & -3/20 & -1/20 \\ 0 & 0 & -27/20 & -9/20 \\ 0 & 0 & -108/20 & -36/20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1/4 & 3/4 \\ 0 & 1 & -3/20 & -1/20 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\hat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

Question 9.

With the given QR-factorization of A, compute the least square solution of $A\mathbf{x} = \mathbf{b}$:

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

Solution 9.

Let $\epsilon = A\hat{\mathbf{x}} - \mathbf{b}$, in which $A^T\epsilon = 0$.
Then, $R^TQ^T\epsilon = 0$. Since R is invertible, $Q^T\epsilon = 0$.

$$Q^TQR\hat{\mathbf{x}} + Q^T\epsilon = R\hat{\mathbf{x}} = Q^T\mathbf{b},$$

$$Q^T\mathbf{b} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}\hat{\mathbf{x}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} \Rightarrow \hat{\mathbf{x}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Question 10.

Let A be an $m \times n$ matrix. Use the steps below to show that a vector $\mathbf{x} \in \mathbf{R}^n$ satisfies $A\mathbf{x} = 0$ if and only if $A^T A\mathbf{x} = 0$. This implies Nul(A)=Nul($A^T A$).

(1) Show that if $A\mathbf{x} = 0$, then $A^T A\mathbf{x} = 0$.

(2) Suppose that $A^T A\mathbf{x} = 0$. Explain why $\mathbf{x}^T A^T A\mathbf{x} = 0$ and deduce from it that $A\mathbf{x} = 0$.

Deduce from the above results that rk($A^T A$)=rk(A).

Solution 10.

(1) $A\mathbf{x} = 0 \Rightarrow A^T A\mathbf{x} = A^T(A\mathbf{x}) = A^T \cdot 0 = 0$

(2) $A^T A\mathbf{x} = 0 \Rightarrow \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T(A^T A\mathbf{x}) = \mathbf{x}^T \cdot 0 = 0,$
$\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = ||A\mathbf{x}||^2 = 0 \Rightarrow ||A\mathbf{x}|| = 0 \Rightarrow A\mathbf{x} = 0$

By (1)(2), $(\forall x \in \mathbf{R}^n)(\mathbf{x} \in$Nul$(A) \leftrightarrow \mathbf{x} \in$Nul$(A^T A)) \Rightarrow$ dim Nul(A)= dim Nul($A^T A$).
By rank-nullity theorem,
rk($A^T A$)+dim Nul($A^T A$)=n=rk(A)+dim Nul(A)$\Rightarrow$ rk($A^T A$)=rk(A).$\blacksquare$

Question 11.

A certain experiment produces the data $(1, 7.9)$, (2,5.4), (3,-0.9). Describe the model that produces a least squares fit of these points by a function of the form $y = A\cos(x) + B\sin(x)$.

Solution 11.

$$\begin{bmatrix} \cos 1 & \sin 1 \\ \cos 2 & \sin 2 \\ \cos 3 & \sin 3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \vec{\varepsilon} = \begin{bmatrix} 7.9 \\ 5.4 \\ -0.9 \end{bmatrix}$$