# 作业 6 简答

李子钰，肖子达

2021 年 11 月 21 日

**习题 1.** *In* $\mathbb{P}_2$*, find the transition matrix from the basis*

$$\mathcal{B} = \left\{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\right\}$$

*to the standard basis. Then write* $t^2$ *as a linear combination of the polynomials in* $\mathcal{B}$*.*

**解答.** 令

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix},$$

则 $B$ 可以看成把标准基底 $1, t, t^2$ 变为基底 $\mathcal{B}$ 的线性变换. 求得

$$B^{-1} = \begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix}.$$

直接计算

$$\begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix},$$

这说明

$$t^2 = 3\left(1 - 3t^2\right) - 2\left(2 + t - 5t^2\right) + (1 + 2t).$$

**习题 2.** *Let* $P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}$, *and*

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix}$$

1. *Find a basis* $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ *for* $\mathbb{R}^3$ *such that* $P$ *is the transition matrix from* $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ *to the basis* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
2. *Find a basis* $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ *for* $\mathbb{R}^3$ *such that* $P$ *is the transition matrix from* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ *to the basis* $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

**解答.** *1.* 直接计算

$$\begin{bmatrix} \overrightarrow{u_1} & \overrightarrow{u_2} & \overrightarrow{u_3} \end{bmatrix} = \begin{bmatrix} \overrightarrow{v_1} & \overrightarrow{v_2} & \overrightarrow{v_3} \end{bmatrix} P = \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -6 & -5 \\ -5 & -9 & 0 \\ 21 & 32 & 3 \end{bmatrix}.$$

因此 $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\} = \left\{ \begin{bmatrix} -6 \\ -5 \\ 21 \end{bmatrix}, \begin{bmatrix} -6 \\ -9 \\ 32 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} \right\}$.

*2.*

$$P^{-1} = \begin{bmatrix} 5 & 8 & 5 \\ -3 & -5 & -3 \\ -2 & -2 & -1 \end{bmatrix}$$

从而

$$\begin{aligned} \begin{bmatrix} \vec{w}_1 & \overrightarrow{w_2} & \overrightarrow{w_3} \end{bmatrix} &= \begin{bmatrix} \overrightarrow{v_1} & \overrightarrow{v_2} & \overrightarrow{v_3} \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix} \begin{bmatrix} 5 & 8 & 5 \\ -3 & -5 & -3 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 28 & 38 & 21 \\ -9 & -13 & -7 \\ -3 & 2 & 3 \end{bmatrix}. \end{aligned}$$

因此 $\{\overrightarrow{w_1}, \overrightarrow{w_2}, \overrightarrow{w_3}\} = \left\{ \begin{bmatrix} 28 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 38 \\ -13 \\ 2 \end{bmatrix}, \begin{bmatrix} 21 \\ -7 \\ 3 \end{bmatrix} \right\}$.

**习题 3.** *Let* $\mathcal{B}=\left\{\vec{b}_1,\vec{b}_2\right\},\mathcal{C}=\{\vec{c}_1,\vec{c}_2\}$ *and* $\mathcal{D}=\left\{\vec{d}_1,\vec{d}_2\right\}$ *be bases for a two dimensional vector space. Write an equation that relates the matrices* $P_{\mathcal{B}\to\mathcal{C}},P_{\mathcal{C}\to\mathcal{D}}$ *and* $P_{\mathcal{B}\to\mathcal{D}}$*. Justify your answer.*

**解答.** 它们有关系

$$P_{\mathcal{B}\to\mathcal{D}}=P_{\mathcal{C}\to\mathcal{D}}P_{\mathcal{B}\to\mathcal{C}},$$

设向量 $\vec{v}$ 在这三组基底的坐标分别为 $V_B,V_C,V_D$，即有

$$\vec{v}=\left[\begin{array}{cc}\vec{b}_1 & \overrightarrow{b}_2\end{array}\right]V_B=\left[\begin{array}{cc}\vec{c}_1 & \overrightarrow{c}_2\end{array}\right]V_C=\left[\begin{array}{cc}\vec{d}_1 & \overrightarrow{d}_2\end{array}\right]V_D,$$

由定义

$$P_{\mathcal{B}\to\mathcal{D}}V_B=V_D,\quad P_{\mathcal{B}\to\mathcal{C}}V_B=V_C,\quad ,P_{\mathcal{C}\to\mathcal{D}}V_C=V_D,$$

故有

$$P_{\mathcal{B}\to\mathcal{D}}V_B=P_{\mathcal{C}\to\mathcal{D}}P_{\mathcal{B}\to\mathcal{C}}V_B,$$

约去 $V_B$ 即可.

**习题 4.** *Calculate the determinants by cofactor expansion:*

$$\left|\begin{array}{ccc}2&3&-3\\4&0&3\\6&1&5\end{array}\right|,\quad\left|\begin{array}{ccccc}4&0&-7&3&-5\\0&0&2&0&0\\7&3&-6&4&-8\\5&0&5&2&-3\\0&0&9&-1&2\end{array}\right|.$$

**解答.**

$$\left|\begin{array}{ccc}2&3&-3\\4&0&3\\6&1&5\end{array}\right|=-4\left|\begin{array}{cc}3&-3\\1&5\end{array}\right|-3\left|\begin{array}{cc}2&3\\6&1\end{array}\right|=-72+48=-24$$

$$\left|\begin{array}{ccccc}4&0&-7&3&-5\\0&0&2&0&0\\7&3&-6&4&-8\\5&0&5&2&-3\\0&0&9&-1&2\end{array}\right|=-2\left|\begin{array}{cccc}4&0&3&-5\\7&3&4&-8\\5&0&2&-3\\0&0&-1&2\end{array}\right|=-6\left|\begin{array}{ccc}4&3&-5\\5&2&-3\\0&-1&2\end{array}\right|.$$

$$=-6\left(\left|\begin{array}{cc}4&-5\\5&-3\end{array}\right|+2\left|\begin{array}{cc}4&3\\5&2\end{array}\right|\right)=-6(13-14)=6$$

**习题 5.** *What is the determinant of an elementary row replacement matrix? What is the determinant of an elementary scaling matrix with $k$ on the diagonal?*

**解答.** 行倍加变换不改变行列式，因而它等于单位阵的行列式 1. 设乘 $k$ 的初等数乘矩阵为

$$\begin{bmatrix} k & & & & \\ & \ddots & & & \\ & & k & & \\ & & & \ddots & \\ & & & & k \end{bmatrix}_{n\times n},$$

则其行列式是 $k^n$.

**习题 6.** *Combine the methods of row reduction and cofactor expansion to calculate the determiants:*

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix}$$

**解答.**

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & -3 & -2 & 0 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ 0 & 5 & 3 \end{vmatrix} = 6\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} = 6$$

$$\begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 2 & 0 & 0 & 0 \\ -6 & 5 & 5 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & -2 & -4 \\ 2 & 0 & 0 \\ -6 & 5 & 5 \end{vmatrix} = -20 + 40 = 20$$