Topics in Linear Algebra: Homework 6

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Solution 1.6.1.

1. $$L = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$

   L is commonly known as Vandermonde matrix.

2. For any square Vandermonde matrix,

   $$\det(L) = \prod_{1 \le i < j \le n} (a_j - a_i)$$

   $det(L) \neq 0$ iff none of the factors above is zero iff $a_i \neq a_j,\ \forall i, j : 1 \le i < j \le n$

3. $a_i \neq a_j,\ \forall i, j : 1 \le i < j \le n \Leftrightarrow det(L) \neq 0 \Leftrightarrow$ all row vectors of $L(= \mathrm{ev}_{a_i})$ are linearly independent. n independent vectors are required to span V^*, so $\mathrm{ev}_{a_i}, 1 \le i \le n$ are linearly independent $\Leftrightarrow$ $\mathrm{ev}_{a_1}, \cdots, \mathrm{ev}_{a_n}$ form a basis of V^*.

4. The three vectors should satisfy

   $$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = I_3$$

   So inverting that Vandermonde matrix solves the three polynomials.

   $$\begin{bmatrix} 1 & -1 & 1 & 1 & & \\ 1 & 0 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 & & \\ 0 & 1 & -1 & -1 & 1 & \\ 0 & 0 & 1 & 0.5 & -1 & 0.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 & -1 & 0.5 \end{bmatrix}$$

   Therefore,

   $$p_{-1}(x) = -\frac{1}{2}x + \frac{1}{2}x^2, p_0(x) = 1 - x^2, p_1(x) = \frac{1}{2}x + \frac{1}{2}x^2$$

5. This is equivalent to finding the cokernel of the Vandermonde matrix

   $$x^T L = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} = 0$$

   So

   $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -3 & -4 & -3 & 0 \\ 0 & 7 & 8 & 9 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 6 & 12 \\ 0 & 0 & -6 & -12 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}, x = x_5 \cdot \begin{bmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \end{bmatrix}$$

Therefore,

$$\mathrm{ev}_{-2} - 4\mathrm{ev}_{-1} + 6\mathrm{ev}_0 - 4\mathrm{ev}_1 + \mathrm{ev}_2 = 0$$

Solution 1.6.2.

Dual vectors are linear maps, so linearity should be checked.

1. It is.

$$0 \mapsto \mathrm{ev}_5((x+1)\cdot 0) = 0$$

$$\forall p, q : p, q \in V, \forall c_1, c_2 : c_1, c_2 \in F$$

$$\mathrm{ev}_5((x+1)\cdot(c_1 p + c_2 q)(x)) = 6(c_1 p + c_2 q)(5) = c_1 \cdot 6p(5) + c_2 \cdot 6q(5)$$
$$= c_1 \mathrm{ev}_5((x+1)p(x)) + c_2 \mathrm{ev}_5((x+1)q(x))$$

2. It is not. Suppose $p(x) = 1 - x^2$, $q(x) = 1 + x^2$,

$$(p+q) \mapsto \lim_{x\to+\infty} \frac{(p+q)(x)}{x} = 0,$$

But

$$\lim_{x\to+\infty} \frac{p(x)}{x} + \lim_{x\to+\infty} \frac{q(x)}{x} = -\infty + \infty$$

is not defined.

3. It is. Suppose $p(x) = a_1 + b_1 x + c_1 x^2$, $q(x) = a_2 + b_2 x + c_2 x^2$,

$$\lim_{x\to+\infty} \frac{(Ap+Bq)(x)}{x^2} = \lim_{x\to+\infty} \left(\frac{Aa_1 + Ba_2}{x^2} + \frac{Ab_1 + Bb_2}{x} + (Ac_1 + Bc_2) \right) = Ac_1 + Bc_2,$$

$$\lim_{x\to+\infty} \frac{Ap(x)}{x^2} = \lim_{x\to+\infty} \left(\frac{Aa_1}{x^2} + \frac{Ab_1}{x} + (Ac_1) \right) = Ac_1,$$
$$\lim_{x\to+\infty} \frac{Bq(x)}{x^2} = \lim_{x\to+\infty} \left(\frac{Ba_2}{x^2} + \frac{Bb_2}{x} + (Bc_2) \right) = Bc_2$$

Sum up the last two equations yields the first. Moreover,

$$0 \mapsto \lim_{x\to+\infty} \frac{0}{x^2} = 0$$

4. It is not.

$$[(x^2 + x)] \mapsto (3^2 + 3)(2(4) + 1) = 108$$

$$[x^2] \mapsto (3^2)(2(4)) = 72$$

$$[x] \mapsto (3)(1) = 3$$

Sum up the last 2 equations, it does not equal to the first.

5. It is not. Suppose $p(x) = x$, then

$$p \mapsto deg(p) = 1,$$

But

$$2p \mapsto deg(2p) = 1,$$

So double of the first equation cannot yield the second, it is not linear.

Solution 1.6.3.

Since $\nabla f : \mathbf{R}^2 \to \mathbf{R}$, if the map is linear, it is a dual vector.
The following lemma should be proved first:
Lemma: Given differentiable function f, its directional derivative with direction $\mathbf{u}$ is

$$\nabla_{\mathbf{u}}(f) = \nabla(f) \cdot \mathbf{u}$$

where $\nabla(f) = [\ \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\ ]$.
Proof:
Directional derivative of f at $\mathbf{p}$ with direction u is

$$\nabla_u(f) = \lim_{t\to 0} \frac{f(p_x + tu_x, p_y + tu_y) - f(p_x, p_y)}{t}$$

Since f is differentiable, at (x_0, y_0), then

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1, \epsilon_2 \to 0$ when $\Delta x, \Delta y \to 0$ (as $\frac{\epsilon_1 \Delta x + \epsilon_2 \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \le |\epsilon_1| + |\epsilon_2|$). So,

$$\lim_{t\to 0^+} \frac{f(p_x + tu_x, p_y + tu_y) - f(p_x, p_y)}{t}$$

$$= \lim_{t\to 0^+} \left(f'_x(p_x, p_y)\frac{tu_x}{t} + f'_y(p_x, p_y)\frac{tu_y}{t} \right) = f'_x(p_x, p_y)u_x + f'_y(p_x, p_y)u_y \ \square$$

With regards to the zero element,

$$\nabla_0(f) = \nabla(f) \cdot 0 = 0$$

With regards to linearity,

$$\nabla_{au+bv}(f) = \nabla(f) \cdot (au + bv) = a\nabla(f) \cdot u + b\nabla(f) \cdot v = a\nabla_u(f) + b\nabla_v(f)$$

It is trivial from the derivation above that the coordinates of the dual vector, is exactly the gradient of f.

Solution 1.6.4.

Denote U^0 as the annihilator of U.

1. Let $\alpha \in W^*$.

   $$\alpha \in (\mathrm{Ran}(L))^0 \Leftrightarrow \alpha(L(v)) = 0 (\forall v \in V) \Leftrightarrow \alpha \circ L = 0 \Leftrightarrow L^*(\alpha) = 0 \Leftrightarrow \alpha \in \mathrm{Ker}(L^*)$$

   Hence $(\mathrm{Ran}(L))^0 = \mathrm{Ker}(L^*)$.

2. Let $\beta \in V^*$.
   On one hand,

   $$\beta \in \mathrm{Ran}(L^*) \Leftrightarrow \exists \omega \in W^* : \beta = L^*(\omega) \Leftrightarrow \exists \omega \in W^* : \beta(v) = \omega \circ L(v)$$

   In particular, when $v \in \mathrm{Ker}(L), \beta(v) = \omega \circ L(v) = 0$.
   Therefore $\mathrm{Ran}(L^*) \subseteq (\mathrm{Ker}(L))^0$.
   Before proving the backward direction, define $\sim$ as an equivalence relation that:

   $$x \sim y \Leftrightarrow L(x) = L(y)$$

   It is trivial that $\sim$ is reflexive, symmetric and transitive. In particular,

   $$[x] = [y] \Leftrightarrow (x - y) \in \mathrm{Ker}(L)$$

   Let $f : V \to V/\sim$ be a surjective function such that $f : v \mapsto [v]$. Then $L = L_1 \circ f$, where $L_1 : V/\sim \to \mathrm{Ran}(L)$ is bijective. Immediately we have $f = L_1^{-1} \circ L$.
   So on the other hand, if $\beta \in (\mathrm{Ker}(L))^0$, $\mathrm{Ker}(L) \subseteq \mathrm{Ker}(\beta)$. Moreover β can be factorized as $\beta = \xi \circ f$, where $\xi : V/\sim \to \mathbf{R}$.

   $$\beta = \xi \circ f = \xi \circ (T_1^{-1} \circ T) = (\xi \circ T_1^{-1}) \circ T = T^*(\xi \circ T_1^{-1})$$

   So, $(\mathrm{Ker}(L))^0 \subseteq \mathrm{Ran}(L^*)$. Hence $(\mathrm{Ker}(L))^0 = \mathrm{Ran}(L^*)$.