Topics in Linear Algebra: Homework 4

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Solution 1.4.1.

1. $B = \begin{bmatrix} I & X \\ & I \end{bmatrix}$, where X satisfies

$$\begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} X - X \begin{bmatrix} 3 & 5 \\ & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\begin{cases} -2a + 2c & = & 1 \\ -5a - 3b + 2d & = & 2 \\ -2c & = & 3 \\ -5c - 3d & = & 4 \end{cases}$$

So $B = \begin{bmatrix} 1 & 0 & -2 & 31/9 \\ 0 & 1 & -3/2 & 7/6 \\ & & 1 & \\ & & & 1 \end{bmatrix}$

2.

$$\mathcal{B} = \left\{ \begin{bmatrix} -4 \\ -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 62 \\ 21 \\ 0 \\ 18 \end{bmatrix} \right\}$$

is a basis that span $V_3 + V_4$.

Solution 1.4.2.

1.

$$\begin{bmatrix} I & & & \\ & & I & \\ & I & & \\ & & & I \end{bmatrix} \begin{bmatrix} X & & I & \\ & Y & & I \\ & & X & \\ & & & Y \end{bmatrix} = \begin{bmatrix} X & & I & \\ & & X & \\ & Y & & I \\ & & & Y \end{bmatrix}$$

$$\begin{bmatrix} X & I & & \\ & X & & \\ & & Y & I \\ & & & Y \end{bmatrix} \begin{bmatrix} I & & & \\ & & I & \\ & I & & \\ & & & I \end{bmatrix} = \begin{bmatrix} X & & I & \\ & & X & \\ & Y & & I \\ & & & Y \end{bmatrix}$$

Hence $\begin{bmatrix} X & & I & \\ & Y & & I \\ & & X & \\ & & & Y \end{bmatrix}$ and $\begin{bmatrix} X & I & & \\ & X & & \\ & & Y & I \\ & & & Y \end{bmatrix}$ are similar.

2.

$$p_B(x) = (3 - x)(4 - x) = 12 - 7x + x^2,$$

$$p_A(x) = (3 - x)^2(4 - x)^2 = 144 - 168x + 73x^2 - 14x^3 + x^4$$

3.
$$A = \begin{bmatrix} & 1 & 1 & \\ & & & 1 \\ & & & 1 \\ & & & 0 \end{bmatrix}$$

A has two pivot columns, rank$(A) = 2$.

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ & & & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix},$$

So rank$(A^2) = 1$, rank(A^3) =rank$(A^4) = 0$.

$$J = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

4. $p_B(x) = x^2, p_A(x) = x^3$

5. If
$$p_B(x) = \prod_{i=1}^{s} (x - \lambda_i I)^{m_i},$$
where m_i is the algebraic multiplicity of λ_i, and s is number of distinct eigenvalues of B, then
$$p_A(x) = p_B(x) \cdot \prod_{i=1}^{s} (x - \lambda_i I)^{\mathrm{Dim}(\mathrm{Ker}(A - \lambda_i I))}$$

Solution 1.4.3.

1. $\forall a : a \in R$,
$$L(aI) = aX - Xa = aX - aX = 0$$

2.
$$\begin{aligned} L(X)Y + XL(Y) &= (AX - XA)Y + X(AY - YA) = AXY - XAY + XAY - XYA \\ &= A(XY) - (XY)A = L(XY) \end{aligned}$$

3. Here we prove the following lemma first:
$$S(n) : L(X^n) = n \cdot L(X) \cdot X^{n-1}$$
Showing that $S(n)$ holds $\forall n \in Z^+$.
$S(1)$ is trivial.
Suppose $S(k)$ holds, then for $S(k+1)$,
$$\begin{aligned} L(X^{k+1}) &= XL(X^k) + L(X)X^k \\ &= X(kL(X)X^{k-1}) + L(X)X^k = kL(X)X^k + L(X)X^k = (k+1)L(X)X^k, \end{aligned}$$
So $S(k) \Rightarrow S(k+1)$. By first principle of induction, $(\forall n)(n \in Z^+ \Rightarrow S(n))$.
$p(x)$ is a polynomial, i.e. it is analytic, and has finite terms in its Taylor's expansion, so W.L.O.G. let $p(x) = \sum_{i=0}^{n} a_i x^i$
$$L(p(X)) = L(\sum_{i=0}^{n} a_i \cdot X^i) = \sum_{i=0}^{n} a_i \cdot L(X^i) = \sum_{i=1}^{n} a_i \cdot L(X^i) = \sum_{i=1}^{n} a_i \cdot i \cdot L(X) \cdot X^{i-1} = L(X)p'(X)$$

4. Given $L(X) = I$, then for any polynomial p,

$$L(p(X)) = L(X)p'(X) = p'(X),$$

By choosing p as the minimal polynomial of X, $\deg p > 0$, then

$$p'(X) = L(p(X)) = L(0) = 0,$$

So X is one of the zeros of p', that $\deg p' < \deg p$, contradicting the minimality of p, hence $L(X) = I$ is not possible.

5. Given that A is diagonalizable with distinct eigenvalues, so W.L.O.G. let $A = P\Lambda P^{-1}$, then

$$L(X) = 0 \Leftrightarrow AX = XA \Leftrightarrow P\Lambda P^{-1}X = XP\Lambda P^{-1} \Leftrightarrow \Lambda P^{-1}XP = P^{-1}XP\Lambda$$

So $L(X) = 0$ iff $[P^{-1}XP, \Lambda] = 0$ iff $P^{-1}XP$ is diagonal iff A and X shares the same eigenvectors, and hence $\mathrm{Dim}(\mathrm{Ker}(L)) = n$.

6. Let t_1, t_2, t_3 are all distinct. Then $\forall X : X \in M_n(R)$,

$$L(X) = \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix} X - X \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix}$$

$$= \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (t_1 - t_2)a_{12} & (t_1 - t_3)a_{13} \\ (t_2 - t_1)a_{21} & 0 & (t_2 - t_3)a_{23} \\ (t_3 - t_1)a_{31} & (t_3 - t_2)a_{32} & 0 \end{bmatrix}$$

$\mathrm{Ran}(L)$ are hollow matrices. For instance,

$$A = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$