Calculus A(1): Homework 4

November 12, 2021

52.

Prove that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{h \to 0} f(h + c) = L$.

Solution

Let $g(x) = x + c$. Clearly, g is a bijective continuous function. Then, the original proposition can be transformed into

$$\lim_{x \to g(b)} f(x) = L \Leftrightarrow \lim_{h \to b} (f \circ g)(h) = L$$

The following proves "$\Leftarrow$".

$$\lim_{h \to b} (f \circ g)(h) = L \Leftrightarrow$$

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall h)((0 < |h - b| < \delta) \Rightarrow (|(f \circ g)(h) - L| < \epsilon))$$

In addition, g^{-1} is also a bijective continuous function, thus

$$(\forall \epsilon_1 > 0)(\exists \delta_1 > 0)(\forall x)((0 < |x - g(b)| < \delta_1) \Rightarrow (0 < |g^{-1}(x) - b| < \epsilon_1))$$

Choose ϵ_1 such that $\epsilon_1 < \delta$, and $h = g^{-1}(x)$. Then,

$$((0 < |x - g(b)| < \delta_1) \Rightarrow (0 < |g^{-1}(x) - b| < \epsilon_1 < \delta) \Rightarrow (|(f \circ g)(g^{-1}(x)) - L| < \epsilon) \Rightarrow (|(f(x)) - L| < \epsilon))$$

So, by letting $b = 0$,

$$\lim_{x \to g(b)} f(x) = \lim_{x \to c} f(x) = L$$

. "$\Rightarrow$".
Let $s = f \circ g, b = g^{-1}(c)$. Thus we are proving

$$\lim_{x \to c} s(g^{-1}(x)) = L \Leftrightarrow \lim_{x \to g^{-1}(c)} s(x) = L$$

Replace $g(b), b$ involved the proof above with $g^{-1}(c), c$ directly completes the proof. ∎

54

Another wrong statement about limits Show by example that the following statement is wrong.

The number L is the limit of $f(x)$ as x approaches x_0 if, given any $\epsilon > 0$, there exists a value of x for which $|f(x) - L| < \epsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \to x_0$.

Solution

Let $f(x) = x$.
Given any $\epsilon > 0$, exists a value of x for which $|f(x) - 0| = 0 < \epsilon$. In that case above, $x = 0$. However, when x approaches 1,i.e. $x_0 = 1$, $f(x)$ approaches 1.

5.

Let $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$.

a Does $\lim_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b Does $\lim_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c Does $\lim_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

a No.
Assume the limit exists.
Let $n_1 \in \mathbb{N}^+$ such that $n_1 > \frac{1}{\pi\delta} - \frac{1}{2}$. Then, $0 < \frac{2}{(2n_1+1)\pi} < \delta$.
$\forall \epsilon > 0$,
$|\sin \frac{1}{x} - L| = |\sin \frac{(2n_1+1)\pi}{2} - L| = |1 - L| < \epsilon$.
Also let $n_2 \in \mathbb{N}^+$ such that $n_2 > \frac{1}{\delta\pi}$, then $0 < \frac{1}{n_2\pi} < \delta$
$\forall \epsilon > 0$,
$|\sin \frac{1}{x} - L| = |\sin n_2\pi - L| = |L| < \epsilon$.
Hence, $2\epsilon > |1 - L| + |L| \geq |1|$, which fails for $0 < \epsilon < 1/2$
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b Yes.

$$\lim_{x \to 0^-} f(x) = 0$$

c No.

$$\lim_{x \to 0^-} f(x) = 0, \text{but } \lim_{x \to 0^+} f(x) \text{ does not exist.}$$

66.

Suppose that f is an even function of x. Does knowing that $\lim_{x \to 2^-} f(x) = 7$ tell you anything about either $\lim_{x \to 2^-} f(x)$ or $\lim_{x \to -2^+} f(x)$? Give reasons for your answer.

Solution

$$(\lim_{x \to 2^-} f(x) = 7) \Rightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x)((2 - \delta < x < 2) \Rightarrow (|f(x) - 7| < \epsilon))$$

So we have

$$(2 - \delta < -x < 2) \Rightarrow (|f(-x) - 7| < \epsilon), \text{or} (-2 + \delta > x > -2) \Rightarrow (|f(-x) - 7| < \epsilon)$$

Since f is even, i.e. $f(-x) = f(x)$, that implies

$$(-2 < x < -2 + \delta) \Rightarrow (|f(x) - 7| < \epsilon)$$

Thus,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to -2^+} f(x) = 7$$

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How many horizontal asymptotes can the graph of a give rational function have? Give reasons for your answer.

Solution

Horizontal asymptotes of a function $f(x)$ are linear equations in the form of $y = k$,where k satisfies

$$\lim_{x\to-\infty} f(x) = k \text{ or } \lim_{x\to+\infty} f(x) = k$$

Since limit is unique if it exists, thus $f(x)$ can have at most 2 asymptotes.
For a rational function $f(x)$, $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are both polynomials. By polynomial division, $f(x) = A(x) + \frac{R(x)}{Q(x)}$, where $A(x)$ and $R(x)$ are polynomials, and deg R<deg Q. Let n=deg R, k=deg Q-deg R.Then,

$$\lim_{x\to\infty} \frac{R(x)}{Q(x)} = \lim_{x\to\infty} \frac{\sum_{i=0}^{n} r_i x^i}{\sum_{i=0}^{n+k} q_i x^i} = \lim_{x\to\infty} \frac{\sum_{i=0}^{n} r_i x^{i-n}}{\sum_{i=0}^{n+k} q_i x^{i-n}} = 0$$

The fraction does also approach to zero when $x \to -\infty$. Hence, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} A(x)$. The limit exists if and only if $A(x)$ is constant, and has the same value as $x \to -\infty$ So, the graph of a function can have 0 or 1 asymptotes.

A1.

Let $c \in \mathbb{R}$ and f be a function defined on an open interval I containing c, except possibly at c. Show that for $\ell \in \mathbb{R}$, the following assertions are equivalent.

1. $\lim_{x\to c} f(x) = \ell$.
2. For **any** sequence $(x_n)_{n\geq 0}$ converging to c such that $x_n \in I - \{c\}$ for all $n \geq 0$, we have $\lim_{n\to\infty} f(x_n) = \ell$.

Solution

(1)⇒(2).
Suppose

$$\lim_{x\to c} f(x) = \ell \text{ and } \lim_{n\to\infty} x_n = c$$

Then,

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)((0 < |x - c| < \delta) \Rightarrow (|f(x) - \ell| < \epsilon))$$

Also,

$$(\forall\epsilon' > 0)(\exists N \in \mathbb{N}^*)(\forall n)((n > N) \Rightarrow (0 < |x_n - c| < \epsilon'))$$

Hence, choose $\epsilon' = \delta$,

$$(\forall\epsilon > 0)(\exists N \in \mathbb{N}^*)(\forall n)((n > N) \Rightarrow (0 < |x_n - c| < \delta) \Rightarrow (|f(x_n) - \ell| < \epsilon))$$

So

$$\lim_{n\to\infty} f(x_n) = \ell.$$

(2)⇒(1).
Assume $\lim_{x\to c} f(x)$ does not exist or it is not ℓ, but $\lim_{n\to\infty} x_n = c$. Then,

$$(\exists\epsilon_1 > 0)(\forall\delta > 0)(\exists x)((0 < |x - c| < \delta) \Rightarrow (|f(x) - \ell| \geq \epsilon_1))$$

As we have

$$(\forall\epsilon' > 0)(\exists N \in \mathbb{N}^*)(\forall n)((n > N) \Rightarrow (0 < |x_n - c| < \epsilon'))$$

If we choose $\delta = \epsilon', \epsilon'/2, \cdots, \epsilon'/n, \cdots$, then there exists $x_1, x_2, x_3, \cdots, x_n$ such that

$$0 < |x_n - c| < \frac{\epsilon'}{n} \text{ and } |f(x_n) - \ell| \geq \epsilon_1$$

That contradicts the assumption that $\lim_{n\to\infty} f(x_n) = \ell$, hence $\neg(1) \Rightarrow \neg(2)$. ■

B1.

Prove that for any $c \in \mathbb{R}$, we have $\lim_{x\to c} \sin(x) = \sin(c)$. You are allowed to use the following limits, already proved in class: $\lim_{x\to 0} \sin(x) = 0$ and $\lim_{x\to 0} \cos(x) = 1$.

Solution.

$$\begin{aligned}\lim_{x\to c}\sin x = \lim_{x\to 0}\sin\left(x+c\right) &= \lim_{x\to 0}\left(\sin\left(x\right)\cos\left(c\right)+\cos\left(x\right)\sin\left(c\right)\right)\\ &= \cos(c)\lim_{x\to 0}\sin(x)+\sin(c)\lim_{x\to 0}\cos(x)=\sin(c)\end{aligned}$$

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