作业 3 简答

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2021 年 10 月 24 日

问题 1. *Let W be the union of the first and the third quadrants in the xy-plane. That is, let*

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid xy \geq 0 \right\}$$

1. *If $\vec{u}$ is in W and c any scalar, is $c\vec{u}$ in W? Why?*
2. *Find specific vectors $\vec{u}$ and $\vec{v}$ in W such that $\vec{u}+\vec{v}$ is not in W. This is enough to show that W is not a vector space.*

解答.

1. 设 $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in W$，这里 $xy \geq 0$，于是 $c\vec{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}$，直接验证
$$(cx) \cdot (cy) = c^2 \cdot xy \geq 0,$$
因此 $c\vec{u} \in W$.
2. 直接取
$$\vec{u} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
即可.

问题 2. *Determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate n, justify your answer.*

1. *All polynomials of the form $P(t) = at^2$, with $a \in \mathbb{R}$.*
2. *All polynomials of the form $P(t) = a + t^2$, with $a \in \mathbb{R}$.*

3. *All polynomials of degree at most 3 , with integers as coefficients.*

4. *All polynomials in $\mathbb{P}_n$ such that $P(0) = 0$.*

解答. 1. 对的.$\forall c_1, c_2 \in \mathbb{R}$，有

$$c_1 P_1(t) + c_2 P_2(t) = c_1 a_1 t^2 + c_2 a_2 t^2 = (c_1 a_1 + c_2 a_2) t^2 \in \mathbb{P}_2$$

2. 错.$2t^2 = t^2 + t^2$ 不在里面.

3. 错.$\sqrt{2} t^2 = \sqrt{2} \cdot t^2$ 不在里面.

4. 对的.$\forall c_1, c_2 \in \mathbb{R}$，$c_1 P_1(t) + c_2 P_2(t)$ 也在 0 处取 0.

问题 3. *Let H and K be subspaces of a vector space V. The intersection of H and K is defined to be*

$$H \cap K = \{\vec{v} \in V \mid \vec{v} \in H \text{ and } \vec{v} \in K\}$$

Show that $H \cap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is, in general, not a subspace.

解答. $\forall c_1, c_2 \in \mathbb{R}, \forall v_1, v_2 \in H \cap K$，注意

$$v \in H \cap K \Longleftrightarrow v \in H \quad \text{and} \quad v \in K,$$

于是由于 H 是线性空间：

$$v_1, v_2 \in H \Longrightarrow c_1 v_1 + c_2 v_2 \in H,$$

同理

$$v_1, v_2 \in K \Longrightarrow c_1 v_1 + c_2 v_2 \in K,$$

故有

$$c_1 v_1 + c_2 v_2 \in H \cap K.$$

考虑

$$S = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) \in \mathbb{R}\},$$

这里

$$(1, 1) = (1, 0) + (0, 1) \notin S.$$

问题 4. *Consider the polynomials* $P_1(t) = 1 + t^2$ *and* $P_2(t) = 1 - t^2$*. Is* $\{P_1, P_2\}$ *a linear independent set in* $\mathbb{P}_3$*? Why or why not?*

解答. 若存在 $a, b \in \mathbb{R}$ 使得

$$a \cdot P_1(t) + b \cdot P_2(t) = 0$$

即有

$$a \cdot P_1(t) + b \cdot P_2(t) = (a+b) + (a-b)t^2 = 0,$$

解出 $a = b = 0$，从而它们线性无关.

问题 5. *Use coordinate vectors to test the linear independence of the set of polynomials. Explain your work.*

1. $1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3$
2. $1 - 2t^2 - t^3, t + 2t^3, 1 + t - 2t^2$

解答. 把多项式对应到 $\mathbb{R}^n$ 中的向量，这两个问题只需要说明矩阵

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \quad 与 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$$

的列向量是线性无关的即可.

问题 6. *The first four Laguerre polynomials are* $1, 1-t, 2-4t+t^2$ *and* $6 - 18t + 9t - t^3$*. Show that these polynomials form a basis of* $\mathcal{P}_{\leq 3}$*.*

解答. 把多项式对应到 $\mathbb{R}^n$ 中的向量：

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix},$$

只要它们是线性无关的，就自然是 4 维线性空间 $\mathcal{P}_{\leq 3}$ 的一组基底. 我们可以用解线性方程组

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix} \cdot \boldsymbol{x} = \mathbf{0}$$

的方式判断它们的线性无关性. 这一线性方程组只有零解.

问题 7. *Show that the space $C(\mathbb{R})$ of all continuous functions on the real line is an infinite dimensional vector space.*

解答. 考虑

$$1, t, t^2, \ldots, t^n, \cdots \in C(\mathbb{R})$$

它们任意有限多个都是线性无关的. 如果 $C(\mathbb{R})$ 是有限维的，那么 $C(\mathbb{R})$ 中线性无关的向量个数不超过 $C(\mathbb{R})$ 的维数[1]. 矛盾!

In the following two questions, V is a finite dimensional vector space, and the vectors listed belong to V. Mark each statement true of false, and justify your answer.

问题 8.

1. *If there exists a set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ that spans V, then* $\dim(V) \leq p$.
2. *If there exists a linearly independent set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ in V, then* $\dim(V) \geq p$.
3. *If* $\dim(V) = p$, *then there exists a spanning set of $p+1$ vectors in V.*

解答. 全是对的.

1. $\{\vec{v}_1, \cdots, \vec{v}_p\}$ 的极大线性无关组 I，I 也张成 V. 由于 I 还是线性无关的, 因此 I 是 V 的基底, $\dim(V)$ 是 I 的元素个数. 而 $I \subset \{\vec{v}_1, \cdots, \vec{v}_p\}$, 从而 I 的元素个数不超过 p.
2. $\{\vec{v}_1, \cdots, \vec{v}_p\}$ 张成了 V 的子空间，由于它们线性无关，故张成子空间的维数就是它们的元素个数. 而子空间的维数不超过大空间的维数.
3. 找到 V 中 $p+1$ 个互异的向量即可. 取 V 的基底 $\{\vec{v}_1, \cdots, \vec{v}_p\}$ 再加上零向量 $\mathbf{0}$ 即可.

问题 9.

1. *If there exists a linearly dependent set $\{\vec{v}_1, \cdots, \vec{v}_p\}$ in V, then* $\dim(V) \leq p$.

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[1] 设 $dimC(\mathbb{R}) = m$，我们可以在

$$1, t, t^2, \ldots, t^n, \cdots \in C(\mathbb{R})$$

里轻松找到 $m+1$ 个线性无关的向量.

2. *If every set of p elements in V fails to span V, then $\dim(V) > p$.*

3. *If $p \geq 2$ and $\dim(V) = p$, then every set of $p-1$ nonzero vectors is linearly independent.*

解答. 1. 错. 考虑 $\mathbb{R}^3$ 中的 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$.

2. 对. 它的逆否命题是若 $\dim(V) \leq p$，则存在 V 中的 p 个元素张成 V；

3. 错. 考虑第一问的反例.

问题 10. *Let* $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

1. *Solve the equation $A\vec{x} = 0$.*

2. *Let* $\vec{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, *is $\vec{b}$ in the range of the linear transformation $\vec{x} \mapsto A\vec{x}$? Why or why not?*

解答. 1. 解得 $\vec{x} = t \cdot \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}$;

2. 解方程 $A\vec{x} = \vec{b}$ 即可，发现它无解，即 $\vec{b}$ 不在 $\vec{x} \mapsto A\vec{x}$ 的像里.

问题 11. *An affine transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\vec{x}) = A\vec{x} + \vec{b}$, with A an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$. Show that T is not a linear transformation if $\vec{b} \neq 0$.*

解答. 利用线性变换一定把零向量映为零向量.

问题 12. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent.*

解答. 由于 $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ 线性相关，故存在不全为零的 a_i 使得

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0,$$

把 T 作用上去得到

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + a_3T(\vec{v}_3) = 0,$$

得到 $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ 线性相关.

问题 13. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects points through the horizontal x_1-axis and then reflects points through the line $x_1 = x_2$. What is the standard matrix of T.*

解答. 取标准的基底，以 x_1 轴为反射轴的反射是 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$，以 $x_1 = x_2$ 为轴的反射是 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$，它们的复合 *(注意顺序)* 是

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

问题 14. *A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of the rotation?*

解答. 取标准的基底，这两个反射分别是

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

它们的复合是

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

这是绕原点旋转 π 对应的矩阵.

问题 15. *1. Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.*

2. *The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.*

3. *The standard matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that reflects points through the horizontal axis, the vertical axis, or the origin has the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where a and d are ± 1.*

4. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onc-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.*

5. *If A is a 3×2 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ can not map $\mathbb{R}^2$ onto $\mathbb{R}^3$.*

解答. *1.* 错. 任意线性映射 $\mathcal{A}$ 都能写成矩阵的形式. 设

$$\mathcal{A}\vec{e}_i = a_{1i}\vec{e}_i + \cdots + a_{mi}\vec{e}_m$$

把它写成

$$\mathcal{A}\vec{e}_i = (\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_m) \begin{pmatrix} a_{1i} \\ a_{2i} \\ \cdots \\ a_{mi} \end{pmatrix}$$

再将所有的 $\vec{e}_i$ 拼一起, 就有

$$\mathcal{A}(\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n) = (\mathcal{A}\vec{e}_1, \cdots, \mathcal{A}\vec{e}_n) = (\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_m) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

2. 错. 需要 $m = n$ 才可以.

3. 对. 使用前面的方法转化.

4. 不对. 都映成一个向量的线性映射肯定不是单射.

5. 对. 否则 $\mathbb{R}^3$ 的基底在 $\mathbb{R}^2$ 的原象是线性无关的, 矛盾!