Calculus A(1): Homework 8

December 11, 2021

4.4.

80.

Horizontal tangents True, or false? Explain.

a. The graph of every polynomial of even degree (largest exponent even) has at least one horizontal tangent.

b. The graph of every polynomial of odd degree (largest exponent odd) has at least one horizontal tangent.

Solution.

a. **True**.
Let $f(x)$ be the polynomial.
(1). It is clear that the degree of the derivative of a polynomial would have a decrement of 1 w.r.t. the polynomial itself, so $f'(x)$ is a polynomial with odd degree.
$f'(x)$ is odd-degree-polynomial, so by intermediate value theorem, $f'(x)$ must have at least one real root.
(2). Since any point that its tangent is a horizontal line must be an extremum, the point must have the derivative of the polynomial zero.
(1) and (2) together which proved the existence of a horizontal tangent.

b. **False**. $f(x) = x$ is a polynomial of odd degree, however $f'(x) = 1$ is never a horizontal tangent.

84.

Cubic curves What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

Solution.

$(x_0, y|_{x=x_0})$ is a inflection point if x_0 satisfies $y'' = 0$.

$$y'' = 6ax + 2b$$

So

$$x_0 = -\frac{b}{3a}$$

This point exists and is unique. As y'' is shown to be continuous in $\mathbf{R}$, and is linear, $x_0 = -b/(3a)$ is the only inflection point of y.

91.

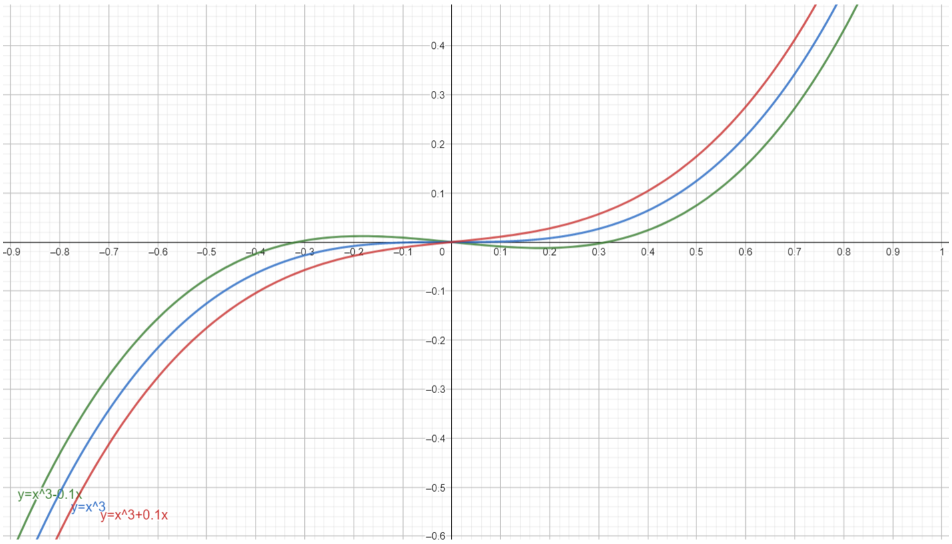
a. On a common screen, graph $f(x) = x^3 + kx$ for $k = 0$ and nearby positive and negative values of k. How does the value of k seem to affect the shape of the graph?

b. Find $f'(x)$. As you will see, $f'(x)$ is a quadratic function of x. Find the discriminant of the quadratic (the discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$). For what values of k is the discriminant positive? Zero? Negative? For what values of k does f' have two zeros? One or no zeros? Now explain what the value of k has to do with the shape of the graph of f.

c. Experiment with other values of k. What appears to happen as $k \to -\infty$? as $k \to \infty$?

Solution.

a.



As seen by the graphs of the three functions, it seems that k manipulates the existence and the magnitude of the extrema of the polynomial.
For negative values of k, it appears that magnitude of the extrema increases with the magnitude of k. For positive value of k, the graph appears to be steep, which has positive slope everywhere.

b.

$$f'(x) = 3x^2 + k$$

$$\Delta = -4 * 3 * k = -12k$$

Let x_0 satisfies $f'(x_0) = 0$. Then

$$x_0 = \pm\sqrt{-\frac{k}{3}}$$

$$f' \text{ have two zeros } \Leftrightarrow \Delta > 0 \Leftrightarrow k < 0$$

$$f' \text{ have one zero } \Leftrightarrow \Delta = 0 \Leftrightarrow k = 0$$

$$f' \text{ have no zeros } \Leftrightarrow \Delta < 0 \Leftrightarrow k > 0$$

c. By continuing the manipulation of k from (a), when $k \to +\infty$, the locus of points of the graph seems to satisfy $x = 0$.
When $k \to -\infty$, the two extrema tend to behave like points at infinity on the second and forth quadrants.

4.6.

33.

Continuous extension Find a value of c that makes the function

$$f(x) = \begin{cases} \frac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

Solution.

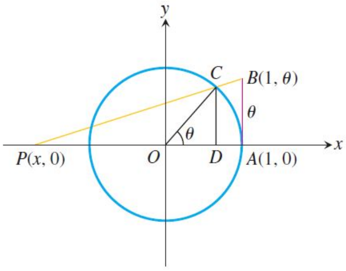
f is continuous at $x = 0$

$$\Leftrightarrow \lim_{x\to 0} f(x) = f(0) = c$$

$$\Leftrightarrow c = \lim_{x\to 0} \frac{9x - 3\sin 3x}{5x^3} = \lim_{x\to 0} \frac{9 - 9\cos 3x}{15x^2} = \lim_{x\to 0} \frac{27\sin 3x}{30x} = \lim_{x\to 0} \frac{81\cos 3x}{30} = \frac{81}{30} = 2.7$$

39.

In the accompanying figure,



the circle has radius OA equal to 1, and AB is tangent to the circle at A. The arc AC has radian measure θ and the segment AB also has length θ. The line through B and C crosses the x-axis at $P(x, 0)$.

a. Show that the length of PA is

$$1 - x = \frac{\theta(1-\cos\theta)}{\theta - \sin\theta}$$

b. Find $\lim_{\theta\to 0}(1-x)$.

c. Show that $\lim_{\theta\to\infty}[(1-x)-(1-\cos\theta)] = 0$.

Solution.

a. Let Q be the projection of C on segment AB. Clearly, $\Delta PAB \sim \Delta CQB$. So it can be seen that

$$\frac{\overline{PA}}{\overline{AB}} = \frac{\overline{CQ}}{\overline{QB}}$$

$$\Leftrightarrow \frac{1-x}{\theta} = \frac{1-\cos\theta}{\theta - \sin\theta}$$

$$\Leftrightarrow 1 - x = \frac{\theta(1-\cos\theta)}{\theta-\sin\theta} \quad \square$$

b.

$$\lim_{\theta\to 0}(1-x) = \lim_{\theta\to 0}\frac{\theta(1-\cos\theta)}{\theta-\sin\theta} = \lim_{\theta\to 0}\frac{1-\cos\theta+\theta\sin\theta}{1-\cos\theta} = 1 + \lim_{\theta\to 0}\frac{\sin\theta+\theta\cos\theta}{\sin\theta} = 2 + \lim_{\theta\to 0}\frac{\theta}{\tan\theta} = 3$$

c.

$$\lim_{\theta\to\infty}[(1-x)-(1-\cos\theta)] = \lim_{\theta\to\infty}\frac{\theta(1-\cos\theta)-(\theta-\sin\theta)(1-\cos\theta)}{\theta-\sin\theta} = \lim_{\theta\to\infty}\frac{\sin\theta(1-\cos\theta)}{\theta-\sin\theta}$$

Since $\left|\frac{\sin\theta}{\theta}\right| \le \frac{1}{|\theta|}$, we have

$$\lim_{\theta\to\infty}\frac{\sin\theta}{\theta} = 0,$$

Moreover,

$$0 \le \frac{\sin\theta(1-\cos\theta)}{\theta-\sin\theta} \le \frac{2\sin\theta}{\theta-\sin\theta}$$

so

$$0 \le \lim_{\theta\to\infty}\frac{\sin\theta(1-\cos\theta)}{\theta-\sin\theta} \le \lim_{\theta\to\infty}\frac{2\frac{\sin\theta}{\theta}}{1-\frac{\sin\theta}{\theta}} = \frac{0}{1-0} = 0$$

Thus

$$\lim_{\theta\to\infty}[(1-x)-(1-\cos\theta)] = 0 \ \blacksquare$$

Additional/Advanced Exercises.

8.

An inequality

a. Show that $-1/2 \le x/(1+x^2) \le 1/2$ for every value of x.

b. Suppose that f is a function whose derivative is $f'(x) = x/(1+x^2)$. Use the result in part (a) to show that

$$|f(b) - f(a)| \le \frac{1}{2}|b-a|$$

for any a and b.

Solution.

a.

$$\begin{aligned} &-1/2 \le x/(1+x^2) \le 1/2 \\ \Leftrightarrow &\left|\frac{x}{1+x^2}\right| \le \frac{1}{2} \\ \Leftrightarrow &\ 2|x| \le 1+x^2 \\ \Leftrightarrow &\ (|x|-1)^2 \ge 0 \end{aligned}$$

The last statement holds as $\forall a \in \mathbf{R}, a^2 \ge 0$. $\square$

b. f' is continuous over $\mathbf{R}$, hence Lagrange's mean value theorem can be used.
If $a = b$, then the inequality clearly holds. W.L.O.G., assume $a < b$. Then $\exists x_0 \in (a,b)$ s.t.

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

It is proved in part (a) that $\forall x \in \mathbf{R}, |f'(x)| = \left|\frac{x}{1+x^2}\right| \le 1/2$, so

$$\left|\frac{f(b)-f(a)}{b-a}\right| = |f'(x_0)| = \left|\frac{x_0}{1+x_0^2}\right| \le \frac{1}{2}$$

$$\Rightarrow |f(b)-f(a)| \le \frac{1}{2}|b-a| \ \blacksquare$$

14.

Proving the second derivative test The Second Derivative Test for Local Maxima and Minima (Section 4.4) says:

a. f has a local maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

b. f has a local minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$

To prove statement (a), let $\epsilon = (1/2)|f''(c)|$. Then use the fact that

$$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h}$$

to conclude that for some $\delta > 0$,

$$0 < |h| < \delta \Rightarrow \frac{f'(c+h)}{h} < f''(c) + \epsilon < 0.$$

Thus, $f'(c+h)$ is positive for $-\delta < h < 0$ and negative for $0 < h < \delta$. Prove statement (b) in a similar way.

Solution.

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h}$$

Hence,

$$\forall \epsilon > 0, \exists \delta > 0, \forall h \in \mathring{U}(0, \delta), |\frac{f'(c+h)}{h} - f''(c)| < \epsilon$$

Let $\epsilon = \frac{1}{2}|f''(c)|$.

$$\Rightarrow -\frac{1}{2}|f''(c)| + f''(c) < \frac{f'(c+h)}{h} < \frac{1}{2}|f''(c)| + f''(c)$$

a. $f''(c) < 0$. Then

$$\frac{3}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{1}{2}f''(c) < 0$$

For any h in the right neighborhood of 0, $f'(c+h) < 0$.
For any h in the left neighborhood of 0, $f'(c+h) > 0$. So $f(c)$ is a local maximum.

b. $f''(c) > 0$. Then

$$0 < \frac{1}{2}f''(c) < \frac{f'(c+h)}{h} < \frac{3}{2}f''(c)$$

For any h in the right neighborhood of 0, $f'(c+h) > 0$.
For any h in the left neighborhood of 0, $f'(c+h) < 0$. So $f(c)$ is a local minimum.

26.

Let $f(x)$, $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, find $h(10)$.

Solution.

$$\forall x \in \mathbf{R}, g'(x) = f''(x) = -f(x)$$

$$h'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x)g(x) - 2g(x)f(x) = 0$$

Hence h is a constant function, so $h(10) = h(0) = 5$.