作业 5 简答

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问题 1. *The inverse of* $\begin{bmatrix} I & & \\ C & I & \\ A & B & I \end{bmatrix}$ *is* $\begin{bmatrix} I & & \\ Z & I & \\ X & Y & I \end{bmatrix}$. *Find* X, Y, Z *in terms of* A, B, C.

解答. 列方程

$$\begin{bmatrix} I & & \\ C & I & \\ A & B & I \end{bmatrix}\begin{bmatrix} I & & \\ Z & I & \\ X & Y & I \end{bmatrix}=\begin{bmatrix} I & & \\ C+Z & I & \\ A+BZ+X & B+Y & I \end{bmatrix}=\begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix},$$

解出 $Y=-B, Z=-C, X=-A-BZ=-A+BC$.

问题 2. *Suppose that* A_{11} *is invertible. Find* X *and* Y *such that*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\begin{bmatrix} I & \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & \\ & S \end{bmatrix}\begin{bmatrix} I & Y \\ & I \end{bmatrix}$$

where $S=A_{22}-A_{21}A_{11}^{-1}A_{12}$.

解答. 注意

$$\begin{bmatrix} I & \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & \\ & S \end{bmatrix}\begin{bmatrix} I & Y \\ & I \end{bmatrix}=\begin{bmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y+S \end{bmatrix},$$

解出 $X=A_{21}A_{11}^{-1}, Y=A_{11}^{-1}A_{12}$.

问题 3. *Use partitioned matrices to prove by induction that for* $n=2,3,\cdots,$

the $n\times n$ matrix A shown below is invertible and B is its inverse.

$$A=\begin{bmatrix}1&&&&\\1&1&&&\\1&1&1&&\\\vdots&&\ddots&\ddots&\\1&1&\cdots&1&1\end{bmatrix},\quad B=\begin{bmatrix}1&&&&\\-1&1&&&\\0&-1&1&&\\\vdots&&\ddots&\ddots&\\0&0&\cdots&-1&1\end{bmatrix}.$$

解答. 令

$$A_n=\begin{bmatrix}1&&&&\\1&1&&&\\1&1&1&&\\\vdots&&\ddots&\ddots&\\1&1&\cdots&1&1\end{bmatrix}_{n\times n}\qquad B_n=\begin{bmatrix}1&&&&\\-1&1&&&\\&-1&1&&\\&&\ddots&\ddots&\\&&&-1&1\end{bmatrix}_{n\times n}$$

显然 $n=2$ 时有 $A_n^{-1}=B_n$. 假设对 $n=k$ 成立，注意

$$A_{n+1}=\begin{bmatrix}A_n&\\\alpha&1\end{bmatrix},\qquad \alpha=(1,\dots,1)_{n\times 1}.$$

并且

$$\begin{bmatrix}I_n&\\-\alpha A_n^{-1}&1\end{bmatrix}\begin{bmatrix}A_n&\\\alpha&1\end{bmatrix}=\begin{bmatrix}A_n&\\&1\end{bmatrix},$$

取逆后得到

$$\begin{bmatrix}A_n^{-1}&\\&1\end{bmatrix}=A_{n+1}^{-1}\begin{bmatrix}I_n&\\-\alpha A_n^{-1}&1\end{bmatrix}^{-1},$$

这推出

$$A_{n+1}^{-1}=\begin{bmatrix}A_n^{-1}&\\&1\end{bmatrix}\begin{bmatrix}I_n&\\-\alpha A_n^{-1}&1\end{bmatrix}=\begin{bmatrix}A_n^{-1}&\\-\alpha A_n^{-1}&1\end{bmatrix}=\begin{bmatrix}B_n&\\-\alpha B_n&1\end{bmatrix},$$

现在我们计算

$$-\alpha B_n=-(1,\dots,1)\begin{bmatrix}1&&&&\\-1&1&&&\\&-1&1&&\\&&\ddots&\ddots&\\&&&-1&1\end{bmatrix}=(0,\dots,0,-1),$$

这说明

$$A_{n+1}^{-1} = \begin{bmatrix} B_n & \\ -\alpha B_n & 1 \end{bmatrix} = \begin{bmatrix} B_n & \\ (0,\dots,0,-1) & 1 \end{bmatrix} = B_{n+1}.$$

问题 4. *Without using row reductions, find the inverse of* $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$.

解答. 注意

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -4 \\ -5/2 & 7/2 \end{bmatrix},$$

从而

$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} & & \\ & 2^{-1} & \\ & & \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}^{-1} \end{bmatrix} = \begin{bmatrix} -5 & 2 & & & \\ 3 & -1 & & & \\ & & 1/2 & & \\ & & & 3 & -4 \\ & & & -5/2 & 7/2 \end{bmatrix}.$$

问题 5. *Find the LU-factorization of the matrices:*

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix}.$$

解答. 直接计算

$$A = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ \frac{3}{2} & -2 & 1 & & \\ -3 & 2 & 0 & 1 & \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

问题 6. *Suppose $A = UDV^t$, where U and V are $n \times n$ matrices with the property that $U^tU = I$ and $V^tV = I$, and where D is a diagonal matrix with positive numbers $\sigma_1, \cdots, \sigma_n$ on the diagonal. Show that A is invertible and find a formula for A^{-1}.*

解答. D 是可逆的，从而直接验证矩阵 $VD^{-1}U^t$ 是 A 的逆.

问题 7. *Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the diagonal matrix*

$$D = \begin{bmatrix} 1 & & \\ & 1/2 & \\ & & 1/3 \end{bmatrix}$$

Show that this factorization is useful when computing high powers of A. Find fairly simple formulas for A^2, A^3 and A^k, using P and the entries of D

解答. $A^k = PD^kP^{-1}$，这里 $D^k = \begin{bmatrix} 1 & & \\ & 2^{-k} & \\ & & 3^{-k} \end{bmatrix}$.

问题 8. *Find bases for the column space and the null space of* $A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$.

解答. 直接计算 A 的简化阶梯型 $\begin{bmatrix} 1 & 0 & 5/2 & -9/2 & 7/2 \\ 0 & 1 & 3/2 & -5/2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$，得到零空间一组基底

$$(-5/2, -3/2, 1, 0, 0)^T, \qquad (9/2, 5/2, 0, 1, 0)^T, \qquad (-7/2, -3/2, 0, 0, 1)^T.$$

列空间基底可由 A 的第一列和第二列组成.

问题 9. *Determine whether $\vec{w}$ is in the column space of A, the null space* *of A, or both, where*

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \quad A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}.$$

解答. 直接计算 $A\vec{w} = \begin{bmatrix} 14 \\ 0 \\ 0 \\ 0 \end{bmatrix}$，所以不在零空间里面. 考虑线性方程组 $AX = \vec{w}$ 的解，化简增广矩阵得到 $\begin{bmatrix} 1 & 0 & 0 & -1/95 & 1/95 \\ 0 & 1 & 0 & 39/19 & -20/19 \\ 0 & 0 & 1 & 267/95 & -172/95 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$，从而方程组有解，$\vec{w}$ 在 A 的列空间中.

问题 10. *Let* $\vec{a}_1, \cdots, \vec{a}_5$ *be the column vectors of* $A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix}$, *let* $B = [\vec{a}_1\vec{a}_2\vec{a}_4]$.

1. *Explain why* $\vec{a}_3$ *and* $\vec{a}_5$ *are in the column space of* B.
2. *Find a set of vectors that* spans $\mathrm{Nul}(A)$.
3. *Let* $T : \mathbb{R}^5 \to \mathbb{R}^4$ *be defined by* $T(\vec{x}) = A\vec{x}$. *Explain why* T *is neither one-to-one nor onto.*

解答.

1. 把 A 化简成阶梯型 $\begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$，由于行化简不改变列与列之间的线性关系，因而阶梯型的第 3, 5 列在其余各列张成的线性空间这一事实可以推出 $\vec{a}_3$ 与 $\vec{a}_5$ 在 B 的列空间中.

2. 由阶梯型，得到一组基底

$$\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -\frac{10}{3} \\ \frac{26}{3} \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

3. 这里利用这样一个事实，如果线性映射 $T: V_1 \to V_2$ 为一一映射，则 V_1 与 V_2 维数相同. 这是因为 T 把 V_1 的基底变为 V_2 的线性无关向量，T^{-1} 把 V_2 的基底变为 V_1 的线性无关向量.

问题 11. *It is known that a linear independent set $\{\vec{v}_1, \cdots, \vec{v}_k\}$ in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$. One way to do this is to consider the matrix $A = [\vec{v}_1 \cdots \vec{v}_k \vec{e}_1 \cdots \vec{e}_n]$ with $\vec{e}_1, \cdots, \vec{e}_n$ the standard basis of $\mathbb{R}^n$. The pivot columns of A form a basis for $\mathbb{R}^n$.*

1. *Use the method described to extend the following vectors to a basis for $\mathbb{R}^5$:*

$$\vec{v}_1 = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}$$

2. *Explain why the method works in general: Why are the original vectors $\vec{v}_1, \cdots, \vec{v}_k$ included in the basis found for $\mathrm{Col}(A)$? Why is $\mathrm{Col}(A) = \mathbb{R}^n$?*

解答. 1. 把 $A = [\vec{v}_1 \cdots \vec{v}_3 \vec{e}_1 \cdots \vec{e}_5]$ 化简为阶梯型

$$\begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 & 0 & 1 & 3/7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 5/7 \\ 0 & 0 & 1 & -1/3 & 0 & 0 & 0 & -3/7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 22/7 \\ 0 & 0 & 0 & 0 & 0 & 1 & -9 & -53/7 \end{bmatrix},$$

从而可取 $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{e}_2, \vec{e}_3$ 作为一组基底;

2. 这是因为它们是线性无关的，而行简化不改变列向量间的线性关系. 考虑最简阶梯形，它的第一列是 e_1，第二列如果和前面的列线性无关，那么它应该是 e_2，第三列如果与前面的列线性无关，那么它应该是 e_3 ……以此类推，前 i 列 $(1 \leq i \leq k)$ 必然是 e_i，我们再找阶梯型中 $e_i, k+1 \leq i \leq n$ 对应的 A 中列，所有的这些向量构成 $Col(A) \subset \mathbb{R}^n$ 的一组基底. 注意他们有 n 个，故 $Col(A) = \mathbb{R}^n$.