Week 10a.
p.845. 1,5,9,13, 19,25,29,39
Week 10b
p.858 1,3,5,31
p.864. 1,23,41.

1. Sol:

$f(x,y,z)=x^2+y^2+z^2=3$. (at $P_0(1,1,1)$).

$\Rightarrow \Pi: f'_x(x-1)+f'_y(y-1)+f'_z(z-1)=0$.

$\Rightarrow 2(x-1)+2(y-1)+2(z-1)=0$.

Tangent plane: $x+y+z=3$.

Normal: $\begin{cases} x=1+t \\ y=1+t \\ z=1+t \end{cases}$, $t\in\mathbb{R}$.

5. Sol:

$f(x,y,z)=\cos\pi x - x^2y + e^{xz}+yz=4$, at $P_0(0,1,2)$.

$\Rightarrow \Pi: f'_x(x) + f'_y(y-1)+f'_z(z-2)=0$.

$\Rightarrow (-\pi\sin\pi x_0 - 2x_0y_0 + z_0e^{x_0z_0})x + (-x_0^2+z_0)(y-1) + (x_0e^{x_0z_0}+y_0)(z-2)\Big|_{P_0}=0$.

$\Rightarrow 2x+2(y-1)+(z-2)=0$.

Tangent plane: $2x+2y+z=4$

Normal: $\begin{cases} x=2t \\ y=1+2t \\ z=2+t. \end{cases}$, $t\in\mathbb{R}$.

9. Sol:

$z=\ln(x^2+y^2)$, $(1,0,0)$

$\Leftrightarrow \ln(x^2+y^2)-z=0$.

$\Pi: f'_x(x-1)+f'_y y+f'_z z=0$.

$\Leftrightarrow \frac{2x_0}{x_0^2+y_0^2}(x-1)+\frac{2y_0}{x_0^2+y_0^2}(y)-z=0$.

$\Rightarrow 2(x-1)-z=0$.

Tangent plane: $2x-z=2$.

13. Sol:

$\begin{cases} x+y^2+2z=4 \\ x=1. \end{cases}$ at $(1,1,1)$.

Tangent line orthogonal to two surfaces, having direction // to

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2y & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\hat{j}-2y\hat{k}$,

at $(1,1,1)$ it is $2\hat{j}-2\hat{k}$,

so tangent line: $\begin{cases} x=1 \\ y=1+2t \\ z=1-2t \end{cases}$, $t\in\mathbb{R}$

19. Sol:

$df = f'_x dx + f'_y dy + f'_z dz$

$\Delta f = f'_x\Delta x + f'_y\Delta y + f'_z\Delta z$

$= \frac{2(x_0\hat{i}+y_0\hat{j}+z_0\hat{k})}{2(x_0^2+y_0^2+z_0^2)}\cdot\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{3^2+6^2+(-2)^2}}\cdot 0.1$

$= \frac{3\hat{i}+4\hat{j}+12\hat{k}}{169}\cdot\frac{3\hat{i}+6\hat{j}-2\hat{k}}{7}\cdot 0.1$

$= \frac{9}{11830}$ #.

25. Sol:.

$f(x,y) = x^2+y^2+1$, $\begin{cases}(0,0)\\(1,1).\end{cases}$

$f'_x = 2x$, $f'_y = 2y$.

$L(x_0,y_0) = f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$

$= 2x_0(x-x_0) + 2y_0(y-y_0) + x_0^2+y_0^2+1$

At $(0,0)$ $\Rightarrow L(x,y) = 1$

At $(1,1)$ $L(x,y) = 2(x-1)+2(y-1)+3$

$= 2x+2y-1$

29. Sol:.

$f(x,y) = e^x\cos y$ at $\begin{cases}(0,0)\\(0,\frac{\pi}{2})\end{cases}$

$f'_x = f(x,y)$, $f'_y = -e^x\sin y$.

$L(x_0,y_0) = e^{x_0}\cos y_0(x-x_0) - e^{x_0}\sin y_0(y-y_0) + f(x_0,y_0)$.

At $(0,0)$, $L(x,y) = x+1$, At $(0,\pi/2)$, $L(x,y) = -(y-\frac{\pi}{2}) = \frac{\pi}{2}-y$.

39. Sol:

$f(x,y,z) = xy+yz+zx$ $\begin{cases}(1,1,1)\\(1,0,0)\\(0,0,0)\end{cases}$

$L = f'_x(x-x_0) + f'_y(y-y_0) + f'_z(z-z_0) + f(x_0,y_0,z_0)$

$= (y_0+z_0)(x-x_0) + (z_0+x_0)(y-y_0) + (x_0+y_0)(z-z_0) + f(x,y,z)$

At $(1,1,1)$, $L(x,y,z) = 2(x-1+y-1+z-1)+3 = 2x+2y+2z-3$.

At $(1,0,0)$, $L(x,y,z) = y+z$

At $(0,0,0)$. $L(x,y,z) = 0$.

p. ~~858~~ 855.

1. Sol:

$f(x,y) = x^2+xy+y^2+3x-3y+4$

$f'_x = 2x+y+3$, $f'_y = 2y+x-3$.

$f''_{xx} = 2$, $f''_{yy} = 2$, $f''_{xy} = f''_{yx} = 1$.

The extrema satisfies $f'_x = f'_y = 0$

$\Leftrightarrow \begin{cases}2x_0+y_0+3=0\\x_0+2y_0-3=0\end{cases} \Rightarrow \begin{cases}x_0=-3\\y_0=3,\ f(-3,3) = -5\end{cases}$

$H(f) = \begin{bmatrix} f''_{xx} & f''_{yx} \\ f''_{xy} & f''_{yy}\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$ is positive definite, so $(-3,3)$ is ~~too~~ local minimum

f has no local maxima, neither does it have saddle points.

3. Sol:.

$f(x,y) = x^2+xy+3x+2y+5$.

$f'_x = 2x+y+3$, $f'_y = x+2$.

$f''_{xx} = 2$, $f''_{yy} = 0$

$f''_{xy} = f''_{yx} = 1$.

Extrema satisfies $f'_x = f'_y = 0$.

$\Rightarrow x_0 = -2$, $y_0 = 1$, $f(-2,1) = 3$

$H(f) = \begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}$ is ~~positive~~ non-definite, so $(-2,1)$ is ~~local minimum~~ a saddle point.

5. Sol:.

$f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$.

$f'_x = 2y-2x+3$, $f'_y = 2x-4y$

$f''_{xx} = -2$, $f''_{yy} = -4$, $f''_{xy} = f''_{yx} = 2$.

Extrema satisfies $f'_x = f'_y = 0$.

$\Rightarrow \begin{cases}2y_0-2x_0+3=0\\2x_0-4y_0=0\end{cases} \Rightarrow \begin{cases}x_0=3\\y_0=3/2\end{cases}$, $f(3,\frac{3}{2}) = \frac{17}{2}$.

$H(f) = \begin{bmatrix}-2 & 2\\2 & -4\end{bmatrix}$ is negative definite,

so $(3,3/2)$ is local maximum.

31. Sol:

$f(x,y) = 2x^2-4x+y^2-4y+1$,

$D = \{(x,y): x>0 \wedge y<2 \wedge 2x-y<0\}$.

$\partial D = \{(x,y): x=0 \vee y=2 \vee 2x-y=0\}$.

$\bar{D} = D \cup \partial D$.

$\forall (x,y): (x,y) \in D$.

(x,y) is extrema $\Leftrightarrow f'_x = f'_y = 0$.

$f'_x = 4x-4$, $f'_y = 2y-4$.

$f''_{xx} = 4$, $f''_{yy} = 2$, $f''_{xy} = f''_{yx} = 0$.

~~So~~ However $f'_x = f'_y = 0$.

yields $x_0 = 1$, $y_0 = 2$, $(x_0,y_0) \notin D$

So the maxima, minima are ~~on~~ ∂D:

$\forall (x,y) \in \partial D$.

1) for $x=0$: $(0 \leq y \leq 2)$.

$f(0,y) = y^2-4y+1$, so $(0, $ ~~[illegible]~~ 2 is ~~[illegible]~~

~~extrema~~ minimum: $f(0,$ ~~[illegible]~~ 2$) = $ ~~[illegible]~~ -3,

$f(0,0) = 1$ is "maximum".

2) for $y=2$: $(0 \leq x \leq 1)$.

$f(x,2) = 2x^2-4x-3$, ~~[illegible] minimum.~~

~~[illegible] extrema:~~

$f(1,2)$

~~[illegible]~~ $= 2(x-1)^2-5$

$= -5$. (is minimum)

$f(0,2) = -3$ is "maximum".

3) for $y=2x$:

$f(x,2x) = 2x^2-4x+4x^2-8x+1$ $(0 \leq x \leq 1)$.

$= 6x^2-12x+1$, minimum

So $(1,2)$ is again the ~~extrema~~:

$f(1,2) = -5$.

$f(0,0) = 1$ is maximum.

To sum up: In $\bar{D}$, f attains absolute maximum at $(0,0)$, $f(0,0)=1$

f attains absolute minimum at $(1,2)$, $f(1,2) = -5$.

p.864.

1. Sol:

~~[illegible]~~ $\mathcal{L} = xy - \lambda(x^2+2y^2-1)$

$\mathcal{L}'_x = y - 2\lambda x$, $\mathcal{L}'_y = x - 4y\lambda$

$\mathcal{L}'_\lambda = -(x^2+2y^2-1)$

$\mathcal{L}'_x = \mathcal{L}'_y = \mathcal{L}'_\lambda = 0$ yields.

$y = 2\lambda(4\lambda y) = 8\lambda^2 y$.

$\Rightarrow (1-8\lambda^2)y = 0$.

case 1: $y=0$. then $x = \pm 1$, $xy = 0$.

case 2: $1-8\lambda^2 = 0$

then $\lambda = \pm\frac{\sqrt{2}}{4}$, $y = \pm\frac{\sqrt{2}}{2}x$.

~~[illegible]~~

$\Rightarrow x^2 + 2(\frac{1}{2})x^2 - 1 = 0$. $\Rightarrow x = \pm\frac{1}{\sqrt{2}}$, so.

$y = $ $(\frac{1}{\sqrt{2}}, \frac{1}{2})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{2})$,

$(-\frac{1}{\sqrt{2}}, \frac{1}{2})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$,

$(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2})$ corresponds to $\frac{\sqrt{2}}{4}$,

$(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2})$ corresponds to $-\frac{\sqrt{2}}{4}$.

23. $x^2+y^2+z^2=30.$

$f(x,y,z) = x-2y+5z.$

Sol:

Extrema:

By Cauchy - Schwarz inequality.

$(x-2y+5z)^2 \le [1^2+(-2)^2+5^2](x^2+y^2+z^2)$

$= 30 \cdot 30 = 900.$

$\Rightarrow -30 \le x-2y+5z \le 30.$

$x-2y+5z=30$ (equality holds when)

$\Leftrightarrow \frac{x}{1} = \frac{y}{-2} = \frac{z}{5}.$

$\Rightarrow \boxed{\begin{array}{c} x=1, y=-2, z=5 \\ f(1,-2,5)=30 \end{array}}$ is maximum

$x^2+(-2x)^2+(5x)^2=30.$

$\Rightarrow x = 1, -1$

$y=-2, z=5.$

$x-2y+5z=-30$ holds

when $x=-1, y=2, z=-5.$

$\Rightarrow \boxed{\begin{array}{c} f(-1,2,-5)=-30 \\ \text{is minimum} \end{array}}$

41. Sol:

$f(x,y,z) = x^2yz+1.$

subject to $z=1,$

$x^2+y^2+z^2=10.$

~~$\mathcal{L} = x^2yz+1-\lambda_1(z-1)-\lambda_2(x^2+y^2+z^2-10)$~~

~~$\frac{\partial \mathcal{L}}{\partial x} = 2xyz - 2\lambda_2 x = 0$ (41.1)~~

~~$\frac{\partial \mathcal{L}}{\partial y} = x^2z - 2\lambda_2 y = 0$ (41.2)~~

~~$\frac{\partial \mathcal{L}}{\partial z} = x^2y - \lambda_1 - 2\lambda_2 z = 0$ (41.3)~~

~~$\frac{\partial \mathcal{L}}{\partial \lambda_1} = -z+1 = 0$ (41.4)~~ ~~$\frac{\partial \mathcal{L}}{\partial \lambda_2} = -(x^2+y^2+z^2-10) = 0$ (41.5)~~

~~By (41.4), the other 4 equations simplified to~~

~~$\frac{\partial \mathcal{L}}{\partial x} = -2\lambda_2 x = 0.$~~

~~$\frac{\partial \mathcal{L}}{\partial y} = -2\lambda_2 y = 0.$~~

~~$\frac{\partial \mathcal{L}}{\partial z} = x^2y - \lambda_1 = 0.$~~

~~$\frac{\partial \mathcal{L}}{\partial \lambda_2} = (x^2+y^2-10) = 0.$~~

Since $z=1$, it becomes optimize

$f(x,y,1) = x^2y+1$ subject to $x^2+y^2=9.$

$\Rightarrow$ optimize

$g(y) = (9-y^2)y+1, \quad -3 \le y \le 3.$

$g'(y) = 9-3y^2,$

$g''(y) = -6y.$

$g'(y)=0 \Rightarrow y = \pm\sqrt{3}.$

$g''(\sqrt{3}) < 0$, $y=\sqrt{3}$ is maximum

$g''(-\sqrt{3}) > 0$. $y=-\sqrt{3}$ is minimum

So when $(x,y,z) = (\pm\sqrt{6}, \sqrt{3}, 1)$,

f attains its maximum:

$f(\pm\sqrt{6}, \sqrt{3}, 1) = 6\sqrt{3}+1.$

when $(x,y,z) = (\pm\sqrt{6}, -\sqrt{3}, 1)$

f attains minimum:

$f(\pm\sqrt{6}, -\sqrt{3}, 1) = -6\sqrt{3}+1$ #.