- Corollary: Let A be an invertible matrix, then $det(A^{-1}) = det(A)^{-1}$.
- Proof: It follows from $det(A)det(A^{-1}) = det(AA^{-1}) = det(I_n) = 1$.
- Corollary: Let A be an $n \times n$ matrix, let B be an invertible $n \times n$ matrix, then $det(BAB^{-1}) = det(A)$.
- Proof: $det(BAB^{-1}) = det(B)det(A)det(B)^{-1} = det(A)$.

Determinant and the volume

- Theorem: Let $A = [a_1 \cdots a_n]$ be an $n \times n$ matrix, the volume of the parallelotope spanned by the vectors $a_1, \cdots, a_n$ equals $|det(A)|$.
- Proof: Notice that the volume of the parallelotope changes in the same way as $|det(A)|$. Indeed,

A. The volume and $|det(A)|$ don' t change under $(a_i, a_j) \mapsto (a_i + \lambda a_j, a_j)$.

B. The volume and $|det(A)|$ don' t change under $(a_i, a_j) \mapsto (a_j, a_i)$.

C. The volume and $|det(A)|$ change to its $|\lambda|$ multiple under $a_i \mapsto \lambda a_i$.

- Hence it is enough to consider the case when A is lower triangular. In this case, both the volume and $|det(A)|$ equals $|a_{11} \cdot \cdots \cdot a_{nn}|$.
- For $|det(A)|$, this has been proven before.
- For the volume, let P be the parallelotope spanned by the column vectors of A, let P_{11} be the parallelotope spanned by the column vectors of A_{11}. Take A_{11} as the base, the height is $|a_{11}|$, hence $\text{vol}(P) = \text{area}(P_{11})|a_{11}|$. Use this relation iteratively, we get $\text{vol}(P) = |a_{11} \cdot \cdots \cdot a_{nn}|$.
- This finishes the proof.

- Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\} \tag{5}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\} \tag{6}$$

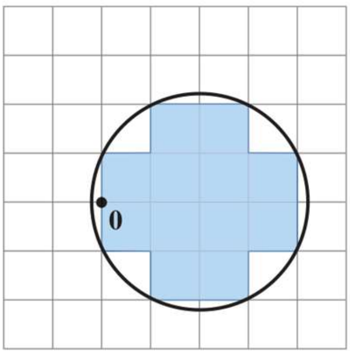
Proof After a suitable translation, we can suppose that S has the origin as one of its vertices, and it is spanned by the vectors $\vec{b}_1, \vec{b}_2 \in \mathbb{R}^2$ in dim 2 case (or $\vec{b}_1, \vec{b}_2, \vec{b}_3 \in \mathbb{R}^3$ in dim 3).

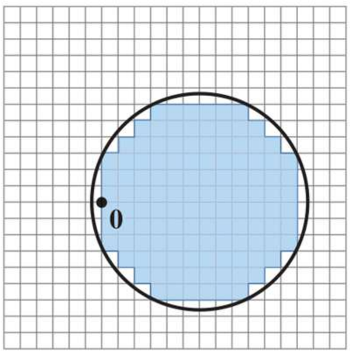
Let $B = [\vec{b}_1 \ \vec{b}_2]$ or $[\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3]$ accordingly. Then $T(S)$ is the parallelogram spanned by $A\vec{b}_1$ and $A\vec{b}_2$ (or the parallelotope spanned by $A\vec{b}_1$, $A\vec{b}_2$ and $A\vec{b}_3$).

Then
$$\begin{aligned} \text{Area}(T(S)) &= \left|\det[A\vec{b}_1 \ A\vec{b}_2]\right| \\ &= |\det(AB)| = |\det(A)| \cdot |\det(B)| \\ &= |\det(A)| \cdot \text{Area}(S). \end{aligned}$$

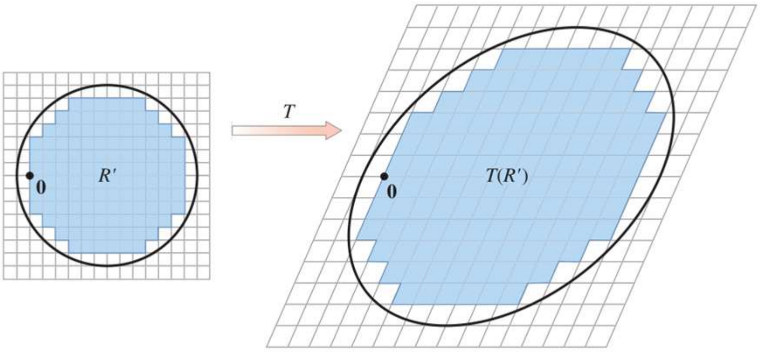
(or $\text{vol}(T(S)) = \left|\det[A\vec{b}_1 \ A\vec{b}_2 \ A\vec{b}_3]\right| = |\det(AB)|$
$= |\det(A)| \cdot |\det(B)| = |\det(A)| \cdot \text{vol}(S).$).

- In general, the conclusion of the theorem holds whenever S is a region in R^2 with finite area or a region in R^3 with finite volume.
- Recall that the volume of such a region can be calculated as the limit of the more and more refine approximation:





- Under the linear transformation T, each unit changes by a factor $|det(A)|$, hence the total volume changes also by the factor $|det(A)|$.



EXAMPLE 5 Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

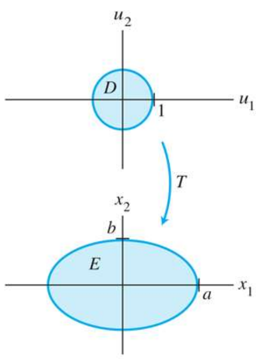
Solution Notice that

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1,$$

With the change of variable

$$y_1 = \frac{x_1}{a}, \quad y_2 = \frac{x_2}{b},$$

we get $\quad y_1^2 + y_2^2 = 1$



Hence E is the image of the unit disk under the linear transformation

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a y_1 \\ b y_2 \end{bmatrix} = \begin{bmatrix} a & \\ & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

By the previous theorem,

$$\text{Area}(E) = \left| \det \begin{bmatrix} a & \\ & b \end{bmatrix} \right| \cdot \text{Area}(\text{unit disk})$$

$$= ab \cdot \pi$$

Cramer's rule and the inversion formula

Cramer's rule

- Recall that we introduce the determinant of a 2×2 matrix when solving the linear system

$$\begin{cases} ax + by = r_1 \\ cx + dy = r_2. \end{cases}$$

- The linear system has unique solution if and only if the determinant

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0$$

- In this case, we have

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

And

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} dr_1 - br_2 \\ -cr_1 + ar_2 \end{bmatrix}.$$

- Notice that

$$dr_1 - br_2 = \det\begin{bmatrix} r_1 & b \\ r_2 & d \end{bmatrix} \quad \text{and} \quad -cr_1 + ar_2 = \det\begin{bmatrix} a & r_1 \\ c & r_2 \end{bmatrix}.$$

- These results can be generalized.

- Definition: Let $A = [a_1 \cdots a_n]$ be an $n \times n$ matrix, let $b \in R^n$, we define $A_i(b)$ as the matrix obtained from A by replacing a_i by b, I.e.

$$A_i(b) = [a_1 \cdots b \cdots a_n]$$

- Theorem

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Proof Notice the following simple fact:

$$\begin{array}{cc} i & \begin{vmatrix} 1 & & & x_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \vdots & & & \\ & & & x_i & & & \\ & & & \vdots & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{vmatrix} \\ & \quad\quad\quad i \end{array} = x_i$$

[expansion along the i-th row.]

and that

$$A \begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \vdots & & & \\ & & & x_i & & & \\ & & & \vdots & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix} = \Big[\, \underbrace{A\vec{e}_1 \ \cdots \ A\vec{e}_{i-1} \ \ \overset{\vec{b}}{\overset{\|}{A\vec{x}}} \ \ A\vec{e}_{i+1} \ \cdots \ A\vec{e}_n}_{A_i(\vec{b})} \,\Big]$$

Take determinant, we get

$$\det(A)\cdot x_i = \det A_i(\vec{b})$$

$$\Rightarrow \quad x_i = \frac{\det A_i(\vec{b})}{\det(A)} \qquad \square$$

- This generalize the solution for the two variable case.

- Remark: The above proof is very tricky. We can give a more elementary proof as follows:

Recall that to calculate $A^{-1}\vec{b}$, we have the algorithm

$$[A \mid \vec{b}] \sim [I_n \mid A^{-1}\vec{b}].$$

This can be rephrased as

$$[\vec{a}_1 \cdots \vec{a}_n \mid \vec{a}_1 \cdots \vec{b} \cdots a_n] \overset{A^{-1}}{\sim} [I_n \mid \vec{e}_1 \cdots A^{-1}\vec{b} \cdots \vec{e}_n]$$

$\vec{b}$: i-th column; $A^{-1}\vec{b} = \vec{x}$: i-th column.

i.e. $A^{-1} \cdot A_i(\vec{b}) = \begin{bmatrix} 1 & & x_1 & & \\ & \ddots & \vdots & & \\ & & x_i & & \\ & & \vdots & \ddots & \\ & & x_n & & 1 \end{bmatrix}$. Take determinants, get

$$\frac{\det A_i(\vec{b})}{\det(A)} = x_i. \quad \boxed{S}$$

An inverse formula

- Definition: Let A be an invertible $n \times n$ matrix, recall that its (i,j)-cofactor is $C_{ij} = (-1)^{i+j}det(A_{ij})$, we define its adjugate(伴随矩阵), denoted $\mathrm{adj}(A)$, as

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- Attention: The (i,j)-entry of $\mathrm{adj}(A)$ is C_{ji}.

- Theorem

An Inverse Formula

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Proof Notice that $A^{-1} = [A^{-1}\vec{e}_1 \cdots A^{-1}\vec{e}_n]$ and that $A^{-1}\vec{e}_j$ is the solution to the linear system $A\vec{x} = \vec{e}_j$. By Cramer's rule, we get the i-th column of $A^{-1}\vec{e}_j$, i.e. the (i,j)-entry of A^{-1}, equals $\dfrac{\det A_i(\vec{e}_j)}{\det(A)}$.

We calculate

$$\det A_i(\vec{e}_j) = \begin{array}{c} \\ \end{array}\begin{array}{ccccccccc} & & & & i & & & & \\ \end{array}$$

$$\det A_i(\vec{e}_j) = \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & 0 & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{j1} & \cdots & a_{j,i-1} & 1 & a_{j,i+1} & \cdots & a_{jn} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & 0 & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= (-1)^{j+i} \det(A_{ji})$$ expansion along the i-th column

$$= C_{ji}.$$

$\Rightarrow$ The (i,j)-entry of A^{-1} equals C_{ji}. $\boxed{\text{↯}}$