Topics in Linear Algebra: Homework 5

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Solution 1.5.1.

First, examine the differentiability of f.
For $f : C \to C$, $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$ iff

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_0} = \left.\frac{\partial v}{\partial y}\right|_{y=y_0}, \left.\frac{\partial u}{\partial y}\right|_{y=y_0} = -\left.\frac{\partial v}{\partial x}\right|_{x=x_0}$$

For $f(z) = z|z|$, $u(x,y) = x\sqrt{x^2+y^2}$, $v(x,y) = y\sqrt{x^2+y^2}$, then

$$u_x = \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}}, u_y = \frac{xy}{\sqrt{x^2+y^2}}$$

$$v_x = \frac{xy}{\sqrt{x^2+y^2}}, v_y = \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}.$$

Solving $u_x = v_y$, $v_x = -u_y$ yields

$$\begin{cases} x^2 = y^2 \\ 2xy = 0 \\ x^2+y^2 > 0 \end{cases}$$

However, no such tuple (x_0, y_0) satisfies all three equations above, and hence f is nowhere complex differentiable.
But if it is the case $f|_R : R \to R$, then it is real differentiable everywhere.

$f(x) = x|x| = x\sqrt{x^2}$, then $f'(x) = \sqrt{x^2} + x \cdot \frac{x}{\sqrt{x^2}} = 2|x|$

1.
$$A_t = \begin{bmatrix} 1 & 1 \\ & 1+t \end{bmatrix} = \begin{bmatrix} 1 & -x \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & 1+t \end{bmatrix}\begin{bmatrix} 1 & -x \\ & 1 \end{bmatrix}$$

   So $1 = -xt$, $x = -1/t$.

$$f(A_t) = \begin{bmatrix} 1 & 1/t \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & (1+t)^2 \end{bmatrix}\begin{bmatrix} 1 & -1/t \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & t+2 \\ & (1+t)^2 \end{bmatrix}, \forall t : t \geq -1$$

   So
$$\lim_{t\to 0} f(A_t) = \lim_{t\to 0}\begin{bmatrix} 1 & t+2 \\ & (1+t)^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}$$

2. f is real differentiable, but not complex differentiable, which is obviously not analytic, so f cannot equal to its Taylor series, hence f is not defined for matrices with complex eigenvalues.
   Moreover,
$$p_{A_t}(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -t^2 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 + t^2$$
   so for A_t, $\lambda = 1 \pm it$.
   As the limit is taking values from the punctured neighborhood of zero, the limit does not exist, so f is not defined.

Solution 1.5.2.

1. $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, so

$$\sin(tA) = \frac{e^{itA}-e^{-itA}}{2i}, \frac{d}{dt}\sin(tA) = \frac{iAe^{itA}-(-iA)e^{-itA}}{2i} = \frac{A(e^{itA}+e^{-itA})}{2} = A\cos(tA)$$

2. W.L.O.G., assume f is analytic at $z=0$ for $f:\mathbf{C}\to\mathbf{C}$.
Then,

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k,$$

where

$$a_k = \frac{1}{2\pi i}\oint_{|z|=\epsilon}\frac{f(z)dz}{z^{n+1}}$$

Before seeking for the solution, first prove a lemma:

$$S(n): \begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}^n = \begin{bmatrix} (2A)^n & n\cdot 2^{n-1}A^n \\ & (2A)^n \end{bmatrix}, \forall n: n\in\mathbf{Z}^+$$

$S(1)$ is trivial.
Assume $S(k)$ holds, then for $S(k+1)$,

$$\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}^{k+1} = \begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}^k \begin{bmatrix} 2A & A \\ & 2A \end{bmatrix} = \begin{bmatrix} (2A)^k & k\cdot 2^{k-1}A^k \\ & (2A)^k \end{bmatrix}\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}$$

$$= \begin{bmatrix} (2A)^{k+1} & (2^k+k\cdot 2^k)A^{k+1} \\ & (2A)^{k+1} \end{bmatrix} = \begin{bmatrix} (2A)^{k+1} & (k+1)\cdot 2^k A^{k+1} \\ & (2A)^{k+1} \end{bmatrix}, S(k)\Rightarrow S(k+1)$$

By first principle of induction, $\forall n: n\in\mathbf{Z}^+ \Rightarrow S(n)$. ∎
So

$$f\left(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}\right) = \sum_{k=0}^{+\infty} a_k \begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}^k = \sum_{k=0}^{+\infty}\begin{bmatrix} a_k(2A)^k & a_k\cdot k\cdot 2^{k-1}A^k \\ & a_k(2A)^k \end{bmatrix}$$

Let $X = BJB^{-1}$, then for $p(x) = x^n$, $p'(X) = Bp'(J)B^{-1}$, $p'(J) = nJ^{n-1}$, so $p'(X) = nX^{n-1}$

$$\Rightarrow f'(X) = \sum_{k=0}^{+\infty} a_k\cdot k\cdot X^{k-1}$$

$$\Rightarrow f\left(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}\right) = \begin{bmatrix} f(2A) & f'(A)A \\ & f(2A) \end{bmatrix} \Rightarrow B = f'(A)\cdot A.$$

3. It can be disproved by letting $f(x) = x^2$, $A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}$

$$f'(A)B = \left(\begin{bmatrix} f'(1) & f''(1) \\ & f'(1) \end{bmatrix}\right)\begin{bmatrix} 2 & \\ & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ & 6 \end{bmatrix}$$

$$f'\left(\begin{bmatrix} 1+2t & 1 \\ & 1+3t \end{bmatrix}\right) = \begin{bmatrix} 1 & -1/t \\ & 1 \end{bmatrix}\begin{bmatrix} 2+4t & \\ & 2+6t \end{bmatrix}\begin{bmatrix} 1 & 1/t \\ & 1 \end{bmatrix} = \begin{bmatrix} 2+4t & -2 \\ & 2+6t \end{bmatrix}$$

Plugging in $t=0$ yields $\begin{bmatrix} 2 & -2 \\ & 2 \end{bmatrix}$.
Actually, the proposition holds only when $[A,B]=0$. In this case, binomial theorem is suitable for matrices A and B.

$$[f(A+tB)]' = \left(\sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k}(Bt)^k\right)' = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_n C_n^k A^{n-k} k B^k t^{k-1}$$

Plugging in $t=0$,

$$[f(A+tB)]'|_{t=0} = \sum_{n=1}^{+\infty} a_n C_n^1 A^{n-1}B = f'(A)\cdot B$$

Solution 1.5.3.

1. Let v be the common eigenvector of A and B, λ_A and λ_B are the two eigenvalues associated with v for A and B. V_2 is a matrix such that the column vectors are orthonormal basis of orthogonal complement of span of v. Then for A,

$$\begin{bmatrix} v & V_2 \end{bmatrix}^H A \begin{bmatrix} v & V_2 \end{bmatrix} = \begin{bmatrix} v & V_2 \end{bmatrix}^H \begin{bmatrix} \lambda_A v & AV_2 \end{bmatrix} = \begin{bmatrix} \lambda_A & v^H AV_2 \\ 0 & V_2^H AV_2 \end{bmatrix}$$

Similarly, for B,

$$\begin{bmatrix} v & V_2 \end{bmatrix}^H B \begin{bmatrix} v & V_2 \end{bmatrix} = \begin{bmatrix} \lambda_B & v^H BV_2 \\ 0 & V_2^H BV_2 \end{bmatrix}$$

So $A_1 = V_2^H AV_2$, $B_1 = V_2^H BV_2$.

$$A_1 B_1 = V_2^H AV_2 V_2^H BV_2 = V_2^H ABV_2$$
$$B_1 A_1 = V_2^H BV_2 V_2^H AV_2 = V_2^H BAV_2$$

The above two equations are equal as A commutes with B, i.e.

$$[A, B] = 0 \Rightarrow [A_1, B_1] = 0$$

2. From (1), $[A_1, B_1] = 0$, so A_1, B_1 should have a common eigenvector. We may assume that it is one of the column vectors of V_2, otherwise V_2 can be reconstructed in the previous step to contain that common eigenvector.
Moveover, when that common eigenvector is placed on the first column of V_2 in (1), then

$$A_1 = \begin{bmatrix} \lambda_{A_1} & v'^H BV_{22} \\ 0 & V_{22}^H A_1 V_{22} \end{bmatrix},$$

where v' is the common eigenvector of A_1 and B_1. Same for B_1.
Hence we can iteratively apply that transformation to $(A_i)_{22}$ and $(B_i)_{22}$ for all i.
In addition, for each iteration, the size of A_i, B_i strictly decreases.
Given that A and B are finite dimensional, the process of iteration eventually terminates.
The iterations terminate when i is such that A_i and B_i has dimension one.
That gives

$$\begin{bmatrix} v & V_2 \end{bmatrix}^H A \begin{bmatrix} v & V_2 \end{bmatrix}$$

an upper triangular matrix. Same for B. ■