

Linear Algebra: Homework 8

December 4, 2021

Question 1.

Without writing A , find an eigenvalue of A and describe the eigenspace:

- (1) T is the transformation on \mathbf{R}^2 that reflects points across some line through the origin.
- (2) T is the transformation on \mathbf{R}^3 that rotates points about some line through the origin.

Solution 1.

- (1) A would have eigenvalues of -1 and 1 , where the eigenspace correspond to $\lambda = -1$ are all vectors with direction perpendicular to the line, and the eigenspace correspond to $\lambda = 1$ are all vectors parallel to the line.
- (2) A would have eigenvalues of $1, e^{i\theta}$ and $e^{-i\theta}$, where θ is the angle of rotation. The eigenspace correspond to $\lambda = 1$ are all vectors parallel to the line axis of rotation. The eigenspaces correspond to the two other eigenvalues are not discussed, since these eigenvectors do not belong to \mathbf{R}^3 .

Question 2.

Let \mathbf{u} and \mathbf{v} be eigenvectors of a matrix A , with eigenvalues λ and μ respectively. Let c_1, c_2 be scalars. Define $\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}, k = 0, 1, 2, \dots$.

- (1) What is \mathbf{x}_{k+1} , by definition?
- (2) Compute $A\mathbf{x}_k$ from the formula for \mathbf{x}_k , and show that $A\mathbf{x}_k = \mathbf{x}_{k+1}$.

Solution 2.

- (1) $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$
- (2) $A\mathbf{x}_k = A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) = c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} = c_1\lambda^k(\lambda\mathbf{u}) + c_2\mu^k(\mu\mathbf{v}) = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v} = \mathbf{x}_{k+1}$

Question 3.

Find the characteristic polynomial and eigenvalues of the matrices

$$\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}, \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}.$$

Solution 3.

Let $A = \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}, B = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$. Then, the characteristic polynomials are

$$p_A(\lambda) = \begin{vmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{vmatrix} = 20 - 9\lambda + \lambda^2 + 12 = \lambda^2 - 9\lambda + 32$$

and

$$p_B(\lambda) = \begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = 21 - 10\lambda + \lambda^2 + 4 = \lambda^2 - 10\lambda + 25$$

respectively.

Eigenvalues λ_1, λ_2 of A satisfies $p_A(\lambda_1) = p_A(\lambda_2) = 0$, in which

$$\lambda_1 = \frac{9 - \sqrt{81 - 128}}{2} = 4.5 - \frac{1}{2}\sqrt{47}i, \lambda_2 = 4.5 + \frac{1}{2}\sqrt{47}i$$

Eigenvalue λ_1 satisfies $p_B(\lambda_1) = (\lambda_1 - 5)^2 = 0 \Rightarrow \lambda_1 = 5$, with algebraic multiplicity of 2.

Question 4.

Find the characteristic polynomial of the following matrices, with the cofactor expansion of the determinant:

$$\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}, \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}.$$

Solution 4.

Let $A = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$. Then, the characteristic polynomials are

$$p_A(\lambda) = \begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 9-\lambda & 0 \\ 5 & 8 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = (3-\lambda)(50-15\lambda+\lambda^2) = -\lambda^3+18\lambda^2-95\lambda+150$$

$$p_B(\lambda) = \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix} = (1-\lambda)(-28-3\lambda+\lambda^2) = -\lambda^3+4\lambda^2+25\lambda-28$$

Question 5.

Diagonalize the following matrices if possible, it is known that the first matrix has eigenvalues $\lambda = 2, 8$, and that the second matrix has eigenvalues $\lambda = 2, 1$:

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}.$$

Solution 5.

Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$.

For A ,

eigenvectors \vec{v}_1 correspond to $\lambda_1 = 2$ satisfies

$$(A - 2I)\vec{v}_1 = 0$$

Solve the above equation by augmented matrix, we have

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\vec{v}_1 = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, where $t_1, t_2 \in \mathbf{R}$ are parameters.

eigenvectors \vec{v}_2 correspond to $\lambda_2 = 8$ satisfies

$$(A - 8I)\vec{v}_2 = 0$$

Solve the above equation by augmented matrix, we have

$$\begin{bmatrix} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\vec{v}_2 = t_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ where $t_3 \in \mathbf{R}$ is a parameter.

Let $A = PDP^{-1}$, where $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 8 \end{bmatrix}$. Then, one possible way to construct P is $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

$$\begin{aligned} [P|I_3] &\sim \begin{bmatrix} 1 & 1 & 1 & 1 & & \\ -1 & 0 & 1 & & 1 & \\ 0 & -1 & 1 & & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & & \\ 0 & 1 & 2 & 1 & 1 & \\ 0 & -1 & 1 & & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & & \\ 0 & 1 & 2 & 1 & 1 & \\ 0 & 0 & 3 & 1 & 1 & 1 \end{bmatrix} \sim \\ &\begin{bmatrix} 1 & 1 & 1 & 1 & & \\ 0 & 1 & 2 & 1 & 1 & \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 & -2/3 & -1 \\ 0 & 1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{bmatrix} \end{aligned}$$

Hence,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 2 & \\ & & 8 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & -1 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

For B ,

eigenvectors \vec{v}_1 correspond to $\lambda_1 = 2$ satisfies

$$(B - 2I)\vec{v}_1 = 0$$

Solve the above equation by augmented matrix, we have

$$\begin{bmatrix} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ -1 & -2 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\vec{v}_1 = t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, where $t_1, t_2 \in \mathbf{R}$ are parameters.

eigenvectors \vec{v}_1 correspond to $\lambda_1 = 1$ satisfies

$$(B - I)\vec{v}_1 = 0$$

Solve the above equation by augmented matrix, we have

$$\begin{bmatrix} -1 & -4 & -6 & 0 \\ -1 & -1 & -3 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -4 & -6 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -4 & -6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\vec{v}_2 = t_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, where $t_3 \in \mathbf{R}$ is a parameter.

Let $A = PDP^{-1}$, where $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$. Then, one possible way to construct P is $P = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$.

$$\begin{aligned} [P|I_3] &\sim \begin{bmatrix} 2 & 3 & 2 & 1 & & \\ -1 & 0 & 1 & & 1 & \\ 0 & -1 & -1 & & & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 2 & 1 & & \\ 0 & 3/2 & 2 & 1/2 & 1 & \\ 0 & -1 & -1 & & & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 2 & 1 & & \\ 0 & 3/2 & 2 & 1/2 & 1 & \\ 0 & 0 & 1/3 & 1/3 & 2/3 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 2 & 3 & 0 & -1 & -4 & -6 \\ 0 & 3/2 & 0 & -3/2 & -3 & -6 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & 2 & 2 & 6 \\ 0 & 1 & 0 & -1 & -2 & -4 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 & -2 & -4 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

Hence,

$$B = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ -1 & -2 & -4 \\ 1 & 2 & 3 \end{bmatrix}$$

Question 6.

- (1) A is a 5×5 matrix with two eigenvalues, one eigenspace is 3-dimensional, and the other eigenspace is 2-dimensional. Is A diagonalizable? Why?
- (2) A is a 4×4 matrix with three eigenvalues. One eigenspace is 1-dimensional, and one of the other eigenspaces is 2-dimensional. Is it possible that A is not diagonalizable? Why?

Solution 6.

- (1) Yes.

The algebraic multiplicities of eigenvalues must not less than the dimension of the eigenspace correspond to that eigenvalue. Thus the first eigenvalue must have algebraic multiplicity not less than 3, and the latter not less than 2.

Moreover, the sum of algebraic multiplicities of the roots of a polynomial is the degree of it. The characteristic polynomial of A is degree 5, hence we can conclude that the algebraic multiplicity of the first eigenvalue is 3 and the latter is 2.

Thus, for all eigenvalues of A , the dimensions of eigenspaces correspond to the eigenvalues is equal to the algebraic multiplicities of the eigenvalues in the characteristic polynomial.

- (2) No

(The solution below assumes the three eigenvalues are distinct.)

By using the conclusion in part 1, if the sum of the dimensions of eigenspaces is the dimension of the size of the matrix, i.e. 4 for A , then A is diagonalizable.

The remaining eigenspace must have dimension 1,

since eigenspace, directly from the definition of eigenvectors, must have its dimension not less than 1,

nor more than the algebraic multiplicity of the eigenvalue that the eigenspace corresponds to.

For this 4×4 matrix, it has 3 eigenspaces, two of which are one-dimensional and correspond to the two eigenvalues with algebraic multiplicity one for each in the characteristic polynomial of A .

The other eigenspace is two-dimensional, eigenvalue corresponds to it has algebraic multiplicity of 2 in the characteristic polynomial.

Question 7.

Define $T : \mathbb{P}_2 \rightarrow \mathbf{R}^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

- (1) Find the image under T of $\mathbf{p}(t) = 5 + 3t$.
- (2) Show that T is a linear transformation.
- (3) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for \mathbb{P}_2 and the standard basis for \mathbf{R}^3 .

Solution 7.

1. $T(\mathbf{p}) = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

2.

$$T\left(\sum_{i=1}^n c_i \mathbf{p}_i\right) = \begin{bmatrix} \left(\sum_{i=1}^n c_i \mathbf{p}_i\right)(-1) \\ \left(\sum_{i=1}^n c_i \mathbf{p}_i\right)(0) \\ \left(\sum_{i=1}^n c_i \mathbf{p}_i\right)(1) \end{bmatrix} = \sum_{i=1}^n c_i \begin{bmatrix} \mathbf{p}_i(-1) \\ \mathbf{p}_i(0) \\ \mathbf{p}_i(1) \end{bmatrix} = \sum_{i=1}^n c_i T(\mathbf{p}_i)$$

3. Let A be the matrix for transformation T .

$$A = \begin{bmatrix} T(1) & T(t) & T(t^2) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Question 8.

Let A and B be similar square matrices, show that they have the same rank.

Solution 8.

$A \sim B \Rightarrow (\exists P)(AP = PB)$. Let $\mathbf{w} \in \text{Ran}(B)$, and \mathbf{v} is the preimage of \mathbf{w} . Then we have

$$AP\mathbf{v} = PB\mathbf{v} = P\mathbf{w}$$

Hence $P : \text{Ran}(B) \rightarrow \text{Ran}(A)$.

As the linear transformation represented by P is injective,

$\text{rank}(B) = \dim \text{Ran}(B) \leq \dim \text{Ran}(A) = \text{rank}(A)$. (1)

As matrix similarity is an equivalence relation, we also have $\text{rank}(A) \leq \text{rank}(B)$. (2)

With (1) and (2), $\text{rank}(A) = \text{rank}(B)$. ■

Question 9.

(1) The *trace* of a square matrix A is the sum of the diagonal entries of A , and is denoted $\text{Tr}(A)$. It can be verified that $\text{Tr}(AB) = \text{Tr}(BA)$ for any two $n \times n$ matrices A and B . Show that if A and B are similar, then $\text{Tr}(A) = \text{Tr}(B)$.

(2) Suppose that A is diagonalizable, show that $\text{Tr}(A)$ equals the sum of the eigenvalues of A .

Solution 9.

(1) $A \sim B \Rightarrow (\exists P)(A = P^{-1}BP) \Rightarrow \text{Tr}(A) = \text{Tr}(P^{-1}(BP)) = \text{Tr}((BP)P^{-1}) = \text{Tr}(B)$.

(2) A is diagonalizable, so $A = P\Lambda P^{-1}$, where P is a matrix with columns of eigenvectors of A and Λ is a matrix with eigenvalues of A on the main diagonal and zero elsewhere. Hence,

$$\text{Tr}(A) = \text{Tr}((P\Lambda)P^{-1}) = \text{Tr}(P^{-1}(P\Lambda)) = \text{Tr}(\Lambda) = \sum_{i=1}^n (\Lambda)_{ii} = \sum_{i=1}^n \lambda_i$$

Question 10.

(1) Diagonalize the matrix $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ in the realm of complex matrices if possible.

(2) Express the matrix $\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ in the form PCP^{-1} , with P an invertible real matrix and C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with a, b real.

Solution 10.

(1)

$$\begin{vmatrix} 5 - \lambda & -2 \\ 1 & 3 - \lambda \end{vmatrix} = 17 - 8\lambda + \lambda^2 = 0 \Rightarrow \lambda_{1,2} = 4 \pm i$$

Finding eigenvector corresponds to $\lambda_1 = 4 - i$ leads to the calculation of augmented matrix

$$\begin{bmatrix} 1 + i & -2 & 0 \\ 1 & -1 + i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 + i & 0 \\ 1 + i & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 + i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence that eigenvector will be $t_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$, where $t_1 \in \mathbf{C}$ is a parameter. Finding eigenvector corresponds to $\lambda_1 = 4 + i$ leads to the calculation of augmented matrix

$$\begin{bmatrix} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1-i & 0 \\ 1-i & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1-i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence that eigenvector will be $t_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$, where $t_2 \in \mathbf{C}$ is a parameter.

Let $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = PDP^{-1}$, where $D = \begin{bmatrix} 4-i & \\ & 4+i \end{bmatrix}$. One possible way to construct P is $P = \begin{bmatrix} 1-i & 1+i \\ 1 & 1 \end{bmatrix}$.

$$\begin{aligned} [P|I_2] &\sim \begin{bmatrix} 1-i & 1+i & 1 & \\ & 1 & & 1 \end{bmatrix} \sim \begin{bmatrix} 1-i & 1+i & 1 & 0 \\ & 0 & -1 & -1+i \end{bmatrix} \sim \begin{bmatrix} 1 & i & (1+i)/2 & 0 \\ 0 & 1 & 1-i & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -(1+i)/2 & i \\ 0 & 1 & 1-i & -1 \end{bmatrix} \end{aligned}$$

Hence,

$$\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1-i & 1+i \\ & 1 \end{bmatrix} \begin{bmatrix} 4-i & \\ & 4+i \end{bmatrix} \begin{bmatrix} -(1+i)/2 & i \\ & 1-i \end{bmatrix}$$

(2) For any $A \in \mathbf{R}^{2 \times 2}$, Let $\lambda \in \mathbf{C}$ be one of the eigenvalue of A . By definition,

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A(\Re(\mathbf{v}) + i \cdot \Im(\mathbf{v})) = A\Re(\mathbf{v}) + i \cdot A\Im(\mathbf{v})$$

$$= (\Re(\lambda) + i \cdot \Im(\lambda))(\Re(\mathbf{v}) + i \cdot \Im(\mathbf{v})) = (\Re(\lambda)\Re(\mathbf{v}) - \Im(\lambda)\Im(\mathbf{v})) + i \cdot (\Im(\lambda)\Re(\mathbf{v}) + \Re(\lambda)\Im(\mathbf{v})),$$

where \mathbf{v} is the eigenvector correspond to λ .

$$\begin{aligned} \Rightarrow A \begin{bmatrix} \Re(\mathbf{v}) & \Im(\mathbf{v}) \end{bmatrix} &= \begin{bmatrix} \Re(\lambda)\Re(\mathbf{v}) - \Im(\lambda)\Im(\mathbf{v}) & \Im(\lambda)\Re(\mathbf{v}) + \Re(\lambda)\Im(\mathbf{v}) \end{bmatrix} \\ &= \begin{bmatrix} \Re(\mathbf{v}) & \Im(\mathbf{v}) \end{bmatrix} \begin{bmatrix} \Re(\lambda) & \Im(\lambda) \\ -\Im(\lambda) & \Re(\lambda) \end{bmatrix} \end{aligned}$$

Consider the given matrix, we have

$$\begin{vmatrix} 1-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} = 13 - 4\lambda + \lambda^2 = 0 \Rightarrow \lambda = 2 \pm 3i$$

Choose $\lambda = 2 - 3i$, then

$$\begin{bmatrix} -1+3i & 5 \\ -2 & 1+3i \end{bmatrix} \vec{v} = 0 \Rightarrow \vec{v} = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$$

Hence,

$$\begin{aligned} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1/3 & -1/6 \end{bmatrix} \end{aligned}$$