Topics in Linear Algebra: Homework 2

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In the following, "linearly independent" is used interchangeably with "independent", and "linearly dependent" is used interchangeably with "dependent".

Solution 1.2.1.

1. True.
   Suppose that any three of the four given subspaces are linearly independent, but the four subspaces are not. Then assume
   $$v_i \in V_i, i = 1, 2, 3, 4$$
   Since the four subspaces are dependent, W.L.O.G. assume $\dim(V_4 \cap (V_1 + V_2 + V_3)) > 0$, then
   $$v_4 = -(v_1 + v_2 + v_3)$$
   So $||v_1|| + ||v_2|| + ||v_3|| > 0$, that means V_4 has intersection with at least one subspace, contradicting the assumption that V_4 and any other two of the three spaces are linearly independent. ■

2. False.
   Consider
   $$V_1 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, V_2 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}, V_3 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}, V_4 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$
   Here $V_4 = V_1 + V_2$, so V_1, V_2, V_4 are linearly dependent.

3. True.
   Assume
   $$v_i \in V_i, i = 1, 2, 3, 4.$$
   If the four subspaces are dependent, then
   $$v_1 + v_2 + v_3 + v_4 = 0$$
   has non-trivial solutions. Say
   $$v_4 = -(v_1 + v_2 + v_3)$$
   Hence $||v_1|| + ||v_2|| + ||v_3|| > 0$. So at least one of the three vectors is non-zero. If v_3 is non-zero, then it contradicts that "V_3, V_4 are independent". If v_1 or v_2 is non-zero, then

   (a) If $v_1 + v_2 = 0$, then $v_1 = -v_2$, and both of which are non-zero. It violates that "V_1, V_2 are independent".

   (b) Else, since $v_4 \in (V_3 + V_4)$, and $v_1 + v_2 \neq 0$, then $(V_1 + V_2) \cap (V_3 + V_4)$ is a non-trivial space, which is another contradiction.

   Switch v_4 to the vectors from the three other subspaces, and the same conclusion would be drawn. ■

Solution 1.2.2.

By definition,

$$T : A \mapsto A^T$$

and

$$T^2(A) = (A^T)^T = A.$$

By observation,

$$A \text{ is symmetric } \Leftrightarrow T(A) = A$$

$$A \text{ is skew-symmetric } \Leftrightarrow T(A) = -A.$$

T is bijective, so if a matrix A_0 is both symmetric and anti-symmetric, then

$$T(A_0) + T(A_0) = A_0 + (-A_0) = 0 \Rightarrow A_0 = 0$$

Thus, the space of symmetric and skew-symmetric matrices are independent.
Denote the former as Sym, and the latter as Skew. Consider

$$A = A_1 + A_2,$$

where

$$A_1 = \frac{1}{2}(A + T(A)), A_2 = \frac{1}{2}(A - T(A))$$

$$T(A_1) = \frac{1}{2}T(A + T(A)) = \frac{1}{2}(T(A) + T^2(A)) = \frac{1}{2}(T(A) + A) \in \text{Sym}$$

$$T(A_2) = \frac{1}{2}T(A - T(A)) = \frac{1}{2}(T(A) - T^2(A)) = \frac{1}{2}(T(A) - A) \in \text{Skew}$$

So every matrix can be expressed as a linear combination of two matrices, one from Sym, and one from Skew. Hence

$$V = \text{Sym} \bigoplus \text{Skew}$$

In particular, the former has dimension $\frac{1}{2}n(n+1)$, and the latter has $\frac{1}{2}n(n-1)$.

With regards to block form of T,

$$T = \begin{bmatrix} e_1 & \\ & -e_2 \end{bmatrix},$$

where e_1 is the block representing the transformation:

$$T_1 : \text{Sym} \to \text{Sym}, T_1 \text{ is identity transformation}$$

e_2 is the block representing the transformation:

$$T_2 : \text{Skew} \to \text{Skew}, T_2 \text{ is identity transformation}$$

Solution 1.2.3.

1. Ker(B):
   $\forall v \in \text{Ker}(B)$,

   $$B(Av) = BAv = ABv = A(Bv) = A \cdot 0 = 0,$$

   so $v \in \text{Ker}(B) \Rightarrow Av \in \text{Ker}(B)$ $\square$
   Ran(B) :
   $\forall w \in \text{Ran}(B)$, $\exists v$ s.t. $Bv = w$.

   $$A(w) = A(Bv) = ABv = BAv = B(Av),$$

   so $w \in \text{Ran}(B) \Rightarrow Aw \in \text{Ran}(B)$.

2. $$Ap(A) = A \cdot \sum_{i=0}^{n-1} p_i A^i = \sum_{i=0}^{n-1} p_i A^{i+1} = (\sum_{i=0}^{n-1} p_i A^i)A = p(A)A$$

3. Let k be the smallest positive integer satisfies $\mathrm{Dim}(\mathrm{Ker}((A-\lambda I)^k)) = n$, where $k \le n$. Then $\mathrm{Ker}((A-\lambda I)^k) = N_\infty(A-\lambda I)$, $\mathrm{Ran}((A-\lambda I)^k) = R_\infty(A-\lambda I)$. As

$$(A-\lambda I)^k = \sum_{i=0}^{k} \binom{k}{i} A^i(-\lambda)^{k-i}$$

is a polynomial of A, by (2) $A(A-\lambda I)^k = (A-\lambda I)^k A$, and by (1), and by the fact that $\lambda \in \mathbf{C}$ has no restriction, the conclusion is drawn.

Solution 1.2.4.

1. Suppose W is an A-invariant subspace, but none of the eigenvectors of A belongs to W. Then $\forall \lambda \in \mathbf{C}, \forall v \in W$, $v \neq 0$

$$Av \neq \lambda v \Leftrightarrow (A-\lambda I)v \neq 0$$
$$\Leftrightarrow |A|_W - \lambda I| \neq 0$$

Hence

$$p_{A|_W}(\lambda)$$

has no roots. By the fundamental theorem of algebra, $\mathrm{deg}p_{A|_W} = 0$. Since $p_A(\lambda) = p_{A|_w}(\lambda) \cdot p_{A|_{W^\perp}}(\lambda)$,
$\dim(W) = \mathrm{deg}p_{A|_W} = 0 \Rightarrow W = \{0\}$, which contradicts the assumption that $v \neq 0$. □

2. $$B(A-\lambda I) = BA - \lambda B = AB - \lambda B = (A-\lambda I)B,$$

so by 1.2.3.1. $\forall \lambda \in \mathbf{C} : \mathrm{Ker}(A-\lambda I)$ is B-invariant. □

3. In (1), let $W = \mathrm{Ker}(A-\lambda_1 I)$, and fix λ_1 as an eigenvalue of A, then B has an eigenvector in $\mathrm{Ker}(A-\lambda_1 I)$, which is the λ_1-eigenspace of A, hence that eigenvector of B is also an eigenvector of A. ■