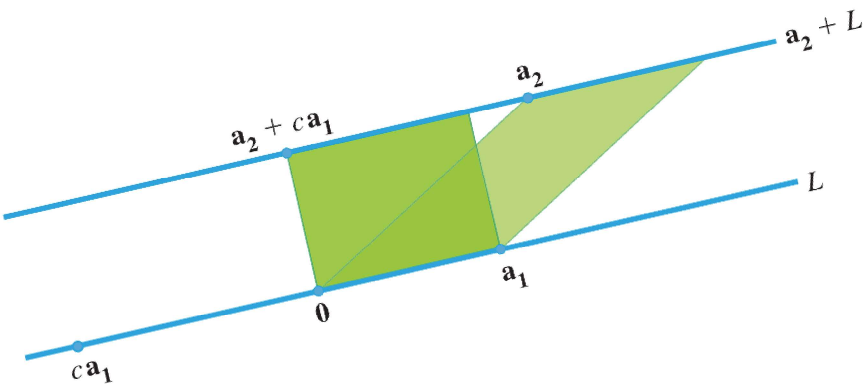
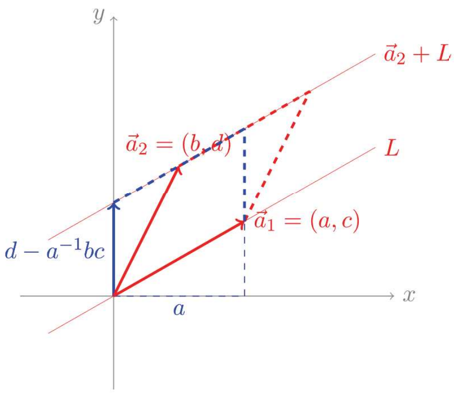
Determinant (行列式)

- Recall that the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$, and we have the formula
$$A^{-1} = \det(A)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
- Geometrically, the determinant calculates the volume.
- Theorem: Let $A = [a_1 \ a_2]$ be a 2×2 matrix, then the area of the parallelogram spanned by a_1, a_2 equals the absolute value of $\det(A)$.

- Proof: Observe that the parallelogram spanned by a_1, a_2 and that spanned by $a_1, a_2 + \lambda a_2$ has the same area. Notice that here we are performing elementary column operation to A.



- Performing elementary column operation, suppose that $a \neq 0$, we see that A can be column reduced to $\begin{bmatrix} a & 0 \\ c & d - a^{-1}bc \end{bmatrix}$.



- Notice that the parallelogram spanned by (a, c) and $(0, d - a^{-1}bc)$ has base of length $|d - a^{-1}bc|$ and height $|a|$, it is then of area $|ad - bc|$.
- In case that $a = 0, b \neq 0$, we can column exchange A and get the same answer.
- In case that $a = b = 0$, the volume is clearly 0, it equals $|\det(A)|$ as well.
- This finishes the proof.
- Question: Let $A = [a_1 \cdots a_n]$ be a $n \times n$ matrix, what is the volume of the parallelotope (平行体)?

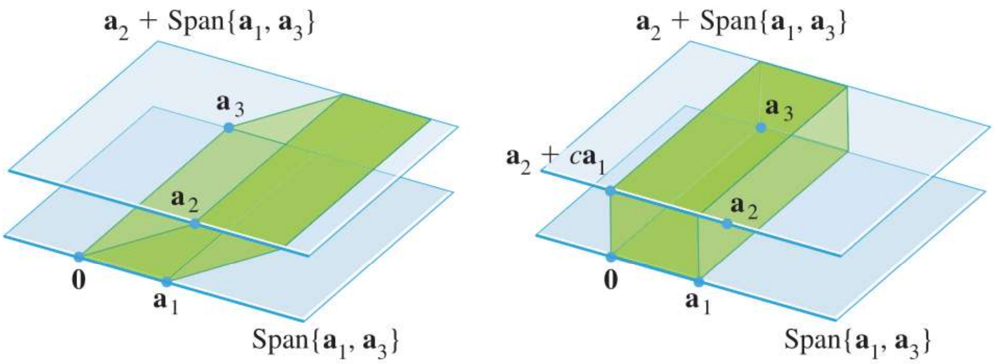
- Theorem: Let $A = [a_1\ a_2\ a_3]$ be a 3×3 matrix, then the volume of the parallelotope spanned by a_1, a_2, a_3 is the absolute value of

$$\Delta = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

- Proof: The key observation is still that the transformation

$$(a_1, a_2, a_3) \mapsto (a_1, a_2 + \lambda_2 a_1, a_3 + \lambda_3 a_1)$$

doesn't change the volume of the parallelotope. In other words, performing elementary column operation doesn't change the volume.



- Exchange the columns doesn't change the parallelotope, hence also the volume. Suppose that $a_{11} \neq 0$, performing elementary column operations to A, we get

$$A \sim A' = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} - \frac{a_{12}}{a_{11}} a_{21} & a_{23} - \frac{a_{13}}{a_{11}} a_{21} \\ a_{31} & a_{32} - \frac{a_{12}}{a_{11}} a_{31} & a_{33} - \frac{a_{13}}{a_{11}} a_{31} \end{bmatrix}.$$

- Let $A' = [a_1\ a_2'\ a_3']$. Notice that the vectors a_2', a_3' lie in the subspace $x_1 = 0$. Let $B = [b_2\ b_3]$ be the submatrix of A' with the first row and column deleted. Regard the parallelogram spanned by a_2', a_3' as the base, the parallelotope has height $|a_{11}|$, hence it is of volume $|a_{11}| \cdot |\det(B)|$.

- Expand it out, we see that the volume is the absolute value of

$$\Delta = a_{11} \left[\left(a_{22} - \frac{a_{12}}{a_{11}} a_{21} \right) \left(a_{33} - \frac{a_{13}}{a_{11}} a_{31} \right) - \left(a_{23} - \frac{a_{13}}{a_{11}} a_{21} \right) \left(a_{32} - \frac{a_{12}}{a_{11}} a_{31} \right) \right]$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

- Collecting the sums, we get

$$\Delta = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

- In case that $a_{11} = a_{12} = a_{13} = 0$, the parallelotope has volume 0, so does Δ. The theorem holds in this case. This finishes the proof.
- The expression for Δ suggests the following definition: Let A be a $n \times n$ matrix, for $i, j = 1, \cdots, n$, let A_{ij} be the submatrix of A with the i-th row and j-th column deleted

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

- Definition

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

- We call $C_{ij} = (-1)^{i+j} \det(A_{ij})$ the (i, j)-cofactors (代数余子式) of A, then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

- Theorem

> If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Proof The determinant is defined recursively, so we will prove the theorem inductively.

1) A is lower triangular, i.e. $A = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

A_{11}

The assertion is clearly true if $n=1$, as $\det(A) = a_{11}$.

Suppose that the assertion is true for $n-1$,

then by definition,

$$\det(A) = a_{11} \cdot \det(A_{11}) - \underset{\overset{\shortparallel}{0}}{a_{12}} \cdot \det(A_{12}) + \cdots + (-1)^{n+1} \underset{\overset{\shortparallel}{0}}{a_{1n}} \cdot \det A_{1n}$$

$$= a_{11} \cdot \det(A_{11})$$

induction hypothesis $\rightarrow$ $= a_{11} \cdot (a_{22} \cdots a_{nn})$

$$= a_{11} \cdot a_{22} \cdots a_{nn}.$$

2) A is upper triangular, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$.

Notice that for $j = 2, 3, \cdots, n$, the first column of A_{1j} is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

Claim Let B be an $n \times n$ matrix with the first column being $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, then $\det(B) = 0$. [The assertion is clearly true for $n=1$. In general, notice that the first column of B_{1j} is also $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, hence

$$\det(B) = b_{11} \cdot \underset{0}{\underset{\shortparallel}{\det(B_{11})}} - b_{12} \underset{0}{\underset{\shortparallel}{\det(B_{12})}} + \cdots + (-1)^{n+1} b_{1n} \underset{0}{\underset{\shortparallel}{\det(B_{1n})}}$$

$= 0$]. by induction hypothesis.

So
$$\det(A) = a_{11} \cdot \det(A_{11}) - a_{12} \underset{0}{\underset{\shortparallel}{\det(A_{12})}} + \cdots + (-1)^{n+1} a_{1n} \underset{0}{\underset{\shortparallel}{\det(A_{1n})}}$$
$$= a_{11} \cdot \det(A_{11})$$

as first column is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

induction hypothesis $\rightarrow$
$$= a_{11} \cdot (a_{22} \cdots a_{nn})$$
$$= a_{11} \cdot a_{22} \cdots a_{nn}. \quad \square$$

The complete expansion

- We have defined the determinant recursively, it can be expanded out.
- Let $\sigma = (\sigma_1 \cdots \sigma_n)$ be a permutation of $(1 \cdots n)$. If it interchanges only i and j, then we call it a transposition (对换), denoted $(i\ j)$. It is clear that any permutation can be written as the composition of a sequence of transpositions

$$\sigma = (i_1\ j_1) \circ (i_2\ j_2) \circ \cdots \circ (i_r\ j_r).$$

- The expression is not unique, but the parity of r is the same for any such expression. We define the signature of the permutation σ as $(-1)^r$, it is well defined, denoted $\mathrm{sgn}(\sigma)$.

- Theorem: Let A be a $n \times n$ matrix, then

$$\det(A) = \sum_{\sigma} \mathrm{sgn}(\sigma) a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n},$$

where σ runs through all the permutations of $(1\ 2\ \cdots\ n)$.

- Proof by induction, omitted.
- Example: For 2×2 matrix, $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$.
- Example: For 3×3 matrix A, we have

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

- With the above complete expansion, we can prove the theorem:

> The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is
>
> $$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
>
> The cofactor expansion down the jth column is
>
> $$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

- The complete expansion also implies the theorem

> If A is an $n \times n$ matrix, then $\det A^T = \det A$.

EXAMPLE 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution: Notice that the third column has only 1 non-zero entry, we will expand along it.

$$\det(A) = \underset{\overset{\shortparallel}{0}}{a_{13}} \cdot C_{13} + a_{23} \cdot C_{23} + \underset{\overset{\shortparallel}{0}}{a_{33}} C_{33}$$

$$= (-1) \cdot (-1)^{2+3} \det \begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix} = -2 .$$

EXAMPLE 3 Compute det A, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Solution Notice that the first column of A has only one non-zero entry, expand along it, get

$$\det(A) = 3 \cdot \det(A_{11}) .$$

Expand along the first column of A_{11}, get

$$\det(A_{11}) = 2 \cdot \begin{vmatrix} 1 & 5 & \boxed{0} \\ 2 & 4 & \boxed{-1} \\ 0 & -2 & \boxed{0} \end{vmatrix}$$

expand along last column

$$= 2 \cdot (-1) \cdot (-1)^{2+3} \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix}$$

$$= -4$$

So $\det(A) = 3 \cdot \det(A_{11}) = -12.$ $\square$