Topics in Linear Algebra: Homework 1

March 11, 2022

Solution 1.1.1.

1. To begin, inspect the case when $n = 2$.
When $n = 2$, it is equivalent to finding a linear transformation that, when it is a composite transformation of its own, that composite transformation maps any vector on the plane to another vector in its opposite direction. That can be done by rotation matrix of $\pm\frac{\pi}{2}$, or any matrix similar to that rotation matrix. Then we have

$$\text{Rot}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{Rot}\left(\frac{\pi}{2}\right)^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Let $A_1 = \text{Rot}\left(\frac{\pi}{2}\right)$. The following proves that when $n = 2k$, the matrix would be in the form that the main diagonal of the matrix is filled with blocks of A_1, and elsewhere zero, or any matrix similar to that.
Let $S(k) : (A_k)^2 = -I_{2k\times 2k}$.
The proof of $S(1)$ is shown above.
Assume $S(k)$ is true, then for $S(k+1)$,

$$A_{k+1} = \begin{bmatrix} A_k & \\ & A_1 \end{bmatrix}, A_{k+1}^2 = \begin{bmatrix} A_k & \\ & A_1 \end{bmatrix}\begin{bmatrix} A_k & \\ & A_1 \end{bmatrix} = \begin{bmatrix} -I_{2k} & \\ & -I_2 \end{bmatrix} = -I_{2(k+1)\times 2(k+1)}$$

So we have $S(k) \Rightarrow S(k+1)$. By the first principle of mathematical induction, $S(k)$ is true for any positive integer k, and hence

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_1 \end{bmatrix}$$

or any other matrices similar to that are possible solutions.

2. First, consider a Jordan block. Every Jordan block would be in the form of

$$J = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}, J^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 & & & \\ & \lambda^2 & 2\lambda & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \lambda^2 & 2\lambda & 1 \\ & & & & \lambda^2 & 2\lambda \\ & & & & & \lambda^2 \end{bmatrix}$$

Assume that $A \in \mathbf{R}^{n\times n}$, then its characteristic polynomial would be a polynomial with real coefficients. By Vieta's theorem, if a complex number z is an eigenvalue of A, then $\bar{z}$ is another eigenvalue of A, and hence when n is odd, there exists real eigenvalue to A.
Factorize A in the form that $A = XJX^{-1}$, where X are generalized eigenvectors w.r.t. Jordan blocks in J, then we have

$$A^2 = XJ^2X^{-1} = -I \Leftrightarrow J^2 = -I,$$

so A should be diagonalizable, and nothing other than $\pm i$ can be eigenvalues of A, contradicting the fact that there exists real eigenvalue to A. ∎

Solution 1.1.2.

1. A is known as a linear complex structure.
   Let $k = x + iy$, where $x, y \in \mathbf{R}, k \in \mathbf{C}$.
   Let $P = B(kv) - kB(v)$, then

   $$P = B(kv) - kB(v) = B(x + iy)v - (x + iy)Bv = B(xI + Ay)v - (xI + Ay)Bv$$

   Since $A, B \in M_n(\mathbf{R})$, A and B are both $\mathbf{R}$−linear, then

   $$P = xBv + yBAv - xBv - yABv = yBAv - yABv$$

   As v and y are both arbitary,

   $$B(kv) = k(Bv) \Leftrightarrow P = 0 \Leftrightarrow AB = BA$$

2. No. Consider the case when

   $$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

   Then

   $$AX - XA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \neq 0$$

3. Set A=$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $C = \mathrm{Ref}(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$, $\theta \in [0, \pi)$.
   Clearly, $\mathrm{Ref}(\theta)$ represents a reflection matrix, which implies $C^2 = I$.

   $$\begin{aligned} CA + AC &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\ &= \begin{bmatrix} \sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & -\sin 2\theta \end{bmatrix} + \begin{bmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{bmatrix} = 0 \end{aligned}$$

   For example,

   $$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Solution 1.1.3.

1. C is real linear, but not complex linear. $\forall a \in \mathbf{R}$, $\forall v \in \mathbf{C}^n$,

$$C(a\mathbf{v}) = C\begin{bmatrix} ax_1 + iay_1 \\ ax_2 + iay_2 \\ \vdots \\ ax_n + iay_n \end{bmatrix} = \begin{bmatrix} ax_1 - iay_1 \\ ax_2 - iay_2 \\ \vdots \\ ax_n - iay_n \end{bmatrix}$$

$$a(C\mathbf{v}) = a\left(C\begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \vdots \\ x_n + iy_n \end{bmatrix}\right) = a\begin{bmatrix} x_1 - iy_1 \\ x_2 - iy_2 \\ \vdots \\ x_n - iy_n \end{bmatrix} = \begin{bmatrix} ax_1 - iay_1 \\ ax_2 - iay_2 \\ \vdots \\ ax_n - iay_n \end{bmatrix}$$

However, C is not complex linear. For instance,

$$C\left(i \cdot \begin{bmatrix} 1+i \\ 2 \\ 3+i \end{bmatrix}\right) = C\begin{bmatrix} -1+i \\ 2i \\ -1+3i \end{bmatrix} = \begin{bmatrix} -1-i \\ -2i \\ -1-3i \end{bmatrix},$$

but then

$$i \cdot C\begin{bmatrix} 1+i \\ 2 \\ 3+i \end{bmatrix} = i \cdot \begin{bmatrix} 1-i \\ 2 \\ 3-i \end{bmatrix} = \begin{bmatrix} 1+i \\ 2i \\ 1+3i \end{bmatrix}$$

2. **C**-linear $\Rightarrow$ **R**-linear. Sufficiency is proved by the fact that reals are complex number with zero imaginary component. Necessity is disproved in (1).

3. **R**-basis: $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\}$, **C**-basis: $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.
Real-dimension of $\mathbf{C}^2$=size of **R**-basis=4, complex-dimension of $\mathbf{C}^2$=size of **C**-basis=2.

4. **C**-linearly independent $\Rightarrow$ **R**-linearly independent. Consider matrix B with basis column vectors. Let $\mathbf{v} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. B is filled by basis, so $B\mathbf{v} = B\mathbf{x} + Bi\mathbf{y} = 0$ has $\mathbf{x} = \mathbf{y} = 0$, proving sufficiency. Necessity is disproved by considering (3) that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \cdot \begin{bmatrix} i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ i \end{bmatrix} = 0$$

5. **R**-spanning $\Rightarrow$ **C**-spanning. Any vector in a space w.r.t. the set of spanning vectors can be expressed as a linear combination of the spanning vectors. **R**-spanning means the coefficients corresponding to the linear combination are real, which is a special case of expressing the vectors in a linear combination with coefficients of complex numbers, in the sense that the imaginary part is zero. Necessity is disproved by (3) that $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ failed to span $\mathbf{C}^2$ as non real vectors in $\mathbf{C}^2$ are missed out.

Solution 1.1.4.

By given conditions,

$$P = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}$$

1.

$$P\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, P\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}$$

2.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}, F_4^{-1}PF_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

D is similar to P, so $\lambda_k = i^k, (k = 0, 1, 2, 3)$ are eigenvalues of P, corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}.$$

3.

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$$

$$C\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 + c_2 + c_3 \end{bmatrix}, C\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} (c_0 - c_2) + (c_1 - c_3)i \\ (c_3 - c_1) + (c_0 - c_2)i \\ (c_2 - c_0) + (c_3 - c_1)i \\ (c_1 - c_3) + (c_2 - c_0)i \end{bmatrix}$$

4.

$$C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3$$

As C is similar to $F_4^{-1}CF_4$, C and $F_4^{-1}CF_4$ shares the same eigenvalues.

$$F_4^{-1}CF_4 = c_0 I + c_1 F_4^{-1}PF_4 + c_2 F_4^{-1}P^2F_4 + c_3 F_4^{-1}P^3F_4$$

$$= c_0 I + c_1 D + c_2 D^2 + c_3 D^3$$

$$= \begin{bmatrix} c_0 + c_1 + c_2 + c_3 & 0 & 0 & 0 \\ 0 & (c_0 - c_2) + (c_1 - c_3)i & 0 & 0 \\ 0 & 0 & c_0 - c_1 + c_2 - c_3 & 0 \\ 0 & 0 & 0 & (c_0 - c_2) + (c_3 - c_1)i \end{bmatrix}$$

Thus $\lambda_0 = c_0 + c_1 + c_2 + c_3$,
$\lambda_1 = (c_0 - c_2) + (c_1 - c_3)i$,
$\lambda_2 = c_0 - c_1 + c_2 - c_3$ and
$\lambda_3 = (c_0 - c_2) + (c_3 - c_1)i$ are the eigenvalues associated with C, the four associated eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}.$$