Linear Algebra: Homework 7

November 27, 2021

Question 1.

Compute $\det(B^4)$ for $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution 1.

$$\det(B^4) = \det(B)^4 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}^4 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix}^4 = (-2)^4 = 16$$

Question 2.

Explain briefly the assertion.

(1) If A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$.

(2) Let A and B be square matrices. Even thougth AB and BA may not be equal, it is always true that $\det(AB) = \det(BA)$.

(3) Let A and P be square matrices with P invertible, then $\det(PAP^{-1}) = \det(A)$.

(4) Let U be a square matrix such that $U^T U = I_n$, then $\det(U) = \pm 1$.

Solution 2.

Suppose $A, B, P, U \in \mathbf{R}^{n \times n}$

(1)
$$A^{-1}A = I_n \Rightarrow \det(A^{-1}A) = \det(A^{-1})\det(A) = \det(I_n) = 1 \Leftrightarrow \det(A^{-1}) = \det(A)^{-1} \quad \blacksquare$$

(2)
$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA) \quad \blacksquare$$

(3)
$$\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)\det(P)^{-1} = \det(A) \quad \blacksquare$$

(4)
$$U^T U = I_n \Rightarrow \det(U^T U) = \det(U^T)\det(U) = \det(U)\det(U) = \det(I_n) = 1 \Rightarrow \det(U) = \pm 1 \quad \blacksquare$$

Question 3.

Compute the adjugate of the matrix $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, and use the inversion formula to calculate its inverse.

Solution 3.

Denote C as the cofactor matrix of the matrix above,then

$$adj \begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} = C^T = \begin{bmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ - \begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} & + \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} & - \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} \\ + \begin{vmatrix} 5 & 4 \\ 0 & 1 \end{vmatrix} & - \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} & + \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ -1 & -5 & 7 \\ 5 & 1 & -5 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$$

$$\begin{vmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -5 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

Hence,

$$\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}^{-1} = \frac{adj \begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}}{\begin{vmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}} = \begin{bmatrix} -1/6 & -1/6 & 5/6 \\ 1/6 & -5/6 & 1/6 \\ 1/6 & 7/6 & -5/6 \end{bmatrix}$$

Question 4.

Suppose that all the entries of A are integers and $\det(A) = 1$. Explain why all the entries of A^{-1} are integers.

Solution 4.

Denote C as the cofactor matrix of A.

$$(A^{-1})_{ij} = \left(\frac{A^*}{\det(A)} \right)_{ij} = (A^*)_{ij} = (C)_{ji} = (-1)^{i+j} M_{ji},$$

where M_{ji} is the determinant of part of the matrix. Determinant involves addition and multiplication of matrix entries only. Since integers are closed under addition and multiplication, all entries A^{-1} must be integers. ■

Question 5.

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$ and $(5, 1, 0)$.

Solution 5.

$$\text{Volume of that parallelepiped} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 2 & 1 \\ -3 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 7 & 15 \end{vmatrix} = 23$$

Question 6.

Let R be the triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3). Show that the area of R equals the absolute value of

$$\frac{1}{2} \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

Solution 6.

$$\frac{1}{2}\left|\left|\begin{array}{ccc} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{array}\right|\right| = \frac{1}{2}\left|\left|\begin{array}{ccc} x_1 & y_1 & 1 \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_1 & 0 \end{array}\right|\right| = \frac{1}{2}\left|\left|\begin{array}{cc} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{array}\right|\right| \tag{1}$$

Alternatively, translate R by a constant vector of $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, and denote it as R', in which (x_2-x_1, y_2-y_1), (x_3-x_1, y_3-y_1) and the origin are the vertices.
We can create a linear transformation that maps $\vec{e_1}$ to $\begin{bmatrix} x_2-x_1 \\ y_2-y_1 \end{bmatrix}$, and $\vec{e_2}$ to $\begin{bmatrix} x_3-x_1 \\ y_3-y_1 \end{bmatrix}$.
The determinant of the matrix representation of a transformation in $\mathbf{R}^2$ is defined as the enlargement factor of the area of the unit square determined by $\vec{e_1}$ and $\vec{e_2}$ to the area of the parallelogram determined by $\begin{bmatrix} x_2-x_1 \\ y_2-y_1 \end{bmatrix}$ and $\begin{bmatrix} x_3-x_1 \\ y_3-y_1 \end{bmatrix}$.
Hence,

$$\text{area of } R' = \frac{1}{2}\left|\left|\begin{array}{cc} x_2-x_1 & x_3-x_1 \\ y_2-x_1 & y_3-y_1 \end{array}\right|\right| = \frac{1}{2}\left|\left|\begin{array}{cc} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{array}\right|\right| \tag{2}$$

Hence, by (1) and (2), area of $R = \frac{1}{2}\left|\left|\begin{array}{ccc} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{array}\right|\right|$.■

Question 7.

Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$, with a, b, c positive. Let S be the unit ball, bounded by the surface $x_1^2 + x_2^2 + x_3^2 = 1$.

(1) Show that $T(S)$ is bounded by the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$.

(2) It is known that the volume of the unit ball is $\frac{4}{3}\pi$. Calculate the volume of $T(S)$.

Solution 7.

Since

(1) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $\vec{x}$ is inside the sphere if and only if $||\vec{x}|| \leq 1$. $T(\vec{x}) = \begin{bmatrix} ax_1 \\ bx_2 \\ cx_3 \end{bmatrix}$. Substitute $T(\vec{x})$ in the ellipsoid, then $\frac{(ax_1)^2}{a^2} + \frac{(bx_2)^2}{b^2} + \frac{(cx_3)^2}{c^2} = x_1^2 + x_2^2 + x_3^2 \leq 1$, thus it lies inside the ellipsoid.

(2) Volume of $T(S)$=det$(A) \cdot \frac{4\pi}{3} = \frac{4}{3}\pi abc$.

Question 8.

Let S be the tetrahedron in $\mathbf{R}^3$ with vertices at the vectors $0, \vec{e_1}, \vec{e_2}, \vec{e_3}$. Let S' be the tetrahedron with vertices at the vectors $0, \vec{v_1}, \vec{v_2}, \vec{v_3}$.

(1) Describe a linear transformation that maps S onto S'.

(2) Find a formula for the volume of S', using the fact that the volume of S equals

$$\frac{1}{3} \cdot \{\text{area of the base}\} \cdot \{\text{height}\}$$

Solution 8.

(1) $[\ \vec{v_1}\quad \vec{v_2}\quad \vec{v_3}\ ]$.

(2) Volume of S=(1/2)(1/3)=1/6
Volume of $S' = |\ \vec{v_1}\quad \vec{v_2}\quad \vec{v_3}\ | \cdot$ (Volume of S) $= \frac{1}{6}|\ \vec{v_1}\quad \vec{v_2}\quad \vec{v_3}\ |$