Part 1.

1. D

2. A $\quad f(x) = \ln(e^{e^x}) = e^x \quad , \quad f^{(99)}(x) = e^x$.

3. C $\quad f(x) = (4-3x)^4 \qquad f'''(x) = (-3)^3 \cdot 4 \cdot 3 \cdot 2 (4-3x)$

$$f'''(1) = -648$$

4. C $\quad$ We need $f(x) = y = \dfrac{ax+b}{cx+d} \neq$ constant

So $\dfrac{a}{c} \neq \dfrac{b}{d} \iff ad - bc \neq 0$

5. B $\quad$ Notice that $\dfrac{d}{dy}(4y^3 - y^2 + 4y + 1) = 12y^2 - 2y + 4$

Let $t = t(y) = 4y^3 - y^2 + 4y + 1$, $\quad t(1) = 8 \quad t(0) = 1$

So $I = \displaystyle\int_1^8 t^{-\frac{2}{3}}\, dt = 3t^{\frac{1}{3}} \Big|_{t=1}^{8} = 3 \cdot 2 - 3 \cdot 1 = 3$

Part 2

2a. $\dfrac{d}{dx}\left(\displaystyle\int_{\sin x}^{x} \frac{1}{\sqrt{\sin^2(t)+1}}\, dt \right)$

$= \dfrac{d}{dx}\left(\displaystyle\int_0^x \cdots dt - \int_0^{\sin x} \cdots dt \right)$

$= \dfrac{1}{\sqrt{\sin^2(x)+1}} - \cos x \cdot \dfrac{1}{\sqrt{\sin^2(\sin x)+1}}$.

(chain rule)

2b

Notice that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

so $\lim_{x\to 0} \frac{\sin(a_i x)}{x} = a_i$

$$\lim_{x\to 0} \frac{1 - \cos(a_1 x)\cos(a_2 x)\cdots\cos(a_n x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\sum_{i=1}^{n} a_i \cos(a_1 x)\cdots\cos(a_{i-1} x)\sin(a_i x)\cos(a_{i+1} x)\cdots\cos(a_n x)}{2x}$$

$$= \sum_{i=1}^{n} a_i \cdot \frac{a_i}{2}$$

$$= \sum_{i=1}^{n} \frac{a_i^2}{2}$$

2c

(1) $\int f(x)\,dx = \int x \cdot e^{-x^2}\,dx$

Let $t = t(x) = x^2$, $t'(x) = 2x$

so $\int f(x)\,dx = \int \frac{1}{2} e^{-t}\,dt = -\frac{1}{2}e^{-t} + c$

$= -\frac{1}{2}e^{-x^2} + c$

(2) $\int g(x)\,dx = \int \frac{1}{1+e^x}\,dx = \int \frac{e^x}{(1+e^x)e^x}\,dx$

$\overset{s=e^x}{=\!=} \int \frac{1}{(1+s)s}\,ds = \int \left(\frac{1}{s} - \frac{1}{s+1}\right) ds$

$= \ln s - \ln(s+1)$

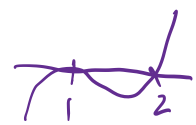
$= x - \ln(e^x+1)$.

2d.

(1) $y'(x) = (x-1)^2 (x-2)$

$y''(x) = 2(x-1)(x-2) + (x-1)^2 = (x-1)(3x-5)$

	$(-\infty, 1)$	1	$(1, 2)$	2	$(2, +\infty)$
$y'(x)$	$-$	0	$-$	0	$+$



so $f = y(x)$ has a local minimum at $x=2$.

doesn't have any local maximum.

has a point of inflection $x=1$.

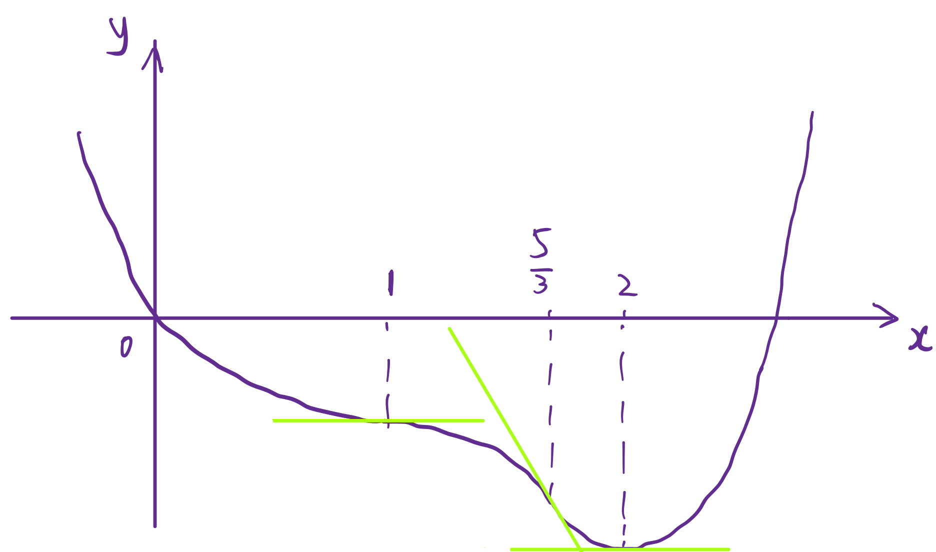
(2)

since $y''(x) = (x-1)(3x-5)$

so $f(x)$ is concave up on $(-\infty, 1)$ and $(\frac{5}{3}, +\infty)$ respectively.

concave down on $(1, \frac{5}{3})$

graph:



(Actually, you can calculate $y(x)$ by integration.)

2e

pf:

Define $g(x) = \dfrac{f(c)}{(c-a)^2(c-b)}(x-a)^2(x-b) - f(x)$

then $g(a) = g(b) = 0$, $g(c) = f(c) - f(c) = 0$

By Rolle's Thm. $\exists\ \sigma_1 \in (a,c)$, $\sigma_2 \in (c,b)$. s.t

$$g'(\sigma_1) = g'(\sigma_2) = 0$$

Since

$$g'(x) = \frac{f(c)}{(c-a)^2(c-b)}(x-a)(3x-2b-a) - f'(c)$$

$$g'(a) = 0 - f'(a) = 0$$

By Rolle's Thm. $\exists\ \eta_1 \in (a, \sigma_1)$, $\eta_2 \in (\sigma_1, \sigma_2)$. s.t

$$g''(\eta_1) = g''(\eta_2) = 0$$

By Rolle's Thm. $\exists\ \xi \in (\eta_1, \eta_2)$. s.t $g'''(\xi) = 0$

$$\Rightarrow \quad g'''(\xi) = 6 \cdot \frac{f(c)}{(c-a)^2(c-b)} - f'''(\xi)$$

i.e $\exists\ \xi \in (a,b)$. s.t $f(c) = \frac{1}{6} f'''(\xi)(c-a)^2(c-b)$

$\square$

Part 3.

pf: WLOG. $x > x_0$

Define $F(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$

$G(x) = \frac{1}{2}(x - x_0)^2$

Then $F'(x) = f'(x) - f'(x_0)$

$G'(x) = x - x_0$

Notice that $F(x_0) = 0 \qquad G(x_0) = 0$

$F'(x_0) = 0 \qquad G(x_0) = 0$

By Cauchy's Mean Value Thm. $\exists\, \eta \in (x_0, x),\ \xi \in (x_0, \eta)$ s.t.

$$\frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\frac{1}{2}(x - x_0)^2} = \frac{F(x)}{G(x)} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)}$$

$$= \frac{F'(\eta)}{G'(\eta)} = \frac{F'(\eta) - F'(x_0)}{G'(\eta) - G'(x_0)} = \frac{F''(\xi)}{G''(\xi)} = f''(\xi)$$

$\square$