第二章习题解答

6（2）．错误。例如，$f(z)=|z|^2=x^2+y^2$, 则$f(z)=u(x,y)=x^2+y^2$, $v(x,y)\equiv 0$, $\frac{\partial u}{\partial x}(0,0)=\frac{\partial v}{\partial y}=0$, $\frac{\partial u}{\partial y}(0,0)=-\frac{\partial v}{\partial x}=0$, 故$f(z)$在$z_0=0$处可导且$f'(0)=0$，但易见$f(z)$在$z_0\neq 0$处不可导，故$f(z)$处处不解析。

(3). 错误。例如，由上题的例子知$f(z)=|z|^2=x^2+y^2$在$z_0=0$可导且$f'(0)=0$，但易见$f(z)$在$z_0\neq 0$处不可导，故$f(z)$处处不解析，故$z_0=0$是$f(z)$的一个奇点，但$f(z)$在$z_0=0$可导。

(5). 错误。例如，$f(z)=\overline{z}=x-iy$, 则$u(x,y)=x, v(x,y)=-y\in C^1$, $\frac{\partial u}{\partial x}(=1)\neq\frac{\partial v}{\partial y}(=-1)$。故$f(z)$处处不可导。

(6). 因$f(z)=u+iv$解析， 由Cauchy-Riemann 条件知当v为常数时$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0$, 因而由$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\equiv 0$, 知$f(z)=$常数。

7．因$f(z)$解析， 知$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=: u_x+iv_x$处处存在且有

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}, \tag{1}$$

处处成立. 由$|f(z)|^2=u^2+v^2$, 得

$$\frac{\partial|f(z)|^2}{\partial x}=2|f(z)|\frac{\partial|f(z)|}{\partial x}=2\left[u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right],$$

$$\frac{\partial|f(z)|^2}{\partial y}=2|f(z)|\frac{\partial|f(z)|}{\partial y}=2\left[u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}\right].$$

以上两式除以 2，再平方后相加得

$$|f(z)|^2\left[\left(\tfrac{\partial|f(z)|}{\partial x}\right)^2+\left(\tfrac{\partial|f(z)|}{\partial y}\right)^2\right]=$$

$$\left(u^2u_x^2+2uvu_xv_x+v^2v_x^2+u^2u_y^2+2uvu_yv_y+v^2v_y^2\right).$$

将等式(1)带入右边的等式中并利用$f'(z) = u_x + iv_x$, 化简后可得

$$|f(z)|^2 \left[\left(\frac{\partial |f(z)|}{\partial x} \right)^2 + \left(\frac{\partial |f(z)|}{\partial y} \right)^2 \right] = |f(z)|^2 |f'(z)|^2.$$

当$f(z) \equiv 0$时，所要证明的等式显然成立。当$f(z) \neq 0$时， 上式两边除以$|f(z)|^2$, 则得所要证的等式。

9. 由$x = r\cos\theta$, $y = r\sin\theta$, 知$u(x,y) = u(r\cos\theta, r\sin\theta)$, $v(x,y) = v(r\cos\theta, r\sin\theta)$, 因而有

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$

及

$$\begin{aligned} \tfrac{\partial v}{\partial \theta} &= \tfrac{\partial v}{\partial x}\tfrac{\partial x}{\partial \theta} + \tfrac{\partial v}{\partial y}\tfrac{\partial y}{\partial \theta} = \tfrac{\partial v}{\partial x}(-r\sin\theta) + \tfrac{\partial v}{\partial y} r\cos\theta \\ &= r\left[\tfrac{\partial u}{\partial y}\sin\theta + \tfrac{\partial u}{\partial x}\cos\theta\right] = r\tfrac{\partial u}{\partial r}. \end{aligned}$$

这里用到了Cauchy-Riemann等式。同理可证第二个等式。

10(1). 这时$f(z) = u(x,y), v(x,y) \equiv 0$, 由$f(z)$解析，知$f'(z) = \frac{\partial u}{\partial x}$且由Cauchy-Riemann条件知$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv 0$, 因而有$f'(z) \equiv 0$, 从而有$f(z) =$常数。

(2). 因$f(z) = u + iv$解析， 知$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, 又因$\overline{f}(z) = u - iv$解析，知$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$, 得$\frac{\partial u}{\partial x} = 0$. 同理有$\frac{\partial v}{\partial x} = 0$, 故有$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$, 因而有$f(z) =$常数。

(3), (4).

当$f(z) \equiv 0$时， (3)显然正确。 （这时(4)中的$argf(z)$不定义。）

当$f(z) \neq 0$时，令$h(z) = lnf(z) = ln|f(z)| + iargf(z) = u + iv$. 则易见$h(z)$解析， 且有$u = ln|f(z)|$, $v = argf(z)$. 由6(6)可知， 当$argf(z) = v$ 为常数时， $u = ln|f(z)|$ 也是常数， 从而有$h(z) = lnf(z)$为常数，即$f(z) = e^{h(z)}$为常数。

同理可证当$u = ln|f(z)|$为常数时，　$v = argf(z)$为常数，　从而有$h(z)$为常数，　即$f(z) = e^{h(z)}$为常数。

15. $Ln(-i) = ln(-i)+2k\pi i = ln|-i|+iarg(-i)+2k\pi i = ln1+i\frac{3\pi}{2}+2k\pi i = \frac{(4k+3)\pi}{2}i,\ k \in Z.$

$Ln(-3+4i) = ln(-3+4i)+2k\pi i = ln|-3+4i|+iarg(-3+4i)+2k\pi i = ln5+i(\pi - arctan\frac{4}{3})+2k\pi i,\ k \in Z.$

18 答案见书。