Linear Algebra: Homework 6

November 27, 2021

Question 1.

In $\mathbb{P}_2$, find the transition matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Solution 1.

The transition matrix is actually the "column vectors" of the polynomials, as $\mathcal{B}$ is already shown in standard basis, which is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

Express t^2 in $\mathcal{B}$ requires the inverse of the transition matrix.

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -3 & -5 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & -6 & 3 & -2 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10 & -5 & 3 \\ 0 & 1 & 0 & -6 & 3 & -2 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Therefore,

$$t^2 = 3(1 - 3t^2) - 2(2 + t - 5t^2) + (1 + 2t)$$

Question 2.

Let $P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}$, and

$$\vec{v_1} = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix}.$$

(1) Find a basis $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ in $\mathbf{R}^3$ such that P is the transition matrix from $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ to the basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

(2) Find a basis $\{\vec{w_1}, \vec{w_2}, \vec{w_3}\}$ in $\mathbf{R}^3$ such that P is the transition matrix from $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ to the basis $\{\vec{w_1}, \vec{w_2}, \vec{w_3}\}$.

Solution 2.

Calculate P^{-1} first.

$$\left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ -3 & -5 & 0 & 0 & 1 & 0 \\ 4 & 6 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & -2 & 5 & -4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & -1 & 2 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & 2 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & -3 & -5 & -3 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{rrrrrr} 1 & 0 & 0 & 5 & 8 & 5 \\ 0 & 1 & 0 & -3 & -5 & -3 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{array}\right]$$

(1) Then entries of each column in P are the coefficients of the linear combination of $\vec{v_i}$ $(i = 1, 2, 3)$ of vectors $\vec{u_i}$ $(i = 1, 2, 3)$. So,

$$\left[\begin{array}{ccc} \vec{u_1} & \vec{u_2} & \vec{u_3} \end{array}\right] = \left[\begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{array}\right] P$$

$$= \left[\begin{array}{rrr} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{array}\right] \left[\begin{array}{rrr} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{array}\right] = \left[\begin{array}{rrr} -6 & -6 & -5 \\ -5 & -9 & 0 \\ 21 & 32 & 3 \end{array}\right]$$

Hence, $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \left\{ \left[\begin{array}{r} -6 \\ -5 \\ 21 \end{array}\right], \left[\begin{array}{r} -6 \\ -9 \\ 32 \end{array}\right], \left[\begin{array}{r} -5 \\ 0 \\ 3 \end{array}\right] \right\}$

(2) Similar to (1), we have

$$\left[\begin{array}{ccc} \vec{w_1} & \vec{w_2} & \vec{w_3} \end{array}\right] P = \left[\begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{array}\right]$$

$$\left[\begin{array}{ccc} \vec{w_1} & \vec{w_2} & \vec{w_3} \end{array}\right] = \left[\begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{array}\right] P^{-1}$$

$$= \left[\begin{array}{rrr} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{array}\right] \left[\begin{array}{rrr} 5 & 8 & 5 \\ -3 & -5 & -3 \\ -2 & -2 & -1 \end{array}\right] = \left[\begin{array}{rrr} 28 & 38 & 21 \\ -9 & -13 & -7 \\ -3 & 2 & 3 \end{array}\right]$$

Hence, $\{\vec{w_1}, \vec{w_2}, \vec{w_3}\} = \left\{ \left[\begin{array}{r} 28 \\ -9 \\ -3 \end{array}\right], \left[\begin{array}{r} 38 \\ -13 \\ 2 \end{array}\right], \left[\begin{array}{r} 21 \\ -7 \\ 3 \end{array}\right] \right\}$

Question 3.

Let $\mathcal{B} = \{\vec{b_1}, \vec{b_2}\}$, $\mathcal{C} = \{\vec{c_1}, \vec{c_2}\}$ and $\mathcal{D} = \{\vec{d_1}, \vec{d_2}\}$ be bases for a two dimensional vector space. Write an equation that relates the matrices $P_{\mathcal{B}\to\mathcal{C}}$, $P_{\mathcal{C}\to\mathcal{D}}$ and $P_{\mathcal{B}\to\mathcal{D}}$. Justify your answer.

Solution 3.

Matrix $P_{\mathcal{B}\to\mathcal{C}}$ converts a vector represented by basis $\mathcal{B}$ to basis $\mathcal{C}$. Matrix $P_{\mathcal{C}\to\mathcal{D}}$ converts a vector represented by basis $\mathcal{C}$ to basis $\mathcal{D}$. Matrix $P_{\mathcal{B}\to\mathcal{D}}$ converts a vector represented by basis $\mathcal{B}$ to basis $\mathcal{D}$. Hence, to convert a vector represented by basis $\mathcal{B}$ to $\mathcal{D}$, we can alternatively convert it to representation by basis $\mathcal{C}$ then to $\mathcal{D}$. Therefore,

$$P_{\mathcal{B}\to\mathcal{D}} = P_{\mathcal{C}\to\mathcal{D}} P_{\mathcal{B}\to\mathcal{C}}$$

Question 4.

Calculate the determinants by cofactor expansion:

$$\left|\begin{array}{rrr} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{array}\right|, \left|\begin{array}{rrrrr} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{array}\right|.$$

Solution 4.

$$\begin{vmatrix} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{vmatrix} = -4\begin{vmatrix} 3 & -3 \\ 1 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix} = -72 + 48 = -24$$

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = -2\begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -6\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= -6\left(\begin{vmatrix} 4 & -5 \\ 5 & -3 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}\right) = -6(13 - 14) = 6$$

Question 5.

What is the determinant of an elementary row replacement matrix? What is the determinant of an elementary scaling matrix with k on the diagonal?

Solution 5.

A row replacement matrix E is a lower triangular matrix with all 1 in the entries of the main diagonal, hence $\det E = 1$.

A scaling matrix transforming vectors in $\mathbf{R}^n$ has the form of $\begin{bmatrix} k & & & \\ & k & & \\ & & \ddots & \\ & & & k \end{bmatrix}_{n\times n}$,its determinant is k^n.

Question 6.

Combine the methods of row reduction and cofactor expansion to calculate the determinants.

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix}$$

Solution 6.

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & -3 & -2 & 0 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ 0 & 5 & 3 \end{vmatrix} = 6\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} = 6$$

$$\begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 3 & 5 & 4 & 1 \\ -6 & 5 & 5 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 4 & 1 \\ 0 & -2 & -4 & 0 \\ 2 & 0 & 0 & 0 \\ -6 & 5 & 5 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & -2 & -4 \\ 2 & 0 & 0 \\ -6 & 5 & 5 \end{vmatrix} = 2(-10 + 20) = 20$$