Policy

I encourage collaborations on homeworks, projects and even the takehome midterm. However, you must obey the following rule:

1. You MUST each hand in your own work individially in your own words.
2. You MUST understand everything you wrote. (Say you copied your friend's WRONG answer without thinking, and that will most likely be in violation of this rule.)
3. You need to write down the names of your collaborator.
4. Failure to comply rule 2 and rule 3 will be treated as plagiarism.
5. Collaboration with people not in this class (such as a math grad student) is not forbidden but not recommended. If you choose to, then write down their names as well.

Contents

Chapter 1

Topics in Linear Algebra

1.1 HW1 (Due 3.11) Complex Stuff

Exercise 1.1.1. *We would like to find a real $n \times n$ matrix A such that $A^2 = -I$.*

1. *For each even number n, find a real solution.*

2. *If odd n, show that there is no real solution.*

Exercise 1.1.2. *Suppose $A^2 = -I$ for a real $n \times n$ matrix A. For each vector $\boldsymbol{v} \in \mathbb{R}^n$, we write $\mathrm{i}\boldsymbol{v}$ to mean $A\boldsymbol{v}$. For any $n \times n$ matrix B, we say it is complex linear if $B(k\boldsymbol{v}) = kB\boldsymbol{v}$ for any complex number $k \in \mathbb{C}$.*

1. *Show that B is complex linear if and only if $AB = BA$.*

2. *If X also satisfies $X^2 = -I$, then must X be complex linear? Prove or provide a counter example.*

3. *For $n = 2$, pick any A such that $A^2 = -I$, and pick two distinct C such that $CA = -AC$ and $C^2 = I$.*

4. *(Read only) Consider an $n \times n$ real matrix C such that $CA = -AC$ and $C^2 = I$. This C is called a **complex conjugate operator**. Then such C must be diagonalizable, must have only eigenvalues 1 and -1, and its eigenspaces for 1 and -1 have the same dimension. The eigenspace for 1 is the space of "real vectors" while the eigenspace for -1 is the space of "imaginary vectors". As you can see, the "real part" and "imaginary part" of a vector is NOT defined by the complex structure A alone. In particular, for abstract arguments, it might be a good idea to AVOID arguments that split complex things into real parts and imaginary parts.*

Exercise 1.1.3. *If V is an abstract vector space over $\mathbb{C}$, then for each vector $\boldsymbol{v}$ and each $k \in \mathbb{C}$, obviously $k\boldsymbol{v}$ is well-defined. But as a result, for each vector $\boldsymbol{v}$ and each $k \in \mathbb{R}$, $k\boldsymbol{v}$ must be well-defined. So any complex vector space must also be a real vector space (but NOT vice versa). This gives rise to some very tricky distinctions.*

Given abstract vector spaces V, W over $\mathbb{C}$, we say a map $L : V \to W$ is complex linear if $L(k\boldsymbol{v}) = kL\boldsymbol{v}$ for all $k \in \mathbb{C}$ and $\boldsymbol{v} \in V$. We say it is real linear if $L(k\boldsymbol{v}) = kL\boldsymbol{v}$ for all $k \in \mathbb{R}$ and $\boldsymbol{v} \in V$. Note that it is possible to be real linear but NOT complex linear.

Given a bunch of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in a complex vector space, we say they are $\mathbb{R}$-linearly independent if $\sum a_i\boldsymbol{v}_i = \boldsymbol{0}$ for $a_1, \ldots, a_k \in \mathbb{R}$ implies all $a_i = 0$. We say they are $\mathbb{C}$-linearly independent if $\sum a_i\boldsymbol{v}_i = \boldsymbol{0}$ for $a_1, \ldots, a_k \in \mathbb{C}$ implies all $a_i = 0$. Similarly, we can define $\mathbb{R}$-spanning, $\mathbb{C}$-spanning, $\mathbb{R}$-basis, $\mathbb{C}$-basis and so on.

1. *Consider the map $C : \mathbb{C}^n \to \mathbb{C}^n$ of taking complex conjugates, i.e., $C \begin{bmatrix} 1+\mathrm{i} \\ 2 \\ 3\mathrm{i} \end{bmatrix} = \begin{bmatrix} 1-\mathrm{i} \\ 2 \\ -3\mathrm{i} \end{bmatrix}$. Is this real linear? Is this complex linear?*

2. *Which implies which:* $\mathbb{R}$*-linear and* $\mathbb{C}$*-linear.*

3. *Find an* $\mathbb{R}$*-basis for* $\mathbb{C}^2$ *and then find a* $\mathbb{C}$*-basis for* $\mathbb{C}^2$*. What is the real dimension of* $\mathbb{C}^2$*? What is its complex dimension?*

4. *Which implies which:* $\mathbb{R}$*-linearly independent and* $\mathbb{C}$*-linearly independent.*

5. *Which implies which:* $\mathbb{R}$*-spanning and* $\mathbb{C}$*-spanning.*

Exercise 1.1.4 (adapted from Gilbert Strang 9.3.11-15). *Take the permutation matrix P that sends* $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ *to* $\begin{bmatrix} b \\ c \\ d \\ a \end{bmatrix}$*. Let F_4 be the 4×4 Fourier matrix.*

1. *Compute* $P\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $P\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$.

2. *Show that $PF_4 = F_4D$ for some diagonal matrix D. Find all eigenvalues and eigenvectors of P.*

3. *Let* $C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix}$*. Compute* $C\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $C\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$.

4. *Write C as a polynomial of P. Find the eigenvalues and eigenvectors of C.*

5. *(Read Only) Matrices such as C are called circulant matrices. Just like periodic functions can be simplified using Fourier series, circulant matrices can be simplified using Fourier matrices. One can compute $C\mathbf{v}$ using fast Fourier transform, and it will be slightly faster than computing $C\mathbf{v}$ directly.*

1.2 HW2 (Due 3.18) Invariant Decomposition

Exercise 1.2.1. *Prove or find counter examples.*

1. *For four subspaces, if any three of them are linearly independent, then the four subspaces are linearly independent.*

2. *If subspaces V_1, V_2 are linearly independent, and V_1, V_3, V_4 are linearly independent, and V_2, V_3, V_4 are linearly independent, then all four subspaces are linearly independent.*

3. *If V_1, V_2 are linearly independent, and V_3, V_4 are linearly independent, and $V_1 + V_2, V_3 + V_4$ are linearly independent, then all four subspaces are linearly independent.*

Exercise 1.2.2. *Let V be the space of $n \times n$ real matrices. Let $T : V \to V$ be the transpose operation, i.e., T sends A to A^{T} for each $A \in V$. Find a non-trivial T-invariant decomposition of V, and find the corresponding block form of T.*

(Here we use real matrices for your convenience, but the statement is totally fine for complex matrices and conjugate transpose.)

Exercise 1.2.3. *Let $p(x)$ be any polynomial, and define $p(A)$ in the obvious manner. E.g., if $p(x) = x^2 + 2x + 3$, then $p(A) = A^2 + 2A + 3I$. We fix some $n \times n$ matrix A.*

1. *If $AB = BA$, show that $\mathrm{Ker}(B), \mathrm{Ran}(B)$ are both A-invariant subspaces.*

2. *Prove that $Ap(A) = p(A)A$.*

3. *Conclude that $N_\infty(A - \lambda I), R_\infty(A - \lambda I)$ are both A-invariant for any $\lambda \in \mathbb{C}$.*

Exercise 1.2.4. *Note that any linear map must have at least one eigenvector. (You may try to prove this yourself, but it is not part of this homework.) You may use this fact freely in this problem.*

Fix any two $n \times n$ square matrices A, B. Suppose $AB = BA$.

1. *If W is an A-invariant subspace, show that A has an eigenvector in W.*

2. *Show that $\mathrm{Ker}(A - \lambda I)$ is always B-invariant for all $\lambda \in \mathbb{C}$. (Hint: Last problem.)*

3. *Show that A, B has a common eigenvector. (Hint: Last two sub-problems.)*

1.3 HW3 (Due 3.25) Jordan Canonical Form

Exercise 1.3.1. *Find a basis in the following vector space so that the linear map involved will be in Jordan normal form. Also find the Jordan normal form.*

1. *$V = \mathbb{C}^2$ is a 4 dimensional real vector space, and $A : V \to V$ that sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} \overline{x} - \Re(y) \\ (1+i)\Im(x) - y \end{bmatrix}$ is a real linear map. (Here $\overline{x}$ means the complex conjugate of a complex number x, and $\Re(x), \Im(x)$ means the real part and the imaginary part of a complex number x.)*

2. *$V = P_4$, the real vector space space of all real polynomials of degree at most 4. And $A : V \to V$ is a linear map such that $A(p(x)) = p'(x) + p(0) + p'(0)x^2$ for each polynomial $p \in P_4$.*

3. *$A = \begin{bmatrix} & & & a_1 \\ & & a_2 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix}$. Be careful here. Maybe we have many possibilities for its Jordan normal form depending on the values of a_1, a_2, a_3, a_4.*

Exercise 1.3.2. *A partition of integer n is a way to write n as a sum of other positive integers, say $5 = 2 + 2 + 1$. If you always order the summands from large to small, you end up with a dot diagram, where each column represent an integer: $\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \end{bmatrix}$. Similarly, $7 = 2 + 4 + 1$ should be represented as $\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & & \\ \cdot & & \end{bmatrix}$.*

Again, note that we always first re-order the summands from large to small.

1. *If the Jordan normal form of an $n \times n$ nilpotent matrix A is diag$(J_{a_1}, J_{a_2}, ..., J_{a_k})$, then we have a partition of integer $n = a_1 + ... + a_k$. However, we also have a partition of integer $n = [\dim \operatorname{Ker}(A)] + [\dim \operatorname{Ker}(A^2) - \dim \operatorname{Ker}(A)] + [\dim \operatorname{Ker}(A^3) - \dim \operatorname{Ker}(A^2)] + ...$, where we treat the content of each bracket as a positive integer. Can you find a relation between the two dot diagrams?*

2. *A partition of integer $n = a_1+...+a_k$ is called self-conjugate if, for the matrix $A = $ diag$(J_{a_1}, J_{a_2}, ..., J_{a_k})$, the two dot diagrams you obtained above are the same. Show that, for a fixed integer n, the number of self-conjugate partition of n is equal to the number of partition of n into distinct odd positive integers.(Hint: For a self-conjugate dot diagram, count the total number of dots that are either in the first column or in the first row or in both. Is this always odd?)*

3. *Suppose a 4 by 4 matrix A is nilpotent and upper trianguler, and all (i, j) entries for $i < j$ are chosen randomly and uniformly in the interval $[-1, 1]$. What are the probabilities that its Jordan canonical form corresponds to the partitions $4 = 4, 4 = 3 + 1, 4 = 2 + 2, 4 = 2 + 1 + 1, 4 = 1 + 1 + 1 + 1$?*

4. *(NOT part of the HW.) If you want a challenge, show that the number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts. Perferably you should do this via some construction of one-to-one correspondence.*

5. *(NOT part of the HW.) As a side remark, two matrices A, B are similar in GL_n if $A = CBC^{-1}$ for some invertible C. But in physics sometimes we are interested in the case when two matrices A, B are similar in SO_n, i.e., if $A = CBC^{-1}$ for some orthogonal C with determinant 1. You may also require C to be symplectic or whatever. The similarity classes in each case usualy corresponds with some special kind of partitions of integers (although they no longer necessarily be related to Jordan normal forms). Partitions of integers also connect with physics DIRECTLY by providing an estimate for the density of energy levels for a heavy nucleus. (I don't really know how, so don't ask me.) Curiously, many properties of partitions of integers are still OPEN PROBLEMS of mathematics. We don't really know enough about them.*

1.4 HW4 (Due 4.1, this is NOT a joke....) Minimal polynomials, Sylvester's equation

Exercise 1.4.1. *Consider the matrix* $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

1. *Find a matrix B such that* $BAB^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.
2. *Find a basis for the subspace $V_3 + V_4$, where V_λ is the eigenspace of A for the eigenvalue λ.*

Exercise 1.4.2. *Suppose we have a complex matrix* $A = \begin{bmatrix} B & I \\ & B \end{bmatrix}$. *We know the characteristic polynomial of A is just the square of the characteristic polynomial of B. Is the minimal polynomial of A the square of minimal polynoial of B? Let us investigate this.*

1. *Show that for any square matrices X, Y, the matrices* $\begin{bmatrix} X & & I & \\ & Y & & I \\ & & X & \\ & & & Y \end{bmatrix}$ *and* $\begin{bmatrix} X & I & & \\ & X & & \\ & & Y & I \\ & & & Y \end{bmatrix}$ *are similar.*
2. *Suppose* $B = \begin{bmatrix} 3 & \\ & 4 \end{bmatrix}$, *find the minimal polynomial for B and for A.*
3. *Suppose* $B = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$, *find the rank of A, A^2, A^3, A^4, and deduce the Jordan canonical form J of A. (You are not required to find the X such that $A = XJX^{-1}$. Finding the J is enough.)*
4. *Suppose* $B = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$, *find the minimal polynomial for B and for A.*
5. *Guess (no need to prove) the general relation between the minimal polynomial of A and of B. (The proof is not too bad, just lenthy. It is the usual proof: first prove it for nilpotent Jordan blocks, then for λ-Jordan blocks, then for matrices whose eigenvalues are all the same, and finally for an arbitrary matrix.)*

Exercise 1.4.3. *In class we see that for Sylvester's equations $AX - XB = C$, if A, B have no common eigenvalue, then there is always a unique solution. What if A, B have common eigenvalues? Let us take an extreme case, and assume that $A = B$. So we are looking at an equation $AX - XA = C$ for constant $n \times n$ matrices A, C. Let V be the space of $n \times n$ matrices, and consider the linear map $L : V \to V$ such that $L(X) = AX - XA$.*

1. *Show that $L(X) = 0$ always has infinitely many solutions.*
2. *Show that L satisfy the Leibniz rule (or the product rule for derivatives), i.e., $L(XY) = L(X)Y + XL(Y)$. (This is an indication of the usefulness of this map in physics.)*
3. *Suppose $L(X)X = XL(X)$. Show that for any polynomial $p(x)$, $L(p(X)) = L(X)p'(X)$, where $p'(x)$ is the derivative of $p(x)$.*

4. *Show that* $L(X) = I$ *has no solution by choosing* $p(x)$ *in the last subproblem to be the minimal polynomial. (Maybe you have seen another proof of this fact before. To show that* $AX - XA = I$ *has no solution, you can also just take trace on both sides. Hence now you have two proofs, and the new one is more insightful with potential meanings in physics.)*

5. *If* A *is diagonalizable with distinct eigenvalues, find* $\dim \operatorname{Ker}(L)$.

6. *Find a* 3×3 *matrix* A *such that* $\operatorname{Ran}(L)$ *is exactly the subspace of matrices with zero entries on the diagonal.*

1.5 HW5 (Due 4.8)

Exercise 1.5.1. *Let* $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$*. We have a function* $f(x) = x|x|$*. Note that as a real function,* $f(x)$ *is everywhere differentiable. (However, as a complex function, it is not differentiable.)*

1. *Let* $A_t = \begin{bmatrix} 1 & 1 \\ 0 & 1+t \end{bmatrix}$*. Note that* $\lim A_t = J$*. Find* $\lim f(A_t)$*.*
2. *Let* $A_t = \begin{bmatrix} 1 & 1 \\ -t^2 & 1 \end{bmatrix}$*. Note that* $\lim A_t = J$*. Find* $\lim f(A_t)$*. Is* $f(J)$ *well-defined? (No credit but fun to think about: Why is real differentiability not enough?)*
3. *No credit, for fun challenge problem: If all* A_t *have real eigenvalues and* $\lim A_t = J$*, would* $\lim f(A_t)$ *always converge to the same matrix? If always converge to the same matrix, find this matrix. If they could converge to different things, given two sequences with different limits.*

Exercise 1.5.2. *Compute the following*

1. *Find the derivative of* $\sin(tA)$ *as a function of* t*.*
2. *For the formula* $f(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}) = \begin{bmatrix} f(2A) & B \\ & f(2A) \end{bmatrix}$*, what is the block matrix* B *in terms of* f *and* A*? (You may assume that* f *equals to its Taylor expansion. Maybe try polynomials first to get some clue.)*
3. *Prove or find counter example: The derivative to* $f(A + tB)$ *as a differentiable function of* t *at* $t = 0$ *is* $f'(A)B$*.*

Exercise 1.5.3. *Suppose* $AB = BA$*. In previous homework, we see that this implies that* A, B *must have a common eigenvector.*

1. *Show that we can find invertible* X_1*, such that* $X_1 A X_1^{-1} = \begin{bmatrix} a_1 & * \\ & A_1 \end{bmatrix}, X_1 B X_1^{-1} = \begin{bmatrix} b_1 & * \\ & B_1 \end{bmatrix}$*, and* $A_1 B_1 = B_1 A_1$*. (Hint: Use the common eigenvector.)*
2. *Show that* A, B *can be simultaneously triangularized. (Hint: Look at* A_1, B_1 *and use induction.)*

1.6 HW6 (Due 4.29) Dual Stuff

Exercise 1.6.1. *Let V be the space of real polynomials of degree less than n. So $\dim V = n$. Then for each $a \in \mathbb{R}$, the evaluation ev_a is a dual vector.*

For any real numbers $a_1, \ldots, a_n \in \mathbb{R}$, consider the map $L : V \to \mathbb{R}^n$ such that $L(p) = \begin{bmatrix} p(a_1) \\ \vdots \\ p(a_n) \end{bmatrix}$.

1. *Write out the matrix for L under the basis $1, x, \ldots, x^{n-1}$ for V and the standard basis for $\mathbb{R}^n$. (Do you know the name for this matrix?)*

2. *Prove that L is invertible if and only if $a_1, \ldots, a_n$ are distinct. (If you can name the matrix L, then you may use its determinant formula without proof.)*

3. *Show that $\mathrm{ev}_{a_1}, \ldots, \mathrm{ev}_{a_n}$ form a basis for V^* if and only if all $a_1, \ldots, a_n$ are distinct.*

4. *Set $n = 3$. Find polynomials p_{-1}, p_0, p_1 such that $p_i(j) = \delta_{ij}$ for $i, j \in \{-1, 0, 1\}$.*

5. *Set $n = 4$, and consider $\mathrm{ev}_{-2}, \mathrm{ev}_{-1}, \mathrm{ev}_0, \mathrm{ev}_1, \mathrm{ev}_2 \in V^*$. Since $\dim V^* = 4$, these must be linearly dependent. Find a non-trivial linear combination of these which is zero.*

Exercise 1.6.2. *Let V be the space of real polynomials of degree less than 3. Which of the following is a dual vector? Prove it or show why not.*

1. $p \mapsto \mathrm{ev}_5((x+1)p(x))$.

2. $p \mapsto \lim_{x\to\infty} \frac{p(x)}{x}$.

3. $p \mapsto \lim_{x\to\infty} \frac{p(x)}{x^2}$.

4. $p \mapsto p(3)p'(4)$.

5. $p \mapsto \deg(p)$, *the degree of the polynomial p.*

Exercise 1.6.3. *Fix a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, and fix a point $\boldsymbol{p} \in \mathbb{R}^2$. For any vector $\boldsymbol{v} \in \mathbb{R}^2$, then the directional derivative of f at $\boldsymbol{p}$ in the direction of $\boldsymbol{v}$ is defined as $\nabla_{\boldsymbol{v}} f := \lim_{t\to 0} \frac{f(\boldsymbol{p}+t\boldsymbol{v})-f(\boldsymbol{p})}{t}$. Show that the map $\nabla f : \boldsymbol{v} \mapsto \nabla_{\boldsymbol{v}}(f)$ is a dual vector in $(\mathbb{R}^2)^*$, i.e., a row vector. Also, what are its "coordinates" under the standard dual basis?*

(Remark: In calculus, we write ∇f as a column vector for historical reasons. By all means, from a mathematical perspective, the correct way to write ∇f is to write it as a row vector, as illlustrated in this problem. (But don't annoy your calculus teachers though.... In your calculus class, you use whatever notation your calculus teacher told you.)

(Extra Remark: If we use row vector, then the evaluation of ∇f at $\boldsymbol{v}$ is purely linear, and no inner product structure is needed, which is sweet. But if we HAVE TO write ∇f as a column vector (for historical reason), then we would have to do a dot product between ∇f and $\boldsymbol{v}$, which now requires an inner product structure. That is an unnecessary dependence on an extra structure that actually should have no influence.)

Exercise 1.6.4. *Consider a linear map $L : V \to W$ and its dual map $L^* : W^* \to V^*$. Prove the following.*

1. $\mathrm{Ker}(L^*)$ *is exactly the collection of dual vectors in W^* that kills* $\mathrm{Ran}(L)$.

2. $\mathrm{Ran}(L^*)$ *is exactly the collection of dual vectors in V^* that kills* $\mathrm{Ker}(L)$.

1.7 HW7 (Due 5.6) Tangent vectors and Cotangent vectors

Exercise 1.7.1. *On the space $\mathbb{R}^n$, we fix a symmetric positive-definite matrix A, and define $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^{\mathrm{T}} A \boldsymbol{w}$.*

1. *Show that this is an inner product.*
2. *The Riesz map (inverse of the bra map) from V^* to V would send a row vector $\boldsymbol{v}^{\mathrm{T}}$ to what?*
3. *The bra map from V to V^* would send a vector $\boldsymbol{v}$ to what?*
4. *The dual of the Riesz map from V^* to V would send a row vector $\boldsymbol{v}^{\mathrm{T}}$ to what?*

Exercise 1.7.2 (What is a derivative)**.** *The discussions in this problem holds for all manifolds M. But for simplicities sake, suppose $M = \mathbb{R}^3$ for this problem.*

Let V be the space of all analytic functions from M to $\mathbb{R}$. Here analytic means $f(x, y, z)$ is a infinite polynomial series (its Taylor expansion) with variables x, y, z. Approximately $f(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 xy + a_6 xz + a_7 y^2 + \dots$, and things should converge always.

Then a dual vector $v \in V^*$ is said to be a "derivation at $\boldsymbol{p} \in M$" if it satisfy the following Leibniz rule (or product rule):

$$v(fg) = f(\boldsymbol{p})v(g) + g(\boldsymbol{p})v(f).$$

(Note the similarity with your traditional product rule $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$.)

Prove the following:

1. *Constant functions in V must be sent to zero by all derivations at any point.*
2. *Let $x, y, z \in V$ be the coordinate function. Suppose $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, then for any derivation v at $\boldsymbol{p}$, then we have $v((x - p_1)f) = f(\boldsymbol{p})v(x)$, $v((y - p_2)f) = f(\boldsymbol{p})v(y)$ and $v((z - p_3)f) = f(\boldsymbol{p})v(z)$.*
3. *Let $x, y, z \in V$ be the coordinate function. Suppose $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, then for any derivation v at $\boldsymbol{p}$, then we have $v((x - p_1)^a (y - p_2)^b (z - p_3)^c) = 0$ for any non-negative integers a, b, c such that $a + b + c > 1$.*
4. *Let $x, y, z \in V$ be the coordinate function. Suppose $\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, then for any derivation v at $\boldsymbol{p}$, $v(f) = \frac{\partial f}{\partial x}(\boldsymbol{p})v(x) + \frac{\partial f}{\partial y}(\boldsymbol{p})v(y) + \frac{\partial f}{\partial z}(\boldsymbol{p})v(z)$. (Hint: use the Taylor expansion of f at $\boldsymbol{p}$.)*
5. *Any derivation v at $\boldsymbol{p}$ must be exactly the directional derivative operator $\nabla_{\boldsymbol{v}}$ where $\boldsymbol{v} = \begin{bmatrix} v(x) \\ v(y) \\ v(z) \end{bmatrix}$.*

(Remark: So, algebraically speaking, tangent vectors are exactly derivations, i.e., things that satisfy the Leibniz rule.)

Exercise 1.7.3 (What is a vector field)**.** *The discussions in this problem holds for all manifolds M. But for simplicities sake, suppose $M = \mathbb{R}^3$ for this problem. Let V be the space of all analytic functions from M to $\mathbb{R}$ as usual.*

We say $X : V \to V$ is a vector field on X if $X(fg) = fX(g) + gX(f)$, i.e., the Leibniz rule again!

Prove the following:

1. *Show that $X_{\boldsymbol{p}} : V \to \mathbb{R}$ such that $X_{\boldsymbol{p}}(f) = (X(f))(\boldsymbol{p})$ is a derivation at $\boldsymbol{p}$. (Hence X is indeed a vector field, since it is the same as picking a tangent vector at each point.)*

2. *Note that each f on M induces a covector field $\mathrm{d}f$. Then at each point $\boldsymbol{p}$, the cotangent vector $\mathrm{d}f$ and the tangent vector X would evaluate to some number. So $\mathrm{d}f(X)$ is a function $M \to \mathbb{R}$. Show that $\mathrm{d}f(X) = X(f)$, i.e., the two are the same. (Hint: just use definitions and calculate directly.)*

3. *If $X, Y : V \to V$ are vector fields, then note that $X \circ Y : V \to V$ might not be a vector field. (Leibniz rule might fail.) However, show that $X \circ Y - Y \circ X$ is always a vector field.*

4. *On a related note, show that if A, B are skew-symmetric matrices, then $AB - BA$ is still skew-symmetric. (Skew-symmetric matrices actually corresponds to certain vector fields on the manifold of orthogonal matrices. So this is no coincidence.)*

1.8 HW8 (Due 5.13) Multilinear maps

Exercise 1.8.1 (Elementary layer operations for tensors)**.** *Note that, for "2D" matrices we have row and column operations, and the two kinds of operations corresponds to the two dimensions of the array.*

For simplicity, let M be a $2 \times 2 \times 2$ "3D matrix". Then we have "row layer operations", "column layer operations", "horizontal layer operations". The three kinds corresponds to the three dimensions of the array.

We interpret this as a multilinear map $M : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. Let $((\mathbb{R}^2)^*)^{\otimes 3}$ be the space of all multilinear maps from $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}$.

1. *Given $\alpha, \beta, \gamma \in (\mathbb{R}^2)^*$, what is the (i, j, k)-entry of the "3D matrix" $\alpha \otimes \beta \otimes \gamma$ in terms of the coordinates of α, β, γ? Here $\alpha \otimes \beta \otimes \gamma$ is the multilinear map sending $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ to the real number $\alpha(\boldsymbol{u})\beta(\boldsymbol{v})\gamma(\boldsymbol{w})$.*

2. *Let E be an elementary matrix. Then we can send $\alpha \otimes \beta \otimes \gamma$ to $(\alpha E) \otimes \beta \otimes \gamma$. Why can this be extended to a linear map $M_E : ((\mathbb{R}^2)^*)^{\otimes 3} \to ((\mathbb{R}^2)^*)^{\otimes 3}$? (This gives a formula for the "elementary layer operations" on "3D matrices", where the three kinds of layer operations corresponds to applying E to the three arguments respectively.)*

3. *Show that elementary layer operations preserve rank. Here we say M has rank r if r is the smallest possible integer such that M can be written as the linear combination of r "rank one" maps, i.e., maps of the kind $\alpha \otimes \beta \otimes \gamma$ for some $\alpha, \beta, \gamma \in (\mathbb{R}^2)^*$.*

4. *Show that, if some "2D" layer matrix of a "3D matrix" has rank r, then the 3D matrix has rank at least r.*

5. *Let M be made of two layers, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find its rank.*

6. *(Read only) Despite some practical interests, finding the tensor rank in general is NOT easy. In fact, it is NP-complete just for 3-tensors over finite field. Furthermore, a tensor with all real entries might have different real rank and complex rank.*

Exercise 1.8.2. *Let M be a $3 \times 3 \times 3$ "3D matrix" whose (i, j, k)-entry is $i + j + k$. We interpret this as a multilinear map $M : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$.*

1. *Let $\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then $M(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v})$ is a polynomial in x, y, z. What is this polynomial?*

2. *Let $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ be any bijection. Show that $M(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = M(\boldsymbol{v}_{\sigma(1)}, \boldsymbol{v}_{\sigma(2)}, \boldsymbol{v}_{\sigma(3)})$. (Hint: brute force works. But alternatively, try find the (i, j, k) entry of the multilinear map M^σ, a map that sends $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ to $M(\boldsymbol{v}_{\sigma(1)}, \boldsymbol{v}_{\sigma(2)}, \boldsymbol{v}_{\sigma(3)})$.)*

3. *Show that the rank r of M is at least 2 and at most 3. (It is actually exactly three.)*

4. *(Read only) Any study of polynomial of degree d on n variables is equivalent to the study of some symmetric d tensor on $\mathbb{R}^n$.*

1.9 HW9 (Due 5.20) Tensor Calculations

Exercise 1.9.1 (Kronecker product?)**.** *Consider two linear maps $X : V \to V$ and $Y : W \to W$ over finite dimensional spaces. We define $X \otimes Y$ to be the map from $V \times W$ to $V \otimes W$, such that $(\boldsymbol{v}, \boldsymbol{w}) \mapsto X\boldsymbol{v} \otimes Y\boldsymbol{w}$.*

1. *Verify that $X \otimes Y$ is bilinear. Therefore we can think of it as a linear map $X \otimes Y : V \otimes W \to V \otimes W$*
2. *Show that* $\operatorname{trace} X \otimes Y = (\operatorname{trace} X)(\operatorname{trace} Y)$.

Exercise 1.9.2 (Trace as a tensor)**.** *Note that a linear map from $\mathbb{R}^n \to \mathbb{R}^n$ is an element of $\mathbb{R}^n \otimes (\mathbb{R}^n)^*$. Hence $\mathbb{R}^n \otimes (\mathbb{R}^n)^*$ is exactly the space of $n \times n$ matrices.*

The trace map send matrices to numbers. So we have $\operatorname{trace} : \mathbb{R}^n \otimes (\mathbb{R}^n)^* \to \mathbb{R}$. *It is also linear. So what space is* trace *an element of? What are the entries of the tensor* trace*?*

Exercise 1.9.3 (Quantum Entanglement)**.** *[(Optional Background) Quantum physics usually start like this. Given a particual A, its "possible states" are unit vectors $\boldsymbol{v}$ in some (usually infinite dimensional) inner product space H_A. (We use unit vectors, because $\|\boldsymbol{v}\|$ usually means total probability, which should be one.) When we make an observation on A, the observation is encoded by a (usually self-adjoint) linear map $L : H_A \to H_A$. If $L = I$ the identity map, then this means we do not observe at all.]*

For simplicity, suppose we have two inner product spaces H_A, H_B. (You may think that they describe the states of two different particles A and B.) Then on their tensor space $H_A \otimes H_B$, we define $\langle \boldsymbol{v}_1 \otimes \boldsymbol{w}_1, \boldsymbol{v}_2 \otimes \boldsymbol{w}_2 \rangle = \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle$ for rank one elements, and extend this bilinearly on all elements of $H_A \otimes H_B$.

For simplicity, we assume that $H_A = H_B = \mathbb{R}^2$.

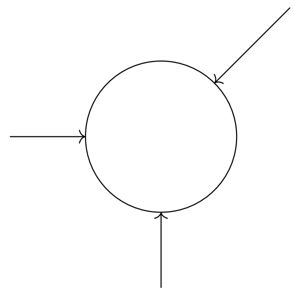
1. *Show that $\langle -, - \rangle$ on $H_A \otimes H_B$ is indeed an inner product. (You don't need to prove bilinearity, but prove that it is symmetric and positive definite.)*
2. *Show that $a\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + b\boldsymbol{e}_2 \otimes \boldsymbol{e}_2$ has rank two when a, b are both non-zero. (Rank one states in $H_A \otimes H_B$ are called non-entangled states. And these rank two states here are entangled states.)*
3. *Given a pair of linear maps $L_A : H_A \to H_A$ and $L_B : H_B \to H_B$, we define the expected observation result of $L_A \otimes L_B$ to be $\langle \omega, L_A \otimes L_B(\omega) \rangle$. Now let $I_A : H_A \to H_A$ and $I_B : H_B \to H_B$ be the identity maps on H_A and H_B respectively. Suppose $L = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$. Show that if ω is non-entangled, say $\omega = \boldsymbol{v} \otimes \boldsymbol{w}$, then $\langle \omega, L \otimes I_B(\omega) \rangle$ and $\langle \omega, I_A \otimes L(\omega) \rangle$ could be any pair of real numbers. (Here $L_A \otimes I_B$ means that we perform observation L_A on A while we do not observe B at all. As we can see here, the obervation of only A and the observation of only B has no relation to each other.) (You may use dot product for the inner product in this question.)*
4. *Suppose $L = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, and let $a, b \in \mathbb{R}$ be real numbers such that $a^2 + b^2 = 1$, then calculate $\langle \omega, L \otimes I_B(\omega) \rangle$ and $\langle \omega, I_A \otimes L(\omega) \rangle$ when $\omega = a\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + b\boldsymbol{e}_2 \otimes \boldsymbol{e}_2$. (This is an entangled-state, where observing A is identical to observing B.) (You may use dot product for the inner product in this question.)*

1.10 HW10 (Due 5.27) More Tensors

Exercise 1.10.1 (Squeezing a Ping Pong)**.** *Tensors are basically just recording multiple vectors simultaneously, in a multilinear way. This problem applies this idea in the situation of squeezing a Ping Pong ball. The traditional problem is like this: 10 forces are applied on a ping pong ball. Where would the ping pong ball most likely crack? We hereby reduced the problem to a 2-dim version with only three forces to simplify calculation.*

If we apply a force $\boldsymbol{f} \in \mathbb{R}^2$ on the unit circle at point $\boldsymbol{p}$, then we record this information as $\boldsymbol{p}\otimes\boldsymbol{f} \in \mathbb{R}^2\otimes\mathbb{R}^2$.

Suppose we apply three forces $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{v}$ where $\boldsymbol{v} = -\boldsymbol{e}_1 - \boldsymbol{e}_2$, and the forces are applied to the circle perpendicularly, as shown in the graph below.



1. *The three forces are recorded via three tensors. Calculate their sum T in terms of the standard basis $\boldsymbol{e}_{11}, \boldsymbol{e}_{12}, \boldsymbol{e}_{21}, \boldsymbol{e}_{22} \in \mathbb{R}^2 \otimes \mathbb{R}^2$.*

2. *Find two mutually orthogonal unit vector $\boldsymbol{x}, \boldsymbol{y}$ such that T is a linear combination of $\boldsymbol{x} \otimes \boldsymbol{x}$ and $\boldsymbol{y} \otimes \boldsymbol{y}$.*

3. *Let us say that the circle is squeezed such that its shape has changed a tiny bit, into an ellipse. Which directions would the long axis and short axis be?*

4. *Find a tensor $T \in \mathbb{R}^2 \otimes \mathbb{R}^2$ that cannot be obtained via squeezing the circle perpendicularly, with no matter how many forces. Here squeezing perpendicularly means the forces are always inward. Explain why squeezing perpendicularly cannot produce your tensor. (Hint: Suppose T is a sum of tensors for perpendicular squeezing forces. Write T^{ij} into a matrix. Is it positive definite, positive semi-definite, negative definite, negative semi-definite or indefinite?)*

Exercise 1.10.2 (Change of basis)**.** *Suppose a change of basis happens in V, from the old basis $\mathcal{B} = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ to a new basis $\mathcal{C} = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\}$. For each $\boldsymbol{v} \in V$, we denote its coordinates in the basis $\mathcal{B}$ as $\boldsymbol{v}_{\mathcal{B}} \in \mathbb{R}^n$, and we similarly define $\boldsymbol{v}_{\mathcal{C}} \in \mathbb{R}^n$.*

Let M be the change of coordinate matrix such that $\boldsymbol{v}_{\mathcal{C}} = M\boldsymbol{v}_{\mathcal{B}}$. In the following questions, we shall try to find the change-of-basis formula for various (a, b) tensors over V, i.e., in the space $\mathcal{T}^a_b(V)$.

1. *Suppose $a = 0, b = 1$. Then $\mathcal{T}^a_b(V) = V^*$. For any $\alpha \in V^*$, let $\alpha_{\mathcal{B}} \in (\mathbb{R}^n)^*$ be the matrix for the linear map $\alpha : V \to \mathbb{R}$ under the basis $\mathcal{B}$ for V, and let $\alpha_{\mathcal{C}}$ be defined similarly. (So these $\alpha_{\mathcal{B}}$ and $\alpha_{\mathcal{C}}$ are row vectors.) How to express the new coordinates of the row vector $\alpha_{\mathcal{C}}$ in terms of the old coordinates of the row vector $\alpha_{\mathcal{B}}$ and the matrix M? (Hint: The value $\alpha(\boldsymbol{v})$ must be independent of basis. So we should have $\alpha_{\mathcal{B}}(\boldsymbol{v}_{\mathcal{B}}) = \alpha_{\mathcal{C}}(\boldsymbol{v}_{\mathcal{C}})$.)*

2. *Given a $(0, 2)$-tensor T on V, we can write $T = \sum x_{ij}\boldsymbol{b}^*_i \otimes \boldsymbol{b}^*_j$, and get a matrix $T_{\mathcal{B}}$ with entries x_{ij}. (Here the stars mean that we are using the corresponding dual basis.) Show that $T(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}^{\mathrm{T}}_{\mathcal{B}} T_{\mathcal{B}} \boldsymbol{w}_{\mathcal{B}}$.*

3. *Given a $(0, 2)$-tensor T on V, we can write $T = \sum x_{ij}\boldsymbol{b}^*_i \otimes \boldsymbol{b}^*_j$ and $T = \sum y_{ij}\boldsymbol{c}^*_i \otimes \boldsymbol{c}^*_j$. Then we have two matrices $T_{\mathcal{B}}, T_{\mathcal{C}}$ with entries x_{ij} and y_{ij} correspondingly. Express $T_{\mathcal{C}}$ in terms of $T_{\mathcal{B}}$ and the matrix M. (Hint: Again, note that the value $T(\boldsymbol{v}, \boldsymbol{w})$ is independent of basis. So we should always have $\boldsymbol{v}^{\mathrm{T}}_{\mathcal{B}} T_{\mathcal{B}} \boldsymbol{w}_{\mathcal{B}} = \boldsymbol{v}^{\mathrm{T}}_{\mathcal{C}} T_{\mathcal{C}} \boldsymbol{w}_{\mathcal{C}}$.)*

4. *Given a* $(2,0)$*-tensor* T *on* V*, we can write* $T = \sum x_{ij}\boldsymbol{b}_i \otimes \boldsymbol{b}_j$ *and* $T = \sum y_{ij}\boldsymbol{c}_i \otimes \boldsymbol{c}_j$*. Then we have two matrices* $T_{\mathcal{B}}, T_{\mathcal{C}}$ *with entries* x_{ij} *and* y_{ij} *correspondingly. Express* $T_{\mathcal{C}}$ *in terms of* $T_{\mathcal{B}}$ *and the matrix* M*. (Hint: Same as the last one. Note that* $T(\alpha,\beta) = \alpha_{\mathcal{B}} T_{\mathcal{B}} \beta_{\mathcal{B}}^{\mathrm{T}}$*.)*

5. *Given a* $(1,1)$*-tensor* T *on* V*, we can write* $T = \sum x_{ij}\boldsymbol{b}_i \otimes \boldsymbol{b}_j^*$ *and* $T = \sum y_{ij}\boldsymbol{c}_i \otimes \boldsymbol{c}_j^*$*. Then we have two matrices* $T_{\mathcal{B}}, T_{\mathcal{C}}$ *with entries* x_{ij} *and* y_{ij} *correspondingly. Express* $T_{\mathcal{C}}$ *in terms of* $T_{\mathcal{B}}$ *and the matrix* M*. (Hint: Same as the last one. Note that* $T(\alpha,\boldsymbol{v}) = \alpha_{\mathcal{B}} T_{\mathcal{B}} \boldsymbol{v}_{\mathcal{B}}$*. Feel free to double check this with your old change-of-basis formula for linear maps, from last semester.)*

6. *(Read only) Note that* $(2,0)$*-tensors,* $(1,1)$*-tensors and* $(0,2)$*-tensors can all be written as a* 2*-dimensional arrays of numbers, i.e., matrices. However, they are not the same, since the change of basis formula is vastly different. This is the most important reason why we should NOT simply think of an* (a,b)*-tensor as some* $a+b$ *dimensional array of numbers.*

Exercise 1.10.3 (Why should the "gradient" be a row vector)**.** *Sometimes, you might think you have a vector. But actually, you have a dual vector. One way to detect this is via a potential change-of-basis, and see what the formula corresponds to. This problem is an example of this.*

Take a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$*, say* $f(x,y,z) = x^2 + y^2 + z^2$*. Calculate the gradient of* f*.*

Now take a change of basis process such that $\boldsymbol{v}_{new} = M(\boldsymbol{v}_{old})$ *where* $M = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$*. Find the function* $f_{new} : \mathbb{R}^3 \to \mathbb{R}$ *such that* $f_{new}(\boldsymbol{v}_{new}) = f(\boldsymbol{v}_{old})$*.*

Calculate the gradient of f_{new}*, and verify that* $\nabla f_{new}(a,b,c) = (M^{-1})^{\mathrm{T}}(\nabla f(x,y,z))$ *when* $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}$*.*

(In particular, this is NOT the column vector's change-of-basis formula. On the other hand, if we write ∇f *as a row vector, then this would actually be the row vector's change-of-basis formula, as determined by the last problem. This indicates that, alas, people should always write* ∇f *as a row vector, and think of it as a "dual vector" to our domain* $\mathbb{R}^3$*.)*

(However, please keep writing ∇f *as a column vector in your calculus class. Otherwise your calculus teacher might get angry at me.)*

(Another thing to keep in mind. In physics, if we think of "forces" as gradients of some potential functions, then "forces" should also be dual vectors. They are indeed dual to "velocity" since they evaluate each other to the real number "power". Now, consider Newton's law, which states that $F = ma$*. Here* F *is a row vector, but acceleration* a *is the derivative of the column vector "velocity", hence it is a column vector. So, this law basically gives an identification between row vectors and column vectors. As you may recall from class, an identification between* V *and* V^* *means we have an inner product structure on* V*. Again, we see that most classical laws of physics are basically just pointing out various inner product structures used in various mechanical systems.)*

1.11 HW11 (Due 6.10) Alternating Tensors

Notice: We have 11 HW in total. In the end, your lowest HW score will be dropped, and only your highest 10 HW score would count.

Exercise 1.11.1. *All the tensor below is over $\mathbb{R}^n$. We take standard basis $e_1, ..., e_n$ for $\mathbb{R}^n$ and the corresponding standard dual basis, and the induced standard basis on all related tensor spaces. Compute the following:*

1. $e^1 \otimes e^2(e_1 \otimes e_2)$;
2. $Alt(e^1 \otimes e^2)(e_1 \otimes e_2)$;
3. $e^1 \otimes e^2(Alt(e_1 \otimes e_2))$;
4. $Alt(e^1 \otimes e^2)(Alt(e_1 \otimes e_2))$;
5. $e^1 \wedge e^2(Alt(e_1 \otimes e_2))$;
6. $e^1 \wedge e^2(e_1 \otimes e_2)$;
7. $e^1 \otimes e^2(e_1 \wedge e_2)$;
8. $e^1 \wedge e^2(e_1 \wedge e_2)$;
9. *Using dot product on $\mathbb{R}^n$ as inner product, find $\langle e_1 \otimes e_2, e_1 \otimes e_3 \rangle$ and $\langle e_1 \wedge e_2, e_1 \wedge e_3 \rangle$;*
10. *Using dot product on $\mathbb{R}^n$ as inner product, find $||e_1 \wedge e_2 \wedge e_3||$ and $||Alt(e_1 \otimes e_2 \otimes e_3)||$ and $||Alt(e_1 \otimes e_2) \wedge e_3||$.*

Exercise 1.11.2. *We define the Levi-Civita Symbol as the following: $e_{i_1,...,i_n} = 0$ if $(1, ..., n) \mapsto (i_1, ..., i_n)$ is not a permutation, and 1 if it is an even permutation, -1 if it is an odd permutation. We also define $e^{i_1,...,i_n}$ to have the same value as $e_{i_1,...,i_n}$. We also define δ^i_j to be the coordiantes of the identity map as a $(1,1)$ tensor.*

1. *Show that $\det = e_{ijk}e^i \otimes e^j \otimes e^k$ in $\bigwedge_3(\mathbb{R}^3)$. Here det is the usual determinant tensor. In particular, we have $e_{ijk}u^iv^jw^k = \det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. (So these Levi-Civita Symbol are representing the determinant tensor.)*
2. *Show that $e_{ij}e^{mn} = \delta^m_i\delta^n_j - \delta^n_i\delta^m_j$.*
   (Hint: There are two proofs. One is to check this for all possible values of i, j, m, n, and there are only 16 possibilities, and 12 of them are zero.
   Alternatively, we can be fancy. Both sides of the equation is representing some multilinear maps. So they are equal if they agree on all inputs. So we only need to check if $e_{ij}e^{mn}v^iw^j\alpha_m\beta_n = \delta^m_i\delta^n_jv^iw^j\alpha_m\beta_n - \delta^n_i\delta^m_jv^iw^j\alpha_m\beta_n$. Now, note that $e_{ij}v^iw^j = \det(\begin{bmatrix}\boldsymbol{v} & \boldsymbol{w}\end{bmatrix})$ by definition, and $e^{ij}\alpha_i\beta_j = \det(\begin{bmatrix}\alpha \\ \beta\end{bmatrix})$. So the left hand side is $\det(\begin{bmatrix}\alpha \\ \beta\end{bmatrix}\begin{bmatrix}\boldsymbol{v} & \boldsymbol{w}\end{bmatrix})$. Now check the right hand side. Note that δ represent the identity map, and you have tools such as $\delta^m_iv^i = v^m$. So on the right hand side, you see $\alpha(\boldsymbol{v})\beta(\boldsymbol{w}) - \alpha(\boldsymbol{w})\beta(\boldsymbol{v})$ after simplification. Then you should see the results.)
3. *Show that $e_{ijk}e^{imn} = \delta^m_j\delta^n_k - \delta^n_j\delta^m_k$.*
   (Hint: Note that i is auto-summarized. Again, you can check the 16 possibilities of j, k, m, n where 12 of them are zero. No shame in a down-to-earth approach. Or you can try to be fancy and work something out.)
4. *Calculate $e_{ijk}e^{ijn}$ and $e_{ijk}e^{ijk}$, express them in terms of the δ's. (Hint: Beware that $\delta^i_i = 3$, since the trace of identity matrix on $\mathbb{R}^3$ is three.)*

5. *(Read Only) Note that e^i_{jk} is the cross product tensor. Here e^i_{jk} has the same value as e_{ijk}. You may also think of cross product as "raising the index" from the determinant tensor, using dot product. (I.e., $e^i_{jk} := e_{mjk}\delta^{im}$.)*

Exercise 1.11.3 (Bell's Inequality)**.** *This problem is part of Bell's theorem that shows how, according to quantum mechanics, informations could transfer instantaneously between entangled particles even if they are very very far away from each other.*

Here are some interpretations (but they are not really important for solving this problem.... They are just backgrounds for those that are curious.) For the moment, let us use classical physic's way of thinking. Suppose we have two particles. Alice and Bob are very very far away, each has one of the particles. Alice can make measurements M_A or N_A, and Bob can make measurements M_B or N_B. These measurements are designed such that, if we just perform M_A or N_A or M_B or N_B, we always obtain value 1 or -1. So in particular, by checking all possibilities, you can see that exactly one of $M_B + N_B$ and $M_B - N_B$ must be zero, and the other must be ± 2. Combining this with the fact that M_A, N_A are also ± 1, we would always have $M_A(M_B + N_B) + N_A(M_B - N_B) = \pm 2$ for any possible results of these measurement. If we repeat measurements many times and take average, the expected value of $M_A(M_B + N_B) + N_A(M_B - N_B) = M_A M_B + M_A N_B + N_A M_B - N_A N_B$ must be between 2 and -2. This is called Bell's inequality. So far it seems true, right?

However, the following calculations show that this is only true if the two particles are NOT entangled. Under quantum entanglement, this inequality could be violated. Currently, experiments by physicists shows that Bell inequality is NOT true in real life, hence proving that quantum entanglements are real.

We perform some tensor calculations for $\mathbb{R}^2$. We use standard basis e_1, e_2 for $\mathbb{R}^2$ and dual basis e^1, e^2 for $(\mathbb{R}^2)^*$ Let $\tau = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)$, and define $\tau^2 = \frac{1}{2}(e_1 \otimes e_1 \otimes e_2 \otimes e_2 - e_1 \otimes e_2 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 \otimes e_1)$. (This is essentially $\tau \otimes \tau$, but we change the order of $(a \otimes b) \otimes (c \otimes d)$ into $(a \otimes c) \otimes (b \otimes d)$, so that things about the first particle are still all in the front, and things about the second particle are all to the back.)

1. *Show that τ^2 has unit length, and τ cannot be written as $v \otimes w$ for some $v, w \in \mathbb{R}^2$. (So it is an entangled state. Usually "un-entangled" means "rank 1".)*

2. *Recall that $(2, 0)$ tensors, $(1, 1)$ tensors, and $(0, 2)$ tensors all have matrix representations. Find a $(0, 2)$-tensor M_A whose matrix representation is a symmetric matrix with eigenvalues $1, -1$ and eigenvectors e_1, e_2 respectively. Express this tensor in terms of the standard basis. (In quantum mechanics, these eigenvalues are possible measurement outcomes.)*

3. *Find a $(0, 2)$-tensor N_A whose matrix representation is a symmetric matrix with eigenvalues $1, -1$ and eigenvectors $e_1 + e_2, e_1 - e_2$ respectively. Express this tensor in terms of the standard basis.*

4. *Let $M_B = \frac{1}{\sqrt{2}}(M_A + N_A)$ and $N_B = \frac{1}{\sqrt{2}}(M_A - N_A)$. Show that their corresponding matrices are symmetric with eigenvalues ± 1. (I expressed M_B, N_B in terms of M_A, N_A not because of any dependence. Rather, it is for numerical convenience to save your time.... If properly done, they should have be defined via eigenvectors and eigenvalues, like in previous problems for M_A, N_A.)*

5. *Compute $(M_A \otimes M_B + M_A \otimes N_B + N_A \otimes M_B - N_A \otimes N_B)(\tau^2)$. (This is how to do expected value in quantum mechanics. This should NOT be between 2 and -2, therefore violating Bell's inequality.) (Hint: Brute force works. Or, simplify the tensor $M_A \otimes M_B + M_A \otimes N_B + N_A \otimes M_B - N_A \otimes N_B$ first.)*

6. *Show that $(M_A \otimes M_B + M_A \otimes N_B + N_A \otimes M_B - N_A \otimes N_B)(v \otimes v \otimes w \otimes w)$ for unit vectors v, w must be between $-\sqrt{2}$ and $\sqrt{2}$. (So for untangled states, Bell's inequality is not violated. Here we do not reach ± 2 because of the "angle" between M_A, M_B and N_A, N_B.) (Hint: Show that $M_A(v, v)^2 + N_A(v, v)^2 = 1$ for all unit vector v by direct computation.)*