作业 10 简答

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习题 1. *Let* $\mathbf{y}=\begin{bmatrix}5\\-9\\5\end{bmatrix}, \mathbf{u}_1=\begin{bmatrix}-3\\-5\\1\end{bmatrix}, \mathbf{u}_2=\begin{bmatrix}-3\\2\\1\end{bmatrix}$. *Find the distance from* $\mathbf{y}$ *to the plane in* $\mathbf{R}^3$ *spanned by* $\mathbf{u}_1$ *and* $\mathbf{u}_2$.

解答. 取 $A=\begin{bmatrix}-3&-3\\-5&2\\1&1\end{bmatrix}$，利用投影矩阵的计算公式

$$dist=\left\|\mathbf{y}-A\left(A^TA\right)^{-1}A^T\mathbf{y}\right\|=2\sqrt{10}.$$

习题 2. *Mark each statement true or false, and justify your answer.*

1. *If* W *is a subspace of* $\mathbf{R}^n$ *and if* $\mathbf{v}$ *is both in* W *and* $W^\perp$, *then* $\mathbf{v}$ *must be the zero vector.*

2. *In the orthogonal decomposition theorem, each term in the formula for* $\mathrm{Proj}_W(\mathbf{y})$ *is itself an orthogonal projection of* $\mathbf{y}$ *onto a subspace of* W.

3. *If* $\mathbf{y}=\mathbf{z}_1+\mathbf{z}_2$, *with* $\mathbf{z}_1\in W$ *and* $\mathbf{z}_2\in W^\perp$, *then* $\mathbf{z}_1$ *must be the orthogonal projection of* $\mathbf{y}$ *onto* W.

4. *The best approximation to* $\mathbf{y}$ *by elements of a subspace* W *is given by the vector* $\mathbf{y}-\mathrm{Proj}_W(\mathbf{y})$.

5. *If an* $n\times p$ *matrix* U *has orthonormal columns, then* $UU^t\mathbf{x}=\mathbf{x}$ *for all* $\mathbf{x}\in\mathbf{R}^n$.

解答. 1. 对的. 此时 $\mathbf{v}$ 与自己垂直，从而必定为零向量.

2. 对的. 这是正交投影定义的等价说法.

3. 对的. 注意 $\mathbf{z}_1 \in W$ 时, $\mathrm{Proj}_W(\mathbf{z}_1) = \mathbf{z}_1$, 而 $\mathbf{z}_2 \in W^\perp$ 时, $\mathrm{Proj}_W(\mathbf{z}_2) = 0$. 因此
$$\mathrm{Proj}_W(\mathbf{y}) = \mathrm{Proj}_W(\mathbf{z}_1 + \mathbf{z}_2) = \mathbf{z}_1.$$

4. 不对. 用 W 中的元素去近似 y, 自然要保证近似元素在 W 里. 但是题中给出的近似元素 $\mathbf{y} - \mathrm{Proj}_W(\mathbf{y})$ 未必在 W 中.

5. 不对. 例如取 $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, 于是 $UU^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 导出矛盾.

习题 3. *Find an orthogonal basis for the column space of the matrix*
$$\begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$$
and a QR-factorization of it.

解答.
$$Q = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{2}}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}}{4} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{2}}{4} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{4} & \frac{1}{2} \end{bmatrix}, R = \begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 6 \end{bmatrix}$$

习题 4. *Mark each statement true or false, and justify your answer.*

1. *If $A = QR$ and Q has orthonormal columns, then $R = Q^t A$.*

2. *Let $W = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ linearly independent, let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthogonal set in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.*

3. *If $\mathbf{x}$ is not in the subspace W, then $\mathbf{x} - \mathrm{Proj}_W(\mathbf{x})$ is non-zero.*

4. *In a QR-factorization $A = QR$, where A has linearly independent columns, the column vectors of Q form an orthonormal basis for* $\mathrm{Col}(A)$.

解答. 1. 对的. 利用 $Q^TQ = I$.

2. 错误. 可能包含零向量. 非零正交集一定线性无关，但是正交集可以有零向量.

3. 对的. 注意 Proj_W 的像集一定包含在 W 里. 因而若 $\mathbf{x} - \mathrm{Proj}_W(\mathbf{x}) = 0$，则
$$\mathbf{x} = \mathrm{Proj}_W(\mathbf{x}) \in W.$$

4. 对的. 因为正交化的过程中每一步都是可逆的, 相当于 R 是可逆矩阵[1]，因而 A 变为 Q 的时候，秩是不改变的.

习题 5. 1. *Let $A = QR$, where Q is $m \times n$ and R is $n \times n$. Show that if the columns of A are linearly independent, then R must be invertible.*

2. *Let $A = QR$ with R invertible, show that A and Q have the same column space.*

解答. 1. 若 R 不可逆, 则存在 $\mathbf{x} \in \mathrm{R}^n$ 使得 $Rx = 0$, 此时 $A\mathbf{x} = QR\mathbf{x} = 0$，与 A 的列线性无关矛盾.

2. 注意矩阵恒等式 $X = YC$ 可以推出 $Col(X) \subset Col(Y)$，于是这里 $Col(A) \subset Col(Q)$，再由于 $AR^{-1} = Q$，得到 $Col(A) = Col(Q)$.

习题 6. *Let $A = QR$ be a QR-factorization of an $m \times n$ matrix A with linearly independent columns. Partition A as $[A_1 A_2]$, where A_1 has p columns. Show how to obtain a QR-factorization of A_1 and explain the reason.*

解答. 这相当于 QR 分解仅完成了前 p 步，令 $Q = [Q_1 Q_2]$，这里 Q_1 只有 p 列，
$$R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix},$$
R_1 是 p 阶矩阵[2]，则 $A_1 = Q_1 R_1$.

[1] 可以在下一题的证明中见到.

[2] 由构造，这里 $R_3 = 0$.

习题 7. *Find the least-square solution of* $A\mathbf{x}=\mathbf{b}$ *by solving the corresponding normal equation:*

1. $A=\begin{bmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{bmatrix}, \mathbf{b}=\begin{bmatrix}3\\ 1\\ -4\\ 2\end{bmatrix}$

2. $A=\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}, \mathbf{b}=\begin{bmatrix}1\\ 3\\ 8\\ 2\end{bmatrix}.$

解答. 1. 直接计算

$$A^TA=\begin{bmatrix}6 & 6\\ 6 & 42\end{bmatrix}, A^T\mathbf{b}=\begin{bmatrix}6\\ -6\end{bmatrix},$$

解方程 $A^TA\hat{\mathbf{x}}=A^T\mathbf{b}$ 得到 $\hat{\mathbf{x}}=\begin{bmatrix}4/3\\ -1/3\end{bmatrix}$.

2. 直接计算

$$A^TA=\begin{bmatrix}4 & 2 & 2\\ 2 & 2 & 0\\ 2 & 0 & 2\end{bmatrix}, A^T\mathbf{b}=\begin{bmatrix}14\\ 4\\ 10\end{bmatrix}.$$

解方程 $A^TA\hat{\mathbf{x}}=A^T\mathbf{b}$ 得到

$$\hat{\mathbf{x}}=\begin{bmatrix}5\\ -3\\ 0\end{bmatrix}+a\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}.$$

习题 8. *Find the orthogonal projection of* $\mathbf{b}$ *onto* $\mathrm{Col}(A)$ *and the least square solution of* $A\mathbf{x}=\mathbf{b}$ *:*

1. $A=\begin{bmatrix}1 & 2\\ -1 & 4\\ 1 & 2\end{bmatrix}, \mathbf{b}=\begin{bmatrix}3\\ -1\\ 5\end{bmatrix}$

2. $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

解答. *1.* 直接计算[3]

$$\mathrm{Proj}_{\mathrm{Col}(A)}(\mathbf{b}) = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}.$$

2. 直接计算

$$\mathrm{Proj}_{\mathrm{Col}(A)}(\mathbf{b}) = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}.$$

习题 9. *With the given QR-factorization of A, compute the least square solution of* $A\mathbf{x} = \mathbf{b}$ *:*

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

解答. 把 $A = QR$ 代入 $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$, 得到

$$R^T Q^T QR\hat{\mathbf{x}} = R^T Q^T \mathbf{b},$$

即 $R\hat{\mathbf{x}} = Q^T\mathbf{b}$, 代入得

$$\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 7 \\ -1 \end{bmatrix},$$

解出 $\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

习题 10. *Let A be an $m \times n$ matrix. Use the steps below to show that a vector $\mathbf{x} \in \mathbf{R}^n$ satisfies $A\mathbf{x} = 0$ if and only if $A^t A\mathbf{x} = 0$. This implies that* $\mathrm{Nul}(A) = \mathrm{Nul}\left(A^t A\right)$.

[3]注意列向量正交，所以可以不必标准正交化. 但利用投影矩阵计算工作量也不大. 下同.

1. *Show that if* $A\mathbf{x}=0$, *then* $A^tA\mathbf{x}=0$.
2. *Suppose that* $A^tA\mathbf{x}=0$. *Explain why* $\mathbf{x}^tA^tA\mathbf{x}=0$ *and deduce from it that* $A\mathbf{x}=0$.

Deduce from the above results that $\operatorname{rk}(A^tA)=\operatorname{rk}(A)$.

解答. 1. 如果 $A\mathbf{x}=0$，那么 $A^TA\mathbf{x}=A^T(A\mathbf{x})=A^T\cdot 0=0$.

2. 这是因为 $\mathbf{x}^TA^TA\mathbf{x}=(A\mathbf{x})^T(A\mathbf{x})=\|A\mathbf{x}\|^2$, 而 $\|A\mathbf{x}\|=0$ 等价于 $A\mathbf{x}=0$. 于是

$$\operatorname{Nul}(A)=\operatorname{Nul}\left(A^TA\right),$$

取维数立刻得到

$$\operatorname{rk}\left(A^TA\right)=\operatorname{rk}(A).$$

习题 11. *A certain experiment produces the data* $(1,7.9),(2,5.4),(3,-0.9)$. *Describe the model that produces a least squares fit of these points by a function of the form* $y=A\cos(x)+B\sin(x)$.

解答. 对应的数学问题是解出 $\begin{bmatrix}\cos 1 & \sin 1\\ \cos 2 & \sin 2\\ \cos 3 & \sin 3\end{bmatrix}\begin{bmatrix}A\\ B\end{bmatrix}=\begin{bmatrix}7.9\\ 5.4\\ -0.9\end{bmatrix}$ 的最小二乘解.