Topics in Linear Algebra: Homework 8

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* Credit to Fan Sunqi for some parts in Sol. 1.8.2.

Solution 1.8.1.

1. $(\alpha \otimes \beta \otimes \gamma)_{ijk} = \alpha_i \beta_j \gamma_k$, where subscript n is the index of component of the row vector.

2. **Lemma1 :**
   Let $U = \alpha \otimes \beta \otimes \gamma$, $\mathcal{K}(u, v, w) = u \otimes v \otimes w$, L be a 1×8 row vector that its entries are in the form of $\alpha_i \beta_j \gamma_k$, where i, j, k are put in lexicographic order. Then, $U = L \circ \mathcal{K}$.
   Proof 1:
   $U(u, v, w) = \alpha(u)\beta(v)\gamma(v)$. Expand the R.H.S. directly follows.
   Lemma 2:
   $M_E(U) = M'_E(L) \circ \mathcal{K}$, where M'_E is just a matrix, whenever E is an elementary matrix.
   Proof 2:
   Case 1: $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $M'_E(L) = L \cdot \begin{bmatrix} 0 & I_4 \\ I_4 & 0 \end{bmatrix}$.
   Case 2: $E = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$, then $M'_E(L) = L \cdot \begin{bmatrix} s_1 I_4 & 0 \\ 0 & s_2 I_4 \end{bmatrix}$
   Case 3: $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, then $M'_E(L) = L \cdot \begin{bmatrix} I_4 & 0 \\ kI_4 & I_4 \end{bmatrix}$
   Let $U_1 = L_1 \circ \mathcal{K}$, $U_2 = L_2 \circ \mathcal{K}$, then $aU_1 + bU_2 = aL_1 \circ \mathcal{K} + bL_2 \circ \mathcal{K} = (aL_1 + bL_2) \circ \mathcal{K}$,

   $$M_E(aU_1 + bU_2) = M'_E(aL_1 + bL_2) \circ \mathcal{K} = aM'_E(L_1) \circ \mathcal{K} + bM'_E(L_2) \circ \mathcal{K} = aM_E(U_1) + bM_E(U_2)$$

   Other matrices in $((R^2)^*)^{\otimes 3}$ are just linear combinations of the U 's, the proof directly follows.

3. Suppose $M = \sum_{i=1}^{r} a_i U_i$, where $U_i =$, $a_i \neq 0$ and $\sum_{i=1}^{r} t_i U_i = 0$ yields $t_1 = t_2 = \cdots = t_r = 0$, then

   $$M_E(M) = M_E\left(\sum_{i=1}^{r} a_i U_i\right) = \sum_{i=1}^{r} a_i M_E(U_i)$$

   By the way, M_E is bijective, since elementary operations E are always invertible, so $M_E(M)$ is a rank r tensor.

4. Assume the 3D matrix M to have rank less than r, then $M = \sum_{i=1}^{r'} a_i U_i$, $r' < r$ and U_i are simple tensors. Then for each $a_i U_i$, that layer has rank 1, and the sum yields that, that layer can only have rank less than r, which is a contradiction.

5. It is rank two, since $M = ([1, -1] \otimes [1, -1] \otimes [1, -1] + [1, 1] \otimes [1, 1] \otimes [1, 1])/2$.

Solution 1.8.2.

1. $$M(v, v, v) = v^T \left(x \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} + y \begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{bmatrix} + z \begin{bmatrix} 5 & 6 & 7 \\ 6 & 7 & 8 \\ 7 & 8 & 9 \end{bmatrix} \right) v$$

   It can be treated as a "cubic form", analogous to quadratic forms, so it is

   $$M(v, v, v) = 3x^3 + 6y^3 + 9z^3 + 12x^2y + 15xy^2 + 15x^2z + 21xz^2 + 21y^2z + 24yz^2 + 36xyz$$

2. Similar to 1.8.1, convert M to be a composition of a 1×27 row vector and the Kronecker product of the three input vectors, $M = L \circ \mathcal{K}$. Then, for each entry in $\mathcal{K}(v_1, v_2, v_3)$, it must be in the form of $v_1^i v_2^j v_3^k$, where superscript denotes the position of entry in its own vector.
Let σ' be an element in a group that isomorphic to S_6, that $\sigma'(v_1, v_2, v_3) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$, then σ' induces σ'' which is another element that comes from another group isomorphic to S_6 and σ'' correspond to σ',
$$\mathcal{K}(\sigma'(v_1, v_2, v_3)) = \sigma'' \circ \mathcal{K}(v_1, v_2, v_3)$$
$L \circ \sigma''$ gives a row vector that permutates around the entries of L. As entries of L implicitly refers to the entries of M, $L = [3, 4, 5, 4, 5, 6, \cdots]$, that $L_1 = (1, 1, 1)$ entry, $L_2 = (1, 1, 2)$ entry and so on, $L \circ \sigma''$ permutates the subscript of the entry that the entry of L refers to in M. However, in M, $M_{ijk} = i + j + k$ is invariant over permutation, so $L \circ \sigma'' = L$.
Intuitively, M can be thought as a cube that is invariant over "rotations of triangles that results rotational symmetry", on the axis that pass through $(1, 1, 1)$ and $(3, 3, 3)$. In addition, straighten up the axis. Watch the cube from the top, then it has two-fold reflectional symmetry.

3. On one hand, rank of M is not less than 2 by 1.8.1.4.
On the other hand, the upper layer and the lower layer can be treated as adding or subtracting a layer of matrix with all ones.
Further crack down the layer of $\begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{bmatrix}$: it is a rank two matrix,
$$\begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{bmatrix} = [5, 6, 7]^T [1, 1, 1] + [1, 1, 1]^T [-1, 0, 1]$$
Moreover,
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = [1, 1, 1]^T [1, 1, 1]$$
So,
$$M = [5, 6, 7] \otimes [1, 1, 1] \otimes [-1, 0, 1] + [1, 1, 1] \otimes [-1, 0, 1] \otimes [-1, 0, 1] + [1, 1, 1] \otimes [1, 1, 1] \otimes [-1, 0, 1]$$
M is expressed as sum of three simple tensors, so its rank is at most 3.