Calculus A(1): Homework 9

December 19, 2021

5.3.

74.

It would be nice if average values of integrable function obeyed the following rules on an interval $[a, b]$.

a.

$$\mathrm{av}(f+g) = \mathrm{av}(f) + \mathrm{av}(g)$$

b.

$$\mathrm{av}(kf) = k\mathrm{av}(f), \forall k \in \mathbb{R}$$

c.

$$\mathrm{av}(f) \leq \mathrm{av}(g), (\forall x)(x \in [a,b] \to f(x) \leq g(x))$$

Do these rules ever hold? Give reasons for your answers.

Solution.

All of these rules hold.

a.

$$\mathrm{av}(f+g) = \frac{1}{b-a}\int_a^b (f(x)+g(x))dx = \frac{1}{b-a}\left(\int_a^b f(x)dx + \int_a^b g(x)dx\right)$$
$$= \frac{1}{b-a}\int_a^b f(x)dx + \frac{1}{b-a}\int_a^b g(x)dx = \mathrm{av}(f) + \mathrm{av}(g)\blacksquare$$

b.

$$\mathrm{av}(kf) = \frac{1}{b-a}\int_a^b (kf(x))dx = k\cdot\frac{1}{b-a}\int_a^b f(x)dx = k\mathrm{av}(f)\blacksquare$$

c.

$$g(x) - f(x) \geq 0 \Rightarrow \frac{1}{b-a}\int_a^b (g(x)-f(x))dx \geq 0 \Rightarrow \mathrm{av}(g-f) \geq 0 \Rightarrow \mathrm{av}(f) \leq \mathrm{av}(g)\blacksquare$$

77.

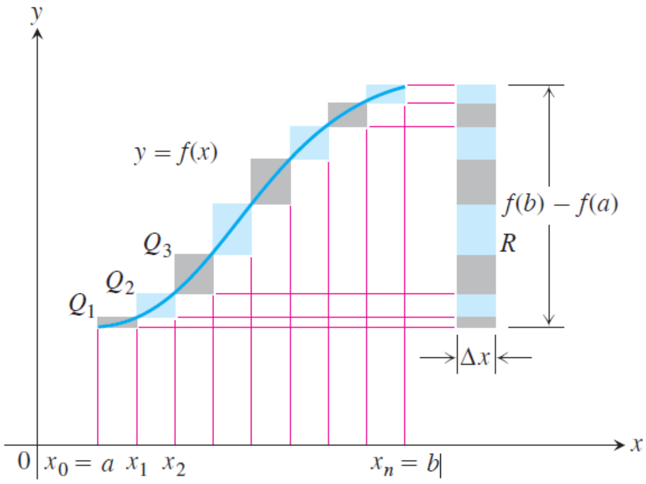
Upper and lower sums for increasing functions

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of length $\Delta x = (b-a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \dots, Q_{n-1}Q_n$ lie along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition $[a, b]$ vary in size. Show that

$$U - L \leq |f(b) - f(a)|\Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P,and hence that $\lim_{||P||\to 0}(U - L) = 0$.



Solution.

a. With regards to each segment, its area is $(f(x_{i+1}) - f(x_i)) \cdot \Delta x$, so

$$U - L = \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i)) \cdot \Delta x$$

$$= (-f(x_0) + f(x_1) - f(x_1) + \cdots + f(x_n))\Delta x = (f(b) - f(a))\Delta x$$

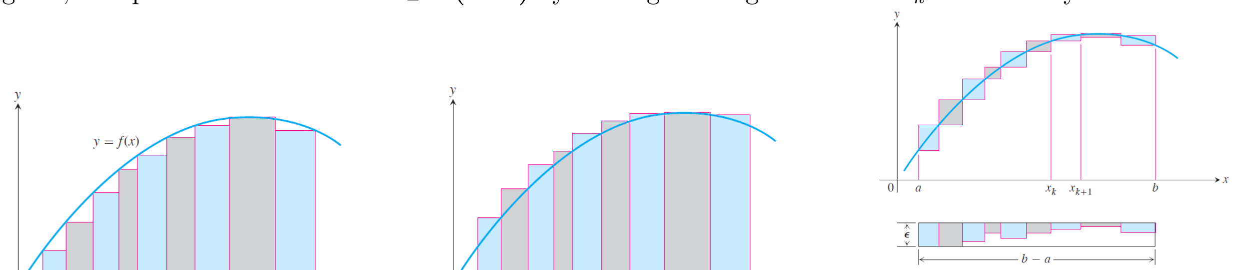
b.

$$U - L = \left| \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i)) \cdot \Delta x_i \right| \le \left| \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i)) \right| \cdot \Delta x_{\max} = |(f(b) - f(a))| \Delta x_{\max}$$

$$\lim_{\|P\| \to 0} (U - L) = \lim_{\Delta x_{\max} \to 0} |f(b) - f(a)| \Delta x_{\max} = 0$$

81.

We say f is **uniformly continuous** on $[a, b]$ if given any $\epsilon > 0$ there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure at the right to show that if f is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \le \epsilon \cdot (b - a)$ by making the largest of the Δx_k's sufficiently small.



Pleacher, *The Mathematics Teacher*, Vol. 85, No. 6, pp. 445–446, September 1992.)

Solution.

Let $x_{m_i}, x_{M_i} \in [x_{i-1}, x_i]$ such that

$$(\forall x \in [x_{i-1}, x_i])(f(x_{m_i}) \le f(x) \wedge f(x_{M_i}) \ge f(x))$$

Then

$$U - L = \sum_{i=1}^{n} (f(x_{M_i}) - f(x_{m_i}))\Delta x_i.$$

$[a, b]$ is possible to be partitioned in a way that $\forall i \in \{1, 2, \cdots, n\}, \Delta x_i < \delta$. That implies

$$U - L = \sum_{i=1}^{n} (f(x_{M_i}) - f(x_{m_i}))\Delta x_i < \sum_{i=1}^{n} \epsilon \cdot \Delta x_i = \epsilon \cdot (b - a)$$

5.4.

68.

Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)dt?$$

Give reasons for your answers.

- **a.** h is a twice-differentiable function of x.
- **b.** h and dh/dx are both continuous.
- **c.** The graph of h has a horizontal tangent at $x = 1$.
- **d.** h has a local maximum at $x = 1$.
- **e.** h has a local minimum at $x = 1$.
- **f.** The graph of h has an inflection point at $x = 1$.
- **g.** The graph of dh/dx crosses the x-axis at $x = 1$.

Solution.

By definition, $h'(x) = f(x)$. Also, $\forall x \in \mathbb{R}, f'(x) < 0$, hence h'' exists and $\forall x \in \mathbb{R}, h''(x) < 0$.
$h'(1) = f(1) = 0$, and $h''(1) = f'(1) < 0$, so $x = 1$ is a maximum in some neighborhood of it, and has a horizontal tangent, and clearly it is not an inflection point. That means h' changes sign when x transverses the neighborhood of $x = 1$.
h is twice-differentiable implies that its first derivative and its own are both continuous. Therefore, **a.,b.,c.,d.,g.** are true, while **e.,f.** are false.

5.5.

52.

Evaluate the integral

$$\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta \cos^3 \sqrt{\theta}}} d\theta$$

Solution.

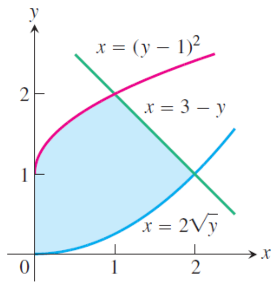
$$\int \frac{\sin \sqrt{\theta}}{\sqrt{\theta \cos^3 \sqrt{\theta}}} d\theta = 4 \int \left(\left(-\frac{1}{2} \cos^{-\frac{3}{2}} \sqrt{\theta} \right) (-\sin \sqrt{\theta}) \left(\frac{1}{2\sqrt{\theta}} \right) \right) d\theta = \frac{4}{\sqrt{\cos \sqrt{\theta}}} + C,$$

where C is a constant.

5.6.

78.

Find the area of the region in the first quadrant bounded on the left by the y–axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3 - y$.



Solution.

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x \geq (y-1)^2 \wedge x \leq 3-y \wedge x \geq 0 \wedge x \leq 2\sqrt{y}\}$.
By inspection, (0,0),(2,1),(1,2) and (0,1) are the vertices of Ω.

$$\iint\limits_{\Omega} dxdy = \int_0^1 2\sqrt{y}dy + \int_1^2 (3-y-(y-1)^2)dy = \frac{4}{3}y^{3/2}\bigg|_0^1 + \left(3y - \frac{1}{2}y^2 - \frac{1}{3}(y-1)^3\right)\bigg|_1^2 = \frac{5}{2}$$

87.

If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)dx}{f(x)+f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

Solution.

$$I = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du) = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}dx$$

So,

$$I = \frac{1}{2}\int_0^a \left(\frac{f(x)}{f(x)+f(a-x)} + \frac{f(a-x)}{f(x)+f(a-x)}\right)dx = \frac{1}{2}\int_0^a dx = \frac{1}{2}a$$

Additional and Advanced Exercises.

8.

Prove that

$$\int_0^x \left(\int_0^u f(t)dt\right)du = \int_0^x f(u)(x-u)du$$

(*Hint:* Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

Solution.

Let $F(x) = \int_0^x f(t)dt$.
By the fundamental theorem of Calculus,

$$F'(x) = \frac{d}{dx}\int_0^x f(t)dt = f(x),$$

So on one hand,

$$\int_0^x \left(\int_0^u f(t)dt \right) du = \int_0^x F(u)du.$$

On the other hand,

$$\int_0^x f(u)(x-u)du = F(u)(x-u)|_0^x - \int_0^x F(u)(-1)du = \int_0^x F(u)du$$

Hence,

$$\int_0^x \left(\int_0^u f(t)dt \right) du = \int_0^x F(u)du = \int_0^x f(u)(x-u)du \blacksquare$$

Bonus.

1.

Show that if $f : [a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

Solution.

Fix the $\epsilon > 0$ given.
By continuity of f on $[a,b]$, let U_x be a set such that it is the $\frac{\delta_x}{2}$-neighborhood of x for $x \in [a,b]$, i.e.

$$U_x = \left\{ y : |x-y| < \frac{\delta_x}{2} \right\}$$

satisfying the property that if $x_0 \in U_x$, then

$$|f(x) - f(x_0)| < \epsilon/2$$

With this definition, we can see that

$$[a,b] \subset \bigcup_{x \in [a,b]} U_x$$

As $[a,b]$ is compact (closed and bounded), then there is a finite number of $x_i \in [a,b]$ that the union of their neighborhood contains $[a,b]$.

$$[a,b] \subset \bigcup_{i=1}^{n} U_{x_i}$$

Pick δ as the smallest of $\frac{1}{2}\delta_{x_i}$.
Then, $\forall x, y \in [a,b]$, $x \in U_{x_i}$ for some i. If given that $|x-y| < \delta$, then

$$|x_i - y| \leq |x_i - x| + |x-y| < \frac{1}{2}\delta_{x_i} + \delta \leq \delta_{x_i}$$

So $|x - x_i| < \delta_{x_i}$, $|y - x_i| < \delta_{x_i}$. As f is continuous on $[a,b]$, the above implies that

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(y) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$