Linear Algebra: Homework 4

October 22, 2021

Question 1.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.

Solution 1.

$$AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$$

$$DA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$$

When A is multiplied by D on the right, the i-th column of A is enlarged by a scalar $(D)_{ii}$, while multiplying D on the left yields the enlargement of the i-th row of A by scalar $(D)_{ii}$.
Since A is not singular, one possible matrix B is the inverse of A.

$$[A|B] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 4 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 & -3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

Thus, $B = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ -1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ is possible.

Question 2.

Show that if the columns of B are linearly dependent, then so are the columns of AB.

Solution 2.

B is singular, so $\Rightarrow B\vec{x} = 0$ has non-trivial solutions.
Consider $A(B\vec{x})$,where $\vec{x}$ is the solution of the equation above. Since matrix multiplication is associative, $A(B\vec{x}) = 0 \Rightarrow (AB)\vec{x} = 0$, thus AB is singular.

Question 3.

Suppose that $CA = I_n$. Show that the equation $A\vec{x} = 0$ has only the trivial solution. Explain A not have more columns than rows.

Solution 3.

$\forall \vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = 0$, $C(A\vec{x}) = C \cdot 0 = 0$
Also, $C(A\vec{x}) = (CA)\vec{x} = I_n\vec{x} = 0$, thus $\vec{x} = 0$. Since $\vec{x} = 0$ is the only solution, there should not be any free variable, and every column is pivot column.

Question 4.

Suppose that A is an $m \times n$ matrix and there exists $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$.

Solution 4.

$$\begin{cases} CA = I_n \Rightarrow m \geq n \\ AD = I_m \Rightarrow m \leq n \end{cases} \Rightarrow m = n \text{ }, C = I_nC = (DA)C = D(AC) = DI_m = D$$

Question 5.

Let A be an invertible $n \times n$ matrix, let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

Solution 5.

$X = A^{-1}B$ is a solution, since $AX = A(A^{-1}B) = (AA^{-1})B = I_nB = B$. To prove uniqueness, use left multiplication. Let X_1 below be any solution of $AX = B$. Since $\exists A^{-1}$, thus $A^{-1}(AX_1) = A^{-1}B \Rightarrow (AA^{-1})X_1 = A^{-1}B \Rightarrow I_nX_1 = A^{-1}B \Rightarrow X_1 = A^{-1}B$ is unique.

Question 6.

Let A be an invertible $n \times n$ matrix, let B be an $n \times p$ matrix. Explain why$A^{-1}B$ can be computed by row reduction: If $[A\ B] \sim \cdots \sim [I_n\ X]$, then $X = A^{-1}B$.

Solution 6.

Let $B = [\ \vec{b_1} \quad \vec{b_2} \quad \cdots \quad \vec{b_p}\ ]$, where $\vec{b_1}, \vec{b_2}, \cdots, \vec{b_p} \in \mathbb{R}^n$ are the column vectors of B, and $X = [\ \vec{x_1} \quad \vec{x_2} \quad \cdots \quad \vec{x_p}\ ]$, where $\vec{x_1}, \vec{x_2}, \cdots, \vec{x_p} \in \mathbb{R}^n$ are the column vectors of X. Then $AX = B$ can be decomposed into $A\vec{x_i} = \vec{b_i}, i \in \{1, 2, \cdots, p\}$
Use $[A\ \ \vec{b_i}]$ to solve for $\vec{x_i}$. Since the manipulation of the components of $\vec{b_i}$ depends on A only, and must yield $[I_n\ \ \vec{x_i}]$, this manipulation can be done simultaneously. That is, $[\ A \quad \vec{b_1} \quad \vec{b_2} \quad \cdots \quad \vec{b_p}\ ]$ yields $[\ I_n \quad \vec{x_1} \quad \vec{x_2} \quad \cdots \quad \vec{x_p}\ ] = [I_n \quad X]$. By solution 5, this $X = A^{-1}B$ is the unique solution of $AX = B$.

Question 7.

Use the algorithm of this section to find the inverse of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let A be an $n \times n$ matrix of the same form, find its inverse A^{-1}.

Solution 7.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Thus, the inverses of the matrices are $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

Question 8.

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 \\ 0 & 2 & 0 & & 0 \\ 0 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 2 & 3 & \cdots & n \end{bmatrix}$.

Solution 8.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 & 0 & 1 & 0 & & 0 \\ 1 & 2 & 3 & & 0 & 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & & 0 & -1 & 1 & 0 & & 0 \\ 0 & 2 & 3 & & 0 & -1 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 2 & 3 & \cdots & n & -1 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & & 0 & -1 & 1 & 0 & & 0 \\ 0 & 0 & 3 & & 0 & 0 & -1 & 1 & & 0 \\ \vdots & & \ddots & & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 3 & \cdots & n & 0 & -1 & 0 & \cdots & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & & 0 & -1 & 1 & 0 & & 0 \\ 0 & 0 & 3 & & 0 & 0 & -1 & 1 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 & -1/2 & 1/2 & 0 & & 0 \\ 0 & 0 & 1 & & 0 & 0 & -1/3 & 1/3 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1/n & 1/n \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1/2 & 1/2 & 0 & & 0 \\ 0 & -1/3 & 1/3 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1/n & 1/n \end{bmatrix}$$

Question 9.

Show that if AB is invertible, so is A and B.

Solution 9.

Suppose $K = (AB)^{-1}$. Then, $(AB)K = A(BK) = I_n$. The converse of the proposition in question 2 is: If the columns of AB are independent, then so are the columns of A and B. Since AB is invertible, columns of AB are independent. From solution 2 and solution 3, both A and B are square. Otherwise, either A or B would have more columns than rows, making the columns of AB linearly dependent.Columns of A and B are both independent, they are both invertible.