Topics in Linear Algebra: Homework 11

June 4, 2022

Note: sgn of a permutation is $(-1)^{\{\text{number of inversions of the permutation}\}}$.

Solution 1.11.1.

1.
$$e^1 \otimes e^2(e_1 \otimes e_2) = e^1(e_1)e^2(e_2) = 1$$

2.
$$\mathrm{Alt}(e^1 \otimes e^2)(e_1 \otimes e_2) = \frac{1}{2}(e^1 \otimes e^2 - e^2 \otimes e^1)(e_1 \otimes e_2) = \frac{1}{2}$$

3.
$$e^1 \otimes e^2(\mathrm{Alt}(e_1 \otimes e_2)) = e^1 \otimes e^2(\frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)) = \frac{1}{2}$$

4.
$$\mathrm{Alt}(e^1 \otimes e^2)\mathrm{Alt}(e_1 \otimes e_2) = \frac{1}{4}(e^1 \otimes e^2 - e^2 \otimes e^1)(e_1 \otimes e_2 - e_2 \otimes e_1) = \frac{1}{2}$$

5.
$$e^1 \wedge e^2(\mathrm{Alt}(e_1 \otimes e_2)) = 2\mathrm{Alt}(e^1 \otimes e^2)\mathrm{Alt}(e_1 \otimes e_2) = 1$$

6.
$$e^1 \wedge e^2(e_1 \otimes e_2) = 2\mathrm{Alt}(e^1 \otimes e^2)(e_1 \otimes e_2) = 1$$

7.
$$e^1 \otimes e^2(e_1 \wedge e_2) = 2e^1 \otimes e^2(\mathrm{Alt}(e_1 \otimes e_2)) = 1$$

8.
$$e^1 \wedge e^2(e_1 \wedge e_2) = 4\mathrm{Alt}(e^1 \otimes e^2)\mathrm{Alt}(e_1 \otimes e_2) = 2$$

9.
$$\langle e_1 \otimes e_2, e_1 \otimes e_3 \rangle = \langle e_1, e_1 \rangle \langle e_2, e_3 \rangle = 0$$
$$\langle u_1 \wedge v_1, u_2 \wedge v_2 \rangle = \langle u_1 \otimes v_1 - v_1 \otimes u_1, u_2 \otimes v_2 - v_2 \otimes u_2 \rangle$$
$$= \langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle - \langle u_1 \otimes v_1, v_2 \otimes u_2 \rangle - \langle v_1 \otimes u_1, u_2 \otimes v_2 \rangle + \langle v_1 \otimes u_1, v_2 \otimes u_2 \rangle$$
$$= \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle - \langle u_1, v_2 \rangle \langle v_1, u_2 \rangle - \langle v_1, u_2 \rangle \langle u_1, v_2 \rangle + \langle v_1, v_2 \rangle \langle u_1, u_2 \rangle$$
$$= 2(\langle u_1, u_2 \rangle \langle v_1, v_2 \rangle - \langle u_1, v_2 \rangle \langle v_1, u_2 \rangle)$$
$$\langle e_1 \wedge e_2, e_1 \wedge e_3 \rangle = 2(\langle e_1, e_1 \rangle \langle e_2, e_3 \rangle - \langle e_1, e_3 \rangle \langle e_1, e_2 \rangle) = 0$$

10.
$$\langle e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3 \rangle$$
$$= \sum_{\sigma, \tau \in S_3} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \langle \sigma(e_{123}), \tau(e_{123}) \rangle$$
$$= \sum_{\sigma, \tau \in S_3} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \delta_{il}\delta_{jm}\delta_{ln} e^i_{\sigma(1)} e^j_{\sigma(2)} e^k_{\sigma(3)} e^l_{\tau(1)} e^m_{\tau(2)} e^n_{\tau(3)}$$

Any term in this sum is 1 iff $\sigma = \tau$, and 0 otherwise. So,

$$= \sum_{\sigma, \tau \in S_3} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \langle \sigma(e_{123}), \tau(e_{123}) \rangle$$

$$= \sum_{\sigma \in S_3} \delta_{il}\delta_{jm}\delta_{ln} e^i_{\sigma(1)} e^j_{\sigma(2)} e^k_{\sigma(3)} e^l_{\sigma(1)} e^m_{\sigma(2)} e^n_{\sigma(3)} = 6$$

$$\Rightarrow \|e_1 \wedge e_2 \wedge e_3\| = \sqrt{6}$$

$$\|\mathrm{Alt}(e_1 \otimes e_2 \otimes e_3)\| = \|\frac{1}{3!} e_1 \wedge e_2 \wedge e_3\| = \frac{\sqrt{6}}{6}$$

$$\|\mathrm{Alt}(e_1 \otimes e_2) \wedge e_3\| = \frac{1}{2!}\|e_1 \wedge e_2 \wedge e_3\| = \frac{\sqrt{6}}{2}$$

Solution 1.11.2.

1. In $\bigwedge_3 R^3$, $\forall u, v, w \in R^3$

$$\det(u, v, w) = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) u^{\sigma(1)} v^{\sigma(2)} w^{\sigma(3)}$$

   By definition of Levi-Civita notation,

$$e_{ijk} = \begin{cases} 0, & (i-j)(j-k)(k-i) = 0 \\ \mathrm{sgn}(\sigma), & \text{otherwise} \end{cases}$$

   where for the latter case, in particular, $\sigma(1) = i, \sigma(2) = j, \sigma(3) = k$. That immediately yields

$$\det(u, v, w) = e_{ijk} u^i v^j w^k = e_{ijk} e^i(u) e^j(v) e^k(w) = (e_{ijk} e^i \otimes e^j \otimes e^k)(u, v, w)$$

2. $\forall u, v \in R^2, \alpha, \beta \in (R^2)^*$,

$$e_{ij} e^{mn} u^i v^j \alpha_m \beta_n = e_{ij} u^i v^j e^{mn} \alpha_m \beta_n = \det\left(\begin{bmatrix} \alpha(u) & \alpha(v) \\ \beta(u) & \beta(v) \end{bmatrix}\right) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

$$(\delta_i^m \delta_j^n - \delta_i^n \delta_j^m)(u^i v^j \alpha_m \beta_n) = \delta_i^m u^i \alpha_m \delta_j^n v^j \beta_n - \delta_i^n u^i \beta_n \delta_j^m v^j \alpha_m = \alpha(u)\beta(v) - \beta(u)\alpha(v)$$
$$\Rightarrow \delta_i^m \delta_j^n - \delta_i^n \delta_j^m = e_{ij} e^{mn}$$

3. For a component of $e_{ijk} e^{imn}$ to be nonzero, only $j = m, k = n$ or $j = n, k = m$ is possible.
   For the former case, the parities of the two permutations are the same, which yields 1.
   For the latter case, the parities of the two permutations are different, which yields -1.
   Therefore, $e_{ijk} e^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m$ $\square$.

4. Set $m = j$, then $e_{ijk} e^{ijn} = \delta_j^j \delta_k^n - \delta_j^n \delta_k^j = 3\delta_k^n - \delta_k^n = 2\delta_k^n$
   Set $n = k$, then $e_{ijk} e^{ijk} = 2\delta_k^k = 6$

Solution 1.11.3.

1. $\tau^2 = \frac{1}{2}(e_{1122} - e_{1221} - e_{2112} + e_{2211})$,

$$\langle a \otimes b \otimes c \otimes d, e \otimes f \otimes g \otimes h \rangle = \langle a \otimes b, e \otimes f \rangle \langle c \otimes d, g \otimes h \rangle = \langle a, e \rangle \langle b, f \rangle \langle c, g \rangle \langle d, h \rangle$$

   So expansion of the inner product by enumerating all combinations (through distributing properties) ignores non-identical terms.

$$\langle \tau^2, \tau^2 \rangle = \frac{1}{4}\left(\langle e_{1122}, e_{1122} \rangle + \langle e_{1221}, e_{1221} \rangle + \langle e_{2112}, e_{2112} \rangle + \langle e_{2211}, e_{2211} \rangle\right) = 1 \Rightarrow \|\tau^2\| = 1$$

   Suppose $\tau = u \otimes v$. Then W.L.O.G. let $u = u^1 e_1 + u^2 e_2, v = v^1 e_1 + v^2 e_2$.

$$\tau = u \otimes v = (u^1 e_1 + u^2 e_2) \otimes (v^1 e_1 + v^2 e_2) = u^1 v^1 e_1 \otimes e_1 + u^1 v^2 e_1 \otimes e_2 + u^2 v^1 e_2 \otimes e_1 + u^2 v^2 e_2 \otimes e_2$$

   Hence, $u^1 v^1 = u^2 v^2 = 0, u^1 v^2 = \frac{1}{\sqrt{2}}, u^2 v^1 = -\frac{1}{\sqrt{2}}$, and $0 = u^1 v^1 u^2 v^2 = -\frac{1}{2}$, which is impossible. $\square$

2. Trivially its matrix representation is $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, so $M_A = e_1^T \otimes e_1^T - e_2^T \otimes e_2^T$.

3. Its matrix representation is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$,
   $N_A = e_1^T \otimes e_2^T + e_2^T \otimes e_1^T$.

4. Matrix multiplication is associative, and is distributive over addition, so the matrix representation of the sum of the two bilinear map is just the sum of the matrix:
   For M_B, its representation is
   $$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
   That matrix has determinant -1 and trace 0, so ±1 are the eigenvalues.
   For N_B, its representation is
   $$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$
   That matrix has determinant -1 and trace 0, so ±1 are the eigenvalues.

5.
$$M_A \otimes M_B + M_A \otimes N_B + N_A \otimes M_B - N_A \otimes N_B$$
$$= \frac{1}{\sqrt{2}}(M_A \otimes M_A + M_A \otimes N_A + M_A \otimes M_A - M_A \otimes N_A + N_A \otimes M_A + N_A \otimes N_A - N_A \otimes M_A + N_A \otimes N_A)$$
$$= \sqrt{2}(M_A \otimes M_A + N_A \otimes N_A) = \sqrt{2}((e^{11} - e^{22}) \otimes (e^{11} - e^{22}) + (e^{12} + e^{21}) \otimes (e^{12} + e^{21}))$$
$$= \sqrt{2}(e^{1111} - e^{1122} - e^{2211} + e^{2222} + e^{1212} + e^{1221} + e^{2112} + e^{2121}),$$
$$(M_A \otimes M_B + M_A \otimes N_B + N_A \otimes M_B - N_A \otimes N_B)(\tau^2)$$
$$= \frac{\sqrt{2}}{2}(-e^{1122}(e_{1122}) + e^{1221}(-e_{1221}) + e^{2112}(-e_{2112}) - e^{2211}(e_{2211})) = -2\sqrt{2}$$

6. Let $v = \cos\theta e_1 + \sin\theta e_2$, $w = \cos\phi e_1 + \sin\phi e_2$.
$$v \otimes v = \cos^2\theta e_{11} + \cos\theta \sin\theta(e_{12} + e_{21}) + \sin^2\theta e_{22}$$
$$v \otimes v \otimes w \otimes w = \cos^2\theta \cos^2\phi e_{1111} + \cos^2\theta \sin^2\phi e_{1122}$$
$$+ \cos\theta \sin\theta \cos\phi \sin\phi(e_{1212} + e_{1221} + e_{2112} + e_{2121}) + \sin^2\theta \cos^2\phi e_{2211} + \sin^2\theta \sin^2\phi e_{2222} + \cdots$$
   The 8 other terms are neglected as they are irrelevant in calculation below.
$$(M_A \otimes M_B + M_A \otimes N_B + N_A \otimes M_B - N_A \otimes N_B)(v \otimes v \otimes w \otimes w)$$
$$= \sqrt{2}(\cos^2\theta \cos^2\phi - \cos^2\theta \sin^2\phi + 4\cos\theta \sin\theta \cos\phi \sin\phi - \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi)$$
$$= \sqrt{2}((\cos^2\theta - \sin^2\theta)(\cos^2\phi - \sin^2\phi) + \sin(2\theta)\sin(2\phi))$$
$$= \sqrt{2}(\cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi)) = \sqrt{2}\cos(2\theta - 2\phi),$$
   It is trivial that $-\sqrt{2} \le \sqrt{2}\cos(2\theta - 2\phi) \le \sqrt{2}$. ■