Calculus A(1): Homework 5

November 19, 2021

2.6

39.

For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

Solution

Polynomials $P(x)$ are continuous at every x. Thus, it does only require $f(x)$ to be continuous at $x = 3$ for f to be continuous in $\mathbb{R}$.

$$\lim_{x \to 3} f(x) = f(3)$$

$$\Leftrightarrow \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

Given $f(x) = x^2 - 3, x < 3$, thus

$$\lim_{x \to 3^-} f(x) = 3^2 - 1 = 8 = f(3) = 2a(3)$$

Hence $a = 4/3$.

46.

Explain why the equation $\cos x = x$ has at least one solution.

Solution

Let $y = f(x) = \cos x - x$, and consider $x \in [0, \frac{\pi}{2}]$.
Clearly, f is continuous over $[0, \frac{\pi}{2}]$, and $f(0) = 1, f(\frac{\pi}{2}) = -\frac{\pi}{2}$.
$f(0) > 0 > f(\frac{\pi}{2})$, so by intermediate value theorem, $\exists x_0 \in (0, \frac{\pi}{2})$ such that $f(x_0) = 0.\blacksquare$

59.

A fixed point theorem Suppose that a function f is continuous on the closed interval $[0,1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0,1]$. Show that there must exist a number c in $[0,1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

Solution

$a := f(0)$, $b := f(1)$
If $a = 0$, then $c = 0$ is a possible solution. If $b = 1$, then c=1 is a possible solution.
Else, we have $0 < a, b < 1$
$g(x) := f(x) - x$. Then $g(0) = a > 0$ and $g(1) = b - 1 < 0$, thus $g(0) > 0 > g(1)$. By intermediate value theorem, $\exists c \in (0,1)$ such that $g(c) = 0.\blacksquare$

60.

The sign-preserving property of continuous functions Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$. Notice how remarkable this conclusion is. Although f is defined throughout (a, b), it is not required to be continuous at any point except c. That and the condition $f(c) \neq 0$ are enough to make f different from zero (positive or negative) throughout an entire interval.

Solution

f is continuous at $x = c$, i.e.

$$(\forall \epsilon > 0)(\exists \delta > 0)(|x - c| < \delta \to |f(x) - f(c)| < \epsilon)$$

Choose $\epsilon = \frac{|f(c)|}{2}$. Then we have

$$-\frac{|f(c)|}{2} + f(c) < f(x) < \frac{|f(c)|}{2} + f(c)$$

1. $f(c) < 0$. Then $\frac{3}{2} f(c) < f(x) < \frac{1}{2} f(c) < 0$.
2. $f(c) > 0$. Then $0 < \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c)$.

■

2.7

33.

Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

has a vertical tangent at the origin? Give reasons for your answer.

Solution

With the given function f, we have

$$\lim_{\Delta x \to 0^-} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-1}{\Delta x} = +\infty, \lim_{\Delta x \to 0^+} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{1}{\Delta x} = +\infty$$

Hence

$$\lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = +\infty,$$

the graph of $f(x)$ has a vertical tangent at $x = 0$, and it is the y-axis.

43.

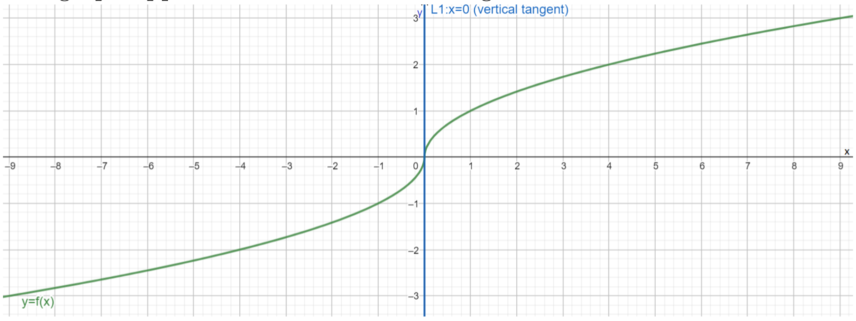
$$y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$$

(a) Graph the curve. Where does the graph appear to have vertical tangents?

(b) Confirm your findings in part(a) with limit calculations. But before you do, read the introduction to Exercise 33 and 34.

Solution

(a) The graph appears to have a vertical tangent at $x = 0$.



(b) (i)

$$\lim_{\Delta x \to 0^-} \frac{y|_{x=\Delta x} - y|_{x=0}}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\sqrt{|\Delta x|}}{\Delta x} = \lim_{\Delta x \to 0^-} \sqrt{\frac{-\Delta x}{(-\Delta x)^2}} = +\infty$$

(ii)

$$\lim_{\Delta x \to 0^+} \frac{y|_{x=\Delta x} - y|_{x=0}}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\sqrt{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0^+} \sqrt{\frac{\Delta x}{(\Delta x)^2}} = +\infty$$

(i)$\wedge$(ii) $\Rightarrow \lim_{\Delta x \to 0} \frac{y|_{x=\Delta x} - y|_{x=0}}{\Delta x} = +\infty \Rightarrow$ The graph has a vertical tangent at $x = 0$.

2. Additional and Advanced Exercises

19.

Antipodal points Is there any reason to believe that there is always a pair of antipodal (diametrically oppositve) points on Earth's equator where the temperatures are the same? Explain.

Solution

Surface temperature of the Earth can be considered as a scalar field on a sphere.
Assume $\vec{r}$ and $-\vec{r}$ as a pair of antipodal points on the Earth, where $\vec{0}$ is the positional center of earth.
Define f as the "directed" temperature difference between the antipodal points, i.e. $f(\vec{r}) = T(\vec{r}) - T(-\vec{r})$, where $T(\vec{r})$ is the surface temperature of position $\vec{r}$.
Let $d = f(\vec{r_0})$, where r_0 is arbitary position on Earth. Then for its antipodal point, $f(-\vec{r}) = -d$
As temperature is continuous on every position $\vec{r}$, the range of f includes positive and negative real numbers. By intermediate theorem, $\exists \vec{r_1}$ such that $f(\vec{r_1}) = 0$.■

18.

The Dirichlet ruler function If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n has no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, 6/4 wirtten in lowest terms is 3/2.) Let $f(x)$ be defined for all x in the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1/n, & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

a. Show that f is discontinuous at every rational number in $[0, 1]$.

b. Show that f is continuous at every irrational number in $[0, 1]$. (Hint: If ϵ is a given positve number, show that there are only finitely many rational numbes r in $[0, 1]$ such that $f(r) \geq \epsilon$.)

c. Sketch the graph of f. Why do you think f is called the "ruler function"?

Solution

a. Proof:

Assume $\exists q \in \mathbb{Q}$ such that f is continuous at $x = q$. Let $q = m/n$. Then

$$(\forall \epsilon > 0)(\exists \delta > 0)(|x - q| < \delta \rightarrow |f(x) - f(q)| < \epsilon)$$

Choose $\epsilon = 1/(2n)$, and x be a irrational number in the δ−neighborhood of q. Then,

$$|f(x) - f(q)| = |\frac{1}{n}| > \frac{1}{2n} = \epsilon$$

Hence such q does not exist.

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b. Proof:

Let $x \in \mathbb{R} \setminus \mathbb{Q}$.
$(\forall \epsilon > 0)(\exists N \in \mathbb{N}^*)(\frac{1}{N} < \epsilon)$.
Let $M = \{q \in \mathbb{Q} : f(q) > \epsilon\}$, where N is the smallest positive integer such that $\frac{1}{N} < \epsilon$

$$|M| = 1 + \sum_{i=2}^{N} \varphi(i),$$

where $\varphi(n)$ is Euler's totient function.(number of positive integers less than n and coprime with n)
Let $q_i \in M (1 \leq i \leq |M|)$.
Let $\delta = \min\{|x - q_i| : q_i \in M\}$ Hence, $\neg(\exists q \in \mathbb{Q})(|x - q| < \delta \wedge |f(q)| \geq \epsilon)$

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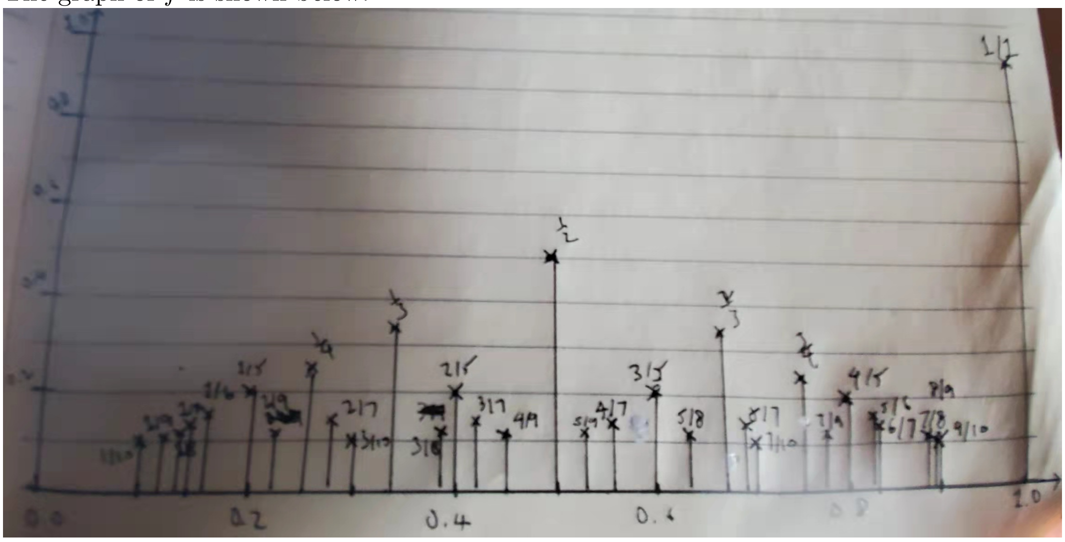
c. The graph of f is shown below.



The graph looks like the markings on a ruler, when line segments with ends of $(x, 0)$ and $(x, f(x))$ are constructed.