Topics in Linear Algebra: Homework 7

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Solution 1.7.1.

1. To show $(\cdot,\cdot)$ is inner product, bilinearity, symmetry and positive definitiveness of it should be shown. It is trivial that $A \in M_n(R)$.

   i) Bilinearity:
   $\forall u, u_1, u_2, v, v_1, v_2 \in R^n,\ a, b \in R,$

   $$(au_1 + bu_2, v) = (au_1 + bu_2)^T Av = au_1^T Av + bu_2^T Av = a(u_1, v) + b(u_2, v)$$

   $$(u, av_1 + bv_2) = u^T A(av_1 + bv_2) = a(u^T Av_1) + b(u^T Av_2) = a(u, v_1) + b(u, v_2)$$

   ii) Symmetry:

   $$(u, v) = u^T Av = (u^T Av)^T = v^T A^T u = v^T Au = (v, u)$$

   iii) Positive definitiveness:
   By definition, A is positive definite, therefore $\forall x \in R^n$,

   $$x^T Ax \geq 0$$

   In addition, $x^T Ax = 0 \Leftrightarrow x = 0$.

2. Suppose u is the pre-image of v^T of the bra map. Then,

   $$v^T = u^T A \Leftrightarrow v^T A^{-1} = u^T \Leftrightarrow u = (A^{-1})^T v$$

   So the Riesz map is

   $$v^T \mapsto A^{-1}v$$

3. Immediately from (2),

   $$v \mapsto v^T A$$

4. Riesz is $V^* \to V$, so dual of Riesz is $V^* \to V^{**}$, i.e. v^T would be sent to an image such that the image is a linear functional: $V^* \to R$.

   $$[\text{Riesz}^*(\alpha)](\beta)] = \alpha \circ \text{Riesz}(\beta) = \beta A^{-1}\alpha^T$$

   For $\alpha, \beta \in V^*$. So the dual will send $\alpha \in V^*$ in the following way:

   $$\alpha \mapsto (\beta \mapsto \beta A^{-1}\alpha^T)$$

Solution 1.7.2.

1. v is a linear functional, hence showing v sends constant to zero requires v sending one to zero. Let $f(x) = a, g(x) = b$, then $v(fg) = abv(1)$, so we may let $f(x) \equiv g(x) \equiv 1$,

   $$v(1) = f(p)v(g) + v(f)g(p) = v(1) + v(1)$$

   The only possible solution for the above equation is $v(1) = 0$.

2. Let $x_1 = x, x_2 = y, x_3 = z$, then

   $$v((x_i - p_i)f) = (x_i - p_i)(\mathbf{p})v(f) + v(x_i - p_i)f(\mathbf{p}) = (p_i - p_i)v(f) + v(x_i)f(\mathbf{p}) - v(p_i)f(\mathbf{p}) = v(x_i)f(\mathbf{p})$$

   $(i = 1, 2, 3)$

3. By applying Leibniz's formula repeatedly,
$$v(fgh) = v(f)g(\mathbf{p})h(\mathbf{p}) + f(\mathbf{p})v(g)h(\mathbf{p}) + f(\mathbf{p})g(\mathbf{p})v(h)$$
a, b, c are non-negative integers, and $a + b + c > 1$, so W.L.O.G. let
$$(w_1 - p_i)(w_2 - p_j)Q = (x - p_1)^a(y - p_2)^b(z - p_3)^c,$$
where Q is a function of x, y, z and the two factors prior to it comes from the factor of the non-zero powers of $(x - p_1)^a(y - p_2)^b(z - p_3)^c$.
For instance, when $a = 2$, then $w_1 = w_2 = x, p_i = p_j = p_1$, when $a = b = 1$, $w_1 = x, w_2 = y, p_i = p_1, p_j = p_2$. Then
$$v((x - p_1)^a(y - p_2)^b(z - p_3)^c) = v((w_1 - p_i)(w_2 - p_j)Q)$$
$$= v(w_1 - p_i)(w_2 - p_j)(\mathbf{p})Q(\mathbf{p}) + (w_1 - p_i)(\mathbf{p})v(w_2 - p_j)Q(\mathbf{p}) + (w_1 - p_i)(\mathbf{p})(w_2 - p_j)(\mathbf{p})v(Q)$$
By setting $f \equiv 1$ in (2), the last expression immediately yields zero.
4. Taylor expansion of f at p is
$$f(x, y, z) = \sum_{n=0}^{+\infty} \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} \left.\frac{\partial^{(i+j+k)} f}{\partial x^i \partial y^j \partial z^k}\right|_{\mathbf{p}} (x - p_1)^i(y - p_2)^j(z - p_3)^k$$
Since applying v to constant yields zero, so does those polynomial with degree ≥ 2, therefore
$$v(f) = v(f'_x(\mathbf{p})(x - p_1)) + v(f'_y(\mathbf{p})(y - p_2)) + v(f'_z(\mathbf{p})(z - p_3))$$
$$= f'_x(\mathbf{p})v(x) + f'_y(\mathbf{p})v(y) + f'_z(\mathbf{p})v(z)$$
5. It is proved in solution 1.6.3. that, for any differentiable f (which is done by generalizing the proof to higher dimension), $\nabla_u(f) = \nabla f \cdot u$, so for any $v = v(x)\hat{\mathbf{i}} + v(y)\hat{\mathbf{j}} + v(z)\hat{\mathbf{k}}$, and for any analytic f,
$$\nabla_v(f) = \nabla f \cdot v = f'_x(\mathbf{p})v(x) + f'_y(\mathbf{p})v(y) + f'_z(\mathbf{p})v(z) = v(f)$$
where f is arbitrary analytic function, so $\nabla_v = v$. ■

Solution 1.7.3.

1. $$X(fg) = X(f)g + fX(g),$$
So $\forall p : p \in M$,
$$X(fg)(p) = X(f)(p) \cdot g(p) + f(p) \cdot X(g)(p)$$
$$\Leftrightarrow X_p(fg) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g)$$
The last equation satisfies Leibniz's rule. □
2. $\forall p \in M$,
$$[X(f)](p) = X_p(f) = \nabla f|_p(X_p) = df|_p(X_p) = [df(X)](p)$$
Hence, $X(f) = df(X)$. ■
3. Let f and g be two analytic functions, then the following should be proved:
$$(X \circ Y - Y \circ X)(fg) = [(X \circ Y)(f)]g + f[(X \circ Y - Y \circ X)(g)]$$
L.H.S.
$$= (X \circ Y - Y \circ X)(fg) = (X \circ Y)(fg) - (Y \circ X)(fg)$$
$$= X(Y(f)g + fY(g)) - Y(X(f)g + fX(g)) = X(Y(f)g) + X(fY(g)) - Y(X(f)g) - Y(fX(g))$$
$$= X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y(g)) - Y(X(f))g - X(f)Y(g) - Y(f)X(g) - fY(X(g))$$
$$= X(Y(f))g + fX(Y(g)) - Y(X(f))g - fY(X(g))$$
R.H.S.
$$= [(X \circ Y - Y \circ X)(f)]g + f[(X \circ Y - Y \circ X)(g)]$$
$$= X(Y(f))g - Y(X(f))g + fX(Y(g)) - fY(X(g))$$
Therefore L.H.S. = R.H.S. ■
4. A and B are both skew-symmetric, so
$$(AB-BA)^T = (AB)^T-(BA)^T = B^TA^T-A^TB^T = (-B)(-A)-(-A)(-B) = BA-AB = -(AB-BA)$$ ■