NAME :

STUDENT ID:

NOTE: There are 3 Parts to this FINAL EXAM (total of 7 pages). For Part 1 (multiple choice) be sure to indicate your answers clearly as no partial credit will be awarded. Each question has a unique right answer.

Part 3 consists of a bonus exercise, which is not compulsory, and can give you some extra points. It is advised to finish all the compulsory questions before attempting the bonus question. If your total is > 100 points, your assigned grade will be 100.

In order to receive full credit for Parts 2 and 3, you must show work to explain your reasoning. If you require additional sheets for Parts 2 or 3, be sure to put your name and ID to each additional page that you turn in with this exam. Use of calculators, computers, etc. will NOT be permitted. You have 120 minutes complete this test.

Part 1 (30pts)

1. Let $y : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be the unique differentiable function such that $y(0) = 0$ and for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $y'(x) = \sin(x) \cdot \exp(2\cos(x)) + \tan(x)^2$. What is $y(\frac{\pi}{4})$?

   (A) $e^{\sqrt{2}} - e^2 + \frac{\pi}{4}$

   (B) $\frac{e^{\sqrt{2}}}{2} + 1 - \frac{\pi}{4} - \frac{e^2}{2}$

   (C) $-\frac{e^{\sqrt{2}}}{2} + 1 - \frac{\pi}{4} + \frac{e^2}{2}$

   (D) 0

   (E) None of the above

2. Let $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ be defined by $f(x) = (\sin(x))^2$. We denote as usual by f^{-1} the inverse function of f. What is $(f^{-1})'(\frac{1}{2})$?

   (A) 1

   (B) $\frac{1}{2\cos(\frac{1}{2})\sin(\frac{1}{2})}$

   (C) 0

   (D) $-\sqrt{2}$

   (E) None of the above.

3. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1-x}$. What is $f^{(4)}(\frac{1}{2})$? ($f^{(4)}$ is the 4th derivative of f.)

   (A) -756

   (B) 768

   (C) 96

   (D) -32

   (E) None of the above.

4. What is the value of

$$\int_1^{e^2} \frac{(\ln(x))^{\frac{1}{3}}}{x} dx \ ?$$

   (A) $3 \cdot 2^{\frac{1}{3}}$

   (B) $2^{-\frac{2}{3}}$

   (C) $3 \cdot 2^{-\frac{2}{3}}$

   (D) $3 \cdot 2^{-\frac{1}{3}}$

   (E) None of the above.

5. What is the value of $\lim_{x \to 2} \frac{x^3-8}{x^4-16}$?

(A) The limit does not exist.

(B) 0

(C) ∞

(D) $\frac{3}{8}$

(E) None of the above.

Part 2a. (15pts) Find anti-derivatives for the functions $f, g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $f(x) = \tan(x)$ and $g(x) = \frac{\exp(\tan(x))}{\cos(x)^2}$.

Part 2b. (30 pts) Sketch the graph of the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ given by $f(x) = \frac{x^2-1}{x^3}$. Sketching the graph includes finding the local and global extrema, inflection points, horizontal or vertical asymptotes, intervals on which f is concave up or down, and intervals on which f is increasing or decreasing. Please justify carefully your answer.

Part 2c. (25pts) Let $f(x)$ be a degree 4 polynomial, i.e. $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ where the coefficients a_i (for $0 \leq i \leq 4$) are in $\mathbb{R}$ and $a_4 \neq 0$. Show that f has either exactly 2 points of inflection or f has no points of inflection.

Part 3 (BONUS QUESTION).
Prove that $\sum_{k=1}^{2021} \frac{1}{k^2} < 2$. (One can actually show that the infinite sum $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ is equal to $\frac{\pi^2}{6}$, but you are not allowed to use that result).
Hint: use Riemann sums.