Q1

06:12

(1) True. 此时$\exists$可逆的P, s.t. $A=PB$, 有 $rk(A)=rk(PB)=rk(B)$

(2) False. 取 $A=I_n, B=2I_n$, $\det(A)=1\neq 2^n=\det(B)$

(3) False. 取 $A=\begin{pmatrix}2&\\&1\end{pmatrix}$, $B=\begin{pmatrix}1&1\\&1\end{pmatrix}$, $AB=\begin{pmatrix}2&2\\&1\end{pmatrix}\neq\begin{pmatrix}2&1\\&1\end{pmatrix}=BA$

但 $\det(AB)=\det(BA)=2$

(4) True. 反证, 若$\exists P$, s.t. $P^{-1}AP=D_1$, $P^{-1}BP=D_2$, D_1, D_2 为对角阵.

则 $AB=PD_1P^{-1}PD_2P^{-1}=PD_1D_2P^{-1}=PD_2D_1P^{-1}=BA$

(5) False. 取 $A=I_n$, $B=\begin{pmatrix}0&I_{n-1}\\0&\end{pmatrix}$, $AB=BA$. 但B不可对角化.

Q2

11:32

(1)

(a) 若 $\begin{pmatrix}2&1\\&2\end{pmatrix}\sim\begin{pmatrix}\lambda_1&\\&\lambda_2\end{pmatrix}\xRightarrow[\text{相同}]{\text{特征值}}\begin{pmatrix}2&1\\&2\end{pmatrix}\sim\begin{pmatrix}2&\\&2\end{pmatrix}$

$\Rightarrow\begin{pmatrix}0&1\\&0\end{pmatrix}\sim\begin{pmatrix}0&\\&0\end{pmatrix}$. 矛盾! 故不可

(b) $\begin{vmatrix}\lambda-2&-1\\-1&\lambda-2\end{vmatrix}=(\lambda-2)^2-1=(\lambda-1)(\lambda-3)$, 有互异特征值.

故可　或使用对称矩阵必可正交对角化.

(c) $\begin{vmatrix}\lambda-2&-1\\1&\lambda-2\end{vmatrix}=(\lambda-2)^2+1$, 在$\mathbb{R}$上无解 在$\mathbb{C}$上有互异解

故在$\mathbb{R}$上不可对角化, 在$\mathbb{C}$上可以.

(2)

可使用PPT第87页定理, 计算$\det(A)$及a_{11}正负性

(a) $Q_a(x)=2x_1^2+2x_1x_2+x_2^2=(x_1+x_2)^2+x_1^2+x_2^2\geq 0$

且 $Q_a(x)=0\Leftrightarrow x_1=x_2=x_1+x_2=0$. 故$Q_a(x)$正定

(b) $Q_b(x)=x_1^2+4x_1x_2+x_2^2$, 取 $x_1=1, x_2=-1$. $Q(x)=-2<0$

$x_1=x_2=1$, $Q(x)=6>0$.

故$Q_b(x)$不定

(c) $Q_c(x)=-Q_a(x)$, 故 $Q_c(x)$负定.

(d) $Q_d(x)=-Q_b(x)$, 故 $Q_b(x)$不定.

Q3

16:08

(1) $\det(A)=\begin{vmatrix}1&-1&\\-1&2&-1\\&-1&2\end{vmatrix}=\begin{vmatrix}1&-1&\\0&1&-1\\&-1&2\end{vmatrix}=\begin{vmatrix}1&-1&\\0&1&-1\\&0&1\end{vmatrix}=1$

$$\det(B)=\begin{vmatrix}2&-1&\\-1&2&-1\\&-1&2\end{vmatrix}=\begin{vmatrix}2&-1&\\0&\frac{3}{2}&-1\\&-1&2\end{vmatrix}=\begin{vmatrix}2&-1&\\0&\frac{3}{2}&-1\\&0&\frac{4}{3}\end{vmatrix}=4$$

(2) $$\det(A_n)=\begin{vmatrix}1&-1&&&\\-1&2&&&\\&-1&\ddots&&\\&&\ddots&\ddots&-1\\&&&-1&2\end{vmatrix}_{n\times n}=\begin{vmatrix}1&-1&&&\\0&1&-1&&\\&-1&2&\ddots&\\&&\ddots&\ddots&-1\\&&&-1&2\end{vmatrix}_{n\times n}=\det(A_{n-1})$$

(3) 第一列 $\begin{pmatrix}2\\-1\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\0\\\vdots\\0\end{pmatrix}+\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$.

$$\det(B_n)=\begin{vmatrix}1&-1&&&\\&2&-1&&\\&-1&\ddots&\ddots&\\&&\ddots&\ddots&-1\\&&&-1&2\end{vmatrix}+\begin{vmatrix}1&-1&&&\\-1&2&\ddots&&\\&\ddots&\ddots&\ddots&\\&&\ddots&\ddots&-1\\&&&-1&2\end{vmatrix}=\det(B_{n-1})+\det(A_{n-1})$$

$$=\det(B_{n-1})+1$$

$\Rightarrow \det(B_n)=n+1$ （由 $\det(B_3)=4$）.

Q4 23:43

(1) $$|A-\lambda I|=\begin{vmatrix}-\lambda&&1\\1&-\lambda&\\&1&-\lambda\end{vmatrix}=-\lambda^3+1$$

$\Rightarrow \lambda_1=1,\ \lambda_2=w,\ \lambda_3=w^2,\quad w=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad i^2=-1$

解 $(A-I)\gamma_1=0 \Rightarrow \gamma_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ 令 $T=\begin{pmatrix}1&1&1\\1&w^2&w\\1&w&w^2\end{pmatrix}$

$(A-wI)\gamma_2=0 \Rightarrow \gamma_2=\begin{pmatrix}1\\w^2\\w\end{pmatrix}$ $T^{-1}AT=\begin{pmatrix}1&&\\&w&\\&&w^2\end{pmatrix}$

$(A-w^2I)\gamma_3=0 \Rightarrow \gamma_3=\begin{pmatrix}1\\w\\w^2\end{pmatrix}$

(2) 取 $V_1=\mathrm{Re}\,\gamma_2=\begin{pmatrix}1\\-\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}$. $V_2=\mathrm{Im}\,\gamma_2=\begin{pmatrix}0\\-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}$, $V_3=\gamma_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$.

$P=[V_1\ V_2\ V_3]$

注意 若 $Ax=\lambda x$, $\lambda\in\mathbb{C}$, $x\in\mathbb{C}^n$, $A\in\mathbb{R}^{n\times n}$

则 $A(\mathrm{Re}\,x, \mathrm{Im}\,x) = (\mathrm{Re}\,x, \mathrm{Im}\,x)\begin{pmatrix}\mathrm{Re}\,\lambda, & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda, & \mathrm{Re}\,\lambda\end{pmatrix}$

$$\Rightarrow A\begin{pmatrix}1 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2} & \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \\ & & 1\end{pmatrix}$$

故 P满足要求

(3) A是以 $\begin{pmatrix}1\\1\\1\end{pmatrix}$ 为轴，角度为 $\frac{2\pi}{3}$ 的旋转.

Q5

35:49

(1) $\|v_1\| = \sqrt{1^2+1^2} = \sqrt{2}$, 类似 $\|v_2\| = \|v_3\| = \sqrt{2}$

$\langle v_1, v_2\rangle = \arccos\frac{v_1\cdot v_2}{\|v_1\|\cdot\|v_2\|} = \arccos\frac{1}{2} = \frac{\pi}{3}$

类似, $\langle v_1, v_3\rangle = \langle v_2, v_3\rangle = \frac{\pi}{3}$

(2) 记 $A_1 = \begin{pmatrix}1 & 0 \\ 1 & 1 \\ 0 & 1\end{pmatrix}$, $A_1^T A_1 = \begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 1 & 1 \\ 0 & 1\end{pmatrix} = \begin{pmatrix}2 & 1 \\ 1 & 2\end{pmatrix}$

$(A_1^T A_1)^{-1} = \begin{pmatrix}\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$.

计算 $A_1(A_1^T A_1)^{-1}A_1^T v_3 = \begin{pmatrix}1 & 0 \\ 1 & 1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}$

$= \begin{pmatrix}\frac{2}{3}, & -\frac{1}{3} \\ \frac{1}{3}, & \frac{1}{3} \\ -\frac{1}{3}, & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}$

$= \begin{pmatrix}\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{pmatrix}$

另解: $V_2' = V_2 - \frac{V_2 \cdot V_1}{\|V_1\| \cdot \|V_1\|} \cdot V_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$

$\|V_2'\| = \sqrt{1+\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{6}}{2}$

$Proj_W(V_3) = \frac{V_3 \cdot V_1}{\|V_1\|^2} \cdot V_1 + \frac{V_3 \cdot V_2'}{\|V_2'\|^2} \cdot V_2'$

$= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{\frac{1}{2}}{\frac{3}{2}}\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

$dist(V_3, W) = \|V_3 - Proj_W(V_3)\| = \sqrt{(\frac{2}{3})^2 \times 3} = 2 \cdot \frac{\sqrt{3}}{3}$

(3) $V_1 \times V_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1, -1, 1)^T$

$S_P = \|V_1 \times V_2\| = \sqrt{3}$

$S_\Sigma = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} = 2.$

Q6 (1) $B = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

45:36

$|B - \lambda I| = \lambda^2 - 10\lambda + 16 = (\lambda-2)(\lambda-8) \Rightarrow \lambda_1 = 2, \lambda_2 = 8$

$(B-2I)x_1 = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. 正交规范化, 得

$(B-8I)x_2 = 0 \Rightarrow x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $V_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $V_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$B(V_1, V_2) = (V_1, V_2)\begin{pmatrix} 2 & \\ & 8 \end{pmatrix}$

$\Rightarrow B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & \\ & 8 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(2) $\sigma_1=\sqrt{2}$, $\sigma_2=\sqrt{8}=2\sqrt{2}$

$$u_1=\frac{1}{\sigma_1}Av_1=\frac{1}{\sqrt{2}}\begin{pmatrix}2&2\\-1&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$u_2=\frac{1}{\sigma_2}Av_2=\frac{1}{2\sqrt{2}}\begin{pmatrix}2&2\\-1&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\sqrt{2}&\\&2\sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}},&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}},&\frac{1}{\sqrt{2}}\end{pmatrix}$$

(3) 令 $\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$

$$Q(y)=2y_1^2+8y_2^2=\frac{y_1^2}{(\frac{1}{\sqrt{2}})^2}+\frac{y_2^2}{(\frac{1}{2\sqrt{2}})^2}$$

$$a=\frac{1}{\sqrt{2}},\ b=\frac{1}{2\sqrt{2}},\ S=\pi ab=\pi\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2\sqrt{2}}=\frac{\pi}{4}$$

(4) 由于A的最小奇异值为$\sqrt{2}$，对应 A^TA的特征向量 $\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$

故在 $\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$ 取得最小值 $m=\sqrt{2}$

类似，在 $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$ 取得最大值 $M=\sqrt{8}$.

Q7

52:29

(1) $Q(x)=(Ax)^TAx\geq 0$, $Q(x)=0 \Leftrightarrow Ax=0$

若A有线性无关的列，则 $Ax=0 \Leftrightarrow x=0$，即$Q(x)$正定

若否，则取x_0为$Ax=0$非零解，则 $Q(x_0)=0$，$Q(x)$不是正定的

(2) 设 $x\neq 0$，满足 $A^TAx=\lambda x$，则 $(A^TA)^TA^TAx=(A^TA)^2x=\lambda^2x$.

设λ_1为A的最大奇异值，x_1满足 $A^TAx_1=\lambda_1x_1$,

则 $M_1=\lambda_1$ 在 x_1 处取到. $M_2=\lambda_1^2$ 在 x_1 处取到.

$\Rightarrow M_2=M_1^2$.

对极小值也有类似性质

(3) 设 A 有奇异值分解, $A=U\Sigma V^T$. U,V 正交, Σ 对角

$A^tA=V\Sigma^2V \sim \Sigma^2$, $AA^t=U\Sigma^2U^T \sim \Sigma^2$

故 $A^tA\sim AA^t$