Calculus A(1): Homework 7

December 5, 2021

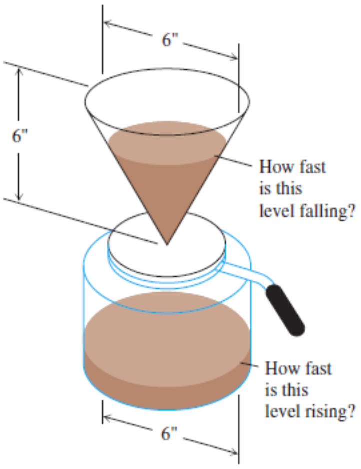
3.7.

24.

Making coffee Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.

a. How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

b. How fast is the level in the cone falling then?



Solution.

Denote $V(t)$ as volume of coffee in the pot, $h_1(t)$ as the height of level in the pot measured from the bottom, $h_2(t)$ as the height of level int the cone measured from the bottom of the cone.

a. Consider the pot.

$$\frac{dV(t)}{dt} = \frac{9\pi h_1(t)}{dt} = 9\pi\frac{dh_1(t)}{dt} = 10$$

Hence, the level is rising at $\frac{10}{9\pi}$ in/min when the coffee in the cone is 5 in deep.

b. Consider the cone.

$$\frac{dV(t)}{dt} = \frac{d(\pi(h_2(t)/2)^2 \cdot h_2(t)/3)}{dt} = \frac{\pi}{12}\frac{d(h_2(t)^3)}{dt} = \frac{\pi}{4}h_2(t)^2\frac{dh_2(t)}{dt} = -10$$

$$\Rightarrow \left.\frac{dh_2(t)}{dt}\right|_{h_2(t)=5} = -\frac{40}{25\pi} = -\frac{8}{5\pi}$$

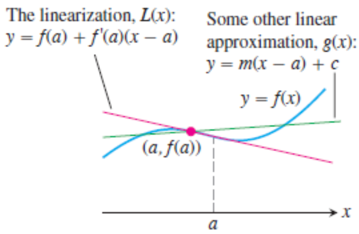
3.8.

61.

The linearization is the best linear approximation (This is why we use the linearization.) Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x-a)+c$ is a linear function in which m and c are constants. If the error $E(x) = f(x)-g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of linearization $L(x) = f(a) + f'(a)(x-a)$. Show that if we impose on g the conditions

1. $E(a) = 0$
2. $\lim_{x\to a} \frac{E(x)}{x-a} = 0$

then $g(x) = f(a) + f'(a)(x-a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparsion with $x-a$.



Solution.

The first condition implies that $f(a) = g(a) = m(a-a)+c = c$.
The second condition can be rewritten as

$$\lim_{\Delta x\to 0} \frac{E(a+\Delta x)-E(a)}{\Delta x} = E'(a) = 0$$

$$\Rightarrow f'(a) = g'(a) = m$$

Hence, enforcing the conditions immediately gives

$$g(x) = f(a) + f'(a)(x-a) = L(x) \approx f(x), x \to a$$

is a real number that is small enough.

4.1.

54.

Let $f(x) = |x^3 - 9x|$.

a. Does $f'(0)$ exist?

b. Does $f'(3)$ exist?

c. Does $f'(-3)$ exist?

d. Determine all extrema of f.

Solution.

$$f(x) = \begin{cases} -x^3 + 9x, & x < -3 \vee 0 \le x < 3 \\ x^3 - 9x, & -3 \le x < 0 \vee x \ge 3 \end{cases}$$

a.

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{h^3-9h}{h}=\lim_{h\to 0^-}(h^2-9)=-9$$

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{-h^3+9h}{h}=\lim_{h\to 0^+}(-h^2+9)=9$$

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}\neq\lim_{h\to 0^+}\frac{f(h)-f(0)}{h},$$

so $f'(0)$ does not exist.

b.

$$\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^-}\frac{-(3+h)^3+9(3+h)}{h}=\lim_{h\to 0^-}\frac{-h^3-9h^2-18h}{h}=\lim_{h\to 0^-}(-h^2-9h-18)=-18$$

$$\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^+}\frac{h^3+9h^2+18h}{h}=\lim_{h\to 0^+}(h^2+9h+18)=18$$

$$\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}\neq\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h},$$

so $f'(3)$ does not exist.

c.

$$\lim_{h\to 0^-}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\to 0^-}\frac{-(-3+h)^3+9(-3+h)}{h}=\lim_{h\to 0^-}\frac{-h^3+9h^2-18h}{h}=\lim_{h\to 0^-}(-h^2+9h-18)=-18$$

$$\lim_{h\to 0^+}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\to 0^+}\frac{h^3-9h^2+18h}{h}=\lim_{h\to 0^+}(h^2-9h+18)=18$$

$$\lim_{h\to 0^-}\frac{f(-3+h)-f(-3)}{h}\neq\lim_{h\to 0^+}\frac{f(-3+h)-f(-3)}{h},$$

so $f'(-3)$ does not exist.

d.

$$f'(x)=\begin{cases}-3x^2+9, & x<-3\vee 0<x<3\\ 3x^2-9, & -3<x<0\vee x>3\end{cases}$$

$$f''(x)=\begin{cases}-6x, & x<-3\vee 0<x<3\\ 6x, & -3<x<0\vee x>3\end{cases}$$

On one hand,let $x_0\in\mathbb{R}$ satisfies $f'(x_0)=0\Rightarrow 3x_0^2-9=0\Rightarrow x_0=\pm\sqrt{3}$.

$$f''(-\sqrt{3})=6(-\sqrt{3})<0, f''(\sqrt{3})=-6(\sqrt{3})<0$$

Hence, f has two extrema, both of which are maxima. The maxima are $x=\sqrt{3}$ and $x=-\sqrt{3}$.
On the other hand, $f(x)\geq 0$, and equality holds iff $x=-3\vee x=0\vee x=3$.
In conclusion, $f(-3)=0$, $f(0)=0$, $f(3)=0$ are global minima, and $f(-\sqrt{3})=6\sqrt{3}$, $f(\sqrt{3})=6\sqrt{3}$ are local maxima.

61.

Area of a right triangle What is the largest possible area for a right triangle whose hypotenuse is 5 cm long?

Solution.

Let $a,b>0$ be the two other sides of the triangle. So, it is a problem of maximizing $ab/2$ given $a^2+b^2=25$. By arithmetic-mean geometric-mean inequality,

$$\text{Area of the triangle}=\frac{1}{2}ab\leq\frac{1}{4}(a^2+b^2)=\frac{25}{4}$$

70.

Functions with no extreme values at endpoints

a. Graph the function

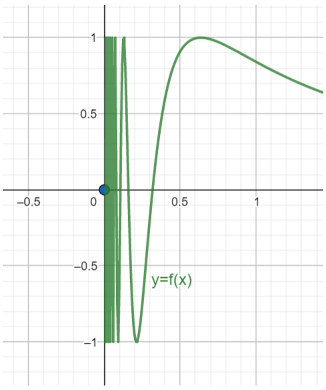
$$f(x) = \begin{cases} \sin\frac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Explain why $f(0) = 0$ is not a local extreme value of f.

b. Construct a fucntion of your own that fails to have an extreme value at a domain endpoint.

Solution.

a.



A local extreme value $y_0 = f(x_0)$ satisfies

$$f(x) \le y_0$$

for all x in some neighborhood of x_0, if it is a local maximum, and

$$f(x) \ge y_0$$

for all x in some neighborhood of x_0, if it is a local minimum.
As f is defined on $[0, +\infty)$, we only discuss $x \in (0, \delta)$, $\delta > 0$, that is all the points in the right δ-neighborhood of 0.

$$(\forall \delta > 0)(\exists k \in \mathbb{Z}^+)(x_1 = \frac{2}{(4k+1)\pi} \wedge x_2 = \frac{2}{(4k-1)\pi} \wedge 0 < x_1 < x_2 < \delta).$$

However,

$$f(x_1) = 1, f(x_2) = -1$$

Hence, there exists x_1, x_2 in any right neighborhood of 0, that $f(x_1) = -1 < f(0) = 0 < f(x_2) = 1$, and $f(0) = 0$ is not a local extreme value.

b. Let $g : [0, +\infty) \to [-1, 1]$.

$$g(x) = \begin{cases} \sin\frac{2}{x}, & x > 0 \\ \frac{1}{3}, & x = 0. \end{cases}$$

Again, $g(0) = \frac{1}{3}$ is not a local extreme value of g, The proof is similar to (a), by switching all f to g, and by letting $x_1 = \frac{1}{(4x+1)\pi}, x_2 = \frac{1}{(4k-1)\pi}$

4.2.

10.

For what values of a, m and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Solution.

f is continuous on $[0,2]$. Thus,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-x^2 + 3x + a) = f(0) = 3 \Rightarrow a = 3$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x^2 + 3x + 3) = 5 = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (mx + b) = m + b$$

f is differentiable on $(0,2)$. Thus,

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = -2 + 3 = 1,$$

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = m$$

Thus, $m = 1, b = 5 - m = 4$. In conclusion,

$$a = 3, b = 4, m = 1$$

14.

Show that a cubic polynomial can have at most three zeros.

Solution.

Let $P(x) = ax^3 + bx^2 + cx + d, a \neq 0$. By definition, deg $P = 3$, thus $P(x)$ is a cubic polynomial.

$$P'(x) = 3ax^2 + 2bx + c$$

$$P''(x) = 6ax + 2b$$

$$P'''(x) = 6a$$

Hence $P'''(x)$ is a non-zero constant. Assume $x_1, x_2, x_3, x_4 \in \mathbb{R}$ are all distinct that statisfies $P(x_1) = P(x_2) = P(x_3) = P(x_4) = 0$.
By applying Rolle's theorem for several times,

$$(\exists x_{11} \in (x_1, x_2))(\exists x_{12} \in (x_2, x_3))(\exists x_{13} \in (x_3, x_4))(P'(x_{11}) = P'(x_{12}) = P'(x_{13}) = 0)$$

$$\Rightarrow (\exists x_{21} \in (x_{11}, x_{12}))(\exists x_{22} \in (x_{12}, x_{23}))(P''(x_{21}) = P''(x_{22}) = 0)$$

$$\Rightarrow (\exists x_{31} \in (x_{21}, x_{22}))(P'''(x_{31}) = 0)$$

However, the last proposition is a contradiction to the fact that the third derivative of P is a non-zero constant, hence a cubic polynomial cannot have more than 3 roots.

A1.

Let $a < b$ and $f : [a,b] \to \mathbb{R}$ such that the derivative f' of f exists, f' is continuous on $[a,b]$ and f' is differentiable on (a,b), and $f(a) = f(b) = 0$. In particular, the second derivative f'' exists on (a,b). Show that for any $x \in (a,b)$, there exists $c \in (a,b)$ such that

$$f(x) = \frac{f''(c)}{2} \cdot (x-a)(x-b)$$

Solution.

Define $g : [a, b] \to \mathbb{R}$ such that

$$g : x \mapsto A(x - a)(x - b),$$

where g has g'' exists on (a, b), and g' is continuous on $[a, b]$. $A \in \mathbb{R}$ is a constant to be determined from the behavior of f.
Then, $\forall y \in (a, b)$, A can be chosen that satisfies

$$f(y) = g(y)$$

$$\Rightarrow A = \frac{f(y)}{(y - a)(y - b)}$$

Define $h : [a, b] \to \mathbb{R}$ such that

$$h(x) = f(x) - g(x)$$

Clearly, h'' exists on (a, b), and h' is continuous on $[a, b]$. With the given conditions,

$$h(a) = h(y) = h(b) = 0$$

Hence, as a consequence of Rolle's theorem, $\exists c_1 \in (a, y), c_2 \in (y, b)$ such that

$$h'(c_1) = h'(c_2) = 0$$

$$\Rightarrow (\exists c \in (c_1, c_2))h''(c) = 0$$

$$\Rightarrow f''(c) = g''(c) = 2A = \frac{2f(y)}{(y - a)(y - b)}$$

$$\Rightarrow f(y) = \frac{1}{2}f''(c)(y - a)(y - b) \blacksquare$$

B1.

Let $f : [1, +\infty) \to \mathbb{R}$ be continuous on $[1, +\infty)$ and differentiable on $(1, +\infty)$. Determine if the following statements are true or false. If true, provide a proof and if false, give a counter-example.

1. If $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to+\infty} f'(x)$ exists, then $\lim_{x\to+\infty} f'(x) = 0$.(Hint: apply the MVT on each segment $[n, n + 1]$ for $n \in \mathbb{N}^*$.)
2. If $\lim_{x\to+\infty} f(x) = 0$, then $\lim_{x\to+\infty} f'(x) = 0$.

Solution.

1. True.
   f is continuous on $[1, +\infty)$ and differentiable on $(1, +\infty)$, hence $\forall t \in \mathbb{N}^*$, $\exists c_t \in (t, t + 1)$ such that

   $$f'(c_t) = \frac{f(t + 1) - f(t)}{t + 1 - t} = f(t + 1) - f(t) \Leftrightarrow f(t + 1) = f'(c_t) + f(t)$$

   Now we restrict t such that for any given $M \in \mathbb{R}$, $t \geq M$. Then, by the existence of

   $$\lim_{x\to+\infty} f'(t),$$

   $$\lim_{t\to+\infty} f(t + 1) = \lim_{t\to+\infty} f'(c_t) + \lim_{t\to+\infty} f(t) \Rightarrow \lim_{t\to+\infty} f'(c_t) = 0$$

   Therefore,

   $$f'(c_t), f'(c_{t+1}), f'(c_{t+2}), \cdots \to 0 \text{ when } t \to +\infty$$

   The existence of this sequence implies

   $$\lim_{x\to+\infty} f'(x) = 0 \blacksquare$$

2. False.
Consider $f(x) = e^{-x}\cos(e^x)$.
By definition, $f \in C^1$. Moreover,

$$0 = \lim_{x\to+\infty} -e^{-x} \le \lim_{x\to+\infty} f(x) \le \lim_{x\to+\infty} e^{-x} = 0 \Rightarrow \lim_{x\to+\infty} f(x) = 0$$

Its derivative is

$$f'(x) = -e^{-x}\cos(e^x) - e^{-x}\sin(e^x)\cdot e^x = -e^{-x}\cos(e^x) - \sin(e^x)$$

Thus if the limit exists, then

$$(\exists A)(\forall \epsilon > 0)(\exists M)(\forall x)(x > M \to |f'(x) - A| < \epsilon.$$

Below chooses $k \in \mathbb{N}^*$ such that k satisfies $x = \ln\frac{(4k-1)\pi}{2} > M$. Clearly, k exists.

$$f'(x) = -\frac{2}{(4k-1)\pi}\cos\frac{(4k-1)\pi}{2} - \sin\frac{(4k-1)\pi}{2} = 1$$

Choose another x, for instance $x = \ln\frac{(4k+1)\pi}{2}$. Then,

$$f'(x) = -\frac{2}{(4k+1)\pi}\cos\frac{(4k+1)\pi}{2} - \sin\frac{(4k+1)\pi}{2} = -1$$

Hence, A does not exist, which disproved the statement. ■