清华大学本科生考试试题

考试课程　Calculus (A2)　　2022年6月　　（**B 卷**）

系_______ 班_______ 学号______________ 姓名______________

Please write a letter “B” on top of the first page of your submission.

1. Fill in the following blanks (每空4分，共40分)

(1) If $f(x, y) = x^2 - y$, then the gradient vector field $\nabla f =$ ____________

(2) Suppose the line C is: $x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 1$, then the line integral $\int_C xye^{yz}dy =$ ____________

(3) Evaluate the iterated integral $\int_0^4 \int_0^{\sqrt{y}} xy^2 dxdy =$ ____________

(4) Let A denote the area of the surface: the part of the plane $3x + 2y + z = 6$ that lies in the first octant $(x \ge 0,\ y \ge 0,\ z \ge 0)$, then $A =$ ____________

(5) Suppose $w = xy + yz + zx,\ x = r\cos\theta,\ y = r\sin\theta,\ z = r\theta$, then $\dfrac{\partial w}{\partial r}(r = 2, \theta = \pi/2) =$ ____________

(6) Let $(D_{\mathbf{u}}f)_P$ denote the directional derivative of the function $f(x, y, z) = x^2yz - xyz^3$ at the given point $P(2, -1, 1)$ in the direction of vector $\mathbf{u} =< 0, 4/5, -3/5 >$, then $(D_{\mathbf{u}}f)_P =$ ____________

(7) Suppose the directional derivative $(D_{\mathbf{u}}f)_P$ gives the maximum rate of change of $f(x, y) = \sin(xy)$ at the given point $P(1, 0)$ in the unit vector direction $\mathbf{u}$, then $(D_{\mathbf{u}}f)_P =$ ____________, $\mathbf{u} =$ ____________

(8) If $f(x, y) = 16 - 4x^2 - y^2$, then $f_x(1, 2) =$ ____________

(9) Suppose S is the tangent plane to the given surface $z = 3y^2 - 2x^2 + x$ at the specified point $(2, -1, -3)$, give an equation of the plane S: ____________

(continued at next page)

2. Answer the following questions in detail（共60分）

(10) (12分) Find all the local maxima, local minima, and saddle points of the function $f(x, y) = 4 + x^3 + y^3 - 3xy$, and use the Second Derivative Test to confirm your answer.

(11) (12分) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 2x+2y+z$, subject to the given constraint $x^2+y^2+z^2 = 9$.

(12) (5分) Evaluate $\iiint_E zdV$, where E is bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = 4$.

(13) (10分) Suppose C is the 'counterclockwise' circle $x^2 + y^2 = 4$, evaluate the line integral $\oint_C (x - y)dx + (x + y)dy$, by two methods: (a) directly with the parametric equation of C; (b) using Green's Theorem.

(14) (5分) Evaluate the surface integral $\iint_S (x^2 z + y^2 z)d\sigma$, where S is the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$.

(15) (10分) Suppose the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$, the surface S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant $(x \geq 0, y \geq 0, z \geq 0)$, and $\mathbf{n}$ is the unit normal vector of S toward the origin, evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$.

(16) (6分) Suppose $\mathbf{f}$ is a vector-valued function $\mathbf{f}(\mathbf{X}) = (f_1(\mathbf{X}), f_2(\mathbf{X})) \in \mathbb{R}^2$ where $\mathbf{X} = (x_1, x_2) \in \mathbb{R}^2$， and there is a number $q \in (0, 1)$ such that $\forall \mathbf{X}, \mathbf{Y} \in \mathbb{R}^2$,

$$|\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{Y})| \leq q|\mathbf{X} - \mathbf{Y}|.$$

Prove that there exists a unique $\mathbf{A} \in \mathbb{R}^2$ such that $\mathbf{f}(\mathbf{A}) = \mathbf{A}$.