Calculus A2: Homework 14

May 29, 2022

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1	2	3	4	5	6	7	8
c	e	g	a	d	b	f	h

11.

$$C : \mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2-2t)\mathbf{k}$$

$$\int_C (xy+y+z)ds = \int_0^1 ((2t)(t)+t+(2-2t)\sqrt{2^2+1^2+(-2)^2}dt = 3\int_0^1 (2t^2-t+2)dt = 3(2/3-1/2+2) = 13/2$$

19.

(a)

$$C : \mathbf{r}(t) = t\mathbf{i} + (t/2)\mathbf{j}$$

$$\int_C xds = \int_0^4 t\sqrt{1+(1/2)^2}dt = 4\sqrt{5}$$

(b)

$$C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

$$\int_C xds = \int_0^2 t\sqrt{1+(2t)^2}dt = \frac{1}{12}((1+4(2)^2)^{3/2}-1) = \frac{17^{3/2}-1}{12}$$

33.

$$\mathbf{r}(t) = (t^2-1)\mathbf{j} + 2t\mathbf{k}$$

$$\int_\gamma dm = \int_\gamma \delta(x,y)ds = \int_0^1 (3/2)t\sqrt{(2t)^2+2^2}dt = 3\int_0^1 t\sqrt{1+t^2}dt = (1+t^2)^{3/2}|_0^1 = 2\sqrt{2}-1$$

p.967

1.

$$\nabla f = -(x^2+y^2+z^2)^{-3/2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$

3.

$$\nabla f = -\frac{2x}{x^2+y^2}\mathbf{i} - \frac{2y}{x^2+y^2}\mathbf{j} + e^z\mathbf{k}$$

7.

(a) $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

$$\int_{C_1} (3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}) \cdot d\mathbf{r} = \int_0^1 (3t + 2t + 4t)dt = 9/2$$

(b) $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$

$$\int_{C_2} (3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}) \cdot d\mathbf{r} = \int_0^1 (3t^2 + (2t)(2t) + 4t^4(4t^3))dt = 7/3 + 2 = 13/3$$

(c) $C_3 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, $C_4 : \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}$

$$\int_{C_3 \cup C_4} (3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}) \cdot d\mathbf{r} = \int_0^1 (3t + 2t)dt + \int_0^1 4tdt = 5/2 + 2 = 9/2$$

19.

$$C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^3(1) + (t^2)(2t) - (t^3)(1))dt = 1/2$$

23.

$$C : y = x^2 (\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j})$$

$$\int_C xydx + (x+y)dy = \int_{-1}^2 ((t^3)(1) + (t+t^2)(2t))dt = \int_{-1}^2 (3t^3 + 2t^2)dt = (3/4)(16-1) + (2/3)(8+1) = 69/4$$

29.

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}, \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

The following is used for solving flux.

$$\mathbf{n}ds = \mathbf{T} \times \mathbf{k}ds = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} \times \mathbf{k} \cdot \|\dot{\mathbf{r}}(t)\|dt = \dot{\mathbf{r}}(t) \times \mathbf{k}dt$$

(a) $C : \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$.

$$\dot{\mathbf{r}}(t) \times \mathbf{k} = ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j}) \times \mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

Circulation:

$$\oint_C \mathbf{F}_1 \cdot d\mathbf{r} = \int_0^{2\pi} [(\cos t)(-\sin t) + (\sin t)(\cos t)]dt = 0$$

$$\oint_C \mathbf{F}_2 \cdot d\mathbf{r} = \int_0^{2\pi} [(-\sin t)(-\sin t) + (\cos t)(\cos t)]dt = 2\pi$$

Flux:

$$\oint_C \mathbf{F}_1 \cdot \mathbf{n}ds = \int_0^{2\pi} \mathbf{F}_1(\mathbf{r}(t)) \cdot (\dot{\mathbf{r}}(t) \times \mathbf{k})dt = \int_0^{2\pi} [(\cos t)(\cos t) + (\sin t)(\sin t)]dt = 2\pi$$

$$\oint_C \mathbf{F}_2 \cdot \mathbf{n}ds = \oint_0^{2\pi} \mathbf{F}_2(\mathbf{r}(t)) \cdot (\dot{\mathbf{r}}(t) \times \mathbf{k})dt = \int_0^{2\pi} [(-\sin t)(\cos t) + (\cos t)(\sin t)]dt = 0$$

(b) $C : \mathbf{r}(t) = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$.

$$\dot{\mathbf{r}}(t) \times \mathbf{k} = ((-\sin t)\mathbf{i} + (4\cos t)\mathbf{j}) \times \mathbf{k} = (4\cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

Circulation:

$$\oint_C \mathbf{F}_1 \cdot d\mathbf{r} = \int_0^{2\pi} [(\cos t)(-\sin t) + (4\sin t)(4\cos t)]dt = (15/2)\int_0^{2\pi} \sin(2t)dt = 0$$

$$\oint_C \mathbf{F}_2 \cdot d\mathbf{r} = \int_0^{2\pi} [(-4\sin t)(-\sin t) + (\cos t)(4\cos t)]dt = 8\pi$$

Flux:

$$\oint_C \mathbf{F}_1 \cdot \mathbf{n}ds = \int_0^{2\pi} \mathbf{F}_1(\mathbf{r}(t)) \cdot (\dot{\mathbf{r}}(t) \times \mathbf{k})dt = \int_0^{2\pi} (\cos t)(4\cos t) + (4\sin t)(\sin t)dt = 8\pi$$

$$\oint_C \mathbf{F}_2 \cdot \mathbf{n}ds = \int_0^{2\pi} \mathbf{F}_2(\mathbf{r}(t)) \cdot (\dot{\mathbf{r}}(t) \times \mathbf{k})dt = \int_0^{2\pi} (-4\sin t)(4\cos t) + (\cos t)(\sin t)dt = 0$$

51.

$$C_1 : \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, t \in [0, \pi/2],$$
$$C_2 : \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1-t)\mathbf{k}, t \in [0, 1],$$
$$C_3 : \mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}, t \in [0, 1]$$
$$\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$$

Circulation:

$$\oint_{C_1 \cup C_2 \cup C_3} \mathbf{F} \cdot d\mathbf{r}$$
$$= \int_0^{\pi/2} \mathbf{F} \cdot ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k})dt + \int_0^1 \mathbf{F} \cdot (-(\pi/2)\mathbf{k})dt + \int_0^1 \mathbf{F} \cdot (\mathbf{i} - \mathbf{j})dt$$
$$= \int_0^{\pi/2} ((-2\cos t \sin t) + (2t\cos t) + 2\sin t)dt + \int_0^1 (-2(\pi/2))dt + \int_0^1 (2t)dt$$
$$= \int_0^{\pi/2} (-\sin(2t) + 2t\cos t + 2\sin t)dt - \pi + 1 = 2\int_0^{\pi/2} t\cos t dt - \pi + 2$$
$$= 2(t\sin t)_0^{\pi/2} - 2\int_0^{\pi/2} \sin t dt - \pi + 2 = 0$$

p.978

1.

The natural domain D of $\mathbf{F}$ is simply connected,

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$
$$\frac{\partial(yz)}{\partial y} = z, \frac{\partial(xz)}{\partial x} = z, \frac{\partial(xz)}{\partial z} = x, \frac{\partial(xy)}{\partial y} = x, \frac{\partial(xy)}{\partial x} = y, \frac{\partial(yz)}{\partial z} = y$$

Therefore $\mathbf{F}$ is conservative.

3.

The natural domain D of $\mathbf{F}$ is simply connected,

$$\mathbf{F} = y\mathbf{i} + (x+z)\mathbf{j} - y\mathbf{k}$$
$$\frac{\partial(x+z)}{\partial z} = 1, \frac{\partial(-y)}{\partial y} = -1$$

Therefore $\mathbf{F}$ is not conservative.

7.

$$\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$$
$$f'_x = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z)$$
$$f'_y = 3y = g'_y(y,z) \Rightarrow g(y,z) = (3/2)y^2 + h(z)$$
$$f'_z = 4z = h'_z(z) \Rightarrow h(z) = 2z^2 + C$$
$$\Rightarrow f(x,y,z) = x^2 + (3/2)y^2 + 2z^2 + C, C \in R \text{ is an arbitrary constant}$$

9.

$$\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$$
$$f'_x = e^{y+2z} \Rightarrow f(x,y,z) = xe^{y+2z} + g(y,z)$$
$$f'_y = xe^{y+2z} = xe^{y+2z} + g'_y(y,z) \Rightarrow g(y,z) = h(z)$$
$$f'_z = 2xe^{y+2z} = 2xe^{y+2z} + h'_z(z) \Rightarrow h(z) = C$$
$$\Rightarrow f(x,y,z) = xe^{y+2z} + C, C \in R \text{ is an arbitrary constant}$$

13.

$$\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

is conservative, because when $f(x,y,z) = x^2 + y^2 + z^2 + C, \nabla f = \mathbf{F}$, so the integral is path independent.

$$\int_{(0,0,0)}^{(2,3,-6)} (2xdx + 2ydy + 2zdz) = 4 + 9 + 36 = 49$$

p.990

The following is for (1) and (3):

$$\partial R = C : \mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$
$$\dot{\mathbf{r}}(t) = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$$
$$\dot{\mathbf{r}}(t) \times \mathbf{k} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$

1.

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$$

Circulation:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)]dt = 2\pi a^2$$
$$\iint_R (\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y})dxdy = \iint_R 2dxdy = 2\pi a^2$$

□

Flux:

$$\oint_C \mathbf{F} \cdot \mathbf{n}ds = \int_0^{2\pi} [(-a\sin t)(a\cos t) + (a\cos t)(a\sin t)]dt = 0$$
$$\iint_R (\frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y})dxdy = 0$$

□

3.

$$\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$$

Circulation:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [(2a\cos t)(-a\sin t) + (-3a\sin t)(a\cos t)]dt = -5a^2 \int_0^{2\pi} \sin t\cos t dt = 0$$
$$\iint_R (\frac{\partial(-3y)}{\partial x} - \frac{\partial(2x)}{\partial y})dxdy = 0$$

□

Flux:

$$\oint_C \mathbf{F} \cdot \mathbf{n}ds = \int_0^{2\pi} [(2a\cos t)(a\cos t) + (-3a\sin t)(a\sin t)]dt = a^2 \int_0^{2\pi} [(1 + \cos 2t) - \frac{3}{2}(1 - \cos 2t)]dt = -\pi a^2$$
$$\iint_R (\frac{\partial(2x)}{\partial x} + \frac{\partial(-3y)}{\partial y})dxdy = \iint_R (-1)dxdy = -\pi a^2$$

□

5.

$$\mathbf{F} = (x-y)\mathbf{i} + (y-x)\mathbf{j}, D = [0,1]^2$$

Circulation:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D [(-1) - (-1)] dxdy = 0$$

Flux:

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D [(1) + (1)] dxdy = 2$$

19.

$$\mathbf{F} = (2xy^3)\mathbf{i} + (4x^2y^2)\mathbf{j}, D = \{(x,y) : 0 \le x \le 1, 0 \le y \le x^3\}$$

$$W = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D [8xy^2 - (6xy^2)] dxdy = 2\int_0^1 x \left(\int_0^{x^3} y^2 dy \right) dx = \frac{2}{3} \int_0^1 x^{10} dx = \frac{2}{33}$$

21.

$$D = \{(x,y) : x, y \ge 0, x + y \le 1\}$$

$$\oint_{\partial D} (y^2 dx + x^2 dy) = \iint_D (2x - 2y) dxdy = \int_0^1 \int_0^{1-x} (2x - 2y) dxdy = \int_0^1 (2x(1-x) - (1-x)^2) dx =$$

$$\int_0^1 (-3x^2 + 4x - 1) dx = -1 + 2 - 1 = 0$$