# Calculus A2: Homework 13

May 24, 2022

## p.912

**1.**

$\Omega = \{(x, y, z) : 0 \le y \le 1 \wedge 0 \le x \le y \wedge 0 \le z \le y - x\}$

$$\iiint_{\Omega} dV = \int_0^1 \int_0^y \int_0^{y-x} dz dx dy = \int_0^1 \int_0^y (y - x) dx dy = \int_0^1 (y^2/2) dy = 1/6$$

**7.**

$$\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx = 3 \int_0^1 dx \int_0^1 dy \int_0^1 z^2 dz = 3(1)(1)(1/3) = 1$$

**23.**

$$\text{Volume} = \int_0^1 \int_{-1}^1 \int_0^{y^2} dz dy dx = \int_0^1 dx \int_{-1}^1 y^2 dy = (1)(2)(1/3) = 2/3$$

**25.**

By given constraints, let $\Omega = \{(x, y, z) : 0 \le y \le 2 \wedge 0 \le x \le 4 - y^2 \wedge 0 \le z \le 2 - y\}$ be that region, then volume

$$= \iiint_{\Omega} dV = \int_0^2 \left( \int_0^{4-y^2} dx \int_0^{2-y} dz \right) dy = \int_0^2 (4 - y^2)(2 - y) dy$$

$$= \int_0^2 (8 - 4y - 2y^2 + y^3) dy = 8(2) - 2(2)^2 - 2(2)^3/3 + (2)^4/4 = 20/3$$

**27.**

By given constraints, let $\Omega = \{(x, y, z) : x \ge 0 \wedge y \ge 0 \wedge z \ge 0 \wedge x + y/2 + z/3 \le 1\}$ be that region, then volume

$$= \iiint_{\Omega} dV = \int_0^1 \int_0^{2(1-x)} \int_0^{3(1-x-y/2)} dz dy dx = \int_0^1 \int_0^{2(1-x)} 3(1 - x - y/2) dy dx$$

$$= 3 \int_0^1 (2(1 - x) - x(2(1 - x)) - (2(1 - x))^2/4) dx = 3 \int_0^1 (2 - 4x + 2x^2 - (1 - x)^2) dx$$

$$= 3(2(1) - 2(1)^2 + 2(1)^3/3) - 3((1 - 0)^3/3) = 1$$

**33.**

$$\Omega = \{(x, y, z) : x \ge 0 \wedge y \ge 0 \wedge x/2 + y/2 + z \ge 1 \wedge x/2 + y/2 + z/4 \le 1\},$$

Volume

$$= \iiint_{\Omega} dV = \int_0^2 \int_0^{2-x} \int_{1-x/2-y/2}^{4(1-x/2-y/2)} dz dy dx = 3 \int_0^2 \int_0^{2-x} (1 - x/2 - y/2) dy dx$$

$$= 3 \int_0^2 ((2 - x) - \frac{x}{2}(2 - x) - \frac{1}{4}(2 - x)^2) dx = 3 \int_0^2 (2 - 2x + \frac{1}{2}x^2 - \frac{1}{4}(2 - x)^2) dx$$

$$= 3(2(2) - (2)^2 + \frac{1}{6}(2)^3) - 3(\frac{1}{12}(2)^3) = 2$$

**37.**

$$\text{Average} = \frac{1}{8}\iiint\limits_{[0,2]^3}(x^2+9)dV = \frac{1}{8}\int_0^2 (x^2+9)dx\int_0^2 dy\int_0^2 dz = \frac{1}{2}\left(\frac{1}{3}x^3+9x\right)_0^2 = 31/3$$

## p.920

**1.**

Let $D$ be the thin plate, then $D = \{(x,y) : 0 \le x \le 1 \wedge x \le y \le 2-x^2\}$

$$\mathbf{r_c} = \frac{\int \mathbf{r}\cdot dm}{\int dm}$$

$$\begin{aligned}\mathbf{r_c} &= \left(\iint\limits_D (\mathbf{r}\cdot 3dS)\right)/\left(\iint\limits_D 3dS\right) = \left(\int_0^1\int_x^{2-x^2}(x\mathbf{i}+y\mathbf{j})dydx\right)/\left(\int_0^1\int_x^{2-x^2}dydx\right)\\ &= \left(\int_0^1 (x(2-x^2-x)\mathbf{i}+\frac{1}{2}((2-x^2)^2-x^2)\mathbf{j})dx\right)/\left(\int_0^1(2-x^2-x)dx\right)\\ &= \left(\int_0^1((2x-x^2-x^3)\mathbf{i}+\frac{1}{2}(4-5x^2+x^4)\mathbf{j})dx\right)/(7/6)\\ &= \frac{6}{7}((1-1/3-1/4)\mathbf{i}+(4-5/3+1/5)\mathbf{j}/2) = \frac{5}{14}\mathbf{i}+\frac{38}{35}\mathbf{j}\end{aligned}$$

**3.**

$$D = \{(x,y) : 0 \le y \le 2 \wedge y^2/2 \le x \le 4-y\}$$

$$\begin{aligned}\mathbf{r_c} &= \left(\iint\limits_D \mathbf{r}dS\right)/\left(\iint\limits_D dS\right) = \left(\int_0^2\int_{y^2/2}^{4-y}\mathbf{r}dxdy\right)/\left(\int_0^2\int_{y^2/2}^{4-y}dxdy\right)\\ &= \left(\int_0^2(\frac{1}{2}((4-y)^2-y^4/4)\mathbf{i}+(4-y-y^2/2)y\mathbf{j})dy\right)/\left(\int_0^2(4-y-y^2/2)dy\right)\\ &= \left(\frac{1}{2}(-\frac{1}{3}(4-y)^3-\frac{1}{20}y^5)\mathbf{i}+(2y^2-\frac{1}{3}y^3-\frac{1}{8}y^4)\mathbf{j}\right)_0^2/(14/3) = \frac{3}{14}(\frac{128}{15}\mathbf{i}+\frac{10}{3}\mathbf{j}) = \frac{64}{35}\mathbf{i}+\frac{5}{7}\mathbf{j}\end{aligned}$$

**7.**

$D = \{(x,y) : 0 \le x^2+y^2 \le 4\}$.
Alternatively, $D = \{(r,\theta) : 0 \le r \le 2 \wedge 0 \le \theta \le 2\pi\}$.
So

$$I_x = \iint\limits_D y^2\cdot dS = \int_0^2 r^3dr\int_0^{2\pi}\sin^2\theta d\theta = 4\cdot\int_0^{2\pi}\frac{1-\cos(2\theta)}{2}d\theta = 4\pi$$

$$I_y = 4\cdot\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}d\theta = 4\pi$$

$$I_0 = I_x + I_y = 8\pi$$

## p.930

**1.**

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}}dzrdrd\theta = \int_0^{2\pi}d\theta\int_0^1(\sqrt{2-r^2}-r)rdr = 2\pi\left(-\frac{1}{3}(2-r^2)^{3/2}-\frac{1}{3}r^3\right)_0^1 = \frac{4\pi}{3}(\sqrt{2}-1)$$

**13.**

$$\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2}f(r,\theta,z)dzrdrd\theta$$

**15.**

$$\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r,\theta,z)dzrdrd\theta$$

**21.**

$$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi}\rho^2\sin\phi d\rho d\phi d\theta = \int_0^{\pi}d\theta\int_0^{\pi}\sin\phi\frac{8}{3}\sin^3\phi d\phi = \frac{8\pi}{3}\int_0^{\pi}\sin^4\phi d\phi$$

$$\frac{2\pi}{3}\int_0^{\pi}(\frac{3}{2}-2\cos 2\phi+\cos 4\phi)d\phi = \pi^2$$

**33.**

(a)

$$\Omega = \{(\rho,\phi,\theta) : 0 \le \theta \le 2\pi \wedge 0 \le \phi \le \pi/2 \wedge \cos\phi \le \rho \le 2\}$$

(b)

$$\iiint_{\Omega} dV = \int_{-\pi/2}^{\pi/2}\int_0^{\pi/2}\int_{\cos\phi}^{2}\rho^2\sin\phi d\rho d\phi d\theta = \frac{2\pi}{3}\int_0^{\pi/2}\sin\phi(8-\cos^3\phi)d\phi$$

$$= \frac{2\pi}{3}\left(-8\cos\phi+\frac{1}{4}\cos^4\phi\right)_0^{\pi/2} = \frac{31\pi}{6}$$

**45.**

$$\iiint_{\Omega} dV = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} rdzdrd\theta = -\int_{3\pi/2}^{2\pi}\sin\theta\int_0^{3\cos\theta}r^2drd\theta = -9\int_{3\pi/2}^{2\pi}\sin\theta\cos^3\theta d\theta$$

$$= \frac{9}{4}\cos^4\theta|_{3\pi/2}^{2\pi} = \frac{9}{4}$$

**49.**

$$\iiint_{\Omega} dV = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a} r^2\sin\phi drd\phi d\theta = \frac{2\pi a^3}{3}(-\cos\phi)_{\pi/3}^{2\pi/3} = \frac{2\pi}{3}a^3$$

**63.**

$$\text{Average} = \frac{1}{2\pi}\iiint_D rdV = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{1}\int_{-1}^{1} rdzrdrd\theta = \frac{2}{3}$$

## p.942

**1.**

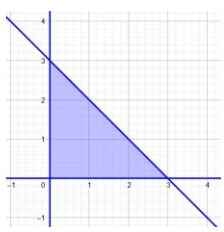
(a)

$$x = (u+v)/3, y = (v-2u)/3$$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = 1/3$$

(b)

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$



**9.**

$R = \{(x,y) : 1 \le xy \le 9 \wedge x \le y \le 4x \wedge x \ge 0 \wedge y \ge 0\}$.

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$$

$x = u/v, y = uv$.

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = 2u/v$$

W.R.T. $R$, $1 \le u^2 \le 9 \wedge 1 \le v^2 \le 4$

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy = \int_1^2 \int_1^3 (v+u)\cdot(2u/v)dudv = \int_1^2 \left(8 + \frac{52}{3v}\right) dv = 8 + \frac{52}{3}\ln 2$$

**19.**

$x = au, y = bv, z = cw,$

$$\iiint |xyz|dxdydz = \iiint\limits_{u^2+v^2+w^2\le 1} a^2b^2c^2|uvw|dudvdw$$

Let $u = r\sin\phi\cos\theta, v = r\sin\phi\sin\theta, w = r\cos\phi$, where $0 \le r \le 1, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$, then the above becomes

$$a^2b^2c^2 \int_0^{2\pi}\int_0^{\pi}\int_0^1 |r^3\sin^2\phi\cos\phi\cos\theta\sin\theta| r^2\sin\phi drd\phi d\theta$$

$$= 8a^2b^2c^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^5\sin^3\phi\cos\phi\cos\theta\sin\theta drd\phi d\theta$$

$$= 4a^2b^2c^2 \int_0^{\pi/2} \sin 2\theta d\theta \int_0^{\pi/2} \sin^3\phi\cos\phi d\phi \int_0^1 r^5 dr = 4a^2b^2c^2 \cdot 1 \cdot \frac{1}{4}\cdot\frac{1}{6} = \frac{a^2b^2c^2}{6}$$