作业 9 简答

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习题 1. *Determine which pair of vectors are orthogonal:*

1. $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

2. $\mathbf{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

解答. 1. 直接计算

$$\langle \mathbf{u}, \mathbf{v} \rangle = 12 \times 2 + 3 \times (-3) + (-5) \times 3 = 24 - 9 - 15 = 0,$$

发现正交；

2. 直接计算

$$\langle \mathbf{u}, \mathbf{v} \rangle = (-3) \times 1 + 7 \times (-8) + 4 \times 15 + 0 \times (-7) = -3 - 56 + 60 + 0 = 1,$$

发现不正交.

习题 2. *Mark each statement true or false, and justify your answer.*

1. $\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2$.

2. *For any scalar* c, $\langle \mathbf{u}, c\mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$.

3. *If the distance from* $\mathbf{u}$ *to* $\mathbf{v}$ *equals the distance from* $\mathbf{u}$ *to* $-\mathbf{v}$, *then* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal.*

4. *For a square matrix A, vectors in $\mathrm{Col}(A)$ are orthogonal to vectors in $\mathrm{Nul}(A)$.*

5. *If vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p$ span a subspace W, and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_i$ for $i = 1, \cdots, p$, then $\mathbf{x}$ is in $W^\perp$.*

解答. 本题只考虑 $\mathbb{R}^n$ 中的内积.

1. 对的. 利用 $\langle \mathbf{v}, \mathbf{v} \rangle = \mathbf{v}^T\mathbf{v} = \sum_{i=1}^n v_i^2 = \|\mathbf{v}\|^2$.

2. 对的. 利用 $\langle \mathbf{u}, c\mathbf{v} \rangle = \mathbf{u}^T(c\mathbf{v}) = c\mathbf{u}^T\mathbf{v} = c\langle \mathbf{u}, \mathbf{v} \rangle$.

3. 对的，利用

$$\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle - \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle = 4\langle \mathbf{u}, \mathbf{v} \rangle.$$

4. 错误，例如取 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$，注意列向量 $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathrm{Nul}(A)$，但是 $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \neq 0$，矛盾!

5. 对的. 因为 W 中任何一个向量都能写成 $\sum_{i=1}^p c_i\mathbf{v}_i$ 的形式，从而

$$\left\langle \mathbf{x}, \sum_{i=1}^p c_i\mathbf{v}_i \right\rangle = \sum_{i=1}^p c_i \langle \mathbf{x}, \mathbf{v}_i \rangle = 0.$$

习题 3. *Verify the parallelogram law for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$:*

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$$

解答. 直接展开

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle + \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle = 2\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{v}, \mathbf{v} \rangle = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) = 0.$$

习题 4. *Show that if $\mathbf{x}$ is in both W and $W^\perp$, then $\mathbf{x} = 0$.*

解答. 回忆 $W^\perp$ 的定义

$$W^\perp = \{\mathbf{v} \in \mathbf{R}^n : \forall \mathbf{w} \in W, \langle \mathbf{v}, \mathbf{w} \rangle = 0\},$$

所以若 $\forall \mathbf{x} \in W \cap W^\perp$，则 x 作为 $W^\perp$ 的元素一定与 W 的元素 x 正交，即 $\langle \mathbf{x}, \mathbf{x} \rangle = 0$，这推出 $x = 0$.

习题 5. *Determine which set of vectors are orthogonal:*

1. $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$;

2. $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$.

解答. *1.* 由于

$$\begin{bmatrix} -6 & -3 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = -18 - 3 - 9 = -30 \neq 0,$$

故不是正交集;

2. 直接计算

$$\begin{bmatrix} 3 & -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} = -3 - 6 - 3 + 12 = 0,$$

$$\begin{bmatrix} 3 & -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = 9 - 16 + 7 + 0 = 0,$$

$$\begin{bmatrix} -1 & 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = -3 + 24 - 21 + 0 = 0,$$

得到正交集.

习题 6. *Show that* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *or* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is an orthogonal basis for* $\mathbf{R}^2$ *or* $\mathbf{R}^3$ *respectively. Then express* $\mathbf{x}$ *as a linear combination of the* $\mathbf{u}_i$ *'s.*

1. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, *and* $\mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$

2.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}.$$

解答. *1.* 正交性可以直接验证 $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 12 - 12 = 0$. 设 $\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2$, 分别与 $\mathbf{u}_1, \mathbf{u}_2$ 作内积可以解出 $a = 3, b = \frac{1}{2}$.

2. 正交性可以直接验证

$$\begin{aligned} \langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= -1 + 1 = 0 \\ \langle \mathbf{u}_1, \mathbf{u}_3 \rangle &= 2 - 2 = 0 \quad , \\ \langle \mathbf{u}_2, \mathbf{u}_3 \rangle &= -2 + 4 - 2 = 0 \end{aligned}$$

设 $\mathbf{x} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$, 分别与 $\mathbf{u}_i$ 做内积后可得

$$a = \frac{\langle \mathbf{x}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} = \frac{5}{2}, \quad b = \frac{\langle \mathbf{x}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} = -\frac{3}{2}, \quad c = \frac{\langle \mathbf{x}, \mathbf{u}_3 \rangle}{\|\mathbf{u}_3\|^2} = 2.$$

习题 7. *1. Let* $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ *and* $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. *Write* $\mathbf{y}$ *as the sum of two orthogonal vectors, one in* $\mathrm{Span}\{\mathbf{u}\}$ *and the other orthogonal to* $\mathbf{u}$.

2. Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ *and* $\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. *Compute the distance from* $\mathbf{y}$ *to the line through* $\mathbf{u}$ *and the origin.*

解答. *1.* 取与 $\mathbf{u}$ 正交的 $\mathbf{v} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$, 仿造前一题的方法，直接说明 $\mathbf{y} = -\frac{1}{5}\mathbf{u} + \frac{2}{5}\mathbf{v}$.

2. 直接计算

$$dist = \left\| \mathbf{y} - \frac{\langle \mathbf{u}, \mathbf{y} \rangle}{\|\mathbf{u}\|^2} \mathbf{u} \right\| = 1.$$

习题 8. *1. Let* U *and* V *be* $n \times n$ *orthogonal matrices. Explain why* UV *remains an orthogonal matrix.*

2. *Let U be an orthogonal matrix, and construct V by interchanging some of the columns of U. Explain why V is an orthogonal matrix.*

解答. 1. 直接验证

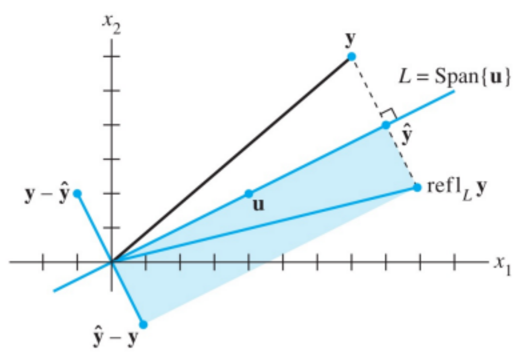
$$(UV)^T(UV) = V^TU^TUV = V^TI_nV = V^TV = I_n.$$

2. 此时我们可以把 U 写成 $V = UP$ 的形式，这里

$$P = \begin{bmatrix} \mathbf{e}_{\pi(1)} & \mathbf{e}_{\pi(2)} & \cdots & \mathbf{e}_{\pi(n)} \end{bmatrix},$$

$\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ 为一个置换. 这样的 P 也是正交矩阵，然后由前一问立刻得到.

习题 9. *Given $\mathbf{u} \neq 0$ in $\mathbf{R}^n$, let $L = \mathrm{Span}\{\mathbf{u}\}$. For $\mathbf{y} \in \mathbf{R}^n$, let $\mathrm{Refl}_L(\mathbf{y})$ be the reflection of $\mathbf{y}$ with respect to L as shown in the figure*



Show that

$$\mathrm{Refl}_L(\mathbf{y}) = 2 \cdot \mathrm{proj}_L(\mathbf{y}) - \mathbf{y}$$

and that $\mathbf{y} \mapsto \mathrm{Refl}_L(\mathbf{y})$ defines a linear transformation.

解答. 利用 proj_L 是一个线性变换，直接验证

$$\mathrm{Refl}_L(a\mathbf{x} + b\mathbf{y}) = a \cdot \mathrm{Refl}_L(\mathbf{x}) + b \cdot \mathrm{Refl}_L(\mathbf{y})$$

即可.