Calculus A(1): Homework 6

November 19, 2021

3.1.

58.

a. Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b. Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

Solution.

a. $f(x)$ satisfies $|f(x)| \le x^2$, so $|f(0)| \le 0 \Rightarrow f(0) = 0$. Also,

$$0 \le \lim_{x\to 0} |f(x)| \le \lim_{x \to 0} x^2 = 0$$

Hence f is continuous at $x = 0$. f is differentiable at $x = 0$ iff

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

exists.

$$\Leftrightarrow (\exists A)(\forall \epsilon > 0)(\exists \delta > 0)(\forall h)(0 < |h| < \delta \to \left| \frac{f(h) - f(0)}{h} - A \right| < \epsilon)$$

We also have

$$\left| \frac{f(h) - f(0)}{h} \right| \le |h^2/h| = |h| < \delta$$

by choosing $A = 0$. Set $\delta = \epsilon$ proves the existence of δ, and hence $f'(0) = 0$. ■

b.

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0 \Rightarrow f'(0) = 0$$

■

3.2.

53.

Generalizing the Product Rule The product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

for derivative of the product uv of two differentaible functions of x.

a. What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x?

b. What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of x?

c. What is the formula for the derivative of a product $u_1u_2u_3\ldots u_n$ of a finite number n of differentiable functions of x?

Solution.

a.

$$\frac{d}{dx}(uvw) = \frac{d}{dx}((uv)w) = w\frac{d}{dx}(uv) + uv\frac{dw}{dx} = w(u\frac{dv}{dx} + v\frac{du}{dx}) + uv\frac{dw}{dx} = wu\frac{dv}{dx} + vw\frac{du}{dx} + uv\frac{dw}{dx}$$

b.

$$\frac{d}{dx}(u_1u_2u_3u_4) = \frac{d}{dx}((u_1u_2u_3)u_4)$$

$$= u_4(u_1u_2\frac{du_3}{dx}+u_2u_3\frac{du_1}{dx}+u_3u_1\frac{du_2}{dx})+u_1u_2u_3\frac{du_4}{dx} = \frac{du_1}{dx}u_2u_3u_4+\frac{du_2}{dx}u_1u_3u_4+\frac{du_3}{dx}u_1u_2u_4+\frac{du_4}{dx}u_2u_3u_1$$

c. The following proves

$$\frac{d}{dx}\left(\prod_{i=1}^{n} u_i\right) = \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\neq i}}^{n} u_j\right)\frac{du_i}{dx}$$

by induction.
Let predicate $P(n)$ defined for all $n \in \mathbb{N}^*$ where

$$P(n): \frac{d}{dx}\left(\prod_{i=1}^{n} u_i\right) = \sum_{i=1}^{n}\left(\prod_{\substack{j=1\\ j\neq i}}^{n} u_j\right)\frac{du_i}{dx}$$

Then $(\forall n)(n \in \mathbb{N}^* \to P(n))$.

i) $P(1)$ is clearly true.

ii) Assume $P(k)$ is true for some $k \in \mathbb{N}^*$, i.e.

$$\frac{d}{dx}\left(\prod_{i=1}^{k} u_i\right) = \sum_{i=1}^{k}\left(\prod_{\substack{j=1\\ j\neq i}}^{k} u_j\right)\frac{du_i}{dx}$$

$P(k+1)$:

$$\frac{d}{dx}\left(\prod_{i=1}^{k+1} u_i\right) = \frac{d}{dx}\left(\left(\prod_{i=1}^{k} u_i\right)u_{k+1}\right) = u_{k+1}\frac{d}{dx}\left(\prod_{i=1}^{k} u_i\right) + \left(\prod_{i=1}^{k} u_i\right)\frac{du_{k+1}}{dx}$$

$$= u_{k+1}\sum_{i=1}^{k}\left(\prod_{\substack{j=1\\ j\neq i}}^{k} u_j\right)\frac{du_i}{dx} + \left(\prod_{i=1}^{k} u_i\right)\frac{du_{k+1}}{dx} = \sum_{i=1}^{k}\left(\prod_{\substack{j=1\\ j\neq i}}^{k+1} u_j\right)\frac{du_i}{dx} + \left(\prod_{i=1}^{k} u_i\right)\frac{du_{k+1}}{dx}$$

$$= \sum_{i=1}^{k+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{k+1} u_j\right)\frac{du_i}{dx}$$

Hence, $P(k) \Rightarrow P(k+1)$.

By i),ii) and the principle of the first mathematical induction, $(\forall n)(n \in \mathbb{N}^* \to P(n))$.■

3.4.

47.

Is there a value of c that will make

$$f(x) = \begin{cases} \frac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

Solution.

$c = 9$ works.
$x = 0$ is the removable discontinuity of $\frac{\sin^2 3x}{x^2}$, as

$$\lim_{x \to 0} \frac{\sin^2 3x}{x^2} = \lim_{x \to 0} 9 \left(\frac{\sin 3x}{3x} \right)^2 = 9.$$

If $c = 9$, then

$$\lim_{x \to 0} f(x) = f(0) = 9.$$

48.

Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

Solution.

$g(x)$ is continuous at $x = 0$ iff

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x) = g(0)$$

By definition, $g(0) = 1$, and $\lim_{x \to 0^+} g(x) = 1$. Hence $g(x)$ is continuous at $x = 0$ iff

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x + b) = 1 \Leftrightarrow b = 1$$

$g(x)$ is not diferentiable at $x = 0$, since

$$\lim_{h \to 0^-} \frac{g(h) - g(0)}{h} = \lim_{h \to 0^-} \frac{h + b - b}{h} = 1$$

But

$$\lim_{h \to 0^+} \frac{g(h) - g(0)}{h} = \lim_{h \to 0^+} \frac{\cos h - 1}{h} = \lim_{h \to 0^+} \frac{\cos^2 h - 1}{h(\cos h + 1)} = \lim_{h \to 0^+} -\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1} = 0$$

Hence $g(x)$ is continuous at $x = 0$ iff $b = 1$, and such value of b making $g(x)$ differentiable at $x = 0$ does not exist.

3.5.

80.

Find parametric equations and a parameter interval for the motion of a particle that starts at (a,0) and traces the ellipse $(x^2/a^2) + (y^2/b^2) = 1$

- **a.** once clockwise.
- **b.** once counterclockwise.
- **c.** twice clockwise.
- **d.** twice counterclockwise.

Solution.

In this solution, $t \in \mathbb{R}$ is the parameter.

a. $\begin{cases} \cos t \\ -\sin t \end{cases}, t \in [0, 2\pi]$

b. $\begin{cases} \cos t \\ \sin t \end{cases}, t \in [0, 2\pi]$

c. $\begin{cases} \cos t \\ -\sin t \end{cases}, t \in [0, 4\pi]$

d. $\begin{cases} \cos t \\ \sin t \end{cases}, t \in [0, 4\pi]$

3.6.

58.

Tangents parallel to the coordinate axes Find points on the curve $x^2 + xy + y^2 = 7$

a. where the tangent is parallel to the x-axis and

b. where the tangent is parallel to the y-axis.

In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?

Solution.

Denote y' as $\frac{dy}{dx}$.

$$(x^2 + xy + y^2)' = (7)' = 0 \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow y' = -\frac{2x + y}{x + 2y}$$

Let $P(x_0, y_0)$ be a point of the locus of the curve.

a. Tangent at P is parallel to the x-axis iff $y' = 0$. As the origin does not belong to the locus, we have

$$\begin{cases} x_0^2 + x_0 y_0 + y_0^2 = 7 \\ 2x_0 + y_0 = 0 \end{cases}$$

By substitution, $x_0^2 + x_0(-2x_0) + (-2x_0)^2 = 3x_0^2 = 7$ Hence, $\left(\pm\frac{\sqrt{21}}{3}, \mp\frac{2\sqrt{21}}{3}\right)$ are points that its tangent to the curve is parallel to the x-axis.

b. Tangent at P is parallel to the y-axis iff y' is not defined. As the origin does not belong to the locus, we have

$$\begin{cases} x_0^2 + x_0 y_0 + y_0^2 = 7 \\ x_0 + 2y_0 = 0 \end{cases}$$

By substitution, $(-2y_0)^2 + (-2y_0)y_0 + y_0^2 = 3y_0^2 = 7$ Hence, $\left(\pm\frac{2\sqrt{21}}{3}, \mp\frac{\sqrt{21}}{3}\right)$ are points that its tangent to the curve is parallel to the y-axis. At these two points, $1/y' = 0$.

71.

Normals to a parabola Show that if it is possible to draw three normals from the point $(a, 0)$ to the prabola $x = y^2$, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?

Solution.

$$x = y^2 \Rightarrow \frac{dx}{dx} = \frac{d}{dx}(y^2) \Leftrightarrow 2y\frac{dy}{dx} = 1 \Leftrightarrow \frac{dy}{dx} = \frac{1}{2y}$$

For any point (x_0, y_0) that satisfies $x_0 = y_0^2$, its normal to the curve is the locus of

$$\begin{aligned} y &= y_0 - \left(\frac{dy}{dx}\bigg|_{(x,y)=(x_0,y_0)} \right)^{-1} (x - x_0) \\ &= y_0 - 2y_0(x - x_0) \\ &= -2y_0 x + 2x_0 y_0 + y_0 \end{aligned}$$

The normal of the point intersects x-axis at a, so

$$0 = -2y_0 a + 2x_0 y_0 + y_0 = y_0(-2a + 2x_0 + 1)$$

$$\Rightarrow y_0 = 0 \vee -2a + 2x_0 + 1 = 0$$

As (x_0, y_0) is on the parabola, $x_0 = y_0^2 \geq 0$

$$-2a + 2x_0 + 1 = 0 \Leftrightarrow 2x_0 = 2a - 1 \geq 0 \Rightarrow a \geq 1/2.$$

The parabola is symmetric on x-axis, thus if $(a, 0)$ is on the normal of (x_0, y_0) to the curve, then $(a, 0)$ is on the normal of $(x_0, -y_0)$ to the curve.
The normals of (x_0, y_0) and $(x_0, -y_0)$ are perpendicular if $(-2y_0)(2y_0) = -1 \Rightarrow y_0 = \pm\frac{1}{2}$
So, $0 = \frac{1}{2}(-2a + \frac{1}{2} + 1) \Rightarrow a = \frac{3}{4}$.