Calculus A Homework 1

30 Sept 2021

Assigned exercises:

1. Prove that $\{ x \in \mathbb{Q} \ : x^2 < 2 \ \vee \ x < 0\}$ is a Dedekind cut.

   Definition:

   A Dedekind cut T is a non-empty proper subset of $\mathbb{Q}$ such that:

   i) $\forall p \in \mathbb{Q}, q \in T, p < q \rightarrow p \in T$

   ii) $\forall p \in T, \exists\, q \in T\ s.t.\ p < q$

   Proof:

   Let $T = \{x \in \mathbb{Q} \ : x^2 < 2 \ \vee \ x < 0\}$

   $(1 \in \mathbb{Q} \wedge 1 \in T)\ but\ (3 \in \mathbb{Q} \wedge\ 3 \notin T), so\ T\ is\ a\ non\ empty\ subset\ of\ \mathbb{Q}.$

   i) $\forall p \in \mathbb{Q},$

   $If\ p\ \leq 0, then\ obviously\ p \in T.$

   $Otherwise, let\ q \in T\ s.t. p < q.$

   $By\ definition, p < q\ and\ q^2 < 2,$

   $thus\ 0 < p^2 < q^2 < 2 \Rightarrow p^2 < 2\ , so\ p \in T.$

   ii) $\forall p \in T,$

   $If\ p \leq 0\ then\ q = 1, q \in T.$

   $Otherwise\ p > 0\ and\ p^2 < 2$

   $$\Leftrightarrow p^2 + 2p < 2 + 2p \Leftrightarrow p < \frac{2p+2}{p+2}$$

   Let $q = \frac{2p+2}{p+2}$

   By definition, p<q.

   $$q^2 < 2\ requires\ \left(\frac{2p+2}{p+2}\right)^2 - 2 < 0$$

   $$\Leftrightarrow \frac{4p^2 + 8p + 4 - 2p^2 - 8p - 8}{(p+2)^2} = \frac{2p^2 - 4}{(p+2)^2} < 0, which\ is\ true\ as\ p^2 < 2.$$

   $$So,\ q = \frac{2p+2}{p+2} \in T, thus\ T\ is\ a\ Dedekind\ cut.$$

   $Q.E.D.$

2. Let $f\colon \mathbb{N} \rightarrow \mathbb{N}$ be a map such that $\forall n \in \mathbb{N}$ we have f(f(n))=f(n). Show that f is injective if and only if f is surjective.

   Proof:

   i) Assume f is injective, but f is not surjective.

   Let $P = \mathbb{N} \backslash (Range\ f), and\ m \in P.$

   It is given that f(f(n))=f(n) so f(m)=m.

   Means that $\exists x \in Domain\ f, f(x) = m.$

Thus m does not exist and $P = \emptyset$, arising a contradiction to that f is not surjective.

ii) Assume f is surjective, but f is not injective.

$Assumptions \Rightarrow \exists\, m_1, m_2 \in Domain\ f\ s.t.\ m_1 \neq m_2\ but\ f(m_1) = f(m_2).$

$By\ assumption, Range\ f = \mathbb{N}\ (codomain\ of\ f).$

$Given\ f\big(f(n)\big) = f(n), so\ \forall a \in \mathbb{N}, f(a) = a \Rightarrow f\ is\ bijective$

That contradicts the assumption of f is not a one-to-one function. Q.E.D.

3. Let $(a_n)_{n\geq1}\ and\ (b_n)_{n\geq1}$ be two Cauchy sequences of rationals. Show that $(a_n \cdot b_n)_{n\geq1}$ is also a Cauchy sequence.

Proof:

$\because (a_n)_{n\geq1}\ and\ (b_n)_{n\geq1}$ are both Cauchy,

$\therefore \forall \epsilon > 0, \exists\, N \in \mathbb{N}\ \ s.t. \forall m, n > N,$

$|a_m - a_n| < \epsilon\ and\ |b_m - b_n| < \epsilon$

Also, $(a_n)_{n\geq1}\ and\ (b_n)_{n\geq1}$ are both bounded, so let M be the sum of the upper bounds of $(a_n)_{n\geq1}\ and\ (b_n)_{n\geq1}$.

$|a_m b_m - a_n b_n| = |a_m b_m - a_n b_m + a_n b_m - a_n b_n| \leq |b_m||a_m - a_n| + |a_n||b_m - b_n|$

$< (|a_n| + |b_m|)\ \epsilon \leq M\epsilon,$

Thus $(a_n b_n)_{n\geq1}$ is also Cauchy.

Q.E.D.

4. Show that if $(a_n)_{n\geq1}$ is a Cauchy sequence of rationals, then $(|a_n|)_{n\geq1}$ is also a Cauchy sequence. Is the converse true (justify)?

Proof:

$(a_n)_{n\geq1}$ is Cauchy, so $\forall \epsilon > 0\ , \exists N \in \mathbb{N}\, s.t. \forall\, m, n > N, |a_m - a_n| < \epsilon$

For $(|a_n|)_{n\geq1}$, $|\ |a_m| - |a_n|\ | \leq |a_m - a_n| < \epsilon$, so $(|a_n|)_{n\geq1}$ is also Cauchy.

Q.E.D.

Assume $(a_n)_{n\geq1}\ s.t. a_n = (-1)^n, obviously\ (|a_n|)_{n\geq1}\ is\ true, however\ (a_n)_{n\geq1}$ is not. So the converse is false.

Bonus exercises:

1. Prove that $\sqrt{2}$ is irrational.

Proof:

$$Assume\ \sqrt{2} \in \mathbb{Q}\ then\ it\ is\ reasonable\ to\ let\ \sqrt{2} = \frac{m}{n}\ such\ that\ \sqrt{2}\ is\ expressed\ as$$

$$a\ fraction\ in\ its\ simpliest\ form, i.e.\ \ m, n \in \mathbb{Z}\ and\ n \neq 0\ and\ \gcd(m, n) = 1\quad .$$

$$\sqrt{2} = \frac{m}{n} \Leftrightarrow m^2 = 2n^2$$

$\therefore 2|m. Let\ m = 2k.$

$(2k)^2 = 4k^2 = 2n^2 \Leftrightarrow n^2 = 2k^2\ (\therefore 2|n)$

Thus, 2|m and 2|n contradicts gcd (m,n)=1, hence $\sqrt{2}$ is irrational.