The complete expansion of determinant

- We have defined the determinant recursively, it can be expanded out.
- Let $\sigma = (\sigma_1 \cdots \sigma_n)$ be a permutation of $(1 \cdots n)$. If it interchanges only i and j, then we call it a transposition (对换), denoted $(i\ j)$. It is clear that any permutation can be written as the composition of a sequence of transpositions

$$\sigma = (i_1\ j_1) \circ (i_2\ j_2) \circ \cdots \circ (i_r\ j_r).$$

- The expression is not unique, but the parity of r is the same for any such expression. We define the sign of the permutation σ as $(-1)^r$, it is well defined, denoted $\mathrm{sgn}(\sigma)$.

- By definition, it is clear that $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)$.
- The sign of a permutation can be calculated conveniently by counting the number of reversed orders (逆序) in the sequence $(\sigma_1 \cdots \sigma_n)$. A pair σ_i, σ_j with $i < j$ is said to be in reversed order if $\sigma_i > \sigma_j$. The total number of the pairs in reversed order is denoted $r(\sigma)$.
- Proposition: $\mathrm{sgn}(\sigma) = (-1)^{r(\sigma)}$.
- The basic reason is that

$$(i\ 1 \cdots i-1\ i+1 \cdots n) = (1\ 2) \circ (2\ 3) \circ \cdots \circ (i-1\ i)$$

Is a product of $i-1$ transposition, and the number of reversed orders is also $i-1$.

- Theorem: Let A be a $n \times n$ matrix, then

$$det(A) = \sum_{\sigma} \text{sgn}(\sigma) a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n},$$

where σ runs through all the permutations of $(1\ 2\ \cdots\ n)$.

- Example: For 2×2 matrix, $det(A) = a_{11}a_{22} - a_{12}a_{21}$.
- Example: For 3×3 matrix A, we have $\det(A)$ equals
  $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$.

- Proof:

Since the determinant is defined inductively, we'll prove the theorem by induction.

The assertion is clearly true for $n=1$, as $\det(a) = a$.

Suppose that the assertion holds for n. (Induction hypothesis).

Let A be a matrix of size $(n+1) \times (n+1)$, then

$$\det(A) = a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + \cdots + (-1)^{n+2} a_{1\,n+1} \det(A_{1\,n+1}) \quad (*)$$

Look at each term $(-1)^{j+1} a_{1j} \det(A_{1j})$. By induction hypothesis, it equals

$$(-1)^{j+1} a_{1j} \sum_{\sigma} \operatorname{sgn}(\sigma)\, a_{2\sigma_2} a_{3\sigma_3} \cdots a_{n+1,\sigma_{n+1}} . \qquad (**)$$

$\uparrow$ permutation of $(1\ 2 \cdots \hat{j} \cdots n+1)$

$\leftarrow$ deleted

Let $\sigma' = (j\ \sigma_2\ \sigma_3 \cdots \sigma_{n+1})$, then σ' is a permutation of $(1\ 2 \cdots n+1)$. Moreover, the number of reversed order

$$r(\sigma') = r(\sigma) + j - 1 \qquad j > 1, 2, \cdots, j-1 .$$

$$\Rightarrow \operatorname{sgn}(\sigma') = (-1)^{j-1} \operatorname{sgn}(\sigma) .$$

Now we see that $(**)$ equals

$$(-1)^{j+1} a_{1j} \det(A_{1j}) = \sum_{\sigma} \operatorname{sgn}(\sigma')\, a_{1\sigma'_1} a_{2\sigma'_2} \cdots a_{n+1,\sigma'_{n+1}} .$$

$\uparrow$ permutation of $(1 \cdots \hat{j} \cdots n+1)$

As j runs through $1, \cdots, n+1$, σ' runs through all the permutations of $(1\ 2 \cdots n+1)$, and so $(**)$ equals

$$\sum_{\sigma'} \operatorname{sgn}(\sigma')\, a_{1\sigma'_1} a_{2\sigma'_2} \cdots a_{n+1,\sigma'_{n+1}} .$$

$\uparrow$ permutation of $(1, \cdots, n+1)$

$\boxtimes$

- With the above complete expansion, we can prove the theorem:

> The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is
>
> $$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
>
> The cofactor expansion down the jth column is
>
> $$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

- The complete expansion also implies the theorem

> If A is an $n \times n$ matrix, then $\det A^T = \det A$.

EXAMPLE 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution: Notice that the third column has only 1 non-zero entry, we will expand along it.

$$\det(A) = \underset{\overset{\|}{0}}{a_{13}} \cdot C_{13} + a_{23} \cdot C_{23} + \underset{\overset{\|}{0}}{a_{33}}\, C_{33}$$

$$= (-1)\cdot(-1)^{2+3} \det\begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix} = -2.$$

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EXAMPLE 3 Compute det A, where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Solution Notice that the first column of A has only one non-zero entry, expand along it, get

$$\det(A) = 3 \cdot \det(A_{11}).$$

Expand along the first column of A_{11}, get

$$\det(A_{11}) = 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

expand along last column

$$= 2 \cdot (-1) \cdot (-1)^{2+3} \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix}$$

$$= -4$$

So $\det(A) = 3 \cdot \det(A_{11}) = -12.$ ◻

A linearity property

- Theorem: Let $a_1, \cdots, a_j, a_j', \cdots, a_n$ be $n+1$ vectors in R^n, then for any $\lambda, \mu \in R$, we have

$$det[a_1 \cdots \lambda a_j + \mu a_j' \cdots a_n] = \lambda det[a_1 \cdots a_j \cdots a_n] + \mu det[a_1 \cdots a_j' \cdots a_n]$$

- Same result holds for the row operations by taking transposition.

Proof Let $A = [\vec{a}_1 \cdots \vec{a}_n]$, $A' = [\vec{a}_1 \cdots \vec{a}_j' \cdots \vec{a}_n]$, $B = [\vec{a}_1 \cdots a_j + a_j' \cdots \vec{a}_n]$. By the previous theorem, we can expand along the j-th column,

$$\det(B) = (-1)^{1+j}(\lambda a_{1j} + \mu a_{1j}') \det B_{1j} + (-1)^{2+j}(\lambda a_{2j} + \mu a_{2j}') \det B_{2j}$$

$$+ \cdots + (-1)^{n+j}(\lambda a_{nj} + \mu a_{nj}') \det B_{nj}$$

$$= \lambda \Big[(-1)^{1+j} a_{1j} \underbrace{\det B_{1j}}_{= \det A_{1j}} + \cdots + (-1)^{n+j} a_{nj} \underbrace{\det B_{nj}}_{= \det A_{nj}} \Big] +$$

$$\mu \Big[(-1)^{1+j} a_{1j}' \underbrace{\det B_{1j}}_{= \det A_{1j}'} + \cdots + (-1)^{n+j} a_{nj}' \underbrace{\det B_{nj}}_{= \det A_{nj}'} \Big]$$

$$= \lambda \det(A) + \mu \det(A')$$

expansion of det(A), det(A′) along j-th column

$\boxtimes$

Properties of the determinant

- Recall that we can always row reduce a matrix to echelon form, it is then important to understand how the determinant changes under the elementary r...

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

- Take transposition, we get the same result for the elementary column operations.

- Combined with the fact that the determinant of a triangular matrix is the product of its pivots, this gives an efficient algorithm to calculate the determinant.

Proof c) Expand along the row in question,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ k a_{i1} & k a_{i2} & \cdots & k a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k a_{i1} C_{i1} + k a_{i2} C_{i2} + \cdots + k a_{in} C_{in}$$

$$= k (a_{i1} C_{i1} + \cdots + a_{in} C_{in}) = k \det(A).$$

b) Proof by induction. For $n=2$,

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Suppose that the assertion is proven for n. Let A be a matrix of size $(n+1) \times (n+1)$, let A' be the interchange of the i-th and j-th row of A. Expand $\det(A')$ along k-th row, $k \neq i, j$, get

$$\det(A') = (-1)^{k+1} a_{k1} \det(A'_{k1}) + (-1)^{k+2} a_{k2} \det(A'_{k2}) + \cdots + (-1)^{k+n+1} a_{kn} \det(A'_{k\,n+1}) \qquad (*)$$

Notice that A'_{kj} is obtained from A_{kj} by interchange two rows, by induction hypothesis,

$$\det(A'_{kj}) = -\det(A_{kj}), \quad \forall j = 1, \cdots, n+1,$$

plug into $(*)$, get $\det(A') = -\det(A)$.

a) By the linear property of $\det(A)$, we have

$$\det[\vec{a}_1 \cdots \vec{a}_i + \lambda \vec{a}_j \cdots \vec{a}_n] = \det[\vec{a}_1 \cdots \vec{a}_i \cdots \vec{a}_n] + \lambda \det[\vec{a}_1 \cdots \vec{a}_j \cdots \vec{a}_j \cdots \vec{a}_n]$$

<u>claim</u> $\det[\vec{a}_1 \cdots \vec{a}_j \cdots \vec{a}_j \cdots \vec{a}_n] = 0.$

interchange two columns will change the sign, hence can only be 0.

$$\Rightarrow \det[\vec{a}_1 \cdots \vec{a}_i + \lambda \vec{a}_j \cdots \vec{a}_n] = \det[\vec{a}_1 \cdots \vec{a}_i \cdots \vec{a}_n]$$

- Corollary: Let $A = \begin{bmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{rr} \end{bmatrix}$ be a matrix in block with A_{ii} being a matrix of size $n_i \times n_i$, then

$$det(A) = det(A_{11})det(A_{22}) \cdots det(A_{rr}).$$

Proof The elementary row operation have the same effects to both A and the submatrices $A_{11}, \cdots, A_{rr}$, hence it is enough to consider the triangular case, for which the assertion is clear as determinant is product of diagonal entries. ☒

EXAMPLE 2 Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Solution

$$det(A) = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2 \cdot 2 \cdot \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \cdot 2 \cdot 1 \cdot 3 \cdot (-3) \cdot 1$$

$$= -36.$$

☒

- Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

Proof The elementary row operations doesn't change the rk of A, hence it is enough to consider the triangular

A invertible $\Leftrightarrow$ rk(A) = n

case, for which

$$\det(A) = a_{11} \cdot a_{22} \cdots a_{nn},$$

hence A is invertible iff $\det(A) \neq 0$. $\square$

- Theorem

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Proof We begin by examining the case when A is an elementary matrix:

1) $A = \begin{array}{c} \\ i \\ j \\ \\ \end{array} \begin{pmatrix} 1 & & & \\ & 1 & \lambda & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$, then AB is obtained from B by adding the λ-multiple of j-th row of A to i-th row of A

$$\Rightarrow \det(AB) \underset{\text{previous theorem}}{=} \det(B) = \underset{\overset{\shortparallel}{1}}{\det(A)} \cdot \det(B)$$

2) $A = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array}\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & 1 & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & 1 & & & 0 & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{pmatrix}$, then AB is obtained from B by interchanging the i-th and j-th row

$$\Rightarrow \det(AB) = -\det(B) = \underset{\substack{\| \\ -1}}{\det(A)} \cdot \det(B)$$

as A is obtained from I_n by interchanging i-th and j-th row.

3) $A = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \lambda & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$, then AB is obtained from B by multiplying the i-th row by λ

$$\Rightarrow \det(AB) = \lambda \det(B) = \det(A) \cdot \det(B).$$

Back to the general case. If either A or B is not invertible, then AB can not be invertible (why?) The assertion is clearly true as both sides equals 0.

Suppose that A is invertible, then A can be written as products of elementary matrices

$$A = E_1 \cdots E_p \qquad \Rightarrow \det(A) = \det(E_1) \cdots \det(E_p).$$

as it is now equivalent to I_n. So

$$\det(AB) = \det(E_1 \cdots E_p B) = \det(E_1) \cdots \det(E_p) \cdot \det(B)$$
$$= \det(A) \det(B).$$

$\boxed{\text{Ⓑ}}$