Topics in Linear Algebra: Homework 10

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Solution 1.10.1.

1.
$$T^{11} = T^{22} = -\frac{1}{\sqrt{2}} - 1, T^{12} = T^{21} = -1$$

2. Indeed, by observation, the two columns in $\mathrm{Rot}(\pi/4)$ are possible to be x and y, so let
$$\mathbf{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
Then,
$$\mathbf{x} \otimes \mathbf{x}$$
has all entries 1/2, and
$$\mathbf{y} \otimes \mathbf{y}$$
has entry 1/2 when having equal indices, and $-1/2$ when having distinct indices.
Immediately
$$T = -(\sqrt{2}+1)\mathbf{x} \otimes \mathbf{x} - \mathbf{y} \otimes \mathbf{y}$$

3. By observing the result of (2) and the definiton of T, the resultant magnitude of force in $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ direction is larger than that of $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ direction, so the direction of major axis is $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, while the direction of minor is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

4. As the force are "normal" to the circle,
$$T = \sum_{k=1}^{r} c_k \mathbf{x}_k \otimes \mathbf{x}_k$$
where $\mathbf{x}_k \in R^2$, $c_k \in R$, $\|\mathbf{x}_k\| = 1$(in different directions), $c_k < 0$, $k = 1, 2, \cdots, r$.
(Here only only consider forces on upper half of the circle, and the force on lower part are moved to the position π rad apart of the original position of the circle, with direcction reversed). Let $\mathbf{x_k} = a_k i + b_k j$, then Matrix form of T_{ij} is T^i_j
$$T^i_j = \begin{bmatrix} \sum_{k=1}^r c_k(a_k)^2 & \sum_{k=1}^r c_k a_k b_k \\ \sum_{k=1}^r c_k a_k b_k & \sum_{k=1}^r c_k (b_k)^2 \end{bmatrix}$$
By Cauchy-Schwarz inequality,
$$\det(T^i_j) = \left(\sum_{k=1}^r c_k(a_k)^2\right)\left(\sum_{k=1}^r c_k(b_k)^2\right) - \left(\sum_{k=1}^r c_k a_k b_k\right)^2$$
$$= \left(\sum_{k=1}^r (\sqrt{-c_k}a_k)^2\right)\left(\sum_{k=1}^r (\sqrt{-c_k}b_k)^2\right) - \left(\sum_{k=1}^r (\sqrt{-c_k}a_k)(\sqrt{-c_k}b_k)\right)^2 \geq 0$$

Equality holds iff

$$\frac{\sqrt{-c_k}a_k}{\sqrt{-c_k}b_k} = C$$

However, it is not possible, unless $r = 1$.
And trivially, $\operatorname{tr}(T^i_j) < 0$, so the matrix is negative semi-definite. It is negative definite iff $r > 1$.
Moreover, the matrix is symmetric, when the matrix representation is not symmetric, or when it is not negative semi-definite, there must be force that is not perpendicularly towards the center of circle.
For instance, $T^{ij} = \delta^{ij}$ is not possible.

Solution 1.10.2.

1. $\forall \alpha \in V^*, v \in V$,

$$\alpha_{\mathcal{B}}(v_{\mathcal{B}}) = \alpha_{\mathcal{C}}(v_{\mathcal{C}}) = \alpha_{\mathcal{C}}(Mv_{\mathcal{B}})$$

$$\Leftrightarrow \alpha_{\mathcal{B}} = \alpha_{\mathcal{C}}M \Leftrightarrow \alpha_{\mathcal{C}} = \alpha_{\mathcal{B}}M^{-1}$$

2. $$T(v,w) = (\sum_{i,j} x_{ij} b_i^* \otimes b_j^*)(u,v) = \sum_{i,j} x_{ij} b_i^*(v) b_j^*(w) = x_{ij} v^i_{\mathcal{B}} w^j_{\mathcal{B}} = v^i_{\mathcal{B}} \delta_{ik} x^k_j w^j_{\mathcal{B}} = v^T_{\mathcal{B}} T_{\mathcal{B}} w_{\mathcal{B}}$$

3. Replacing x and b to y and c in the proof above immediately yields

$$v^T_{\mathcal{C}} T_{\mathcal{C}} w_{\mathcal{C}} = T(u,v) = v^T_{\mathcal{B}} T_{\mathcal{B}} w_{\mathcal{B}}$$

Moreover, $v_{\mathcal{C}} = Mv_{\mathcal{B}}$, therefore

$$v^T_{\mathcal{B}} M^T T_{\mathcal{C}} M w_{\mathcal{B}} = v^T_{\mathcal{B}} T_{\mathcal{B}} w_{\mathcal{B}}$$

for any v, w, and hence

$$M^T T_{\mathcal{C}} M = T_{\mathcal{B}} \Rightarrow T_{\mathcal{C}} = (M^T)^{-1} T_{\mathcal{B}} M^{-1}$$

4. From 1, $\alpha_{\mathcal{B}} = \alpha_{\mathcal{C}} M$, so

$$\alpha_{\mathcal{C}} T_{\mathcal{C}} \beta^T_{\mathcal{C}} = T(\alpha,\beta) = \alpha_{\mathcal{B}} T_{\mathcal{B}} \beta^T_{\mathcal{B}} = \alpha_{\mathcal{C}} M T_{\mathcal{B}} M^T \beta^T_{\mathcal{C}}$$

for any α, β, and similarly

$$T_{\mathcal{C}} = M T_{\mathcal{B}} M^T$$

5. Similar to above,

$$\alpha_{\mathcal{C}} T_{\mathcal{C}} v_{\mathcal{C}} = T(\alpha, v) = \alpha_{\mathcal{B}} T_{\mathcal{B}} v_{\mathcal{B}} = \alpha_{\mathcal{C}} M T_{\mathcal{B}} M^{-1} v_{\mathcal{C}}$$

So

$$T_{\mathcal{C}} = M T_{\mathcal{B}} M^{-1}$$

Solution 1.10.3.

Gradient:

$$\nabla f = 2xi + 2yj + 2zk$$

New function:

$$f_{\text{new}}(x+y, y+z, z) = x^2 + y^2 + z^2$$

Let $u = x + y, v = y + z$, then

$$f_{\text{new}}(u,v,z) = (u - v + z)^2 + (v - z)^2 + z^2$$

$$= u^2 + v^2 + z^2 - 2uv - 2vz + 2uz + v^2 + z^2 - 2vz + z^2$$

$$= u^2 + 2v^2 + 3z^2 - 2uv - 4vz + 2zu$$

$$f_{\text{new}}(x,y,z) = x^2 + 2y^2 + 3z^2 - 2xy - 4yz + 2zx$$

Gradient of new function:

$$\nabla f_{\text{new}} = (2x - 2y + 2z)i + (4y - 2x - 4z)j + (6z - 4y + 2x)k$$

$$(M^{-1})^T = ((I+N)^{-1})^T = (I-N+N^2)^T = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 1 & -1 & 1 \end{bmatrix}$$

Verification:

$$\text{R.H.S.} = (M^{-1})^T(\nabla f(x,y,z) = 2xi + (-2x+2y)j + (2x-2y+2z)k$$

$$\text{L.H.S.} = \nabla f(x+y, y+z, z) = (2(x+y)-2(y+z)+2z)i + (4(y+z)-2(x+y)-4z)j + (6z-4(y+z)+2(x+y))k$$

$$= 2xi + (2y-2x)j + (2z-2y+2x)k$$

So L.H.S = R.H.S.