Calculus A(1): Homework 2

October 6, 2021

Assigned Exercises

Exercises 11.1

59.

Convergence of $a_n = \frac{n!}{n^n}$

Solution.

Obviously, $a_n \geq 0 (\forall n \in \mathbb{N})$.
On the other hand,

$$a_n = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} \leq 1 \cdot 1 \cdot 1 \cdots 1 \cdot \frac{1}{n}$$

Hence,

$$0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus, a_n converges, and

$$\lim_{n\to\infty} a_n = 0$$

85.

The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n$$

Write out enough early terms of the sequence to deduce a general formula for x_n that holds for $n \geq 2$.

Solution.

By definition,

$$x_1 = 1, x_2 = x_1 = 1, x_3 = x_1 + x_2 = 2, x_4 = x_1 + x_2 + x_3 = 4, \ldots$$

The following proves (by induction) that $a_n = 2^{n-2}$, $n \in \mathbb{N}$, $n \geq 2$.
Let $P(n) : a_n = 2^{n-2}$. The following proves $(\forall n \in \mathbb{N}, n \geq 2) P(n)$ is true.

Proof:

1. $P(2) : a_2 = 2^0 = 1$, so P(2) is true.
2. Assume $P(2), P(3), \ldots, P(k-1), P(k)$ is true for some integers $k > 2$.
   $P(k+1)$:
   $$a_{k+1} = \sum_{i=1}^{k} a_i = 1 + \sum_{i=2}^{k} 2^{i-2} = 1 + \frac{2^{k-1} - 1}{2 - 1} = 2^{k-1}$$
   Thus $P(k+1)$ is true.

By (1),(2) and the second principle of mathematical induction, $P(n)$ is true for all integers $n \geq 2$.

92.

The zipper theorem Prove the "zipper theorem" for sequences: If $\{a_n\}$ and $\{b_n\}$ both converge to L, then the sequence

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$$

converges to L.

Solution.

Let $\{c_n\}_{n\geq 1}$ such that

$$c_n = \begin{cases} a_{\frac{n+1}{2}} & ,2 \nmid n \\ b_{\frac{n}{2}} & ,2 \mid n \end{cases}$$

By definition, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > N$,

$$|a_n - L| < \epsilon$$

and

$$|b_n - L| < \epsilon$$

Let $N' = 2N - 1, N'' = 2N$
Above implies that for any odd number $n' > N'$,

$$|a_{n'} - L| < \epsilon$$

and for any even number $n'' > N''$,

$$|b_{n''} - L| < \epsilon$$

Thus, $\forall \epsilon > 0, \forall n > N''$,

$$|c_n - L| = \begin{cases} |a_{\frac{n+1}{2}} - L| < \epsilon & ,2 \nmid n \\ |b_{\frac{n}{2}} - L| < \epsilon & ,2 \mid n \end{cases}$$

Which shows that $c_n \to L$.

119.

For a sequence $\{a_n\}$ the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1}. Prove that if $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.

Solution.

Refer to solution 92. The proof follows by letting the sequence of odd indices as $\{a_n\}$, the even indices as $\{b_n\}$ and the sequence itself as $\{c_n\}$.

Bonus Exercises

86.

(Pell's equation) A sequence of rational numbers is defined as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a+2b}{a+b}, \ldots$$

Here, the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be respectively the numerator and the denominator of the fraction $r_n = x_n/y_n$.

a. Verify that $x_1^2 - 2y_1^2 = -1$, $x_2^2 - 2y_2^2 = +1$ and, more generally, that if $a^2 - 2b^2 = \mp 1$, then

$$(a+2b)^2 - 2(a+b)^2 = \pm 1$$

respectively.

b. The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit?

Solution.

a. By definition, $x_1 = 1, x_2 = 3, y_1 = 1, y_2 = 2$,

$$x_1^2 - 2y_1^2 = 1 - 2 = -1, x_2^2 - 2y_2^2 = 9 - 8 = 1$$

$$(a + 2b)^2 - 2(a + b)^2 = a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2 = -a^2 + 2b^2 = -(a^2 - 2b^2) = \pm 1$$

b. By definition of x_n, y_n,

$$\begin{cases} x_n = x_{n-1} + 2y_{n-1} \\ y_n = x_{n-1} + y_{n-1} \end{cases}$$

$$r_n = \frac{x_n}{y_n} = \frac{x_{n-1} + 2y_{n-1}}{x_{n-1} + y_{n-1}} = \frac{\frac{x_{n-1}}{y_{n-1}} + 2}{\frac{x_{n-1}}{y_{n-1}} + 1} = \frac{r_{n-1} + 2}{r_{n-1} + 1}$$

Consider $\frac{r_n - \sqrt{2}}{r_n + \sqrt{2}}$.

$$\frac{r_n - \sqrt{2}}{r_n + \sqrt{2}} = \frac{\frac{r_{n-1}+2}{r_{n-1}+1} - \sqrt{2}}{\frac{r_{n-1}+2}{r_{n-1}+1} + \sqrt{2}}$$

$$= \frac{(1 - \sqrt{2})r_{n-1} + 2 - \sqrt{2}}{(1 + \sqrt{2})r_{n-1} + 2 + \sqrt{2}}$$

$$= \frac{(1 - \sqrt{2})(r_{n-1} + \frac{2-\sqrt{2}}{1-\sqrt{2}})}{(1 + \sqrt{2})(r_{n-1} + \frac{2+\sqrt{2}}{1+\sqrt{2}})}$$

$$= \frac{1 - \sqrt{2}}{1 + \sqrt{2}} \cdot \frac{r_{n-1} - \sqrt{2}}{r_{n-1} + \sqrt{2}}$$

Thus,

$$\frac{r_n - \sqrt{2}}{r_n + \sqrt{2}} = (\frac{1 - \sqrt{2}}{1 + \sqrt{2}})^n$$

$$r_n = \sqrt{2} \cdot \frac{1 + (\frac{1-\sqrt{2}}{1+\sqrt{2}})^n}{1 - (\frac{1-\sqrt{2}}{1+\sqrt{2}})^n}$$

Hence,

$$\lim_{n \to \infty} r_n = \sqrt{2}$$

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