Linear Algebra: Homework 5

October 30, 2021

Question 1.

The inverse of $\begin{bmatrix} I & & \\ C & I & \\ A & B & I \end{bmatrix}$ is $\begin{bmatrix} I & & \\ Z & I & \\ X & Y & I \end{bmatrix}$. Find X, Y, Z in terms of A, B, C.

Solution 1.

$$\begin{bmatrix} I & & \\ C & I & \\ A & B & I \end{bmatrix}\begin{bmatrix} I & & \\ Z & I & \\ X & Y & I \end{bmatrix} = \begin{bmatrix} I & & \\ C+Z & I & \\ A+BZ+X & B+Y & I \end{bmatrix}$$

$$\Rightarrow C+Z = A+BZ+X = B+Y = 0 \Rightarrow Y = -B, Z = -C, X = -A-BZ = -A+BC$$

Question 2.

Suppose that A_{11} is invertible. Find X and Y such that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & \\ & S \end{bmatrix}\begin{bmatrix} I & Y \\ & I \end{bmatrix}.$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Solution 2.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & \\ & S \end{bmatrix}\begin{bmatrix} I & Y \\ & I \end{bmatrix} = \begin{bmatrix} I & \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{11}Y \\ & S \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y+S \end{bmatrix}$$

Thus,

$$A_{12} = A_{11}Y, A_{21} = XA_{11} \Rightarrow X = A_{21}A_{11}^{-1}, Y = A_{11}^{-1}A_{12}$$

Question 3.

Use partitioned matrices to prove by induction that for $n = 2, 3, \cdots$, the $n \times n$ matrix A shown below is invertible and B is its inverse.

$$A = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

Solution 3.

Every column of A is pivot column, thus A is invertible.

Let $P(n) : A_n^{-1} = B_n = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}_{n\times n}$

$(\forall n \in \{2,3,\cdots\})P(n)$ will be proved.
$P(2) : \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, thus $P(2)$ is true.

Assume $P(k)$ is true for some integer $k \geq 2$, i.e. $A_k^{-1} = B_k = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}_{k\times k}$

$P(k+1)$:
Let $A_{k+1}^{-1} = \begin{bmatrix} L_1 & \vec{b_1} \\ \vec{b_2}^t & d \end{bmatrix}$
$A_{k+1} \cdot \begin{bmatrix} L_1 & \vec{b_1} \\ \vec{b_2}^t & d \end{bmatrix} = \begin{bmatrix} A_k & \vec{0} \\ \vec{1}^t & 1 \end{bmatrix} \cdot \begin{bmatrix} L_1 & \vec{b_1} \\ \vec{b_2}^t & d \end{bmatrix} = \begin{bmatrix} I_n & \vec{0} \\ \vec{0}^t & 1 \end{bmatrix}$. Thus,

$$\begin{cases} A_k L_1 = I_n \\ A_k \vec{b_1} = 0 \\ \vec{1}^t L_1 + \vec{b_2}^t = 0 \\ \vec{1}^t \vec{b_1} + d = 1 \end{cases}$$

Hence, $L_1 = B_k, \vec{b_1} = 0, d = 1$,

$$\vec{b_2}^t = -\vec{1}^t L_1 = -\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}_{k\times k} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

Hence $A_{k+1}^{-1} = B_{k+1}$, $P(k) \Rightarrow P(k+1)$
By first principle of mathematical induction, $P(n)$ is true for all integers $n \geq 2$.

Question 4.

Without using row reductions, find the inverse of $A = \begin{bmatrix} 1 & 2 & & & \\ 3 & 5 & & & \\ & & 2 & & \\ & & & 7 & 8 \\ & & & 5 & 6 \end{bmatrix}$

Solution 4.

Let $A_{11} = \begin{bmatrix} 1 & 2 & \\ 3 & 5 & \\ & & 2 \end{bmatrix}$, $A_{12} = (0)_{3\times 2}, A_{21} = (0)_{2\times 3}, A_{22} = \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$
Let $B = A^{-1}$ s.t. $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$
Thus, $AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_2 \end{bmatrix}$.
Clearly, A_{11}, A_{22} are both invertible. Thus B_{12} and B_{21} are zero matrices.
$B_{11} = A_{11}^{-1}, B_{22} = A_{22}^{-1}$. $B_{22} = \frac{1}{7\cdot 6 - 5\cdot 8}\begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -5/2 & 7/2 \end{bmatrix}$.
Let $A_{11} = \begin{bmatrix} A_{111} & \vec{0} \\ \vec{0}^t & 2 \end{bmatrix}$, $B_{11} = \begin{bmatrix} b_{111} & b_{112} \\ b_{121} & b_{122} \end{bmatrix}$. Thus, $b_{122} = 1/2, A_{111}b_{111} = I_2, A_{111}b_{112} = \vec{0}, 2b_{121} = \vec{0}^t$.
Hence, $b_{111} = A_{111}^{-1} = \frac{1}{5-6}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$, $b_{112} = \vec{0}, b_{121} = \vec{0}^t$.

Therefore, $A^{-1} = \begin{bmatrix} -5 & 2 & & & \\ 3 & -1 & & & \\ & & 1/2 & & \\ & & & 3 & -4 \\ & & & -5/2 & 7/2 \end{bmatrix}$

Question 5.

Find the LU-factorization of the matrices:

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix}$$

Solution 5.

$$A = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ 3/2 & 0 & 1 & & \\ -3 & 0 & 0 & 1 & \\ 4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ 3/2 & -2 & 1 & & \\ -3 & 2 & 0 & 1 & \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 6.

Suppose $A = UDV^t$, where U and V are $n \times n$ matrices with the property that $U^tU = I$ and $V^tV = I$, and where D is a diagonal matrix with positive numbers $\sigma_1, \cdots, \sigma_n$ on the diagonal. Show that A is invertible and find a formula for A^{-1}.

Solution 6.

Lemma: $D^{-1} = \begin{bmatrix} \sigma_1^{-1} & & & \\ & \sigma_2^{-1} & & \\ & & \ddots & \\ & & & \sigma_n^{-1} \end{bmatrix}$.

Proof: $\left[\begin{array}{cccc|cccc} \sigma_1 & & & & 1 & & & \\ & \sigma_2 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & \sigma_n & & & & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & & & & \sigma_1^{-1} & & & \\ & 1 & & & & \sigma_2^{-1} & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & \sigma_n^{-1} \end{array}\right]$.

Given also $U^tU = I, V^tV = I$, hence $U^{-1} = U^t, V^{-1} = V^t$, thus U, D, V are invertible.
$AV = UDV^tV = UD \Rightarrow AVD^{-1} = U \Rightarrow AVD^{-1}U^t = UU^t = (U^tU)^t = I^t = I$
Thus, A is invertible, and $A^{-1} = VD^{-1}U^t$.

Question 7.

Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the diagonal matrix

$$D = \begin{bmatrix} 1 & & \\ & 1/2 & \\ & & 1/3 \end{bmatrix}.$$

Show that this factorization is useful when computing high powers of A. Find fairly simple formulas for A^2, A^3 and A^k using P and the entries of D.

Solution 7.

Let $P(n): D^n = \begin{bmatrix} 1 & & \\ & 2^{-n} & \\ & & 3^{-n} \end{bmatrix}$. The following proves $(\forall n \in \mathbb{N}^*)P(n)$.

1. $P(1)$ is clearly true.

2. Assume $P(k)$ is true for some $k \in \mathbb{N}^*$, i.e. $D^k = \begin{bmatrix} 1 & & \\ & 2^{-k} & \\ & & 3^{-k} \end{bmatrix}$.

   $P(k+1): D^{k+1} = \begin{bmatrix} 1 & & \\ & 2^{-k} & \\ & & 3^{-k} \end{bmatrix} \begin{bmatrix} 1 & & \\ & 2^{-1} & \\ & & 3^{-1} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 2^{-(k+1)} & \\ & & 3^{-(k+1)} \end{bmatrix}$, thus $P(k) \Rightarrow P(k+1)$.

By (1),(2) and the first principle of mathematical induction, $P(n)$ is true for all positive integers n.

$A^k = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD^kP^{-1}$, where $D^k = \begin{bmatrix} 1 & & \\ & 2^{-k} & \\ & & 3^{-k} \end{bmatrix}$

Question 8.

Find bases for the column space and the null space of $A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$.

Solution 8.

$$\begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ 0 & -8/3 & -4 & 20/3 & -4 \\ 0 & 10/3 & -2 & 10/3 & -2 \\ 0 & -2 & -3 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ 0 & -2 & -3 & 5 & -3 \\ 0 & 5 & -3 & 5 & -3 \\ 0 & -2 & -3 & 5 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ 0 & -2 & -3 & 5 & -3 \\ 0 & 5 & -3 & 5 & -3 \\ 0 & -2 & -3 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ 0 & -2 & -3 & 5 & -3 \\ 0 & 0 & -21/2 & 35/2 & -21/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ 0 & -2 & -3 & 5 & -3 \\ 0 & 0 & -3 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For the column space of A,

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$$

Let $\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^t$ be any vector in the null space of A. Let $x_4 = t_1', x_5 = t_2$. Then, $x_3 = \frac{5}{3}t_1' - t_2$, $x_2 = \frac{1}{2}(-3x_3 + 5x_4 - 3x_5) = 0$, $x_1 = \frac{1}{3}t_1' - t_2$. Also let $t_1 = t_1'/3$

$$\vec{x} = \begin{bmatrix} \frac{1}{3}t_1' - t_2 \\ 0 \\ \frac{5}{3}t_1' - t_2 \\ t_1' \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ 3 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

For the null space of A,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 9.

Determine whether $\vec{w}$ is in the column space of A, the null space of A, or both, where

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}, A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

Solution 9.

$$A\vec{w} = \begin{bmatrix} 7+6+4-3 \\ -5-1+0+6 \\ 9-11-7+9 \\ 19-9-7-3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq 0, \text{ thus } \vec{w} \notin \mathcal{N}(A).$$

Consider $[A\vec{w}]$.

$$\begin{bmatrix} 7 & 6 & -4 & 1 & 1 \\ -5 & -1 & 0 & -2 & 1 \\ 9 & -11 & 7 & -3 & -1 \\ 19 & -9 & 7 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 7 & 6 & -4 & 1 & 1 \\ 0 & 23/7 & -20/7 & -9/7 & 12/7 \\ 0 & -131/7 & 95/7 & -30/7 & -16/7 \\ 0 & -177/7 & 125/7 & -12/7 & -40/7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 7 & 6 & -4 & 1 & 1 \\ 0 & 23/7 & -20/7 & -9/7 & 12/7 \\ 0 & 0 & -665/161 & -1869/161 & 1204/161 \\ 0 & 0 & -665/161 & -1869/161 & 1204/161 \end{bmatrix} \sim \begin{bmatrix} 7 & 6 & -4 & 1 & 1 \\ 0 & 23/7 & -20/7 & -9/7 & 12/7 \\ 0 & 0 & -665/161 & -1869/161 & 1204/161 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\vec{w} \in \mathcal{R}(A)$

Question 10.

Let $\vec{a_1}, \cdots, \vec{a_5}$ be the column vectors of $A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix}$, let $B = [\ \vec{a_1} \quad \vec{a_2} \quad \vec{a_4}\ ]$.

1. Explain why $\vec{a_3}$ and $\vec{a_5}$ are in the column space of B.
2. Find a set of vectors that spans $\mathrm{Nul}(A)$.
3. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be defined by $T(\vec{x}) = A\vec{x}$. Explain why T is neither one-to-one nor onto.

Solution 10.

1. To begin, compute the row echelon form for A.

$$A \sim \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 0 & 12/5 & 4/5 & -11/5 & -12 \\ 0 & 12/5 & 4/5 & -41/5 & 12 \\ 0 & 3/5 & 1/5 & -4/5 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 0 & 12/5 & 4/5 & -11/5 & -12 \\ 0 & 0 & 0 & -6 & 24 \\ 0 & 0 & 0 & -1/4 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 0 & 12/5 & 4/5 & -11/5 & -12 \\ 0 & 0 & 0 & -6 & 24 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Both $\vec{a_3}$ and $\vec{a_5}$ are free columns in A. Each of these two column vectors can be expressed as a linear combination of $\vec{a_1}, \vec{a_2}$ and $\vec{a_4}$. Hence, $\vec{a_3}, \vec{a_5} \in \mathcal{R}(B)$.

2. Let $\vec{x} \in \mathbb{R}^5$ s.t. $A\vec{x} = 0$. Also treat $t_1, t_2 \in \mathbb{R}$ as parameters and $x_3 = t_1, x_5 = t_2$.
Then, $x_4 = 4t_2$, $12x_2 + 4t_1 - 11(4t_2) - 60t_2 = 0 \Rightarrow x_2 = \frac{1}{3}(-t_1 + 26t_2)$,
$5x_1 + x_2 + 2x_3 + 2x_4 = 0 \Rightarrow x_1 = -\frac{1}{3}(t_1 + 10t_2)$

$$\vec{x} = t_1 \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -\frac{10}{3} \\ \frac{26}{3} \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$$\mathcal{N}(A) = Span \left\{ \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{10}{3} \\ \frac{26}{3} \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

3. The column vectors of A are linearly dependent, so $T(\vec{x}) = A\vec{x} = 0$ has non-trivial solutions.
$\Rightarrow T$ is not one-to-one.
A has 3 pivot columns, so dim $\mathcal{R}(A) = 3$. However, dim $\mathbb{R}^4$=4
$\Rightarrow$ dim $\mathcal{R}(A) <$ dim $\mathbb{R}^4 \Leftrightarrow \mathbb{R}^4 \backslash \mathcal{R}(A) \neq \emptyset \Leftrightarrow T$ is not onto.

Question 11.

It is known that a linear independent set $\{\vec{v_1}, \cdots, \vec{v_k}\}$ in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$. One way to do this is to consider the matrix $A = [\ \vec{v_1} \quad \cdots \quad \vec{v_k} \quad \vec{e_1} \quad \cdots \quad \vec{e_n}\ ]$ with $\vec{e_1}, \cdots, \vec{e_n}$ the standard basis of $\mathbb{R}^n$. The pivot columns of A form a basis for $\mathbb{R}^n$.

1. Use the method described to extend the following vectors to a basis of $\mathbb{R}^5$.

$$\vec{v_1} = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}.$$

2. Explain why the method works in general: Why are the original vectors $\vec{v_1}, \cdots, \vec{v_k}$ included in the basis found for Col(A)? Why is Col(A)=$\mathbb{R}^n$?

Solution 11.

1.

$$A = \begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ -7 & 4 & 7 & 0 & 1 & 0 & 0 & 0 \\ 8 & 1 & -8 & 0 & 0 & 1 & 0 & 0 \\ -5 & 6 & 5 & 0 & 0 & 0 & 1 & 0 \\ 7 & -7 & -7 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 7/3 & -7/9 & 1 & 0 & 0 & 0 \\ 0 & 9 & -8/3 & 8/9 & 0 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & -5/9 & 0 & 0 & 1 & 0 \\ 0 & 0 & -7/3 & 7/9 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 7/3 & -7/9 & 1 & 0 & 0 & 0 \\ 0 & 0 & 13/3 & -13/9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 22/9 & -22/27 & 1/3 & 0 & 1 & 0 \\ 0 & 0 & -7/3 & 7/9 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 7/3 & -7/9 & 1 & 0 & 0 & 0 \\ 0 & 0 & 13/3 & -13/9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -53/39 & -22/39 & 1 & 0 \\ 0 & 0 & 0 & 0 & 21/13 & 7/13 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 7/3 & -7/9 & 1 & 0 & 0 & 0 \\ 0 & 0 & 13/3 & -13/9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -53/39 & -22/39 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7/53 & 63/53 & 1 \end{bmatrix}$$

Hence $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{e_2}, \vec{e_3}\}$ of A forms a basis of $\mathbb{R}^5$.

2. The original set of vectors are assumed to be independent, and these vectors are put on the left, hence these vectors correspond to the first k pivot columns. Columns of A includes all standard basis vectors of $\mathbb{R}^n$, so the columns of A spans the whole $\mathbb{R}^n \Rightarrow \mathcal{R}(A) = \mathbb{R}^n$.