Advanced Linear Algebra: Mid-Term

April 22, 2022

Solution 1

1. Proof:

$$i^2 = \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}^2 = \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

$$j^2 = \begin{bmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{bmatrix}^2 = \begin{bmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{bmatrix}\begin{bmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

$$k^2 = \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}^2 = \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}\begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

$$ijk = \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}\begin{bmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{bmatrix}\begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

$$= \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

2. (Actually, it is known as Hamilton product)

$$L_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix}, R_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & z & -y \\ y & -z & r & x \\ z & y & -x & r \end{bmatrix}$$

$$L_qR_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix}\begin{bmatrix} r & -x & -y & -z \\ x & r & z & -y \\ y & -z & r & x \\ z & y & -x & r \end{bmatrix}$$

$$= \begin{bmatrix} r^2 - x^2 - y^2 - z^2 & -2rx & -2ry & -2rz \\ 2rx & r^2 - x^2 + y^2 + z^2 & -2xy & -2xz \\ 2ry & -2xy & r^2 + x^2 - y^2 + z^2 & -2yz \\ 2rz & -2xz & -2yz & r^2 + x^2 + y^2 - z^2 \end{bmatrix}$$

$$R_qL_q = \begin{bmatrix} r & -x & -y & -z \\ x & r & z & -y \\ y & -z & r & x \\ z & y & -x & r \end{bmatrix}\begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix}$$

$$
= \begin{bmatrix} r^2 - x^2 - y^2 - z^2 & -2rx & -2ry & -2rz \\ 2rx & r^2 - x^2 + y^2 + z^2 & -2xy & -2xz \\ 2ry & -2xy & r^2 + x^2 - y^2 + z^2 & -2yz \\ 2rz & -2xz & -2yz & r^2 + x^2 + y^2 - z^2 \end{bmatrix}
$$

So, $L_q R_q = R_q L_q$.

3.

$$
L_q R_{\bar{q}} = \begin{bmatrix} r^2 + x^2 + y^2 + z^2 & 0 & 0 & 0 \\ 0 & r^2 + x^2 - y^2 - z^2 & 2xy - 2rz & 2xz + 2ry \\ 0 & 2xy + 2rz & r^2 - x^2 + y^2 - z^2 & 2yz - 2xr \\ 0 & 2xz - 2yr & 2yz - 2rx & r^2 - x^2 - y^2 + z^2 \end{bmatrix}
$$

Say $q = r + xi + yj + zk$, its matrix representation is

$$
\begin{bmatrix} r & -x & -y & -z \\ x & r & -z & y \\ y & z & r & -x \\ z & -y & x & r \end{bmatrix}.
$$

Then its conjugate $\bar{q}$ has matrix representation of

$$
\begin{bmatrix} r & x & y & z \\ -x & r & z & -y \\ -y & -z & r & x \\ -z & y & -x & r \end{bmatrix}
$$

which is exactly the transpose of its conjugate form.

By letting $Q = \begin{bmatrix} r^2 + x^2 - y^2 - z^2 & 2xy - 2rz & 2xz + 2ry \\ 2xy + 2rz & r^2 - x^2 + y^2 - z^2 & 2yz - 2xr \\ 2xz - 2yr & 2yz - 2rx & r^2 - x^2 - y^2 + z^2 \end{bmatrix}$ (here it is not necessary orthogonal, but its orthognality will be shown below),

$$
L_q R_{\bar{q}} = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}, (L_q R_{\bar{q}})^T (L_q R_{\bar{q}}) = \begin{bmatrix} 1 & 0 \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^T Q \end{bmatrix}
$$

So Q is orthogonal iff $L_q R_{\bar{q}}$ is.

Due to the fact that q is unit quaternion, the matrix representation of $q\bar{q}$ has to be the identity matrix, therefore

$$
(L_q R_{\bar{q}})^T (L_q R_{\bar{q}}) = R_{\bar{q}}^T L_q^T L_q R_{\bar{q}} = R_q L_{\bar{q}} L_q R_{\bar{q}} = R_q R_{\bar{q}} = I
$$

Hence Q is orthogonal. ■

Solution 2

1. Let $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, then

$$
\begin{bmatrix} R & \\ & N \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} R' & \\ & N' \end{bmatrix}
$$

$$
\Leftrightarrow \begin{bmatrix} RX_{11} & RX_{12} \\ NX_{21} & NX_{22} \end{bmatrix} = \begin{bmatrix} X_{11}R' & X_{12}N' \\ X_{21}R' & X_{22}N' \end{bmatrix}
$$

So $RX_{12} = X_{12}N'$ and $NX_{21} = X_{21}R'$.

By the uniqueness of the solutions of Sylvester's equations, $X_{12} = 0$, $X_{21} = 0$.

$X_{11}R'X_{11}^{-1} = R \Leftrightarrow X_{11}(R')^{-1}X_{11}^{-1} = R^{-1}$.

$$
\Rightarrow X \begin{bmatrix} (R')^{-1} & \\ & 0 \end{bmatrix} X^{-1} = \begin{bmatrix} X_{11}(R')^{-1}X_{11}^{-1} & \\ & 0 \end{bmatrix} = \begin{bmatrix} R^{-1} & \\ & 0 \end{bmatrix}
$$

2.
$$AA^{(D)} = X\begin{bmatrix} A_R & 0 \\ 0 & A_N \end{bmatrix}X^{-1}X\begin{bmatrix} A_R^{-1} & 0 \\ 0 & 0 \end{bmatrix}X^{-1} = X\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}X^{-1}$$
$$A^{(D)}A = X\begin{bmatrix} A_R^{-1} & 0 \\ 0 & 0 \end{bmatrix}X^{-1}X\begin{bmatrix} A_R & 0 \\ 0 & A_N \end{bmatrix}X^{-1} = X\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}X^{-1}$$
$\Rightarrow A^{(D)}A = AA^{(D)}$.
$$A^{(D)}AA^{(D)} = X\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}X^{-1}X\begin{bmatrix} A_R^{-1} & 0 \\ 0 & 0 \end{bmatrix}X^{-1} = X\begin{bmatrix} A_R^{-1} & 0 \\ 0 & 0 \end{bmatrix}X^{-1} = A^{(D)}$$
$$\begin{aligned} A^{(D)}A^{k+1} - A^k &= X\begin{bmatrix} A_R^{-1} & 0 \\ 0 & 0 \end{bmatrix}X^{-1}X\begin{bmatrix} A_R^{k+1} & 0 \\ 0 & A_N^{k+1} \end{bmatrix}X^{-1} - X\begin{bmatrix} A_R^k & 0 \\ 0 & A_N^k \end{bmatrix}X^{-1} \\ &= X\begin{bmatrix} A_R^k & 0 \\ 0 & 0 \end{bmatrix}X^{-1} - X\begin{bmatrix} A_R^k & 0 \\ 0 & 0 \end{bmatrix}X^{-1} = 0 \end{aligned}$$

3. Claim:
$$(\mathbf{ab}^*)^{(D)} = \begin{cases} \frac{\mathbf{ab}^*}{(\mathbf{b}^*\mathbf{a})^2}, & \mathbf{b}^*\mathbf{a} \neq 0 \\ 0, & \mathbf{b}^*\mathbf{a} = 0 \end{cases}$$
Proof: Assume that $\lambda \neq 0$.
$$\det(\lambda I_n - \mathbf{ab}^*) = \lambda^n \det(1 - \frac{1}{\lambda}\mathbf{b}^*\mathbf{a}) = \lambda^{n-1}(\lambda - \mathbf{b}^*\mathbf{a})$$
So $\lambda = \mathbf{b}^*\mathbf{a}$ is the only non-zero eigenvalue, when $\mathbf{b}^*\mathbf{a} \neq 0$.
When $\mathbf{b}^*\mathbf{a} = 0$, the characteristic polynomial has no non-trivial zeros.
Hence, when $\mathbf{b}^*\mathbf{a} = 0$,
$$(\mathbf{ab}^*)^{(D)} = X[0]X^{-1} = 0$$
When $\mathbf{b}^*\mathbf{a} \neq 0$, $\lambda = \mathbf{b}^*\mathbf{a}$ is the only non-zero eigenvalue, so
$$\mathbf{ab}^* = X\begin{bmatrix} \mathbf{b}^*\mathbf{a} & 0 \\ 0 & 0 \end{bmatrix}X^{-1}$$
This implies
$$(\mathbf{ab}^*)^{(D)} = X\begin{bmatrix} \frac{1}{\mathbf{b}^*\mathbf{a}} & 0 \\ 0 & 0 \end{bmatrix}X^{-1} = \frac{1}{(\mathbf{b}^*\mathbf{a})^2}X\begin{bmatrix} \mathbf{b}^*\mathbf{a} & 0 \\ 0 & 0 \end{bmatrix}X^{-1} = \frac{\mathbf{ab}^*}{(\mathbf{b}^*\mathbf{a})^2}$$

4. r in the context below is the size of the block matrix A_R.
To begin, prove a lemma first.
Lemma:
$S(k): A_R^{-k}$ is a polynomial of A_R (The following denote $A_R^{-k} = p_k(A_R)$), $\forall k \in \mathbf{Z}^+$.
$S(1):$
Consider the characteristic polynomial of A_R:
$$p_{A_R}(\lambda) = \sum_{i=0}^{r} a_i \lambda^i,$$
where $a_0 = (-1)^r \det(A_R) \neq 0$. Also,
$$p_{A_R}(A_R) = \sum_{i=0}^{r} a_i A_R^i = a_0 I + \sum_{i=1}^{r} a_i A_R^i = 0$$
$$\Rightarrow \left(\sum_{i=0}^{r-1}\left(-\frac{a_{i+1}}{a_0}\right)A_R^i\right)A_R = I \Rightarrow A_R^{-1} = \sum_{i=0}^{r-1}\left(-\frac{a_{i+1}}{a_0}\right)A_R^i = p_1(A_R)$$
So, A_R^{-1} is a polynomial of A_R, with $\deg p_1 \leq r-1$.
Assume $S(k)$ holds,
$$A_R^{-k} = p_k(A_R) = \sum_{i=0}^{r-1} a_{ik} A_R^i = a_{0k} I + \sum_{i=1}^{r-1} a_{ik} A_R^i$$

Then for $S(k+1)$,

$$A_R^{-(k+1)} = A_R^{-1} p_k(A_R) = a_{0k} A_R^{-1} + \sum_{i=1}^{r-1} a_{ik} A_R^{i-1} = a_{0k} \sum_{i=0}^{r-1} \left(-\frac{a_{i+1}}{a_0}\right) A_R^i + \sum_{i=1}^{r-1} a_{ik} A_R^{i-1} = p_{k+1}(A_R)$$

is again a polynomial, so $S(k) \Rightarrow S(k+1)$ and $S(n)$ holds, $\forall n \in \mathbf{Z}^+$.
Let $p(A) = A^{n-r} p_{n-r+1}(A)$,

$$p(A) = A^{n-r} p_{n-r+1}(A) = X \begin{bmatrix} A_R^{n-r} & 0 \\ 0 & A_N^{n-r} \end{bmatrix} X^{-1} X \begin{bmatrix} p_{n-r+1}(A_R) & 0 \\ 0 & p_{n-r+1}(A_N) \end{bmatrix} X^{-1}$$

$$= X \begin{bmatrix} A_R^{n-r} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_R^{-(n-r+1)} & 0 \\ 0 & p_{n-r+1}(A_N) \end{bmatrix} X^{-1} = X \begin{bmatrix} A_R^{-1} & 0 \\ 0 & 0 \end{bmatrix} X^{-1} = A^{(D)}$$

5. $AB = BA$, and $A^{(D)}$ is a polynomial of A, $e^{-A^{(D)}Bt}$ is a polynomial of $A^{(D)}B$, so all these four matrices commute. Therefore,

$$\begin{aligned} Av' + Bv &= A(e^{-A^{(D)}Bt} AA^{(D)} v_0)' + B(e^{-A^{(D)}Bt} AA^{(D)} v_0) \\ &= -AA^{(D)} B e^{-A^{(D)}Bt} AA^{(D)} v_0 + B e^{-A^{(D)}Bt} AA^{(D)} v_0 \\ &= -B e^{-A^{(D)}Bt} AA^{(D)} AA^{(D)} v_0 + B e^{-A^{(D)}Bt} AA^{(D)} v_0 \\ &= -B e^{-A^{(D)}Bt} AA^{(D)} v_0 + B e^{-A^{(D)}Bt} AA^{(D)} v_0 = 0 \end{aligned}$$

Solution 3

1.

$$(I_m - AB)^{-1} = I_m + AB + ABAB + ABABAB + \cdots$$

2.

$$(I_m - AB)^{-1} = \sum_{k=0}^{+\infty} (AB)^k = I_m + \sum_{k=1}^{+\infty} (AB)^k = I_m + A(\sum_{k=1}^{+\infty} (BA)^{k-1}) B = I_m + A(I_n - BA)^{-1} B$$

3. Let $p(x) = \sum_{k=0}^{n} a_k x^k$, then

$$Ap(BA) = A(\sum_{k=0}^{n} a_k (BA)^k) = \sum_{k=0}^{n} a_k A(BA)^k = \sum_{k=0}^{n} a_k (AB)^k A = p(AB)A$$

4. Assume that g is well-defined (i.e. analytic) on the spectrum of X and Y.
W.L.O.G., let $A = \begin{bmatrix} X & \\ & Y \end{bmatrix}$
Let s be the number of distinct eigenvalues of A, λ_i, $i = 1, 2, \cdots, s$ be the eigenvalues of A.
By Hermite interpolation, it is possible to find a polynomial p such that deg p = size t of the largest Jordan block such that

$$p^{(t')}(\lambda_i) = g^{(t')}(\lambda_i), \forall t' : 0 \le t' < t, \forall i : 1 \le i \le s$$

Then immediately we have,

$$g(J_i) = \begin{bmatrix} g(\lambda_i) & g'(\lambda_i)/1! & g''(\lambda_i)/2! & \cdots & g^{(n-1)}(\lambda_i)/(n-1)! \\ & g(\lambda_i) & g'(\lambda_i)/1! & \cdots & g^{(n-2)}(\lambda_i)/(n-2)! \\ & & \ddots & \ddots & \vdots \\ & & & g(\lambda_i) & g'(\lambda_i) \\ & & & & g(\lambda_i) \end{bmatrix}$$

$$= \begin{bmatrix} p(\lambda_i) & p'(\lambda_i)/1! & p''(\lambda_i)/2! & \cdots & p^{(n-1)}(\lambda_i)/(n-1)! \\ & p(\lambda_i) & p'(\lambda_i)/1! & \cdots & p^{(n-2)}(\lambda_i)/(n-2)! \\ & & \ddots & \ddots & \vdots \\ & & & p(\lambda_i) & p'(\lambda_i) \\ & & & & p(\lambda_i) \end{bmatrix} = p(J_i)$$

Due to the fact that g, X and Y are fixed, A can have its Jordan decomposition:

$$A = BJB^{-1}$$

such that

$$\begin{bmatrix} g(X) & \\ & g(Y) \end{bmatrix} = g(A) = Bg(J)B^{-1} = B\begin{bmatrix} g(J_1) & & \\ & g(J_2) & \\ & & \ddots \end{bmatrix}B^{-1} = \begin{bmatrix} p(J_1) & & \\ & p(J_2) & \\ & & \ddots \end{bmatrix}B^{-1}$$

$$= p(A) = \begin{bmatrix} p(X) & \\ & p(Y) \end{bmatrix}$$

5. f is defined on AB and BA, so $\exists p$ s.t.

$$p(AB) = f(AB), p(BA) = f(BA)$$

Therefore,

$$Af(BA) = Ap(BA) = p(AB)A = f(AB)A$$

6.

$$I_m + Af(BA)B = I_m + f(AB)AB = I_m + (\sum_{k=0}^{+\infty}(AB)^k)(AB) = \sum_{k=0}^{+\infty}(AB)^k = f(AB)$$

Solution 4

1. Let $X = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

   Then $XN = \begin{bmatrix} 0 & a_{11} & \cdots & a_{1(n-1)} \\ 0 & a_{21} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{n(n-1)} \end{bmatrix}$, $NX = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.

   By observation,

$$a_{i1} = 0, (i = 2, 3, \cdots, n), a_{1j} = 0, (j = 1, 2, \cdots, n-1), a_{ij} = a_{(i-1)(j-1)}$$

   Hence, X is an upper triangular matrix, with equal elements on every upper diagonals, which is a linear combination of powers of nilpotent matrices.

2.

$$Y = e^N = \sum_{k=0}^{+\infty}\frac{1}{k!}N^k = \sum_{k=0}^{n-1}\frac{1}{k!}N^k, Y - I = \sum_{k=1}^{n-1}\frac{1}{k!}N^k$$

$$(Y-I)^k = N^k\sum_{i_1=1}^{n-1}\cdots\sum_{i_k=1}^{n-1}\frac{1}{i_1!\cdots i_k!}N^{i_1+\cdots+i_k-k}$$

   Suppose by contradiction that $Y, Y-I, (Y-I)^2, \cdots (Y-I)^{n-1}$ are not independent matrices, then W.L.O.G. we may let

$$Y = \sum_{k=1}^{n-1}a_k(Y-I)^k$$

   That immediately results in

$$I + \sum_{k=1}^{n-1}\frac{1}{k!}N^k = Y = \sum_{k=1}^{n-1}a_kN^k\sum_{i_1=1}^{n-1}\cdots\sum_{i_k=1}^{n-1}\frac{1}{i_1!\cdots i_k!}N^{i_1+\cdots+i_k-k}$$

then, I cannot be represented by linear combinations of the other matrices, which is a contradiction (so Y is independent to other matrices).
Similarly, if we let

$$(Y-I)^j = \sum_{k=j+1}^{n-1} a_k (Y-I)^k$$

then,

$$(N^j)\sum_{i_1=1}^{n-1}\cdots\sum_{i_j=1}^{n-1}\frac{1}{i_1!\cdots i_j!}N^{i_1+\cdots+i_j-j} = (Y-I)^j = \sum_{k=j+1}^{n-1} a_k N^k \sum_{i_1=1}^{n-1}\cdots\sum_{i_k=1}^{n-1}\frac{1}{i_1!\cdots i_k!}N^{i_1+\cdots+i_k-k}$$

With regards to $(Y-I)^j$, the elements of the N^j component cannot be expressed by the linear combinations of the matrices of $(Y-I)^{j+1}$, $(Y-I)^{j+2},\cdots$ which is another contradiction.
Therefore, these matrices are linearly independent.
In addition, since $N^n = 0$, but $N^{n-1} \neq 0$, so dimension of the solution space of X is n, corresponding to these n matrices, therefore these matrices form a basis of the solution space of X.

3. Suppose X is a solution to $e^X = e^N$. Then

$$YX = e^N X = e^X X = \sum_{k=0}^{+\infty}\frac{X^k}{k!}\cdot X = X\cdot\sum_{k=0}^{+\infty}\frac{X^k}{k!} = Xe^X = Xe^N = XY$$

Moreover, $[X, Y-I] = [X,(Y-I)^2] = \cdots = [X,(Y-I)^{n-1}] = 0$.
Also, $N = a_0 Y + \sum_{i=1}^{n-1} a_i (Y-I)^i$, so

$$XN = a_0 XY + \sum_{i=1}^{n-1} a_i X(Y-I)^i = a_0 YX + \sum_{i=1}^{n-1} a_i (Y-I)^i X = NX$$

So, $[X,N]=0$, and

$$e^{-N}e^X = I = e^X e^{-N} \Rightarrow e^{X-N} = I$$

Hence,

$$X - N = 2\pi k i I \Rightarrow X = N + 2\pi k i I, k \in \mathbf{Z}$$

4. Before searching for such a pair of (A,B), the following lemma would be proved first:

$$\forall A : A^2 = -4\pi^2 I \Rightarrow e^A = I$$

Proof:

$$e^A = \sum_{k=0}^{+\infty}\frac{A^{2k}}{(2k)!} + \sum_{k=0}^{+\infty}\frac{A^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{+\infty}\frac{(-1)^k(2\pi)^{2k}}{(2k)!}I + \frac{1}{2\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k(2\pi)^{2k+1}}{(2k+1)!}A = \cos 2\pi I + \frac{1}{2\pi}\sin 2\pi A = I$$

Let $X = 2\pi\cdot\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$, $Y = 2\pi\cdot\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Clearly,

$$[X,Y] = 4\pi^2\left(\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = 4\pi^2\cdot\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} \neq 0$$

, however $e^X = I = e^Y$

5. Suppose X exists, then

$$\sin^2(X) + \cos^2(X) = (\frac{e^{iX}-e^{-iX}}{2i})^2 + (\frac{e^{iX}+e^{-iX}}{2})^2 = \frac{2I - e^{2iX} - e^{-2iX}}{4} + \frac{e^{2iX}+e^{-2iX}+2I}{4} = I$$

Therefore,

$$\cos^2(X) = I - \sin^2(X) = \begin{bmatrix} 0 & -3992 \\ 0 & 0 \end{bmatrix}$$

This implies $\cos^2(X)$ is nilpotent, so is $\cos(X)$. However, it means the entry of -3992 should actually be zero, which is a contradiction. ■

Solution 5

1. Let λ_n be any one of the eigenvalues of X_n, where v is the eigenvector associates with it, i.e. $X_n v = \lambda_n v$.
   Then,this immediately yields $X_n^{-1} v = v/\lambda$.
   That means X_{n+1} and X_n both have the eigenvector v.
   Since $X_{n+1} = \frac{1}{2}(X_n + X_n^{-1})$, if λ_{n+1} satisfies $X_{n+1}v = \lambda_{n+1}v$, then $\lambda_{n+1} = \frac{1}{2}(\lambda_n + 1/\lambda_n)$.
   Let $\lambda_n = x + iy$, $x \neq 0$
   $$\lambda_{n+1} = \frac{x(x^2+y^2+1)}{2(x^2+y^2)} + i\frac{y(x^2+y^2-1)}{2(x^2+y^2)}$$
   So that implies $\Re(\lambda_{n+1}) \neq 0$. ∎

2. $$\frac{X_{n+1}+1}{X_{n+1}-1} = \frac{X_n + X_n^{-1} + 2}{X_n + X_n^{-1} - 2} = \frac{(X_n+1)^2}{(X_n-1)^2} = \frac{(A+1)^{2^{n+1}}}{(A-1)^{2^{n+1}}} \Rightarrow X_n = \frac{\frac{(A+1)^{2^n}}{(A-1)^{2^n}}+1}{\frac{(A+1)^{2^n}}{(A-1)^{2^n}}-1} = \frac{1+\frac{(A-1)^{2^n}}{(A+1)^{2^n}}}{1-\frac{(A-1)^{2^n}}{(A+1)^{2^n}}}$$
   When $\Re(A) > 0$, $|A+1| > |A-1|$,
   $$\lim_{n\to+\infty} X_n = \lim_{n\to+\infty} \frac{1+\frac{(A-1)^{2^n}}{(A+1)^{2^n}}}{1-\frac{(A-1)^{2^n}}{(A+1)^{2^n}}} = \frac{1+0}{1-0} = 1$$
   When $\Re(A) < 0$, $|A+1| < |A-1|$,
   $$\lim_{n\to+\infty} X_n = \lim_{n\to+\infty} \frac{\frac{(A+1)^{2^n}}{(A-1)^{2^n}}+1}{\frac{(A+1)^{2^n}}{(A-1)^{2^n}}-1} = \frac{0+1}{0-1} = -1$$
   So in general, for one by one matrix,
   $$\lim_{n\to+\infty} X_n = \mathrm{sgn}(A)$$

3. A is diagonalizable, so is X_n.
   $$X_n = B\Lambda B^{-1}$$
   Then,
   $$\lim_{t\to+\infty} X_t = B \lim_{t\to+\infty} \mathrm{diag}(\lambda_{1t}, \cdots, \lambda_{nt}) B^{-1}$$
   When $A = B\mathrm{diag}(\lambda_{10}, \cdots, \lambda_{n0})B^{-1}$,
   $$B \lim_{t\to+\infty} \mathrm{diag}(\lambda_{1t}, \cdots, \lambda_{nt}) B^{-1} = B\mathrm{diag}(\mathrm{sgn}(\lambda_{10}), \cdots, \mathrm{sgn}(\lambda_{n0}))B^{-1} = \mathrm{sgn}(A)$$

4. Similar to (2), there is also a formula for X_n:
   $$X_n = [I - (A-I)^{2^n}(A+I)^{-2^n}]^{-1}[I + (A-I)^{2^n}(A+I)^{-2^n}]$$
   Here, $A = I + N$.
   Moreover, $2^{n-1} \geq n, \forall n \in \mathbf{Z}^+$, then
   $$X_{n-1} = [I - (I+N-I)^{2^{n-1}}(I+N+I)^{-2^{n-1}}]^{-1}[I + (I+N-I)^{2^{n-1}}(I+N+I)^{-2^{n-1}}]$$
   $$= [I - N^n(2I+N)^{-2^{n-1}}]^{-1}[I + N^n(2I+N)^{-2^{n-1}}] = I$$

Solution 6

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

1. On one hand,

$$AB = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

On the other hand,

$$f_A(f_B(x)) = \frac{a_1(\frac{a_2x+b_2}{c_2x+d_2}) + b_1}{c_1(\frac{a_2x+b_2}{c_2x+d_2}) + d_1} = \frac{a_1a_2x + a_1b_2 + b_1c_2x + b_1d_2}{c_1a_2x + c_1b_2 + d_1c_2x + d_1d_2} = \frac{(a_1a_2 + b_1c_2)x + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)x + (c_1b_2 + d_1d_2)}$$

Therefore $f_A \circ f_B = f_{AB}$

2. If $k \neq 0$, then

$$f_{kA}(x) = \frac{(ka_1)x + (kb_1)}{(kc_1)x + (kd_1)} = \frac{a_1x + b_1}{c_1x + d_1} = f_A(x)$$

Conversely, when $f_A(x) \equiv f_B(x)$

$$\frac{a_1x + b_1}{c_1x + d_1} \equiv \frac{a_2x + b_2}{c_2x + d_2} \Leftrightarrow a_1c_2x^2 + (a_1d_2 + b_1c_2)x + b_1d_2 \equiv a_2c_1x^2 + (b_2c_1 + d_1a_2)x + b_2d_1$$

As x is an arbitary value, let $k_1 = a_1c_2 = a_2c_1$, $k_2 = b_1d_2 = b_2d_1$, then

$$a_2 = k_1/c_1, b_2 = k_2/d_1, c_2 = k_1/a_1, d_2 = k_2/b_1$$

$$a_1d_2 + b_2c_2 = d_1a_2 + b_2c_1 \Rightarrow \frac{a_1}{b_1}k_2 + \frac{b_1}{a_1}k_1 = \frac{d_1}{c_1}k_1 + \frac{c_1}{d_1}k_2 \Rightarrow \frac{k_1}{k_2} = \frac{a_1c_1}{b_1d_1}$$

With all these,

$$\frac{a_1c_1}{b_1d_1} = \frac{a_1c_2}{b_1d_2} \Rightarrow \frac{c_1}{c_2} = \frac{d_1}{d_2}, \frac{a_1c_1}{b_1d_1} = \frac{a_1c_2}{b_2d_1} \Rightarrow \frac{b_1}{b_2} = \frac{c_1}{c_2}, \frac{a_1c_1}{b_1d_1} = \frac{a_2c_1}{b_2d_1} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

Hence

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2} \Leftrightarrow A = kB$$

for some constant k.

3.

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1x + b_1y \\ c_1x + d_1y \end{bmatrix}$$

is interpreted as $\frac{a_1x+b_1y}{c_1x+d_1y}$, while

$$f_A\left(\frac{x}{y}\right) = \frac{a_1(x/y) + b_1}{c_1(x/y) + d_1} = \frac{a_1x + b_1y}{c_1x + d_1y}$$

So they are equal.

4. Let x be the fixed point of transformation f_A.
If $x \in \mathbf{R}$, then

$$\frac{ax + b}{cx + d} = x \Leftrightarrow cx^2 - (a - d)x - b = 0$$

So excluding ∞, f_A may have every points fixed, two points fixed, one point fixed or none fixed.
Every $x \in \mathbf{R}$ is fixed $\Leftrightarrow cx^2 - (a - d)x - b \equiv 0 \Leftrightarrow \begin{cases} b = c = 0 \\ a = d \neq 0 \end{cases}$
f_A has two fixed points $\Leftrightarrow c \neq 0 \wedge (a - d)^2 + 4bc > 0$
f_A has one single fixed point $\Leftrightarrow (c \neq 0 \wedge (a - d)^2 + 4bc = 0) \vee (c = 0 \wedge a \neq d)$
f_A has no fixed point $\Leftrightarrow (c \neq 0 \wedge (a - d)^2 + 4bc < 0) \vee (c = 0 \wedge a = d \wedge b \neq 0)$
If $x = \infty$, then

$$f_A(x) = x \Leftrightarrow \frac{c + d/x}{a + b/x} = 0 \Leftrightarrow c = 0$$

In conclusion, these are the possibilities of the fixed points of f_A:

number of fixed points	conditions
∞	$\begin{cases} b = c = 0 \\ a = d \neq 0 \end{cases}$
2	$\begin{cases} (a-d)^2 + 4bc > 0, & c \neq 0 \\ a \neq d, & c = 0 \end{cases}$
1	$\begin{cases} (a-d)^2 + 4bc = 0, & c \neq 0 \\ a = d \wedge b \neq 0, & c = 0 \end{cases}$
0	$(a-d)^2 + 4bc < 0 \wedge c \neq 0$