Linear Algebra Homework 2

2 *Oct* 2021

***Question* 1**.

Give a geometric description of Span $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ for the vectors

$$\overrightarrow{v_1} = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix} \; and \; \overrightarrow{v_2} = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}.$$

Here geometric description means a necessary and sufficient condition on the coordinates of the vector for it to be in Span$\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$.

Solution 1.

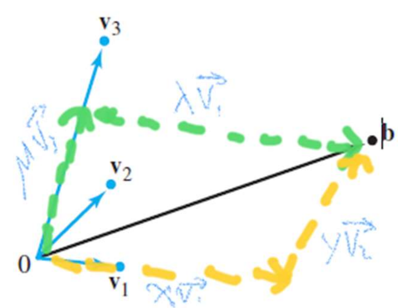
$$\because \overrightarrow{v_2} = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix} = 1.5 \cdot \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix} = 1.5 \cdot \overrightarrow{v_1}$$

$$\therefore Span\{\overrightarrow{v_1}, \overrightarrow{v_2}\} \; is \; a \; line \; in \; \mathbb{R}^3 \; through \; (0,0,0) \; with \; direction \; parallel \; to \; \overrightarrow{v_1}.$$

***Question* 2**.

Consider the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ and $\vec{b}$ in $\mathbb{R}^2$, shown in figure. Does the equation $x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} + x_3\overrightarrow{v_3} = \vec{b}$ have a solution? Is the solution unique? Use the figure to explain your answers.

Solution 2.



$From\ the\ figure, \{\overrightarrow{v_1}, \overrightarrow{v_2}\}, \{\overrightarrow{v_1}, \overrightarrow{v_2}\}\ and\ \{\overrightarrow{v_1}, \overrightarrow{v_2}\}\ are\ all\ independent.$

$Without\ loss\ of\ generality, choose\ \overrightarrow{v_1}\ and\ \overrightarrow{v_2}\ to\ span\ \mathbb{R}^2.$

$Setting\ x_3 = 0, then\ the\ equation\ becomes\ x_1\overrightarrow{v_1} + x_2\overrightarrow{v_2} = \vec{b}$

$Clearly\ \vec{b}\ is\ a\ linear\ combination\ of\ \overrightarrow{v_1}\ and\ \overrightarrow{v_2},$

$so\ \exists x, y, x \neq 0, y \neq 0\ s.t. (x_1, x_2, x_3) = (x, y, 0)\ is\ a\ solution.$

$Similarly, by\ choosing\ \overrightarrow{v_2}\ and\ \overrightarrow{v_3}\ to\ span\ \mathbb{R}^2,$

$then\ \exists \lambda, \mu, \lambda \neq 0, \mu \neq 0\ s.t.\ (x_1, x_2, x_3) = (\lambda, 0, \mu)\ is\ a\ solution.$

$\therefore Solution\ to\ the\ equation\ exists, and\ is\ not\ unique.$

***Question* 3**.

Let $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\vec{u}$ in the plane $\mathbb{R}^3$ spanned by the columns $of\ A$? Why or why not?

Solution 3.

$$If\ \vec{u}\ is\ on\ the\ plane, then\ \exists a, b \in \mathbb{R}\ s.t. \vec{u} = a \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix}$$

$$To\ begin, solve\ (S): \begin{bmatrix} 3 & -5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, in\ which\ a, b\ satsifies\ a + b = 4.$$

$First\ row\ of\ the\ equation\ arises\ 3a = 5b, thus\ the\ second\ row\ becomes$

$$-2a + \frac{18}{5}a = \frac{8}{5}a = 4. \therefore\ a = 2.5 \Rightarrow\ b = 2.5 \cdot 0.6 = 1.5\ and\ a + b = 2.5 + 1.5 = 4.$$

$$Thus, \vec{u} = 2.5 \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + 1.5 \cdot \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix}\ which\ means\ \vec{u} \in C(A).$$

***Question* 4.**

$$Let\ B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}. Do\ the\ columns\ of\ B\ span\ \mathbb{R}^4?$$

Does the equation $B\vec{x} = \vec{y}$ have a solution for each $\vec{y} \in \mathbb{R}^4$?

Solution 4.

$$B\vec{x} = \vec{0}\ has\ non\ trivial\ solutions \Leftrightarrow\ the\ column\ vectors\ of\ B\ are\ linearly\ dependent$$

$$\Leftrightarrow Columns\ of\ B\ cannot\ span\ \mathbb{R}^4 \Leftrightarrow \exists\vec{y} \in \mathbb{R}^4\ s.t.\ B\vec{x} = \vec{y} \text{ has no solution.}$$

$$The\ augmented\ matrix\ corresponds\ to\ B\vec{x} = \vec{0}\ is$$

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 0 \\ 0 & 1 & 1 & -5 & 0 \\ 1 & 2 & -3 & 7 & 0 \\ -2 & -8 & 2 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 0 \\ 0 & 1 & 1 & -5 & 0 \\ 0 & -1 & -1 & 5 & 0 \\ 0 & -2 & -2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 0 \\ 0 & 1 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 \end{bmatrix}$$

$$Row\ 3\ arises\ 0 = 0, so\ \ B\vec{x} = \vec{0}\ has\ non\ trivial\ solutions,$$

$$i.e. column\ vectors\ of\ B\ are\ linearly\ dependent\ that\ do\ not\ span\ \mathbb{R}^4, and$$

$$\exists y \in \mathbb{R}^4\ s.t.\ \ B\vec{x} = \vec{y} \text{ has no solution.}$$

***Question* 5.**

Solve the homogenous linear system of equations and write the set of solutions in parametric vector form.

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 0, \\ x_1 + 4x_2 - 8x_3 = 0, \\ -3x_1 - 7x_2 + 9x_3 = 0. \end{cases}$$

Solution 5.

$$The\ augmented\ matrix\ corresponds\ to\ the\ system\ is$$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Choose\ x_3\ as\ free\ variable\ and\ let\ x_3 = t, where\ t \in \mathbb{R}\ is\ the\ parameter.$$

$$Then\ x_2 = 3t, x_1 = 5x_3 - 3x_2 = -4t, or\ \begin{bmatrix} x_1 \\ x_2 \\ x3 \end{bmatrix} = t \cdot \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

***Question* 6.**

Solve the nonhomogeneous linear system and write the set of solutions in parametric vector form.

Provide a geometric comparison with the solution set in the above question.

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4, \\ x_1 + 4x_2 - 8x_3 = 7, \\ -3x_1 - 7x_2 + 9x_3 = -6. \end{cases}$$

Solution 6.

$$\text{The augmented matrix corresponds to the system is}$$

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Choose\ x_3 as\ the\ free\ variable\ again, and\ let\ x_3 = t, where\ t \in \mathbb{R}\ is\ the\ parameter.$$

$$Then\ x_2 = 3 + 3t, x_1 = 4 + 5x_3 - 3x_2 = -5 - 4t, or \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

$Geometric\ comparsion: The\ solution\ of\ the\ latter\ question\ is\ the\ line\ corresponds\ to\ the\ locus\ of$ $points\ of\ solutions\ to\ the\ former\ question\ translated\ by\ a\ constant\ vector \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}.$

***Question* 7**.

Find the value(s) of h for which the vectors are linearly dependent. Justify the answer.

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}.$$

Solution 7.

$$The\ vectors\ are\ linearly\ dependent, so\ \exists a, b \in \mathbb{R}\ s.t.\ a \cdot \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

$$Solving \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}\ by\ requiring - 4a + 7b = h.$$

$$\because \begin{bmatrix} 2 & -6 & 8 \\ 1 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore Treat\ b\ as\ the\ free\ variable, and\ b = t, t \in \mathbb{R}\ is\ parameter.$$

$$Then\ a = 4 + 3t \Rightarrow h = -4a + 7b = -16 - 5t$$

$$Conclusion: Let\ t \in \mathbb{R}\ as\ the\ parameter. If\ a = 4 + 3t, b = t\ satisfies\ a \cdot \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix},$$

$$then\ h = -16 - 5t$$

Question 8.

(1) The columns of a matrix A are linearly independent, if the equation $A\vec{x} = \vec{0}$ has the trivial solution.

(2) If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

(3) The columns of any 4×5 matrix are linearly dependent.

(4) If $\vec{x}$ and $\vec{y}$ are linearly indenpendent, and if $\vec{x}, \vec{y}, \vec{z}$ is linearly dependent, then $\vec{z}$ is in $Span\{\vec{x}, \vec{y}\}$.

Solution 8

(1) $\boldsymbol{True}. If\ A\vec{x} = \vec{0}\ has\ trivial\ solution\ as\ the\ unique\ solution\ , then\ for\ matrix\ A_{n \times n},$
$its\ kernel\ N(A)\ does\ only\ have\ zero\ vector. As\ \dim C(A) + \dim N(A) = n, \dim C(A) = n,$
$every\ vector\ in\ \mathbb{R}^n\ can\ be\ expressed\ as\ a\ linear\ combination\ of\ the\ columns\ of\ A.$
$For\ n\ vectors\ to\ span\ \mathbb{R}^n, they\ must\ be\ linearly\ independent.$

(2) $\boldsymbol{True}$. $Let\ S = \{\overrightarrow{v_i} : 1 \leq i \leq n, n \in \mathbb{Z}^+\}. As\ \overrightarrow{v_i}\ are\ linear\ dependent,$

$then\ \exists\ x_i, x_j \in \mathbb{R}\ s.t.\ x_i \neq 0, x_j \neq 0,$

$$and\ \sum_{i=0}^{n} x_i\, \overrightarrow{v_i} = \vec{0}\,.$$

$$Without\ loss\ of\ generality\ let\ x_n = x_j.$$

$$\overrightarrow{v_n} = -\frac{1}{x_n} \cdot \sum_{i=1}^{n-1} x_i\, \overrightarrow{v_i}$$

$In\ which\ \overrightarrow{v_n}\ is\ a\ linear\ combination\ of\ the\ other\ vectors.\ Since\ ordering\ is\ arbitary, other\ vectors$ $in\ S\ would\ have\ similar\ results.$

(3) $\boldsymbol{True}$. $The\ matrix\ has\ 4\ rows, so\ the\ dimension\ of\ the\ column\ space,$ $\dim C(A) \leq 4$.

$However, A\ has\ 5\ column\ vectors, and\ 4\ independent\ vectors\ are\ enough\ to\ span\ \mathbb{R}^4, so\ the\ 5^{th}\ vector$ $must\ be\ a\ linear\ combination\ of\ the\ previous\ 4\ vectors.$

(4) $\boldsymbol{True}$. $\vec{x}, \vec{y}, \vec{z}$ is linearly dependent, so $\exists$a, b, c $\in \mathbb{R}$ s.t. $a\vec{x} + b\vec{y} + c\vec{z} = \vec{0}, and\ c \neq 0.$

$Then\ \vec{z} = -\frac{a}{c}\vec{x} - \frac{b}{c}\vec{y}\,, which\ is\ a\ linear\ combination\ of\ \vec{x}\ and\ \vec{y}\,.\therefore\ \vec{z} \in Span\{\,\vec{x}\,, \vec{y}\,\}.$

Question 9.

(1) Two vectors are linearly dependent if and only if they lie on a line through the origin.

(2) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

(3) If $\vec{x}$ and $\vec{y}$ are linearly independent, and if $\vec{z}$ is in Span$\{\vec{x}, \vec{y}\}$, then $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent.

(4) If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more vectors than there are entries in each vector.

Solution 9

(1) $\boldsymbol{True}$. $Assume\ the\ line\ has\ a\ parametric\ equation\ of\ \vec{r} = \lambda\overrightarrow{r_0}, where\ \lambda\ is\ the\ parameter.$

$If\ \vec{a}\ and\ \vec{b}\ are\ linearly\ dependent, then\ \exists c_1, c_2, c_1 \neq 0, c_2 \neq 0\ s.t.\ c_1\vec{a} + c_2\vec{b} = \vec{0}$

$$\Rightarrow \vec{a} = -\frac{c_2}{c_1}\,\vec{b}\,, so\ they\ both\ lie\ on\ the\ same\ line\ that\ passes\ through\ origin.$$

$If\ two\ vectors\ \vec{a}, \vec{b}\ are\ both\ on\ the\ same\ line, then\ \vec{a} = \lambda_1\overrightarrow{r_0}, \vec{b} = \lambda_2\overrightarrow{r_0}\ for\ some\ \lambda_1\ and\ \lambda_2. Clearly,$ $\lambda_2\vec{a} - \lambda_1\vec{b} = \vec{0}, so\ \vec{a}\ and\ \vec{b}\ are\ linearly\ dependent.$

(2) $\boldsymbol{False}$. $\overrightarrow{v_1}\ and\ \overrightarrow{v_2}\ in\ Question\ 1\ is\ a\ counter\ example\ that\ the\ vectors\ have\ 3\ entries, and\ the\ set\ has$ $2\ vectors, but\ \overrightarrow{v_1}, \overrightarrow{v_2}\ are\ linearly\ dependent.$

(3) $\boldsymbol{True}$. $\vec{z} \in Span\{\vec{x}, \vec{y}\}, so\ \exists\ a, b\ s.t.\ a \neq 0\ or\ b \neq 0\ and\ \vec{z} = a\vec{x} + b\vec{y} \Leftrightarrow a\vec{x} + b\vec{y} - \vec{z} = \vec{0},$ $thus\ \{\vec{x}, \vec{y}, \vec{z}\}\ is\ linearly\ dependent.$

(4) $\boldsymbol{False}$. $The\ set\ in\ Question\ 1\ is\ in\ \mathbb{R}^3, in\ which\ the\ vectors\ are\ linearly\ dependent. However, in\ each\ vector,$ $there\ are\ 3\ entries, but\ only\ 2\ vectors\ are\ in\ the\ set.$

Question 10.

Suppose A is an $m \times n$ matrix with the property that for all $\vec{b}$ in $\mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has at most one solution. Use the definition of linear independence to explain why the columns of A must be linearly independent.

Solution 10.

$Since\ \vec{b}\ is\ arbitary, let\ \vec{b} = \vec{0}, so\ A\vec{x} = \vec{0}\ has\ at\ most\ one\ solution, i.e. the\ trivial\ solution,$

$which\ is\ the\ definition\ of\ linear\ independence.$

Question 11.

Suppose an $m \times n$ matrix A has n pivot columns.

Explain why $\forall \vec{b} \in \mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has at most one solution.

Solution 11.

$The\ matrix\ has\ n\ columns\ and\ pivot\ columns, so\ all\ variables\ w.r.t.\ A\vec{x} = \vec{b}\ are\ basic\ variables.$

$Thus\ if\ A\vec{x} = \vec{b}\ is\ solvable, the\ solution\ must\ be\ unique.$