Un-weighted least-squares fitting in HFRPtoolbox

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1 Method

This is the part of this manuscript (([*Kim et al.*(2007)*Kim, Terrill, and Cornuelle*])). The weighted least-squares fitting (WLS, [*Wunch*(1996)]) method to estimate the vector current ($\mathbf{u}$, 2×1 elements) at a vector grid point (x_k, y_k) assumes that each radial velocity is a linear combination of the vector current components. The radial velocities ($\mathbf{r}$, $L \times 1$ elements) within the search radius (d_0) of the vector grid point and the projection matrix ($\mathbf{G_a}$, $L \times 2$ elements) are:

$$\mathbf{r} = \mathbf{G_a u} + \mathbf{n_a} \,, \tag{1.1}$$

where

$$\mathbf{G_a} = \begin{bmatrix} \mathbf{g}_1^{\mathrm{T}} \\ \mathbf{g}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{g}_L^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \\ \vdots & \vdots \\ \cos\theta_L & \sin\theta_L \end{bmatrix} \,, \ \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} \,, \tag{1.2}$$

$\mathbf{g}_i = [\cos\theta_i \ \sin\theta_i]^{\mathrm{T}}$, θ_i denotes the bearing angle at the ith radar grid point (x_i, y_i) measured counter-clockwise from East, and $\mathbf{n_a}$ is the residual when the radial velocities are fit by a single vector current ($^{\mathrm{T}}$ denotes the matrix transpose). The least-squares estimate for the vector current is

$$\hat{\mathbf{u}} = \left(\mathbf{G_a^{\mathrm{T}}} \mathbf{R}^{-1} \mathbf{G_a} + \mathbf{P}^{-1}\right)^{-1} \mathbf{G_a^{\mathrm{T}}} \mathbf{R}^{-1} \mathbf{r} \,. \tag{1.3}$$

The model covariance matrix, $\mathbf{P} \equiv \langle \mathbf{u}\mathbf{u}^{\mathrm{T}} \rangle$, is the expected covariance of the unknown velocity, where $\langle \cdot \rangle$ denotes the expected value. The measurement uncertainty covariance matrix is $\mathbf{R} \equiv \langle \mathbf{n_a} \mathbf{n_a}^{\mathrm{T}} \rangle$. In simple least-squares fitting, both $\mathbf{P}$ and $\mathbf{R}$ are assumed to be a scaled identity matrix:

$$\mathbf{P} = \sigma_s^2 \mathbf{I} \,, \ \ \mathbf{R} = \sigma_r^2 \mathbf{I}. \tag{1.4}$$

With the assumption of infinite signal variance ($\sigma_s^2 = \infty$) and unit error variance ($\sigma_r^2 = 1$), the WLS method matches the un-weighted least-squares fitting (UWLS) method, which is the standard method in the estimate of vector currents ([*Lipa and Barrick*(1983)], [*Gurgel*(1994)], [*Graber et al.*(1997)*Graber, Haus, Chapman, and Shay*]). The UWLS method also assumes constant vector currents within the search radius (d_0) of the vector grid point, and the estimate is

$$\hat{\mathbf{u}} = \left(\mathbf{G_a^{\mathrm{T}}} \mathbf{G_a}\right)^{-1} \mathbf{G_a^{\mathrm{T}}} \mathbf{r} \,, \tag{1.5}$$

where $\mathbf{G_a^T G_a}$ is called the geometric covariance matrix (2×2 elements), of which inverse matrix is geometric dilution of precision (GDOP).

The *a posterior* uncertainty matrices in the WLS and UWLS methods are

$$\hat{\mathbf{P}} = \left(\mathbf{G_a^T R^{-1} G_a} + \mathbf{P}^{-1}\right)^{-1} , \tag{1.6}$$

$$\hat{\mathbf{P}} = \left(\mathbf{G_a^T G_a}\right)^{-1} , \tag{1.7}$$

respectively.

The normalized uncertainty matrix ($\boldsymbol{\varphi}$) is introduced in order to compare the uncertainty in the estimate between the UWLS and WLS methods. The corresponding normalized uncertainty matrix in the OI method is the *a posteriori* uncertainty matrix (Equation 1.6) divided by the observational error variance (σ_r^2):

$$\boldsymbol{\varphi} = \left(\mathbf{G_a^T G_a} + \mathbf{P}^{-1}\sigma_r^2\right)^{-1} , \tag{1.8}$$

recall that $\mathbf{R} = \sigma_r^2 \mathbf{I}$. On the other hand, the uncertainty matrix in the UWLS method is already scaled with the observational error variance to the implied error variance 1 cm^{2}s^{-2}:

$$\boldsymbol{\varphi} = \left(\mathbf{G_a^T G_a}\right)^{-1} . \tag{1.9}$$

References

[*Graber et al.*(1997)*Graber, Haus, Chapman, and Shay*] Graber, H. C., B. K. Haus, R. D. Chapman, and L. K. Shay (1997), HF radar comparisons with moored estimates of current speed and direction: Expected differences and implications, *J. Geophys. Res.*, *102(C8)*, 18,749–18,766.

[*Gurgel*(1994)] Gurgel, K.-W. (1994), Shipborne measurements of surface current fields by HF radar, *L'Onde Electrique*, *74*(5).

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[*Wunch*(1996)] Wunch, C. (1996), *The ocean circulation inverse problem*, Cambridge University Press.

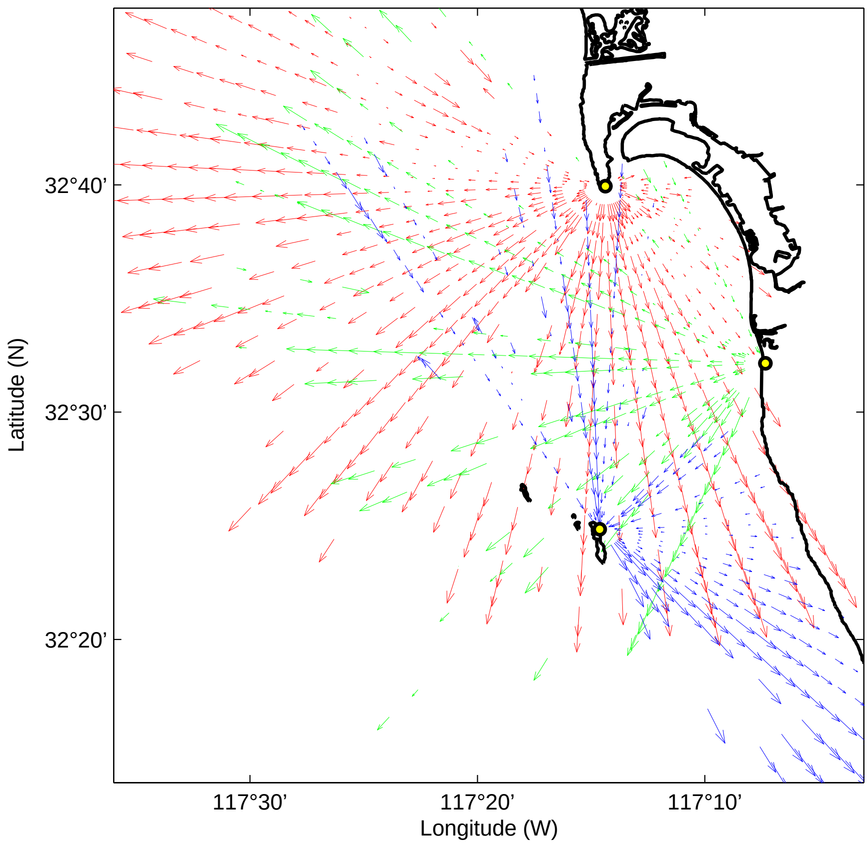

Figure 1: An example of the radial velocity.

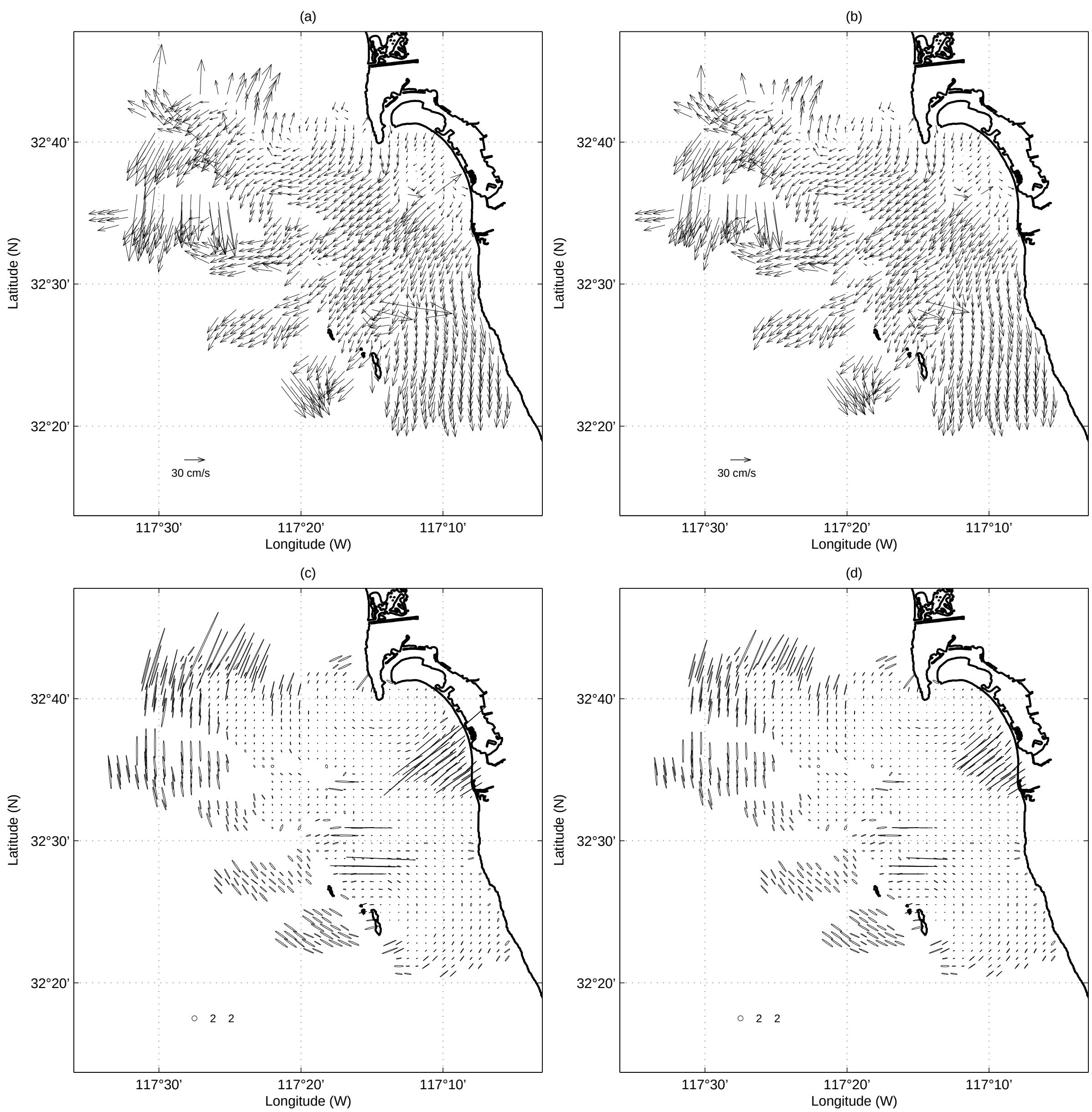

Figure 2: An example of the surface current vector and a posteriori uncertainty normalized by the observational error variance (a) UWLS (b) WLS (c) UWLS (φ) (d) WLS (φ)